

D-INSTANTONS, STRINGS AND M-THEORY

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Abstract

The R^4 terms in the effective action for M-theory compactified on a two-torus are motivated by combining one-loop results in type II superstring theories with the $Sl(2, \mathbf{Z})$ duality symmetry. The conjectured expression reproduces precisely the tree-level and one-loop R^4 terms in the effective action of the type II string theories compactified on a circle, together with the expected infinite sum of instanton corrections. This conjecture implies that the R^4 terms in ten-dimensional string type II theories receive no perturbative corrections beyond one loop and there are also no non-perturbative corrections in the ten-dimensional IIA theory. Furthermore, the eleven-dimensional M-theory limit exists, in which there is an R^4 term that originates entirely from the one-loop contribution in the type IIA theory and is related by supersymmetry to the eleven-form $C^{(3)}R^4$. The generalization to compactification on T^3 as well as implications for non-renormalization theorems in D-string and D-particle interactions are briefly discussed.

1. Introduction

The interconnections between apparently distinct superstring theories and their connection to eleven-dimensional M-theory provide strong constraints on their non-perturbative structure. Any of the various string ‘theories’ is defined as a perturbative expansion in powers of the string coupling, e^ϕ , where ϕ is the dilaton. Its low energy behaviour is determined by an effective action that is a function of background massless fields obtained by integrating out all (massless and massive) quantum fields. The effective action has an expansion in powers of space-time derivatives (inverse powers of the string tension). The absence of a scalar field in eleven-dimensional M-theory means that it does not possess a loop expansion but it does have an effective action that may, in principle, be expressed as a low energy expansion that begins with the standard supergravity action of [1], which is a supersymmetric extension of the Einstein–Hilbert action,

$$S_R = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G^{(11)}} R, \quad (1)$$

where $G^{(11)}$ is the determinant of the eleven-dimensional metric.¹

It was argued in [2] that this may be viewed as the strong coupling limit of ten-dimensional type IIA superstring theory with the identification

$$R_{11} = (\alpha')^{1/2} \lambda^A, \quad (2)$$

where λ^A is the type IIA coupling constant and R_{11} is the radius of the eleventh dimension in the M-theory metric.

The next gravitational terms in the low-energy expansion of type II superstring actions beyond the Einstein–Hilbert term are fourth order in the Riemann curvature. In this paper we shall use the symbol $t_8 t_8 R^4$ to indicate these terms, in which the contractions of the four Riemann tensors are defined by

$$t_8 t_8 R^4 \equiv t^{\mu_1 \dots \mu_8} t_{\nu_1 \dots \nu_8} R_{\mu_1 \mu_2}^{\nu_1 \nu_2} \dots R_{\mu_7 \mu_8}^{\nu_7 \nu_8}, \quad (3)$$

and the tensor $t^{\mu_1 \dots \mu_8}$ ($\mu_r = 0, 1, \dots, 9$) will be defined later. These are the leading terms in the low energy limit of one-loop amplitudes [3] and the first non-leading corrections to the low-energy limit at tree level [4,5]. Furthermore, it was shown in [6] that terms of this form are induced in ten-dimensional type IIB superstring theory by integration over the fermionic zero modes in a D-instanton background. The arguments of this paper will

¹ In the following the ten-dimensional string metric will be denoted by lower-case g while the M-theory metric will be denoted by an upper-case G . In both cases our convention is that the metric is dimensionless.

strongly suggest that the exact form of the R^4 terms in the effective nine-dimensional action for M-theory compactified on \mathcal{T}^2 is

$$S_{R^4} = \frac{1}{3 \cdot (4\pi)^7 l_{11}} \int d^9 x \sqrt{-G^{(9)}} \left(\mathcal{V}_2^{-1/2} f(\Omega, \bar{\Omega}) + \frac{2\pi^2}{3} \mathcal{V}_2 \right) t_8 t_8 R^4, \quad (4)$$

where $G_{mn}^{(9)}$ is the nine-dimensional metric, Ω is the complex structure of \mathcal{T}^2 and $4\pi^2 l_{11}^2 \mathcal{V}_2$ is its volume. The modular function $f(\Omega, \bar{\Omega})$ is the same non-holomorphic Eisenstein series as the one that was conjectured in [6] to determine the R^4 term in ten-dimensional type IIB superstring theory, with Ω replaced by

$$\rho^B \equiv \rho_1^B + i\rho_2^B = C^{(0)} + ie^{-\phi^B}, \quad (5)$$

where $C^{(0)}$ is the $R \otimes R$ pseudoscalar and ϕ^B the IIB dilaton. In equation (4) the eleven-dimensional Planck length, l_{11} is related to κ_{11} by $\kappa_{11}^2 = (2\pi)^8 l_{11}^9 / 2$. We will later use the normalization of the ten-dimensional string theory coupling in which

$$\kappa_{10}^2 = \frac{\kappa_{11}^2}{2\pi R_{11} (\lambda^A)^2} = 2^6 \pi^7 \alpha'^4, \quad (6)$$

which makes the tension of the D-string equal to $e^{-\phi} \times$ (tension of the fundamental string) [7]. From now on we will set $\alpha' = 1$ to simplify the expressions.

The expression (4) will reinforce the conjecture in [6] that the ten-dimensional IIB theory satisfies a perturbative non-renormalization theorem — there are no contributions beyond one loop and the non-perturbative contributions are determined by multiply-charged D-instantons. The $Sl(2, \mathbf{Z})$ duality symmetry of the IIB theory will be related to the geometry of the torus as in [8,9]. All non-perturbative effects will be seen to disappear in the ten-dimensional type IIA theory, essentially because there are no finite-action instantons, and the R^4 term is then given entirely by the sum of the perturbative tree-level and one-loop term. The decompactified eleven-dimensional M-theory effective action ($\mathcal{V}_2 \rightarrow \infty$) has an R^4 term that comes entirely from the one-loop type IIA term with a coefficient that is fixed precisely by the one-loop string calculation. The complete effective action for M-theory could then be determined, in principle, by eleven-dimensional supersymmetry which should relate the new term to the Einstein–Hilbert term.

