Matrices coupled in a chain. I. eigenvalue correlations

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Abstract. The general correlation function for the eigenvalues of p complex hermitian matrices coupled in a chain is given as a single determinant. For this we use a slight generalization of a theorem of Dyson.

1. Introduction

Random matrices were introduced in physics by Wigner [1] in the 1950's to elucidate the statistical properties of nuclear spectra. They later found applications to study the distribution of resonance frequencies in random electro-magnetic cavities and in metallic beams, the conductance of heterogeneous wires, quantum chaos, ... The models of coupled random matrices appeared in the study of planar diagrams of quantum field theory [2] and were used later in two-dimensional quantum gravity. In any statistical study of random phenomena it is natural to consider the probability densities of - and correlations among - various quantities of physical interest. In the model of random matrices, single or coupled, the quantities of interest are the eigenvalues and the study of their correlation functions is justified in their own right.

The probability density $\exp[-\mathrm{tr}\ V(A)]$ for the elements of an $n \times n$ matrix A is known to give rise to the probability density [3]

$$F(x_1, \dots, x_n) \propto \exp\left[-\sum_{i=1}^{n} V(x_i)\right] \prod_{1 \le i < j \le n} |x_i - x_j|^\beta \tag{1.1}$$

for its eigenvalues x_1, ..., x_n. Here $V(x)$ is a real polynomial of even order, the coefficient of the highest power being positive and β is the number of real components of a general element of A, i.e. $\beta = 1$ if A is real symmetric, $\beta = 2$ if A is complex hermitian and $\beta = 4$ if A is quaternion self-dual.

The case of coupled matrices may be represented by a graph where each matrix is represented by a point, and two points representing matrices A and B are joined by a line if the coupling factor $\exp[c\ \mathrm{tr}(AB)]$ is present in the probability density. When several

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matrices are coupled the probability density for the eigenvalues is known only in the case where these matrices are complex hermitian and the graph has a tree structure, i.e. does not have a closed path.

In what follows we will consider the simplest case of a tree, i.e. that of a chain of p complex hermitian $n \times n$ matrices with the joint probability density for their elements

$$F(A_1, \cdots, A_p) \propto \exp\left[-\mathrm{tr}\left\{\frac{1}{2}V_1(A_1) + V_2(A_2) + \cdots + V_{p-1}(A_{p-1}) + \frac{1}{2}V_p(A_p)\right\}\right] \\ \times \exp\left[\mathrm{tr}\left\{c_1 A_1 A_2 + c_2 A_2 A_3 + \cdots + c_{p-1} A_{p-1} A_p\right\}\right], \tag{1.2}$$

where $V_j(x)$ are real polynomials of even order with positive coefficients of their highest powers and c_j are real constants. For each j, the eigenvalues of the matrix A_j are real and will be denoted by $\mathrm{x_j} := \{x_{j1}, x_{j2}, \ldots x_{jn}\}$. The probability density for the eigenvalues of all the p matrices resulting from Eq. (1.2) is [4-7]

$$\begin{aligned} &F(\mathrm{x_1}; ...; \mathrm{x_p}) \\ &= C \exp\left[-\sum_{r=1}^{n}\left\{\frac{1}{2}V_1(x_{1r}) + V_2(x_{2r}) + \cdots + V_{p-1}(x_{p-1r}) + \frac{1}{2}V_p(x_{pr})\right\}\right] \\ &\times \prod_{1\le r<s\le n} (x_{1r} - x_{1s})(x_{pr} - x_{ps}) \\ &\times \det\left[e^{c_1 x_{1r} x_{2s}}\right] \det\left[e^{c_2 x_{2r} x_{3s}}\right] \cdots \det\left[e^{c_{p-1} x_{p-1r} x_{ps}}\right] \qquad (1.3) \\ &= C\left[\prod_{1\le r<s\le n} (x_{1r} - x_{1s})(x_{pr} - x_{ps})\right]\left[\prod_{k=1}^{p-1} \det\left[w_k(x_{kr}, x_{k+1s})\right]_{r,s=1,\ldots n}\right], \qquad (1.4) \end{aligned}$$

