

**More on the exact solution of the $O(n)$ model on a random
lattice and an investigation of the case $|n| > 2$**

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Abstract

For $n \in [-2, 2]$ the $O(n)$ model on a random lattice has critical points to which a scaling behaviour characteristic of 2D gravity interacting with conformal matter fields with $c \in [-\infty, 1]$ can be associated. Previously we have written down an exact solution of this model valid at any point in the coupling constant space and for any n . The solution was parametrized in terms of an auxiliary function. Here we determine the auxiliary function explicitly as a combination of θ -functions, thereby completing the solution of the model. Using our solution we investigate, for the simplest version of the model, hitherto unexplored regions of the parameter space. For example we determine in a closed form the eigenvalue density without any assumption of being close to or at a critical point. This gives a generalization of the Wigner semi-circle law to $n \neq 0$. We also study the model for $|n| > 2$. Both for $n < -2$ and $n > 2$ we find that the model is well defined in a certain region of the coupling constant space. For $n < -2$ we find no new critical points while for $n > 2$ we find new critical points at which the string susceptibility exponent γ_{str} takes the value $+\frac{1}{2}$.

1 Introduction

For $n \in [-2, 2]$ the $O(n)$ model on a random lattice [1] has critical points to which a scaling behaviour characteristic of 2D gravity interacting with conformal matter fields with central charge $c \in [-\infty, 1]$ can be associated [1, 2]. In particular, with $n = 2 \cos(\nu\pi)$, by choosing ν rational and fine-tuning the potential of the model one can reach any rational conformal matter field [4, 5]. Until recently a solution of the model away from its critical points was known only for $n = \pm 2$ [6, 4] and for various rational values of ν and potentials of low degree [4, 5]. In reference [7] we wrote down an exact solution of the model valid for any potential and any value of n . The solution was parametrized in terms of an auxiliary function. In the present paper we will determine this function explicitly as a combination of θ -functions, thereby completing our solution of the model. Eventually the determination of the auxiliary function might in addition lead to a better understanding of the underlying continuum physics. In the one matrix model ($n = 0$) case the auxiliary function is the well known “Wigner square root” which was essential in revealing the Virasoro structure of the model and establishing its connection with the τ -function of the kdV hierarchy [8]. In the general case information about the integrable structure underlying the $O(n)$ model should likewise be encoded in the auxiliary function.

Here we will in particular be concerned with using our exact solution to investigate so far unexplored regions of the parameter space of the model. Although our solution allows us to investigate any version of the model we will here restrict ourselves to the simplest “gaussian” one. In addition we shall consider only genus zero. The results can easily be extended to higher genera by means of the iterative procedure described in reference [7]. We shall derive exact expressions for a number of quantities, including the eigenvalue distribution, away from the traditionally studied critical points. In particular we will use our solution to study the case $|n| > 2$. As we shall see the model is well defined in a certain region of the coupling constant space both for $n < -2$ and for $n > 2$. For $n < -2$ we find no new critical points while for $n > 2$ we find new critical points at which the string susceptibility exponent γ_{str} takes the value $+\frac{1}{2}$, the value characteristic of branched polymers.

We start by, in section 2, presenting the model and those of our results which will be of importance for what follows. We then proceed to, in section 3, determining explicitly the above mentioned auxiliary function. In section 4 we outline the strategy for analysing completely any given version of the model and in section 5 we derive a closed expression for the second derivative of the free energy, an expression which proves very convenient when it comes to the investigation of the critical behaviour of the model. Hereafter we specialize to the simplest version of the model and section 6 contains a

detailed analysis of this version for all values of $n \in]-\infty, +\infty[$. Finally, section 7 contains our conclusion and a discussion of possible future directions of investigation.

2 The Model

The $O(n)$ -model coupled to 2D gravity is described by the following partition function [1]

$$Z = e^{N^2 F} = \int_{N \times N} dM \prod_{i=1}^n dA_i \exp \left(-N \operatorname{Tr} \left[V(M) + M \sum_{i=1}^n A_i^2 \right] \right) \quad (2.1)$$

where M and A_i , $i = 1, \dots, n$ are hermitian $N \times N$ matrices. We shall take the potential to be completely general, i.e.¹

$$V(M) = \sum_{j=1}^{\infty} \frac{g_j}{j} M^j \quad (2.2)$$

and use the following parametrization of n

$$n = 2 \cos(\nu\pi). \quad (2.3)$$

Let us summarize those of our previous results which are needed for the following. In the solution of the model the central object is the genus zero 1-loop function $W(p)$ (or equivalently the distribution function $\rho(\lambda)$ for the eigenvalues $\{\lambda\}$ of the matrix M at the stationary point of the integral (2.1))

$$W(p) = \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \operatorname{tr} \frac{1}{p - M} \right\rangle = \int \frac{\rho(\lambda)}{p - \lambda} d\lambda. \quad (2.4)$$

Once $W(p)$ is known any other correlator of the M -field as well as the free energy associated with surfaces of arbitrary topology can be found by an (in principle) straightforward iterative procedure. In the following we consider the situation where the eigenvalues are restricted to only one interval $[a, b]$ with $a > 0$ and assume that the corresponding distribution function is normalized to 1. Under these circumstances the 1-loop function $W(p)$ is analytic in the complex plane except for one cut $[a, b]$ and the eigenvalue distribution $\rho(\lambda)$ can be found from [9]

$$\rho(\lambda) = \frac{1}{2\pi i} \{W(\lambda - i0) - W(\lambda + i0)\}. \quad (2.5)$$

The 1-loop function fulfills the saddle point equation [1]

$$W(p + i0) + W(p - i0) + nW(-p) = V'(p) \quad (2.6)$$

¹We note that the traditional form of the partition function where there is no term linear in M in the potential and a mass term for the A -field can be obtained by a linear shift of M .

which together with the requirement

$$W(p) \rightarrow \frac{1}{p}, \quad p \rightarrow \infty \quad (2.7)$$

determine uniquely $W(p)$. The key idea which allowed us to solve (2.6) was a transformation of all functions involved in the problem; namely corresponding to a function $h(p)$ we introduced (following reference [5]) $h_{\pm}(p)$ by

$$h_+(p) = \frac{e^{i\nu\pi/2}h(p) + e^{-i\nu\pi/2}h(-p)}{2\sin(\nu\pi)}, \quad h_-(p) = h_+(-p). \quad (2.8)$$

Then (2.6) could be written in a more manageable form, $W_{\pm}(p)$ could be found and $W(p)$ reconstructed using the inverse of (2.8)

$$h(p) = -i \left(e^{i\nu\pi/2}h_+(p) - e^{-i\nu\pi/2}h_-(p) \right). \quad (2.9)$$

The p -dependence of any multi-loop correlator of the M -field on surfaces of any genus could be described using a set of basis functions $G_a^{(k)}(p)$ (and $G_b^{(k)}(p)$) defined by

1. $G_a^{(k)}(p)$ and $G_b^{(k)}(p)$ satisfy the homogeneous saddle point equation

$$G^{(k)}(p+i0) + G^{(k)}(p-i0) + nG^{(k)}(-p) = 0, \quad p \in [a, b] \quad (2.10)$$

2. $G_a^{(k)}(p)$ and $G_b^{(k)}(p)$ behave near the end points of the cut $[a, b]$ as

$$G_a^{(k)}(p) \sim (p-a)^{-k-1/2}(p-b)^{-1/2}, \quad G_b^{(k)}(p) \sim (p-b)^{-k-1/2}(p-a)^{-1/2}.$$

3. $G_a^{(k)}(p)$ and $G_b^{(k)}(p)$ are analytical outside the cut (especially near $-a$ and $-b$).
4. $G_a^{(k)}(p)$ and $G_b^{(k)}(p)$ have the following asymptotic behaviour

$$G_a^{(k)}(p), G_b^{(k)}(p) \sim \frac{1}{p^{k+1}}, \quad p \rightarrow \infty.$$

To determine $G_a^{(k)}(p)$ it suffices to know $G_a^{(0)}(p) = G_b^{(0)}(p) \equiv G^{(0)}(p)$; namely using the important observation that any solution of the saddle point equation (2.10) can be parametrized in terms of any two other independent solutions, a recursive strategy for determining $G_a^{(k)}(p)$ from the knowledge of $G^{(0)}(p)$ was formulated. Needless to say that $G_b^{(k)}(p)$ appears from $G_a^{(k)}(p)$ by the interchange $a \leftrightarrow b$. In section 3 we shall determine the function $G_0(p)$ explicitly as a combination of θ -functions. The basis functions contain no explicit reference to the matrix model coupling constants, $\{g_i\}$. They depend on these only implicitly via the endpoints of the cut, a and b . The nature of this dependence is different for different values of n but this difference can

be hidden by working with a parameter, e , which is the only point (apart from ∞) at which $G_+^{(0)}(p)$ vanishes, i.e.

$$G_+^{(0)}(e) = G_-^{(0)}(-e) = 0. \quad (2.11)$$

The explicit dependence of the observables on the matrix model coupling constants could conveniently be described using a set of moment variables generalizing those introduced in reference [10]

$$M_k = \oint_{C_1} \frac{d\omega}{2\pi i} V'(\omega) \tilde{G}_a^{(k)}(\omega), \quad J_k = M_k(a \leftrightarrow b) \quad (2.12)$$

where for $n = 2 \cos(\nu\pi)$ the \tilde{G} -functions are the G -functions corresponding to $n = 2 \cos((1 - \nu)\pi)$ and where C_1 is a curve which encircles the cut $[a, b]$. (We use the convention that all contours are oriented counter-clockwise.) As in the 1-matrix model case the moment variables have the advantage that for all observables of the model except the genus zero contribution to the one-loop correlator and the free energy the dependence on the infinite series of coupling constants $\{g_i\}$ arranges into a dependence on only a finite number of moment variables. Furthermore the moment variables reflect more directly than the coupling constants the possible critical behaviour of the model.