2. R^4 terms in type II superstring perturbation theory.

Consideration of the on-shell scattering of four gravitons in either type IIA or IIB string perturbation theory at tree level [5,2] and one loop [3] leads to terms in the low

energy Lagrangian of the form² $\sqrt{-g}t_8t_8R^4$ which are $O(\alpha'^{-1})$ whereas the leading term is the Einstein–Hilbert action given by

$$\frac{1}{2\kappa_{10}^2} \int d^{10}x e^{-2\phi} \sqrt{-g} R. \quad (7)$$

After compactification on an n -torus, \mathcal{T}^n , the sum of the tree-level and one-loop contributions to the four-graviton amplitude has the form [10]

$$A = K_{in} \frac{\kappa_{10}^2}{12 \cdot 2^8} \left[-\lambda^{-2} \mathcal{V}_n \frac{\Gamma(-s/4)\Gamma(-t/4)\Gamma(-u/4)}{\Gamma(1+s/4)\Gamma(1+t/4)\Gamma(1+u/4)} + \frac{\kappa_{10}^2}{2^5 \pi^6} d_1 \right] \quad (8)$$

where the coupling λ , is determined by the expectation value, $\langle\phi\rangle$, of the dilaton,

$$\lambda = e^{\langle\phi\rangle}, \quad (9)$$

and $\mathcal{V}_2 = \sqrt{-g^{(n)}}$ is the volume of the compactified space with metric $g_{ij}^{(n)}$. The relative normalization between the tree level and the one-loop term in (8) was determined by arguments based on unitarity in [11].

The kinematic factor in (8) is eighth order in the momenta and can be written as

$$K_{in} \sim \hat{t}_R^{\mu_1\mu_2\cdots\mu_8} \hat{t}_S^{\nu_1\nu_2\cdots\nu_8} \zeta_{\mu_1\nu_1}^{(1)} k_{\mu_2}^{(1)} k_{\nu_2}^{(1)} \cdots \zeta_{\mu_7\nu_7}^{(4)} k_{\mu_8}^{(4)} k_{\nu_8}^{(4)}, \quad (10)$$

where $k_\mu^{(r)}$ is the momentum of the graviton labelled r and $\zeta_{\mu_r\nu_r}^{(r)}$ is its polarization tensor. The constant κ_{10} depends on the definition of the dilaton. The eighth-rank tensors \hat{t}_R ($R = 1, 2$) are conveniently defined in a light-cone frame by

$$\begin{aligned} \epsilon_{a_1a_2\cdots a_8} \gamma_{a_1j_1}^{i_1j_1} \cdots \gamma_{a_7a_8}^{i_4j_4} &= \hat{t}_1^{i_1j_1\cdots i_4j_4} = t^{i_1j_1\cdots i_4j_4} + \frac{1}{2} \epsilon^{i_1j_1\cdots j_4j_4} \\ \epsilon_{\dot{a}_1\dot{a}_2\cdots \dot{a}_8} \gamma_{\dot{a}_1j_1}^{i_1j_1} \cdots \gamma_{\dot{a}_7\dot{a}_8}^{i_4j_4} &= \hat{t}_2^{i_1j_1\cdots i_4j_4} = t^{i_1j_1\cdots i_4j_4} - \frac{1}{2} \epsilon^{i_1j_1\cdots j_4j_4}, \end{aligned} \quad (11)$$

where a and \dot{a} are $SO(8)$ indices labelling the $\mathbf{8}_s$ and $\mathbf{8}_c$ representations and $i_r, j_r = 1, \dots, 8$ label the $\mathbf{8}_v$ representation. The vector indices are covariantized in the ten-dimensional expression, (10). In the type IIB theory the two \hat{t}_8 's are the same ($R = S$) in (10), leading to an irrelevant ambiguity in the sign of the $\epsilon_8\epsilon_8$ term, whereas they are different ($R \neq S$) in the IIA theory. The $\epsilon^{i_1j_1\cdots j_4j_4}$ terms are total derivatives and we will discard them in the following, in which case there is no perturbative distinction between the R^4 terms in the two theories. Since, at linearized level, $R_{\rho\sigma}^{\mu\nu} = \kappa_{10} k^{[\mu} \zeta^{\nu]} k_{[\rho} \zeta_{\sigma]}$ the kinematical factor is $K_{in} = t_8 t_8 R^4 / (24 \kappa_{10}^2)$.

² Expressions in which the superscript/subscript A or B is omitted applies to either type II theory

The coefficient, d_1 , of the one-loop term in (8) is given by the integral of a modular function over the fundamental domain of $Sl(2, \mathbf{Z})$. When the theory is compactified on an n -torus it takes the form,

$$d_1 = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{lat} F(\tau, \bar{\tau}) \quad (12)$$

where Z_{lat} is the partition function associated with the lattice, $\Gamma^{n,n}$,

$$Z_{lat} = \mathcal{V}_n \sum_{m,n \in \mathbf{Z}} e^{-\frac{\pi}{\tau_2} \sum_{i,j} (g+B)_{ij} (m_i + n_i \tau)(m_j + n_j \bar{\tau})} \quad (13)$$

and $i, j = 1, \dots, n$ label the directions in the lattice. This sum can be interpreted as the sum of contributions to the functional integral from fundamental string world-sheets in which the two world-sheet coordinates wind m_i and n_i times around the compact dimension. The dynamical factor in (12) is given by

$$F(\tau, \bar{\tau}) = \frac{1}{\tau_2^3} \int_{\mathcal{T}^2} \prod_{i=1}^3 d^2\nu_i \left[\frac{\chi_{12}\chi_{34}}{\chi_{13}\chi_{24}} \right]^{-s} \left[\frac{\chi_{14}\chi_{23}}{\chi_{13}\chi_{24}} \right]^{-t}, \quad (14)$$

where $\ln \chi_{ij}$ is the scalar Green function between the vertices labelled i and j on the toroidal world-sheet, \mathcal{T}^2 (and $\int_{\mathcal{T}^2} \prod d^2\nu_i = \tau_2^3$).

The leading low energy contributions obtained by expanding (8) at in powers of momenta, are the massless pole terms and the contact term that are associated with the (linearized) Einstein–Hilbert action. After subtracting these terms the remainder of (8) gives the R^4 terms which are obtained by setting the momenta to zero inside the square brackets. The tree contribution to the effective action is,

$$S_{R^4}^{tree} = \frac{\zeta(3)}{3 \cdot 2^7 \kappa_{10}^2} \mathcal{V}_n \int d^{10-n}x \sqrt{-g^{(10-n)}} (\rho_2)^2 t_8 t_8 R^4, \quad (15)$$

in a normalization consistent with (7). We have replaced λ by $\rho_2^{-1} = e^\phi$ in this expression as we will do in the following. The loop contribution (the toroidal world-sheet) in (8) depends on details of the the compactification, which will now be considered.

