



## Large Orders for Self-Avoiding Membranes

François David<sup>\*†</sup>

SPhT, C.E.A. Saclay, 91191 Gif-sur-Yvette Cedex, France

Kay Jörg Wiese<sup>‡</sup>

Fachbereich Physik, Universität GH Essen, 45117 Essen, Germany

### Abstract

We derive the large order behavior of the perturbative expansion for the continuous model of tethered self-avoiding membranes. It is controlled by a classical configuration for an effective potential in bulk space, which is the analog of the Lipatov instanton, solution of a highly non-local equation. The  $n$ -th order is shown to have factorial growth as  $(-cst)^n (n!)^{(1-\epsilon/D)}$ , where  $D$  is the “internal” dimension of the membrane and  $\epsilon$  the engineering dimension of the coupling constant for self-avoidance. The instanton is calculated within a variational approximation, which is shown to become exact in the limit of large dimension  $d$  of bulk space. This is the starting point of a systematic  $1/d$  expansion. As a consequence, the  $\epsilon$ -expansion of self-avoiding membranes has a factorial growth, like the  $\epsilon$ -expansion of polymers and standard critical phenomena, suggesting Borel summability. Consequences for the applicability of the 2-loop calculations are examined.

Submitted to Nucl. Phys. **B**

---

\*Email: david@spht.saclay.cea.fr

†Physique Théorique CNRS

‡Email: wiese@theo-phys.uni-essen.de

# 1 Introduction

Flexible polymerized tethered membranes (also called polymerized membranes or tethered membranes) exhibit fascinating statistical properties [1,2]. Tethered membranes with only short-range repulsive interactions (“self-avoidance”) may be seen as the 2-dimensional analog of polymer chains in a good solvent. They are expected to exist in two phases: (1) a flat (or rigid) phase with an infinite persistence length but non-classical elastic properties, not found for polymers which have a finite persistence length [3–5], and (2) a crumpled phase similar to that of polymers [6,7]. At variance with polymers, the crumpled phase is always swollen (whatever the dimension  $d$  of space is) and has a configuration exponent  $\nu$  larger than the mean-field exponent  $\nu_{\text{mf}} = 0$ , whereas for polymers it is well known that for  $d \geq 4$  one has  $\nu = \nu_{\text{mf}} = 1/2$  while for  $d < 4$  one has  $\nu > \nu_{\text{mf}}$ . This follows from a simple dimensional argument, which shows that self-avoidance is always relevant (in the renormalization group sense) [6].

The properties of the crumpled swollen phase have been much studied theoretically. In particular, a systematic framework for renormalization group (RG) calculations has been proposed in [8–10]. It consists in: (1) a self-avoiding membrane model (SAM) which is a non-trivial extension of the continuous Edwards model for polymers [11,12], and (2) a renormalization procedure which generalizes the direct renormalization [13] for the Edwards model. For 2-dimensional tethered membranes, the upper critical dimension  $d_{\text{uc}}$  is infinite, but the SAM model can be extended to membranes with non-integer internal dimension  $D < 2$ , which have a finite  $d_{\text{uc}}$ . Then the framework of [8,9] allows to perform a double expansion both in  $D$  (the internal dimension of the membrane) and  $d$  (the dimension of bulk space) to study the physical case ( $D = 2, d < \infty$ ) of a 2-dimensional membrane in finite dimension  $d$ . The starting point for the  $\epsilon$ -expansion (characterized by the dimension  $D_0 < 2$  for the membrane) can be chosen arbitrarily. This provides an additional parameter for the expansion, which can be used to optimize the calculations [14,21].

This approach, which amounts to perform perturbative renormalization group calculations for  $D$ -dimensional membranes, has raised considerable challenges. Whereas for polymers (i.e. when  $D=1$ ) the Edwards model can be mapped onto the local  $\Phi^4$  field theory with  $n = 0$  components [15], and direct renormalization is equivalent to the standard renormalization (in the MS scheme) [16], this equivalence does not hold for  $D \neq 1$ . It was finally shown in [17,18] that the perturbative RG calculations are mathematically consistent, by proving that the SAM model is renormalizable to all orders in perturbation theory. If perturbative calculations are simple at first order (with some subtle points [19]), they present considerable difficulties at second order, and require a lot of analytical and numerical work [20,21].

An important issue is to understand if these calculations make sense beyond perturbation theory, or if non-perturbative effects destroy the consistency of the approach. A first step is to understand the large order behavior of perturbation theory for the SAM model. To our knowledge, nothing was known about this problem up to now, except for  $D = 1$ , where one can use the equivalence to the  $n \rightarrow 0$  limit of  $\Phi^4$ -theory, for which a non-trivial solution of the equation of motion at negative coupling constant, the so-called instanton, governs the large order behavior [22,23]; however, this analogy does not provide a physical

picture of which polymer-configurations dominate the large orders. In this paper we shall formulate the problem of the large order behavior for the Edwards model, in a way which is directly applicable both to polymers and to membranes.

Let us summarize the organization and the results of this paper. In Sect. 2 we recall the definition of the SAM model. Then we develop a general semiclassical argument to compute the large orders of the perturbative series for the SAM model. Using the formulation of the SAM model as a model of a “phantom” membrane (without self-avoidance) in a random imaginary external potential  $V$ , we show that the large orders are controlled by a real classical configuration for this potential  $V$ , which is the analog for SAM of the instanton for  $\Phi^4$ . This “SAM instanton” potential  $V$  is the extremum of a non-local functional  $\mathcal{S}[V]$ , which cannot be calculated exactly. We obtain the general form for the asymptotics of the term of order  $n$ , which is

$$n^{d/2} (-\mathcal{C})^n (n!)^{1-\epsilon/D} , \quad (1)$$

where  $D$  is the internal dimension of the membrane,  $d$  the dimension of bulk space,  $\epsilon = 2D - d(2 - D)/2$  the engineering dimension for the self-avoidance coupling in the SAM model and  $\mathcal{C}$  a positive constant depending on  $D$  and  $\epsilon$  (or  $d$ ). This behavior is universal: the constant  $\mathcal{C}$  obtained from the instanton does not depend on the internal shape or topology of the membrane.

In Sect. 3 we show that for polymers ( $D = 1$ ) our results fully agree with the classical results on the large orders for the  $\Phi^4$  theory, as obtained from instanton calculus. We then give a simple physical interpretation of our instanton for the SAM model: the SAM instanton describes the metastable equilibrium configuration of a membrane submitted to two competing forces, a contact attractive interaction and a global entropic repulsion due to thermal fluctuations.

The SAM instanton equations are in general not solvable. In Sect. 4 we propose a Gaussian variational approximation scheme and find the instanton potential  $V$  and the large order constant  $\mathcal{C}$  within this approximation. By construction, this is a lower bound on the exact result.

The variational results are discussed in Sect. 5. They qualitatively agree with the exact results for polymers ( $D = 1$ ). We estimate the “optimal order” for the  $\epsilon$ -expansion, and we argue that the estimates for the configuration exponent  $\nu$  obtained from second order calculations [20,21] are reasonable, especially for large bulk dimension  $d$ . Finally we observe that they imply that the perturbative series for the SAM model should become “quasi-convergent” in the limit  $d \rightarrow \infty$ .

In Sect. 6 we study more carefully the validity of the variational approach. We show that it becomes exact in the limit of large space dimension  $d \rightarrow \infty$ . Technically one has to keep  $\epsilon$  fixed, that is to let the internal dimension  $D$  of the membrane go to 2. To go beyond the leading order, we construct a systematic perturbative expansion around the variational solution, and show that this expansion can be organized as a systematic expansion in powers of  $1/d$ , by resumming infinite classes of diagrams. We compute explicitly the first correction in  $1/d$  for the instanton  $V$  and the large order constant  $\mathcal{C}$ . In particular, we find that along the critical line  $\epsilon = 0$ , the instanton potential  $V$  has  $\log(\epsilon)$  singularities, but that the large order constant  $\mathcal{C}$  is finite.

In Sect. 7 we present our conclusions and discuss open problems.

## 2 Large orders and instantons for the SAM model

### 2.1 The SAM model

We consider a  $D$ -dimensional manifold  $\mathcal{M}$  with size  $L$  and volume  $\mathcal{V} = L^D$  (typically the  $D$ -dimensional torus  $\mathbb{T}_D = [0, L]^D$ ) in  $d$ -dimensional Euclidean bulk space. A configuration of the manifold is described by the continuous function

$$x \in \mathcal{M} \rightarrow \vec{r}(x) \in \mathbb{R}^d . \quad (2)$$

The partition function is

$$\mathcal{Z}[b; L] = \int \mathcal{D}[\vec{r}] e^{-\mathcal{H}[\vec{r}; b, L]} \quad (3)$$

with the Hamiltonian

$$\mathcal{H}[\vec{r}; b, L] = \int_L d^D x \frac{1}{2} (\nabla \vec{r}(x))^2 + \frac{b}{2} \int_L d^D x \int_L d^D y \delta^d(\vec{r}(x) - \vec{r}(y)) . \quad (4)$$

$b > 0$  is the repulsive 2-point interaction coupling which describes self-avoidance. The functional integration measure  $\mathcal{D}[\vec{r}] = \prod_x d^d \vec{r}(x) / \mathcal{Z}_0$  is normalized such that the partition function of the free Gaussian manifold  $\mathcal{Z}[b = 0, L] = 1$ . The canonical dimensions of  $x$ ,  $\vec{r}$  and  $b$  are

$$[x] = -1 \quad , \quad [\vec{r}] = \frac{D-2}{2} \quad , \quad [b] = 2D - \frac{d(2-D)}{2} = \epsilon . \quad (5)$$

We know that the partition function  $\mathcal{Z}$  and expectation values of in bulk space translationally invariant operators  $\langle \mathcal{O} \rangle$  are well defined as a perturbative series in the coupling constant  $b$ , as long as the dimension of the manifold  $D$  is less than 2 and as long as  $\epsilon$  is positive

$$0 < D < 2 \quad , \quad \epsilon > 0 \quad (6)$$

and provided that the size  $\mathcal{V}$  of the manifold is finite<sup>1</sup>. For  $\epsilon > 0$ , perturbation theory suffers from strong Infra-Red (IR) divergences when the size of the manifold becomes large  $L \rightarrow \infty$ , which signal a breakdown of mean-field theory and the appearance of anomalous dimensions in the scaling properties of large self-avoiding manifolds.

For  $\epsilon \rightarrow 0$ , physical UV singularities appear at all orders in perturbation expansion. They can be absorbed in a renormalization of the coupling constant  $b$  and a rescaling of bulk space (wave function renormalization)

$$b = b_R Z_b(b_R) \quad , \quad \vec{r} = Z^{1/2}(b_R) \vec{r}_R \quad (7)$$

so that physical observables are UV finite at  $\epsilon = 0$  when expressed in terms of renormalized quantities  $b_R$  and  $\vec{r}_R$ . The theory, although non-local in the internal  $x$ -space of the

---

<sup>1</sup>Strictly speaking, for  $0 < \epsilon < D$ , the first terms in the expansion of the partition function suffer from short-distance divergences. These Ultra-Violet (UV) singularities are finite in number and can be recast in a counterterm proportional to the volume  $\mathcal{V}$  of the manifold. Moreover they disappear in physical observables; they are thus of no physical significance and unimportant for the purpose of this paper.

membrane, is renormalizable [17,18]. As a consequence, it is possible to write renormalization group equations which encode how the effective renormalized coupling flows with the length scale, and in particular with the size  $L$  of the membrane. The large  $L$  behavior of the membrane is governed by an IR stable fixed point  $b_R^*$ , which can in principle be calculated order by order in perturbation theory as a series in  $\epsilon$ , similar to the celebrated Wilson-Fisher  $\epsilon$ -expansion. In practice, such calculations are very difficult, and have been performed only up to second order [20,21].

## 2.2 Instanton calculus

By dimensional analysis, the partition function (3) only depends on the dimensionless coupling constant

$$g = b L^\epsilon \quad (8)$$

via

$$\mathcal{Z}[b; L] = \mathcal{Z}[g, L = 1] = \mathcal{Z}[g] \quad (9)$$

and is defined as a series

$$\mathcal{Z}[g] = \sum_{n=0}^{\infty} z_n g^n . \quad (10)$$

Of course,  $\mathcal{Z}[g]$  also depends on the shape of the manifold  $\mathcal{M}$ .

Let us assume that  $\mathcal{Z}[g]$  is analytic around the origin for  $-\pi < \arg(g) < \pi$ , and has a discontinuity along the negative real axis. This assumption is natural, since for  $g < 0$ , the membrane is in a collapsed state and the perturbation expansion is performed around an unstable classical state. Then we can write  $z_n$  as a dispersion integral

$$z_n = \oint \frac{dg}{2i\pi} g^{-n-1} \mathcal{Z}[g] = \int_0^{-\infty} \frac{dg}{\pi} g^{-n-1} \text{Im}(\mathcal{Z}[g + i0^+]) . \quad (11)$$

To obtain the behavior for large  $n$ , it turns out that it is sufficient to evaluate the integral in (11) in a saddle point approximation. Indeed, we shall show that, at least for  $0 < \epsilon < D$ , the integral is at large  $n$  dominated by the discontinuity of  $\mathcal{Z}[g]$  at small negative  $g$ . Moreover,  $\mathcal{Z}[g]$  is dominated by a saddle point, when re-expressed as a functional integral over properly defined auxiliary fields.

The Hamiltonian (4) is non-local and involves a distribution of the field  $\vec{r}$ . It is convenient to introduce as an auxiliary field the density of the membrane in bulk space

$$\rho(\vec{r}) = \int d^D x \delta^d(\vec{r} - \vec{r}(x)) \quad (12)$$

and to write the interaction term as

$$\int d^D x \int d^D y \delta^d(\vec{r}(x) - \vec{r}(y)) = \int d^d \vec{r} \rho(\vec{r})^2 . \quad (13)$$

Introducing a potential  $V(\vec{r})$  conjugate to  $\rho(\vec{r})$ , we can re-express the interaction term in the partition function through a Hubbard-Stratonovich transformation as

$$\begin{aligned} & \exp \left[ -\frac{b}{2} \int d^D x \int d^D y \delta^d(\vec{r}(x) - \vec{r}(y)) \right] \\ & = \int \mathcal{D}[\rho] \int \mathcal{D}[V] \exp \left[ \int d^d \vec{r} \left( V(\vec{r}) \left[ \rho(\vec{r}) - \int d^D x \delta^d(\vec{r} - \vec{r}(x)) \right] - \frac{b}{2} \rho(\vec{r})^2 \right) \right] \end{aligned} \quad (14)$$

where

$$\int \mathcal{D}[\rho] = \int_{-\infty}^{+\infty} \prod_{\vec{r} \in \mathbb{R}^d} d\rho(\vec{r}) , \quad \int \mathcal{D}[V] = \int_{-\infty}^{+\infty} \prod_{\vec{r} \in \mathbb{R}^d} \frac{dV(\vec{r})}{2i\pi} . \quad (15)$$

Inserting Eq. (14) into Eq. (3) and integrating over  $\rho$  yields

$$\mathcal{Z}(b; L) = \int \mathcal{D}[\vec{r}] \int \mathcal{D}[V] e^{-\mathcal{H}[\vec{r}, V; b, L]} \quad (16)$$

with the new effective Hamiltonian

$$\mathcal{H}'[\vec{r}; V; b, L] = \int_L d^D x \left( \frac{1}{2} (\nabla \vec{r}(x))^2 + V(\vec{r}(x)) \right) - \frac{1}{2b} \int d^d \vec{r} V(\vec{r})^2 . \quad (17)$$

This representation is nothing but the generalization of the well known formulation of the Edwards model as a model of free random walks in an (imaginary) annealed random potential. As above,  $\mathcal{Z}$  is a function of the dimensionless coupling  $g$  and we replace  $b \rightarrow g$  and  $L \rightarrow 1$  as in Eqs. (8) and (9).