where

$$w_k(\xi, \eta) := \exp\left[-\frac{1}{2}V_k(\xi) - \frac{1}{2}V_{k+1}(\eta) + c_k \xi \eta\right], \tag{1.5}$$

and C is a normalisation constant such that the integral of F over the np variables x_{ir} is 1. We will be interested in the correlation functions

$$\begin{aligned} &R_{k_1,\ldots,k_p}(x_{11}, ..., x_{1k_1}; ...; x_{p1}, ..., x_{pk_p}) \\ &:= \int F(\mathrm{x_1}; ...; \mathrm{x_p}) \prod_{j=1}^{p} \frac{n!}{(n-k_j)!} \prod_{r_j=k_j+1}^{n} dx_{jr_j}. \end{aligned} \tag{1.6}$$

This is the density of ordered sets of k_j eigenvalues of A_j within small intervals around x_{j1}, ..., x_{jk_j} for $j = 1, 2, ..., p$. Here and in what follows, all the integrals are taken over $-\infty$ to ∞.

The case $p = 2$ was considered earlier [8,12]; the expressions given in [8], though correct, can be put in a much simpler form: the general answer can be written as a single $m \times m$ determinant with $m = k_1 + k_2 + \cdots + k_p$. The result is given in section 2 and the proof in section 3. Our result is a generalization of Dyson's for a single hermitian matrix [9], (the case $p = 1$), according to which the correlation function of k eigenvalues is given by a $k \times k$ determinant of the form:

$$R_k(x_1, \ldots, x_k) = \det\left[K(x_i, x_j)\right]_{i,j=1,k}, \tag{1.7}$$

the kernel $K(x, y)$ depending on the polynomial $V(x)$.

2. General correlation function.

To express our result we need some notations. Recall that a polynomial is called monic when the coefficient of the highest power is one. With a monic polynomial $P_j(\xi)$ of degree j let us write

$$P_{1j}(\xi) := P_j(\xi), \tag{2.1}$$

and recursively,

$$P_{ij}(\xi) := \int P_{i-1j}(\eta) w_{i-1}(\eta, \xi) d\eta, \quad 2 \le i \le p. \tag{2.2}$$

Similarly, with a monic polynomial $Q_j(\xi)$ of degree j we will write

$$Q_{pj}(\xi) := Q_j(\xi), \tag{2.3}$$

$$Q_{ij}(\xi) := \int w_i(\xi, \eta) Q_{i+1j}(\eta) d\eta, \quad 1 \le i \le p-1. \tag{2.4}$$

With arbitrary monic polynomials $P_j(\xi)$ and $Q_j(\xi)$, $j = 0, 1, 2, \ldots$, we can write the product of differences as $n \times n$ determinants

$$\prod_{1 \le i < j \le n} (\xi_j - \xi_i) = \det[\xi_i^{j-1}] = \det[P_{j-1}(\xi_i)] = \det[Q_{j-1}(\xi_i)]. \tag{2.5}$$

The first equality is known as the Vandermonde determinant [13], while the later equalities are obtained by the observation that a determinant is not changed if we add to any of its rows a linear combination of its other rows, and in particular, if we add to its j-th row an arbitrary linear combination of the rows 1, 2, ..., $j-1$. The idea is to replace the powers ξ^j by arbitrary monic polynomials $P_j(\xi)$ and choose these polynomials in a convenient way to facilitate later computations. If the polynomials V_j and the constants c_j are such that the moment matrix $[M_{ij}]$, $i, j = 0, 1, \ldots, n$ is non-singular for every n, where

$$M_{ij} := \int \xi^i \; (w_1 * w_2 * \ldots * w_{p-1})(\xi, \eta) \; \eta^j \; d\xi d\eta, \tag{2.6}$$

