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Meander, Folding and Arch Statistics

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The statistics of meander and related problems are studied as particular realizations of compact polymer chain foldings. This paper presents a general discussion of these topics, with a particular emphasis on three points: (i) the use of a direct recursive relation for building (semi) meanders (ii) the equivalence with a random matrix model (iii) the exact solution of simpler related problems, such as arch configurations or irreducible meanders.

Keywords: meanders, polymers, folding, matrix models.

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1. Introduction

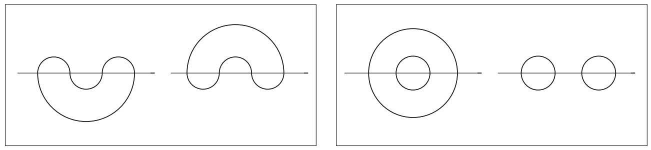
The concept of folding has an important place in polymer physics. Typically one considers the statistical model of a polymer chain made of say n identical constituents, and which may be folded onto itself. The entropy of such a system is obtained by counting the number of inequivalent ways of folding the chain. The combinatorial problem of enumerating all the *compact* foldings of a closed polymer chain happens to be equivalent to another geometrical problem, that of enumerating meanders [1], i.e. configurations of a closed road crossing a river through n bridges. To our knowledge, this is still an open problem, which indeed has been adressed by very few authors.

In the present paper, we study various aspects of this meander problem. In section 2, we define the meander problem itself, as well as a somewhat simpler semi–meander problem corresponding to the compact folding of an open polymer chain. We also gather in this section a number of data on exact enumeration of some meanders, as well as conjectural analytic structure for some of these data. A meander may be viewed as a particular gluing of two arch configurations, representing the configuration of the road respectively above an below the river. Section 3 is devoted to the derivation of exact results for the statistical distribution of arches in arch configurations. In section 4, we introduce an exact recursion relation for constructing all semi–meanders, for which we develop various approximations leading to estimates of the entropy of folding of open polymers. Section 5 presents an alternative description of the meander problems in terms of random matrix models, and gathers a few consequent results. A number of sum rules and inequalities satisfied by meander and semi–meander numbers are displayed in section 6, together with exact solutions for simpler meandric problems. In section 7, we address the question of irreducible meanders [1], i.e. systems of several roads crossing a river, which are interlocked in an irreducible way: in this case, an exact solution is derived for both irreducible meander and semi–meander numbers. A few technical details are gathered in appendix A and B.

2. Definitions and generalities

2.1. Meanders

A **meander** is defined as follows. Let us consider an infinite straight line (river). A meander of order n is a closed self–avoiding connected loop (road) which intersects the line



(a) (b)

Fig. 1: The $M_2 = 2$ meanders of order 2 (a), and the $M_2^{(2)} = 2$ two–component meanders of order 2. The infinite river is represented as a horizontal line.

through $2n$ points[1] (bridges). A meander of order n can clearly be viewed as a compact folding configuration of a closed chain of $2n$ constituents (in one–to–one correspondence with the $2n$ bridges), by putting a hinge on each section of road between two bridges. Note also that the road and the river play indeed similar roles, by appropriately cutting the road and closing the river.

In the following, we will study the number M_n of inequivalent meanders of order n (by inequivalent we mean meanders which cannot be smoothly deformed into each other without changing the order of the bridges). We will also be interested in the numbers $M_n^{(k)}$ of inequivalent meanders of order n with k connected components, i.e. made of k closed connected non–intersecting but possibly interlocking loops, which cross the river through a total of $2n$ bridges. Note that with this last definition, $M_n = M_n^{(1)}$. The $M_2 = M_2^{(1)} = 2$ meanders of order 2 and the $M_2^{(2)} = 2$ two–component meanders of order 2 are depicted in Fig.1 (a) and (b) respectively for illustration.

2.2. Folding a strip of stamps, semi–meanders

As mentioned in the introduction, the meander problem is equivalent to that of compact folding of a closed polymer chain. In this section, we will instead consider the case of an *open* polymer chain, moreover attached by one of its extremities. In another language, the problem is nothing but that of **folding a strip of stamps** [2] [3], and leads to a slightly different version of the meander problem, the *semi–meander* problem, which we describe now.

The problem of **folding a strip of stamps** may be stated as follows. One considers a strip of $n-1$ stamps, the first of which (say the leftmost one) is fixed, and supposedly attached to some support, preventing the strip from winding around the first stamp. A folding of the strip is a complete piling of its stamps, which preserves the (non–intersecting)

[1] The number of intersections between a loop and an infinite line is necessarily an even number.

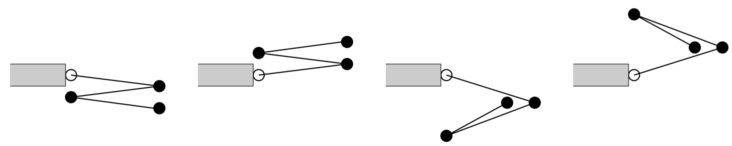


Fig. 2: The 4 inequivalent foldings of a strip of 3 stamps. The fixed stamp is indicated by the empty circle. The other circles correspond to the edges of the stamps. The first stamp is fixed and attached to a support (shaded area).

stamps and their connections, and only affects the relative positions of any two adjacent stamps: each stamp is folded either on top of or below the preceding one in the strip.

The number of inequivalent ways of folding a strip of $n-1$ stamps is denoted by S_n. In Fig.2 (a), we display the $S_4 = 4$ inequivalent foldings of a strip of 3 stamps (stamps are represented in side view, by segments, and also not completely folded to clearly indicate the succession of folds).

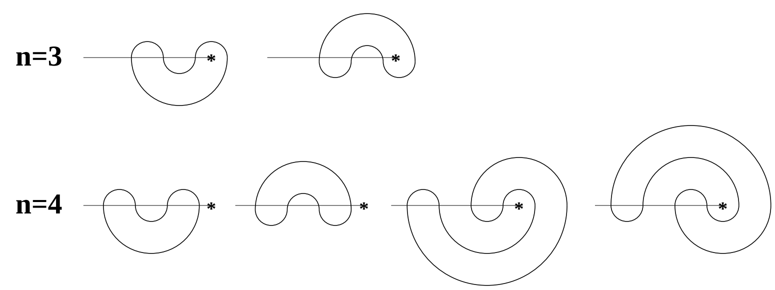


Fig. 3: The $\bar{M}_3 = 2$ and $\bar{M}_4 = 4$ semi–meanders of order 3 and 4. The source of the corresponding semi–infinite river is indicated by an asterisk.

This folding problem turns out to be equivalent to a particular meander problem, which we will refer to as the **semi–meander** problem. Let us consider a half (semi–infinite) straight line (river) starting at a point (source). A semi–meander of order n is a closed self–avoiding connected loop which intersects the half–line through n points[2] (bridges). Let us denote by $\bar{M}_n$ the number of inequivalent semi–meanders of order n. In analogy with the definition of meanders with k connected components, we also denote by $\bar{M}_n^{(k)}$ the number of inequivalent semi–meanders of order n with k connected components. As before, we have in particular $\bar{M}_n^{(1)} = \bar{M}_n$. The $\bar{M}_3 = 2$ and $\bar{M}_4 = 4$ semi–meanders of order 3 and 4 are depicted in Fig.3.

Let us prove that

$$S_n \ = \ \bar{M}_n \ . \tag{2.1}$$

[2] The number n of bridges for a semi–meander need not be even, hence $n = 1, 2, 3, ...$

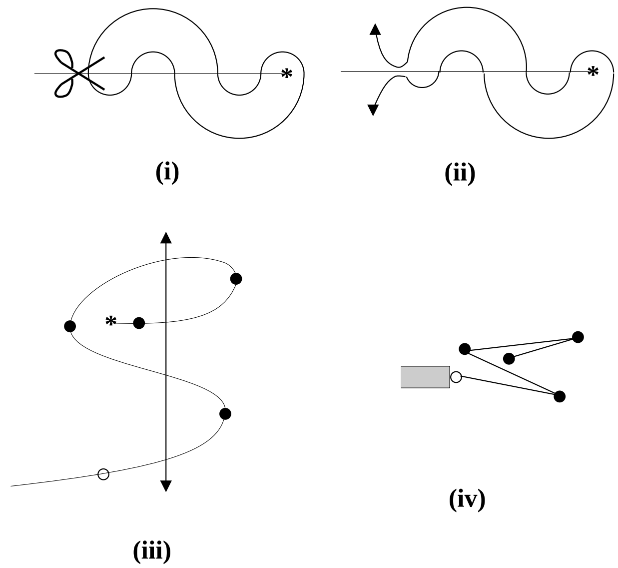


Fig. 4: The mapping from semi–meanders to folded strips of stamps. (i) cut the leftmost arch of the semi–meander along the river. (ii) stretch the open circuit into a line. (iii) identify the segments of bent river with edges of stamps. (iv) draw the stamps according to the relative positions of crossings.

Starting from a semi–meander of order n, let us construct a folding of the strip of $n-1$ stamps as follows. Cut the leftmost arch of the semi–meander along the river as indicated in Fig.4 (i), and stretch the (now open) circuit into an infinite line (ii). The river has been bent in this process, but the structure of its crossings with the line is preserved (except that the leftmost bridge has been erased). In a third step (iii), each segment of the river between two crossings of the infinite line is identified with an edge of stamp. In particular, the last edge of the strip of stamps is associated with the segment of river between its source and its first bridge, whereas the first edge of the fixed stamp corresponds to the infinite segment of river after the last bridge. This last choice singles out the first edge of the first stamp in such a way that no piece of the strip can wind around it. Finally (iv), the stamps can be drawn between the edges, the connection being indicated by the river. Note that in the process one of the bridges has been erased by the initial cut, henceforth we are left with $n-1$ bridges, hence $n-1$ stamps. This gives a one to one mapping from semi–meanders of order n to folded strips of $n-1$ stamps, thus proving (2.1).

An analogous construction allows one to relate the meander number M_n to that of foldings of a *closed* strip of $2n$ stamps. This construction is dual to the direct equivalence mentioned in the previous section, in the sense that the road and the river are exchanged.

2.3. Numerical data and a few conjectures

In this section, we gather numerical data for multicomponent meanders and semi–meanders.

Meanders. A few preleminary remarks are in order. The number of connected components for a meander of order n may not exceed n, as each connected component of the meander must cross the river an even number of times, whereas the number of bridges is $2n$. We display the first values of the multi–component meander numbers $M_n^{(k)}$ in Table I below (some of these numbers can be found in [1] and [4]).

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	8	42	262	1828	13820	110954	933458	8152860	73424650	678390116
2		2	12	84	640	5236	45164	406012	3772008	35994184	351173328	3490681428
3			5	56	580	5894	60312	624240	6540510	69323910	742518832	8028001566
4				14	240	3344	42840	529104	6413784	76980880	919032664	10941339452
5					42	990	17472	271240	3935238	54787208	742366152	9871243896
6						132	4004	85904	1569984	26200468	412348728	6230748192
7							429	16016	405552	8536890	161172704	2830421952
8								1430	63648	1860480	44346456	934582000
9									4862	251940	8356656	222516030
10										16796	994840	36936988
11											58786	3922512
12												208012

Table I: The numbers $M_n^{(k)}$ of inequivalent meanders of order n with k connected components, for $1 \le k \le n \le 12$, obtained by exact enumeration on the computer.

The reader will check on Table I that

$$M_n^{(n)} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{(2n)!}{n!(n+1)!} \ , \qquad (2.2)$$

known as the Catalan number of order n, denoted by c_n.

Encouraged by the relative simplicity of $M_n^{(n)}$, we have tried various guesses for the explicit form of the $M_n^{(n-j)}$, leading to the following conjecture[3] values of $j=0,1,2,3,4,5$

$$
\begin{aligned}
M_n^{(n)} &= \frac{(2n)!}{n!(n+1)!}\\
M_n^{(n-1)} &= \frac{2(2n)!}{(n-2)!(n+2)!}\\
M_n^{(n-2)} &= \frac{2(2n)!}{(n-3)!(n+4)!}(n^2+7n-2)\\
M_n^{(n-3)} &= \frac{4(2n)!}{3(n-4)!(n+6)!}(n^4+20n^3+107n^2-107n+15)\\
M_n^{(n-4)} &= \frac{2(2n)!}{3(n-5)!(n+8)!}(n^6+39n^5+547n^4+2565n^3-5474n^2+2382n-672)\\
M_n^{(n-5)} &= \frac{4(2n)!}{15(n-6)!(n+10)!}(n^8+64n^7+1646n^6\\
&\qquad +20074n^5+83669n^4-323444n^3+257134n^2-155604n+45360)
\end{aligned}
\tag{2.3}
$$

This leads to the more general conjecture that the number $M_n^{(n-j)}$ of $(n-j)$–component meanders of order n has the general form

$$
M_n^{(n-j)} = \frac{2^j(2n)!}{j!(n+2j)!(n-j-1)!} \times P_{2j-2}(n) \ , \tag{2.4}
$$

for $j \geq 1$, where $P_{2j-2}(x)$ is a polynomial of degree $2j-2$, with integer coefficients. The first few coefficients of P_{2j-2} read

$$
P_{2j-2}(x) \ = \ x^{2j-2}+(j-1)(3j+1)x^{2j-3}+\frac{1}{6}(j-1)(27j^3-23j^2-65j-6)x^{2j-4}+\ldots \tag{2.5}
$$

The expression (2.4) is still valid for $j=0$, but the factor is not polynomial, it reads

$$
P_{-2}(x) \ = \ \frac{1}{x(x+1)} \ = \ x^{-2}-x^{-3}+x^{-4}-x^{-5}+\ldots, \tag{2.6}
$$

and agrees with the large x expansion (2.5) for $j=0$.

This structure is strongly suggestive of algebraic recursion relations between the M's. We indeed found recursion relations between $M_n^{(n-j)}$ and the numbers of lower j, for the

[3] We have actually proved the cases $j=0,1,2,3$ recursively. Although some of them are quite tedious, all the proofs are based on the same strategy, which we will explicitly show for the case $j=0$ in the next section. The same strategy should extend to higher values of j.

first few j's. But the growing complexity of these relations when j is increased is not encouraging (typically, the structure of the recursion itself involves the detailed structure of the M_j meanders of order j).

Semi–meanders. Like in the meander case, the number k of connected components of order n semi–meanders cannot exceed n. There is actually only one semi–meander of maximal number of connected components: it corresponds to having n concentric circles, each crossing the half–river exactly once, hence

$$\bar{M}_n^{(n)} = 1 . \tag{2.7}$$

The first values of $\bar{M}_n^{(k)}$ are displayed in Table II below, for $1 \leq k \leq n \leq 14$ (some of these numbers can be found in [3] and [4]).

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	2	4	10	24	66	174	504	1406	4210	12198	37378	111278
2		1	2	6	16	48	140	428	1308	4072	12796	40432	129432	413900
3			1	3	11	37	126	430	1454	4976	16880	57824	197010	675428
4				1	4	17	66	254	956	3584	13256	49052	179552	658560
5					1	5	24	104	438	1796	7238	28848	113518	444278
6						1	6	32	152	690	3028	12996	54812	228284
7							1	7	41	211	1023	4759	21533	95419
8								1	8	51	282	1451	7112	33721
9									1	9	62	366	1989	10227
10										1	10	74	464	2653
11											1	11	87	577
12												1	12	101
13													1	13
14														1

Table II: The numbers $\bar{M}_n^{(k)}$ of inequivalent semi–meanders of order n with k connected components, for $1 \leq k \leq n \leq 14$, obtained by exact enumeration on the computer.

Like in the meander case, it is straightforward (a little less tedious here) to compute the first few $\bar{M}_n^{(n-j)}$, $j=0,1,2,...$ and to guess their general structure as functions of n and j. We find

$$\begin{aligned}
\bar{M}_n^{(n)} &= 1 \\
\bar{M}_n^{(n-1)} &= n-1 \\
\bar{M}_n^{(n-2)} &= \frac{1}{2!}(n^2+n-8) \\
\bar{M}_n^{(n-3)} &= \frac{1}{3!}(n^3+6n^2-31n-24) \\
&+ 2\delta_{n,4} \\
\bar{M}_n^{(n-4)} &= \frac{1}{4!}(n^4+14n^3-49n^2-254n) \\
&+ 15\delta_{n,5}+5\delta_{n,6} \\
\bar{M}_n^{(n-5)} &= \frac{1}{5!}(n^5+25n^4-15n^3-1105n^2-1066n+1680) \\
&+ 87\delta_{n,6}+42\delta_{n,7}+14\delta_{n,8} \\
\bar{M}_n^{(n-6)} &= \frac{1}{6!}(n^6+39n^5+145n^4-2895n^3-10226n^2+8616n+31680) \\
&+ 456\delta_{n,7}+292\delta_{n,8}+126\delta_{n,9}+42\delta_{n,10} \\
\bar{M}_n^{(n-7)} &= \frac{1}{7!}(n^7+56n^6+532n^5-5110n^4-50141n^3-20146n^2+377208n \\
&+ 282240)+2234\delta_{n,8}+1720\delta_{n,9}+1008\delta_{n,10}+396\delta_{n,11}+132\delta_{n,12} \ .
\end{aligned} \tag{2.8}$$

Note that in addition to some polynomial structure as functions of n, the numbers $\bar{M}_n^{(n-j)}$ also incorporate some constant corrections for the $j-1$ first values of $n=j+1, j+2, ..., 2j-1$. The above suggests the following general form for the polynomial part of $\bar{M}_n^{(n-j)}$

$$\begin{aligned}
\bar{M}_n^{(n-j)}\Big|_{\text{pol.}} &= \frac{1}{j!}(n^j+\frac{1}{2}j(3j-5)n^{j-1}+\frac{1}{24}j(j-1)(27j^2-163j+122)n^{j-2} \\
&+ \frac{1}{48}j(j-1)(j-2)(27j^3-354j^2+1163j-1224)n^{j-3}+\cdots) \ .
\end{aligned} \tag{2.9}$$

We also found the beginning of a pattern for the corrections to the polynomial part (2.9), of the form

$$\bar{M}_n^{(n-j)}\Big|_{\text{corr.}} = c_j\delta_{n,2j-2}+3c_j\delta_{n,2j-3}+\cdots \tag{2.10}$$

where $c_j = \binom{2j+1}{j}/(2j+1)$ is the Catalan number of order j, and the sum goes down to a $\delta_{n,j+1}$ term.