As argued before, we aim in calculating the partition function for small negative  $g$ . For that purpose, it is convenient to rescale the coordinates and the potential  $V(\vec{r})$

$$x \rightarrow (-g)^{\frac{1}{D-\epsilon}} x \quad , \quad \vec{r} \rightarrow (-g)^{\frac{2-D}{2(D-\epsilon)}} \vec{r} \quad , \quad V \rightarrow (-g)^{\frac{-D}{D-\epsilon}} V , \quad (18)$$

so that we now consider a membrane with size  $\bar{L} = (-g)^{\frac{-1}{D-\epsilon}}$  and volume

$$\bar{\mathcal{V}} = \bar{L}^D = (-g)^{\frac{-D}{D-\epsilon}} . \quad (19)$$

This yields the rescaled Hamiltonian

$$\mathcal{H}'_{\text{resc}}[\vec{r}; V; \bar{L}] = \int_{\bar{L}} d^D x \left( \frac{1}{2} (\nabla \vec{r}(x))^2 + V(\vec{r}(x)) \right) - \frac{\bar{\mathcal{V}}}{2} \int d^d \vec{r} V(\vec{r})^2 . \quad (20)$$

The integral over  $\vec{r}$  for fixed potential  $V$  defines the free energy *density*  $\mathcal{E}[V]$  of a “phantom” (i.e. non self-avoiding) membrane in the external potential  $V$

$$\mathcal{Z}[V] = e^{-\bar{\mathcal{V}} \mathcal{E}[V]} = \int \mathcal{D}[\vec{r}] e^{-\int_{\bar{L}} d^D x \frac{1}{2} (\nabla \vec{r}(x))^2 + V(\vec{r}(x))} . \quad (21)$$

The partition function finally becomes

$$\mathcal{Z}(g) = \int \mathcal{D}[V] e^{-\bar{\mathcal{V}} [\mathcal{E}[V] + \frac{1}{2} \int d^d \vec{r} V(\vec{r})^2]} . \quad (22)$$

The crucial point of this formulation is that according to Eq. (19), as long as

$$0 < \epsilon < D , \quad (23)$$

the limit  $g \rightarrow 0^-$  corresponds to the thermodynamical limit when the volume  $\bar{\mathcal{V}} \rightarrow \infty$ . In this limit the free energy density  $\mathcal{E}[V]$  has a finite limit (from extensivity) so that the volume appears only as a global prefactor in the exponential of (22). Hence in the large

$\bar{\mathcal{V}}$  limit the integral (22) is dominated by a saddle point  $V_{\text{inst}}$ , which is an extremum of the effective energy  $\mathcal{S}[V]$  for an *infinite and flat* membrane. The latter is defined as

$$\mathcal{S}[V] = \mathcal{E}[V] + \frac{1}{2} \int d^d \vec{r} V(\vec{r})^2, \quad (24)$$

where  $\mathcal{E}[V]$  is defined in (21) as the free energy density of an infinite flat membrane in the potential  $V$ . This saddle point  $V_{\text{inst}}(\vec{r})$  is the non-trivial instanton, since the action  $\mathcal{S}$  of the trivial extremum  $V(\vec{r}) = 0$  is real and does not contribute to the discontinuity of  $\mathcal{Z}(g)$ . Moreover, as the instanton is obtained through the thermodynamical limit  $\bar{L} \rightarrow \infty$ , it is independent of the shape of the initial membrane. This implies that the large order behavior of perturbation theory is *universal*, and does not depend on the internal geometry of the membrane. Let us now derive the saddle-point equations:

The variation of the free energy density is in general

$$\frac{\delta \mathcal{E}[V]}{\delta V(\vec{r})} = \langle \delta[\vec{r}] \rangle_V \quad (25)$$

where  $\delta[\vec{r}]$  is the normalized density of the membrane

$$\delta[\vec{r}] = \frac{\rho[\vec{r}]}{\mathcal{V}} = \frac{1}{\mathcal{V}} \int d^D x \delta^d(\vec{r} - \vec{r}(x)) \quad (26)$$

(which has a finite limit when the volume becomes infinite), and  $\langle \cdot \rangle_V$  denotes the expectation value for the phantom membrane in the potential  $V$ , as defined in Eq. (21). Hence extremizing  $\mathcal{S}[V]$  leads to the variational equation for the instanton potential  $V_{\text{inst}}$

$$0 = \langle \delta[\vec{r}] \rangle_{V_{\text{inst}}} + V_{\text{inst}}(\vec{r}). \quad (27)$$

Let us postpone the solution of Eq. (27) and first ask what the consequences of the existence of an instanton for the large order behavior are. Denoting by  $\mathcal{S}_{\text{inst}}$  the action for the instanton  $\mathcal{S}[V_{\text{inst}}]$ , we deduce from Eqs. (19) and (21) that for small negative  $g$ , the discontinuity of  $\mathcal{Z}(g)$  behaves as

$$\text{Im}(\mathcal{Z}(g)) \approx \exp \left[ -(-g)^{\frac{-D}{D-\epsilon}} \mathcal{S}_{\text{inst}} \right] \quad (28)$$

and the integral representation for  $z_n$  (11) can be evaluated by the saddle point method at large  $n$ . This saddle point is at

$$g_c = - \left[ \frac{\mathcal{S}_{\text{inst}}}{n(1-\epsilon/D)} \right]^{1-\epsilon/D} \quad (29)$$

and replacing the integral in (11) by its value at  $g_c$  gives the large  $n$  behavior at leading order

$$z_n \sim (-\mathcal{C})^n (n!)^{1-\epsilon/D}, \quad \mathcal{C} = \left[ \frac{1-\epsilon/D}{\mathcal{S}_{\text{inst}}} \right]^{1-\epsilon/D} \quad (30)$$

Let us briefly discuss this result. For  $0 \leq \epsilon < D$ , perturbation theory is divergent with alternating signs. For  $\epsilon = 0$ , one recovers the typical factorial behavior  $(-\mathcal{C})^n n!$  of field

theories, provided that  $\mathcal{S}_{\text{inst}}$  remains UV finite. As we shall see in the next section, our result (30) coincides for  $D = 1$  with the large order behavior deduced from the  $\Phi^4$  model with  $n = 0$  components. The reasoning seems to break down at  $\epsilon = D$ , but we shall see that in fact the factor of  $\mathcal{C}$ , when considered as a function of  $D$  and  $\epsilon$ , is regular at  $\epsilon = D$  and can be continued to the region  $\epsilon \geq D$ . Thus, the asymptotics (30), although derived for  $0 < \epsilon < D$ , is valid in the whole physical domain  $0 < \epsilon < 2D$ . A more rigorous argument is as follows: Eqs. (28) and (29) are still valid for  $\epsilon > D$ ; the instanton then governs the behavior of the discontinuity of  $\mathcal{Z}(g)$  at *large*  $g$ . This means that the saddle point of Eq. (30) for large  $n$  now is at large negative  $g$ .

To go beyond this estimates, one must (i) compute the instanton and its action, and (ii) integrate the fluctuations around the instanton in (22). If one assumes that this calculation goes along the same lines as in standard field theory, one must first isolate the zero modes, i.e. the collective coordinates of the instanton. As we shall see later, the instanton  $V_{\text{inst}}$  is rotationally invariant and is characterized by its position in  $d$ -dimensional space only. Thus it has  $d$  zero modes, each of them gives a factor  $\mathcal{V}^{1/2}$  (by a standard collective coordinates argument), and the remaining fluctuations  $\delta_{\perp} V$  (orthogonal to the translational variations  $\delta_{\mu} V \sim \frac{\partial V_{\text{inst}}}{\partial r^{\mu}}$ ) give a finite determinant  $\mathcal{A}$ . Therefore we expect the semiclassical estimate for the discontinuity to be

$$\text{Im}(\mathcal{Z}(g)) \simeq \mathcal{A}^{-1/2} \bar{\mathcal{V}}^{\frac{d}{2}} e^{-\bar{\mathcal{V}} \mathcal{S}_{\text{inst}}} \quad (31)$$

and that the large  $n$  behavior is more precisely

$$z_n \sim \mathcal{A}' n^{d/2} (-\mathcal{C})^n (n!)^{1-\epsilon/D} [1 + \dots] . \quad (32)$$

Finally we shall see that the action of the instanton remains finite in the limit  $\epsilon \rightarrow 0$ . As in standard  $\Phi^4$  theory, one expects UV divergences to appear only for fluctuations around the instanton, and that these divergences are cancelled by the same renormalizations as in perturbation theory. This would imply that our large order estimate (30) is also valid for the renormalized theory at  $\epsilon = 0$ , in particular for the renormalization group functions which enter into the  $\epsilon$ -expansion of the scaling exponents. Renormalization however has to be taken into account when evaluating the constant  $\mathcal{A}'$  in (32).

### 3 The polymer case and physical interpretation of the instanton

Before discussing membranes, let us study in detail the special case  $D = 1$ , where the model reduces to the Edwards model for polymers. Using the well known mapping between the problem of a Brownian walk in a potential  $V(\vec{r})$  and quantum mechanics of a single particle in the same potential, the free energy density  $\mathcal{E}[V]$  of a linear chain fluctuating in a potential  $V(\vec{r})$  is in the thermodynamic limit given by the lowest eigenvalue  $E_0$  of the operator

$$H = -\frac{\Delta}{2} + V(\vec{r}) , \quad (33)$$

where  $\Delta$  is the Laplacian in  $d$  dimensions. Thus we have

$$\mathcal{E}[V] = E_0 . \quad (34)$$



Denoting by  $\Psi_0(\vec{r})$  the ground state wave-function, and using Eq. (27) and the standard result from first order perturbation theory

$$\langle \delta[\vec{r}] \rangle_V = \frac{\delta E_0[V]}{\delta V(\vec{r})} = \langle \Psi_0 | \frac{\delta H}{\delta V(\vec{r})} | \Psi_0 \rangle = |\Psi_0(\vec{r})|^2, \quad (35)$$

we obtain the instanton potential

$$V_{\text{inst}}(\vec{r}) = - (\Psi_0(\vec{r}))^2. \quad (36)$$

The eigenvalue equation  $H\Psi_0 = E_0\Psi_0$  becomes non-linear

$$\frac{1}{2} \Delta \Psi_0 + E_0 \Psi_0 + \Psi_0^3 = 0. \quad (37)$$

Since  $\Psi_0$  obeys the normalization condition

$$\|\Psi_0\|^2 = \int d^d \vec{r} \Psi_0(\vec{r})^2 = 1, \quad (38)$$

the wave function  $\Psi_0$  and the ground state energy  $E_0$  are fully determined by Eqs. (37) and (38). Eq. (37) has nontrivial normalizable solutions for  $2 < d < 4$  and  $E_0 < 0$ . In addition, the ground state  $\Psi_0$  is rotational symmetric, i.e. does not vanish at finite  $\vec{r}$ . The action for the instanton (24) finally reads

$$\mathcal{S}_{\text{inst}} = E_0 + \frac{1}{2} \int d^d \vec{r} \Psi_0^4. \quad (39)$$

To make contact to the instanton analysis in the Landau-Ginsburg-Wilson (LGW)  $\Phi^4$ -theory with  $n = 0$  components, remark that Eqs. (37)-(38) hold *if and only if*  $\Psi_0$  and  $E_0$  are extrema of the action

$$\mathcal{S}'[\Psi, E] = E + \int d^d \vec{r} \left[ \frac{1}{2} (\nabla \Psi)^2 - E \Psi^2 - \frac{1}{2} \Psi^4 \right]. \quad (40)$$

This is the standard Landau Ginsburg Wilson action with negative coupling associated to  $\Psi^4$  and mass  $m^2 = -2E$ . Moreover, *at the extrema*, the two actions are equal

$$\mathcal{S}_{\text{inst}}[\Psi_0, E_0] = \mathcal{S}'[\Psi_0, E_0]. \quad (41)$$

The relation becomes clearer by the change of variables

$$\Psi(\vec{r}) = \left( \frac{-2E}{4-d} \right)^{\frac{1}{2}} \Phi \left( \left( \frac{-2E}{4-d} \right)^{\frac{1}{2}} \vec{r} \right). \quad (42)$$

The action  $\mathcal{S}'$  then reads

$$\mathcal{S}'[\Psi, E] = E + \left( \frac{-2E}{4-d} \right)^{2-\frac{d}{2}} \mathcal{S}_{\text{LGW}}[\Phi] \quad (43)$$

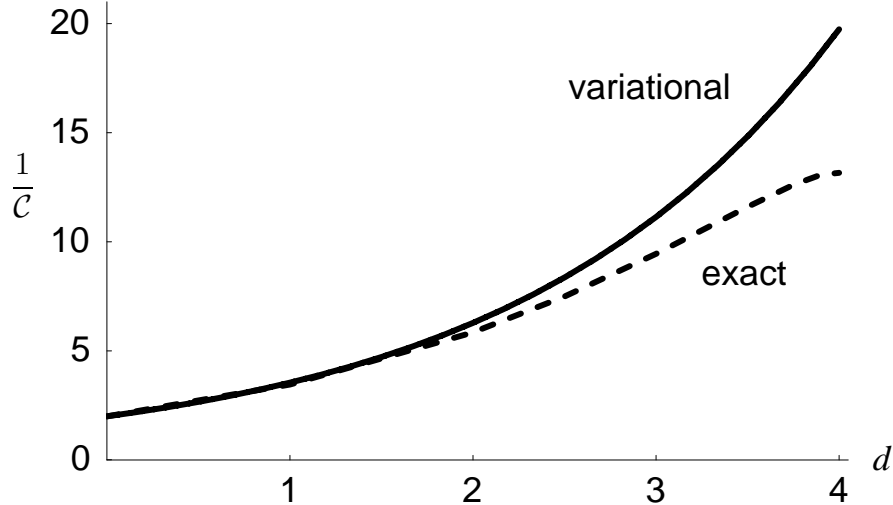


Figure 1:  $1/\mathcal{C}$  as obtained from a numerical solution of Eq. (47), compared to the variational bound derived later in Eq. (64).

with

$$\mathcal{S}_{\text{LGW}}[\Phi] = \int d^d \vec{r} \left[ \frac{1}{2} (\nabla \Phi)^2 + \frac{4-d}{2} \Phi^2 - \frac{1}{2} \Phi^4 \right]. \quad (44)$$

We can extremize (43) with respect to  $E$  and  $\Phi$  independently, and denoting by  $\Phi_0$  and  $E_0$  these extremizing solutions, we get

$$E_0 = \left( \frac{d}{2} - 2 \right) \mathcal{S}_{\text{LGW}}[\Phi_0]^{\frac{1}{d/2-1}}. \quad (45)$$

The change of variables in Eq. (42) was constructed such that the instanton action takes the simple form

$$\mathcal{S}_{\text{inst}} = \mathcal{S}'[\Psi_0, E_0] = \left( \frac{d}{2} - 1 \right) \mathcal{S}_{\text{LGW}}[\Phi_0]^{\frac{1}{d/2-1}}. \quad (46)$$

Since for polymers ( $D = 1$ )  $d/2 - 1 = 1 - \epsilon/D$ , we can use Eq. (30) to write the large order constant  $\mathcal{C}$  of the Edwards model as

$$\frac{1}{\mathcal{C}} = \mathcal{S}_{\text{LGW}}[\Phi_0]. \quad (47)$$

This result could have been derived directly from the standard field theoretical formulation of the Edwards model as a  $n = 0$  component  $(\vec{\Phi}^2)^2$  model.

The equation for the instanton derived from the action (44) admits a regular solution  $\Phi_0(|\vec{r}|)$  for any  $0 \leq d \leq 4$ , so that nothing special occurs at the point  $d = 2$  (i.e.  $\epsilon = D = 1$ ) as one might have expected from Eq. (30). Let us note that since the “mass” in Eq. (44) is equal to  $4 - d$ , it is positive for  $d < 4$  but vanishes at the critical dimension  $d = 4$ , so that the instanton solution  $\Phi_0$  still exists for  $d = 4$ . In Fig. 1 we plot  $\mathcal{C}^{-1}(d)$  for  $0 \leq d \leq 4$ , as obtained from numerical integration.