and

$$(w_{i_1} * w_{i_2} * \cdots * w_{i_k})(\xi, \eta) := \int w_{i_1}(\xi, \xi_1) w_{i_2}(\xi_1, \xi_2) \ldots w_{i_k}(\xi_{k-1}, \eta) d\xi_1 \ldots d\xi_{k-1}, \tag{2.7}$$

then it is always possible to choose the polynomials $P_j(\xi)$ and $Q_j(\xi)$ such that

$$\int P_j(\xi)\ (w_1 * w_2 * \ldots * w_{p-1})(\xi,\eta)\ Q_k(\eta)\ d\xi d\eta = h_j \delta_{jk}, \tag{2.8}$$

i.e. they are orthogonal with a non-local weight. This means that the functions $P_{ij}(\xi)$ and $Q_{ij}(\xi)$, which are not necessarily polynomials, are orthogonal

$$\int P_{ij}(\xi) Q_{ik}(\xi) d\xi = h_j \delta_{jk}, \tag{2.9}$$

for $i = 1, 2, \ldots, p$ and $j, k = 0, 1, 2, \ldots$.
Now define

$$K_{ij}(\xi,\eta) := H_{ij}(\xi,\eta) - E_{ij}(\xi,\eta), \tag{2.10}$$

where

$$H_{ij}(\xi,\eta) := \sum_{\ell=0}^{n-1} \frac{1}{h_\ell} Q_{i\ell}(\xi) P_{j\ell}(\eta), \tag{2.11}$$

$$E_{ij}(\xi,\eta) := \begin{cases} 0, & \text{if } i \geq j, \quad (2.12a) \\ w_i(\xi,\eta), & \text{if } i = j+1, \quad (2.12b) \\ (w_i * w_{i+1} * \cdots * w_{j-1})(\xi,\eta), & \text{if } i < j+1. \quad (2.12c) \end{cases}$$

Theorem. The correlation function (1.6) is equal to

$$R_{k_1,\ldots,k_p}(x_{11},\ldots,x_{1k_1};\ldots;x_{p1},\ldots,x_{pk_p}) \\ = \det\left[K_{ij}\left(x_{ir},x_{js}\right)\right]_{i,j=1,\ldots,p;r=1,\ldots,k_i;s=1,\ldots,k_j}. \tag{2.13}$$

This determinant has $m = k_1 + \cdots + k_p$ rows and m columns; the first k_1 rows and k_1 columns are labeled by the pair of indices $1r$, $r = 1, \ldots, k_1$; the next k_2 rows and k_2 columns are labled by the pair of indices $2r$, $r = 1, \ldots, k_2$ and so on. Each variable x_{ir} appears in exactly one row and one column, this row and column crossing at the main diagonal. If all the eigenvalues of a matrix A_j are not observed (are integrated out), then no row or column corresponding to them appears in Eq. (2.13).

3. Proofs

The theorem is proved by recurrence. We will first prove that Eq. (2.13) holds for the initial case $k_1 = n, \ldots, k_p = n$. Next we will prove that if $R_{k_1,\ldots,k_\ell,\ldots,k_p}$ has the form (2.13) then $R_{k_1,\ldots,k_\ell-1,\ldots,k_p}$ obtained by integrating out one of the $x_{\ell t}$, has the same form. Thus the theorem is a consequence of the following two lemmas:

Lemma 1. The $np \times np$ determinant $\det[K_{ij}(x_{ir},x_{js})]$, $i,j = 1,\ldots,p$; $r,s = 1,\ldots,n$, is, apart from a constant, equal to the probability density $F(\mathrm{x}_1;\ldots;\mathrm{x}_\mathbf{p})$, Eq. (1.4),

$$\det[K_{ij}(x_{ir},x_{js})]_{\substack{i,j=1,\ldots,p \\ r,s=1,\ldots,n}} = \left(\prod_{\ell=0}^{n-1} h_\ell^{-1}\right) C^{-1} F(\mathrm{x}_1;\ldots;\mathrm{x}_\mathbf{p}). \tag{3.1}$$