In our previous paper [7] we wrote down a contour integral representation for the genus zero contribution to the one-loop correlator, the integrand being composed of the functions $G^{(0)}(\omega)$, $\tilde{G}^{(0)}(\omega)$ and the derivative of the potential $V'(\omega)$. For the following analysis it is convenient to rewrite our solution in the moment formulation. Doing so will also highlight the statement made above concerning the advantage of the moment variables. First let us remind the reader that the 1-loop correlator can naturally be decomposed into a regular part $W_r(p)$ and a singular part $W_s(p)$ as

$$W(p) = W_r(p) - W_s(p) \quad (2.13)$$

with $W_r(p)$ being given by

$$W_r(p) = \frac{2V'(p) - nV'(-p)}{4 - n^2}. \quad (2.14)$$

Now, in order to simplify the notation, let us introduce a scalar product by

$$\langle f, \tilde{g} \rangle = (n^2 - 4) \oint_{C_2} \frac{d\omega}{4\pi} \{f_+(\omega)\tilde{g}_+(\omega) - f_-(\omega)\tilde{g}_-(\omega)\} \quad (2.15)$$

where C_2 is a contour which encircles the two intervals $[a, b]$ and $[-b, -a]$. Then it holds that (cf. to reference [7])

$$M_k = \langle W_s, \tilde{G}_a^{(k)} \rangle. \quad (2.16)$$

Furthermore, let us define functions $W_k(p)$ which are solutions of the saddle point equation (2.10), analytic outside the cut $[a, b]$ and fulfill

$$\langle W_k, \tilde{G}_a^{(l)} \rangle = \delta_{k,l}. \quad (2.17)$$

It is easy to see that $W_k(p)$ must have the following asymptotic behaviour

$$W_k(p) \sim \frac{1}{4 \cos^2(\eta_k \pi/2)} p^k, \quad p \rightarrow \infty \quad (2.18)$$

where

$$\eta_{2k} = \nu, \quad \eta_{2k+1} = 1 - \nu \quad (2.19)$$

and that $W_k(p)$ must be finite as $p \rightarrow a$ and as $p \rightarrow b$. The functions $W_k(p)$ are uniquely determined and one can show that

$$W_{k+}(p) = -\frac{1}{2 \sin(\nu \pi)} (a^2 - p^2)^{k-1} \overline{\sqrt{p}} \tilde{G}_{a-}^{(k-1)}(p) \quad (2.20)$$

where

$$\overline{\sqrt{p}} = \sqrt{(p^2 - a^2)(p^2 - b^2)}. \quad (2.21)$$

Here and in the following it is understood that we choose the sign of the square root so that $\overline{\sqrt{p}} \sim p^2$. It now follows that the singular part of the one-loop correlator can be written as

$$W_s(p) = \sum_{k=1}^{\infty} M_k W_k(p) \quad (2.22)$$

and that the two boundary conditions which determine the endpoints of the cut a and b read

$$M_0 = 0 \quad \text{and} \quad M_{-1} = \oint_{C_1} \frac{d\omega}{2\pi i} \omega V'(\omega) G^{(0)}(\omega) = 2 - n. \quad (2.23)$$

In particular from (2.20) and (2.22) we can read off the eigenvalue distribution for any given value of ν and any potential $V(M)$.

It is important to note that the decisive step in solving the model is the determination of the auxiliary function $G^{(0)}(p)$. Once this function is known all other quantities can be found. In our previous paper we derived a first order differential equation for $G^{(0)}(p)$ and formally wrote down its solution. In the next section we will show, using a totally different strategy, how $G^{(0)}(p)$ can be completely explicitated as a combination of θ -functions. This will make complete our solution of the model and explain several of our previous observations. In particular, deriving the explicit expression for $G^{(0)}(p)$ gives as a byproduct a determination of the parameter e in terms of ν , a and b . Furthermore the observation that when ν is rational $G_+^{(0)}(p)$ reduces to an algebraic function is an immediate consequence of the general formulas.

Before proceeding with the determination of $G^{(0)}(p)$, let us spend a few lines showing how the possible types of critical behaviour are very clearly exposed in the moment formulation. For $\nu \in [0, 1]$ critical points are reached when the endpoint a of the eigenvalue distribution touches the origin. In this limit one has (cf. to reference [7])

$$\tilde{G}^{(k)}(p) \sim p^{-k-\eta_k}, \quad p \rightarrow 0 \quad (2.24)$$

which means that

$$W_k(p) \sim p^{k-\eta_{k+1}}. \quad (2.25)$$

Hence we have a series of M 'th multi-critical points characterized by a critical exponent μ_M

$$W_s(p) \sim p^{\mu_M}, \quad \mu_M = M - \eta_{M+1}, \quad (2.26)$$

with the corresponding subspace of the coupling constant space being given by

$$M_0 = M_1 = \dots = M_{M-1} = 0, \quad M_M \neq 0. \quad (2.27)$$

This reproduces the well-known results of references [1, 4, 5].

3 Determination of the function $G^{(0)}(p)$

3.1 Reformulation of the problem

To determine the function $G^{(0)}(p)$ it is convenient to perform a change of variables from p to u given by

$$p = a \operatorname{sn}(u, k), \quad k = \frac{a}{b}. \quad (3.1)$$

This maps the complex p -plane into the rectangle $[-K, K] \times [-iK', iK']$ in the complex u -plane. An important feature of the mapping (3.1) is that it ‘‘opens up’’ the cut $[a, b]$ of $G^{(0)}(p)$, i.e. the part of the complex p -plane which lies above the cut $[a, b]$ is mapped into the first quadrant of the u -plane while the part of the complex p -plane which lies beneath the cut $[a, b]$ is mapped into the fourth quadrant of the u -plane. The analyticity properties of $G^{(0)}(p)$ imply that the function $G^{(0)}(u)$ must fulfill a number of relations on the boundary of its domain. First of all, the saddle point equation (2.10) implies that for $u \in [-iK', iK']$ we have

$$G^{(0)}(K + u) + G^{(0)}(K - u) + nG^{(0)}(-K + u) = 0. \quad (3.2)$$

Furthermore the fact that $G^{(0)}(p)$ has no cut along the interval $[-b, -a]$ gives that for $u \in [-iK', iK']$

$$G^{(0)}(-K + u) + G^{(0)}(-K - u) = 0. \quad (3.3)$$

Finally from the fact that $G^{(0)}(p)$ is analytic along the line segments $[-\infty, -b]$ and $[b, \infty]$ we deduce that for $u \in [-K, K]$

$$G^{(0)}(iK' + u) = G^{(0)}(-iK' + u). \quad (3.4)$$

By means of the three relations (3.2), (3.3) and (3.4) we now analytically continue the function $G^{(0)}(u)$ from its original domain $[-K, K] \times [-iK', iK']$ to the whole complex plane. First we extend the definition of the function $G^{(0)}(u)$ to the vertical band $[-K, K] \times [-i\infty, i\infty]$ by means of the relation (3.4). Next we use the relation (3.3) to extend the definition of the function to the band $[-3K, K] \times [-i\infty, i\infty]$ and then using the equation (3.2) we can define $G^{(0)}(u)$ in the remaining part of the complex plane. This procedure leaves us with a function which is defined in the whole complex plane and which obeys the equations (3.2), (3.3) and (3.4) for all u . Now we would like to determine this function. First, let us note that combining (3.2) and (3.3) we get

$$G^{(0)}(u + 2K) + G^{(0)}(u - 2K) + nG^{(0)}(u) = 0. \quad (3.5)$$

Furthermore the parity condition (3.3) and the periodicity condition (3.4) can be expressed as

$$G^{(0)}(-u - 2K) = G^{(0)}(u), \quad (3.6)$$

$$G^{(0)}(u + 2iK') = G^{(0)}(u). \quad (3.7)$$

Let us next introduce a translation operator \hat{X} by

$$\hat{X} = e^{2K \frac{\partial}{\partial u}}. \quad (3.8)$$

Then we can write (3.5) as

$$\left(\hat{X}^2 + n\hat{X} + 1\right) G^{(0)}(u) = \left(\hat{X} - x_+\right) \left(\hat{X} - x_-\right) G^{(0)}(u) = 0 \quad (3.9)$$

with

$$x_{\pm} = e^{\mp i(1-\nu)\pi}. \quad (3.10)$$

Now, let us decompose $G^{(0)}(u)$ in the following way.