It is interesting to give a physical interpretation of the instanton for the Edwards model, since this interpretation is the same for membranes with  $D \neq 1$ . Let us first recall the standard interpretation of the instanton for the LGW model with action (44), i.e. negative  $\Phi^4$ -coupling. The classical false vacuum  $\Psi(r) = 0$  is separated from the true

vacua  $\Psi(r) = \mp\infty$  by a finite barrier. The instanton solution  $\Psi_0$  describes a metastable droplet of true vacuum (with  $\Psi_0(r) \neq 0$  inside the droplet) in the false vacuum, which is on the verge to nucleate. Indeed, if the droplet is slightly larger, the positive surface energy dominates and the droplet shrinks and finally vanishes, while if it is slightly smaller, the negative volume energy dominates and the droplet expands.

We now consider the energy density  $\mathcal{S}[V]$  given by Eq. (24). It corresponds to the total free energy of a polymer globule trapped in the potential well  $V(\vec{r}) < 0$ , where this effective potential results from the *attractive* 2-point interaction between elements of the polymer (since we are at negative coupling,  $b < 0$ ). To see how  $\mathcal{S}$  varies with the average gyration radius of the polymer, it is convenient to consider the following scale transformation on  $V$

$$V(\vec{r}) \rightarrow V_\lambda(\vec{r}) = \lambda^{\frac{2D}{2-D}} V(\lambda\vec{r}) . \quad (48)$$

Simple dimensional analysis shows that under (48)

$$\mathcal{E}[V] \rightarrow \lambda^{\frac{2D}{2-D}} \mathcal{E}[V] \quad , \quad \int V^2 \rightarrow \lambda^{\frac{2\epsilon}{2-D}} \int V^2 \quad (49)$$

(here  $D = 1$  and  $\epsilon = 2 - d/2$ ). As long as  $\epsilon < D$ , and for large  $\lambda$ , i.e. when shrinking the polymer globule, it is the first term  $\mathcal{E} < 0$  which dominates and the total free energy  $\mathcal{S}$  becomes large and negative; while for small  $\lambda$ , i.e. when expanding the globule, it is the second term on the r.h.s. of (24) which is larger than 0, and which dominates. Thus in this mean field picture, i.e. neglecting thermal fluctuations around the instanton, large globules tend to expand, while small globules tend to collapse. This has a simple physical interpretation: the polymer trapped in its own potential is subject to two opposite forces, (i) attractive forces between its elements which would like to make the polymer collapse, (ii) entropic repulsion which exerts a pressure on the well and would like to expand the polymer (until it becomes a free random walk). What our calculation implies is the simple fact that for large radius (i.e. small  $\lambda$ ) entropic repulsion dominates, while at small radius (large  $\lambda$ ) attraction dominates and the polymer collapses. Thus the instanton solution describes a polymer with attractive interactions on the verge to collapse into its dense (and most stable) phase; this is similar to the instanton in the LGW theory which describes a bubble of true vacuum on the verge to nucleate and to destroy the false vacuum.

## 4 Gaussian variational calculation

For  $D \neq 1$  (and in general for  $0 < D < 2$  non-integer) we know of no exact method to calculate the instanton. A simple and natural approximation is the variational method, i.e. the Hartree-Fock approximation.

To evaluate the free energy density  $\mathcal{E}[V]$  of the free, i.e. non-interacting membrane in a potential  $V$ , and described by the Hamiltonian

$$\mathcal{H}_V = \int d^Dx \left( \frac{1}{2} (\nabla\vec{r})^2 + V(\vec{r}) \right) , \quad (50)$$

we introduce the trial Gaussian Hamiltonian

$$\mathcal{H}_{\text{var}} = \int d^Dx \int d^Dy \frac{1}{2} \vec{r}(x) K(x-y) \vec{r}(y)$$

$$= \int \frac{d^D k}{(2\pi)^D} \frac{1}{2} \vec{r}(k) \tilde{K}(k) \vec{r}(-k) , \quad (51)$$

where  $\tilde{\phantom{x}}$  denotes the Fourier transform. The free energy for the trial Hamiltonian is

$$\begin{aligned} \mathcal{E}_{\text{var}} &= -\frac{1}{\mathcal{V}} \text{Log} \left[ \int \mathcal{D}[\vec{r}] e^{-\mathcal{H}_{\text{var}}} \right] \\ &= \frac{d}{2} \int \frac{d^D k}{(2\pi)^D} \text{Log}[\tilde{K}(k)/k^2] , \end{aligned} \quad (52)$$

and the factor of  $1/k^2$  comes from the normalization of the measure  $\mathcal{D}[\vec{r}]$  taken such that  $\mathcal{E}[V=0] = 0$ .  $\mathcal{V}$  is the total volume of the membrane.

The Hartree Fock approximation amounts in replacing  $\mathcal{E}[V]$  by the best variational estimate  $\mathcal{E}_{\text{var}}[V]$

$$\mathcal{E}[V] \leq \mathcal{E}_{\text{var}}[V] = \mathcal{E}_{\text{var}} + \frac{1}{\mathcal{V}} \langle \mathcal{H}_V - \mathcal{H}_{\text{var}} \rangle_{\text{var}} . \quad (53)$$

$\langle \phantom{x} \rangle_{\text{var}}$  denotes the average with respect to the trial Hamiltonian  $\mathcal{H}_{\text{var}}$  and one must look for the trial Hamiltonian  $\mathcal{H}_{\text{var}}$  (i.e. the kernel  $K$ ) which minimizes  $\mathcal{E}_{\text{var}}[V]$ . Denote by  $\tilde{V}(\vec{p})$  the Fourier transform of the potential  $V(\vec{r})$ . Since the variational Hamiltonian is Gaussian, it is easy to compute the second term on the r.h.s. of (53),  $\mathcal{V}^{-1} \langle \mathcal{H}_V - \mathcal{H}_{\text{var}} \rangle_{\text{var}}$  in the infinite volume limit:

$$\begin{aligned} &\langle V(\vec{r}(0)) \rangle_{\text{var}} + \frac{1}{2} \left[ \langle (\nabla \vec{r}(0))^2 \rangle_{\text{var}} - \int d^D x K(x) \langle \vec{r}(x) \vec{r}(0) \rangle_{\text{var}} \right] \\ &= \int \frac{d^d \vec{p}}{(2\pi)^d} \tilde{V}(\vec{p}) \langle e^{i\vec{p}\vec{r}(0)} \rangle_{\text{var}} + \int \frac{d^D k}{(2\pi)^D} \frac{k^2 - \tilde{K}(k)}{2} \langle \vec{r}(k) \vec{r}(-k) \rangle_{\text{var}} \\ &= \int \frac{d^d \vec{p}}{(2\pi)^d} \tilde{V}(\vec{p}) \exp \left[ -\frac{\vec{p}^2}{2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{\tilde{K}(k)} \right] + \frac{d}{2} \int \frac{d^D k}{(2\pi)^D} \left( \frac{k^2}{\tilde{K}(k)} - 1 \right) . \end{aligned} \quad (54)$$

Combining Eqs. (24), (52) and (54), we finally obtain the variational estimate for the total energy of the instanton

$$\begin{aligned} \mathcal{S}_{\text{var}}[V] &= \mathcal{E}_{\text{var}}[V] + \frac{1}{2} \int d^d \vec{r} V(\vec{r})^2 \\ &= \int \frac{d^d \vec{p}}{(2\pi)^d} \left( \tilde{V}(\vec{p}) \exp \left( -\frac{\vec{p}^2}{2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{\tilde{K}(k)} \right) + \frac{1}{2} \tilde{V}(\vec{p}) \tilde{V}(-\vec{p}) \right) \\ &\quad + \frac{d}{2} \int \frac{d^D k}{(2\pi)^D} \left( \text{Log} \left[ \frac{\tilde{K}(k)}{k^2} \right] + \frac{k^2}{\tilde{K}(k)} - 1 \right) . \end{aligned} \quad (55)$$

We now extremize Eq. (55) *both* with respect to  $\tilde{K}$  (variational approximation) and with respect to  $\tilde{V}$  (to obtain the instanton solution). Extremizing w.r.t.  $\tilde{K}(k)$  yields the equation

$$\tilde{K}(k) = k^2 - \frac{1}{d} \int \frac{d^d \vec{p}}{(2\pi)^d} \vec{p}^2 \tilde{V}(\vec{p}) \exp \left[ -\frac{\vec{p}^2}{2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{\tilde{K}(k)} \right] \quad (56)$$

which implies that the variational Hamiltonian depends just on a mass  $m_{\text{var}}$

$$\tilde{K}(k) = k^2 + m_{\text{var}}^2 . \quad (57)$$

Extremizing Eq. (55) w.r.t.  $\tilde{V}(p)$  gives

$$\tilde{V}_{\text{inst}}^{\text{var}}(\vec{p}) = - \exp \left[ -\frac{\vec{p}^2}{2} A \right] \quad (58)$$

with

$$A = \int \frac{d^D k}{(2\pi)^D} \frac{1}{\tilde{K}(k)} = m_{\text{var}}^{D-2} \frac{\Gamma(1 - \frac{D}{2})}{(4\pi)^{D/2}} . \quad (59)$$

$\Gamma$  is Euler's Gamma function. Thus, in the variational approximation the instanton potential is Gaussian. Inserting Eq. (58) into Eq. (56) yields the self-consistent equation for  $m_{\text{var}}$

$$m_{\text{var}}^2 = \frac{1}{d} \int \frac{d^d \vec{p}}{(2\pi)^d} \vec{p}^2 e^{-\vec{p}^2 A} = \frac{1}{2} (4\pi)^{-d/2} A^{-1-d/2} . \quad (60)$$

We finally get in terms of  $D$ ,  $\epsilon$  and  $d = 2(2D - \epsilon)/(2 - D)$

$$m_{\text{var}} = \sqrt{4\pi} \left[ 2 \Gamma(\frac{2-D}{2})^{1+\frac{d}{2}} \right]^{\frac{1}{D-\epsilon}} . \quad (61)$$

The final result for  $A$  reads

$$A = \frac{1}{4\pi} \Gamma(\frac{2-D}{2})^{\frac{-2}{D-\epsilon}} 2^{\frac{D-2}{D-\epsilon}} . \quad (62)$$

We can now insert these results into Eq. (55), and after straightforward calculations get the variational instanton action

$$\mathcal{S}_{\text{inst}}^{\text{var}} = \mathcal{S}_{\text{var}}[V_{\text{inst}}^{\text{var}}] = \left(1 - \frac{\epsilon}{D}\right) \left[ 2 \Gamma(\frac{2-D}{2})^{\frac{d}{D}} \right]^{\frac{D}{D-\epsilon}} . \quad (63)$$

The corresponding variational estimate for the large order constant  $\mathcal{C}$  defined by Eq. (30) is

$$1/\mathcal{C}^{\text{var}} = 2 \Gamma(\frac{2-D}{2})^{\frac{d}{D}} . \quad (64)$$

As claimed in the previous section, although intermediate results are singular at  $\epsilon = D$ , the final result is regular for all  $\epsilon > 0$ . We shall discuss the physical significance of these results in the next section.

## 5 Discussion of the variational result

### 5.1 $D = 1$

It is interesting to compare the variational estimate with the exact result for polymers, i.e. for the case  $D = 1$ . Let us consider the LGW instanton action, as given by Eq. (44). It is equal to the inverse of the large order constant  $\mathcal{C}$ . On Fig. 1 we have plotted the

variational result for  $1/\mathcal{C}^{\text{var}}$ , as given by Eq. (64) and the exact result for  $1/\mathcal{C}$  obtained by numerical solution, as a function of  $0 < d < 4$ . First we note that always

$$\mathcal{C} \geq \mathcal{C}^{\text{var}} \quad (65)$$

as expected from the variational inequality  $\mathcal{E} \leq \mathcal{E}_{\text{var}}$ . This implies that the variational method gives an underestimate of the large orders.

Second the variational estimate becomes good for small  $d$ , and exact for  $d \rightarrow 0$ . This is not unexpected, since in that limit the membrane  $\mathcal{M}$  has no inner degrees of freedom, and the functional integration over  $V(r)$  reduces to a simple integration over  $V \in \mathbb{R}$ . Since this integral is Gaussian, the variational method becomes exact.

Finally, the variational estimate for  $\mathcal{C}$  is regular when  $d \rightarrow 4$ , and then equals  $1/(2\pi^2)$ ; this is 50% smaller than the exact result  $3/(4\pi^2)$ . Thus the variational method is only qualitatively correct when  $\epsilon = 0$ . This is not so surprising, since the limit  $\epsilon \rightarrow 0$  is somewhat peculiar. Indeed when  $d = 4$  the ground state energy  $E_0$  in the equation (37) for the wave function  $\Psi_0$  is then equal to 0. Then the most general solution to Eq. (37) (for  $d = 4$  and  $E_0 = 0$ ) is

$$\Psi_0(\vec{r}) = \frac{2r_0}{r_0^2 + r^2}, \quad (66)$$

with  $r_0$  an arbitrary scale (the size of the instanton).  $r_0$  is fixed by the normalization condition (38) which cannot be fulfilled at  $d = 4$  for finite  $r_0$ . In fact a more careful analysis of the rotationally invariant solutions of Eqs. (37) and (38) (see Appendix A) shows that as  $d \rightarrow 4$ ,  $E_0$  should scale as  $E_0 \sim 4 - d$  and that for  $0 < 4 - d \ll 1$  the true solution  $\Psi_0$  is well approximated by Eq. (66) (at least as long as  $|\vec{r}|^2(4 - d) \ll 1$ ) with an instanton size  $r_0$  which vanishes as  $d \rightarrow 4$  as

$$r_0 \sim \frac{1}{\sqrt{|\log(4 - d)|}}. \quad (67)$$

The corresponding instanton potential  $V_{\text{inst}} = -|\Psi_0|^2$  is also singular in the  $d \rightarrow 4$  limit (it may be considered as a Dirac-like  $\delta$ -function), and is very poorly approximated by the Gaussian variational solution at  $D = 1$ ,  $d = 4$  for the potential

$$V^{\text{var}} = -16\pi^2 e^{-16\pi^2 \vec{r}^2}, \quad (68)$$

with positive width. As usual with variational methods, the approximation for the ground state energy is much better than that for the wave function.

## 5.2 Consequences for the $\epsilon$ -expansion

Of course, one is interested in the consequences of these large order estimates for the  $\epsilon$ -expansion of the scaling exponents for self-avoiding membranes and polymers. Let us recall that in renormalized perturbation theory one computes the renormalization group  $\beta$ -function  $\beta(g)$ , as a power series in  $g$  of the form <sup>2</sup>

$$\beta(g) = -\epsilon g + B_1 g^2 + \mathcal{O}(g^3). \quad (69)$$

---

<sup>2</sup>Strictly speaking  $g$  is now the renormalized coupling constant  $g_{\text{R}}$ .

Its zero at  $g^* = \epsilon/B_1 + \mathcal{O}(\epsilon^2)$  is the IR fixed point which governs the scaling limit for large membranes. Other anomalous dimensions, like the dimension  $\nu(g)$  of the field  $\vec{r}$  (which gives the fractal dimension of the membrane) can also be computed as a series in  $g$ . Their values at the fixed point  $g^*$  give the scaling exponents of the membrane, and may be expanded as power series in  $\epsilon$ .

By analogy with the ordinary Wilson-Fisher  $\epsilon$ -expansion for LGW field theories, let us assume that the large orders of the function  $\beta(g)$  and of the other anomalous dimensions are given by the instanton estimate, and that they can be resummed by Borel-techniques. We are not able at the moment to give any more precise argument to this last claim (which is still a conjecture even for the LGW theories). Then a simple calculation consists in estimating the “optimal” order  $n_{\text{opt}}$  beyond which the  $\epsilon$ -expansion starts to diverge. If we only know the first  $n$  terms of the expansion, we expect that for  $n < n_{\text{opt}}$  “ordinary” resummation procedures (like Padé) will be sufficient. If  $n > n_{\text{opt}}$ , or if one seeks higher precision, knowledge of the large orders and more sophisticated resummation methods are required. Assuming that for  $\epsilon = 0$  the  $n$ -th coefficient of  $\beta(g)$  is of order  $(-Cg)^n n!$ , and that we can approximate the fixed point  $g^*$  by its first order estimate  $\epsilon/B_1$ , the term of order  $n$  in the  $\epsilon$  expansion should behave as

$$\left(-\frac{\epsilon C}{B_1}\right)^n n! \quad (70)$$

The optimal order  $n_{\text{opt}}$  is obtained when the absolute value of (70) is the smallest, that is for

$$n_{\text{opt}} \epsilon \simeq \frac{B_1}{C}, \quad (71)$$

where  $B_1$  is the one-loop coefficient of the  $\beta$ -function, and  $C$  the large order constant as obtained from the instanton calculus.

