

SELF-AVOIDING POLYMERIZED MEMBRANES

François DAVID

*Service de Physique Théorique, CE de Saclay
91191, Gif-sur-Yvette CEDEX, FRANCE*

Recent progresses in the understanding of the scaling behavior of self-avoiding flexible polymerized membranes (tethered manifolds) are reviewed (joint works with B. Duplantier and E. Guitter, and with K. Wiese). They rely on a new general renormalization group approach for a class of models with non-local singular interactions. This approach allows to prove the existence of a ϵ -expansion for the scaling exponents, and validates the one loop results obtained by direct renormalization methods. Applications of the method to polymerized membranes at the tricritical Θ -point are presented.

1 What are polymerized membranes ?

Polymerized membranes, also called tethered membranes, are an interesting model for two-dimensional flexible polymerized films ¹. In particular, they generalize the well-known problem of the statistical behavior of long chains, i.e. of polymers. However, even simple properties, such as the statistical properties at thermal equilibrium of a single membrane, are quite difficult to study theoretically, and require the development of new techniques. I shall review some recent progresses made with B. Duplantier and E. Guitter ², and some applications of these techniques with K. Wiese ³.

Let me consider an ideal flexible two dimensional network fluctuating in three dimensional space, characterized by: *(i)* a fixed and regular internal structure with a fixed connectivity (for instance a triangular lattice), and *(ii)* negligible bending rigidity. The only forces between elements of the network are the elastic forces (described for instance by a harmonic potential between nearest neighbors). If self-avoiding interactions are not taken into account, one deals with a “phantom” membrane, which is known to be, for vanishing or small enough bending rigidity, in a crumpled phase. The large scale fluctuations are described by a Gaussian model. Such a model is physically different from the

STATPHYS19, Xiamen, China, 31 July - 4 August, 1995

random surfaces used in models of fluid two-dimensional films (for instance for vesicles or micro-emulsions). For these models, the bending rigidity is important and the fluid character of the film is reproduced by a fluctuating connectivity of the lattice.

Taking into account the self-avoiding interactions (which are short ranged in the three dimensional embedding space, but involve elements of the lattice which can be arbitrarily far apart in term of the internal lattice distance) one expects that the long range properties of the membrane will be changed. For instance, the average size $\langle R \rangle$ of the membrane in physical space is expected to scale with the internal size L of the lattice as

$$\langle R \rangle \propto L^\nu \quad 0 < \nu < 1 \quad (1)$$

if the membrane is still crumpled, but swollen by self-avoidance. The membrane may even become flat (this is observed in numerical simulations of self-avoiding tethered membranes in three dimensions); the exponent ν is then equal to 1.

2 An Edwards Model for D -dimensional polymerized membranes:

In the continuous model introduced in ^{4,5}, the field $\vec{r}(x)$ describes the embedding of a D -dimensional space, whose points are labeled by the coordinate x , into the external d -dimensional space, whose points are labeled by the vector \vec{r} . The Hamiltonian H (the free energy for a configuration r) is the sum of a Gaussian elastic energy and of a 2-body repulsive interaction, proportional to the coupling constant b :

$$H[\vec{r}] = \int d^D x \frac{1}{2} (\nabla_x \vec{r})^2 + b \int d^D x \int d^D y \delta^d(\vec{r}(x) - \vec{r}(y)) . \quad (2)$$

The internal dimension D may be taken as a continuous parameter, interpolating between polymers ($D = 1$) and membranes ($D = 2$). The issue is to compute with this model critical exponents describing the scaling behavior of large membranes, for instance the exponent ν (related to the fractal dimension d_f of the membrane by $\nu = D/d_f$), and the configuration exponent γ , related to the scaling of the partition function Z of a finite membrane with internal extent L by

$$Z \propto L^{\gamma-1} \text{constant}^{L^D} . \quad (3)$$

The mean field exponents are obtained by setting $b = 0$ and one obtains $\nu_0 = (2 - D)/2$ and $\gamma_0 = 1 - d(2 - D)/2$.