Lemma 2. Using the convolution $*$ defined in (2.7):

$$(f * g)(\xi, \eta) := \int f(\xi, \zeta) g(\zeta, \eta) d\zeta, \tag{3.2}$$

let's assume that the p^2 functions $K_{ij}(x, y)$, $i, j = 1, ..., p$, are such that:

$$K_{ij} * K_{jk} = \begin{cases} K_{ik}, & \text{if } i \geq j \geq k, \\ -K_{ik}, & \text{if } i < j < k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Then the integral of the $m \times m$ determinant $\det [K_{ij}(x_{ir}, x_{js})]$, $(i, j = 1, ..., p$; $r = 1, ..., k_i$; $s = 1, ..., k_j$; $k_1 \geq 0$, ..., $k_p \geq 0$; $m = k_1 + k_2 + ... + k_p)$, over $x_{\ell t}$ is proportional to the $(m-1) \times (m-1)$ determinant obtained from it by removing the row and the column containing the variable $x_{\ell t}$. The constant of proportionality is $\alpha_\ell - k_\ell + 1$, with

$$\alpha_\ell = \int K_{\ell\ell}(x, x) dx. \tag{3.4}$$

Let us recall here a result of Dyson [9,10].

Dyson's theorem. Let the function $K(x, y)$ be such that $K * K = K$, then

$$\int \det[K(x_i, x_j)]_{i,j=1,\ldots,k} dx_k = (\alpha - k + 1) \det[K(x_i, x_j)]_{i,j=1,\ldots,k-1}, \tag{3.5}$$

with

$$\alpha = \int K(x, x) dx. \tag{3.6}$$

Our lemma 2 above is a generalization of this one when the matrix elements $K(x_i, x_j)$ are replaced by $k_i \times k_j$ matrices $K_{ij}(x_{ir}, x_{js})$.

Proof of lemma 1. Consider the $np \times np$ matrix $[H_{ij}(x_{ir}, x_{js})]$, Eq. (2.11), the rows of which are denoted by the pair of indices ir and the columns by js; $i, j = 1$, ..., p; $r, s = 1$, ..., n. This matrix can be written as the product of two rectangular matrices $[Q_{i\ell}(x_{ir})]$ and $[P_{j\ell}(x_{js})/h_\ell]$ respectively of sizes $np \times n$ and $n \times np$, with $\ell = 0$, 1, ..., $n-1$. The rows of the first matrix $[Q_{i\ell}(x_{ir})]$ are numbered by the pair ir and its columns by ℓ. For $[P_{j\ell}(x_{js})/h_\ell]$ the rows are numbered by ℓ and the columns by js. Cutting the matrix $[H_{ij}(x_{ir}, x_{js})]$ into $n \times n$ blocks, we can write,

$$\begin{aligned} H &= \begin{bmatrix} \bar{Q}_1\bar{P}_1 & \bar{Q}_1\bar{P}_2 & \cdots & \bar{Q}_1\bar{P}_p \\ \bar{Q}_2\bar{P}_1 & \bar{Q}_2\bar{P}_2 & \cdots & \bar{Q}_2\bar{P}_p \\ \cdots & \cdots & \cdots & \cdots \\ \bar{Q}_p\bar{P}_1 & \bar{Q}_p\bar{P}_2 & \cdots & \bar{Q}_p\bar{P}_p \end{bmatrix} \\ &= \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \\ \vdots \\ \bar{Q}_p \end{bmatrix}_{np \times n} \begin{bmatrix} \bar{P}_1 & \bar{P}_2 & \cdots & \bar{P}_p \end{bmatrix}_{n \times np}, \end{aligned} \tag{3.7}$$