With our choice of normalizations for the coupling constant  $b$  in the Hamiltonian (4), the 1-loop coefficient of the  $\beta$ -function is

$$B_1 = \frac{1}{2} \left[ \frac{(2-D)S_D}{4\pi} \right]^{d/2} S_D^2 \left[ 1 + \frac{1}{2-D} \frac{(\Gamma(\frac{D}{2-D}))^2}{\Gamma(\frac{2D}{2-D})} \right], \quad (72)$$

with  $S_D$  the volume of the unit sphere in  $D$  dimensions

$$S_D = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})}. \quad (73)$$

Let us replace  $C$  in Eq. (71) by the variational approximant  $C^{\text{var}}$  given by Eq. (64). Setting finally  $\epsilon = 0$  in  $B_1/C$  (since we are interested in the expansion around  $\epsilon = 0$ ), we obtain the following variational estimate for the r.h.s. of (71)

$$n_{\text{opt}} \epsilon \simeq \frac{16}{(2-D)^2} \left[ \frac{\Gamma(\frac{4-D}{2})}{\Gamma(\frac{D}{2})} \right]^{\frac{4}{2-D}} \left[ 1 + \frac{1}{2-D} \frac{(\Gamma(\frac{D}{2-D}))^2}{\Gamma(\frac{2D}{2-D})} \right] \quad (74)$$

Let us recall that in practice the  $\epsilon$ -expansion is used as follows: in order to compute for instance the scaling exponent  $\nu$  for a membrane with internal dimension  $D = 2$  in

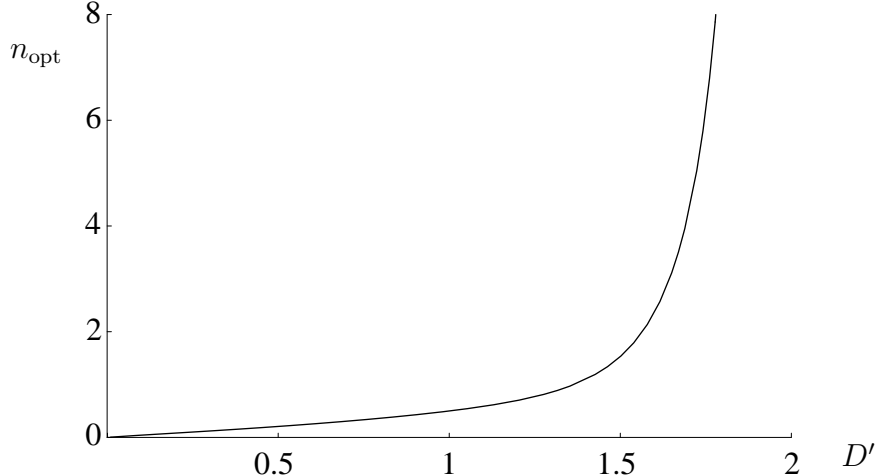


Figure 2: Optimal order  $n_{\text{opt}}(D')$  for the  $\epsilon$ -expansion for membrane as function of the extrapolation (dimension) parameter  $D'$ , as obtained from the variational estimate for the large orders.

$d$ -dimensional space, one starts from some point  $D' \neq D$ ,  $\epsilon = 0$  (i.e.  $d' = 4D'/(2 - D')$ ), uses an expansion in  $\epsilon$  and  $D - D'$  (or some more general expansion parameters) to evaluate  $\nu(D')$ , which thus depends on the expansion point  $D'$ .  $\nu$  is then taken as the best estimate  $\nu(D'_{\text{opt}})$ , as determined for instance by a minimal sensitivity criterium. Membranes ( $D = 2$ ) always correspond to  $\epsilon = 4$ , so setting  $\epsilon = 4$  and replacing  $D$  by  $D'$  in Eq. (74), should give an estimate of the “optimal order”  $n_{\text{opt}}(D')$  for the  $\epsilon$ -expansion at  $D'$ . The result for  $n_{\text{opt}}(D')$  is plotted on Fig. 2.

Some interesting comments can be made on this curve. For  $D' > 1.6$ ,  $n_{\text{opt}}(D') > 2$  and becomes large as  $D' \rightarrow 2$ , while for  $D' < 1.6$ ,  $n_{\text{opt}}(D') < 2$  and becomes small as  $D' \rightarrow 0$ . In the first regime ( $D' \rightarrow 2$ ) we thus expect that the power series in  $\epsilon$  will behave like a convergent series, up to some quite large order  $n_{\text{opt}}$ . In the second regime ( $D'$  small), we expect that the power series in  $\epsilon$  will be divergent from the very first terms. This is in agreement with the calculations at second order in [20,21]. For large  $d$ , the 2-loop results for  $\nu$  can neatly be resummed, and the stability of the various resummation procedures and extrapolation schemes analyzed in [20,21] is good. The final estimates are close to the prediction of a variational approximation  $4/d$  for  $\nu$ . For smaller values of  $d$  stability is less good, but in all cases, the reliable extrapolations are obtained for values of the extrapolation dimension  $D' \simeq 1.6$  or larger. It is not possible to resum safely the 2-loop results if one starts the  $\epsilon$ -expansions from  $D' \leq 1.5$ . Thus it seems that our rough estimates for the large order behavior may explain some general features of the calculation at second order, and corroborate the results of the estimates of [20,21].

### 5.3 Limit $D \rightarrow 2$

Of course these arguments are valid if the variational approximation for the instanton action stays (at least qualitatively) correct in the limit  $D \rightarrow 2$ . First let us note that, although Eqs. (63) and (64) give estimates for  $\mathcal{S}_{\text{inst}}$  and  $\mathcal{C}$  which are singular when  $D \rightarrow 2$ , our variational formula for  $n_{\text{opt}}$  is much less singular, since according to Eq. (74) it behaves



as

$$n_{\text{opt}}(D) \simeq \frac{1}{\epsilon} \frac{16 e^{-4\gamma}}{(2-D)^2} \quad \text{as } D \rightarrow 2, \epsilon \text{ fixed}, \quad (75)$$

with  $\gamma = 0.577216$  the Euler's constant. It has also been noted in [19] that instead of using the simple coupling constant  $b$  in the Hamiltonian (4) it might be more physical and convenient to use as coupling constant the "second virial coefficient"  $z$ , defined as

$$z = \left[ \frac{(2-D)S_D}{4\pi} \right]^{d/2} b L^\epsilon. \quad (76)$$

Using  $z$  as expansion parameter instead of  $g = bL^\epsilon$ , the large order constant  $\mathcal{C}$  in Eqs. (10) and (30) is now

$$\mathcal{C}_z = \mathcal{C} \left[ \frac{(2-D)S_D}{4\pi} \right]^{-d/2}, \quad (77)$$

which in the variational approximation reads

$$\mathcal{C}_z^{\text{var}} = \frac{1}{2} \left[ \frac{(2-D)S_D}{4\pi} \Gamma\left(\frac{2-D}{2}\right)^{\frac{2}{D}} \right]^{-\frac{d}{2}} \simeq \text{Cst} \left( \frac{2-D}{2} \right)^{2-\frac{\epsilon}{2}} \quad \text{as } D \rightarrow 2, \epsilon \text{ fixed}. \quad (78)$$

(with  $\text{Cst} = \frac{1}{2}(\pi e^{-2\gamma})^{2-\epsilon/2}$ . Therefore in this normalization also the singularities as  $D \rightarrow 2$  are simply algebraic. The same remark holds for the coupling constant normalization used in [20,21].

This prompts us to study the consistency of the variational approximation in the limit  $D \rightarrow 2$  (or equivalently  $d \rightarrow \infty$ ) for fixed  $\epsilon$ . As we shall see, in fact the variational approximation becomes exact in that limit. This makes the arguments of this section fully valid.

## 6 Beyond the variational approximation and $1/d$ corrections

In this section we show that the variational result is nothing but the leading term of a systematic expansion in  $1/d$ . In ordinary theories like the one described by the effective Hamiltonian  $\mathcal{H}_V$  which appears in Eq. (50) for a *fixed potential*  $V(\vec{r})$

$$\mathcal{H}_V[\vec{r}] = \int d^D x \left[ \frac{1}{2} (\nabla \vec{r})^2 + V(\vec{r}) \right], \quad (79)$$

this result is not unexpected. For fixed  $D$ , the limit  $d \rightarrow \infty$  is nothing but the limit where the number of components  $n = d$  of the field  $\vec{r}$  becomes large. In this limit, and provided that the model is  $O(n)$  invariant, it is known that the variational approximation becomes exact and a systematic  $1/n$  expansion can be constructed. For the problem considered here, the existence of a  $1/d$  expansion is not that easy to prove for two (related) reasons. First, the potential  $V(\vec{r})$  is not fixed, it is also a variable which has to be determined self-consistently, and it has a singular behavior when  $d \rightarrow \infty$ . In particular one cannot simply take the limit  $d \rightarrow \infty$  while  $D < 2$  is fixed, as can be seen on the exactly solvable case of polymers ( $D = 1$ ) discussed in section 4. Indeed in this case, the equation for the

instanton has no physical solution for  $d > 4$ . Second, the physically meaningful limit is to take  $d \rightarrow \infty$  while  $\epsilon = 2D - d(2 - D)/2$  is kept fixed. In this limit,  $D \rightarrow 2$  and one expects potentially dangerous additional singularities, since the massive free propagator is known to have a logarithmic singularity at short distance for  $D = 2$ . There is a subtle interplay between the corresponding  $1/(2 - D)$  poles and the terms which would naively disappear in the large  $d$  limit.

### 6.1 Expansion around the variational solution

To evaluate the corrections to the variational approximation, we expand around the variational Hamiltonian

$$\mathcal{H}_{\text{var}}[\vec{r}] = \int d^D x \left[ \frac{1}{2} (\nabla \vec{r})^2 + \frac{m_{\text{var}}^2}{2} \vec{r}^2 \right] \quad (80)$$

by writing

$$\mathcal{H}_V[\vec{r}] = \mathcal{H}_{\text{var}}[\vec{r}] - \int d^D x \Delta(\vec{r}) \quad (81)$$

$$\Delta(\vec{r}) = \frac{m_{\text{var}}^2}{2} \vec{r}^2 - V(\vec{r}) . \quad (82)$$

The saddle point equation (27) which defines the instanton potential  $V$  is

$$V(\vec{r}_0) + \langle \delta(\vec{r}_0 - \vec{r}(x_0)) \rangle_V = 0 , \quad (83)$$

where  $\langle \dots \rangle_V$  denotes the expectation value of  $\dots$  taken with respect to the Hamiltonian  $\mathcal{H}_V$

$$\langle \dots \rangle_V = \frac{\int \mathcal{D}[\vec{r}] \dots e^{-\mathcal{H}_V}}{\int \mathcal{D}[\vec{r}] e^{-\mathcal{H}_V}} , \quad (84)$$

for an infinite flat membrane. (Recall that this was the limit we had to take in Eq. (22)). This implies that the point  $x_0$  can be chosen arbitrarily on the membrane. It is simpler to use the Fourier transform of  $V$

$$\tilde{V}(\vec{k}) = \int d^d \vec{r} e^{i\vec{k}\vec{r}} V(\vec{r}) , \quad (85)$$

so that Eq. (83) reads

$$\tilde{V}(\vec{k}_0) + \langle e^{i\vec{k}_0 \vec{r}(x_0)} \rangle_V = 0 . \quad (86)$$

We now expand around the variational solution, using Eq. (81), to rewrite the expectation value on the l.h.s. of Eq. (86) as a *connected* correlation function computed with the variational Hamiltonian

$$\langle e^{i\vec{k}_0 \vec{r}(x_0)} \rangle_V = \langle e^{i\vec{k}_0 \vec{r}(x_0)} \cdot e^{\int_x \Delta} \rangle_{\text{var}}^{\mathbf{C}} , \quad (87)$$

where  $\langle \dots \rangle_{\text{var}}$  denotes the expectation value of  $\dots$  taken with respect to the variational Hamiltonian  $\mathcal{H}_{\text{var}}$

$$\langle \dots \rangle_{\text{var}} = \frac{\int \mathcal{D}[\vec{r}] \dots e^{-\mathcal{H}_{\text{var}}}}{\int \mathcal{D}[\vec{r}] e^{-\mathcal{H}_{\text{var}}}} . \quad (88)$$

The suffix  $\langle \dots \rangle^{\text{C}}$  means the **connected** correlation function in the usual sense: since  $\mathcal{H}_{\text{var}}$  is a free Gaussian Hamiltonian, using Wick's theorem, correlation functions like that in Eq. (87) can be expressed as Feynman diagrams involving the free variational propagator in  $D$  dimensions

$$G_{\text{var}}(x) = \int \frac{d^D q}{(2\pi)^D} \frac{e^{-iq \cdot x}}{q^2 + m_{\text{var}}^2} \quad (89)$$

and vertices obtained by expanding the ‘‘perturbation term’’  $\Delta(\vec{r})$  of Eq. (82) in powers of  $\vec{r}$ ; the connected correlation function in Eq. (87) is just given by the restriction to connected diagrams.

Similarly, the free energy density  $\mathcal{E}[V]$  defined by Eq. (21) can be written as a sum over connected vacuum diagrams.

## 6.2 Resummation of tadpoles and reorganization in terms of normal products

In these calculations, we encounter numerous ‘‘tadpole diagrams’’, which result from the evaluation of  $\langle r(x)r(y) \rangle_{\text{var}}$  at coinciding points  $x = y$ . A standard way to resum these tadpoles is to use *normal products*. This procedure consists in replacing any monomial  $\mathcal{P}[r(x)]$  of the field  $r(x)$  at a single point  $x$  by the corresponding normal product (or normal ordered operator)  $:\mathcal{P}[r(x)]:$ , defined such that the expectation values of any products of normal ordered operators  $\langle : \mathcal{P}_1 : \dots : \mathcal{P}_Q : \rangle_{\text{var}}$  at non-coinciding points  $x_1 \neq \dots \neq x_Q$  is equal to the sum over all Feynman diagrams *without tadpoles* which appear in the evaluation of  $\langle \mathcal{P}_1 \dots \mathcal{P}_Q \rangle_{\text{var}}$ . All normal ordered operators can be obtained<sup>3</sup> from the normal ordered exponential, satisfying the relation

$$e^{i\vec{k}\vec{r}(x)} = e^{-\frac{\vec{k}^2}{2}\mathbb{C}_{\text{var}}} :e^{i\vec{k}\vec{r}(x)}:, \quad (90)$$

where  $\mathbb{C}_{\text{var}}$  is the tadpole diagram amplitude

$$\mathbb{C}_{\text{var}} = G_{\text{var}}(0) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m_{\text{var}}^2} = m_{\text{var}}^{D-2} \mathbb{C} \quad (91)$$

with

$$\mathbb{C} = (4\pi)^{-D/2} \Gamma(1 - D/2). \quad (92)$$

$\mathbb{C}_{\text{var}}$  coincides with the factor of  $A$  in Eq. (59) of Sect. 4. Since  $\mathbb{C}_{\text{var}}$  depends on the mass  $m_{\text{var}}$  in Eq. (91) which is chosen to be the mass appearing in  $\mathcal{H}_{\text{var}}$ , the normal ordered products  $:[\dots]:$  depend explicitly on a mass scale  $m$ , and should be denoted by  $:[\dots]:_m$ . This mass scale dependence will be omitted in this section, since we shall always choose  $m = m_{\text{var}}$ .