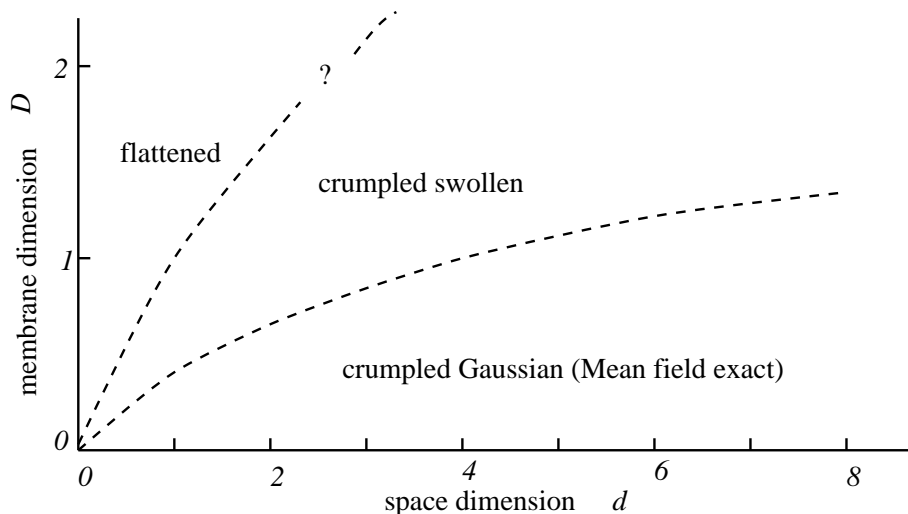


Figure 1: d - D plane

Dimensional analysis shows that mean field theory is invalid if ϵ , the engineering dimension of b , is positive

$$[b] = \epsilon = 2D - d(2 - D)/2 > 0 . \quad (4)$$

The general picture of the expected behavior as a function of the internal dimension D and of the external dimension d is presented in Fig. 1

A natural idea is to compute the corrections to mean field by a ϵ -expansion à la Wilson-Fisher. Using the so-called direct renormalization, inspired from polymer theory ⁶, the first explicit calculations have been performed to first order in ϵ by Aronowitz & Lubensky ⁴ and by Kardar & Nelson ⁵. The basic idea of this method is to perform explicit perturbative calculations for a finite membrane and for $\epsilon > 0$. Perturbation theory is then UV and IR finite, but has UV divergences when $\epsilon \rightarrow 0$. These poles in $1/\epsilon$ can be removed by reexpressing the observables in terms of adequate dimensionless renormalized quantities, such as the 2nd virial coefficient. The internal size L of the membrane plays the role of a renormalization mass⁻¹ scale, and the renormalization group equations can thus be obtained, by writing the L dependence of the renormalized theory. The consistency of this procedure at first order has been checked by explicit calculations by Duplantier, Hwa & Kardar ⁸, but its consistency at all orders cannot be proved by the de Gennes trick valid for polymers (the scaling limit of the self-avoiding walk can be mapped into a local

field theory with $O(n)$ symmetry in external d -dimensional space, in the limit $n \rightarrow 0$, and standard renormalization group theory is then applicable).

3 Renormalization Theory and Renormalization Group for non-local Theories:

Instead of using direct renormalization, we succeeded in proving the renormalizability of the model (2), by considering it as a non-local field theory in D dimensions². Perturbation theory is obtained by expanding the observables as power series in b . The bi-local “interaction vertex” (in field theoretic language I denote it a bilocal operator) is written in Fourier transform as

$$\begin{array}{c} \bullet \\ \hline x \end{array} \begin{array}{c} \bullet \\ \hline y \end{array} = \delta^d(\vec{r}(x) - \vec{r}(y)) = \int d^d k e^{i\vec{k}(\vec{r}(x) - \vec{r}(y))} . \quad (5)$$