$$G^{(0)}(u) = \sqrt{x_+} G_+^{(0)}(u) + \sqrt{x_-} G_-^{(0)}(u) \quad (3.11)$$

with

$$G_+^{(0)}(u) = \frac{1}{\sqrt{x_+}} \frac{1}{x_+ - x_-} \left(\hat{X} - x_-\right) G^{(0)}(u), \quad (3.12)$$

$$G_-^{(0)}(u) = \frac{1}{\sqrt{x_-}} \frac{1}{x_- - x_+} \left(\hat{X} - x_+\right) G^{(0)}(u). \quad (3.13)$$

Then it obviously holds that

$$\hat{X}G_+^{(0)}(u) = x_+G_+^{(0)}(u), \quad \hat{X}G_-^{(0)}(u) = x_-G_-^{(0)}(u) \quad (3.14)$$

and it is easy to show that the decomposition (3.11) of $G^{(0)}(u)$ is unique, i.e. if $G^{(0)} = H_+ + H_-$ with $\hat{X}H_\pm = x_\pm H_\pm$ then $H_\pm = \sqrt{x_\pm}G_\pm^{(0)}$. Exploiting this uniqueness result it follows from the parity condition (3.3) that

$$G_-^{(0)}(u) = G_+^{(0)}(-u). \quad (3.15)$$

It is now clear that the decomposition (3.11) corresponds exactly to the decomposition introduced in equation (2.9) and our original problem of determining $G^{(0)}(p)$ has hence been transformed into the problem of determining a function $G_+^{(0)}(u)$ which fulfills

$$G_+^{(0)}(u + 2K) = e^{-i(1-\nu)\pi}G_+^{(0)}(u), \quad (3.16)$$

$$G_+^{(0)}(u + 2iK') = G_+^{(0)}(u) \quad (3.17)$$

(where the second equation originates from the periodicity condition (3.4)) and which is compatible with the requirements 2, 3 and 4 on page 4. In the following subsection we will show how to solve this problem.

3.2 The explicit expression for $G_+^{(0)}(p)$

Let us start by noting that the condition 2 and 3 on page 4 imply that $G_+^{(0)}(p)$ has simple poles at $u = K$ and $u = K + iK'$ but not other singularities. Furthermore the condition 4 implies that $G_+^{(0)}(u)$ has a simple zero at $u = iK'$. Now we introduce a function $H_+(u)$ by

$$H_+(u) = \frac{\theta\left(\frac{u-iK'}{2K}\right)\theta\left(\frac{u-\varepsilon}{2K}\right)}{\theta\left(\frac{u+K}{2K}\right)\theta\left(\frac{u-(K+iK')}{2K}\right)}e^{-i(1-\nu)\frac{\pi u}{2K}}. \quad (3.18)$$

This function has the same poles as $G_+^{(0)}(u)$. It also has the zero at $u = iK'$ in common with $G_+^{(0)}(u)$ and it fulfills the equations (3.16) and (3.17) provided the parameter ε takes the following value

$$\varepsilon = i(1-\nu)K'. \quad (3.19)$$

Now, $G_+^{(0)}(u)/H_+(u)$ is a doubly periodic function (with periods $2K$ and $2iK'$) which has no singularities. (The function can have no singularities except for poles and of poles it can have only one originating from the zero $u = \varepsilon$ of $H_+(u)$. Hence it has none at all.) This allows us to conclude that

$$G_+^{(0)}(u) = \text{const} \cdot H_+(u). \quad (3.20)$$

In particular we see that in accordance with our previous analysis the function $G_+^{(0)}(u)$ must have exactly one zero in addition to the one at $u = iK'$ ($p = \infty$), namely $u = \varepsilon$. Translating back to p -variables gives us the value of e (cf. to relation (2.11))

$$e = a \operatorname{sn}(i(1 - \nu)K'). \quad (3.21)$$

Now it only remains to determine the constant in relation (3.20). Its value follows from the normalization condition 4 on page 4 and reads

$$\text{const} = \frac{1}{a} \frac{\theta_2^2}{\theta_4 \theta_4 \left(\frac{\varepsilon}{2K}\right)} \frac{1}{2 \cos(\nu\pi/2)}. \quad (3.22)$$

Now we have the explicit expression for $G_+^{(0)}(u)$ and we can write down as explicit expressions for (in principle) any other quantity. Let us note in this connection that with our sign convention for the square root the quantity \sqrt{p} translates to u -variables as

$$\sqrt{p} = -ab \operatorname{cnu} \operatorname{dnu} \quad (3.23)$$

3.3 Rational case

Let us remind the reader that when ν is rational the scaling behaviour at the critical points of the model is that characteristic of 2D gravity interacting with rational conformal matter fields. More precisely, when $\nu = l/q$ with l and q integer and $0 < l < q$ the matter fields which appear are of the type $(q, (2m+1)q \pm l)$. In reference [7] it was shown that when ν is rational a simplification of the function $G_+^{(0)}(p)$ occurs; namely for $\nu = l/q$ one has

$$\left(\sqrt{p} G_+^{(0)}(p)\right)^q = A(p)\sqrt{p} + B(p) \quad (3.24)$$

where $A(p)$ and $B(p)$ are polynomials in p having a definite parity

$$A(-p) = (-1)^{l+1} A(p), \quad B(-p) = (-1)^l B(p) \quad (3.25)$$

and having degree $(q-3)$ and q when $(q+l)$ is even and $(q-2)$ and $(q-1)$ when $(q+l)$ is odd. The existence of a relation like (3.24) is a direct consequence of standard properties of elliptic functions. Consider the identity (3.24) rewritten in the u -variable

$$\left(\operatorname{cnu} \operatorname{dnu} G_+^{(0)}(u)\right)^q = A(\operatorname{snu})\operatorname{cnu} \operatorname{dnu} + B(\operatorname{snu}). \quad (3.26)$$

From (3.16) and (3.17) it follows that the function on the left hand side of (3.26) is an elliptic function with periods $2K$ and $2iK'$ when l is even and periods $4K$ and $2iK'$ when l is odd. The same is true for the function on the right hand side. Furthermore the two functions have the same poles, namely only one of order q at $u = iK'$. Now,

counting the number of adjustable constants in the polynomials A and B it is easily seen that one can always arrange that the two functions also have the same zeros namely one of order q at $p = e$. Exact equality between the two functions can hereafter be ensured by choosing the overall normalization of A and B so that the residue at $u = iK'$ equals 1.

4 Explicit solution of a given model

Our explicit expression for the auxiliary function $G^{(0)}(p)$ gives us the possibility of exploring in detail the coupling constant space of our model. In particular exact results for a large number of quantities can be obtained. Let us briefly describe how one would extract exact results for a given model. Observables generally depend explicitly on the matrix model coupling constants $\{g_i\}$ via the moments and implicitly via the endpoints of the cut a and b which are in turn determined by the boundary conditions (2.23). For a given matrix model potential the moments (including those entering (2.23)) can be expressed in terms of the $\{g_i\}$, a and b by simply carrying out the contour integrals appearing in the definition (2.12). This can either be done using the θ -function representation of the G -functions in which case the contour integration can be reduced to to an integration around the pole $u = iK'$ or using the p -representation in which case the contour can be deformed into one which encircles infinity. We will choose the latter line of action. To proceed along this line one needs to know the large p -expansion of $G^{(0)}(p)$

$$G^{(0)}(p) = \frac{1}{p} \left(1 - \frac{i \tan(\nu\pi/2) \alpha_1}{p} - \frac{\alpha_2}{p^2} - \frac{i \tan(\nu\pi/2) \alpha_3}{p^3} - \dots \right). \quad (4.1)$$