Asymptotics. The data of the Tables I and II (see also [1] for M_n) enable one to evaluate numerically the asymptotic behaviour of M_n and $\bar{M}_n$ which read respectively

$$\begin{aligned} M_n &\sim \text{const} \, \frac{(12.25)^n}{n^{7/2}} \\ \bar{M}_n &\sim \text{const} \, \frac{(3.5)^n}{n^2} \end{aligned} \tag{2.11}$$

The exponent 7/2 for meanders was conjectured to be exact in [1].

Remarkably, the entropy of meanders

$$s = \lim_{n\to\infty} \frac{1}{n} \text{Log} M_n = \text{Log}\, R \sim 2.50 \tag{2.12}$$

is likely to be exactly twice that of semi–meanders

$$\bar{s} = \lim_{n\to\infty} \frac{1}{n} \text{Log} \bar{M}_n = \text{Log}\, \bar{R} \sim 1.25 \, . \tag{2.13}$$

Considering M_n (resp. $\bar{M}_n$) as the number of foldings of a closed (resp. open with a fixed end) strip of $2n$ (resp. $n-1$) stamps, general statistical mechanical considerations suggest that the leading behaviour of these numbers is exactly the same (i.e. $\propto R^N$), when expressed in the total number of stamps $N = 2n$ (resp. $N = n - 1$) for large N. This leads to the relation $\bar{s} = \frac{s}{2}$ ($= \text{Log}\, \bar{R}$) between the corresponding thermodynamical entropies, i.e. $R = \bar{R}^2$ between the leading terms.

3. Arch statistics

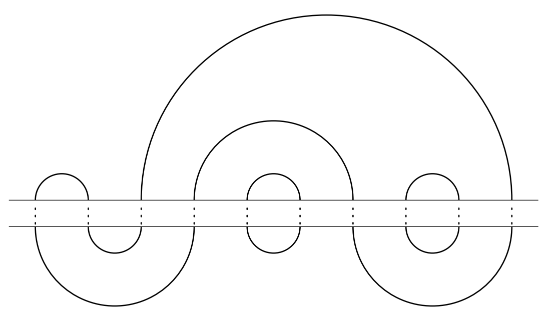


Fig. 5: A generic meander is the superimposition of two arch configurations.

The most general meander of order n with arbitrary number of connected components is specified uniquely by its upper half (above the river) and lower half (below the river), as shown in Fig.5. Both halves form systems of n non–intersecting arches connecting $2n$ bridges by pairs. Any two arches are either disjoint or included, one into the other. Any such system of arches will be referred to as an **arch configuration** of order n.

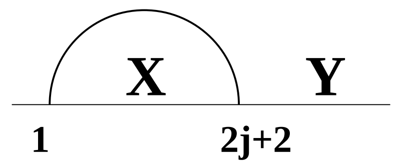


Fig. 6: Recursion principle for arch configurations. One sums over all positions $2j+2$ of the right bridge of the leftmost arch, which separates the initial configuration into two configurations X of order j and Y of order $n-j-1$, respectively below the leftmost arch and to its right.

3.1. Catalan numbers

Let us first compute by recursion the number c_n of arch configurations of order n, linking n pairs of bridges. Starting from one such arch configuration, let us follow the arch linking the leftmost bridge (position 1) to another, say in position $2j+2$ (see Fig.6: this position has to be even). This arch separates the configuration into two sub–configurations of arches, X of order j (below the leftmost arch), and Y of order $n-j-1$ (to the right of the leftmost arch). Summing over the position of the right–hand bridge of the leftmost arch ($2j+2$), we get the following simple recursion relation

$$c_n \;=\; \sum_{j=0}^{n-1} c_j \; c_{n-1-j} \;, \qquad n \geq 1 \;, \tag{3.1}$$

where we have set $c_0 = 1$. This rather simple example is nevertheless archetypical of a general type of reasoning used to count any relevant number associated to arch configurations. The scheme of Fig.6 is quite general.

The relation (3.1) is the defining recursion relation of the celebrated Catalan numbers c_n, which count, among other things, the numbers of parenthesings (with n opening and n closing parentheses) of words of $n+1$ letters. It follows immediately that

$$c_n \;=\; \frac{1}{2n+1}\binom{2n+1}{n} \;=\; \frac{(2n)!}{n!(n+1)!} \;. \tag{3.2}$$

In the following, we will need the generating function $C(x)$ of Catalan numbers,

$$\begin{aligned} C(x) \;&=\; \sum_{n=0}^{\infty} c_n \; x^n \\ &=\; 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + \cdots \end{aligned} \tag{3.3}$$

subject to the algebraic relation

$$xC(x)^2 \;=\; C(x) - 1 \;, \tag{3.4}$$

due to (3.1). One gets

$$C(x) \;=\; \frac{1-\sqrt{1-4x}}{2x} \;. \tag{3.5}$$

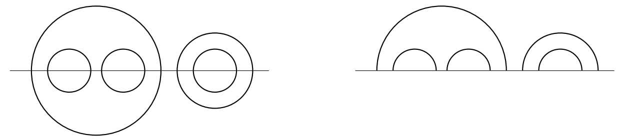


Fig. 7: A particular 5–component meander of order 5, and the corresponding arch configuration.

3.2. Arch configurations and meanders

Our interest in arch configurations is motivated by the fact that both meanders and semi–meanders can be built out of them. For instance, we get a direct one to one correspondence between n–component meanders of order n and arch configurations of order n: any arch configuration is completed by reflection wrt the river (see Fig.7). As a consequence, we have

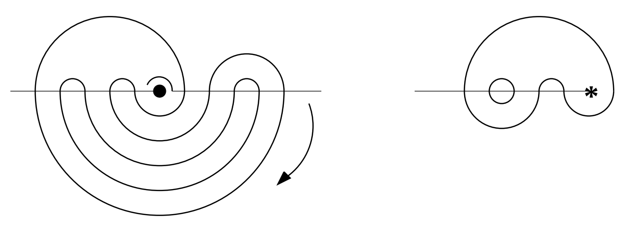
$$M_n^{(n)} \ = \ c_n \ , \tag{3.6}$$

the total number of arch configurations of order n, in agreement with (2.2).

More generally, any multi–component meander of order n is obtained by superimposing any two arch configurations of order n, one above the river, one below the river, and connecting them through the $2n$ bridges. As a consequence, we find the sum rule

$$\sum_{k=1}^{n} M_n^{(k)} \ = \ (c_n)^2 \ , \tag{3.7}$$

expressing the total number of multi–component meanders of order n as the total number of couples of (top and bottom) arch configurations of order n. This is readily checked on the data of Table I: the sums of numbers by columns are equal to the square of the lowest number ($M_n^{(n)} = c_n$) in each column.



(a) **(b)**

Fig. 8: Any semi–meander (b) is obtained from the superimposition of an arbitrary arch configuration (top of (a)) and a rainbow arch configuration (bottom of (a)) connected through the $2n$ bridges ($n = 5$ here), and by folding the river as indicated, thus identifying the bridges by pairs. The process (a) $\leftrightarrow$ (b) is clearly reversible.

It is instructive to note that any multi–component semi–meander of order n may also be obtained as the superimposition of one arbitrary arch configuration of order n above an infinite river, and a "rainbow" arch configuration below it (see Fig.8). The rainbow arch configuration of order n, denoted by $\mathcal{R}_n$, consists of n arches linking the opposite pairs of bridges: $(1, 2n)$, $(2, 2n-1)$, ...,$(n, n+1)$. Note that the number of bridges is doubled in this representation. To recover the semi–meander, one simply has to fold the infinite river into a semi–infinite one, identifying the $2n$ bridges by pairs according to the rainbow arches as indicated in Fig.8. As a consequence, we get the semi–meander version of the sum rule (3.7) for meanders

$$\sum_{k=1}^{n} \bar{M}_n^{(k)} = c_n \ , \tag{3.8}$$

expressing the total number of multi–component semi–meanders of order n as the total number c_n of arch configurations of order n (completed by the lower rainbow $\mathcal{R}_n$ to yield the semi–meanders). This sum rule is readily checked on the data of Table II: the sums of numbers by columns are the Catalan numbers.

The main difficulty in the meander and semi–meander problems is to find a direct way, for given arch configurations, to count the number of connected components of the resulting meander or semi–meander. This however is far beyond reach. Nevertheless, it is instructive to gather more refined statistical informations on the distribution of arches in random configurations, in view of a tentative generalization to the arch statistics of meanders with fixed number of connected components.

3.3. Statistics of arches

Exterior arches. Let us first compute the distribution law of exterior arches. By exterior arches, we mean arches which have no other arch above them. For instance, the arch configuration of Fig.7 has two exterior arches, a rainbow $\mathcal{R}_n$ has only one exterior arch, etc... Let $E(n, k)$ denote the number of arch configurations of order n with exactly k exterior arches. A simple recursion relation can be obtained in the same spirit as for Catalan numbers, by following the general scheme of Fig.6. Starting from an order n arch configuration with k exterior arches, let us consider the leftmost arch, starting at the leftmost bridge. It is clearly an exterior arch. Let $2j+2$ be the position of the right bridge of this arch. Again, the arch separates the configuration into two arch configurations. The one below the arch is an arbitrary configuration among the c_j arch configurations of order

j. The one to the right of the arch is an arch configuration of order $n-j-1$, with $k-1$ exterior arches. This leads to the recursion[4]

$$E(n,k) \;=\; \sum_{j=0}^{n-1} c_j \; E(n-j-1,k-1)\;, \tag{3.9}$$

for $k \geq 1$ and $n \geq 1$, with the initial condition $E(n,0)=\delta_{n,0}$ and $E(n,k)=0$ for $k>n$. This defines the numbers $E(n,k)$ uniquely, and it is easy to prove that

$$E(n,k) \;=\; \frac{k}{2n-k}\binom{2n-k}{n} \;=\; \frac{k(2n-k-1)!}{n!(n-k)!}\;. \tag{3.10}$$

Another way of characterizing the distribution of arch configurations according to their number of exterior arches is through its factorial moments, defined as

$$\langle \binom{k}{l} \rangle_{\rm ext} \;=\; \frac{\sum_{k=1}^{n}\binom{k}{l}E(n,k)}{\sum_{k=1}^{n}E(n,k)}\;, \tag{3.11}$$

where $\binom{k}{l} \equiv k(k-1)...(k-l+1)/l!$. Thanks to the identity

$$\sum_{k=1}^{n}\binom{k}{l}E(n,k) \;=\; \frac{2l+1}{2n+1}\binom{2n+1}{n-l} \;=\binom{2n}{n-l}-\binom{2n}{n-l-1}, \tag{3.12}$$

which is proved in appendix A (note that for $l=0$ (3.12) gives the total number c_n of arch configurations of order n), we deduce that

$$\langle \binom{k}{l} \rangle_{\rm ext} \;=\; (2l+1)\frac{n!(n+1)!}{(n-l)!(n+l+1)!}\;. \tag{3.13}$$

In particular, for $l=1$, the average number of exterior arches for arch configurations of order n reads

$$\langle k \rangle_{\rm ext} \;=\; \frac{3n}{n+2}\;. \tag{3.14}$$

In the limit of infinite order, $n\to\infty$ (the thermodynamic limit), the lth factorial moment of the distribution of exterior arches tends to $2l+1$. In particular, the average number of exterior arches tends to 3.

Interior arches. An interior arch of an arch configuration is an arch containing no other arch below it. For instance the configuration of Fig.7 has 3 interior arches, a rainbow $\mathcal{R}_n$ has only one interior arch, etc... Let $I(n,k)$ denote the number of arch configurations

[4] One can also show that $E(n,k)=E(n,k+1)+E(n-1,k-1)$.

of order n with exactly k interior arches. As before (see Fig.6), we can derive a recursion relation by following the leftmost exterior arch of any order n arch configuration with k interior arches. Three situations may occur:

(i) the arch links the first bridge to the last one. Below this arch, we may have any arch configuration of order $n-1$ with k interior arches.

(ii) the arch links the first bridge to the second, hence it is an interior arch. To its right, we have any arch configuration of order $n-1$, with $k-1$ interior arches.

(iii) the arch links the first bridge to the one in position $2j+2$, $j=1,...,n-2$. It separates the configuration into two configurations of respective orders j (below the arch) and $n-j-1$ (to the right of the arch), with respectively q and $k-q$ interior arches, $q=1,...,k-1$.

This leads to the recursion relation

$$I(n,k) \;=\; I(n-1,k)+I(n-1,k-1)+\sum_{j=1}^{n-2}\sum_{q=1}^{k-1} I(j,q)I(n-j-1,k-q) \;, \tag{3.15}$$

for $n \geq k \geq 1$, and with the initial condition $I(1,k)=\delta_{k,1}$, and $I(j,q)=0$ for $q>j$. This allows one to solve for

$$I(n,k) \;=\; \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \;=\; \frac{n!(n-1)!}{k!(k-1)!(n-k)!(n-k+1)!} \;. \tag{3.16}$$

With these numbers, we compute

$$\sum_{k=1}^{n}\binom{k}{l} I(n,k) \;=\; \frac{1}{n}\binom{n}{l}\binom{2n-l}{n-l+1} \;. \tag{3.17}$$

A proof of this identity is given in appendix A. Hence the factorial moments of the distribution of interior arches read

$$\begin{aligned} \langle\binom{k}{l}\rangle_{\rm int} \;&=\; \frac{\sum_{k=1}^{n}\binom{k}{l}I(n,k)}{\sum_{k=1}^{n}I(n,k)} \\ &=\; \frac{\binom{n}{l}\binom{2n-l}{n-l+1}}{\binom{2n}{n-1}} \;. \end{aligned} \tag{3.18}$$

For instance the average number of interior arches on an arch configuration of order n is

$$\langle k\rangle_{\rm int} \;=\; \frac{n+1}{2} \;. \tag{3.19}$$

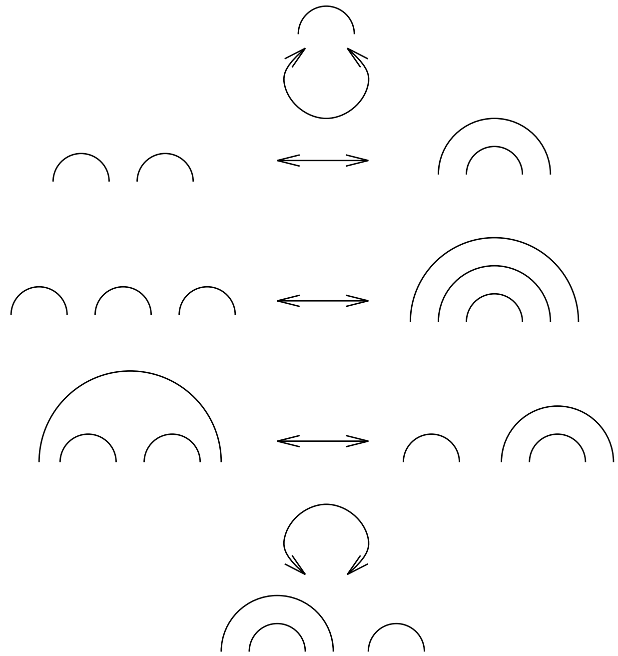


Fig. 9: The S duality between arch configurations is indicated by arrows, for orders $n=1,2,3$.

From (3.18), it is easy to see that the lth factorial moment behaves as

$$\langle \binom{k}{l} \rangle_{\rm int} \ \sim \ \frac{n^l}{2^l l!} \tag{3.20}$$

for large n. All these numbers tend to infinity when n is large, hence diverge in the thermodynamic limit.

A last remark is in order. The numbers $I(n,k)$ (3.16) satisfy a symmetry relation

$$I(n,k) \ = \ I(n,n+1-k) \ , \tag{3.21}$$

which is in fact the expression of a duality between arch configurations. Singling out the leftmost exterior arch, any arch configuration may be expressed as on Fig.6. The duality transformation is defined recursively as

$$\begin{aligned} S(\emptyset) \ &= \ \emptyset \\ S(\overset{\frown}{\ X\ }\ Y) \ &= \ \overset{\frown}{S(Y)}\ S(X) \end{aligned} \tag{3.22}$$

This transformation is involutive ($S \circ S = \mathbf{I}$) and exchanges disjoint and nested arches as exemplified in Fig.9. The transformation S changes the number of interior arches of a given configuration of order n from k to $n+1-k$, as is readily shown by recursion. This explains the symmetry (3.21).