It can be viewed in a Coulomb gas representation as the integral over the “charge” \vec{k} of a neutral “dipole” with charge $+\vec{k}$ at x and charge $-\vec{k}$ at y . The term of order b^K in the perturbative expansion of the partition function (as well as of other observables) involves K dipoles $(x_1, y_1), \dots, (x_K, y_K)$. The integration over the charges $\vec{k}_1, \dots, \vec{k}_K$ gives an integral over the positions of the dipoles of the determinant of the “dipole energy” quadratic form Q

$$\int \dots \int \prod_{i=1}^K d^D x_i d^D y_i \det [Q[x_i, y_i]]^{-d/2} . \quad (6)$$

Q is a $K \times K$ matrix such that $\sum_{i,j=1}^K \vec{k}_i Q_{ij} \vec{k}_j$ is the Coulomb energy (in D -dimensions) of the K dipoles. Each Q_{ij} is a linear combination

$$Q_{ij} = G_0(x_i, x_j) + G_0(y_i, y_j) - G_0(x_i, y_j) - G_0(x_j, y_i) . \quad (7)$$

of the Coulomb potentials G_0 between endpoints of the dipoles i and j

$$G_0(x, x') = \langle r(x)r(x') \rangle_0 = \frac{\Gamma((D-2)/2)}{4\pi^{D/2}} |x - x'|^{2-D} . \quad (8)$$

The Coulomb potential is properly defined for $0 < D < 2$ by analytic continuation. It is then negative, it vanishes for $x = x'$ (for $D > 2$ it diverges), but as for $D > 2$ it decreases at large distances. The integration over the $2K$ points in non-integer D -dimensional space can also be defined properly by analytic continuation in D and the use of distance geometry. This amounts to

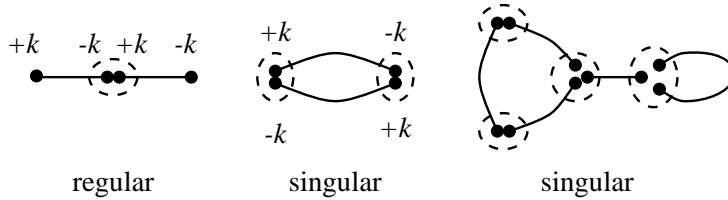


Figure 2: A non-neutral regular configuration and two neutral UV singular configurations

replace the integration over the $2K \times D$ coordinates by an integration over the $K \times (2K - 1)$ scalar distances between these points.

One can show that short distance UV singularities occur in the integrals, when the determinant $\det[Q]$ vanishes. This occurs if and only if some of the end-points of (not necessarily the same) dipoles coincide, forming “atoms”, and so that the dipoles form “molecules, and moreover if one can assign non-zero charges \vec{k}_i to the dipoles while the atoms stay neutral. This condition is more easily depicted graphically on Figure 2.

The associated singularities of these integrals are related to the behavior at short distance of the expectation value (with respects to the free Gaussian model) of products of bilocal operators as given by Equ. 5. One can show that this short distance behavior is encoded in a multilocal product expansion (MOPE), which generalizes Wilson’s operator product expansion. Let me give two examples:

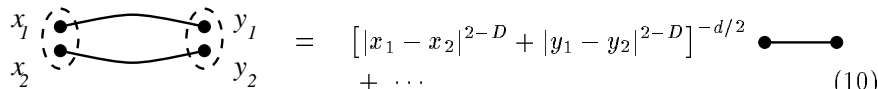
When the two points x and y of the bi-local interaction operator tend towards a single point, this operator can be expanded in terms of local operators involving derivatives of the field \vec{r} . The first terms of the expansion are explicitly (omitting unimportant coefficients)

$$\begin{aligned}
 \begin{array}{c} \text{loop} \\ \text{---} \\ \bullet \quad \bullet \\ x \quad y \end{array} &= |x - y|^{\epsilon - 2D} \mathbf{1} \\
 &+ |x - y|^{\epsilon - D - 2} (x^\alpha - y^\alpha) (x^\beta - y^\beta) : \nabla_\alpha \vec{r} \nabla_\beta \vec{r} : \\
 &+ \dots \tag{9}
 \end{aligned}$$

$\mathbf{1}$ is the identity operator (its expectation value is 1), the $: : \nabla_\alpha \vec{r} \nabla_\beta \vec{r} :$ denotes the normal ordering subtraction prescription required to deal properly with the UV singularities contained in $\nabla_\alpha \vec{r} \nabla_\beta \vec{r}$.