As explained earlier once $G^{(0)}(p)$ is known the remaining G -functions can be found by a straightforward iterative procedure. The α -coefficients are most easily found from the differential equation for $G_+^{(0)}(p)$ derived in reference [7]

$$\frac{\partial}{\partial p} \left(\sqrt{p} G_+^{(0)}(p) \right) = \left(\alpha_1 - \frac{\sqrt{e}}{e} \right) G_+^{(0)}(p) + p g_+(p) G_+^{(0)}(p) \quad (4.2)$$

where

$$g_+(p) = \frac{\sqrt{p} + p \frac{\sqrt{e}}{e}}{p^2 - e^2}. \quad (4.3)$$

Here the constant α_1 is an integration constant which is not determined by the differential equation itself but can be found by other means [7]. It is given by

$$\alpha_1 = -\frac{e\sqrt{e}}{a^2 - e^2} + \frac{\partial e^2}{\partial a^2} \frac{a^2(b^2 - a^2)}{e\sqrt{e}} \quad (4.4)$$

and fulfills the following relation which we shall make use of later

$$\frac{\partial \alpha_1}{\partial a^2} = -\frac{1}{2} \frac{e}{\sqrt{e}} \frac{e^2 - a^2}{a^2} \rho_a, \quad \rho_a = \frac{a^2}{e^2} \frac{\partial e^2}{\partial a^2}. \quad (4.5)$$

In the following we will use the notation that the expansion coefficients of the function $\tilde{G}^{(0)}(p)$ are denoted as $\{\tilde{\alpha}_i\}$. Obviously the $\{\tilde{\alpha}_i\}$ appear from the $\{\alpha_i\}$ by the replacement $\nu \rightarrow 1 - \nu$. The relationship between the two sets of parameters can also be expressed in another often very useful way, namely via the identity (cf. to reference [7])

$$\tilde{G}_-^{(0)}(p) = -\cot\left(\frac{\nu\pi}{2}\right) g_+(p) G_+^{(0)}(p). \quad (4.6)$$

For instance this identity implies

$$\tilde{\alpha}_1 = -\left(\alpha_1 - \frac{\sqrt{e}}{e}\right). \quad (4.7)$$

Let us in this connection also note that we have the relations (with the obvious notation)

$$\tilde{e} = -\frac{ab}{e}, \quad \frac{\sqrt{e}}{e} = \frac{\sqrt{\tilde{e}}}{\tilde{e}} \quad (4.8)$$

as well as

$$\tilde{\rho}_a = 1 - \rho_a = \rho_b. \quad (4.9)$$

Here the first two relations are simple consequences of the identity (3.21). (We note that the sign of \sqrt{e} is determined by equation (3.23)). The first equality sign in the second relation follows from the relation $\tilde{e} = -ab/e$ while the second follows from the fact that the parameter α_1 should be symmetric in a and b . Inserting the expression (3.21) for e in (4.4) we get for $\frac{\tilde{\alpha}_1}{b}$

$$\frac{\tilde{\alpha}_1}{b} = E(i\nu K') + i\nu(E' - K') = Z(i\nu K') + i\nu \frac{\pi}{2K}. \quad (4.10)$$

Furthermore we find for $\tilde{\rho}_a$

$$\tilde{\rho}_a = \frac{1}{k'^2} \frac{\text{dn}(\nu K', k')}{\text{sn}(\nu K', k') \text{cn}(\nu K', k')} Z(\nu K', k'). \quad (4.11)$$

For our later consideration we shall also need to know α_2 . It reads

$$\alpha_2 = \frac{1}{2} (e^2 - \alpha_1^2 - a^2 - b^2). \quad (4.12)$$

In principle the boundary equations can be explicitly solved and the moments determined for any matrix model potential but of course the boundary equations become more and more involved as the degree of the potential increases. Let us point out that

our expressions are valid not only for any potential but also, at least formally, for any value of ν . Hence our formulas can be used to explore the hitherto unexplored regions of the parameter space, $n < -2$ and $n > 2$. While a priori nothing prevents us from choosing these unconventional ranges for n there is of course no guarantee that results obtained for $|n| > 2$ are physically meaningful. For example it would not be acceptable (at least immediately) if the endpoints of the cut turned out to be complex or if the eigenvalue distribution were not real and positive.

5 The string susceptibility

In this section we will derive a closed expression for the string susceptibility which we will make use of later when investigating the critical behaviour of the model. First let us introduce an overall coupling constant in front of our potential, i.e. let us replace $V(p)$ by $V(p)/T$ where T can be thought of as the cosmological constant or the temperature. The string susceptibility, $U(T)$, is then given as $U(T) = \frac{d^2}{dT^2}(T^2 F_0)$ where F_0 is the genus zero contribution to the free energy defined in equation (2.1). Knowing the string susceptibility we can, in case the model has a critical point at $T = T_c$, extract the value of the critical index γ_{str}

$$U(T) = \frac{d^2}{dT^2}(T^2 F_0) \sim (T_c - T)^{-\gamma_{str}} \quad (5.1)$$

To proceed with the derivation we first note that

$$T^2 \frac{dF_0}{dT} = \left\langle \frac{1}{N} \text{tr} V(M) \right\rangle_0 = \oint_{C_1} \frac{dp}{2\pi i} W(p) V(p) \quad (5.2)$$

where the subscript 0 refers to the genus zero contribution. Next we use the identity (proven in reference [7])

$$\frac{d}{dT}(TW(p)) = G^{(0)}(p) \quad (5.3)$$

to conclude that

$$\frac{d}{dT}(T^3 F_0) = \oint_{C_1} \frac{dp}{2\pi i} G^{(0)}(p) V(p) \quad (5.4)$$

and consequently that

$$\frac{d^2}{dT^2} \left(T^3 \frac{dF_0}{dT} \right) = T \frac{dU(T)}{dT} = \oint_{C_1} \frac{dp}{2\pi i} V(p) \left\{ \frac{\partial G^{(0)}(p)}{\partial a^2} \frac{da^2}{dT} + \frac{\partial G^{(0)}(p)}{\partial b^2} \frac{db^2}{dT} \right\}. \quad (5.5)$$

Let us rewrite $\frac{\partial G^{(0)}(p)}{\partial a^2}$ in the following way

$$\frac{\partial G^{(0)}(p)}{\partial a^2} = \frac{\partial}{\partial p} \left(c_1 \tilde{G}^{(0)}(p) + c_2 p G^{(0)}(p) \right). \quad (5.6)$$

The existence of such a relation follows from the fact that any solution of the homogeneous saddle point equation (2.10) can be parametrized in terms of any two other independent solutions (cf. to reference [7]). The two constants c_1 and c_2 can be determined using the fact that the right hand side of equation (5.6) should have the same asymptotic behaviour and analyticity structure as the left hand side. The asymptotic behaviour is determined by (4.1) and (4.4) and as regards the analyticity structure the requirement to be imposed is a behaviour of the type $(p - b)^{-1/2}$ as $p \rightarrow b$. We note that the determination of c_1 and c_2 is facilitated by the use of the relation (4.6). The two constants read

$$c_1 = \frac{1}{2} i \tan\left(\frac{\nu\pi}{2}\right) \frac{1}{b^2 - a^2} \frac{e^2 - a^2}{a^2} e\sqrt{e} \quad (5.7)$$

$$c_2 = -\frac{1}{2} \frac{1}{b^2 - a^2} \frac{e^2 - a^2}{a^2} \quad (5.8)$$

Now we can actually evaluate the integral (5.5); namely inserting the expression (5.6) for $\frac{\partial G^{(0)}(p)}{\partial a^2}$ and its equivalent for $\frac{\partial G^{(0)}(p)}{\partial b^2}$ in (5.5), integrating by parts and making use of the boundary equations we find

$$\frac{dU(T)}{dT} = \left(1 - \frac{n}{2}\right) \frac{1}{b^2 - a^2} \left\{ \frac{e^2 - a^2}{a^2} \frac{da^2}{dT} - \frac{e^2 - b^2}{b^2} \frac{db^2}{dT} \right\}. \quad (5.9)$$

This quantity is amazingly universal. First, it contains no explicit reference to the matrix model coupling constants $\{g_i\}$. Secondly, the given expression is valid for all values of n . This universality is another manifestation of the universality of the two-loop correlator of the model observed earlier [7]. For $n = 0$ the universality with respect to the coupling constants was discovered in reference [11]. In that case the above equation can be integrated exactly which gives

$$U(T) = 2 \log(b - a), \quad n = 0. \quad (5.10)$$

We note that for $n = 2$ special care must be taken. The parameter e diverges and the prefactor $\left(1 - \frac{n}{2}\right)$ goes to zero. The case $n = 2$ is not by any means a singular case, however. Only our parametrization is not well suited for this value of n . Actually as shown in references [6, 4, 7] for $n = \pm 2$ a simpler parametrization can be chosen. The limit $n \rightarrow 2$ of all expressions in the present parametrization are well defined and reproduce the results obtained in the simpler parametrization.