With these notations, we can rewrite the operators in  $\mathcal{H}_V$  and  $\mathcal{H}_{\text{var}}$  in terms of normal products. This gives

$$(\vec{r}(x))^2 = d\mathbb{C}_{\text{var}} \mathbf{1} + :(\vec{r}(x))^2: \quad (93)$$

---

<sup>3</sup>by differentiating with respect to  $\vec{k}$

with the identity operator  $\mathbf{1}$ . For  $V(\vec{r})$  we have

$$\begin{aligned} V(\vec{r}) &= \int \frac{d^d \vec{k}}{(2\pi)^d} \tilde{V}(\vec{k}) e^{-\frac{\vec{k}^2}{2} \mathbb{C}_{\text{var}}} : e^{-i\vec{k}\vec{r}} : \\ &= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int \frac{d^d \vec{k}}{(2\pi)^d} e^{-\frac{\vec{k}^2}{2} \mathbb{C}_{\text{var}}} \tilde{V}(\vec{k}) : (\vec{k}\vec{r})^m : . \end{aligned} \quad (94)$$

If we make the additional assumption that  $V(\vec{r})$  is rotational invariant, then  $\tilde{V}(\vec{k})$  depends only on  $|\vec{k}| = k$  and only the even terms  $m = 2n$  are non zero in Eq. (94). Integrating over  $\vec{k}$  and after some algebra we obtain

$$V(\vec{r}) = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{\Gamma(d/2)}{\Gamma(n+d/2)} \frac{\mathcal{M}_n}{n!} : (\vec{r}^2)^n : \quad (95)$$

with the moments  $\mathcal{M}_n$  given by

$$\begin{aligned} \mathcal{M}_n &= \int \frac{d^d \vec{k}}{(2\pi)^d} (\vec{k}^2)^n e^{-\frac{\vec{k}^2}{2} \mathbb{C}_{\text{var}}} \tilde{V}(\vec{k}) \\ &= 2 \frac{(4\pi)^{-d/2}}{\Gamma(d/2)} \int_0^{\infty} dk k^{d+2n-1} e^{-\frac{k^2}{2} \mathbb{C}_{\text{var}}} \tilde{V}(k) . \end{aligned} \quad (96)$$

Thus we finally can write Eq. (86) for  $\tilde{V}(k)$  as

$$\tilde{V}(k) + e^{-\frac{k^2}{2} \mathbb{C}_{\text{var}}} \langle : e^{i\vec{k}\vec{r}(x_0)} : e^{\int_x \Delta} \rangle_{\text{var}}^{\mathbb{C}_{\text{var}}} = 0 , \quad (97)$$

with the ‘‘perturbation’’  $\Delta(\vec{r})$  written in terms of normal products as

$$\begin{aligned} \Delta(\vec{r}) &= \left[ \frac{d}{2} m_{\text{var}}^2 \mathbb{C}_{\text{var}} - \mathcal{M}_0 \right] \mathbf{1} + \left[ \frac{1}{2} m_{\text{var}}^2 + \frac{1}{2d} \mathcal{M}_1 \right] : \vec{r}^2 : \\ &\quad - \sum_{n=2}^{\infty} \left( \frac{-1}{4} \right)^n \frac{\Gamma(d/2)}{\Gamma(n+d/2)} \frac{\mathcal{M}_n}{n!} : (\vec{r}^2)^n : . \end{aligned} \quad (98)$$

Let us first evaluate  $\Delta(\vec{r})$  when we simply take for  $V$  the variational estimate  $V_{\text{var}}$  given by Eqs. (58) and (59)

$$\tilde{V}(k) \rightarrow \tilde{V}_{\text{var}}(k) = -e^{-\frac{k^2}{2} \mathbb{C}_{\text{var}}} . \quad (99)$$

We get for the moments

$$\mathcal{M}_n \rightarrow \mathcal{M}_n^{\text{var}} = -\frac{\Gamma(n+d/2)}{\Gamma(d/2)} (4\pi \mathbb{C}_{\text{var}})^{-d/2} \mathbb{C}_{\text{var}}^{-n} \quad (100)$$

and we can use the self-consistent equation (60) for  $m_{\text{var}}$  which is in our notation

$$2 m_{\text{var}}^2 \mathbb{C}_{\text{var}} = (4\pi \mathbb{C}_{\text{var}})^{-d/2} \quad (101)$$

to get for  $\Delta(\vec{r})$

$$\Delta(\vec{r}) \rightarrow \Delta_{\text{var}}(\vec{r}) = m_{\text{var}}^2 \mathbb{C}_{\text{var}} \left[ \left( \frac{d}{2} + 2 \right) \mathbf{1} + \sum_{n=2}^{\infty} \left( \frac{-1}{4\mathbb{C}_{\text{var}}} \right)^n \frac{2}{n!} :(\vec{r}^n)^2: \right]. \quad (102)$$

The coefficient of the  $:\vec{r}^2:$  term is zero, since  $\mathcal{M}_1 = -d m_{\text{var}}^2$ .

This means that when we take  $V_{\text{var}}$  as potential  $V$ ,  $\Delta$  is indeed an interaction term, which contains no “mass renormalization”, but only higher order interaction terms.

### 6.3 A convenient rescaling

To check whether these terms are unimportant in the large  $d$  limit, it is better to rescale all quantities in terms of the variational mass  $m_{\text{var}}$ , that we take as unit of scale. Thus let us rescale

$$\begin{aligned} x &\rightarrow (m_{\text{var}})^{-1} x & q &\rightarrow m_{\text{var}} q \\ \vec{r} &\rightarrow (m_{\text{var}})^{D/2-1} \vec{r} & \vec{k} &\rightarrow (m_{\text{var}})^{1-D/2} \vec{k} \\ V &\rightarrow (m_{\text{var}})^D V \end{aligned} \quad (103)$$

In these units, all previous results obtained in this section are given by the same equations<sup>4</sup>, provided that we replace

$$m_{\text{var}} \rightarrow 1 \quad , \quad \mathbb{C}_{\text{var}} \rightarrow \mathbb{C} \quad , \quad (104)$$

that we *do not rescale*  $\tilde{V}$  in Eq. (97), and that we rescale the free energy density  $\mathcal{E}[V]$  as

$$\mathcal{E}[V] \rightarrow m_{\text{var}}^D \mathcal{E}[V]. \quad (105)$$

Thus we got rid of the complicated  $d$  and  $D$  dependence of  $m_{\text{var}}$  and keep only the simple factor of  $\mathbb{C}$  given by Eq. (92) in the calculations.

### 6.4 The variational solution as $D \rightarrow 2$

In order to estimate “how close” the variational potential  $V_{\text{var}}$  is from the exact instanton potential, let us consider the instanton equation (97), where we replace  $V$  by  $V_{\text{var}}$  and  $\Delta$  by  $\Delta_{\text{var}}$ . Then of course Eq. (97) is not satisfied, since

$$\langle :e^{i\vec{k}\vec{r}(x_0)}: e^{\int_x \Delta_{\text{var}}} \rangle_{\text{var}}^{\mathbb{C}} \neq 1. \quad (106)$$

This would be true only if all the  $n \geq 2$  terms in the expansion (102) for  $\Delta_{\text{var}}$  were zero.

We can compute the l.h.s. of Eq. (106) in perturbation theory. The first interaction vertices with their coefficients are depicted on Fig. 3. Let us note that the 2-point vertex, corresponding to the  $:\vec{r}^2:$  coefficient in Eq. (102), is zero. The “0-point” vertex, corresponding to the coefficient of the identity operator  $\mathbf{1}$ , disappears in the connected correlation functions (since by definition  $\langle [\dots] \mathbf{1} \rangle_{\text{var}}^{\mathbb{C}} = 0$ ), but will be present in the free energy density  $\mathcal{E}$ . The propagator is simply  $(q^2 + 1)^{-1}$ . The 0-th order term in the expansion of Eq. (106) is

$$\langle :e^{i\vec{k}\vec{r}(x_0)}: \rangle = 1 \quad (107)$$

---

<sup>4</sup>except of course Eq. (101).

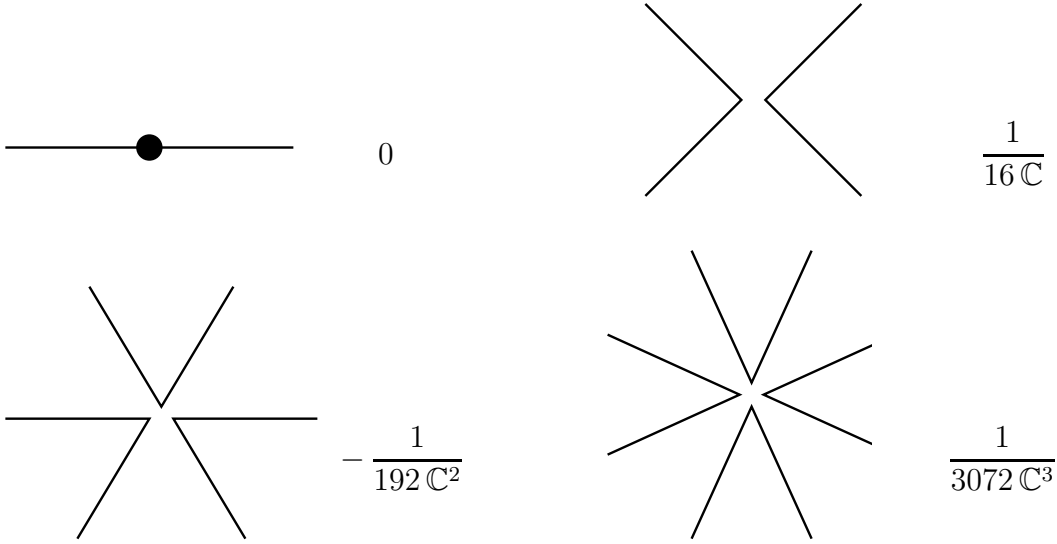


Figure 3: The interaction vertices from  $\Delta(\vec{r})$  in the variational approximation.

because all tadpoles are subtracted by the normal product prescription. On Fig. 4 we have depicted the only diagram which appears at order  $\mathbb{C}^{-1}$  with its combinatorial weight resulting from its symmetry factor, the interaction vertex coefficients and the contractions of the  $d$ -dimensional indices of the  $\vec{r}$ 's. We have also depicted the other possible diagrams which do not contribute, since they contain a tadpole which is subtracted by the normal order prescription. The diagrams at order  $\mathbb{C}^{-2}$  are depicted on Fig. 5, together with their weight. In general, these diagrams contain internal closed loops, which give a factor of  $d$ , and open chains which must end at some  $\vec{k}\vec{r}(x_0)$  in the exponential  $:\exp(i\vec{k}\vec{r}(x_0)):$ , thus giving a factor of  $\vec{k}^2$ .

We can now look at the limit when the internal dimension of the membrane  $D$  goes to 2, while  $\epsilon$  is fixed. In this limit, the bulk dimension of space  $d$  goes to infinity as

$$d = \frac{2(2D - \epsilon)}{2 - D} \simeq \frac{8 - 2\epsilon}{2 - D} \quad (108)$$

and the tadpole amplitude  $\mathbb{C}$  given by Eq. (92) diverges like  $d$ , since

$$\mathbb{C} = (4\pi)^{-D/2} \Gamma(1 - D/2) \simeq \frac{1}{2\pi} \frac{1}{2 - D} \simeq d \frac{1}{4\pi(4 - \epsilon)}. \quad (109)$$

On the other hand, the Feynman amplitude of the first diagram depicted on Fig. 4 and of those of Fig. 5 are finite when  $D \rightarrow 2$ , since they do not have any long distance (infra-red) or short distance (ultra-violet) divergences for  $0 \leq D \leq 2$ . This implies that as  $D \rightarrow 2$ , the contributions of diagrams that we are considering vanish at least as fast as  $2 - D \simeq 1/d$ .

This result is in fact valid for *all the diagrams* which appear at higher orders in the evaluation of the l.h.s. of Eq. (106). All diagrams vanish individually at least as  $1/d$ ,

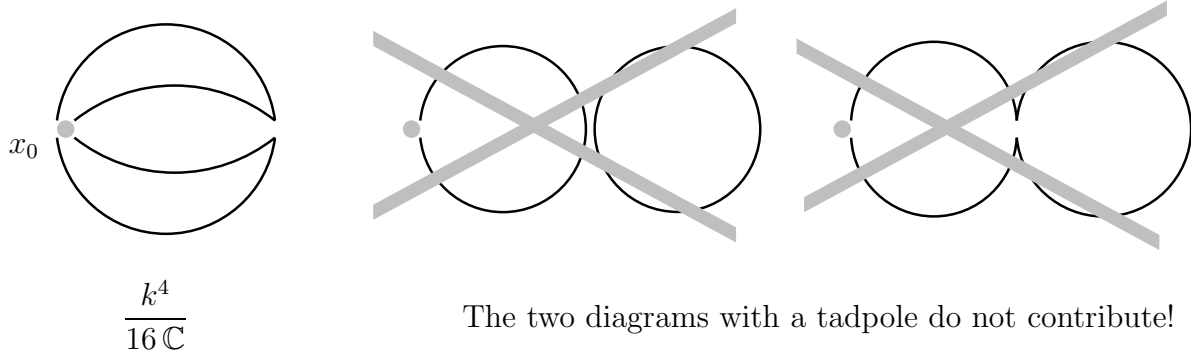


Figure 4: First order graphs in the variational approximation.

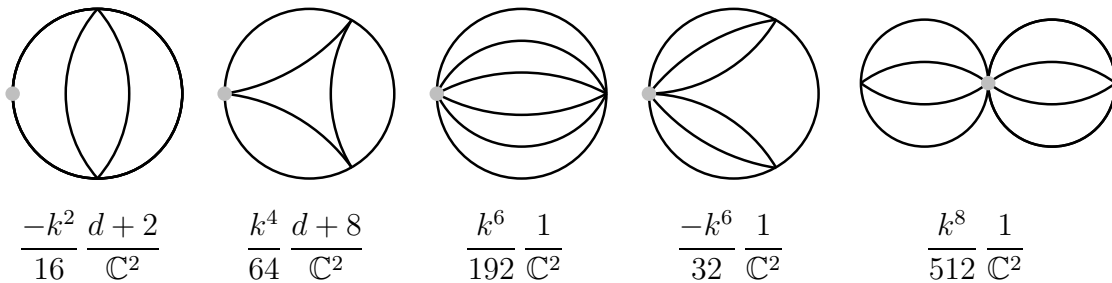


Figure 5: Second order graphs in the variational approximation.

and it is possible to generate a systematic expansion in powers of  $1/d$ . Of course there will be an infinite number of diagrams which contribute at a given order in  $1/d$ , that we shall characterize later. Let us assume that the sum of all diagrams which contribute at a given order is convergent. (This turns out to be true at least as long as  $\epsilon > 0$ .) Then this implies that the saddle point equation (97) holds at leading order in  $1/d$  for the variational solution

$$\tilde{V}_{\text{var}}(k) + e^{-\frac{k^2}{2}\mathbb{C}_{\text{var}}} \langle :e^{i\vec{k}\vec{r}(x_0)} : e^{\int_x \Delta_{\text{var}}} \rangle_{\text{var}}^{\mathbb{C}} = e^{-\frac{k^2}{2}\mathbb{C}_{\text{var}}} \mathcal{O}\left(\frac{1}{d}\right), \quad (110)$$

when we take the limit  $(2 - D) \sim d^{-1} \rightarrow 0$ ,  $\epsilon$  and  $k$  fixed.

A similar statement can be made for the vacuum diagrams which appear in the perturbative expansion of the free energy density. The first terms of this expansion are depicted on Fig. 6. The first two graphs denote the tree and one-loop contributions given respectively by (minus) the first term in the expansion of  $\Delta_{\text{var}}$

$$\mathcal{E}_{\text{var}}^{(0)} = -\mathbb{C} \left(2 + \frac{d}{2}\right) \quad (111)$$

and by the loop integral

$$\mathcal{E}_{\text{var}}^{(1)} = \frac{d}{2} \int \frac{d^D q}{(2\pi)^D} \log [1 + 1/q^2] = \frac{d}{D} \mathbb{C}. \quad (112)$$

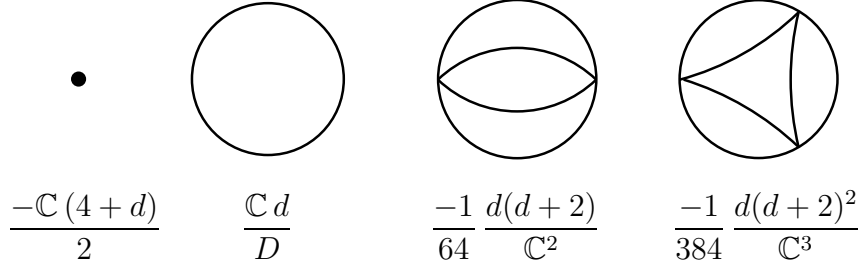


Figure 6: Free energy density.