3. Compactification to nine dimensions on S^1 .

The one-loop term in the nine-dimensional theory is obtained by setting $g_{10\,10} = r^2$ and $B = 0$ in (13), where r is the circumference of the tenth dimension in sigma model (string frame) units so that $\mathcal{V}_1 = r$. After performing a Poisson resummation on one integer (12) gives

$$d_1 = r \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{(m,n) \in \mathbf{Z}^2} e^{-\pi r^2 |m\tau + n|^2 / \tau_2} F(\tau, \bar{\tau}). \quad (16)$$

Following a standard procedure [12,13] it is useful to separate the term with $(m, n) = (0, 0)$ and set $m = sp$ and $n = sq$ in the other terms where p and q are coprime integers and s is an unconstrained integer. The sum over p, q is a sum over fundamental domains of $Sl(2, \mathbf{Z})$ which is equivalent to extending \mathcal{F} to the semi-infinite strip, $0 \leq \tau_2 \leq \infty$, $-1/2 \leq \tau_1 \leq 1/2$, so that (16) can be expressed as [14],

$$d_1 = r \left[\int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} F(\tau) + \int_{Strip} \frac{d^2 \tau}{\tau_2^2} \sum_{s \in \mathbf{Z} \setminus \{0\}} e^{-\pi r^2 s^2 / \tau_2} F(\tau) \right]. \quad (17)$$

The integrals converge and using the low energy limit in (14), which sets $F = 1$, the result of the integrations is

$$d_1 = \frac{\pi}{3} \left[r + \frac{1}{r} \right]. \quad (18)$$

The total contribution of the R^4 terms to the effective action therefore has the form

$$\mathcal{S}_{R^4} = \frac{1}{3 \cdot 2^8 \kappa_{10}^2} \int d^9 x \sqrt{-g^{(9)}} t_8 t_8 R^4 r \left[2\zeta(3)(\rho_2)^2 + \frac{2\pi^2}{3} \left(1 + \frac{1}{r^2} \right) + \dots \right], \quad (19)$$

where \dots represents potential higher-order perturbative and non-perturbative terms.

The same expressions apply to the type IIA and the type IIB theories which are related by the T-duality transformations,

$$r_A = \frac{1}{r_B}, \quad r_A e^{-\phi_A} = e^{-\phi_B}, \quad C^{(1)} = C^{(0)}, \quad (20)$$

where the subscript A or B indicates which theory the relevant quantities are defined in and $C^{(1)} \equiv C_{10}^{(1)}$ is the component of the IIA $R \otimes R$ vector potential in the tenth direction. The value of $C^{(0)}$ in the IIB theory does not enter into the fundamental string amplitudes but it is related to the component of the $R \otimes R$ vector of IIA via (20) and hence to the complex structure of the torus in the compactification of M-theory to nine dimensions on \mathcal{T}^2 to be described later. The $Sl(2, \mathbf{Z})$ symmetry of type IIB implies that the effective action is invariant under integer shifts of $C^{(0)}$ which implies that the IIA action must be invariant under $C^{(1)} \rightarrow C^{(1)} + 1$. The complex scalar,

$$\rho^A \equiv \rho_1^A + i\rho_2^A = C^{(1)} + i r_A e^{-\phi_A}, \quad (21)$$

is equated with ρ^B by the T-duality transformation,

$$\rho^A = \rho^B. \quad (22)$$

We must also consider non-perturbative contributions to the $t_8 t_8 R^4$ term due to the effects of D-instantons. The type II theories have a total of 32 components in their supercharges. Since a D-instanton breaks half of the supersymmetries, there are at least sixteen fermionic zero modes in the fluctuations around instanton configurations [14,15]. However, the $t_8 t_8 R^4$ term arises only from the sector with sixteen fermionic zero modes [6]. Therefore, we only need to consider configurations with single D-instantons carrying multiple charges. This will turn out to be consistent with the various duality symmetries in the problem and with the coefficients of the perturbative terms in (19). From the point of view of the IIA theory in nine dimensions the only instantons are configurations in which the Euclidean world-line of a ten-dimensional $D0$ -brane winds around the tenth dimension [6]. Since the $D0$ -branes are Kaluza–Klein modes of M-theory there must be a single normalizable $D0$ -brane state with charge n and mass proportional to n (this is the basis of the as yet unproven conjecture that there is precisely one threshold bound state of n minimally charged $D0$ -branes [16]). In the Euclidean compactification to nine dimensions the world-line of such a particle can wind m times so that its action is $2\pi m n \rho^A$. The consequent non-perturbative terms in the effective action have the form [6],

$$\sum_{m,n>0} c_{mn}^A(\rho_2^A, r_A) \left(e^{2\pi i m n \rho^A} + e^{-2\pi i m n \bar{\rho}^A} \right), \quad (23)$$

which is consistent with the shift symmetry of $C^{(1)}$ and the coefficients c_{mn}^A are to be determined. These coefficients can be determined directly by evaluating the functional integral for a supersymmetric D-particle world-line that is wrapped around the compact Euclidean direction. Further details will be given in [17] but here we will determine the coefficients c_{mn}^A by a duality argument.

T-duality equates the series (23) with the series of D-instanton contributions to the type IIB theory,

$$\sum_{m,n>0} c_{mn}^B(\rho_2^B, r_B) \left(e^{2\pi i m n \rho^B} + e^{-2\pi i m n \bar{\rho}^B} \right), \quad (24)$$

which was discussed in [6] in the $r_B \rightarrow \infty$ limit. In that limit the complete R^4 term of the ten-dimensional IIB effective action has the form,

$$S_{R^4} = \frac{1}{3 \cdot 2^8 \kappa_{10}^2} \int d^{10}x \sqrt{-g^B} (\rho_2^B)^{1/2} f(\rho^B, \bar{\rho}^B) t_8 t_8 R^4, \quad (25)$$

where the function $f(\rho^B, \bar{\rho}^B)$ must be modular invariant since the group $Sl(2, \mathbf{Z})$ is a duality symmetry of type IIB superstrings in the Einstein frame [18] under which

$$\rho^B \rightarrow \frac{a\rho^B + b}{c\rho^B + d}, \quad (26)$$

where $ad - bc = 1$ and the coefficients are integers (and R is inert). We are here using the fact that

$$\sqrt{-g_E^B} t_8 t_8 R^4 = \sqrt{-g^B} (\rho_2^B)^{1/2} t_8 t_8 R^4, \quad (27)$$

where g_E^B is the Einstein-frame metric.