6 The gaussian potential

We will now solve in detail the $O(n)$ model in the case of a gaussian potential $V(M)$. Due to the interaction term in the action this is by no means a trivial case. On the

contrary one expects that the major features of the general model are reflected already in this simplest version. We choose our potential to be of the form

$$V(p) = \frac{1}{2T} (p - p_0)^2, \quad p_0 > 0 \quad (6.1)$$

and we remind the reader that in addition to the potential $V(p)$ the eigenvalues of the matrix M feel an attractive ($n > 0$) or repulsive ($n < 0$) force from their mirror images with respect to zero and that when one of the eigenvalues touches the origin the partition function (2.1) ceases to exist (cf. to reference [1]). With a potential like (6.1) one has a well around the point $p = p_0$ and one expects that by choosing T sufficiently small one can obtain a stable situation where the eigenvalues are confined to the well and located at a finite distance from the origin. Stated differently one expects that a stable 1-cut solution of the matrix model exists in some region of the coupling constant space. In the following we shall explore in detail the coupling constant space of the model and determine when such a stable solution exists. In particular we shall look for singular points, i.e. points where the solution ceases to exist or changes its nature. In order for a solution to make sense the endpoints of the cut a and b which are determined by the boundary equations (2.23) should come out real and positive. Let us take a look at these boundary equations. Deforming the contours in (2.23) to infinity and inserting the large p expansion for $G^{(0)}(p)$ and $\tilde{G}^{(0)}(p)$ we find that the boundary equations with the potential (6.1) read

$$-\tilde{\alpha}_1 + ip_0 \tan\left(\frac{\nu\pi}{2}\right) = 0, \quad (6.2)$$

$$p_0^2 \tan^2\left(\frac{\nu\pi}{2}\right) + \tilde{e}^2 = 2(2-n)T \quad (6.3)$$

where we on our way have made use of the relations (4.7) and (4.12). For given values of the parameters of the potential these equations determine a and b . However, due to the complexity of the equations (cf. to (3.21) and (4.10)) trying to solve directly for $a(p_0, T)$ and $b(p_0, T)$ is not a practicable way of proceeding. We shall hence take another line of action. Inserting the expressions (4.7) and (3.21) for α_1 and e in (6.2) and (6.3) and performing a few additional manipulations we arrive at the following two relations

$$b = \frac{ip_0 \tan\left(\frac{\nu\pi}{2}\right)}{E(i\nu K') + i\nu(E' - K')}, \quad (6.4)$$

$$T = \frac{1}{2(2-n)} \left\{ p_0^2 \tan^2\left(\frac{\nu\pi}{2}\right) + a^2 \operatorname{sn}^2(i\nu K') \right\}. \quad (6.5)$$

Let us fix the parameter p_0 . This corresponds to fixing the position of the potential well and does not influence qualitatively the features of the model. Then we have that for a

given value of the parameter $k = \frac{a}{b}$ the quantities b , T and a are uniquely determined. Our strategy for studying the above equations will hence be the following. We let k vary between 1 and 0 and determine for each of its values the corresponding values of a , b and T . In doing so we find $a(T)$ and $b(T)$ and miss no real solutions. It is easy to show that the relations (6.4) and (6.5) give real values for a , b and T for any $k \in [0, 1]$ and any $n \in] - \infty, \infty[$. These quantities are in addition positive except for certain ranges of k values when $n \in] - \infty, -2[$. We shall explain the situation in more detail later when we consider separately various ranges of n . Let us just note that bearing in mind the parametrization of $G^{(0)}(p)$ in section 3 the use of the quantity $k = \frac{a}{b}$ as the fundamental parameter is actually quite natural.

Let us now turn to determining the eigenvalue distribution. From (2.22) and (2.12) it follows that we have

$$W_s(p) = \frac{1}{T} W_1(p) \quad (6.6)$$

and hence according to (2.5) and (2.13)

$$\rho(\lambda) = \frac{1}{2\pi i} \frac{1}{T} W_1(\lambda) \Big|_{\lambda-i0}^{\lambda+i0} \quad (6.7)$$

Using (2.9), (2.20) and the rotated version of the saddle point equation (2.10) (cf. to reference [7]) we can rewrite (6.7) in the following way

$$\rho(\lambda) = \frac{i}{2\pi T} e^{i\nu\pi/2} \sqrt{\lambda} \tilde{G}_+^{(0)}(\lambda) \Big|_{\lambda-i0}^{\lambda+i0} \quad (6.8)$$

Now bearing in mind the properties of the mapping (3.1) this identity can also be expressed as

$$\rho(v(\lambda)) = \frac{i}{2\pi T} e^{i\nu\pi/2} (ab) \operatorname{cnu} \operatorname{dnu} \tilde{G}_+^{(0)}(u) \Big|_{K+iv}^{K-iv} \quad (6.9)$$

where v is related to λ by

$$\lambda = a \operatorname{sn}(K + iv), \quad v \in [0, K'] \quad (6.10)$$

Finally if we insert the explicit expression for $\tilde{G}_+^{(0)}(u)$ found in section 3 we arrive at the following expression for the eigenvalue distribution

$$\begin{aligned} \rho(v(\lambda)) = & \frac{1}{2\pi T} (ab)^{1/2} \frac{1}{2 \sin\left(\frac{\nu\pi}{2}\right)} \frac{\theta_4}{\theta_4\left(\frac{i\nu K'}{2K}\right)} \times \\ & \frac{1}{\theta_3\left(\frac{iv}{2K}\right)} \left\{ \theta_2\left(\frac{iv + i\nu K'}{2K}\right) e^{-\nu \frac{\pi v}{2K}} - \theta_2\left(\frac{iv - i\nu K'}{2K}\right) e^{\nu \frac{\pi v}{2K}} \right\} \quad (6.11) \end{aligned}$$

It is worthwhile noting that the expression (6.11) is valid for any value of ν . In particular (6.11) together with the two boundary equations (6.2) and (6.3) give a generalization

of the Wigner semi-circle law which is reproduced when n is set equal to zero ($\nu = 1/2$). As mentioned earlier we can of course not be sure that the solution makes sense for any range of the parameters ν and T . We shall address this aspect in detail later.

Let us finish this section by writing down the expressions for da^2/dT and db^2/dT for the gaussian potential (6.1). These can be found by differentiating (6.2) and (6.3) with respect to T and making use of the relation (4.5) and (4.9). They read

$$\frac{da^2}{dT} = 2(2-n) \frac{a^2 - e^2}{a^2 - b^2} \frac{1}{1 - \rho_a}, \quad \frac{db^2}{dT} = 2(2-n) \frac{b^2 - e^2}{b^2 - a^2} \frac{1}{\rho_a} \quad (6.12)$$

which implies that $\frac{dU(T)}{dT}$ takes the form

$$\frac{dU(T)}{dT} = (2-n)^2 \frac{1}{(b^2 - a^2)^2} \left\{ \frac{(e^2 - a^2)^2}{a^2} \frac{1}{1 - \rho_a} + \frac{(e^2 - b^2)^2}{b^2} \frac{1}{\rho_a} \right\} \quad (6.13)$$

From this expression we can readily read off where in the coupling constant space we can expect to encounter critical points; namely we see that the free energy can become singular when ρ_a is equal to zero or one, when it diverges, when $e^2 \rightarrow \infty$ and due to the relation (3.21) when $a \rightarrow 0$.

6.1 $n \in] - 2, 2[$

For $n \in] - 2, 2[$ we have $0 < \nu < 1$ and we can rewrite the equations (6.4) and (6.5) as

$$b = \frac{p_0 \tan\left(\frac{\nu\pi}{2}\right)}{\nu E' - E(\nu K', k') + \operatorname{dn}(\nu K', k') \operatorname{sc}(\nu K', k')}, \quad (6.14)$$

$$T = \frac{1}{2(2-n)} \left\{ p_0^2 \tan\left(\frac{\nu\pi}{2}\right) - k^2 b^2 \operatorname{sc}^2(\nu K', k') \right\}. \quad (6.15)$$

It is straightforward to show that the expressions (6.14) and (6.15) give real non negative values for b , T and $a (= k \cdot b)$ for all $k \in [0, 1]$. In figure 1 we have plotted $a(T)$ and $b(T)$ for various values of ν . The curves are obtained by letting k vary between zero and one. We have set p_0 equal to one. Note that the point $a = b = p_0$ corresponds to $k = 1$ and the point $a = 0$ to $k = 0$. In accordance with the nature of the interaction between an eigenvalue and its mirror image, we see that the values of a and b increase with ν for fixed T . (This can also easily be inferred from the boundary equations.) Let us stress that for any given value of k we have a closed expression for the eigenvalue distribution, namely (6.11).