Arches which are both interior and exterior. Let us compute the total number $EI(n)$ of arches which are both interior and exterior in all the arch configurations of order n. We have $EI(1) = 1$. By the recursion process of Fig.6, we can relate the number $EI(n)$ to the numbers $EI(j)$, $j < n$ in the following manner. Summing over the position $2j+2$ of the right bridge of the leftmost exterior arch in any order n configuration, we find that only the configuration Y of order $n-1-j$, to the right of this arch, contributes to $EI(n)$ through the total of $EI(n-1-j)$ arches which are both interior and exterior. This number comes c_j times, corresponding to the arbitrary choice of the configuration X of order j below the leftmost exterior arch. When $j=0$, the leftmost arch is both interior and exterior, and there are c_{n-1} such configurations, hence in addition to the $EI(n-1)$ arches which are both interior and exterior, we also have c_{n-1} leftmost arches, which are both interior and exterior. There is no contribution from $j=(n-1)$ (for $n \geq 2$) as the only exterior arch cannot be interior. This yields

$$EI(n) \ = \ \sum_{j=1}^{n-2} c_j \ EI(n-j-1) \ + EI(n-1) + c_{n-1} \ , \tag{3.23}$$

valid for $n \geq 2$. If we define $EI(0) = 1$, this takes the form

$$EI(n) \ = \ \sum_{j=0}^{n-1} c_j \ EI(n-1-j) \ , \tag{3.24}$$

hence

$$EI(n) \ = \ c_n \ . \tag{3.25}$$

For all n, there is therefore exactly one arch in average which is both exterior and interior.

Arches of given depth. The depth of an arch in an arch configuration is defined as follows. The exterior arches are the arches of depth 1. The arches of depth $k+1$ are the exterior arches of the arch configuration obtained by erasing the arches of depth 1, 2, ...,k. For instance the rainbow configuration $\mathcal{R}_n$ of order n has one arch of each depth between 1 and n; the arch configuration of Fig.7 has 2 arches of depth 1, and 3 arches of depth 2, which are also its interior arches. Let us compute the *total* number $A(n,k)$ of arches of depth k in *all* the arch configurations of order n, by the usual reasoning of Fig.6. Starting from any order n arch configuration, consider as usual the leftmost arch. It links the first bridge to the one in position $2j+2$, $j=0,...,n-1$. The arch separates the configuration into one of order j (below the arch, X in Fig.6), whose depth $k-1$ arches give rise to depth k arches of the initial configuration, and one of order $n-j-1$ (to the right of the arch, Y in Fig.6), whose depth k arches are depth k arches of the initial configuration. This translates into the recursion relation

$$A(n,k) \;=\; \sum_{j=0}^{n-1} \left(A(j,k-1)c_{n-j-1} + A(n-j-1,k)c_j \right) \;, \tag{3.26}$$

for $n \geq 0$ and $k \geq 2$, with $A(n,k)=0$ if $k>n$. The initial condition is the total number $A(n,1)$ of external arches of all order n arch configurations[5], namely

$$A(n,1) \;=\; \sum_{j=1}^{n} jE(n,j) \;=\; \frac{3}{2n+1}\binom{2n+1}{n-1} \;, \tag{3.27}$$

according to (3.12) for $l=1$. This fixes the $A(n,k)$ completely, and we find

$$A(n,k) \;=\; \frac{2k+1}{2n+1}\binom{2n+1}{n-k} \;=\; \binom{2n}{n-k} - \binom{2n}{n-k-1}. \tag{3.28}$$

This is nothing but the rhs of the identity (3.12), with the substitution $l \to k$ (indeed we have taken the $l=1$ value of this expression as initial condition). We still lack of a good explanation for this phenomenon. Note that the formula (3.28) can be used as the definition of $A(n,0)=c_n$, the total number of arch configurations, thus interpreting the depth 0 arch number to be 1 for each configuration. We can define the average number of depth k arches as

$$\langle A(n,k) \rangle \;=\; \frac{A(n,k)}{A(n,0)} \;=\; (2k+1)\frac{n!(n+1)!}{(n-k)!(n+k+1)!} \;. \tag{3.29}$$

These numbers coincide with the factorial moments (3.13) of the distribution of external arches. In particular, in the thermodynamic limit ($n \to \infty$), the average number of depth k arches tends to $2k+1$. So the average arch configuration has 3 exterior arches, 5 arches of depth 2, 7 arches of depth 3, etc ...

The total number of arches on a given configuration is n. Therefore, we have the following sum rule

$$\sum_{j=1}^{\infty} A(n,j) \;=\; n\; A(n,0) \;. \tag{3.30}$$

---

[5] Indeed the initial condition $A(1,1)=1$ would be sufficient, as the expression (3.26) is recursive wrt $k+n$.

The above result for the thermodynamic averages enables us to define an average maximal depth in arch configurations of order n, denoted by $d(n)$, by

$$\begin{aligned} n &= \sum_{j=1}^{d(n)} \lim_{n\to\infty} \frac{A(n,j)}{A(n,0)} \\ &= \sum_{j=1}^{d(n)} (2j+1) \ = \ d(n)(d(n)+2) \ . \end{aligned} \tag{3.31}$$

This gives

$$d(n) \ = \ \sqrt{n+1}-1 \ . \tag{3.32}$$

4. Recursion relations for semi–meanders

In the following, we will mainly concentrate our efforts on semi–meanders. However, we will mention whenever possible the extensions of our results to meanders.

4.1. The main recursion

Let us present a simple algorithm for enumerating the semi–meanders of order n with k connected components. In this section, we will work in the infinite river/lower rainbow arch–framework for semi–meanders (see Fig.8 (a)), namely consider a semi–meander as the superimposition of the lower rainbow configuration $\mathcal{R}_n$, and some upper arch configuration.

Starting from a semi–meander of order n with k connected components, let us construct several semi–meanders of order $n+1$ with k connected components in the following way, as indicated in Fig.10 (I) (i)-(ii).

(i) Pick any exterior arch of the upper arch configuration of the semi–meander and cut it.

(ii) Pull the two edges of the cut across the river (the left part of the exterior arch to the left, the right one to the right), and paste them around the lower rainbow, thus increasing the rainbow configuration by one arch ($\mathcal{R}_n \to \mathcal{R}_{n+1}$), and the number of bridges by 2.

The result is a certain semi–meander of order $n+1$ with the same number k of connected components. So for each semi–meander $\mathcal{M}$ of order n with k components, we can construct $E(\mathcal{M})$ semi–meanders of order $n+1$ with k components, where $E(\mathcal{M})$ denotes the number of exterior arches of $\mathcal{M}$. All these new semi–meanders are clearly

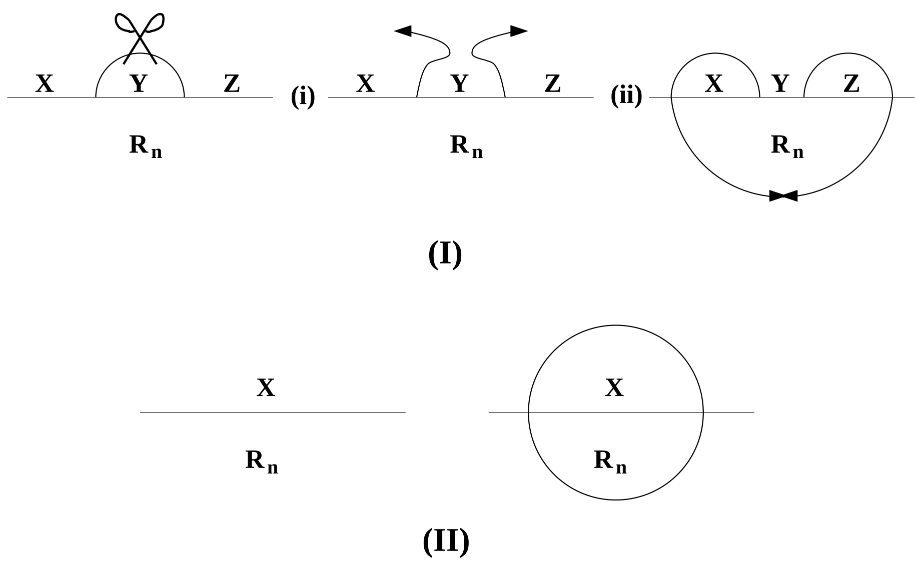


Fig. 10: The construction of all the semi–meanders of order $n+1$ with arbitrary number of connected components from those of order n. Process (I): (i) pick any exterior arch and cut it (ii) draw its edges around the semi–meander and paste them below. The lower part becomes the rainbow configuration $\mathcal{R}_{n+1}$ of order $n+1$. The process (I) preserves the number of connected components. Process (II): draw a circle around the semi–meander of order n. The process (II) adds one connected component.

distinct. There is however another way of generating more semi–meanders of order $n+1$, indicated in Fig.10 (II). Starting with any semi–meander of order n with $k-1$ connected components, one just adds an extra circular loop around it, which transforms the lower rainbow of order n into $\mathcal{R}_{n+1}$, and adds 2 bridges. The resulting semi–meander of order $n+1$ has clearly k connected components. Such a semi–meander cannot be obtained from some order n semi–meander by the procedure (I), because it has only one exterior upper arch, whereas the process (I) produces at least two exterior arches.

So all the order $n+1$ semi–meanders constructed by (I)-(II) are distinct. Conversely, given a semi–meander of order $n+1$ with k components, two cases may occur:

(a) it has only one exterior upper arch. In this case, it is surrounded by one circle, and therefore arises from the order n semi–meander with $k-1$ components inside the circle, through (II).

(b) it has at least two exterior upper arches. Cutting the lower exterior arch of the lower rainbow $\mathcal{R}_{n+1}$, pulling the edges of this arch above the upper configuration (the left edge by the left side, the right one by the right side, both suppressing the left and right–most bridges), and finally pasting the two edges on the upper side of the river (thus creating an exterior arch on the upper configuration), one gets an order n semi–meander with k

components, which leads to the initial semi–meander of order $n+1$ through (I)–(i)-(ii), using the exterior arch constructed above.

This proves that the procedures (I)-(II) give a recursive algorithm for constructing *all* the semi–meanders of order $n+1$ with k connected components from the k and $k-1$–component meanders of order n. More precisely, these procedures can also be viewed as a recursive algorithm for constructing all the arch configurations of order n from those of order $n-1$. When completed by a lower rainbow configuration $\mathcal{R}_n$, so as to give multi–component semi–meanders, we have the interesting property that (I) preserves the number of connected components of the semi–meander, while (II) increases it by 1, which allows us in principle to follow this number of components throughout the construction.

The case $k=1$ of connected semi–meanders is special, in the sense that the corresponding arch configurations are obtained by successive actions of the procedure (I) *only*. As a consequence, the number of connected semi–meanders of order $n+1$ is equal to the total number of exterior upper arches of all the semi–meanders of order n.

4.2. Expression in terms of arch numbers

We now want to translate into some recursion relation the transformations (I)–(II). It is best expressed in terms of the numbers of arches of given depth in the upper configurations of the semi–meanders. Recall that to build a semi–meander, one just has to close an upper arch configuration using a lower rainbow arch configuration. Therefore, we can decompose the set of arch configurations of order n into a partition, according to the number $k=1,2,...,n$ of connected components of the corresponding semi–meander. Denoting respectively by $\mathcal{A}_n$ and $SM_{n,k}$ the sets of arch configurations of order n and of semi–meanders of order n with k connected components, we have therefore

$$\mathcal{A}_n \; = \; \cup_{k=1}^{n} \; SM_{n,k} \; . \tag{4.1}$$

The statistics of arches extensively studied in sect.3 only give some global information on semi–meanders, irrespectively of their number of connected components. Taking the latter into account is a subtle refining of the above study. In particular we do not expect the statistics of arches to be the same when the number of connected components is fixed. We rather want to use sect.3 as a guideline for the study of semi–meander arch statistics. In particular, we will use analogous definitions for observable quantities, and derive recursion relations between them.

For a given semi–meander $\mathcal{M}$ of order n with k components, let us denote by $A(\mathcal{M},j)$ the number of arches of depth j in its upper configuration, and let

$$A_k(n,j) \ = \sum_{\mathcal{M}\in SM_{n,k}} A(\mathcal{M},j) \ , \tag{4.2}$$

denote the total number of arches of depth j of the upper configurations of the semi–meanders of order n with k connected components. To make the contact with the numbers $A(n,j)$ of arches of depth j in the arch configurations of order n (eq.(3.28)), let us mention the obvious sum rule

$$\sum_{k=1}^{n} A_k(n,j) \ = \ A(n,j) \ . \tag{4.3}$$

This simply states that summing the numbers of upper arches of depth j over all (multi–component) semi–meanders of order n amounts to counting the total number of arches of depth j in all the arch configurations of order n. This sum rule is parallel to the set decomposition (4.1), and simply evaluates the number of depth j arches on both sides.

To write a recursion relation, we must follow the evolution of the depth of arches through the processes (I) and (II). In the process (II), it is clear that the arches of depth $j-1$ in the order n semi–meander become arches of depth j in the order $n+1$ one, hence we get the contribution

$$A_{k-1}(n,j-1) = \sum_{\mathcal{M}\in SM_{n,k-1}} A(\mathcal{M},j-1) \tag{4.4}$$

to the total number of arches of depth j. The process (I) of Fig.10 is more subtle. We start from a meander of order n with k components, $\mathcal{M}_{n,k}$. For each of its $E(\mathcal{M}_{n,k})$ exterior arches, the process (I) affects the depth of its arches in the following way. The arches of depth $j+1$ below the specified exterior arch (configuration denoted by Y on Fig.10) become arches of depth j. The arches of depth $j-1$ to the left (X) and to the right (Z) of this exterior arch become arches of depth j. Summing over all the $A(\mathcal{M}_{n,k},1)$ exterior arches of $\mathcal{M}_{n,k}$, we get the contribution

$$A(\mathcal{M}_{n,k},j+1) + [A(\mathcal{M}_{n,k},1)-1] \ A(\mathcal{M}_{n,k},j-1) \tag{4.5}$$

to the total number of arches of depth j. Adding the contributions (4.4) and (4.5) of the processes (II) and (I), we finally get the main recursion relation, for $k\geq 2$ and $j\geq 2$

$$A_k(n+1,j) \ = \ A_{k-1}(n,j-1) + A_k(n,j+1) + \sum_{\mathcal{M}\in SM_{n,k}} [A(\mathcal{M},1)-1] \ A(\mathcal{M},j-1) \ . \tag{4.6}$$

Like in the arch configuration case, we can introduce the number of arches of depth 0 in a semi–meander, in such a way that

$$A(\mathcal{M},0) \;=\; 1 \qquad \forall \mathcal{M} \in SM_{n,k}\ . \tag{4.7}$$

In particular,

$$A_k(n,0) \;=\; \mathrm{card}(SM_{n,k}) \;=\; \bar{M}_n^{(k)}\ . \tag{4.8}$$

For $j=1$, the recursion reads

$$A_k(n+1,1) \;=\; A_{k-1}(n,0) + A_k(n,2) + 2A_k(n,1)\ . \tag{4.9}$$

For $j=0$, the recursion relation becomes

$$A_k(n+1,0) \;=\; A_{k-1}(n,0) + A_k(n,1)\ . \tag{4.10}$$

Eqs.(4.6)(4.8)(4.9) are all valid for $k \geq 2$.

The case $k=1$ is different since it uses the procedure (I) only. However, the above equations remain valid upon setting $A_0(n,j)=0$ for any n,j. This leads to

$$A_1(n+1,j) \;=\; A_1(n,j+1) + \sum_{\mathcal{M}\in SM_{n,1}} \left[A(\mathcal{M},1)-1\right] A(\mathcal{M},j-1)\ , \tag{4.11}$$

valid for $j \geq 2$. For $j=1,0$, we have

$$\begin{aligned} A_1(n+1,1) \;&=\; A_1(n,2) + 2A_1(n,1) \\ A_1(n+1,0) \;&=\; A_1(n,1)\ . \end{aligned} \tag{4.12}$$

The last equation expresses the fact that the total number $\bar{M}_{n+1}$ of connected semi–meanders of order $n+1$ is equal to the total number of exterior (depth 1) arches on all the semi–meanders of order n.

The abovementioned leading behaviour (2.11) of the connected semi–meander number

$$\bar{M}_n \;\propto\; \bar{R}^n\ , \tag{4.13}$$

enables one to recover the value of $\bar{R}$ from the thermodynamic limit of various averages over semi–meanders.