A conjecture was made in [6] that $f(\rho^B, \bar{\rho}^B) = \zeta(3) E_{\frac{3}{2}}(\rho^B)$, where ζ is the Riemann zeta function and $E_s(\rho)$ is a non-holomorphic Eisenstein series (or Maass waveform) defined by ([19], [20])

$$E_s(\rho) = \sum_{\gamma \in \Gamma/\Gamma_\infty} [\text{Im}(\gamma\rho)]^s, \quad (28)$$

where γ indicates a transformation in $\Gamma = Sl(2, \mathbf{Z})$ modded out by the subgroup defined by $\Gamma_\infty = \begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix}$. Such Eisenstein series are eigenfunctions of the Laplace operator on the fundamental domain of $Sl(2, \mathbf{Z})$,

$$\Delta E_s(\rho) \equiv \rho_2^2 \left(\frac{\partial^2}{\partial \rho_1^2} + \frac{\partial^2}{\partial \rho_2^2} \right) E_s(\rho) = s(s-1) E_s(\rho). \quad (29)$$

The function f can be expressed in various ways as

$$\begin{aligned} f(\rho^B, \bar{\rho}^B) &= \zeta(3) \sum_{\gamma \in \Gamma/\Gamma_\infty} [\text{Im}(\gamma\rho^B)]^{3/2} = \sum_{(p,n) \neq (0,0)} \frac{(\rho_2^B)^{3/2}}{|p + n\rho^B|^3} \\ &= 2\zeta(3)(\rho_2^B)^{3/2} + \frac{2\pi^2}{3} (\rho_2^B)^{-1/2} + 8\pi(\rho_2^B)^{1/2} \sum_{m \neq 0, n \geq 1} e^{2i\pi n m \rho_1} \left| \frac{m}{n} \right| K_1(2\pi |m| n \rho_2^B) \\ &= 2\zeta(3)(\rho_2^B)^{3/2} + \frac{2\pi^2}{3} (\rho_2^B)^{-1/2} \\ &\quad + 4\pi \sum_{m,n \geq 1} \left(\frac{m}{n^3} \right)^{1/2} \left(e^{2i\pi m n \rho^B} + e^{-2i\pi m n \bar{\rho}^B} \right) \left(1 + \sum_{k=1}^{\infty} (4\pi m n \rho_2^B)^{-k} \frac{\Gamma(k-1/2)}{\Gamma(-k-1/2)k!} \right), \end{aligned} \quad (30)$$

where K_1 is a Bessel function. Intriguingly, the first expression on the right-hand side of (30) has the form of the tree-level term of [4,5] summed over all its $Sl(2, \mathbf{Z})$ images – in other words, $(\rho_2^B)^{\frac{3}{2}} \sum_{p,q} T_{pq}^{-3}$, where p, q are coprime and T_{pq} is the tension in the (p, q) dyonic string. The first two terms in the last expression for f in (30) should be compared with the ten-dimensional perturbative tree-level term and one-loop terms in (8). The conjectured agreement requires the precise relative normalization of the tree-level and one-loop terms [11]. The remaining infinite series represents the sum over a dilute gas of multiply-charged D-instantons and anti D-instantons that converges for large ρ_2^B (small coupling). Various motivations for this expression were described in [6].

We will here generalize this description and obtain more insight by making use of the fact that in the nine-dimensional theory the $Sl(2, \mathbf{Z})$ symmetry of the IIB string theory

can be interpreted as a geometric symmetry of M-theory compactified on a torus [8,9]. To see this it is necessary to translate the coordinates to the M-theory frame – the frame in which eleven-dimensional supergravity is naturally formulated. Following [2] the eleven-dimensional metric may be parameterized by

$$ds^2 = G_{mn}^{(10)} dx^m dx^n + R_{11}^2 (dx^{11} - C_m^{(1)} dx^m)^2, \quad (31)$$

where the ten-dimensional part of the metric is $G_{mn}^{(10)} = R_{11}^{-1} g_{mn}^A$ (recalling that g_{mn}^A is the ten-dimensional IIA metric in the string frame). Compactifying this on a circle of radius R_{10} leads to the equivalences,

$$\begin{aligned} g_{10\,10}^A = r_A^2 = R_{10}^2 R_{11} = G_{10\,10} R_{11}, \quad \rho_2^A = R_{11}^{-3/2} \\ \rho_2^B = \frac{R_{10}}{R_{11}}, \quad r_B = \frac{1}{R_{10} \sqrt{R_{11}}}. \end{aligned} \quad (32)$$

Using a block diagonal ansatz for the eleven-dimensional metric it can be written so that

$$\sqrt{-G^{(11)}} = \sqrt{G^T} \sqrt{-G^{(9)}} = l_{11}^{-2} R_{10} R_{11} \sqrt{-G^{(9)}} = \mathcal{V}_2 \sqrt{-G^{(9)}}, \quad (33)$$

where $4\pi^2 l_{11}^2 \mathcal{V}_2 = 4\pi^2 R_{10} R_{11}$ is the volume of \mathcal{T}^2 . The metric on the two-torus,

$$G^T = \frac{1}{l_{11}^2} \begin{pmatrix} R_{10}^2 + R_{11}^2 (C^{(1)})^2 & -R_{11}^2 C^{(1)} \\ -R_{11}^2 C^{(1)} & R_{11}^2 \end{pmatrix} \quad (34)$$

can be expressed in terms of string frame quantities so its complex structure is given by

$$\begin{aligned} \Omega = \Omega_1 + i\Omega_2 = C^{(1)} + i \frac{R_{10}}{R_{11}} \\ = C^{(1)} + i r_A e^{-\phi^A} = \rho_A \\ = C^{(0)} + i e^{-\phi^B} = \rho_B. \end{aligned} \quad (35)$$

Equation (19) can now be rewritten in coordinates appropriate for the type IIA, IIB and M-theory as,³

$$\begin{aligned} \mathcal{S}_{R^4} &= \frac{1}{3 \cdot 2^8 \kappa_{10}^2} \int d^9 x \sqrt{-g^{A(9)}} t_8 t_8 R^4 r_A \left[2\zeta(3) (\rho_2^A)^2 + \frac{2\pi^2}{3} \left(1 + \frac{1}{r_A^2}\right) + \dots \right] \\ &= \frac{1}{3 \cdot 2^8 \kappa_{10}^2} \int d^9 x \sqrt{-g^{B(9)}} t_8 t_8 R^4 r_B \left[2\zeta(3) (\rho_2^B)^2 + \frac{2\pi^2}{3} \left(1 + \frac{1}{r_B^2}\right) + \dots \right] \\ &= \frac{l_{11}^6}{3 \cdot 2^8 \kappa_{11}^2} \int d^9 x \sqrt{-G^{(9)}} t_8 t_8 R^4 2\pi R_{11} R_{10} \left[2\zeta(3) \frac{l_{11}^3}{R_{11}^3} + \frac{2\pi^2}{3} + \frac{2\pi^2}{3} \frac{l_{11}^3}{R_{10}^2 R_{11}} + \dots \right] \end{aligned} \quad (36)$$

³ Recall that we are setting $\alpha' = 1$ in (6).

where $g^{A(9)}$, $g^{B(9)}$ are the nine-dimensional metrics in the IIA and IIB theories and the non-perturbative terms, represented by \dots , are given by a power series in $e^{2\pi i\rho^A}$, $e^{2\pi i\rho^B}$ and $e^{2\pi i\Omega}$, respectively.