Let us show how our formalism allows us to recover immediately the results of references [1, 2, 3, 4, 5] regarding the critical behaviour of the model for the here considered range of n -values. As mentioned earlier the free energy can become singular if ρ_a becomes equal to zero or one, if it diverges, if $e^2 \rightarrow \infty$ or if $a \rightarrow 0$. From the

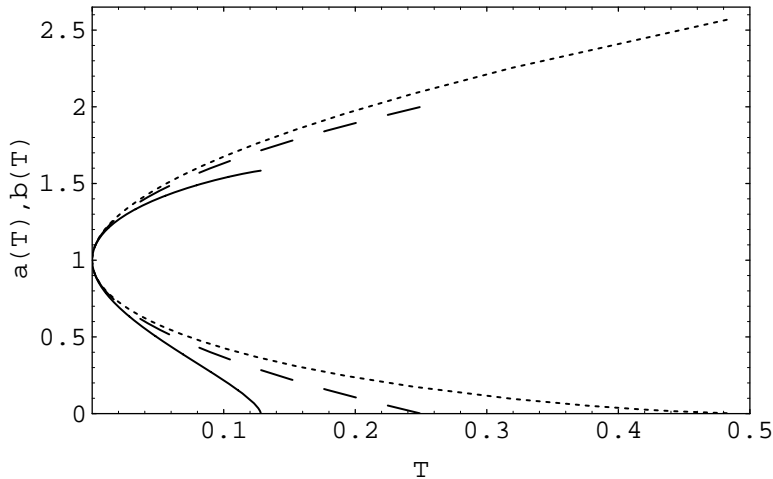


Figure 1: The variation of a (lower curves) and b (upper curves) as a function of T for $\nu = 1/10$ (full line), $\nu = 1/2$ (dashed line) and $\nu = 2/3$ (dotted line).

parametrization (3.21) it is obvious that e^2 remains finite for $\nu \in]0, 1[$ and from the expression (4.11) for ρ_a it follows that

$$0 < \rho_a < 1, \quad \text{for } 0 < \nu < 1. \quad (6.16)$$

(Actually $\rho_a = 1$ for $k = 1$ for all values of $\nu \in]0, 1[$ but this case is not interesting from the point of view of critical behaviour since it corresponds to $a = b = p_0, T = 0$.) Hence the only possible type of singular behaviour is associated with $a = 0$. Using (6.14) and (6.15) we find the corresponding critical values of b and T

$$b_c = p_0 \frac{\tan\left(\frac{\nu\pi}{2}\right)}{\nu}, \quad T_c = \frac{1}{2(2+n)} p_0^2. \quad (6.17)$$

From equation (3.21) it follows that in the limit $a \rightarrow 0$, e behaves as

$$e \sim 2ib \left(\frac{a}{4b}\right)^\nu \quad (6.18)$$

which according to (5.9) implies that

$$U(T) \sim a^{2\nu}. \quad (6.19)$$

Now, since $\rho_a \sim \nu$ (cf. to equation (6.18)) we get using (6.12)

$$(T_c - T) \sim a^{2-2\nu}. \quad (6.20)$$

Note that this scaling law is very clearly exposed in figure 1. The relation (6.20) together with (6.19) allow us to extract the value of γ_{str} associated with the critical point $a = 0$. We find

$$\gamma_{str} = -\frac{\nu}{1-\nu} \quad (6.21)$$

which indeed coincides with the result of references [1, 2, 3, 4, 5]. In addition, using the expression (6.11) one can also easily recover the expression for the eigenvalue density at the critical point found in reference [3].

6.2 $n = \pm 2$

For $n = \pm 2$ the equations (6.14) and (6.15) contain divergent factors and special care must be taken. As mentioned earlier for these values of n the present parametrization is not the optimal one and a simpler one can be chosen. However, the limits $n \rightarrow \pm 2$ of (6.14) and (6.15) are well defined and lead to the same equations as are obtained in the simpler parametrization [6, 1]

6.2.1 $n = -2$

For $n = -2$ we find from (6.14) and (6.15) taking the appropriate limit

$$b = \frac{2p_0}{\pi} K', \quad T = \frac{1}{2} \frac{p_0^2}{\pi^2} K' \left((k^2 + 1)K' - 2E' \right). \quad (6.22)$$

Furthermore we see from (3.21) that the parameter e is equal to zero and hence we can integrate exactly the expression in equation (5.9) which gives

$$U(T) = 2 \log(b^2 - a^2). \quad (6.23)$$

This is in accordance with the observation that the solution of the $n = -2$ case can be read off from the solution of the $n = 0$ case (cf. to references [4, 7]). From (6.23) and (6.12) it follows that singular behaviour can only occur if $\tilde{\rho}_a$ becomes equal to zero or one. For $\tilde{\rho}_a$ we find using (4.11)

$$\tilde{\rho}_a = \frac{k}{k'^2} \left[\frac{E'}{K'} - k^2 \right]. \quad (6.24)$$

This quantity is always less than one but becomes equal to zero as $k \rightarrow 0$. However, as is seen from (6.22) the point $k = 0$ corresponds to $b = T = \infty$ and is not interesting from the point of view of continuum behaviour. We note that $b_c = T_c = \infty$ is also what

one would expect from the formulas (6.17). We also note that it is due to our choice of a gaussian potential that the model has no critical point. It is well known that for potentials of degree larger than two there exist critical points which are characterized by having logarithmic scaling violations [3, 4].

6.2.2 $n = 2$

For $n = 2$ the expressions (6.14) and (6.15) turn into

$$b = \frac{\pi p_0}{2E'}, \quad T = \frac{1}{8} p_0^2 \left\{ 1 - \left(k \frac{K'}{E'} \right)^2 \right\}. \quad (6.25)$$

Furthermore, taking the limit $n \rightarrow 2$ of equation (6.13) leaves one with an expression for $\frac{dU(T)}{dT}$ which can be integrated exactly and leads to the following result

$$U(T) = 2 \left(\frac{\pi}{2K'} \right)^2 \log \left(\frac{a^2}{b^2 - a^2} \right). \quad (6.26)$$

Obviously $a = 0$ is a singular point. The corresponding critical values of b and T read

$$b_c = \frac{\pi p_0}{2}, \quad T_c = \frac{1}{8} p_0^2 \quad (6.27)$$

which is again in compliance with the general results (6.17). For the quantity $\tilde{\rho}_a$ we find using (4.11)

$$\tilde{\rho}_a = \frac{1}{k'^2} \left[1 - \frac{E'}{K'} \right]. \quad (6.28)$$

It is easy to see that $0 \leq \tilde{\rho}_a \leq 1/2$ and that $\tilde{\rho}_a = 0$ iff $k = 0$. Taking the limit $a \rightarrow 0$ we find

$$U(T) \sim \frac{1}{\log a} \quad (6.29)$$

where according to (6.25)

$$T_c - T \sim a^2 (\log a)^2 \quad (6.30)$$

6.3 $n > 2$

For $n > 2$ we set $\nu = i\bar{\nu}$ with $\bar{\nu}$ real. Hence we have

$$n = 2 \cosh(\bar{\nu}\pi). \quad (6.31)$$

With this parametrization we can write our two boundary equations (6.4) and (6.5) as

$$b = \frac{p_0 \tanh\left(\frac{\bar{\nu}\pi}{2}\right)}{E(\bar{\nu}K') + \bar{\nu}(E' - K')}, \quad (6.32)$$

$$T = \frac{1}{2(n-2)} \left\{ p_0^2 \tanh\left(\frac{\bar{\nu}\pi}{2}\right) - k^2 b^2 \operatorname{sn}^2(\bar{\nu}K') \right\}. \quad (6.33)$$

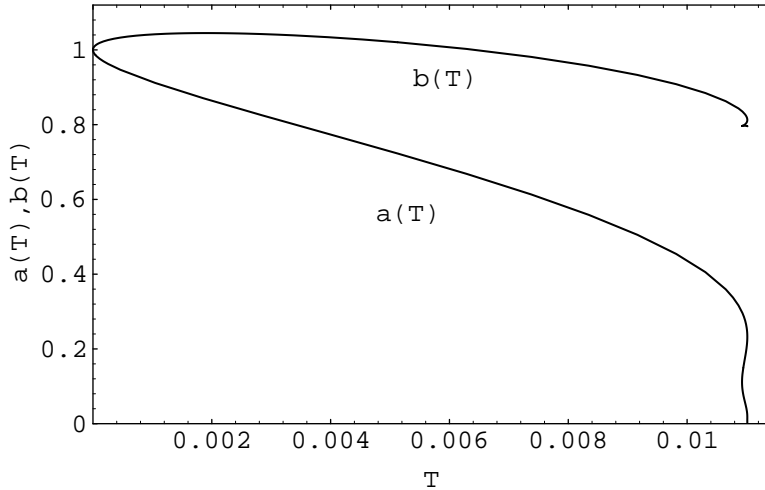


Figure 2: The parametric curves $(T(k), a(k))$ and $(T(k), b(k))$ for $k \in]0, 1]$ and $\bar{\nu} = 1.2$.