We now denote by

$$\langle f(\mathcal{M})\rangle_{n,k} = \frac{1}{A_k(n,0)} \sum_{\mathcal{M}\in SM_{n,k}} f(\mathcal{M}) \tag{4.14}$$

the average of a function f over the semi—meanders of order n with k connected components. Dividing eq.(4.6) by $A_k(n,0)$, we find a relation between averages

$$\begin{aligned} \frac{A_k(n+1,0)}{A_k(n,0)}\langle A(\mathcal{M},j)\rangle_{n+1,k} &= \frac{A_{k-1}(n,0)}{A_k(n,0)}\langle A(\mathcal{M},j-1)\rangle_{n,k-1} \\ &+ \langle A(\mathcal{M},j+1) + [A(\mathcal{M})-1]A(\mathcal{M},j-1)\rangle_{n,k} \ , \end{aligned} \tag{4.15}$$

for $k \geq 2$. For $k=1$, we have

$$\frac{A_1(n+1,0)}{A_1(n,0)}\langle A(\mathcal{M},j)\rangle_{n+1,1} = \langle A(\mathcal{M},j+1) + [A(\mathcal{M})-1]A(\mathcal{M},j-1)\rangle_{n,1} \ , \tag{4.16}$$

for $j \geq 2$ and for $j = 1, 0$

$$\begin{aligned} \frac{A_1(n+1,0)}{A_1(n,0)}\langle A(\mathcal{M},1)\rangle_{n+1,1} &= \langle A(\mathcal{M},2) + 2A(\mathcal{M},1)\rangle_{n,1} \\ \frac{A_1(n+1,0)}{A_1(n,0)} &= \langle A(\mathcal{M},1)\rangle_{n,1} \ . \end{aligned} \tag{4.17}$$

This last equation enables us to interpret $\bar{R}$ as the average thermodynamic number of exterior arches

$$\bar{R} = \lim_{n\to\infty} \frac{A_1(n+1,0)}{A_1(n,0)} = \lim_{n\to\infty} \frac{\bar{M}_{n+1}}{\bar{M}_n} = \langle A(\mathcal{M},1)\rangle_{\infty,1} \ . \tag{4.18}$$

4.3. Mean field approximation

In the thermodynamic limit, in view of the result for the statistics of arches, we expect all these averages to tend to some finite values. A standard thermodynamical approximation when dealing with averages over some statistical distribution is the **mean field** approximation. It expresses that the observable quantities (here the arch numbers) are fixed to their mean value. In particular, in this approximation, the term correlating the arches of depth 1 and $j-1$ is replaced by

$$\langle [A(\mathcal{M})-1]A(\mathcal{M},j-1)\rangle_{n,k} \underset{n\to\infty}{\longrightarrow} \langle [A(\mathcal{M})-1]\rangle_{\infty,k}\langle A(\mathcal{M},j-1)\rangle_{\infty,k} \ . \tag{4.19}$$

Denoting by $a_k(j) = \langle A(\mathcal{M},j)\rangle_{\infty,k}$, and concentrating on the case $k=1$, we find the recursion relation

$$\bar{R}\ a_1(j) = a_1(j+1) + [a_1(1)-1]a_1(j-1) \ , \tag{4.20}$$

valid for $j \geq 2$. On the other hand, $a_1(1) = \bar{R}$, from eq.(4.18). We are thus left with the recursion relation

$$a_1(j+1) \;=\; \bar{R} a_1(j) - (\bar{R}-1) a_1(j-1) \;, \tag{4.21}$$

for $j \geq 2$, whereas

$$a_1(2) \;=\; a_1(1)\;(\bar{R}-2) \;. \tag{4.22}$$

Note that necessarily $\bar{R} > 2$ for the average number $a_2(2)$ to be positive. With the initial conditions $a_1(0) = 1$ and $a_1(1) = \bar{R}$, we find in the 1–component case ($k = 1$) that

$$a_1(j) \;=\; \frac{\bar{R}}{\bar{R}-2}(1 + (\bar{R}-3)(\bar{R}-1)^{j-1}) \;. \tag{4.23}$$

This is the mean field solution for the numbers of depth j arches in connected semi–meanders. These numbers are further constrained by the fact that the average numbers of arches have to be non–negative, hence

$$\bar{R} \;\geq\; 3 \;. \tag{4.24}$$

In fact, if we had $\bar{R} > 3$, the solution (4.23) would grow exponentially with the depth j. This is in fact impossible, as can be shown by studying the stability of the fixed point (4.23) of the evolution equations (4.16)-(4.17) in the mean field approximation, and for $\bar{R} \geq 3$ (see appendix B for a detailed proof). This leaves with

$$\bar{R} \;=\; 3 \quad \Rightarrow \quad a_1(j) \;=\; 3 \quad \forall j \geq 0 \;. \tag{4.25}$$

Note that in this case the recursion (4.23) is satisfied for $j = 0, 1$ as well. Of course this is only a rough approximation of the exact $\bar{R}$ observed in (2.11), as is usually expected from a mean field approximation.

4.4. Mean field is wrong for arch configurations

The recursion relation (4.6) can be used to recover some information on arch configurations, by summing it over all the numbers k of connected components. Recall that $\sum_k A_k(n,j) = A(n,j)$, the total number of arches of depth j in the arch configurations of order n. Hence, summing over $k = 1, 2, ..., n+1$ in (4.6), we get

$$A(n+1,j) \;=\; A(n,j-1) + A(n,j+1) + \sum_{\mathcal{M} \in \mathcal{A}_n} [A(\mathcal{M},1) - 1]\; A(\mathcal{M}, j-1) \;, \tag{4.26}$$

where $\mathcal{A}_n$ denotes the set of arch configurations of order n. Using the explicit formula (3.28) for the number of arches, we deduce that

$$\sum_{\mathcal{M}\in\mathcal{A}_n} [A(\mathcal{M},1)-1]\ A(\mathcal{M},j-1) \ = \ 2A(n,j)\ . \tag{4.27}$$

This is obtained by noticing that the numbers $A(n,j)$ satisfy the linear recursion relation

$$A(n+1,j) \ = \ A(n,j-1)+A(n,j+1)+2A(n,j)\ , \tag{4.28}$$

for $n,j\geq 1$, as a consequence of the expression (3.28) of $A(n,j)$ as a difference between two binomial coefficients. This gives the value of the correlation function between the numbers of arches of depth 1 and those of depth $j-1$ in all arch configurations. Dividing (4.27) by the number of arch configurations $A(n,0)=c_n$, we get a relation for averages. If we denote by

$$\langle f(\mathcal{M})\rangle_n \ = \ \frac{1}{A(n,0)}\sum_{\mathcal{M}\in\mathcal{A}_n} f(\mathcal{M})\ , \tag{4.29}$$

this relation reads

$$\langle [A(\mathcal{M},1)-1]\ A(\mathcal{M},j-1)\rangle_n \ = \ 2\langle A(\mathcal{M},j)\rangle_n\ . \tag{4.30}$$

In the thermodynamic limit $n\to\infty$, as $\langle A(\mathcal{M},j)\rangle_n\to 2j+1$ (see eq.(3.29)), we find that

$$\begin{aligned} &\langle [A(\mathcal{M},1)-1]\ A(\mathcal{M},j-1)\rangle_n - \langle [A(\mathcal{M},1)-1]\rangle_n \langle A(\mathcal{M},j-1)\rangle_n \\ &\qquad \underset{n\to\infty}{\longrightarrow} 2(2j+1)-(3-1)(2j-1)=4\ . \end{aligned} \tag{4.31}$$

We see that the mean field approximation used in the previous section for semi–meanders is clearly wrong in the case of arch configurations. Note also that the result (4.31) is independent of j.

However, it is easy to solve the mean field equations for arch configurations, which lead to a growth $(2+\sqrt{2})^n$ for the total number of arch configurations, whereas the exact behaviour is 4^n (as readily seen from the large n expression of the Catalan number c_n by use of the Stirling formula).

4.5. Improved mean field approximation for semi–meanders

This suggests to try a deformed thermodynamic mean field–type ansatz for the numbers of arches in semi–meanders, of the form

$$\langle [A(\mathcal{M})-1]A(\mathcal{M},j-1)\rangle_{n,k} \underset{n\to\infty}{\longrightarrow} K_k + \langle [A(\mathcal{M})-1]\rangle_{\infty,k}\langle A(\mathcal{M},j-1)\rangle_{\infty,k} \ , \tag{4.32}$$

where K_k are some constants to be determined. For $k=1$, the main recursion relations (4.11) and (4.12) read, in the thermodynamic limit

$$\begin{aligned} \bar{R}\ a_1(j) &= a_1(j+1)+K_1+[a_1(1)-1]a_1(j-1) \quad \text{for } j\geq 2, \\ a_1(2) &= a_1(1)(\bar{R}-2) \\ a_1(1) &= \bar{R}\ a_1(0) \\ a_1(0) &= 1\ . \end{aligned} \tag{4.33}$$

We determine the constant K_1 by imposing that the main recursion be satisfied for the value $j=1$ as well, in which case we get

$$\bar{R}\ a_1(1)=\bar{R}^2=\bar{R}(\bar{R}-2)+K_1+(\bar{R}-1) \quad \Rightarrow \quad K_1=\bar{R}+1\ . \tag{4.34}$$

Next the recursion is easily solved by noticing that

$$a_1(j+1)-\alpha(j+1) \ = \ \bar{R}\ (a_1(j)-\alpha j)-(\bar{R}-1)\ (a_1(j-1)-\alpha(j-1))\ , \tag{4.35}$$

where

$$\alpha=\frac{\bar{R}+1}{\bar{R}-2}\ . \tag{4.36}$$

This is exactly the same recursion as in the mean field case (4.21), except that in the new variables $\alpha_1(j)=a_1(j)-\alpha j$, the initial conditions read

$$\alpha_1(0)=1 \qquad \alpha_1(1)=\frac{\bar{R}^2-3\bar{R}-1}{\bar{R}-2}\ . \tag{4.37}$$

The solution reads

$$a_1(j) \ = \ \frac{\bar{R}+1}{\bar{R}-2}j+\frac{1}{(\bar{R}-2)^2}(3+(\bar{R}^2-4\bar{R}+1)(\bar{R}-1)^j)\ . \tag{4.38}$$

Like in the mean field case, the solution should not grow exponentially with the depth j. This imposes that the prefactor of $(\bar{R}-1)^j$ in (4.38) vanishes, i.e.

$$\bar{R}^2-4\bar{R}+1 \ = \ 0 \quad \Rightarrow \quad \bar{R}=2+\sqrt{3}=3.732... \tag{4.39}$$

hence

$$a_1(j) \;=\; (1+\sqrt{3})j+1 \ . \tag{4.40}$$

The value of $\bar{R}$ found in (4.39) is now above the observed value (2.11).

When applied to the arch configurations, this improved mean field ansatz turns out to yield the exact thermodynamic solution

$$\langle A(\mathcal{M},j)\rangle_\infty \;=\; 2j+1 \ , \tag{4.41}$$

which lead to the correct growth 4^n for the number of arch configurations of order n.

It is possible that even more refined improvements of the mean field ansatz give better approximations for semi–meanders. One can for instance think of replacing the constant K_1 above (4.32) by some specific function of j, but such a function is no longer fixed by initial conditions.

4.6. Other approximations using the main recursion

Starting from a particular arch configuration, let us generate its "descendents" by repeatedly using the processes (I) and (II) of Fig.10. After a given sequence of n ((I) or (II)) steps, let us denote by (m,e) respectively the *total* number of arch configurations generated and their average number of exterior arches. Then let us adopt the following "mean field type" recursive algorithm for an extra action by (I) or (II)

$$\begin{aligned} (m,e) &\underset{(I)}{\longrightarrow} (me,e) \\ (m,e) &\underset{(II)}{\longrightarrow} (m,1) \ . \end{aligned} \tag{4.42}$$

The second line of (4.42) is clear, as (II) builds one arch configuration out of each inital arch configuration, with only one exterior arch. The first line incorporates two suppositions. First, the number of arch configurations generated is approximated by a mean field value, namely the product of the initial number m of arch configurations by the *average* number e of exterior arches. Second, the average number of exterior arches is supposed to be unchanged.

Let us now choose as starting point $(m=1,e)$ the upper arch configuration of a semi–meander. Repeated actions of (I) only will generate its descendents which are themselves semi–meanders. Implicitly, (4.42) supposes a certain number of properties on the starting point of the recursion, like for instance the fact that repeated actions of (I) keep the average number of exterior arches e unchanged, while the number of semi–meanders obtained is multiplied by e each time. Thus in this scheme, e will be identified with $\bar{R}$, governing the large n behaviour of the number of semi–meanders $\bar{M}_n \sim \bar{R}^n$. The scheme (4.42) is only intended as an approximation and for that purpose the initial semi–meander is supposed to be very large, with a number e of exterior arches equal to the average $\bar{R}$. Let us try to evaluate e by identifying the total number $p_n(e)$ of configurations generated after all possible sequences of n actions of (I) or (II) with the total number $c_n \sim 4^n$ (this assumes that c_n gives the correct large n behaviour of the total number of descendents of a given configuration, independently of this configuration). We start with only one semi–meander, hence $p_0(e) = 1$. After one step, according to eq.(4.42), we have generated $p_1(e) = e + 1$ arch configurations. Let us decompose the number $p_n(e)$ into

$$p_n(e) \ = \ p_n^{(I)}(e) \ + \ p_n^{(II)}(e) \ , \tag{4.43}$$

where we make the distinction between the total number of arch configurations obtained by the process (I) (resp. (II)) in the last step (4.42) from the $p_{n-1}(e)$ previous ones. The algorithm (4.42) leads to the recursion relations

$$\begin{aligned} p_n^{(I)}(e) \ &= \ e \ p_{n-1}^{(I)}(e) \ + \ p_{n-1}^{(II)}(e) \\ p_n^{(II)}(e) \ &= \ p_{n-1}^{(I)}(e) \ + \ p_{n-1}^{(II)}(e). \end{aligned} \tag{4.44}$$

In terms of the vectors $P_n(e) = (p_n^{(I)}(e), p_n^{(II)}(e))^t$, this takes the matrix form $P_n(e) = M(e)P_{n-1}(e)$, with

$$M(e) \ = \ \begin{pmatrix} e & 1 \\ 1 & 1 \end{pmatrix} \ . \tag{4.45}$$

For the total number of arch configurations generated to behave like 4^n, we simply have to write that the largest eigenvalue of this matrix is 4, namely that the determinant of $M(e) - 4\mathbf{I}$ vanishes (this also fixes the other eigenvalue of $M(e)$ to be 2/3). This gives

$$e \ = \ \frac{11}{3} \ = \ 3.666... \tag{4.46}$$

This estimate of $\bar{R}$ is the closest we can get to the numerical estimate (2.11). But the sequence of approximations used to get (4.46) should certainly be refined.

A last remark is in order. One could wonder how much the above estimate depends on the starting point of the algorithm. In particular, starting with any semi–meander with a finite number k of connected components and a large order, we end up with the same estimate (4.46) for the average number of exterior arches. This in turn infers an estimate for $\bar{R}_k = \bar{R}_1 = \bar{R} = e = 11/3$ for the numbers $\bar{R}_k$ governing the large order behaviour of $\bar{M}_n^{(k)} \sim (\bar{R}_k)^n$. Moreover the average number of exterior arches for all the arch configurations (generated in our scheme by both (I) and (II)) reads, after n steps,

$$\langle \text{ext} \rangle_n = \frac{e.p_n^{(I)} + 1.p_n^{(II)}}{p_n^{(I)} + p_n^{(II)}} . \tag{4.47}$$

For large n, the vector $(p_n^{(I)}, p_n^{(II)})$ tends, up to a global normalization, to the eigenvector $(3,1)$ associated to the largest eigenvalue $\lambda_{\max} = 4$ of the matrix $M(11/3)$ (4.45). Therefore the average number of exterior arches for all arch configurations is estimated as

$$\langle \text{ext} \rangle = \frac{3e+1}{4} = 3 , \tag{4.48}$$

which coincides with the exact value, as given by (3.14).

5. Matrix model for meanders

Field theory, as a computational method, involves expansions over graphs weighted by combinatorial factors. In this section, we present a particular field theory which precisely generates planar graphs with a direct meander interpretation. The planarity of theses graphs is an important requirement, which ensures that the arches of the meander do not intersect each other, when drawn on a planar surface. The topology of the graphs in field theoretical expansions is best taken into account in matrix models, where the size N of the matrices governs a topological expansion in which the term of order N^{2-2h} corresponds to graphs with genus h. The planar graphs (with $h = 0$) are therefore obtained by taking the large N limit of matrix models (see for instance [5] for a review on random matrices).

5.1. The matrix model as combinatorial tool

Random matrix models are useful combinatorial tools for the enumeration of (connected) graphs [5]. Typically, one considers the following integral over Hermitian matrices of size $N \times N$

$$Z(g,N) = \frac{\int dM\ e^{-N\mathrm{Tr}(\frac{M^2}{2} - g\frac{M^4}{4})}}{\int dM\ e^{-N\mathrm{Tr}(\frac{M^2}{2})}} \tag{5.1}$$

where the integration measure is

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\mathrm{Re}M_{ij} d\mathrm{Im}M_{ij} \ . \tag{5.2}$$

The rules of Gaussian matrix integration are simple enough to provide us with a trick for computing the expression (5.1) as a formal series expansion in powers of g. For instance,

$$\langle M_{ij} M_{kl} \rangle_{\mathrm{Gauss}} = \frac{\int dM \ e^{-N\mathrm{Tr}(\frac{M^2}{2})} M_{ij} M_{kl}}{\int dM \ e^{-N\mathrm{Tr}(\frac{M^2}{2})}} = \frac{\delta_{il}\delta_{jk}}{N} \ . \tag{5.3}$$

Expanding $Z(g,N)$ in powers of g, we are left with the computation of

$$Z(g,N) = \sum_{V=0}^{\infty} \frac{(Ng)^V}{V!} \langle (\mathrm{Tr}\frac{M^4}{4})^V \rangle_{\mathrm{Gauss}} \ . \tag{5.4}$$

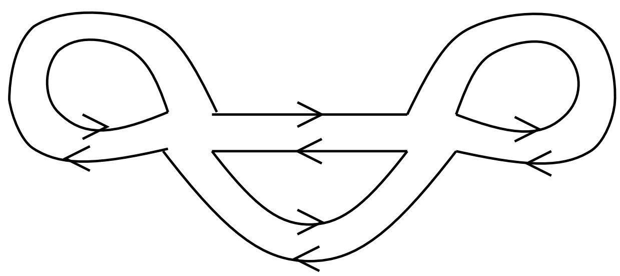


Fig. 11: A ribbon graph with $V = 2$ vertices, $E = 4$ edges, and $L = 4$ oriented loops.