The second example is less simple, and shows that when the end-points of two bilocal operators tend pairwise towards two different points, this generates

again bilocal operators



$$\begin{array}{c} x_1 \\ \bullet \\ \bullet \\ x_2 \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} y_1 \\ \bullet \\ \bullet \\ y_2 \end{array} = \left[|x_1 - x_2|^{2-D} + |y_1 - y_2|^{2-D} \right]^{-d/2} \bullet \text{---} \bullet + \dots \quad (10)$$

This structure is generic, and products of local and bilocal operators generate multilocal operators of the general form

$$\Phi\{x_1, \dots, x_P\} = \int d^d \vec{r}_0 \prod_{i=1}^P [(\nabla_{\vec{r}_0})^{m_i} \delta^d(\vec{r}_0 - \vec{r}(x_i)) A_i(x_i)] . \quad (11)$$

where the $A_i(x_i)$ are local operators, which can be decomposed into products of multiple x -derivatives of \vec{r} . The m_i are integers. For $P = 1$ and $m = 0$ one recovers local operators $A(x)$ ($m > 0$ gives 0). For $P = 2$, $m_1 = m_2 = 0$ and $A_1 = A_2 = \mathbf{1}$ one recovers the bilocal interaction operator, etc. . . These operators have a very special form: they can be viewed as a local convolution in the external d -dimensional \vec{r} space of a non-local product (in the internal D -dimensional space) of the P local operators A_i .

The MOPE implies that the formalism of renormalization theory and of renormalization group equations, which has been developed for local quantum field theories, can be adapted for this model. One is in fact interested in the IR scaling behavior of the lattice model, when some length scale L goes to ∞ . This lattice model is described by the Hamiltonian (2), with a short distance lattice cut-off a . To study this IR limit it is equivalent to look at the UV continuum limit of the model when the physical length scale L is kept fixed, while the UV cut-off a goes to 0. In this limit one can construct, via renormalization, a finite renormalized theory with $a = 0$, which obeys renormalization group equations. From these equations, one recovers the large distance behavior of the lattice model I started from. The procedure works well in perturbation theory when one is close to the upper critical dimension, i.e. for ϵ small, and it leads to the ϵ -expansion.

In our case, the MOPE can be used to determine, by power counting, which multilocal operators are relevant and give UV singularities (poles in $1/\epsilon$). Then one can also show that these poles can be subtracted by adding to the Hamiltonian (2) counterterms proportional to the marginally relevant multilocal operators, leading to the UV finite renormalized theory. For the model of self-avoiding surfaces, this analysis shows that the UV divergences are associated only with local and bilocal operators, as depicted on Fig. 3, and that only three operators are relevant: the identity operator $\mathbf{1}$, the elastic energy operator $(\nabla \vec{r})^2$ and the bilocal operator $\delta^d(\vec{r}(x) - \vec{r}(y))$. $\mathbf{1}$ is strongly