The sum of these two terms gives the variational free energy density

$$\mathcal{E}_{\text{var}} = \mathbb{C} \left( -2 - \frac{d}{2} + \frac{d}{D} \right) = -\frac{\epsilon}{D} \mathbb{C} \quad (113)$$

which is of order  $\mathcal{O}(d)$ , while higher order corrections are at least of order  $\mathcal{O}(1)$  (in our normalization where  $m_{\text{var}} = 1$ ).

What are the consequences of this remarkable fact? First it will be possible to solve the saddle point equation (97) order by order in  $1/d$ , and to show that the exact instanton potential  $V$  differs from the variational solution only by corrections of order  $1/d$ ,

$$\tilde{V}(k) = \tilde{V}_{\text{var}}(k) [1 + \mathcal{O}(d^{-1})] \quad (114)$$

which are finite as long as  $\epsilon > 0$ . Using this fact we can show that the variational instanton action, which is in our normalizations

$$\mathcal{S}_{\text{var}} = \mathcal{E}_{\text{var}} + \frac{1}{2} \int V_{\text{var}}^2 = \left( 1 - \frac{\epsilon}{D} \right) \mathbb{C}, \quad (115)$$

is of order  $\mathcal{O}(d)$ , and differs from the exact instanton action by subdominant terms of order  $\mathcal{O}(1)$ . This justifies the use of the variational method as well as the large order analysis of the  $\epsilon$ -expansion results made in the previous Section.

### 6.5 Leading $1/d$ correction for the instanton potential

In order to study the  $1/d$  corrections, let us rewrite the normal product expansion (95) for the exact instanton potential  $V$  as

$$V(\vec{r}) = 2\mathbb{C} \sum_{n=0}^{\infty} \left( \frac{-1}{4\mathbb{C}} \right)^n \frac{\mu_n}{n!} :(\vec{r}^2)^n: . \quad (116)$$

With our rescaling in Eq. (103), starting from Eq. (96), the moments  $\mu_n$  are given by

$$\mu_n = \frac{2\mathbb{C}^{n+d/2}}{\Gamma(n+d/2)} \int_0^{\infty} dk k^{d+2n-1} e^{-k^2\mathbb{C}/2} \tilde{V}(k) . \quad (117)$$

We assume that  $V$  differs from  $V_{\text{var}}$  by  $\mathcal{O}(d^{-1})$ , or equivalently that

$$\mu_n = -1 + \frac{\delta_n}{d}, \quad \delta_n = \mathcal{O}(1) . \quad (118)$$



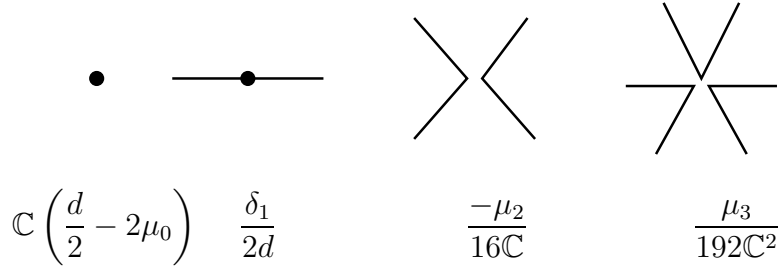


Figure 7: The vertices from  $\Delta(\vec{r})$ , Eq. (119).

The perturbation  $\Delta(\vec{r})$  given by Eq. (98) is then

$$\Delta(\vec{r}) = \mathbb{C} \left( \frac{d}{2} - 2\mu_0 \right) + \frac{\delta_1}{2d} : \vec{r}^2 : + (-2\mathbb{C}) \sum_{n=2}^{\infty} \left( \frac{-1}{4\mathbb{C}} \right)^n \frac{\mu_n}{n!} : (\vec{r}^2)^n : . \quad (119)$$

This generates the  $2n$ -point interaction vertices for the perturbative expansion around the variational Hamiltonian depicted on Fig. 7.

The first vertex with  $n = 0$  is just the correction to the free energy. The additional 2-point vertex is a mass correction and is of order  $1/d$ . The  $2n$ -point vertices ( $n \geq 2$ ) are similar to those of Fig. 3 and are of order  $1/d^{n-1}$ .

Before embarking onto the detailed calculations, let us recall what we are about to do. Our equation for the *exact* instanton potential  $V$  is a self-consistent equation (since both sides contain  $V$ ), and was written as a perturbative expansion with respect to the variational solution as

$$-\tilde{V}(k) e^{\mathbb{C}k^2/2} = \left\langle e^{ikr(x_0)} : e^{\int_x \Delta} \right\rangle_{\text{var}}^{\mathbb{C}} . \quad (120)$$

$\Delta$ , given in Eq. (119), contains all terms of the exact potential  $V$  with the exception of those already taken into account in the variational Hamiltonian.

It is then easy to see that the perturbative expansion generates diagrams, which can be organized in a  $1/d$  expansion in terms of the chain of bubbles depicted on Fig. 8. Indeed, each 4-point vertex carries a factor of  $\mathbb{C} \sim 1/d$  and each bubble carries a factor of  $d$ , so that the whole chain is of order  $1/d$ . A careful but simple analysis shows that only four different classes of diagrams contribute to the r.h.s. of Eq. (120) at order  $1/d$ ; they are depicted on Fig. 9. The suffix  $\boxed{n}$  ( $n = 0, 1$ ) refers to the minimal number of bubbles in the chain in order not to have tadpoles. The situation is thus quite similar to the  $1/n$  expansion in models with a  $n$ -component field with  $O(n)$  symmetry.

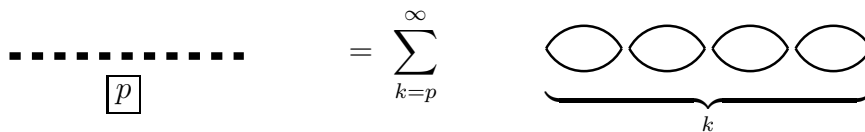


Figure 8: The chain of bubbles

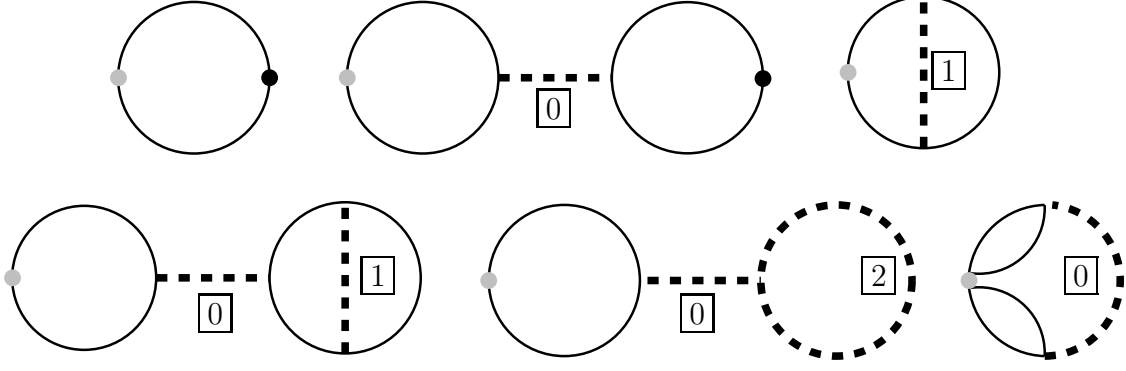


Figure 9:  $1/d$  diagrams for the expansion of  $\langle : e^{i\vec{k}\vec{r}(x_0)} : e^{\int_x \Delta_{\text{var}}} \rangle_{\text{var}}^{\mathbf{C}}$

The amplitude for a single bubble diagram with external momentum  $p$  is

$$\text{bubble} = \mathbb{B}(p) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + 1} \frac{1}{(p+q)^2 + 1} = \frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}} \mathbb{J}(p) \quad (121)$$

with

$$\mathbb{J}(p) = \int_0^1 dx [1 + x(1-x)p^2]^{\frac{D}{2}-2} \quad (122)$$

and can be expressed in terms of hypergeometric functions. In particular one has

$$\mathbb{B}(0) = (1 - D/2) \mathbf{C} . \quad (123)$$

Taking into account the symmetry factor, the amplitude for the  $\boxed{n}$ -truncated chain of Fig. 8 is the geometric function

$$\text{dashed chain } \boxed{n} = \mathbb{H}^{(n)}(p) = \sum_{m \geq n} \left[ (-\mu_2) \frac{d}{4\mathbf{C}} \mathbb{B}(p) \right]^m \quad (124)$$

so that in particular the untruncated chain is

$$\text{dashed chain } \boxed{0} = \mathbb{H}^{(0)}(p) = \left[ 1 + \mu_2 \frac{2D - \epsilon}{4} \mathbb{J}(p) \right]^{-1} . \quad (125)$$

Similarly, the free energy density  $\mathcal{E}$  can be expanded in  $1/d$  in terms of diagrams involving chains of bubbles. The diagrams which are present at order  $\mathcal{O}(1)$  are depicted on Fig. 10

The last diagram is the  $\boxed{2}$ -truncated closed chain of bubbles, whose amplitude is (taking into account the symmetry factors)

$$\text{dashed circle } \boxed{n} = \mathbb{G}^{(n)} = \int \frac{d^D p}{(2\pi)^D} \sum_{m \geq n} \frac{1}{m} \left[ (-\mu_2) \frac{d}{4\mathbf{C}} \mathbb{B}(p) \right]^m . \quad (126)$$

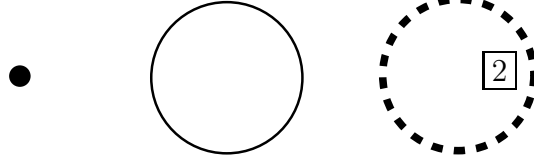


Figure 10:  $1/d$  diagrams for the expansion of the free energy density  $\mathcal{E}$  at order  $\mathcal{O}(1)$

With these notations, we now compute the  $1/d$  correction for  $V$ . With the symmetry factors for the diagrams of Fig. 9, Eq. (97) for  $\tilde{V}$  is at that order

$$\begin{aligned}
 \tilde{V}(k) = & -e^{-\frac{k^2}{2}\mathbb{C}} \left[ 1 + \text{diagram} \frac{-k^2 \delta_1}{2d} + \right. \\
 & + \text{diagram} \frac{k^2 \mu_2 \delta_1}{8\mathbb{C}} + \text{diagram} \frac{k^2 \mu_2}{4\mathbb{C}} + \\
 & + \text{diagram} \frac{-k^4 \mu_2}{16\mathbb{C}} + \text{diagram} \frac{-k^2 d (\mu_2)^2}{16\mathbb{C}^2} \\
 & \left. + \text{diagram} \frac{k^2 \mu_3}{8\mu_2 \mathbb{C}} + \mathcal{O}\left(\frac{1}{d^2}\right) \right]. \tag{127}
 \end{aligned}$$

Since we are interested in computing  $\tilde{V}$  at order  $1/d$  only, and since the diagrams on the r.h.s. of Eq. (127) are of order  $\mathcal{O}(1)$ , we can use the ansatz (118) and replace  $\mu_2$  and  $\mu_3$  by  $-1$  on the r.h.s. of (127) (but keep  $\delta_1$ ). Now we use Eq. (123), which implies that

$$\text{diagram} = (4 - \epsilon) \frac{\mathbb{C}}{d} + \mathcal{O}\left(\frac{1}{d}\right), \tag{128}$$

and from Eq. (124) we have

$$\text{diagram} \Big|_{p=0} = \frac{4}{\epsilon} + \mathcal{O}\left(\frac{1}{d}\right) \tag{129}$$



Inserting this result into Eq. (131), we obtain the potential  $\tilde{V}$ , and we can calculate the moments  $\mu_n$  from Eq. (117). Since for  $n \ll d$ , the moments are independent of  $n$ , we find that

$$\mu_n = -1 + \frac{\delta_1}{d} + \mathcal{O}\left(\frac{1}{d^2}\right) \quad \forall n \geq 0. \quad (136)$$

Therefore the ansatz (118) is consistent and our calculation at order  $1/d$  makes sense, provided that the closed chain of bubbles  $\mathbb{G}^{(2)}$ , which is a function of  $D$ ,  $\epsilon$  and  $\mu_2$ , has a finite limit as  $D \rightarrow 2$  (and  $\mu_2 = -1$ ). This is the case as long as  $\epsilon > 0$ . Indeed, the chain  $\mathbb{G}^{(2)}$  is given by Eq. (126) and using Eqs. (121) and (122), it is given in the limit  $D \rightarrow 2$  by

$$\begin{aligned} G(\epsilon) &= \left. \text{Diagram: a dashed circle with a square labeled '2' inside} \right|_{D=2} \\ &= \int \frac{d^2q}{(2\pi)^2} \left[ -\log \left[ 1 - \left(1 - \frac{\epsilon}{4}\right) J(q) \right] - \left(1 - \frac{\epsilon}{4}\right) J(q) \right] \end{aligned} \quad (137)$$

where

$$\begin{aligned} J(q) &= \mathbb{J}(q)|_{D=2} = \int_0^1 dx [1 + x(1-x)q^2]^{-1} \\ &= \frac{2}{q\sqrt{q^2+4}} \log \left[ \frac{\sqrt{q^2+4}+q}{\sqrt{q^2+4}-q} \right]. \end{aligned} \quad (138)$$

Indeed, we have for  $J(q)$

$$0 < J(q) < 1 \quad \forall q \neq 0; \quad J(q) \simeq 1 - \frac{q^2}{6} \text{ as } q \rightarrow 0; \quad J(q) \simeq \frac{2}{q^2} \log(q^2) \text{ as } q \rightarrow \infty, \quad (139)$$

hence the integral (137) defining the function  $G(\epsilon)$  is convergent for any  $\epsilon \geq 0$ .

Similarly, the chain with a marked point is given in the limit  $D \rightarrow 2$  by

$$\begin{aligned} \bar{G}(\epsilon) &= \left. \text{Diagram: a dashed circle with a square labeled '2' inside and a dot on the boundary} \right|_{D=2} \\ &= \int \frac{d^2q}{(2\pi)^2} \left[ \frac{1}{1 - \left(1 - \frac{\epsilon}{4}\right) J(q)} - 1 - \left(1 - \frac{\epsilon}{4}\right) J(q) \right] \\ &= (4 - \epsilon) \frac{d}{d\epsilon} G(\epsilon). \end{aligned} \quad (140)$$

It is finite for  $\epsilon > 0$ , but has a logarithmic divergence at  $\epsilon = 0$ , since from (139) the integrand in (140) behaves as  $q^{-2}$  as  $q \rightarrow 0$  when  $\epsilon = 0$ . A simple calculation shows that  $\bar{G}$  has a logarithmic singularity of the form

$$\bar{G}(\epsilon) \simeq \frac{3}{2\pi} \log\left(\frac{1}{\epsilon}\right) \quad \text{as } \epsilon \rightarrow 0 \quad (141)$$

With these notations, the final result for the instanton potential is at order  $1/d$

$$\tilde{V}(k) = e^{-k^2 \mathbb{C}/2} \left[ -1 + \frac{1}{d} \left( k^2 \left( \frac{1}{2-\epsilon} G(\epsilon) + \frac{4-\epsilon}{2} G'(\epsilon) \right) - k^4 \frac{1}{4\pi} G'(\epsilon) \right) + \mathcal{O} \left( \frac{1}{d^2} \right) \right]. \quad (142)$$

We note that the corrections to the variational potential are finite as long as  $0 < \epsilon < 2$ , but are singular at  $\epsilon = 0$  and  $\epsilon = 2$ . The singularity at  $\epsilon = 2$  is not surprising. We have seen from Eq. (61) that the variational mass  $m_{\text{var}}$  itself is singular at  $\epsilon = D$ , which corresponds in the limit  $D \rightarrow 2$  to  $\epsilon = 2$ . Therefore it is expected that the corrections to the variational result will also be singular. The singularity at  $\epsilon = 0$  is new, since  $m_{\text{var}}$  is regular at  $\epsilon = 0$ . We shall come back to this issue later.