The result (5.3) suggests to represent any term in the small g expansion as a sum over graphs constructed as follows.

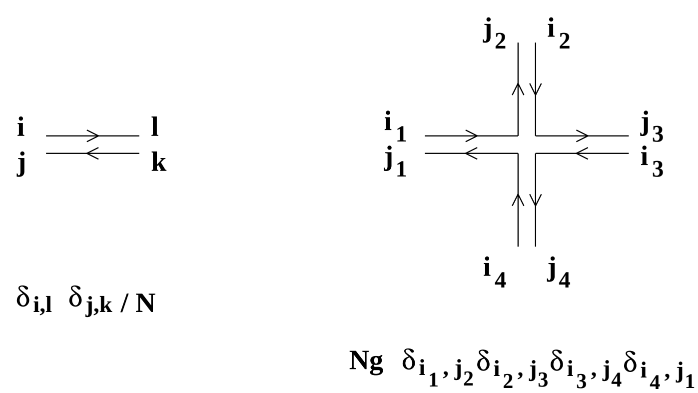


Fig. 12: The Feynman rules for the one matrix model: the matrix indices are conserved along the oriented lines, which form closed loops. Edges receive a factor $1/N$, vertices (Ng), and all the matrix indices have to be summed over.

The graphs considered are ribbon graphs, i.e. with edges made of double–lines, oriented with opposite orientations (see Fig.11 for an example). These two oriented lines represent the circulation of matrix indices $i, j = 1, 2, ..., N$. Namely each end of an edge is associated with a matrix element M_{ij}, the index i being carried by the line pointing from (resp. j by the line pointing to) this end of the edge, as shown in Fig.12. An edge is therefore interpreted as the propagator between the states sitting at its extremities, according to (5.3), hence each edge will be weighed by a factor $1/N$ (hence an overall factor N^{-E}, where E is the total number of edges of the graph). To compute the term of order V in the small g expansion (5.4), one simply has to sum over all the ribbon graphs connecting the V four–valent vertices corresponding to the V terms $\text{Tr}M^4$ (an obvious 4–fold cyclic symmetry absorbs the factors $1/4$). Each vertex has to be weighed by a factor Ng. These Feynman rules are summarized in Fig.12. An overall weight also comes from the summation over all the matrix indices $i = 1, 2, ..., N$ running on the oriented loops of the graph. This gives a global factor N^L for each graph, where L is the number of loops of the graph. For instance, the ribbon graph of Fig.11 receives a total weight $(Ng)^2 \times N^{-4} \times N^4 = (Ng)^2$.

The second trick is the fact that this sum can restricted to *connected* graphs only, by taking the logarithm of the function (5.1)

$$F(g, N) \ = \ \text{Log}\ Z(g, N) \ = \sum_{\text{conn. graphs } \Gamma} g^V N^{V-E+L} \times \frac{1}{|\text{Aut}(\Gamma)|} \ , \tag{5.5}$$

where $|\text{Aut}(\Gamma)|$ denotes the order of the automorphism group of Γ, i.e. the number of permutations of its (supposedly labelled) vertices leaving the graph invariant. This symmetry factor results from the incomplete compensation of the factor $1/V!$ by the number of equivalent graphs (with different labelling of the vertices). Finally, we identify the power of N as the Euler–Poincaré characteristic of the graph Γ

$$\chi(\Gamma) \ = \ V - E + L \ = \ 2 - 2h \ , \tag{5.6}$$

which can be taken as the definition of the genus h of the graph. The number of oriented loops is indeed equal to that of faces F of the cellular complex induced by the graph, hence we can use the more standard definition of the Euler–Poincaré characteristic $\chi = V - E + F = V - E + L$.

Various techniques for direct computation of the integral (5.1) have made it possible to enumerate connected graphs with arbitrary genus, and derive many of their properties. In this section, we consider a matrix model adapted to the meander enumeration problem.

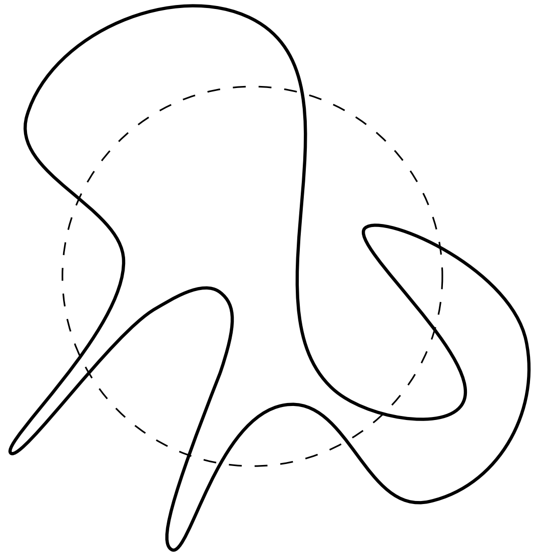


Fig. 13: A sample black and white graph. The white loop is represented in thin dashed line. There are 10 intersections.

5.2. The model

The enumeration of (planar) meanders is very close to that of 4–valent (genus 0) graphs made of two self–avoiding loops (say one black and one white), intersecting each other at simple nodes [1]. The white loop stands for the river, closed at infinity. The black loop is the road. Such a graph will be called a black and white graph. An example is given in Fig.13. The fact that the river becomes a loop replaces the order of the bridges by a cyclic order, and identifies the regions above the river and below it. Hence the number of meanders M_n is $2 \times 2n$ (2 for the up/down symmetry and $2n$ for the cyclic symmetry) times that of inequivalent black and white graphs with $2n$ intersections, weighed by the symmetry factor $1/|\mathrm{Aut}(\Gamma)|$. The same connection holds between $M_n^{(k)}$ and the black and white graphs where the black loop has k connected components.

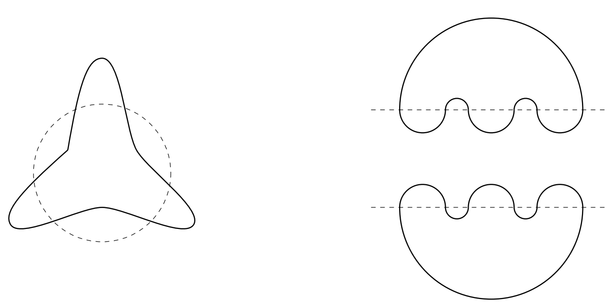


Fig. 14: A particular black and white graph with 6 intersections, and its two associated meanders. The automorphism group of the black and white graph is $\mathbb{Z}_6$.

For illustration, we display a particular black and white graph Γ in Fig.14, together with its two corresponding meanders of order 3. The automorphism group of this black and white graph is $\mathbb{Z}_6$, with order $\mathrm{Aut}(\Gamma) = |\mathbb{Z}_6| = 6$. The two meanders come with an overall factor $1/(2 \times 6)$, hence contribute a total $2 \times 1/12 = 1/6$, which is precisely the desired symmetry factor.

In analogy with the ordinary matrix model (5.1), a simple way of generating black and white graphs is the use of the multi–matrix integral (with $m+n$ hermitian matrices of size N denoted by B and W)

$$Z(m,n,c,N) = \frac{1}{\kappa_N} \int \prod_{\alpha=1}^{m} dB^{(\alpha)} \prod_{\beta=1}^{n} dW^{(\beta)} e^{-N\mathrm{Tr}\ P(B^{(\alpha)},W^{(\beta)})} , \tag{5.7}$$

where the matrix potential reads

$$P(B^{(\alpha)},W^{(\beta)}) = \sum_{\alpha} \frac{(B^{(\alpha)})^2}{2} + \sum_{\beta} \frac{(W^{(\beta)})^2}{2} - \frac{c}{2} \sum_{\alpha,\beta} B^{(\alpha)} W^{(\beta)} B^{(\alpha)} W^{(\beta)} , \tag{5.8}$$

and the normalization constant κ_N is such that $Z(m,n,c=0,N) = 1$. In the following, the α and β indices will be referred to as color indices.

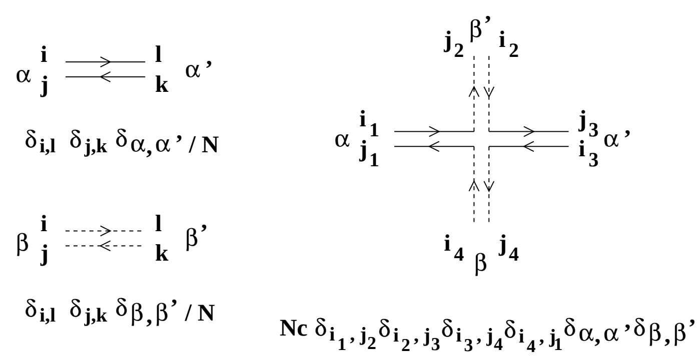


Fig. 15: The Feynman rules for the black and white matrix model. Solid (resp. dashed) double–lines correspond to black (resp. white) matrix elements, whose indices run along the two oriented lines. An extra color index α (resp. β) indicates the number of the matrix in its class, $B^{(\alpha)}$, $\alpha = 1,2,...,m$ (resp. $W^{(\beta)}$, $\beta = 1,2,...,n$). The only allowed vertices are 4–valent, and have alternating black and white edges: they describe simple intersections of the black and white loops.

Like in the case (5.1), the logarithm of the function (5.7) can be evaluated pertubatively as a series in powers of c. A term of order V in this expansion is readily evaluated as a Gaussian multi–matrix integral. It can be obtained as a sum over 4–valent connected graphs, whose V vertices have to be connected by means of the two types of edges

$$\begin{aligned} \text{black edges} \quad \langle [B^{(\alpha)}]_{ij} [B^{(\alpha')}]_{kl} \rangle &= \frac{\delta_{il}\delta_{jk}}{N} \delta_{\alpha\alpha'} \\ \text{white edges} \quad \langle [W^{(\beta)}]_{ij} [W^{(\beta')}]_{kl} \rangle &= \frac{\delta_{il}\delta_{jk}}{N} \delta_{\beta\beta'} \ , \end{aligned} \tag{5.9}$$

which have to alternate around each vertex. The corresponding Feynman rules are summarized in Fig.15. This is an exact realization of the desired connected black and white graphs, except that any number of loops[6] of each color is allowed. In fact, each graph receives a weight

$$N^{2-2h} \ c^V \ m^b \ n^w \ , \tag{5.10}$$

where b (resp. w) denote the total numbers of black (resp. white) loops.

A simple trick to reduce the number of say white loops w to one is to send the number n of white matrices W to 0, and to retain only the contributions of order 1 in n. Hence

$$f(m,c,N) \ = \ \lim_{n \to 0} \frac{1}{n} \text{Log } Z(m,n,c,N) \ = \sum_{\substack{b.\&w. \text{ conn. graphs } \Gamma \\ \text{with one } w \text{ loop}}} N^{2-2h} \ c^V \ m^b \ \frac{1}{|\text{Aut}(\Gamma)|} \ . \tag{5.11}$$

If we restrict this sum to the leading order N^2, namely the genus 0 contribution ($h = 0$), we finally get a relation to the meander numbers in the form

$$\begin{aligned} f_0(m,c) &= \lim_{N \to \infty} \frac{1}{N^2} f(m,c,N) \\ &= \sum_{p=1}^{\infty} \frac{c^{2p}}{4p} \sum_{k=1}^{p} M_p^{(k)} m^k \end{aligned} \tag{5.12}$$

where the abovementioned relation between the numbers of black and white graphs and multi–component meanders has been used to rewrite the expansion (5.11).

5.3. Meander numbers as Gaussian averages of words

The particular form of the matrix potential (5.8) allows one to perform the exact integration over say all the W matrices (the dependence of P on W is Gaussian), with the result

$$Z(m,n,c,N) \ = \ \frac{1}{\theta_N} \int \prod_{\alpha=1}^{m} dB^{(\alpha)} \det \left[\mathbf{I} \otimes \mathbf{I} - c \sum_{\alpha} B^{(\alpha) \ t} \otimes B^{(\alpha)} \right]^{-n/2} e^{-N \text{Tr} \sum_{\alpha} \frac{(B^{(\alpha)})^2}{2}} \ , \tag{5.13}$$

[6] The reader must distinguish between these loops, made of double–lines of a definite color, from the oriented loops along which the matrix indices run.

where $\mathbf{I}$ stands for the $N \times N$ identity matrix, $\otimes$ denotes the usual tensor product of matrices, and the superscript t stands for the usual matrix transposition. The prefactor θ_N is fixed by the condition $Z(m,n,c=0,N)=1$. With this form, it is easy to take the logarithm and to let n tend to 0, with the result

$$\begin{aligned} f(m,c,N) &= -\frac{1}{2\theta_N}\int \prod_{\alpha=1}^{m} dB^{(\alpha)} \mathrm{Tr}(\mathrm{Log}\big[\mathbf{I}\otimes\mathbf{I} - c\sum_{\alpha} B^{(\alpha)\ t}\otimes B^{(\alpha)}\big]) e^{-N\mathrm{Tr}\sum_\alpha \frac{(B^{(\alpha)})^2}{2}} \\ &= \sum_{p=1}^{\infty} \frac{c^p}{2p} \langle \mathrm{Tr}(\sum_{\alpha=1}^{m} B^{(\alpha)\ t}\otimes B^{(\alpha)})^p\rangle_{\mathrm{Gauss}} \\ &= \sum_{p=1}^{\infty} \frac{c^p}{2p} \sum_{1\le\alpha_1,\ldots,\alpha_p\le m} \langle \big|\mathrm{Tr}(B^{(\alpha_1)}\ldots B^{(\alpha_p)})\big|^2\rangle_{\mathrm{Gauss}}\ , \end{aligned} \tag{5.14}$$

where we still use the notation $\langle\ldots\rangle_{\mathrm{Gauss}}$ for the multi–Gaussian average over the matrices $B^{(\alpha)}$, $\alpha = 1,2,\ldots,m$. The modulus square simply comes from the hermiticity of the matrices $B^{(\alpha)}$, namely

$$\mathrm{Tr}(\prod B^{(\alpha_i)\ t}) \ = \ \mathrm{Tr}(\prod B^{(\alpha_i)\ *}) \ = \ \mathrm{Tr}(\prod B^{(\alpha_i)})^*\ . \tag{5.15}$$

Taking the large N limit (5.12), it is a known fact [5] that correlations should factorize, namely

$$\langle \big|\mathrm{Tr}(\prod_{i=1}^{p} B^{(\alpha_i)})\big|^2\rangle_{\mathrm{Gauss}} \underset{N\to\infty}{\longrightarrow} \big|\langle \mathrm{Tr}(\prod_{i=1}^{p} B^{(\alpha_i)})\rangle_{\mathrm{Gauss}}\big|^2\ . \tag{5.16}$$

By parity, we see that only even p's give non–vanishing contributions, and comparing with (5.12) we find a closed expression for the meander numbers of order n with k connected components

$$\sum_{k=1}^{n} M_n^{(k)} m^k \ = \ \sum_{1\le\alpha_1,\ldots,\alpha_{2n}\le m} \big| \lim_{N\to\infty} \langle \frac{1}{N}\mathrm{Tr}(\prod_{i=1}^{2n} B^{(\alpha_i)})\rangle_{\mathrm{Gauss}}\big|^2\ . \tag{5.17}$$

This expression is only valid for integer values of m, but as it is a polynomial of degree n in m (with vanishing constant coefficient), the n first values $m=1,2,\ldots,n$ of m determine it completely. So we only have to evaluate the rhs of (5.17) for these values of m to determine all the coefficients $M_n^{(k)}$.

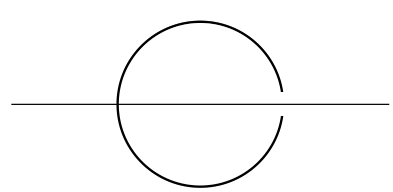


Fig. 16: The connected toric meander of order 1: it has only 1 bridge.

The relation (5.17) suggests to introduce higher genus meander numbers, denoted by $M_p^{(k)}(h)$, with $M_{2n}^{(k)}(0) = M_n^{(k)}$ (note that the indexation is now by the number of intersections, or bridges), through the generating function

$$\sum_{h=0}^{\infty}\sum_{k=1}^{\infty} M_p^{(k)}(h) m^k N^{2-2h} \quad = \quad \sum_{1\leq\alpha_1,\ldots,\alpha_p\leq m} \langle\big|\mathrm{Tr}(\prod_{i=1}^{p} B^{(\alpha_i)})\big|^2\rangle_{\mathrm{Gauss}} \ , \tag{5.18}$$

which incorporates the contribution of all genera in the Gaussian averages. Note that the genus h is that of the corresponding black and white graph and not that of the river or the road alone. In particular, the river (resp. the road) may be contractible or not in meanders of genus $h>0$. As an example the $M_1^{(1)} = 1$ toric meander is represented in Fig.16.