Since we know that the last expression in (36) must be invariant under the action of $Sl(2, \mathbf{Z})$ on Ω it is appealing to write it as an expansion for large Ω_2 ,

$$\mathcal{S}_{R^4} = \frac{1}{3 \cdot (4\pi)^7 l_{11}} \int d^9 x \sqrt{-G^{(9)}} t_8 t_8 R^4 \left\{ \mathcal{V}_2^{-1/2} \left[2\zeta(3)(\Omega_2)^{3/2} + \frac{2\pi^2}{3}(\Omega_2)^{-1/2} + \dots \right] + \frac{2\pi^2}{3} \mathcal{V}_2 \right\}, \quad (37)$$

which should be identified with the expansion of a modular function of Ω . We now compare this with the expansion of the modular function in (4) where the function f is defined in (30). We see that the perturbative terms in (37) are identical to those in (4). Correspondingly, non-perturbative extensions of the effective R^4 actions in the IIA and IIB theories follow by substituting the appropriate variables in (4). Several striking features are apparent from the structure of (4):

- There are only two perturbative terms in the expansion of f , corresponding to the tree and the one-loop terms in the fundamental string calculations. This points to a perturbative non-renormalization theorem beyond one loop in type II string theory. It would be gratifying to demonstrate this explicitly from the expressions for superstring perturbation theory at higher genus but this seems to be difficult.⁴ Such a non-renormalization theorem can be motivated heuristically as follows. The four gravitons attached to a torus are just sufficient to soak up the sixteen zero modes of the space-time fermions. At higher genus there could easily be some extra fermionic zero modes leading to a vanishing result for the effective $t_8 t_8 R^4$ term in the low energy ($\alpha' \rightarrow 0$) limit. The complication is that, at least in the light-cone formalism, there are vertex insertions that might soak these surplus zero modes up.

- In the limit $r_B \rightarrow \infty$ (4) reduces to the sum over non-perturbative terms (in the string frame) conjectured in [6] based on the properties of D-instantons in the ten-dimensional type IIB theory.

- In the limit $r_A \rightarrow \infty$ (4) reduces to the first two terms in square parentheses in the first expression in (36). All the non-perturbative contributions vanish and so the full expression for the ten-dimensional type IIA theory has just the perturbative tree-level and one-loop terms.

- Upon decompactifying the M-theory torus, taking $\mathcal{V}_2 \rightarrow \infty$ in (4), only the term proportional to \mathcal{V}_2 contributes – the constant term in the square parentheses in (36). The

⁴ We are grateful to Nathan Berkovitz for correspondence on this issue.

result is that the R^4 term in the M-theory effective action is determined precisely by the coefficient of the one-loop diagram in the type IIA superstring theory and is given by

$$S_{R^4} = \frac{1}{18 \cdot (4\pi)^7 l_{11}^3} \int d^{11}x \sqrt{-G^{(11)}} t_8 t_8 R^4 \quad (38)$$

in a normalization in which the Einstein–Hilbert action is given by (1).

The range of the indices in t_8 is here extended trivially in the eleventh dimension so that $t_8 \equiv t^{\mu_1 \dots \mu_8}$ with $\mu_r = 0, \dots, 10$.

Since (4) has a finite M-theory limit as $\mathcal{V}_2 \rightarrow \infty$ and reduces to the correct tree and one-loop terms for the IIA and IIB theories as r_A or $r_B \rightarrow \infty$ and also satisfies the correct T-duality relation between the type IIA and type IIB theories, it is a good candidate for the exact R^4 term. We should, however, consider to what extent these conditions determine the solution uniquely.

In principle, we could add a function $h(\mathcal{V}_2; \Omega, \bar{\Omega})$ to the terms in parentheses in (4), which must be a modular function of Ω and must not spoil the above properties. The existence of the M-theory limit means that $h \sim (\mathcal{V}_2)^\alpha k(\Omega, \bar{\Omega})$ (where k is a modular function) as $\mathcal{V}_2 \rightarrow \infty$ with $\alpha < 1$. However, if this limit can be interchanged with the perturbative type IIA limit then $h \sim (\mathcal{V}_2)^\alpha \Omega_2^\beta \sim (r_A)^{\alpha+\beta} (\rho_2^A)^{\beta-\alpha/3}$ as $\rho_2^A \rightarrow \infty$. Taking into account a power of $(R_{11})^{-1/2}$ from the measure in (4) to go to the string frame, the net power of ρ_2^A is $X = \beta - (\alpha - 1)/3$. However, this only contributes in ten dimensions if $\alpha + \beta = 1$, so that $X > 0$ which spoils the known perturbative behaviour. This excludes the presence of terms beyond the tree and one-loop terms in the ten-dimensional IIA theory subject to the (very strong) assumption of the uniformity of the M-theory and perturbative limits.

More generally, the function h may vanish in the M-theory limit ($\alpha < 1$) and have mild enough perturbative behaviour not to spoil the tree or one-loop terms. In that case it is an L^2 function on the fundamental domain of $Sl(2, \mathbf{Z})$ and, following [20], it can be written as a sum of cusp forms and a continuous integral over E_s . Superficially, the constraints imposed by T-duality and by the consistency of the various limits do not exclude the addition of h to (4). They would be eliminated if we had a reason to require f to satisfy the eigenvalue equation, (29), which has f as its unique solution for a given eigenvalue (as follows from theorem 1 section 3.5 of [20]). Although this equation has not been motivated by a direct argument, it has the flavour of a condition that might follow by requiring supersymmetry of the effective action.

Having described the nine-dimensional theory in detail it is of interest to understand the extended U dualities of theories obtained by compactification to lower dimensions. These provide further consistency checks on the validity of the nine-dimensional expression.

4. M-theory on \mathcal{T}^3 or IIB on \mathcal{T}^2 .

Upon compactifying to eight dimensions there is a richer spectrum of instantons. In addition to the direct reduction of the instantons from nine dimensions there are those that arise from the M-theory/IIA point of view from the wrapping of the Euclidean three-volume of the $M2$ -brane and from the IIB side from the wrapping of the $Sl(2, \mathbf{Z})$ multiplet of Euclidean world-sheets of the fundamental and D-strings. The structure of the duality group, $Sl(3, \mathbf{Z}) \times Sl(2, \mathbf{Z})$, is correspondingly richer in eight dimensions.