### 6.6 $1/d$ correction to the instanton action

It is now easy to compute the corrections to the instanton action  $\mathcal{S}$ , defined by Eq. (24). The free energy density  $\mathcal{E}$  has the following graphical expansion

$$\mathcal{E}(V) = -\mathbb{C} \left( \frac{d}{2} - 2\mu_0 \right) \bullet + \frac{d}{2} \bigcirc - \frac{1}{2} \bigcirc\!\!\!\! \bigcirc \boxed{2} + \mathcal{O} \left( \frac{1}{d} \right) \quad (143)$$

where the amplitude for the simple loop is given by Eq. (112)

$$\bigcirc = \int \frac{d^D q}{(2\pi)^D} \log [1 + 1/q^2] = \frac{2}{D} \mathbb{C}. \quad (144)$$

The second term in Eq. (24) is

$$\frac{1}{2} \int d^d \vec{r} V(\vec{r})^2 = \frac{1}{2} \int \frac{d^d \vec{k}}{(2\pi)^d} |\tilde{V}(\vec{k})|^2. \quad (145)$$

Within our rescaling (103) this term is

$$\frac{1}{2} \int V^2 \rightarrow \mathbb{C} \frac{2\mathbb{C}^{d/2}}{\Gamma(d/2)} \int_0^\infty dk k^{d-1} |\tilde{V}(k)|^2 \quad (146)$$

and from Eqs. (117) and (118) it is simply obtained from the moment  $\mu_0$  by

$$\frac{1}{2} \int V^2 = \mathbb{C} \left[ -1 - 2\mu_0 + \mathcal{O}(d^{-2}) \right]. \quad (147)$$

Thus the instanton action, which is the sum of Eqs. (143) and (147) is finally given at order  $1/d$  by

$$\mathcal{S}(V) = \mathbb{C} \left( \frac{d}{2} + 4 \right) \bullet - \frac{d}{2} \bigcirc - \frac{1}{2} \bigcirc\!\!\!\! \bigcirc \boxed{2} + \mathcal{O} \left( \frac{1}{d} \right) \quad (148)$$

$$= \mathbb{C} \left(1 - \frac{\epsilon}{D}\right) - \frac{1}{2} \mathbb{G}^{(2)}(D, \epsilon) + \mathcal{O}\left(\frac{1}{d}\right). \quad (149)$$

This result holds within the rescaling (103). The first term on the r.h.s. of Eq. (149) is the variational result, and is of order  $\mathcal{O}(d)$ . The second term is of order  $\mathcal{O}(1)$ . Thus the first correction to the variational result is given by the closed loop diagram of Eq. (137).

According to Eq. (105), to recover the normalization used in the previous sections, we have simply to multiply Eq. (149) by  $m_{\text{var}}^D$ , where  $m_{\text{var}}$  is the variational mass as given by Eq. (61).

## 6.7 Discussion of the result

(1) The corrections of the variational instanton potential do in fact not contribute to the correction of the instanton action. Indeed, they are entirely contained in the moment  $\mu_0$  which disappears in Eq. (149). This is not surprising, since the instanton potential  $V$  is defined by a variational principle with respect to  $V$ .

(2) The correction to the instanton action is finite as  $\epsilon \rightarrow 0$ , while from Eq. (142) the correction to the potential  $V$  diverges as  $\log(\epsilon)$ . The divergences occur in fact because the chain of bubbles of Fig. 8,  $\mathbb{H}^{(0)}(q)$ , has a singularity at small momentum  $q$  as  $D \rightarrow 2$  and  $\epsilon \rightarrow 0$ . According to Eqs. (125) and (139), it behaves as

$$\left[ \text{---} \right]_{\boxed{0}} \Big|_{q \rightarrow 0} \simeq \frac{1}{q^2/6 + \epsilon/4 + \mathcal{O}(2-D)}. \quad (150)$$

In other word, this chain of bubbles behaves as a massless propagator at  $\epsilon = 0$  and  $D = 2$ . We have no physical interpretation of these IR singularities, and why they cancel in the instanton action. However, the fact that as  $D \rightarrow 2$  the instanton potential is singular as  $\epsilon \rightarrow 0$  is similar to the situation at  $D = 1$  discussed in section 5.1. Here also, we know from the exact solution that the instanton potential is singular when  $\epsilon \rightarrow 0$  ( $d \rightarrow 4$ ), while the instanton action is regular, and in fact Eq. (67) corresponds also to a logarithmic behavior as  $\epsilon \rightarrow 0$ . Therefore, it is reasonable to conjecture that at next orders in  $1/d$ , the IR singularities which occur as  $\epsilon \rightarrow 0$  in the instanton potential still disappear in the instanton action. This would imply that the coefficient  $\mathcal{C}$  has a regular behavior when  $\epsilon \rightarrow 0$  and that the limits  $d \rightarrow \infty$  and  $\epsilon \rightarrow 0$  can be exchanged, providing further consistency to our arguments for the large order behavior of the  $\epsilon$ -expansions for the SAM model.

(3) The  $1/d$  correction to the instanton action  $\mathcal{S}_{\text{inst}}$  is negative, as expected, since it should improve the variational estimate for  $\mathcal{S}_{\text{inst}}$ , which is an upper bound.

(4) It is interesting to see if the  $1/d$  correction can be used to improve the variational estimates for quantities of physical significance, such as the large order constant  $\mathcal{C}$ . Let us start from the result (149) for the instanton action  $\mathcal{S}$  at first order in  $1/d$ , and insert

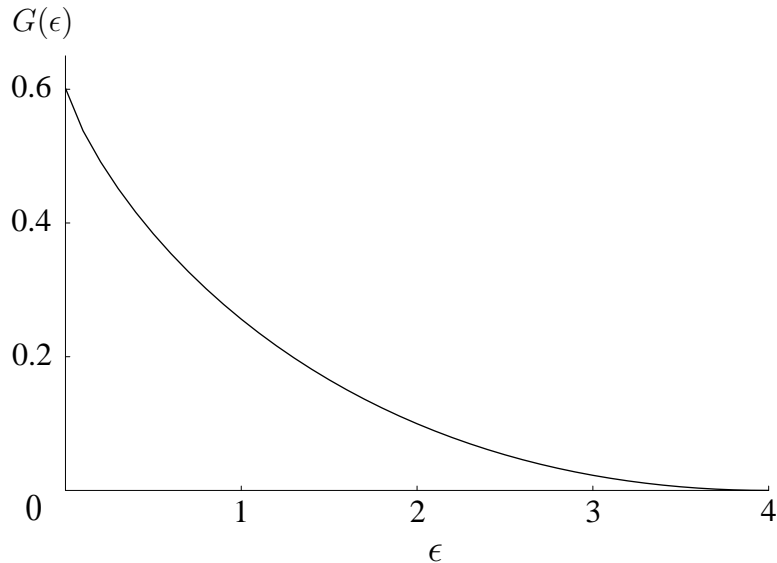


Figure 11: The function  $G(\epsilon)$  defined in Eq. (137) from the chain of bubbles in  $D = 2$ .

it into Eq. (30) for  $\mathcal{C}$ . We obtain

$$\mathcal{C}^{-1} = \mathcal{C}_{\text{var}}^{-1} \left[ 1 - \frac{\mathbb{G}^{(2)}(D, \epsilon)}{2\mathbb{C}(D)} + \mathcal{O}(d^{-2}) \right]. \quad (151)$$

Now we only keep the leading  $1/d$  corrections on the r.h.s. of Eq. (151) in the limit  $d \rightarrow \infty$ , for fixed  $\epsilon$  by using

$$\mathbb{G}^{(2)}(D, \epsilon) = G(\epsilon) + \mathcal{O}(d^{-1}), \quad \mathbb{C}(D) = \frac{d}{4\pi} \frac{1}{4-\epsilon} + \mathcal{O}(1), \quad (152)$$

with  $G(\epsilon)$  given by (137). We thus obtain

$$\mathcal{C}^{-1} = \mathcal{C}_{\text{var}}^{-1} \left[ 1 - 2\pi(4-\epsilon)G(\epsilon) \frac{1}{d} + \mathcal{O}(d^{-2}) \right]. \quad (153)$$

We have plotted on Fig. 11 the function  $G(\epsilon)$  for  $0 < \epsilon < 4$ , as obtained by a straightforward numerical integration.  $G$  is maximal for  $\epsilon = 0$ , where we have  $G(0) = 0.6014\dots$

Let us estimate the first  $1/d$  correction for the case of polymers in  $d = 4$  dimensions. We simply have to set  $d = 4$  and  $\epsilon = 0$  in (153). We find that the  $1/d$  correction is

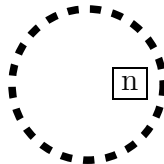
$$2\pi(4-\epsilon)G(\epsilon) \frac{1}{d} \rightarrow 2\pi G(0) = 3.78\dots, \quad (154)$$

which is much larger than 1 ! Thus the  $1/d$  correction is very large for  $d = 4$  and one should take into account the subleading corrections and resum them. We recall that for  $d = 4$  and  $\epsilon = 0$  we have  $\mathcal{C}_{\text{exact}}^{-1}/\mathcal{C}_{\text{var}}^{-1} = 2/3 = 0.666\dots$ . In practice, we expect that for  $\epsilon = 0$  the  $1/d$  corrections will be smaller or equal to the variational  $\mathcal{O}(d^0)$  result for  $d > 16$  ! Therefore we cannot use naively the  $1/d$  corrections that we have calculated to improve the variational estimates. In the next subsection we propose an improved resummation procedure, which takes into account some of the higher order corrections which are contained in the bubble chain diagrams, and which gives much better results.



## 6.8 An attempt to go beyond the first $1/d$ correction

In the last section we have shown that a straightforward  $1/d$ -expansion cannot be applied to small dimension, as e.g. polymers in  $d = 4$ . In this section, we propose a different approximation scheme. It consists in summing exactly the chain of bubbles  $\mathbb{G}^{(2)}$ , and keeping the full  $D, d$ -dependence when extrapolating to low dimensions, instead of expanding this quantity about  $d = \infty$ . First, in Eq. (126), only the leading  $d$ -dependence had been kept. Taking into account the complete  $d$ -dependence, we obtain



$$\boxed{n} = \mathbb{G}^{(n)} = \frac{d}{d+2} \int \frac{d^D p}{(2\pi)^D} \sum_{m \geq n} \frac{1}{m} \left[ (-\mu_2) \frac{d+2}{4\mathbb{C}} \mathbb{B}(p) \right]^m. \quad (155)$$

Eliminating as before  $\mathbb{B}(p)$  in favor of  $\mathbb{J}(p)$ ,  $d$  in favor of  $\epsilon$  and replacing  $\mu_2$  by its leading contribution  $\mu_2 = -1$  (justified later), we obtain for the correction to the variational result

$$\frac{S - S_{\text{var}}}{S_{\text{var}}} = \frac{D(2D - \epsilon)}{(2 + D - \epsilon)(D - \epsilon)} \frac{\sin \frac{\pi D}{2}}{\pi} \times \int_0^\infty dp p^{d-1} \left[ \ln \left( 1 - \frac{2 + D - \epsilon}{4} \mathbb{J}(p) \right) + \frac{2 + D - \epsilon}{4} \mathbb{J}(p) \right], \quad (156)$$

In the remainder, we shall focus on the case  $D = 1$ , for which we can most easily test Eq. (156). In  $D = 1$ ,  $\mathbb{J}(p)$  is exactly given by

$$\mathbb{J}(p) = \frac{4}{4 + p^2}. \quad (157)$$

Eq. (156) is then integrated (using the residue calculus) with the result

$$\frac{S - S_{\text{var}}}{S_{\text{var}}} \Big|_{D=1} = \frac{2d}{(d+2)(d-2)} \left( \sqrt{3 - \frac{d}{2}} + \frac{d}{8} - \frac{7}{4} \right). \quad (158)$$

The large order estimate is finally obtained as

$$\mathcal{C}^{-1}(D = 1) \simeq 2 \pi^{d/2} \left[ 1 + \frac{d}{d+2} \left( \sqrt{3 - \frac{d}{2}} + \frac{d}{8} - \frac{7}{4} \right) + \dots \right], \quad (159)$$

which is plotted on Fig. 12. We see that this corrects 50% of the deviation of the variational result from the exact result in  $d = 4$ , and is even better in lower dimensions. Note that this is not the straightforward  $1/d$ -correction, obtained in section 6.7. In the following, we want to justify that the result given in Eqs. (158) and (159) is meaningful for small dimension, by analyzing the case  $D = 1$  and  $d = 4$ , i.e.  $\epsilon = 0$ , where calculations are most easily done.

The first observation is that  $\mathbb{H}^{(0)}(p)$  is nicely convergent for all values of  $p$ . In contrast to  $D = 2$ , no divergence at  $p = 0$  appears.

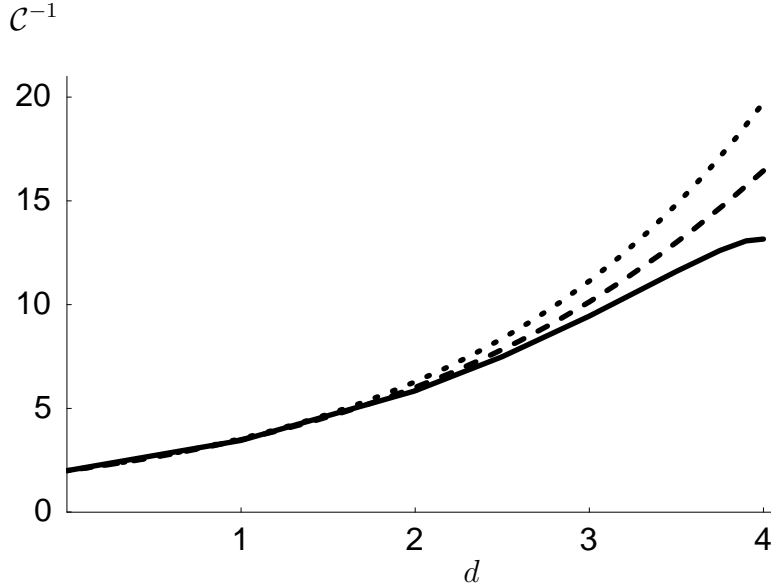
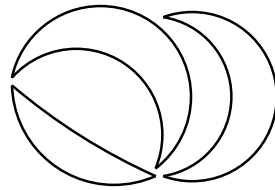


Figure 12: The inverse of the large order constant  $1/\mathcal{C}$  for the Edwards model ( $D = 1$ ) as a function of the bulk dimension  $d$ . The dotted curve is the  $\mathcal{O}(1)$  variational estimate, the dashed curve the estimate from Eq. (159), the continuous curve the exact result.

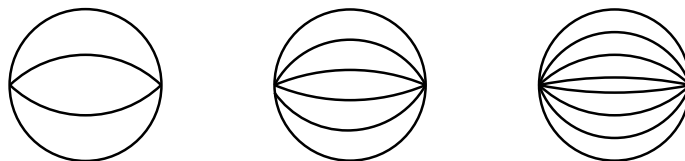
The next observation is that all terms in the chain of bubbles  $\mathbb{G}^{(2)}$  are positive and the sum rapidly converging. (If we denote by 100% the difference between variational estimate and exact result, the contributions of the first terms are 28%, 11%, 4%,..., respectively, which add up to the 50% given above.)

We also observe that most of the missing contributions to  $S(V)$  are positive: Let us call a vertex even, if it possesses an even number of pairs of lines, and odd otherwise. Any even vertex then contributes a factor of 1, whereas any odd vertex contributes a factor of  $(-1)$  (see Eqs. (116) and (118)). Since all integrals are positive, in order to build up a negative contribution, an odd number of odd vertices has to be taken. One sees by inspection, that this is more difficult than for a positive diagram (containing even vertices and an even number of odd vertices). The first diagram of this kind is



(160)

Another hint comes from a second class of diagrams, which can be summed, namely all “watermelon” diagrams with 2 vertices only:



(161)

The first diagram, equivalent to 28%, is already contained in  $\mathbb{G}^{(2)}$ , the following contribute 6%, 1.5%, ... respectively. They can be resummed and are then given by the integral

$$\int_0^\infty dx \left(1 - \frac{e^{-2x}}{4}\right)^{-d/2} - 1 - \frac{d}{8}e^{-2x}, \quad (162)$$

which contributes about 8.2% in  $d = 4$ , when neglecting the first diagram already taken into account in  $\mathbb{G}^{(2)}$ .