where $\left[\bar{Q}_i\right]_{r\ell} := [Q_{i\ell}(x_{ir})]$ and $\left[\bar{P}_j\right]_{\ell s} := [P_{j\ell}(x_{js})/h_\ell]$ are $n\times n$ matrices. Eq. (3.7) implies that the rank of $[H_{ij}(x_{ir}, x_{js})]$ is at most n. The rows of $\bar{P}_1$ and the columns of $\bar{Q}_p$ contain distinct monic polynomials, their ranks are therefore n, thus the rank of $[H_{ij}(x_{ir}, x_{js})]$ is n, and the last $n(p-1)$ columns can be linearly expressed in terms of the first n columns. In view of Eqs. (2.10) and (2.12a) the first n columns of $[H_{ij}(x_{ir}, x_{js})]$ are identical with the first n columns of $[K_{ij}(x_{ir}, x_{js})]$. The determinant of the later is therefore not changed if we subtract from its last $n(p-1)$ columns the corresponding $n(p-1)$ columns of the former. Thus

$$\det [K_{ij}(x_{ir}, x_{js})] = \det \left[\, H_{i1}(x_{ir}, x_{1s}) \quad -E_{ij}(x_{ir}, x_{js})\,\right]_{\substack{i\,=\,1,\ldots,p;\ j\,=\,2,\ldots,p \\ r,s\,=\,1,2,\ldots,n}}$$

$$= \det \begin{bmatrix} H_{11} & -E_{12} & -E_{13} & \cdots & -E_{1p} \\ H_{21} & 0 & -E_{23} & \cdots & -E_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{p-1\,1} & 0 & 0 & \cdots & -E_{p-1\,p} \\ H_{p1} & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{3.8}$$

¿From Eq. (2.12a) the last n rows of this matrix corresponding to $i = p$ have non-zero elements only in the first n columns; also the matrix $[E_{ij}(x_{ir}, x_{js})]$ is block triangular, $E_{ij}(\xi, \eta)$ being zero for $i \geq j$. Therefore,

$$\det [K_{ij}(x_{ir}, x_{js})] = \det [H_{p1}(x_{pr}, x_{1s})] \det [E_{ij}(x_{ir}, x_{js})]_{\substack{i\,=\,1,\ldots,p-1;\ j\,=\,2,\ldots,p-1 \\ r,s\,=\,1,2,\ldots,n}}$$

$$= \det [H_{p1}(x_{pr}, x_{1s})] \prod_{j=2}^{p} \det [E_{j-1\,j}(x_{j-1\,r}, x_{js})]$$

$$= \left(\prod_{\ell=0}^{n-1} h_\ell^{-1}\right) \det [Q_{p\ell}(x_{pr})] \det [P_{1\ell}(x_{1s})] \prod_{j=1}^{p-1} \det [w_j(x_{jr}, x_{j+1\,s})]\,, \tag{3.9}$$

and from Eqs. (1.4), (2.5) one gets Eq. (3.1). This ends the proof.

The learned reader will have recognized that the above E_{ij}'s play the same role as the ε in Dyson's proof in the case of a single matrix of the circular orthogonal ensemble [9,11].

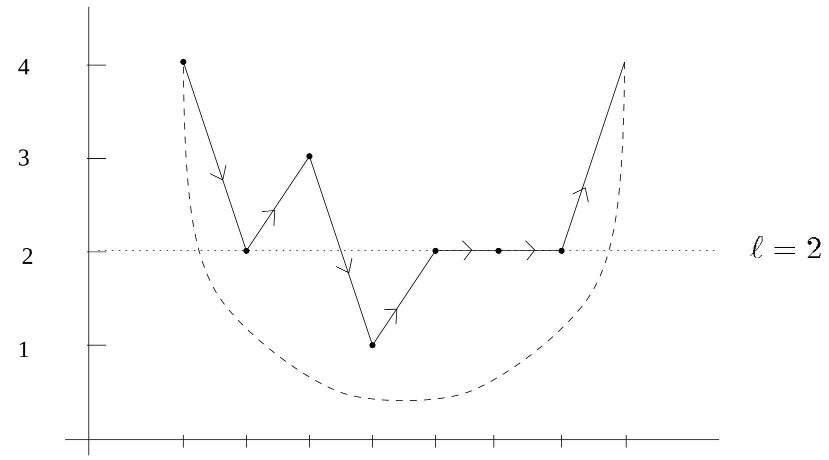

Figure 1: the permutation $\sigma' = (4, 2, 3, 1, 2, 2)$.