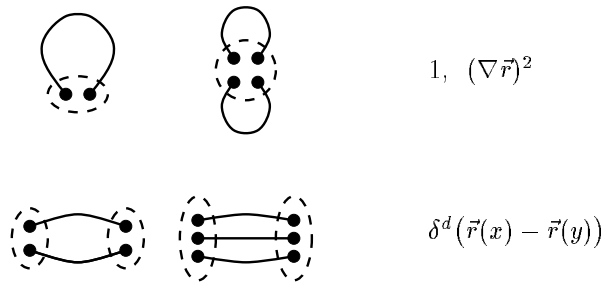


Figure 3: UV divergent configurations and the associated relevant operators

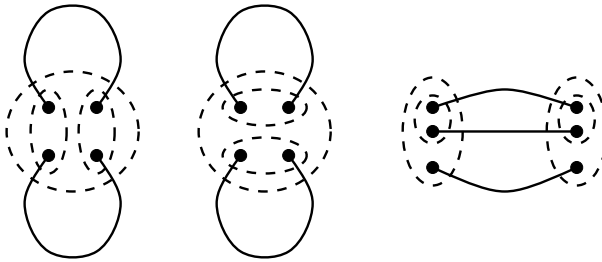


Figure 4: Examples of nested singular configurations

relevant, and gives power-like UV divergences proportional to a^{-D} (a being a short-distance cut-off). The two other operators are superficially relevant, they give logarithmic UV divergences or equivalently poles in $1/\epsilon$ at $\epsilon = 0$.

The fact that the so-called superficial divergences, associated to a global contraction of points towards a singular configuration, can be subtracted by counterterms is a consequence of the MOPE. A complete proof of the renormalizability of the theory is possible, but much more delicate. It requires a control of the subdivergences coming from successive contractions associated to nested singular configurations, such as those depicted of Fig. 3 .

4 Scaling for infinite self-avoiding membranes

A first application of this formalism is the derivation of scaling laws. Since the model is renormalizable (at least perturbatively), it can be made UV finite (for $\epsilon \simeq 0$) by introducing two counterterms in the Hamiltonian. The new

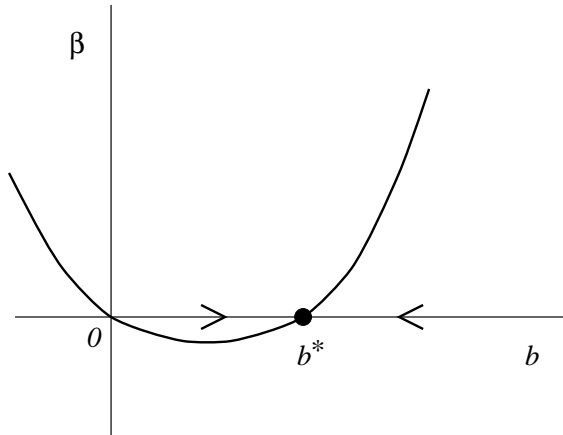


Figure 5: The β -function and the RG flow for $\epsilon > 0$.

renormalized Hamiltonian is of the form

$$H[\vec{r}] = \frac{Z}{2} \int d^D x (\nabla \vec{r})^2 + b_R \mu^\epsilon Z_b \int \int d^D x d^D y \delta(\vec{r}(x) - \vec{r}(y)) . \quad (12)$$

b_R is the dimensionless renormalized coupling constant (the perturbative expansion in b_R is UV finite order by order). Z is a “wave-function” renormalization factor and Z_b a coupling constant renormalization factor, both are perturbative series in b_R , with poles up to degree $1/\epsilon^{K-1}$ at order K . μ is the renormalization momentum scale. As for ordinary local theories, such as the Landau-Ginzburg-Wilson Φ^4 Hamiltonian, one can change b_R and \vec{r} in Equ. 12 into bare quantities in order to rewrite the renormalized Hamiltonian as a bare Hamiltonian given by Equ. 2. The renormalization group β -function and the anomalous dimension γ of the field \vec{r} are now defined in the standard way

$$\beta(b_R) = \mu \frac{\partial}{\partial \mu} b_R \Big|_{\text{bare}} \quad ; \quad \gamma(b_R) = -\frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z \Big|_{\text{bare}} . \quad (13)$$

The β -function is found to be of the form

$$\beta(b_R) = -\epsilon b_R + \mathbf{c} b_R + \mathcal{O}(b_R^2) \quad ; \quad \mathbf{c} \text{ positive constant} , \quad (14)$$

and therefore there is, at least for small $\epsilon > 0$, an IR attractive fixed point $b_R^* = \mathcal{O}(\epsilon)$, which governs the scaling behavior of self-avoiding polymerized surfaces

at large distance. The existence of this fixed point ensures the universality of this non-trivial scaling for $\epsilon > 0$, and that no new interactions, possibly non-local in external space, are generated by the RG transformations.