Also in this case one can easily convince oneself that the expressions (6.32) and (6.33) give rise to real non negative values of b and T for all $k \in [0, 1]$. Furthermore for each value of $\bar{\nu}$ there is a maximal temperature, namely

$$T_{max} = \frac{1}{2(2+n)} p_0^2. \quad (6.34)$$

We note the similarity with the expression for the critical value of T , (6.17) for $n \in]-2, 2[$. However, in the present case the maximum temperature is attained not only once but an infinite number of times corresponding to the infinite number of k values which solve the equations

$$\bar{\nu} K' = 2mK, \quad m = 1, 2, \dots \quad (6.35)$$

In other words for k larger than some critical value k_c given by

$$\frac{K'}{K}(k_c) = \frac{2}{\bar{\nu}} \quad (6.36)$$

an oscillation of the temperature begins. In figure 2 we have shown $a(T)$ and $b(T)$ for $\bar{\nu} = 1.2$. The curves have been produced by letting k vary between zero and one and as before p_0 has been set equal to one. In the parametric plot we see an oscillation of

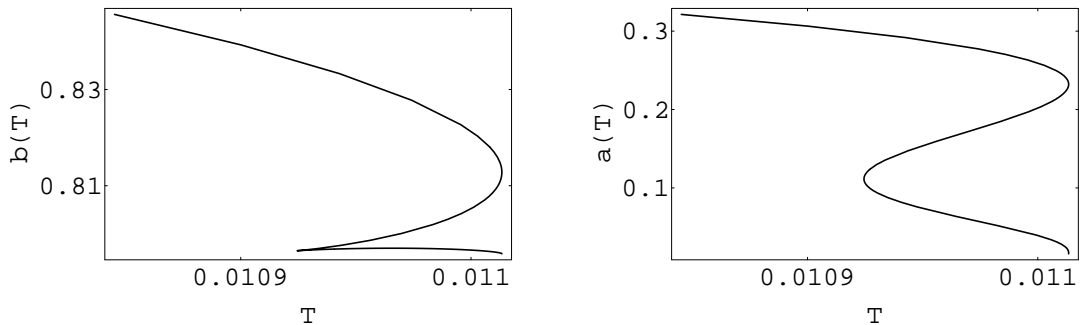


Figure 3: The parametric curves $(T(k), b(k))$ and $(T(k), a(k))$ for $k \in [0.02; 0.38]$ and $\bar{\nu} = 1.2$.

the endpoints a and b as a function of T . (Due to finite precision only a finite number of oscillations appear.) The right turning points for the oscillation correspond to the k -values which fulfill (6.35). At these points both $\frac{da^2}{dT}$ and $\frac{db^2}{dT}$ diverge. This divergence is due to the divergence of e . (We note that ρ_a also diverges at these points.) The left turning points correspond to the situation $\rho_a = 1$. The associated k values are the solutions to the equation

$$E(\bar{\nu}K') + \bar{\nu}(E' - K') = \frac{\text{sn}(\bar{\nu}K') \text{dn}(\bar{\nu}K')}{\text{cn}(\bar{\nu}K')}. \quad (6.37)$$

At these turning points $\frac{da^2}{dT}$ diverges but $\frac{db^2}{dT}$ stays finite. The value of T is the same at all right turning points but the value of T at the left turning points increases as k decreases. We note that there is no divergence of $\frac{db^2}{dT}$ when ρ_a tends to zero. On the contrary these are the points where $\frac{db^2}{dT} = 0$. They are given by

$$\bar{\nu}K' = (2m + 1)K. \quad (6.38)$$

The vanishing of $\frac{db^2}{dT}$ at these points is due to the fact that e becomes equal to b . There is no singular behaviour associated with these points. In figure 3 we have enlarged the region of figure 2 where the oscillations of $a(T)$ and $b(T)$ occur and we see that the features of the curves are in accordance with our analysis. This picture of course implies that at a certain temperature, T_0 , namely the temperature at the first of the left turning points, the boundary equations start to have more than one solution and as T approaches T_{max} the number of possible solutions goes to infinity. One may

wonder if there is a criterion that would allow one to pick out one of these solutions. We will argue that there is; we conjecture that only the solutions corresponding to the upper branch of the curves for $a(T)$ and $b(T)$ give rise to a positive eigenvalue distribution. In other words the model is only well defined if $k_c \leq k \leq 1$ where k_c is given by (6.36). To support this statement, let us determine the eigenvalue distribution at various selected values of k . First, let us consider the k values given by (6.38). As mentioned above at these values of k one has $\frac{db^2}{dT} = 0$. Obviously the first such k gives a set of points $(a(T), b(T))$ which lie on the upper branches of the curves for $a(T)$ and $b(T)$. In general one will encounter one of the points corresponding to (6.38) each time one moves from a left turning point to a right turning point in the direction of decreasing k (cf. to figure 3). Inserting the equality (6.38) in the expression (6.11) for the eigenvalue distribution we get

$$\rho(v(\lambda)) = \frac{1}{\pi T} a (-1)^m \frac{1}{\sinh\left(\frac{\bar{\nu}\pi}{2}\right)} \cos\left(\bar{\nu} \frac{\pi v}{2K}\right) \frac{\text{sn}(v, k')}{\text{dn}(v, k')} \quad (6.39)$$

where λ is related to v by

$$\lambda = a \text{sn}(K + iv), \quad v \in [0, K']. \quad (6.40)$$

Bearing in mind the relation (6.38) it is obvious that the eigenvalue distribution given by (6.39) remains positive for all v in the given interval only if $m = 0$ i.e. only for the value of k which corresponds to a point on the upper branch of the solution. Next, let us determine the eigenvalue distribution corresponding to the right turning points given by (6.35). Inserting (6.35) in (6.11) gives

$$\rho(v(\lambda)) = \frac{1}{\pi T} a (-1)^{m+1} \frac{1}{\sinh\left(\frac{\bar{\nu}\pi}{2}\right)} \sin\left(\bar{\nu} \frac{\pi v}{2K}\right) \frac{1}{\text{dn}(v, k')} \quad (6.41)$$

with v being related to λ as in (6.40). In this case we see that the eigenvalue distribution is positive only if $m = 1$ i.e. only at the first of the right turning points. Furthermore (6.39) and (6.41) show that as k decreases the eigenvalue distribution oscillates more and more rapidly from positive to negative values, the number of oscillations going to infinity as $k \rightarrow 0$. In particular it is obvious that the model makes no sense at the point $k = 0$ which would be the naive analytical continuation of the critical point for $n \in]-2, 2[$ to $n > 2$.

However, now that we have rendered probable that the model is actually well defined for $k_c \leq k \leq 1$ we shall determine the critical index γ_{str} associated with the obviously singular point $k = k_c$. For that purpose we let $k \rightarrow (k_c)_+$ and denote the value of T corresponding to k_c as T_c . (We note that $T_c = T_{max}$ (cf. to equation (6.34)). From the

boundary equation (6.5) we see that

$$T_c - T = \frac{\tilde{e}^2}{2(2-n)}. \quad (6.42)$$

Then introducing \tilde{e} in stead of e in relation (6.13) we get

$$\frac{dU(T)}{dT} = \frac{1}{4} \frac{1}{(b^2 - a^2)^2} \left\{ \frac{(b^2 - \tilde{e}^2)^2}{1 - \rho_a} a^2 + \frac{(a^2 - \tilde{e}^2)^2}{\rho_a} b^2 \right\} \frac{1}{(T_c - T)^2}. \quad (6.43)$$

Now, from the relation (4.7) we find

$$(b^2 - a^2)(1 - \rho_a) = \frac{\sqrt{\tilde{e}}}{\tilde{e}} \tilde{\alpha} + b^2 - \tilde{e}^2 \sim c \cdot (T_c - T)^{-1/2}. \quad (6.44)$$