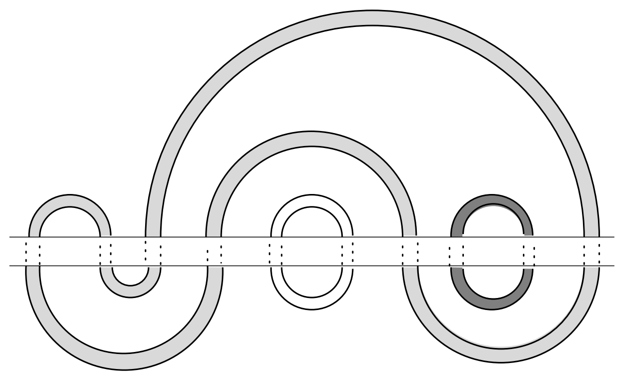


Fig. 17: A typical graph in the computation of the rhs of (5.18). The two p–valent vertices corresponding to the two traces of words are represented as racks of p double legs ($p = 10$ here). The connected components of the resulting meander (of genus $h = 0$ on the example displayed here) correspond to loops of matrices $B^{(\alpha)}$. This is indicated by a different coloring of the various connected components. Summing over all values of α_i yields a factor m per connected component, hence m^3 here.

The relation (5.18) can also be proved directly as follows. Its rhs is a sum over correlation functions of the traces of certain words (products of matrices) with themselves. More precisely, using the hermiticity of the matrices $B^{(\alpha)}$, the complex conjugate of the trace $\mathrm{Tr}(\prod_{1\leq i\leq 2n} B^{(\alpha_i)})$ can be rewritten as

$$\mathrm{Tr}(\prod_{1\leq i\leq 2n} B^{(\alpha_i)})^* \;=\; \mathrm{Tr}(\prod_{1\leq i\leq 2n} B^{(\alpha_{2n+1-i})}) \ , \tag{5.19}$$

i.e. in the form of an analogous trace, with the order of the B's reversed. According to the Feynman rules of the previous section in the case of only black matrices, such a correlation can be computed graphically as follows. The two traces correspond to two p–valent vertices, and the Gaussian average is computed by summing over all the graphs obtained by connecting pairs of legs (themselves made of pairs of oriented double–lines) by means of edges. Re–drawing these vertices as small racks of p legs as in Fig.17, we get a sum over all multi–component, multi–genera meanders (compare Fig.17 with Fig.5). More precisely, the edges can only connect two legs with the *same* matrix label α, which can be interpreted as a color: indeed, we have to sum over all colorings of the graph by means of m colors. But this coloring is constrained by the fact that the colors of the legs of the two racks have to be identified two by two (the color of both first legs is α_1,...,of both p-th legs is α_p). This means that each connected component of the resulting meander is painted with a color $\alpha \in \{1,2,...,m\}$. A graph of genus h comes with the usual weight N^{2-2h}. Summing over all the indices $\alpha_1,...,\alpha_p = 1,2,...,m$, we get an extra factor of m for each connected component of the corresponding meander, which proves the relation (5.18).

In the genus 0 case, we must only consider planar graphs, which correspond to genus 0 meanders by the above interpretation. Due to the planarity of the graph, the two racks of $p = 2n$ legs each are connected to themselves through n edges each, and are no longer connected to each other: they form two disjoint arch configurations of order n. This explains the factorization mentioned in eq.(5.16), and shows that the genus 0 meanders are obtained by the superimposition of two arch configurations. The beauty of eq.(5.17) is precisely to keep track of the number of connected components k in this picture, by the m–coloring of the connected components.

This last interpretation leads to a straightforward generalization of (5.18) to semi–meanders and many meander–related numbers.

5.4. Matrix expressions for semi–meanders and more

In view of the above interpretation, we immediately get the generalization of eq.(5.17) to semi–meanders as

$$\sum_{k=1}^{n} \bar{M}_n^{(k)} m^k \;=\; \sum_{1\leq\alpha_1,...,\alpha_n\leq m} \lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}(B^{(\alpha_1)}B^{(\alpha_2)}...B^{(\alpha_n)}B^{(\alpha_n)}B^{(\alpha_{n-1})}...B^{(\alpha_1)})\rangle_{\mathrm{Gauss}} \; . \tag{5.20}$$

To get this expression, we have used the m–coloring of the matrices to produce the correct rainbow-type connections between the loops of matrices.

More generally, this provides a number of matrix integral identities for the following generalized semi–meanders. Let us label the arch configurations of order n by a permutation $\mu \in S_{2n}$, the symmetric group over $2n$ objects, in such a way that if we label the bridges of the arch configuration $1,2,...,2n$, the permutation μ indicates the pairs of bridges linked by arches, namely, for any $i = 1,2,...,2n$, $\mu(i)$ is the bridge linked to i by an arch. By definition, μ is made of n cycles of length 2, it is therefore an element of the class $[2^n]$ of S_{2n}. Note that an element of this class generally does not lead to an arch configuration, because the most general pairing of bridges has intersecting arches. A permutation $\mu \in [2^n]$ will be called **admissible** if it leads to an arch configuration. Let $\mathcal{A}_\mu$ be the arch configuration associated to some admissible $\mu \in [2^n]$. We can define some generalized semi–meander number $\bar{M}_n^{(k)}(\mathcal{A}_\mu)$ associated to $\mathcal{A}_\mu$ as the number of meanders of order n with k connected components whose lower arch configuration is $\mathcal{A}_\mu$. With this definition,

$$\bar{M}_n^{(k)}(\mathcal{R}_n) \ = \ \bar{M}_n^{(k)} \ , \tag{5.21}$$

where $\mathcal{R}_n$ is the rainbow configuration of order n, associated to the permutation $\mu(i) = 2n+1-i$, for $i = 1,2,...,2n$. In other words, $\bar{M}_n^{(k)}(\mathcal{A}_\mu)$ is the number of closures by some arch configurations of order n of the lower arch configuration $\mathcal{A}_\mu$ which have k connected components. Eq.(5.20) extends immediately to

$$\begin{aligned}\sum_{k=1}^{n} \bar{M}_n^{(k)}(\mathcal{A}_\mu)\ m^k \ = & \sum_{\substack{1\leq \alpha_1,...,\alpha_{2n}\leq m \\ \alpha_i=\alpha_{\mu(i)},\ i=1,...,2n}} \\ & \lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}(B^{(\alpha_1)}B^{(\alpha_2)}...B^{(\alpha_{2n})})\rangle_{\mathrm{Gauss}} \ ,\end{aligned} \tag{5.22}$$

where the structure of the lower arch configuration $\mathcal{A}_\mu$ is encoded in the conditions $\alpha_i = \alpha_{\mu(i)}$, $i = 1,...,2n$, which identifies the colors of the arches according to $\mathcal{A}_\mu$. Some of these numbers will be computed in sect.6.3 below.

Higher genus generalizations are straightforward, by simply removing the large N limit in the above expressions, namely

$$\begin{aligned}\sum_{k\geq 1, h\geq 0} \bar{M}_{2n}^{(k)}(\mathcal{A}_\mu, h)\ m^k\ N^{1-2h} \ = & \sum_{\substack{1\leq \alpha_1,...,\alpha_{2n}\leq m \\ \alpha_i=\alpha_{\mu(i)},\ i=1,...,2n}} \\ & \langle \mathrm{Tr}(B^{(\alpha_1)}B^{(\alpha_2)}...B^{(\alpha_{2n})})\rangle_{\mathrm{Gauss}} \ .\end{aligned} \tag{5.23}$$

Note that the number $2n$ of bridges is even here, as we are considering arbitrary genus closures of a given arch configuration of order n. In genus $h = 0$, we recover the numbers

$\bar{M}_{2n}^{(k)}(\mathcal{A}_\mu, 0) = \bar{M}_n^{(k)}(\mathcal{A}_\mu)$ defined by (5.22). In the particular case $\mathcal{A}_\mu = \mathcal{R}_n$, this defines higher genus semi–meander numbers $\bar{M}_{2n}^{(k)}(h) = \bar{M}_{2n}^{(k)}(\mathcal{R}_n, h)$, with the correspondence $\bar{M}_{2n}^{(k)}(0) = \bar{M}_n^{(k)}$. The resulting higher genus semi–meanders are obtained generically by allowing the 4 ends of two given arches to alternate along the river. As the arches cannot intersect each other, this requires increasing the genus of the graph. On the other hand, the lower rainbow configuration is contractible, hence the genus is also that of the contracted graph obtained by the folding process of Fig.8.

It is instructive to calculate the sum over all genera of these numbers, while keeping track of the numbers of connected components. We simply take $N = 1$, in which case the Gaussian average becomes an ordinary Gaussian average over real scalars $(b^{(1)}, ..., b^{(m)}) \in \mathbb{R}^m$

$$\begin{aligned} \sum_{k\geq 1, h\geq 0} \bar{M}_{2n}^{(k)}(\mathcal{A}_\mu, h)\ m^k &= \langle \Big(\sum_{i=1}^{m} (b^{(i)})^2 \Big)^n \rangle_{\text{Gauss}} \\ &= \lambda_m \int_0^\infty r^{m-1}\ r^{2n}\ e^{-\frac{r^2}{2}} dr \\ &= m(m+2)(m+4)...(m+2n-2)\ , \end{aligned} \tag{5.24}$$

where the normalization constant has been fixed by the $n = 1$ case (the result is m). For $m = 1$, the above simply counts the total number of pairings between $2n$ legs, namely $(2n-1)!! = 1.3.5...(2n-1)$. Note that the result (5.24) is independent of the lower arch configuration, it holds in particular for semi–meanders. The result (5.24) is a polynomial of degree n, with leading coefficient 1 corresponding to the only (genus 0) meander with n connected components, obtained by reflecting the lower arch configuration $\mathcal{A}_\mu$ wrt the river. For meanders, we simply get

$$\begin{aligned} \sum_{k\geq 1, h\geq 0} M_p^{(k)}(h)\ m^k &= \langle \Big(\sum_{i=1}^{m} (b^{(i)})^2 \Big)^p \rangle_{\text{Gauss}} \\ &= m(m+2)(m+4)...(m+2p-2)\ . \end{aligned} \tag{5.25}$$

Note that in this case the polynomial is of degree p, with leading coefficient 1, corresponding to the (genus 1) meander made of a collection of p loops intersecting the river only once.

5.5. Computing averages of traces of words in matrix models

As a warming up, let us first compute the rhs of eq.(5.20) in the case of one matrix $m = 1$, namely

$$\gamma_n = \lim_{N\to\infty} \frac{1}{N} \langle \text{Tr}(B^n) \rangle_{\text{Gauss}}\ . \tag{5.26}$$

By parity, we see that $\gamma_{2s+1} = 0$ for all integer s. A simple method usually applied for computing Gaussian averages uses the so–called **loop equations** of the matrix model. In the case of one matrix, these are obtained as follows. We write that the matrix integral of a total derivative vanishes, namely

$$\begin{aligned} 0 &= \int dB \, \frac{\partial}{\partial B_{ji}} \Big[(B^{s+1})_{kl} e^{-N \mathrm{Tr} \frac{B^2}{2}} \Big] \\ \Rightarrow \quad 0 &= \langle -N (B^{s+1})_{kl} B_{ij} + \sum_{r=0}^{s} (B^r)_{kj} (B^{s-r})_{il} \rangle_{\mathrm{Gauss}} \ . \end{aligned} \tag{5.27}$$

Taking $i = l$ and $j = k$, and summing over $i, j = 1, ..., N$, we finally get

$$\langle \ \mathrm{Tr}(B^{s+2}) \ \rangle_{\mathrm{Gauss}} = \frac{1}{N} \sum_{r=0}^{s} \langle \ \mathrm{Tr}(B^r) \ \mathrm{Tr}(B^{s-r}) \ \rangle_{\mathrm{Gauss}} \ . \tag{5.28}$$

In the large N limit, due to the abovementioned factorization property, only the even powers of B contribute by parity, and setting $s + 2 = 2n$, this becomes

$$\gamma_{2n} = \sum_{r=0}^{n-1} \gamma_{2r} \gamma_{2n-2r-2} \ , \tag{5.29}$$

valid for $n \geq 1$, and $\gamma_0 = 1$. This is exactly the defining recursion (3.1) for the Catalan numbers, hence $\gamma_{2n} = c_n$, whereas $\gamma_{2n+1} = 0$. In this case, the equation (5.20) reduces therefore to the sum rule (3.8) for semi–meanders. Similarly, when $m = 1$, the equation (5.17) reduces to the sum rule (3.7) for meanders. An important remark is in order. It sould not be surprising that the general recursion principle for arches of Fig.6 resembles the large N limit of loop equations in matrix models. The role of the leftmost arch is played in the latter case by the differentiation wrt the matrix element B_{ji}: it can act at all the matrix positions in the word, which cut it into two even words, and can be graphically interpreted as just the right bridge position of the leftmost exterior arch in the pairings of matrices necessary to compute the Gaussian average of the trace of the word.

More generally, the loop equations for the Gaussian m–matrix model enable us to derive a general recursion relation for traces of words. The most general average of trace of word in m matrices in the large N limit is denoted by

$$\begin{aligned} &\gamma^{(m)}_{p_1,p_2,...,p_{mk}} = \\ &\lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}\big((B^{(1)})^{p_1} (B^{(2)})^{p_2} ... (B^{(m)})^{p_m} (B^{(1)})^{p_{m+1}} ... (B^{(m)})^{p_{mk}} \big) \rangle_{\mathrm{Gauss}} \ . \end{aligned} \tag{5.30}$$

In the above, some powers p_j may be zero, but no m consecutive of them vanish (otherwise the word could be reduced by erasing the m corresponding pieces). Of course $2p = \sum_i p_i$ has to be an even number for (5.30) to be non–zero, by the usual parity argument. For $m = 1$, we recover $\gamma_p^{(1)} = \gamma_p$. If $\omega = \exp(2i\pi/m)$ denotes the primitive m–th root of unity, then we have the following recursion relation between large N averages of traces of words, for $m \geq 2$

$$\gamma^{(m)}_{p_1,p_2,\ldots,p_{mk}} = -\sum_{j=1}^{mk-1} \omega^j \gamma^{(m)}_{p_1,\ldots,p_j} \gamma^{(m)}_{p_{j+1},\ldots,p_{mk}} . \tag{5.31}$$

When j is not a multiple of m, it is understood in the above that the multiplets $(p_1,\ldots,p_j)$ and $(p_{j+1},\ldots,p_{km})$ have to be completed by zeros so a to form sequences of m–uplets. For instance, we write $\gamma_3^{(3)} = \gamma_{3,0,0}^{(3)} = \gamma_{0,0,3}^{(3)}$. Note also that if only $q < m$ matrices are actually used to write a word, the corresponding $\gamma^{(m)}$ can be reduced to a $\gamma^{(q)}$ by erasing the spurious zeros (for instance, $\gamma_{3,0,0}^{(3)} = \gamma_3^{(1)}$). Together with the initial condition $\gamma^{(m)}_{0,\ldots,0,2n+1,0,\ldots,0} = \gamma_{2n+1} = 0$ and $\gamma^{(m)}_{0,\ldots,0,2n,0,\ldots,0} = \gamma_{2n} = c_n$, this gives a compact recursive algorithm to compute all the large N averages of traces of words in any multi–Gaussian matrix model.

A direct proof of eq.(5.31) goes as follows. Throughout this argument, we refer to the labels $\alpha = 1, 2, \ldots, m$ of the matrices as colors. The quantity $\gamma^{(m)}_{p_1,\ldots,p_{mk}}$ can be expressed as the sum over all possible planar pairings of matrices of the same color in the corresponding word, i.e. the sum over all arch configurations of order p (encoded in *admissible* permutations μ of $1, 2, \ldots, 2p$), preserving the color of the matrices (the matrices sitting at positions i and $\mu(i)$ have the same color α). Such a color–preserving arch configuration appears exactly once in the lhs of (5.31). We will show the relation (5.31) by proving that each such term also comes with a coefficient 1 in the rhs of (5.31). Let us evaluate the rhs of (5.31) in this language.

The index j may be viewed as the position of a "separator", which cuts the color–preserving arch configurations into two disconnected pieces. The separator positions are labeled by the index $j = 1, 2, \ldots, mk-1$. For a given color–preserving arch configuration in the lhs of (5.31), the only terms of the rhs of (5.31) contributing to it are those where the separator index j takes its values at positions inbetween the exterior arches linking various blocks of the same color. These positions will be referred to as available positions.

Available positions satisfy the two following properties, illustrated in Fig.18.