Proof of Lemma 2. We want to integrate the $m \times m$ determinant $\det\,[K_{ij}(x_{ir}, x_{js})]$ over $x_{\ell t}$. We can write the expansion of the determinant as a sum over $m!$ permutations, writing these permutations as a product of mutually exclusive cycles. The variable $x_{\ell t}$ occurs in the row and the column labeled by ℓt (recall that rows and columns are labeled by a pair of indices). If ℓt forms a cycle by itself, then by Eq. (3.4) integration over $x_{\ell t}$ gives a factor α_ℓ, and its coefficient is just the expansion of the $(m-1) \times (m-1)$ determinant obtained by removing the row and the column containing $x_{\ell t}$. If ℓt occurs in a longer cycle, say in the permutation $\sigma = (ir, \ell t, js, \cdots)(\cdots)\cdots$, then from Eq. (3.3) integration over $x_{\ell t}$ decreases the length of the cycle containing ℓt by one, giving the permutation $\sigma' = (ir, js, \cdots)(\cdots)\cdots$, and multiplies the corresponding term by a factor $+1$ if $i \geq \ell \geq j$, by -1 if $i < \ell < j$ and by 0 otherwise. So the question is, given the permutation σ', in how many ways can one insert ℓt in any of its cycles with the algebraic weights $+1$, -1 and 0 to get a permutation σ, or equivalently, how many properly weighted permutations σ give the same σ' on removing ℓt. Fortunately, it turns out that this number is independent of σ', so that the sum over the $(m-1)!$ permutations σ' gives back the $(m-1) \times (m-1)$ determinant obtained by removing the row and the column containing $x_{\ell t}$.

Let us represent the cycles of permutations by a graph. Since the discussion of the weight $+1$, -1 or 0 depends only on the first index, only this first index i, j, ... of each pair of indices will be plotted against the place number where they occur. For example, the cycle $(42, 26, 36, 15, 24, 22)$ is represented on figure 1, where points at successive heights $4, 2, 3, 1, 2, 2$ are joined successively by line segments or "sides". Note that we identify the 7^{th} point and the first one. Permutation σ' is thus represented by a certain number of closed directed polygons corresponding to its mutually exclusive cycles. Addition of ℓt in one of the cycles of σ' amounts to the addition of a point at a height ℓ in the corresponding polygon. If this added point lies on a non-ascending side, then the weight multiplying the corresponding σ is $+1$, if it lies on an ascending side, the weight is -1, and this weight is zero otherwise. In other words, each downward crossing of the line at height ℓ, with or without stops, contributes a factor $+1$, each point on this height contributes a factor $+1$

and each upward crossing, with or without stops, contributes a factor -1. The graph of σ' consisting of closed loops, the number of upward crossings is equal to the number of downward crossings at any height, and the corresponding contributions cancel out. The algebraic sum of all such coefficients is thus seen to be the number of points at height ℓ in the graph of σ', i.e. it is $k_\ell - 1$. Also the permutations σ and σ' have opposite signs, since only one of their cycle lengths differ by unity. Thus

$$\int \det\left[K_{ij}(x_{ir}, x_{js})\right]_{m\times m} dx_{\ell t} = (\alpha_\ell - k_\ell + 1) \det\left[K_{ij}(x_{ir}, x_{js})\right]_{(m-1)\times(m-1)}, \qquad (3.10)$$

where the integrand on the left hand side is an $m \times m$ determinant, $i, j = 1, ..., p$; $r = 1, ..., k_i$; $s = 1, ..., k_j$; $m = k_1 + k_2 + \cdots + k_p$ and the result on the right hand side is the $(m-1) \times (m-1)$ determinant obtained from the integrand by removing the row and the column containing the variable $x_{\ell t}$. This ends the proof.