We note that the constant, c , entering this relation is positive as it should be. The quantity $\tilde{\alpha}$ is negative as can be seen from the relation (6.4). The parameter \tilde{e} is likewise negative which follows from the parametrization (3.21) and the fact that we study the limit $k \rightarrow (k_c)_+$ with k_c given by (6.36). Finally $\frac{\sqrt{\tilde{e}}}{\tilde{e}}$ is a positive quantity which is a consequence of the recipe (3.23) and the identity (6.36). (We note that the sign of $\frac{\sqrt{\tilde{e}}}{\tilde{e}}$ changes from one right turning point to the next.) Combining (6.44) and (6.43) we get

$$\frac{dU(T)}{dT} \sim (T_c - T)^{-3/2} \quad (6.45)$$

which means that

$$\gamma_{str} = +\frac{1}{2} \quad (6.46)$$

6.4 $n < -2$

For $n < -2$ we set $\nu = 1 + i\tilde{\nu}$ with $\tilde{\nu}$ real. Hence we have

$$n = -2 \cosh(\tilde{\nu}\pi) \quad (6.47)$$

and we can write the two boundary conditions (6.4) and (6.5) as

$$b = \frac{p_0 \coth\left(\frac{\tilde{\nu}\pi}{2}\right)}{E(\tilde{\nu}K') + \operatorname{cn}(\tilde{\nu}K') \operatorname{ds}(\tilde{\nu}K') + \tilde{\nu}(E' - K')}, \quad (6.48)$$

$$T = \frac{1}{2(2-n)} \left\{ -p_0^2 \coth^2\left(\frac{\tilde{\nu}\pi}{2}\right) + \frac{b^2}{\operatorname{sn}^2(\tilde{\nu}K')} \right\}. \quad (6.49)$$

Obviously these equations give real values for a , b and T for all values of $k \in [0, 1]$. In order to describe in more detail the behaviour of a , b and T as a function of k , let us take a look at some selected points. First, let us consider the k values characterized by

$$\tilde{\nu}K' = (2m+1)K, \quad m = 0, 1, \dots \quad (6.50)$$

These points are analogous to the points given by (6.38) for $n > 2$. Only the roles of a and b are interchanged. In the present case we have that $e^2 = a^2$, $1 - \rho_a = 0$ and we find that $\frac{da^2}{dT}$ vanishes while $\frac{db^2}{dT}$ stays finite. As in the case $n > 2$ there is no singular behaviour associated with these points. We note that a , b and T are always positive and finite when k fulfills (6.50). Next, let us consider the points given by

$$\tilde{\nu}K' = 2mK, \quad m = 1, 2, \dots \quad (6.51)$$

When one approaches such a point in the direction of decreasing (increasing) k one finds $b \rightarrow 0_-$, $T \rightarrow 0_+$, ($b \rightarrow 0_+$, $T \rightarrow 0_+$). Thus b changes sign at these points. Obviously b must also change sign at a series of k values each of which lies in between two successive solutions to (6.50) and (6.51). These k values are characterized by the denominator in the expression (6.48) becoming zero, i.e.

$$E(\tilde{\nu}K') + \text{cn}(\tilde{\nu}K') \text{ds}(\tilde{\nu}K') + \tilde{\nu}(E' - K') = 0. \quad (6.52)$$

At these points $\rho_a = 0$ and both a , b and T diverge. It is easy to convince oneself that T always stays positive and that there are no other points than those given by (6.51) and (6.52) where b can change sign. In summary, b starts out positive and equal to p_0 at $k = 1$, goes to $+\infty$ as k approaches the first k which solves (6.52). In the same interval T increases from 0 to ∞ . At the first solution to (6.52) a cycle starts. When k varies between one solution to (6.52) and the next, b and T behave in the following way. Precisely at the solution to (6.52) b jumps from $+\infty$ to $-\infty$. Hereafter it increases as k decreases and tends to zero when k approaches a solution to (6.51). In the same k -interval T decreases from ∞ to 0. Then b and T increase again and both go to $+\infty$ when k approaches the next solution to (6.52). These cycles are the analogues of the oscillations of $b(T)$ and $a(T)$ seen for $n > 2$. In particular we are lead to the conclusion that as in the case $n > 2$, the model makes no sense in the point $k = 0$ which would be the naive analytical continuation of the critical point for $n \in [-2, 2]$ to $n < -2$. However, unlike what was the situation for $n > 2$, in the present case there are no interesting new singular points. The points $1 - \rho_a = 0$ are, as mentioned above, not singular and at the points $\rho_a = 0$ both a , b and T diverge. The final possibility $\tilde{e} \rightarrow \infty$, ($\rho_a \rightarrow -\infty$) corresponds to the points given by (6.51) and here both a , b and T vanish.

Even though we have now made clear that our model has no interesting new critical points for $n < -2$, let us spend a few lines discussing where in the coupling constant space the model has a meaning for this range of n 's. From the analysis above it follows that for each value of T there exists an infinite number of solutions for a and b (even when we reject the obviously unphysical solutions with a and b negative). As in the case $n > 2$ we will argue that only the solutions corresponding to the first

branch of the curves $a(T)$ and $b(T)$ make sense, i.e. the model is only well defined for $k_{min} \leq k \leq 1$ where k_{min} is the solution of the first of the equations (6.52). To support this statement, let us as before determine the eigenvalue distribution at a set of selected k values; namely the k values which solve the equations (6.50). All of these give a and b but only the first one belongs to the interval $[k_{min}, 1]$. Inserting the identity (6.50) in the expression (6.11) for the eigenvalue density we find

$$\rho(v(\lambda)) = \frac{1}{\pi T} (b^2 - a^2)^{1/2} \frac{1}{\cosh\left(\frac{\tilde{v}\pi}{2}\right)} (-1)^m \sin\left(\tilde{v} \frac{\pi v}{2K}\right) \frac{\text{cn}(v, k')}{\text{dn}(v, k')} \quad (6.53)$$

with v related to λ by

$$\lambda = a \text{sn}(K + iv), \quad v \in [0, K'] \quad (6.54)$$

In accordance with our statement the eigenvalue distribution is positive only at the first of the here considered k values.

7 Conclusion

Having determined explicitly the auxiliary function we have completed our solution of the $O(n)$ model on a random lattice. Our solution allows an exhaustive analysis of the model for any value of n and any potential. We have carried out this analysis for the simplest “gaussian” version of the model which corresponds to a collection of self-avoiding non-intersecting loops densely packed on a randomly triangulated surface. Our analysis showed that the model is well defined in a certain region of the coupling constant space both for $n < -2$ and $n > 2$. For $n < -2$ we found no new critical points while for $n > 2$ we found new critical points at which the string susceptibility exponent, γ_{str} takes the value $+\frac{1}{2}$. We expect to encounter the same situation if we include higher order terms in the potential, i.e. we expect that the model will still be well defined in a certain region of the coupling constant space and that there will be new, possibly multi-critical points for $n > 2$. It is tempting to speculate about a connection between $n > 2$ and 2D quantum gravity coupled to matter fields with $c > 1$. Unfortunately there does not exist any mapping of the $O(n)$ model for $n > 2$ onto a model which is known to have $c > 1$ on a regular lattice. Let us note anyway that recent numerical simulations show that γ_{str} changes rather rapidly from 0 to $+\frac{1}{2}$ when one crosses the $c = 1$ barrier [12].

A less speculative unclarified point concerns the relation of the $O(n)$ model on a random lattice with the theory of integrable hierarchies. As mentioned earlier, in the one matrix model ($n = 0$) case the function that we have denoted as our auxiliary function played an essential role in revealing the Virasoro structure of the model and

establishing its connection with the kdV hierarchy [8]. In the general case the integrable structure underlying the $O(n)$ model should likewise be encoded in the auxiliary function. The structure is well known when ν is rational where the kdV_n hierarchies appear but the precise translation from matrix model variables to continuum time variables is lacking. One might also ask whether there are integrable hierarchies associated with the continuum theories corresponding to non-rational values of ν , not to mention imaginary values of ν .

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References

- [1] I. Kostov, Mod. Phys. Lett. A4 (1989) 217
- [2] B. Duplantier and I. Kostov, Phys. Rev. Lett. 61 (1988) 1433
- [3] M. Gaudin and I. Kostov, Phys. Lett. B220 (1989) 200
- [4] I. Kostov and M. Staudacher, Nucl. Phys. B384 (1992) 459
- [5] B. Eynard and J. Zinn-Justin, Nucl. Phys. B386 (1992) 55
- [6] M. Gaudin, *unpublished*
- [7] B. Eynard and C. Kristjansen, Nucl. Phys. B455 (1995) 577
- [8] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6 (1991) 4385
R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. B348 (1991) 435
Yu. Makeenko, A. Marshakov, A. Mironov and A. Morozov, Nucl. Phys. B356 (1991) 574
- [9] E. Brezin, C. Itzykson, G. Parisi and J.-B. Zuber, Comm. Math. Phys. 59 (1978) 35
- [10] J. Ambjørn, L. Chekhov, C. Kristjansen, Yu. Makeenko, Nucl. Phys. B404 (1993) 127
- [11] J. Ambjørn, J. Jurkiewicz and Yu. Makeenko, Phys. Lett. B251 (1990) 517
- [12] J. Ambjørn, J. Jurkiewicz and Y. Watabiki, Nucl. Phys. B454 (1995) 313