The explicit calculation leads to the same results for the scaling exponents ν and γ than the direct renormalization method at first order in ϵ . With this method higher order calculations are in principle feasible, but technically quite difficult. In particular, already at second order the RG functions cannot be expressed analytically, and numerical integration methods have to be developed. Work is in progress to compute the scaling exponents at order ϵ^2 .

5 Finite Size Scaling and Direct Renormalization

The model given by Equ. 2 describes an infinite membrane with flat internal metric, corresponding to an infinite and regular flexible lattice. Finite membranes are described by a similar model, but the D -dimensional membrane M is now embodied with a fixed non-trivial Riemannian metric $g_{\alpha\beta}(x)$, with curvature R (examples are closed membranes with the topology of the sphere \mathcal{S}_D or the torus \mathcal{T}^D), and may have a boundary ∂M (open membrane with the topology of the disk for instance). A similar analysis can be performed for such models, and the MOPE structure of short distance singularities is still valid, but new local operators $A(x)$, which depend on the internal metric on M and on the boundary ∂M , such as the curvature R , appear in the MOPE and in Equ. (11). The renormalized Hamiltonian now contains at least five operators and five independent renormalization factors Z

$$\begin{aligned}
 H[\vec{r}] = & \int_M Z \mathbf{1} + \int_M Z (\nabla \vec{r})^2 + \int \int_M Z b \delta^d(\vec{r} - \vec{r}') \\
 & + \int_M Z R + \int_{\partial M} Z \mathbf{1} .
 \end{aligned} \tag{15}$$

The curvature operator $\int_M R$ is superficially relevant only for $D = 2$ and the boundary operator $\int_{\partial M} \mathbf{1}$ only for $D = 1$. When these additional terms are not relevant, the first three renormalization factors Z are the same for finite curved membranes than for the infinite flat membrane. This property is analogous to the renormalization property of local field theories in finite geometries, which justifies the finite scaling laws for critical systems in finite geometries, and it has two very important consequences: (i) The scaling hypothesis at the basis of the direct renormalization approach, which relies explicitly on calculations with finite membranes, is shown to be valid to all orders in perturbation theory; (ii) for "abstract" membranes with dimension $D < 2$, with the only exception

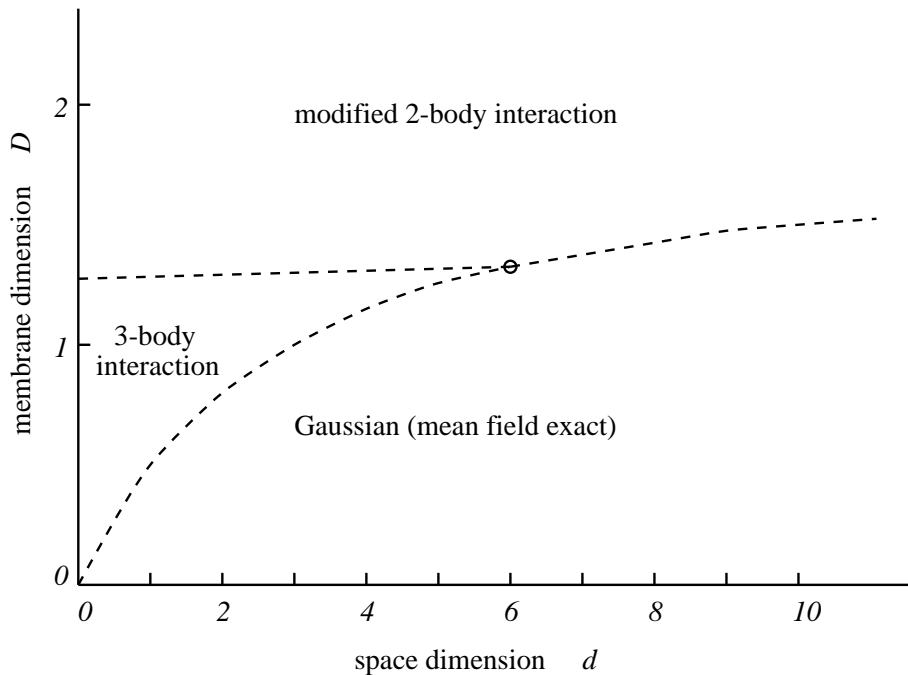


Figure 6: Relevant interactions for the Θ -point in the d - D plane.