Using Eqs. (2.9-2.12), one verifies that the H_{ij} and E_{ij} satisfy the following relations

$$H_{ij} * H_{jk} = H_{ik}, \qquad (3.11)$$

$$H_{ij} * E_{jk} = \begin{cases} H_{ik}, & \text{if } j < k, \\ 0, & \text{if } j \geq k, \end{cases} \qquad (3.12)$$

$$E_{ij} * H_{jk} = \begin{cases} H_{ik}, & \text{if } i < j, \\ 0, & \text{if } i \geq j, \end{cases} \qquad (3.13)$$

$$E_{ij} * E_{jk} = \begin{cases} E_{ik}, & \text{if } i < j < k, \\ 0, & \text{if either } i \geq j \text{ or } j \geq k. \end{cases} \qquad (3.14)$$

This implies for the $K_{ij} = H_{ij} - E_{ij}$, Eq. (2.10), the relations (3.3). Also

$$\alpha_\ell = \int K_{\ell\ell}(\xi, \xi) d\xi = n. \qquad (3.16)$$

Using lemma 1 once and lemma 2 several times one gets the normalization constant C,

$$C = (n!)^{-p} \prod_{\ell=0}^{n-1} h_\ell^{-1}. \qquad (3.17)$$

Again with repeated use of lemma 2 and from the definition, Eq. (1.6), of the correlation function $R_{k_1,...,k_p}(x_{11}, ..., x_{1k_1}; ...; x_{p1}, ..., x_{pk_p})$, one gets Eq. (2.13).

4. Conclusion

The correlation functions of eigenvalues of a chain of random hermitian matrices can thus be written in a very compact form as a single determinant. This result may be used to study the large n limit of correlations between eigenvalues [12,14].

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References

1. See for example, in "Statistical theories of spectra: fluctuations" Edited by C.E. Porter, Academic Press, New York (1965).
2. See for example, D. Bessis, A new method in the combinatoric of the topological expansion, Comm. Math. Phys. 69 (1979) 147-163, or
   C. Itzykson, J.B. Zuber, The planar approximation II, J. Math. Phys. 21 (1980) 411-421 and references given there.
3. See for example, M.L. Mehta, *Random Matrices*, Academic Press, San Diego, CA, U.S.A., 1991, Chapter 3. Here only the case $V(x) = x^2$ is considered, but the same method applies when $V(x)$ is any polynomial of even order.
4. See for example, reference 3, appendix A.5.
5. C. Itzykson and J.B. Zuber, reference 2.
6. M.L. Mehta, A method of integration over matrix variables, Comm. Math. Phys. 79 (1981) 327-340.
7. M.L. Mehta, *Matrix Theory*, Les Editions de Physique, Les Ulis, France, 1989, chapter 13.2
8. M.L. Mehta and P. Shukla, Two coupled matrices: eigenvalue correlations and spacing functions, J. Phys. A: Math Gen. 27 (1994) 7793-7803.
9. F.J. Dyson, Correlations between the eigenvalues of a random matrix, Comm. Math. Phys. 19 (1970) 235-250.
10. See for example reference 3, theorem 5.2.1, or reference 7, theorem 8.10.
11. See for example reference 3, section 6.3, or section 10.3.
12. B. Eynard, Eigenvalue distribution of large random matrices, from one matrix to several coupled matrices, Nuc. Phys. B506 (1997), xxx, cond-mat/9707005.
13. See for example, reference 7, chapter 7.1.
14. B. Eynard, Correlation functions of eigenvalues of multi-matrix models, and the limit of a time depedant matrix, cond-mat/9801075.