(i) any two successive available positions are labelled by *successive* integers modulo m: if an available separator position sits between two blocks of colors α and $\alpha + 1$, hence

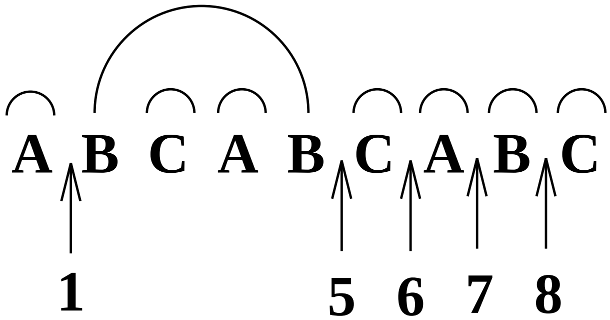


Fig. 18: A sample color–preserving pairing of matrices for $m = 3$ matrices. Each block of matrices is denoted by a single letter A, B, C, according to its color 1, 2, 3. The precise pairing of matrices within blocks is not indicated for simplicity. The available separator positions are indicated by arrows. One checks that (i) the positions are consecutive modulo 3, and (ii) the number of available positions is $5 = 2 \times 3 - 1$.

at a position of the form $j = sm + \alpha$, then the block of color $\alpha + 1$ is linked to other blocks of the same color, and the next available separator position sits between a block of color $\alpha + 1$ and a block of color $\alpha + 2$, hence at a position of the form $j = tm + \alpha + 1$. These two positions are consecutive modulo m.

(ii) the first available separator position is $j = ml + 1$: it sits to the right of the first set of related blocks of color 1. The last available separator position is of the form $j = qm + (m - 1)$, as it sits to the left of the rightmost set of related blocks of color m. Hence, thanks to property (i), the total number of available separator positions is of the form $rm - 1$.

Consequently, the total contribution of a given color–preserving admissible pairing in the rhs of (5.31) is of the form

$$(r - 1) \sum_{j=1}^{m} (-\omega^s) + \sum_{j=1}^{m-1} (-\omega^s) \ = \ 1 \ . \tag{5.32}$$

So we have proved that the total contribution of each admissible pairing in the rhs of (5.31) is 1. This completes the proof of (5.31).

Let us show explicitly how to use the recursion (5.31) to compute the thermodynamic average of the trace of a particular word in $m = 3$ matrices. We wish to compute

$$\gamma^{(3)}_{2,1,2,0,1,0} \ = \ \lim_{N \to \infty} \frac{1}{N} \langle \mathrm{Tr}\big(A^2 B C^2 B\big) \rangle_{\mathrm{Gauss}} \ , \tag{5.33}$$

where we denote by A, B, C the matrices with respective colors 1, 2, 3. Applying (5.31) we get

$$\begin{aligned}\gamma^{(3)}_{2,1,2,0,1,0} &= -j\gamma_2^{(1)}\gamma^{(2)}_{2,2} - j^2\gamma^{(2)}_{2,1}\gamma^{(2)}_{2,1} \\ &\quad - \gamma^{(3)}_{2,1,2}\gamma_1^{(1)} - j\gamma^{(3)}_{2,1,2}\gamma_1^{(1)} - j^2\gamma^{(3)}_{2,1,2,0,1,0} \ ,\end{aligned} \tag{5.34}$$

where $j = \exp(2i\pi/3)$, and the various numbers m of matrices have been reduced to their minimal value. Regrouping the $\gamma^{(3)}_{2,1,2,0,1,0}$'s, and using the fact that $\gamma^{(1)}_{2s+1} = \gamma_{2s+1} = 0$ for integer s, we get

$$(1+j^2)\gamma^{(3)}_{2,1,2,0,1,0} = -j\gamma_2^{(1)}\gamma^{(2)}_{2,2} \ . \tag{5.35}$$

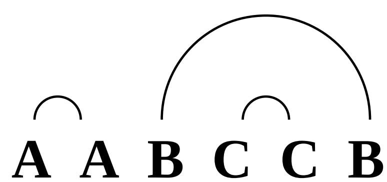
Using the recursion (5.31) for $m=2$, we get

$$\gamma^{(2)}_{2,2} = \gamma_2^{(1)}\gamma_2^{(1)} = (c_1)^2 \ . \tag{5.36}$$

Finally, we get

$$\gamma^{(3)}_{2,1,2,0,1,0} = (c_1)^3 = 1 \ . \tag{5.37}$$

This corresponds to the only way of pairing matrices of the same color in the sequence



5.6. Equivalent coloring problems

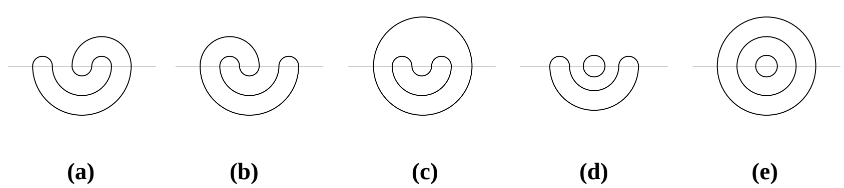


Fig. 19: The 5 semi–meanders of order 3.

The language of sect.5.5 above leads naturally to the introduction of the number $\bar{\mu}_n^{(k)}$ of k–colored multi–component semi–meanders of order n. By k–colored, we mean that exactly k distinct colors are used, but we do not distinguish between colorings differing only by a permutation of the colors. For illustration, let us take $n=3$. The 5 semi–meanders of order 3 are displayed in Fig.19, and denoted (a), (b), (c), (d), (e). Only the last one (e) can be colored with $k=3$ distinct colors, hence $\bar{\mu}_3^{(3)} = 1$. With $k=2$ colors, there are 3 ways of coloring (e), and 1 way of coloring (c) and (d), hence a total of $\bar{\mu}_3^{(2)} = 5$. Finally, each connected semi–meander (a) and (b) can be colored in a unique way with $k = 1$ color, hence $\bar{\mu}_3^{(1)} = 5$.

In terms of these numbers, the rhs of eq.(5.20) is expressed as a sum over the number k of colors used, with a total choice of $m(m-1)(m-2)...(m-k+1)$ colors among m, weighed by the numbers $\bar{\mu}_n^{(k)}$. This implies a relation between coloring numbers $\bar{\mu}$ and meander numbers $\bar{M}$

$$\bar{m}_n(m) \ = \ \sum_{k=1}^{n} \bar{M}_n^{(k)} \ m^k \ = \ \sum_{k=1}^{n} \bar{\mu}_n^{(k)} \ m(m-1)...(m-k+1) \ . \tag{5.38}$$

Remarkably, the numbers $\bar{\mu}$ do not carry direct information about the numbers of connected components of the semi–meanders, but their knowledge is sufficient to know the $\bar{M}$ exactly, by use of the Stirling numbers. Note also that eq.(5.38) expresses the polynomial $\bar{m}_n(x)$ in two different bases, $\{x^k, k = 1, ..., n\}$ and $\{x(x-1)...(x-k+1), k = 1, ..., n\}$, of the space of degree n polynomials with vanishing constant coefficient. For $n = 3$, we simply have

$$x^3 + 2x^2 + 2x \ = \ x(x-1)(x-2) + 5x(x-1) + 5x \ . \tag{5.39}$$

The same type of expression holds for meanders, namely

$$\sum_{k=1}^{n} M_n^{(k)} m^k \ = \ \sum_{k=1}^{n} \mu_n^{(k)} \ m(m-1)...(m-k+1) \ , \tag{5.40}$$

where $\mu_n^{(k)}$ denotes the number of k–colored multi–component meanders of order n.

5.7. Combinatorial expression using symmetric groups

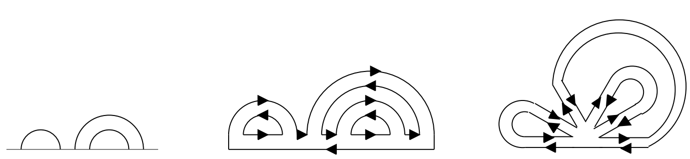


Fig. 20: An arch configuration of order 3 and the corresponding interpretation as a ribbon graph, with $V = 1$ six–valent vertex and $E = 3$ edges. On the intermediate diagram, the arches have been doubled and oriented. These oriented arches indicate the pairing of bridges, i.e. represent the action of μ. Similarly, the oriented horizontal segments indicate the action of the shift permutation σ. Each oriented loop corresponds to a cycle of the permutation $\sigma\mu$.

Admissibility condition. In sect.5.4 above, we have seen how an arch configuration of order n could be encoded in an admissible permutation μ belonging to the class $[2^n]$ of S_{2n}. Let us write the admissibility condition explicitly. This condition states that arches do not intersect each other, namely that the ribbon graph (see Fig.20) with only one $2n$–valent vertex (the $2n$ bridges), whose legs are connected according to the arch configuration, is *planar*, i.e. of genus $h = 0$. This graph has $V = 1$ vertex, and $E = n$ edges (arches). Let us compute its number L of oriented loops in terms of the permutation μ. Let σ denote the "shift" permutation, namely $\sigma(i) = i + 1$, $i = 1, 2, ..., 2n - 1$ and $\sigma(2n) = 1$. Then an oriented loop in the ribbon graph is readily seen to correspond to a *cycle* of the permutation $\sigma\mu$. Indeed, the total number of loops is $L = \text{cycles}(\sigma\mu)$, the number of cycles of the permutation $\sigma\mu$. The admissibility condition reads

$$\begin{aligned} \chi \ &= \ 2 = L - E + V \ = \ 1 - n + \text{cycles}(\sigma\mu) \\ \Leftrightarrow \quad \text{cycles}(\sigma\mu) \ &= \ n + 1 \ . \end{aligned} \tag{5.41}$$

Note that if we demand that the ribbon graph be of genus h, the above condition becomes

$$\text{cycles}(\sigma\mu) \ = \ n + 1 - 2h \ . \tag{5.42}$$

Connected components. Given an admissible permutation $\mu \in [2^n]$, let us now count the number of connected components of the corresponding semi–meander of order n. Let τ be the "rainbow" permutation $\tau(i) = 2n+1-i$. Note that τ changes the parity of the bridge label. On the other hand, the admissible permutation μ is readily seen to also change the parity of the bridge labels. As a consequence, the permutation $\tau\mu$ preserves the parity of bridge labels. In other words, even bridges are never mixed with odd ones. The successive iterations of the permutation $\tau\mu$ describe its cycles. The corresponding meander will be connected iff these cycles are maximal, namely μ has two cycles of length n (one for even bridges, one for odd bridges), i.e. $\tau\mu \in [n^2]$. We get a purely combinatorial expression for connected semi–meander numbers

$$\bar{M}_n \ = \ \text{card}\{\mu \in [2^n] \mid \text{cycles}(\sigma\mu) = n + 1, \ \text{and} \ \tau\mu \in [n^2]\} \ . \tag{5.43}$$

More generally, the semi–meander corresponding to μ will have k connected components iff $\tau\mu$ has exactly k pairs of cycles of equal length (one over even bridges, one over odd ones).

Character expressions. The above conditions on various permutations are best expressed in terms of the characters of the symmetric group. Denoting by $[i^{\nu_i}]$ the class of permutations with ν_i cycles of length i, and labelling the representations of S_{2n} by Young tableaux Y with $2n$ boxes as customary, the characters can be expressed as

$$\chi_Y([i^{\nu_i}]) \;=\; \det\big(p_{i+\ell_i-j}(\theta_.)\big)\Big|_{t_\nu} \;, \tag{5.44}$$

where the Young tableau has ℓ_i boxes in its i-th line, counted from the top, $t_\nu = \prod_i \frac{\theta_i^{\nu_i}}{\nu_i!}$, $p_m(\theta_.)$ is the m-th Schur polynomial of the variables θ_1, θ_2,...

$$p_m(\theta_.) \;=\; \sum_{\substack{k_i\geq 0, i=1,2,\ldots \\ \Sigma i k_i = m}} \prod_i \frac{\theta_i^{k_i}}{k_i!} \;, \tag{5.45}$$

and we used the symbol $f(\theta_.)|_{t_\nu}$ for the coefficient of the monomial $\prod_i \frac{\theta_i^{\nu_i}}{\nu_i!}$ in the polynomial $f(\theta_.)$. As group characters, the χ_Y's satisfy the orthogonality relation

$$\sum_Y \chi_Y([\lambda])\chi_Y([\mu]) \;=\; \frac{(2n)!}{|[\lambda]|}\,\delta_{[\lambda],[\mu]} \;, \tag{5.46}$$

where the sum extends over all Young tableaux with $2n$ boxes, $[\lambda]$ denotes the class of a permutation $\lambda \in S_{2n}$, and $|[\lambda]|$ the order of the class. The order of the class $[i^{\nu_i}]$ is simply

$$|[i^{\nu_i}]| \;=\; \frac{(2n)!}{\prod_i i^{\nu_i}\nu_i!} \;. \tag{5.47}$$

The orthogonality relation (5.46) provides us with a means of expressing any condition on classes of permutations in terms of characters. It leads to the following compact expression for the connected semi–meander numbers

$$\begin{aligned}\bar{M}_n \;&=\; \sum_{\substack{[i^{\lambda_i}]\in S_{2n} \\ \Sigma\lambda_i = n+1}} \sum_{\mu\in[2^n]} \delta_{[\sigma\mu],[i^{\lambda_i}]}\;\delta_{[\tau\mu],[n^2]} \\ &=\; \sum_{\substack{[i^{\lambda_i}]\in S_{2n} \\ \Sigma\lambda_i = n+1}} \sum_{\mu\in S_{2n}} \sum_{Y,Y',Y''} \frac{|[2^n]|\;|[i^{\lambda_i}]|\;|[n^2]|}{((2n)!)^3} \\ &\quad\times \chi_Y([\mu])\chi_Y([2^n])\chi_{Y'}([\sigma\mu])\chi_{Y'}([i^{\lambda_i}])\chi_{Y''}([\tau\mu])\chi_{Y''}([n^2]) \;.\end{aligned} \tag{5.48}$$

Analogous expressions hold for (higher genus) semi–meanders with k connected components and for meanders as well. These make completely explicit the calculation of the various meander–related numbers. Unfortunately, the characters of the symmetric group are not so easy to deal with, and we were not able to use them in an efficient way to enhance our numerical data.

6. Properties

In this section, we derive a number of identities and inequalities involving various meander and semi–meander numbers. We also compute exactly the number of closings of some particular lower arch configurations, made of repetitions of simple motives.

6.1. Sum rules

In addition to the sum rules (3.7) (3.8) and (5.24) (5.25) for (higher genus) meander and semi–meander numbers, let us derive the following

$$\begin{aligned} \sum_{k=1}^{n} M_n^{(k)} \ (-1)^{k-1} &= \begin{cases} 0 & \text{if } n \text{ is even} \\ (c_p)^2 & \text{if } n = 2p+1 \end{cases} \\ \sum_{k=1}^{n} \bar{M}_n^{(k)} \ (-1)^{k-1} &= \begin{cases} 0 & \text{if } n \text{ is even} \\ c_p & \text{if } n = 2p+1 \end{cases} . \end{aligned} \tag{6.1}$$

The reader can easily check these sum rules with the data of tables I and II above.

To prove both sum rules, we introduce the notion of **signature** of an arch configuration. Let us denote by $|\mathcal{A}|$ the order of an arch configuration $\mathcal{A}$. The signature of $\mathcal{A}$ is defined recursively by the initial value on the vacuous configuration and its behaviour under the processes (I) and (II) of sect.4.1, which enable us to construct any arch configuration. We set

$$\begin{aligned} \text{sig}(\emptyset) &= 1 \\ \text{sig}((I).\mathcal{A}) &= (-1)^{|\mathcal{A}|} \ \text{sig}(\mathcal{A}) \\ \text{sig}((II).\mathcal{A}) &= (-1)^{|\mathcal{A}|+1} \ \text{sig}(\mathcal{A}) \ . \end{aligned} \tag{6.2}$$

It is important to notice that the signature behaves differently under (I) and (II), hence depends on the number of connected components in semi–meanders for instance. More precisely, if $\mathcal{M}$ is a semi–meander of order n with k connected components, $\mathcal{M} \in SM_{n,k}$, and $\mathcal{P} \in SM_{n,1}$, then they both originate from the single arch of order 1 (with signature -1) by a total of $(n-1)$ actions of (I) and (II), each adding one arch, except that $\mathcal{P}$ is obtained by acting $(n-1)$ times with (I), whereas $\mathcal{M}$ is obtained by acting $(k-1)$ times with (II), and $(n-k)$ times with (I). Due to (6.2), this implies a difference in the signatures

$$\text{sig}(\mathcal{M}) \ = \ (-1)^{k-1} \ \text{sig}(\mathcal{P}) \ , \tag{6.3}$$

whereas after $(n-1)$ actions of (I), adding one arch each time, the signature of $\mathcal{P}$ reads

$$\text{sig}(\mathcal{P}) \ = \ (-1)^{1+\frac{n(n-1)}{2}} \ . \tag{6.4}$$

Thus, the signature of a semi–meander only depends on its order n and its number of connected components k, namely

$$\mathrm{sig}(\mathcal{M}) \;=\; (-1)^{k+\frac{n(n-1)}{2}}, \quad \forall\, \mathcal{M} \in SM_{n,k}\ . \tag{6.5}$$

In fact this signature can also be easily related to the numbers of arches of given depth in $\mathcal{A}$ as follows

$$\mathrm{sig}(\mathcal{A}) \;=\; (-1)^{\sum_{j\geq 1} jA(\mathcal{A},j)}\ , \tag{6.6}$$

where $A(\mathcal{A},j)$ (see sect.3) denotes the number of arches of depth j in $\mathcal{A}$. One readily checks that eq.(6.6) solves the recursion relations (6.2). For instance, the signature of the rainbow configuration of order n is

$$\mathrm{sig}(\mathcal{R}_n) \;=\; (-1)^{\frac{n(n+1)}{2}}\ , \tag{6.7}$$

in agreement with (6.3) for $k=n$. To conclude, the signature of a semi–meander only depends on its order n and its number of connected components k, namely

$$\mathrm{sig}(\mathcal{M}) \;=\; (-1)^{k+\frac{n(n-1)}{2}}, \quad \forall\, \mathcal{M} \in SM_{n,k}\ . \tag{6.8}$$

The sum rule (6.1) for semi–meanders is obtained by computing the sum of all the signatures of multi–component semi–meanders of order n. Let us introduce

$$\begin{aligned} s(n) \;&=\; \sum_{\mathcal{A}\in A_n} \mathrm{sig}(\mathcal{A}) \\ &=\; (-1)^{1+\frac{n(n-1)}{2}} \sum_{k=1}^{n} \bar{M}_n^{(k)}\, (-1)^{k-1}\ , \end{aligned} \tag{6.9}$$

where the first sum extends over all the arch configurations of order n. Using the usual recursive reasoning of Fig.6, the leftmost arch of a configuration separates the configuration into two configurations of smaller orders, say $\mathcal{X}$ of order j and $\mathcal{Y}$ of order $n-1-j$ (the only configuration of order 0 is just $\emptyset$). As the j arches of $\mathcal{X}$ are shifted downwards in the global configuration, we obtain the recursion relation

$$\begin{aligned} s(n) \;&=\; \sum_{j=0}^{n-1} \sum_{\mathcal{X}\in A_j} \sum_{\mathcal{Y}\in A_{n-1-j}} \mathrm{sig}(\mathcal{X})\mathrm{sig}(\mathcal{Y})(-1)^{j+1} \\ &=\; \sum_{j=0}^{n-1} (-1)^{j+1} s(j)\ s(n-1-j)\ . \end{aligned} \tag{6.10}$$

With $s(0) = 1$, we immediately find that $s(2p) = 0$ for $p = 1, 2, ...$, and that

$$s(2p+1) \ = \ \sum_{j=0}^{p-1} s(2j+1) \ s(2p-2j-1) \ , \tag{6.11}$$

with $s(1) = -1$, hence for $n = 2p+1$

$$s(2p+1) \ = \ (-1)^{p+1} \ c_p \ . \tag{6.12}$$

Comparing with (6.9), and noticing that $1 + n(n-1)/2 = 1 + (2p+1)(2p)/2 = p+1$ modulo 2, this completes the proof of (6.1) for semi–meanders.