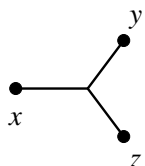
of open polymers ($D = 1$ open surface), the following hyperscaling relation⁹ relating the configuration and the ν exponents holds:

$$\gamma = 1 - \nu d . \quad (16)$$

6 Self-avoiding Membrane at the tricritical Θ -point:

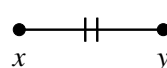
Finally, let me briefly discuss recent results obtained with K. Wiese on the scaling behavior of polymerized membranes at the Θ -point³. This point separates the swollen phase, where the self-avoidance repulsive forces that I considered previously dominate, from the dense collapsed phase, where short ranged attractive forces dominates. At the Θ -point the effective two body repulsive coupling b vanishes, and two different interactions may become relevant. The first one is the 3-body contact repulsion, which is usually considered for poly-

mers



$$= \int \int \int d^D x d^D y d^D z \delta^d(\vec{r}(x) - \vec{r}(y)) \delta^d(\vec{r}(x) - \vec{r}(z)) . \quad (17)$$

The second one is a modified 2-body interaction, repulsive at short range but attractive at larger range ($\Delta_{\vec{r}}$ is the d -dimensional Laplacian)



$$= - \int \int d^D x d^D y \Delta_{\vec{r}} \delta^d(\vec{r}(x) - \vec{r}(y)) . \quad (18)$$

Calculations at first order are not feasible analytically, and already require numerical evaluations of complicated integrals. The results of such one loop calculations are schematically depicted on Fig. 6, where the domains where the 3-body and modified 2-body terms are respectively relevant are shown. This indicates that the last modified 2-body term is the relevant one for 2-dimensional membranes in any external dimension d . There is also a quite interesting and non-trivial crossover between the two terms around $D = 4/3$ $d = 6$, which must be studied by a double ϵ -expansion.

7 Conclusion:

The theoretical study of the scaling behavior of polymerized flexible membranes leads to the development of new multilocal continuum field theories, and to new applications of renormalization group methods. I hope that these methods will lead to a quantitative progress in the understanding of the behavior of real 2-dimensional polymerized membranes. This requires results beyond first order in the ϵ -expansion (recall that $D = 2$ correspond to $\epsilon = 4$), and a better understanding of the relation between this RG approach and more heuristic or approximate methods, such as variational methods or approximate recursion relations. The sophisticated renormalization theory for multilocal models should hopefully also find applications in other problems of statistical physics, or in other areas of theoretical physics.

References:

1. *Statistical Mechanics of Membranes and Surfaces*, Proceedings of the Fifth Jerusalem Winter School for Theoretical Physics (1987), D. R. Nelson, T. Piran and S. Weinberg Eds., World Scientific, Singapore (1989).

2. F. David, B. Duplantier and E. Guitter, Phys. Rev. Lett. **72** (1994) 311.
3. K. Wiese and F. David, *Self-avoiding tethered membranes at the tricritical point*, Saclay preprint cond-mat/9503126, to appear in Nucl. Phys, B.
4. J. A. Aronowitz and T. C. Lubensky, Europhys. Lett. **4** (1987) 395.
5. M. Kardar and D. R. Nelson, Phys. Rev. Lett. **58** (1987) 1289.
6. J. des Cloizeaux, J. de Physique **42** (1981) 635.
7. M. Kardar and D. R. Nelson, Phys. Rev. Lett. **58** (1987) 2280 (E); Phys. Rev. **A 38** (1988) 966.
8. B. Duplantier, T. Hwa and M. Kardar, Phys. Rev. Lett. **64** (1990) 2022.
9. B. Duplantier, Phys. Rev. Lett. **58** (1987) 2733; and in ¹.