We will concentrate on the compactification of the IIB theory on \mathcal{T}^2 but to begin with we will briefly consider the point of view of Euclidean M-theory on \mathcal{T}^3 . The modular group of the torus is $Sl(3, \mathbf{Z})$. There are seven scalar fields that arise from the six moduli of \mathcal{T}^3 and the value of C_{ijk} , the component of the three-form potential in the toroidal directions. Instantons arise from two sources. On the one hand there are three integers associated with the Kaluza–Klein modes. On the other hand the world-volume of the membrane can wrap on the torus with winding numbers associated with the three directions. If the compactification is viewed in two stages it is related to the \mathcal{T}^2 compactification of type IIA theory. In the first stage consider a single compact dimension which gives the Kaluza–Klein modes that are $D0$ -branes of the type IIA theory. In addition the wrapped membrane gives fundamental IIA strings – with tensions that are multiples of the fundamental tension according to the wrapping number. In the second stage the Euclidean IIA theory is compactified on \mathcal{T}^2 . The world-line of a charge- n $D0$ -brane can wind arbitrarily around either cycle, giving two further integers. The world-volume of the fundamental string can also wrap on the torus, giving two further integers in addition to the windings of the membrane around the eleventh dimension.⁵ In the limit in which one direction decompactifies only those configurations with zero winding number in that direction survive and the nine-dimensional result should be recovered.

Now consider the point of view of Euclidean IIB on \mathcal{T}^2 . We shall denote the complex structure of the two-torus by

$$U \equiv U_1 + iU_2 = \frac{1}{g_{11}^B} (g_{12}^B + i\sqrt{-g^B}), \quad (39)$$

and the Kähler structure by

$$T \equiv T_1 + iT_2 = B + i\sqrt{-g^B}, \quad (40)$$

where B is the component of the $NS \otimes NS$ two-form with indices in the directions of the torus and g_{ij}^B ($i = 10, 11$) are components of the string metric in these two directions. The

⁵ A very recent preprint [21] considers Matrix theory on \mathcal{T}^3 which involves some related issues.

determinant of this sub-metric is $g^B = g_{10\ 10}^B g_{11\ 11}^B - (g_{10\ 11}^B)^2$. The seven scalar fields are the six real and imaginary parts of ρ , T and U , together with the components of the $R \otimes R$ two-form in the directions of the torus, $C^{(2)}$. In the IIB language the factor of $Sl(2, \mathbf{Z})$ in the duality group $Sl(3, \mathbf{Z}) \times Sl(2, \mathbf{Z})$ acts on U and none of the other fields. We shall refer to this as the group $Sl_U(2, \mathbf{Z})$. The $Sl(2, \mathbf{Z})$ groups associated with the fields ρ and T ($Sl_\rho(2, \mathbf{Z})$, $Sl_T(2, \mathbf{Z})$) are non-commuting subgroups in $Sl(3, \mathbf{Z})$. The Z_2 transformation $\rho \rightarrow -1/\rho$ in $Sl_\rho(2, \mathbf{Z})$ induces the transformations $B \rightarrow C^{(2)}$ and $\phi \rightarrow -\phi$ (in the Einstein frame) so that its action on T is

$$T \rightarrow \tilde{T}, \quad \tilde{T} \rightarrow T, \quad U \rightarrow U \quad (41)$$

where,

$$\tilde{T} \equiv \tilde{T}_1 + i\tilde{T}_2 = C^{(2)} + ie^{-\phi^B} \sqrt{-g^B} \quad (42)$$

is the Kähler structure of the D-string torus. The Z_2 transformation in $Sl_T(2, \mathbf{Z})$, $T \rightarrow -1/T$, induces

$$\tilde{T} \rightarrow \rho_2, \quad \rho_2 \rightarrow \tilde{T}, \quad U \rightarrow U. \quad (43)$$

The complete duality-invariant eight-dimensional expression will be considered in [17]. Here, we will indicate how those terms that reduce to (25) in the decompactification limit to the ten-dimensional IIB theory may be calculated. The dualities of the IIB theory may be used to map the expression for the one-loop four-graviton amplitude in fundamental string theory into the non-perturbative expression for the R^4 term, as follows. The one-loop amplitude with four external gravitons can be calculated in eight dimensions in a manner very similar to that in [10,13] using (12) and (13). It has the form (in the string frame)

$$\frac{1}{\kappa_{10}^2} \frac{\pi}{3 \cdot 2^8} \int d^8 x \sqrt{-g^{(8)}} t_8 t_8 R^4 (L(T, \bar{T}) + L(U, \bar{U})), \quad (44)$$

where

$$L(T, \bar{T}) = \ln(T_2 |\eta(T)|^4), \quad (45)$$

and $\eta(T)$ is the Dedekind function. This can be expanded into an infinite series of terms,

$$L(T, \bar{T}) \equiv \sum_{m,n=0}^{\infty} L_{mn} = \ln(T_2) + \frac{\pi}{3} T_2 + \sum_{m,n>0} \frac{1}{n} \left(e^{-2i\pi mn \bar{T}} + e^{2i\pi mn T} \right), \quad (46)$$

with a similar expansion for the function of U .

In writing (44) we have subtracted the logarithmic divergence that arises in the low energy eight-dimensional theory ($N = 8$ supergravity in eight dimensions) by imposing the requirement that it be invariant under $Sl_T(2, \mathbf{Z}) \otimes Sl_U(2, \mathbf{Z})$ transformations. The divergent piece is proportional to the β function for the R^4 interaction. This is the same

procedure as the one in [13]. The presence of the factor $\ln T_2$ reflects this logarithmic divergence. The expression (44) also contains a term proportional to T_2 , which is necessary for it to decompactify to the correct ten-dimensional expression. This term arises from string world-sheets that do not wrap around either cycle of the torus. In fact, knowledge of the logarithmic term and the linear term, together with the $Sl(2, \mathbf{Z})$ symmetry and the T-duality symmetry which takes either of the type II theories into itself in eight dimensions, is sufficient to determine (44) completely. The double sum over m and n comes from configurations in the functional integral in which the fundamental string world-sheet winds around each of the two cycles of the torus a non-zero number of times. Upon decompactification ($T_2 \rightarrow \infty$) only the linear T_2 term contributes to the ten-dimensional action. Similarly, the U term in (44) can be interpreted in terms of degenerate wrappings of world-sheets in which the world-sheet coordinates wind around a single cycle of the torus.

This interpretation of the terms in this series can be verified explicitly by evaluation of the path integral for a string wrapped around \mathcal{T}^2 starting with Nambu action and using the ‘Schild gauge’ as in [22]. The Schild gauge is one in which the action is the square of the Nambu action which is invariant under symplectic diffeomorphisms. This will be described in detail in [17].