A more systematic approach to understand why higher order vertices are subdominant, is to look for an additional small parameter. We have seen that  $\mathbb{C} \sim d$  for  $d \rightarrow \infty$ . Comparing the ratio  $d/\mathbb{C}$  at  $d = \infty$  and  $d = 4$ , we obtain (always for  $\epsilon = 0$ )

$$\frac{d/\mathbb{C}|_{d=4}}{d/\mathbb{C}|_{d=\infty}} = \frac{1}{2\pi}. \quad (163)$$

This suggests (as exemplified by the preceding calculation) that higher order vertices, which come with additional factors of  $1/\mathbb{C}$ , indeed contribute an additional small factor.

The last point to verify, is that also the  $1/d$ -correction to the potential is small. In the notation of Eqs. (116) and (118), the parameter  $\delta_1/d$ , given in Eq. (135), reads

$$\left. \frac{\delta_1}{d} \right|_{D=1, \epsilon=0} = \frac{1}{4\mathbb{C}} \left. \left( \text{Diagram: a dashed circle with a square box containing the number 2 inside} \right) \right|_{D=1, \epsilon=0} + \mathcal{O}\left(\frac{1}{d^2}\right) = \frac{1}{12} + \mathcal{O}\left(\frac{1}{d^2}\right) \quad (164)$$

It is indeed small.

Another important limit to verify is  $d \rightarrow 0$ . We have argued in section 5.1 that in that limit the variational result becomes exact. Our correction in Eq. (156) consistently vanishes. Diagrammatically this can be seen from the fact that *any* diagram which contributes to the free energy has at least one closed loop, and therefore comes with a factor of  $d$ . This is also the leading contribution for small  $d$ : due to the “islands and bridges theorem”, which states that if from any island an even number of bridges starts, it is always possible to construct a path which uses any bridge exactly once, the minimal number of closed loops, and thus factors of  $d$ , for any given diagram is one. It is therefore impossible to simply reorganize the perturbative expansion in a  $d$ -expansion. However, the large- $d$  expansion should respect this exact property. This is not assured by the naive  $1/d$ -expansion, but by our modified result discussed here.

All these arguments suggest that Eq. (159) is a sensible correction to the variational result, and should be improved by taking into account higher order vertices, although this expansion is not a systematic expansion in a small parameter.

## 7 Conclusion

Let us first summarize the results of this paper. We have developed a general formulation to estimate the large orders of perturbation theory for the Edwards model of self-avoiding membranes and polymers. We have shown that these large orders are controlled by

a semi-classical effective potential  $V(\vec{r})$ , solution of a non-local extremization problem. This effective potential is the analog to the Lipatov instanton in Landau-Ginsburg-Wilson models. In the case of polymers (membranes with internal dimension  $D = 1$ ), this SAM-instanton corresponds precisely to the Lipatov instanton for the  $n = 0$  components LGW model. The large order behavior for the SAM model is derived in the general case ( $D \neq 1$ ). The equations for the SAM-instanton are solved within a Gaussian variational approximation. The result is for  $D = 1$  compared to the exact results from Lipatov's method, and is found to be qualitatively correct for polymers in  $d = 4$  dimensions. Finally it is shown that the variational result is the first term of a systematic expansion in  $1/d$ , where  $d$  is the dimension of space in which the membrane fluctuates.

All these results are new, and represent a considerable advance in the understanding of self-avoiding membranes beyond the first orders of perturbation theory. A number of interesting issues still has to be addressed:

- We have seen that a systematic expansion in  $1/d$  can be constructed, of which the variational approximation is only the leading term. However, adding the first correction in  $1/d$  to the instanton action does not give reliable results for small  $d$ . This is a numerical problem which comes from the fact that in the interesting cases, the corrections in  $1/d$  are already quite large. We have proposed a modified procedure to resum the first corrections which gives better results when applied to the case of polymers ( $D = 1, d = 4$ ), but more systematic resummation methods are needed in order to improve in a reliable way the variational results.
- We have not calculated the contributions from the fluctuations around the instanton, which should give the value of the global constant  $\mathcal{A}'$  in Eq. (32). This calculation is technically more difficult than the one for the instanton action itself.
- We have only discussed shortly and at a qualitative level the consequences of our results for the  $\epsilon$ -expansion of the scaling exponents. It would be interesting to obtain more precise results. In particular, the discussion relies on the limit  $\epsilon \rightarrow 0$  for the large order estimates. We have shown that this limit exists in the variational approximation, but that the  $1/d$  corrections then suffer from IR divergences. These divergences cancel for the instanton action at first order in  $1/d$ , and we conjectured that this feature persists at higher orders. Further studies are needed to clarify this important problem.
- Finally, let us mention that our approach can be applied to other classes of interactions between membranes.

### **Acknowledgments:**

F. D. thanks J. des Cloizeaux for a useful discussion. K. W. gratefully acknowledges useful discussions with H.W. Diehl. It is equivalently a pleasure for him to thank L. Schäfer for numerous enlightening discussions at an early stage of this work. We thank E. Gutter for his interest and proof reading.

## A The $\Phi^4$ instanton in the limit $d \rightarrow 4$

The instanton equation (37) for  $\Psi_0$  is, assuming rotational invariance i.e.  $\Psi_0(\vec{r}) = \Psi_0(r)$  with  $r = |\vec{r}|$ ,

$$\Psi_0'' + \frac{d-1}{r} \Psi_0' + 2E_0 \Psi_0 + 2\Psi_0^3 = 0. \quad (165)$$

We look for the solution which is regular at the origin, which has no zero for finite  $r$  and which vanishes at infinity, i.e.

$$0 < \Psi_0(r) < \infty \quad \text{for} \quad 0 \leq r < \infty, \quad \Psi_0(r) \rightarrow 0 \quad \text{when} \quad r \rightarrow \infty. \quad (166)$$

$E_0 \leq 0$  is fixed by the normalization (38)

$$\|\Psi_0\|^2 = \int d^d \vec{r} \Psi_0(\vec{r})^2 = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dr r^{d-1} \Psi_0(r)^2 = 1. \quad (167)$$

From Eq. (45) we know that  $E_0$  can be extracted from the action of the instanton for the LGW action (44), which is finite when  $d \rightarrow 4$ , such that  $E_0$  behaves as

$$E_0 \propto d - 4. \quad (168)$$

For  $d = 4$  and  $E_0 = 0$ , the general solution of Eq. (165) is not normalizable and reads

$$\Psi_0(r) = \frac{2r_0}{r^2 + r_0^2}. \quad (169)$$

For  $d \neq 4$ ,  $E_0 < 0$  and  $r \rightarrow \infty$ ,  $\Psi_0$  is a solution of the linearized equation  $\Psi'' + (d-1)/r \Psi' + 2E_0 \Psi = 0$ , given by a Bessel function

$$\Psi_0(r) \simeq r^{1-d/2} K_{d/2-1}(|2E_0|^{1/2} r). \quad (170)$$

Assuming that

$$r_0^2 |E_0| \ll 1 \quad \text{when} \quad d \rightarrow 4, \quad (171)$$

it has the asymptotics

$$\Psi_0(r) \propto \begin{cases} r^{2-d} & \text{for } r^2 |2E_0| \ll 1 \\ r^{\frac{1-d}{2}} e^{-|2E_0|^{1/2} r} & \text{for } r^2 |2E_0| \gg 1 \end{cases}. \quad (172)$$

For  $r_0 \ll r \ll |2E_0|^{-1/2}$ ,  $\Psi_0$  decays algebraically as  $1/r^2$ . It is exponentially decreasing for  $r \gg (-2E_0)^{-1/2}$ . We may thus approximate the integral in Eq. (167) by

$$\|\Psi_0\|^2 \approx 2\pi^2 \int_{r_0}^{\frac{a}{\sqrt{4-d}}} dr r^3 \left(\frac{r_0}{r^2}\right)^2 \approx 2\pi^2 r_0^2 \log \left[ \frac{a}{r_0 \sqrt{4-d}} \right], \quad (173)$$

where  $a$  is the proportionality factor in Eq. (168). From the constraint  $\|\Psi_0\| = 1$  we deduce that the size of the instanton  $r_0$  becomes small as  $d \rightarrow 4$  and scales as

$$r_0^2 \sim \frac{1}{|\log(4-d)|}. \quad (174)$$

This is consistent with the assumption (171).

## B A simple relation

We derive the relation

$$D \text{ (dashed circle with box 2)} = (4-D) \text{ (dashed circle with box 2 and dot)} + \frac{\mu_2 d}{\mathbb{C}} \text{ (solid circle with box 1 and vertical dashed line)} . \quad (175)$$

For that purpose, we write the amplitude for the last diagram as an integral over  $\vec{p}$ , the  $D$ -dimensional momentum which flows through the chain of bubbles

$$\text{ (solid circle with box 1 and vertical dashed line) } = \int \frac{d^D \vec{p}}{(2\pi)^D} \mathbb{D}(p) \mathbb{H}^{(1)}(p) = \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_0^\infty dp p^{D-1} \mathbb{D}(p) \mathbb{H}^{(1)}(p) . \quad (176)$$

The definition (124) for the chain of bubbles was

$$\mathbb{H}^{(1)}(p) = \text{ (dashed line with box 1) } = \left[ 1 + \mu_2 \frac{d}{4\mathbb{C}} \mathbb{B}(p) \right]^{-1} - 1 , \quad (177)$$

and  $\mathbb{D}(p)$  is the amplitude for the bubble with a single mass insertion

$$\mathbb{D}(p) = \vec{p} \rightarrow \text{ (bubble with dot) } = \int \frac{d^D \vec{q}}{(2\pi)^D} \frac{1}{(\vec{q}^2 + 1)^2} \frac{1}{(\vec{p} + \vec{q})^2 + 1} . \quad (178)$$

We are careful in distinguishing the vector  $\vec{p}$  from its modulus  $p = |\vec{p}|$ . Similarly the amplitudes for the first two diagrams are

$$\text{ (dashed circle with box 2) } = \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_0^\infty dp p^{D-1} \left[ -\log \left[ 1 + \mu_2 \frac{d}{4\mathbb{C}} \mathbb{B}(p) \right] + \mu_2 \frac{d}{4\mathbb{C}} \mathbb{B}(p) \right] , \quad (179)$$

and

$$\text{ (dashed circle with box 2 and dot) } = \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_0^\infty dp p^{D-1} \left[ \left( 1 + \mu_2 \frac{d}{4\mathbb{C}} \mathbb{B}(p) \right)^{-1} - 1 + \mu_2 \frac{d}{4\mathbb{C}} \mathbb{B}(p) \right] . \quad (180)$$

We can obtain  $\mathbb{D}(p)$  from  $\mathbb{B}(p)$ . Let us introduce a mass  $m$  and consider

$$\mathbb{D}(p, m) = \int \frac{d^D \vec{q}}{(2\pi)^D} \frac{1}{(\vec{q}^2 + m^2)^2} \frac{1}{(\vec{p} + \vec{q})^2 + m^2} \quad (181)$$

and similarly let us introduce a mass in the bubble  $\mathbb{B}$  of Eq. (121) and define

$$\mathbb{B}(p, m) = \int \frac{d^D \vec{q}}{(2\pi)^D} \frac{1}{(\vec{q}^2 + m^2)} \frac{1}{(\vec{p} + \vec{q})^2 + m^2} . \quad (182)$$

It is easy to see that one has

$$m \frac{d}{dm} \mathbb{B}(p, m) = -4 m^2 \mathbb{D}(p, m) , \quad (183)$$

and since by homogeneity

$$\mathbb{B}(p, m) = m^{D-4} \mathbb{B}(p/m) , \quad (184)$$

we deduce that (setting at the end  $m = 1$ )

$$(D - 4) \mathbb{B}(p) - p \frac{d}{dp} \mathbb{B}(p) = -4 \mathbb{D}(p) . \quad (185)$$

We can use Eq. (185) to express  $\mathbb{D}(p)$  in terms of  $\mathbb{B}(p)$  and its derivative in Eq. (176) and integrate by part in order to eliminate the derivative of  $\mathbb{B}$ . We obtain

$$\begin{aligned} \left( \text{circle with a dot on the left and a vertical dashed line through the center containing a box with '1'} \right) &= \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_0^\infty dp p^{D-1} \left( 1 - \frac{D}{4} \right) \mathbb{B}(p) \left[ \left( 1 + \frac{\mu_2 d}{4\mathbb{C}} \mathbb{B}(p) \right)^{-1} - 1 \right] \\ &\quad - \frac{D}{4} \left[ \frac{4\mathbb{C}}{\mu_2 d} \log \left( 1 + \frac{\mu_2 d}{4\mathbb{C}} \mathbb{B}(p) \right) - \mathbb{B}(p) \right] . \end{aligned} \quad (186)$$

Using Eqs. (179) and (180) we obtain the identity (175).

## References

- [1] D. R. Nelson and L. Peliti, *J. de Physique* **48** (1987) 1085.
- [2] For an introduction and review see: *Statistical Mechanics of Membranes and Surfaces*, Proceedings of the Fifth Jerusalem Winter School for Theoretical Physics (1987), D. R. Nelson, T. Piran and S. Weinberg Eds., World Scientific, Singapore (1989).
- [3] L. Peliti and S. Leibler, *Phys. Rev. Lett.* **54** (1985) 1690.
- [4] M. Paczuski, M. Kardar and D. R. Nelson, *Phys. Rev. Lett.* **60** (1988) 2638.
- [5] F. David and E. Guitter, *Europhys. Lett.* **5** (1988) 709.
- [6] Y. Kantor, M. Kardar and D. R. Nelson, *Phys. Rev. Lett.* **57** (1986) 791.
- [7] Y. Kantor, M. Kardar and D. R. Nelson, *Phys. Rev.* **A 35** (1987) 3056.
- [8] J. A. Aronovitz and T. C. Lubensky, *Phys. Rev. Lett.* **60** (1988) 2634.
- [9] M. Kardar and D. Nelson, *Phys. Rev. Lett.* **58** (1987) 2774.
- [10] M. Kardar and D. Nelson, *Phys. Rev.* **A 58** (1988) 966.
- [11] S. F. Edwards, *Proc. Phys. Soc. Lond.* **85** (1965) 613.

- [12] For a general review see:  
 J. des Cloizeaux and G. Jannink, *Polymers in solution, their modeling and structure*, Clarendon Press, Oxford, (1990).
- [13] J. des Cloizeaux, *J. de Physique* **42** (1981) 635.
- [14] T. Hwa, *Phys. Rev. A* **41** (1990) 1751
- [15] P.G. de Gennes, *Phys. Lett.* **38 A** (1972) 339.
- [16] M. Benhamou and G. Mahoux, *J. de Physique* **47** (1986) 559.  
 see also:  
 B. Duplantier, *J. de Physique* **47** (1986) 569.
- [17] F. David, B. Duplantier and E. Guitter, *Phys. Rev. Lett.* **76** (1996) 4564.
- [18] F. David, B. Duplantier and E. Guitter, *Renormalization Theory for the Self-Avoiding Polymerized membranes*, Saclay Preprint T/97001, cond-mat/9702136.
- [19] B. Duplantier, *Phys. Rev. Lett.* **58** (1987) 2733; and in [2].
- [20] F. David and K. J. Wiese, *Phys. Rev. Lett.* **76** (1996) 4564.
- [21] K. J. Wiese and F. David, *Nucl. Phys. B* **487** (1997) 529.
- [22] L. N. Lipatov, *JETP Lett.* **24** (1976) 157; *Sov. Phys. JETP* **44** (1976) 1055; *JETP Lett.* **25** (1977) 104; *Sov. Phys. JETP* **45** (1977) 216.
- [23] For a general review see:  
 J. Zinn-Justin, *The principles of instanton calculus*, in Recent advances in field theory and statistical mechanics, XXXIX Les Houches Summer School 1982, J.-B. Zuber and R. Stora Eds., North Holland (1984) Amsterdam;  
 J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press (1989), Oxford.