A Z_2 S-duality transformation, $\rho \rightarrow -1/\rho$, (41) converts the fundamental string (the F-string) to a D-string. This converts the terms in (46) with non-zero windings of the fundamental string, $m, n \neq 0$, into corresponding terms for the wrapped D-string. These can again be obtained directly by functional integration over the wrapped D-string world-sheet, starting now with the Euclidean Dirac–Born–Infeld action,

$$\mathcal{L}_{DBI} = \frac{1}{2\pi} \int d^2\xi \left(n e^{-\phi^B} \sqrt{-\det(G + \mathcal{F})} + \frac{i}{2} n C^{(2)} + 2in C^{(0)} \wedge \mathcal{F} \right) \quad (47)$$

where $\mathcal{F} = F - B$ and $F = dA$ is the field strength of the world-volume vector potential, A . This contains the world-sheets for general D-strings with charges (p, n) . For the present argument it is sufficient to keep only the terms with $(0, n)$, which are T-dual to D-instantons. The functional integral now includes integration over \mathcal{F} which gives rise to a nontrivial factor in the measure. The result of the functional integral is that the non-perturbative terms in (46) with $m, n \neq 0$ are replaced by [17]

$$\tilde{L}_{mn} = e^{2\pi i m n C^{(2)}} \left| \frac{m}{n} \right| \frac{e^{-\phi^B}}{\sqrt{e^{-2\phi^B} + (C^{(0)})^2}} K_1 \left(2\pi |m| n T_2 \sqrt{e^{-2\phi^B} + (C^{(0)})^2} \right). \quad (48)$$

Making the T-duality transformation, (43), replaces $e^{-\phi^B}$ by $\sqrt{-g^B} e^{-\phi^B}$ in (48) and the wrapped D-strings by D-instantons. Taking the limit $\sqrt{-g^B} \rightarrow \infty$ decompactifies the dual torus to give

$$\hat{L}_{mn}(\rho^B, \bar{\rho}^B) = (\rho_2^B)^{1/2} e^{2\pi i m n \rho_1^B} \left| \frac{m}{n} \right| K_1(2\pi |m| n \rho_2), \quad (49)$$

which agrees with the conjectured ten-dimensional result in (30) up to an overall constant whose value depends on a detailed calculation of the measure.

5. Comments concerning supersymmetry.

We have presented some evidence that the scalar function of the moduli fields multiplying the R^4 terms of M-theory on a torus is determined by perturbative and non-perturbative duality symmetries. The expression in (4) reproduces the precise coefficients of the tree-level and one-loop perturbation theory results in nine-dimensional IIA and IIB superstring theories, together with the infinite series of instanton terms that are associated with the expected $Sl(2, \mathbf{Z})$ symmetry. Although we do not have a proof that these conditions uniquely determine the function for arbitrary moduli, the decompactification limits to both of the type II ten-dimensional string theories, as well as to eleven-dimensional M-theory, are uniquely determined if these limits are assumed to be uniform. The ten-dimensional IIB limit coincides with the D-instanton sum conjectured in [6]. The scalar function in (4) is an eigenfunction of the Laplace equation on the fundamental domain of $Sl(2, \mathbf{Z})$, (29), so that it is uniquely determined by the tree and one-loop terms in its expansion. Proving that this conjectured function is indeed correct therefore amounts to understanding why it should be an eigenfunction of the Laplace equation — a condition that should follow from the constraints of supersymmetry. The fact that the coefficient of the $t_8 t_8 R^4$ term should be determined by supersymmetry even though it is a non-holomorphic function of the moduli and contributes to two terms in the perturbation expansion is rather unusual.

One consequence of this structure is that there should be a non-renormalization theorem in either of the type II string theories that prevents perturbative contributions to the R^4 terms beyond one loop and prevents non-perturbative contributions to the IIA theory in ten dimensions. Furthermore, the coefficient of the M-theory R^4 term in eleven dimensions is determined by the coefficient of the one-loop term in either of the ten-dimensional type II theories, as in (38). Gratifyingly, this eleven-dimensional expression has an independent motivation based on supersymmetry in ten dimensions. This can be seen to follow from its relation to the term in the M-theory effective action that is an eleven-form $C^{(3)} \wedge X_8$ [23] which is known to be present from a variety of arguments, such as anomaly cancellation [24]. The expression X_8 is the eight-form in the curvatures that is inherited from the term in type IIA superstring theory [25] which is given by

$$-\int \frac{d^{10}x}{(2\pi)^5} B \wedge X_8 = -\frac{1}{2} \int \frac{d^{10}x}{(2\pi)^5} \sqrt{-g^{A(10)}} \epsilon_{10} B X_8. \quad (50)$$

where,⁶

$$X_8 = \frac{1}{192} \left(\text{tr} R^4 - \frac{1}{4} (\text{tr} R^2)^2 \right). \quad (51)$$

⁶ Here, R^n is the outer product of n Riemann curvatures where R is viewed as a two-form matrix in the 10×10 -dimensional representation of $SO(9, 1)$.

This is consistent with the antisymmetric tensor gauge symmetry, as can be seen by the replacement $B \rightarrow B + d\Lambda^{(1)}$ and an integration by parts.

To see how this term is related to the $t_8 t_8 R^4$ term discussed in this paper recall first that in the case of the heterotic and type I superstrings, ten-dimensional $N = 1$ supersymmetry provides powerful constraints on terms that are related to parity-violating anomaly-cancelling terms [26]. An example of the power of these constraints is the explicit determination of highly nontrivial non-perturbative relationships between heterotic $SO(32)$ and type I theories in eight dimensions [14]. These strong constraints follow from the structure of the two independent ten-dimensional $N = 1$ super-invariants which contain an odd-parity term [26,27,28] ,

$$I_3 = t_8 \text{tr} R^4 - \frac{1}{4} \epsilon_{10} B \text{tr} R^4 \quad (52)$$

and

$$I_4 = t_8 (\text{tr} R^2)^2 - \frac{1}{4} \epsilon_{10} B (\text{tr} R^2)^2 \quad (53)$$

(where the notation is that of [28]). Using the fact that $t_8 t_8 R^4 = 24 t_8 \text{tr}(R^4) - 6 t_8 (\text{tr} R^2)^2$, it follows that the particular linear combination,

$$I_3 - \frac{1}{4} I_4 = \frac{1}{24} t_8 t_8 R^4 - 48 \epsilon_{10} B X_8 \quad (54)$$

contains both the ten-form $B \wedge X_8$ and $t_8 t_8 R^4$. Therefore, the part of the effective action that contains (50) must be

$$S'_{R^4} = \frac{1}{\kappa_{10}^2} \frac{\pi^2}{48} \int d^{10}x \sqrt{-g^{A(10)}} (I_3 - \frac{1}{4} I_4) \quad (55)$$

so that S'_{R^4} contains precisely the torus contribution to the $t_8 t_8 R^4$ term in (36). In other words, the torus contribution to the $t_8 t_8 R^4$ term in the IIA theory combines with the ten-form in the linear combination (54) which are bosonic terms in an $N = 2$ super-invariant. At strong coupling this lifts to a super-invariant of the eleven-dimensional theory in which the $t_8 t_8 R^4$ terms have the same coefficient as in (44).

At strong coupling this lifts to the $t_8 t_8 R^4$ terms in eleven dimensions which have the same coefficient as in (44). The full supersymmetric effective action must therefore contain these terms (along with others that we have not considered here).

The other $t_8 t_8 R^4$ terms in (36) depend non-trivially on the dilaton, ρ_2^A , whereas no dilaton dependence can be introduced into the $B \wedge X_8$ term without spoiling the antisymmetric tensor gauge symmetry. For this reason neither the tree-level $t_8 t_8 R^4$ term in the effective ten-dimensional IIA action, nor the instanton corrections in lower dimensions, can be related to the ten-form in an obvious manner. However, once terms beyond the

lowest-order terms in the effective action are known supersymmetry probably determines the whole action.

Our interest in this subject is linked to the related question of whether the F^4 terms that enter into the description of $D0$ -brane scattering [29,30,31] are renormalized. This question has a direct connection with the issues discussed in this paper due to another set of duality relations.⁷ Firstly, upon compactification on a circle to nine dimensions the process in which two $D0$ -branes scatter is mapped by T-duality into the scattering of two D-strings with unit winding number around the compact direction in the Einstein frame. This process is related in turn by S-duality for the type IIB theory into the scattering of two fundamental strings (‘F-strings’) that are also wound around the circle and are BPS states carrying no momentum in the tenth dimension. This scattering amplitude can be calculated directly to any order in string perturbation theory. In order to make this sequence of statements precise there are delicate questions concerning the mapping between the configuration appropriate for the scattering of $D0$ -branes at a fixed impact parameter and the scattering of F-string states at fixed momentum transfer.

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⁷ This argument was formulated in collaboration with Constantin Bachas and will be presented in detail elsewhere.

References

- [1] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in eleven dimensions*, Phys. Lett. **76B** (1978) 409.
- [2] E. Witten, *String theory in various dimensions*, hep-th/9503124, Nucl. Phys. **B443** (1995) 85.
- [3] M.B. Green and J.H. Schwarz, *Supersymmetrical string theories*, Phys. Lett. **109B** (1982) 444.
- [4] D.J. Gross and E. Witten, *Superstring modifications of Einstein's equations*, Nucl. Phys. **B277** (1986) 1.
- [5] M.T. Grisaru, A.E.M. Van de Ven and D. Zanon, *Two-dimensional supersymmetric sigma models on Ricci flat Kahler manifolds are not finite*, Nucl. Phys. **B277** (1986) 388 ; *Four loop divergences for the $N=1$ supersymmetric nonlinear sigma model in two-dimensions*, Nucl. Phys. **B277** (1986) 409.
- [6] M.B. Green and M. Gutperle, *Effects of D-instantons*, hep-th/9701093.
- [7] J. Polchinski, *TASI Lectures on D-branes*, hep-th/9611050.
- [8] P. Aspinwall, *Some Relationships Between Dualities in String Theory* Trieste 1995, Proceedings, S-duality and mirror symmetry, Nucl. Phys. Proc. **46** (1996) 30, hep-th/9508154
- [9] J.H. Schwarz, *An $Sl(2, \mathbf{Z})$ multiplet of type IIB superstrings*, hep-th/9508143 Phys. Lett. **360B** (1995) 13.
- [10] M. Green, J.H. Schwarz and L. Brink, *$N=4$ Yang-Mills and $N=8$ supergravity as limits of string theories*, Nucl. Phys. **B198** (1982) 474.
- [11] N. Sakai and Y. Tanii, *One-loop amplitudes and effective action in superstring theories*, Nucl. Phys. **B287** (1987) 457.
- [12] B. McClain and B.D.B. Roth, *Modular invariance for interacting bosonic strings at finite temperature*, Comm. Math. Phys. **111** (1987) 539.
- [13] L. Dixon, V. Kaplunovsky and J. Louis, *Moduli dependence of String loop corrections to gauge coupling constants*, Nucl. Phys. **B355** (1991) 649
- [14] C. Bachas, C. Fabre, E. Kiritsis, N. Obers, P. Vanhove, *Threshold effects and D-instantons* to be published.
- [15] M.B. Green and M. Gutperle, *Configurations of two D-instantons*, hep-th/9612127.
- [16] E. Witten, *Bound States Of Strings And p-Branes*, hep-th/9510135, Nucl. Phys. **B460** (1996) 335.
- [17] M.B. Green and P. Vanhove, in preparation.
- [18] C.M. Hull and P.K. Townsend, *Unity of Superstring Dualities*, hep-th/9410167, Nucl. Phys. **B438** (1995) 109.
- [19] D. Zagier, *Eisenstein series and the Riemann zeta function*, in Tata Inst. Fund. Res. Studies in Math., 10, Tata Inst. Fundamental Res., Bombay, 1981.

- [20] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications*, vol. I, Springer-Verlag 1985.
- [21] D. Berenstein, R. Corrado and J. Distler, *On the moduli spaces of M(atric)-theory compactifications*, hep-th/9704087.
- [22] M. Lüscher, K. Symanzik and P. Weisz, *Anomalies of the free loop wave equation in the WKB approximation*, Nucl. Phys. **B173** (1980) 365.
- [23] M.J. Duff, J.T. Liu and R. Minasian, *Eleven-dimensional origin of string-string duality: a one loop test*, hep-th/9506126, Nucl. Phys. **B452** (1995) 261.
- [24] P. Horava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, Nucl. Phys. **B475** (1996) 94, hep-th/9603142.
- [25] C. Vafa and E. Witten, *A one loop test of string duality*, hep-th/9505053, Nucl. Phys. **B447** (1995) 261.
- [26] W. Lerche, *Elliptic index and superstring effective actions*, Nucl. Phys. **B308** (1988) 102.
- [27] M. de Roo, H. Suelmann and A. Wiedeman, *Supersymmetric R^4 actions in ten-dimensions*, Phys. Lett. **B280** (1992) 39.
- [28] A.A. Tseytlin, *Heterotic type I superstring duality and low-energy effective actions*, Nucl. Phys. **B467** (1996) 383, hep-th/9512081.
- [29] U.H. Danielsson, G. Ferretti and B. Sundborg, *D-particle Dynamics and Bound States*, hep-th/9603081, Int. J. Mod. Phys. **A11** (1996) 5463.
- [30] D. Kabat and P. Pouliot *A comment on zero-brane quantum mechanics*, hep-th/9603127, Phys. Rev. Lett. **77** (1996) 1004.
- [31] M.R. Douglas, D. Kabat, P. Pouliot and S. Shenker, *D-branes and short distances in string theory*, hep-th/9608024, Nucl. Phys. **B485** (1997) 85.