In the case of meanders, let us define the signature of a given meander $\mathcal{M}$ with upper and lower arch configurations $\mathcal{A}_{\rm up}$ and $\mathcal{A}_{\rm down}$ respectively, as the product[7]

$$\mathrm{sig}(\mathcal{M}) \ = \ \mathrm{sig}(\mathcal{A}_{\rm up}) \ \mathrm{sig}(\mathcal{A}_{\rm down}) \ . \tag{6.13}$$

With this definition, we find that the signature of a meander only depends on its order and on its number of connected components through

$$\mathrm{sig}(\mathcal{M}) \ = \ (-1)^{k+n} \ . \tag{6.14}$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{n} M_n^{(k)} \ (-1)^{k-1} \ &= \ (-1)^{n-1} \sum_{\mathcal{A}_{\rm up} \in A_n} \ \sum_{\mathcal{A}_{\rm down} \in A_n} \mathrm{sig}(\mathcal{A}_{\rm up}) \ \mathrm{sig}(\mathcal{A}_{\rm down}) \\ &= \ (-1)^{n-1} s(n)^2 \\ &= \ \begin{cases} 0 & \text{if } n = 2p \\ (c_p)^2 & \text{if } n = 2p+1 \end{cases} \end{aligned} \tag{6.15}$$

This completes the proof of (6.1) for meanders.

[7] Note that in a previous definition, the signature of a semi–meander was referring to its upper arch configuration only. Considered as a particular meander, its signature is simply multiplied by that of the raibow of order n (6.7).

Combining (3.7) and (3.8) with (6.1), we get the equivalent set of rules

$$\begin{aligned}
\sum_{k=1}^{n} M_{2n}^{(2k-1)} &= \frac{c_{2n}^2}{2} = \sum_{k=1}^{n} M_{2n}^{(2k)} \\
\sum_{k=1}^{n+1} M_{2n+1}^{(2k-1)} &= \frac{c_{2n+1}^2 + c_n^2}{2} \\
\sum_{k=1}^{n} M_{2n+1}^{(2k)} &= \frac{c_{2n+1}^2 - c_n^2}{2} \\
\sum_{k=1}^{n} \bar{M}_{2n}^{(2k-1)} &= \frac{c_{2n}}{2} = \sum_{k=1}^{n} \bar{M}_{2n}^{(2k)} \\
\sum_{k=1}^{n+1} \bar{M}_{2n+1}^{(2k-1)} &= \frac{c_{2n+1} + c_n}{2} \\
\sum_{k=1}^{n} \bar{M}_{2n+1}^{(2k)} &= \frac{c_{2n+1} - c_n}{2} \ .
\end{aligned} \tag{6.16}$$

6.2. Inequalities

The above sum rules (6.16) lead to some immediate inequalities, by simply using the positivity of all the meandric numbers involved, namely

$$\begin{aligned}
M_{2n}^{(k)} &\leq \frac{c_{2n}^2}{2} \\
M_{2n+1}^{(2k-1)} &\leq \frac{c_{2n+1}^2 + c_n^2}{2} \\
M_{2n+1}^{(2k)} &\leq \frac{c_{2n+1}^2 - c_n^2}{2} \\
\bar{M}_{2n}^{(k)} &\leq \frac{c_{2n}}{2} \\
\bar{M}_{2n+1}^{(2k-1)} &\leq \frac{c_{2n+1} + c_n}{2} \\
\bar{M}_{2n+1}^{(2k)} &\leq \frac{c_{2n+1} - c_n}{2} \ ,
\end{aligned} \tag{6.17}$$

for all allowed k's.

In particular, according to the leading behaviour $c_n \sim 4^n$ when $n \to \infty$, this gives an upper bound on the numbers R and $\bar{R}$ (2.12) (2.13)

$$R \leq 16 \qquad \bar{R} \leq 4 \ . \tag{6.18}$$

Defining as before R_k and $\bar{R}_k$ as the leading terms in respectively meander and semi–meander numbers with k connected components for large order n (with $R_1 = R$ and $\bar{R}_1 = \bar{R}$)

$$M_n^{(k)} \sim (R_k)^n \qquad \bar{M}_n^{(k)} \sim (\bar{R}_k)^n \ , \tag{6.19}$$

this also gives the bounds

$$R_k \ \leq \ 16 \qquad \bar{R}_k \ \leq \ 4 \ . \tag{6.20}$$

The matrix model formulation of sect.5 gives rise to more interesting inequalities, relating meander and semi–meander numbers. They are based on the Cauchy–Schwarz inequality for the hermitian product (A,B) over complex matrices defined by

$$(A,B) \ = \ \text{Tr}(AB^\dagger) \ , \tag{6.21}$$

where the superscript $\dagger$ stands for the hermitian conjugate $B^\dagger = B^{t\ *}$, namely

$$|(A,B)|^2 \leq (A,A)\ (B,B) \ . \tag{6.22}$$

Starting from the expression (5.17), let us write

$$A = B^{(\alpha_1)} B^{(\alpha_2)} ... B^{(\alpha_l)} \quad \text{and} \quad B^\dagger = B^{(\alpha_{l+1})} ... B^{(\alpha_{2n})} \ , \tag{6.23}$$

then (6.22) becomes

$$\left|\text{Tr}\big(\prod_{i=1}^{2n} B^{(\alpha_i)}\big)\right|^2 \leq \text{Tr}\big(\prod_{i=1}^{l} B^{(\alpha_i)} \prod_{j=1}^{l} B^{(\alpha_{l+1-j})}\big) \text{Tr}\big(\prod_{i=l+1}^{2n} B^{(\alpha_i)} \prod_{j=1}^{2n-l} B^{(\alpha_{2n+1-j})}\big) \ . \tag{6.24}$$

Taking the thermodynamic Gaussian averages of both sides and summing over the colors $\alpha_i = 1, 2, ..., m$, we finally get

$$\sum_{k=1}^{n} M_n^{(k)}\ m^k \ \leq \ \big(\sum_{i=1}^{l} \bar{M}_l^{(i)}\ m^i\big)\big(\sum_{j=1}^{2n-l} \bar{M}_{2n-l}^{(j)}\ m^j\big) \ , \tag{6.25}$$

where the rhs has been identified by use of (5.20). This inequality between polynomials results in many inequalities for the meander and semi–meander numbers themselves.

In the trivial case $l = 0$, i.e. for the choice $A = \mathbf{I}$, we have

$$\sum_{k=1}^{n} M_n^{(k)}\ m^k \ \leq \ \sum_{k=1}^{2n} \bar{M}_{2n}^{(k)}\ m^k \ . \tag{6.26}$$

This inequality is in fact obviously satisfied coefficient by coefficient for $k=1,2,...,n$, as a meander of order n is a particular case of semi–meander of order $2n$ (which does not wind around the source of the river), namely

$$M_n^{(k)} \leq \bar{M}_{2n}^{(k)} \quad \forall\ k=1,2,...,n. \tag{6.27}$$

This implies the inequalities between the leading terms (6.20)

$$R_k \leq \bar{R}_k^2 \ \forall\ k=1,2,... \tag{6.28}$$

Before taking the large N limit, (6.24) also leads to an inequality involving higher genus meander and semi–meander numbers

$$\begin{aligned}\sum_{h\geq 0,k\geq 1} M_p^{(k)}(h)\ m^k\ N^{2-2h} \leq \Big(&\sum_{h\geq 0,k\geq 1} \bar{M}_{2l}^{(k)}(h)\ m^k\ N^{1-2h}\Big) \\ \times \Big(&\sum_{h\geq 0,k\geq 1} \bar{M}_{2p-2l}^{(k)}(h)\ m^k\ N^{1-2h}\Big)\ .\end{aligned} \tag{6.29}$$

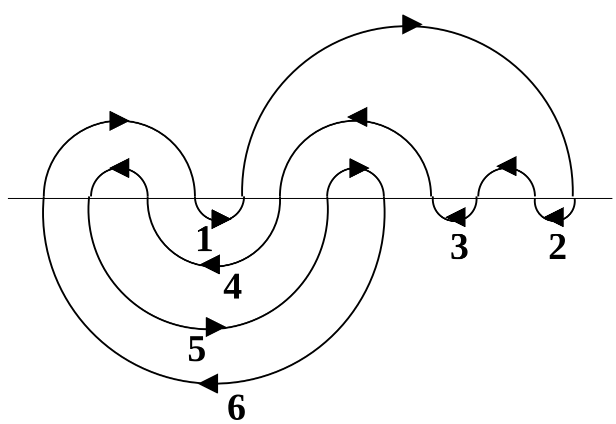


Fig. 21: The algorithm for closing a given upper arch configuration into a connected meander. The upper arch configuration $\mathcal{A}$ is of order 6. The path starts on the leftmost exterior arch of $\mathcal{A}$: the first lower arch connects it to the second exterior arch, and is numbered 1. Then the path goes to the left, and connects the depth 2 arches of $\mathcal{A}$ through lower arches number 2, 3 and 4. The leftmost depth 2 arch of $\mathcal{A}$ is connected to the depth 3 one through the lower arch number 5, and finally the depth 3 arch of $\mathcal{A}$ is connected to the leftmost exterior arch of $\mathcal{A}$ through the lower arch number 6.

A minoration of the connected meander numbers M_n is obtained as follows. For any given upper arch configuration $\mathcal{A}$, we have a simple algorithm for constructing its closing by a lower arch configuration, such that the resulting meander is connected. Consider the exterior arches of $\mathcal{A}$, and draw lower arches connecting any two consecutive exterior arches, as shown in Fig.21: this creates a path (oriented, say from the left to the right) connecting the exterior arches of $\mathcal{A}$. In a second step, the path is continued by connecting from the right to the left the arches of *depth 2* in $\mathcal{A}$, through lower arches which are either disjoint from the previous ones, or contain some of them. The lower arch configuration is further continued by connecting the arches of increasing depth j in $\mathcal{A}$, through lower arches. The resulting path zig–zags through the arches of $\mathcal{A}$, describing those of even depth from right to left and those of odd depth from left to right. Once the last arch of $\mathcal{A}$ is described, a last lower arch is constructed, connecting this last arch to the leftmost exterior arch we started with. The total being closed and going through all the arches of $\mathcal{A}$ as well as the lower ones, we have constructed a connected meander. Therefore the total number of connected meanders of order n, M_n, is larger than that of arch configurations of order n, c_n

$$M_n \ \geq \ c_n \ . \tag{6.30}$$

This implies that

$$R \ \geq \ 4 \qquad \bar{R} \ \geq \ 2 \ , \tag{6.31}$$

where we have used the inequality (6.28) for $k=1$.

6.3. Special cases

In this section, we compute exactly some of the generalized semi–meander numbers of sect.5.4. We will consider the closings of some particular lower arch configurations $\mathcal{A}_\mu$. It turns out that the numbers can always be found by relatively simple recursions provided the arch configuration $\mathcal{A}_\mu$ is the *repetition* (say p times) of some small order arch configuration.

Single arch. The simplest example is $\mathcal{A}_1$, obtained as the repetition (n times) of single arches of order 1, and corresponding to the permutation μ: $\mu(2i-1)=2i$, $i=1,2,...,n$. Let us compute the corresponding numbers $\bar{M}_n^{(k)}(\mathcal{A}_1)$ by the method of loop equations described in sect.5.5. The generating function $\alpha_n(m)$ for these numbers satisfies (5.22), namely

$$\alpha_n(m) \ = \ \sum_{k=1}^{n} \bar{M}_n^{(k)}(\mathcal{A}_1) \ m^k \ = \ \lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}\big(\sum_{j=1}^{m} (B^{(j)})^2 \big)^n \rangle_{\mathrm{Gauss}} \ . \tag{6.32}$$

Let us use the loop equation method of sect.5.4 to derive a recursion relation for $\alpha_n(m)$, by writing that the multi–matrix integral of a total derivative vanishes

$$0 \;=\; \int \prod_{\alpha=1}^{m} dB^{(\alpha)} \frac{\partial}{\partial B_{ji}^{(\beta)}} [[\sum_{r=1}^{m} (B^{(r)})^2]^{n-1} B^{(\beta)}]_{kl} e^{-N\mathrm{Tr}\sum_{s=1}^{m} \frac{B^{(s)}}{2}} \;, \tag{6.33}$$

hence, taking $i=l$ and $j=k$, and summing over $i,j=1,2,...,N$

$$\begin{aligned} N\langle \mathrm{Tr}\big([\sum_{r=1}^{m}(B^{(r)})^2]^{n-1}(B^{(\beta)})^2\big)\rangle_{\mathrm{Gauss}} &= \langle N\mathrm{Tr}\big([\sum_{r=1}^{m}(B^{(r)})^2]^{n-1}\big) \\ &+ \sum_{j=1}^{n-1} \mathrm{Tr}\big([\sum (B^{(r)})^2]^{j-1} B^{(\beta)}\big)\mathrm{Tr}\big([\sum (B^{(r)})^2]^{n-1-j} B^{(\beta)}\big) \\ &+ \mathrm{Tr}\big([\sum (B^{(r)})^2]^{j-1}\big)\mathrm{Tr}\big([\sum (B^{(r)})^2]^{n-1-j}(B^{(\beta)})^2\big)\rangle_{\mathrm{Gauss}} \;. \end{aligned} \tag{6.34}$$

By the factorization property (5.16), the term on the second line vanishes by imparity. Summing then over $\beta=1,2,...,m$ and letting $N\to\infty$, we finally get a recursion relation for $\alpha_n(m)$ (6.32)

$$\alpha_n(m) \;=\; (m-1)\alpha_{n-1}(m) + \sum_{j=0}^{n-1} \alpha_j(m)\;\alpha_{n-1-j}(m) \;, \tag{6.35}$$

whith the initial condition $\alpha_0(m)=1$. This determines uniquely the polynomials $\alpha_n(m)$ of degree n in m. Remarkably, we find

$$\alpha_n(m) \;=\; \sum_{k=1}^{n} I(n,k)\; m^k \;, \tag{6.36}$$

where the numbers $I(n,k)$ are given by (3.16) ! In particular, we find the average number of connected components in closings of $\mathcal{A}_1$

$$\langle \mathrm{comp}_1\rangle_n \;=\; \frac{\sum_{k=1}^{n} kI(n,k)}{c_n} \;=\; \frac{n+1}{2} \;. \tag{6.37}$$

The result (6.36) is readily checked by multiplying both sides of the recursion relation (3.15) by m^k and summing over k, which yields (6.35). So the number of arch configurations of order n with k interior arches is also the number of closings of $\mathcal{A}_1$ with k connected components. Note that there is $I(n,1)=1$ connected closing of $\mathcal{A}_1$ only, which may be obtained by the closing algorithm presented at the end of the previous section.

This remarkable coincidence can be explained as follows. Let us constuct a one–to–one mapping U between arch configurations, which sends arch configurations which close $\mathcal{A}_1$ with k connected components onto arch configurations with k interior arches. The map U is defined recursively as follows

$$
\begin{aligned}
U(\frown) &= \frown \\
U(X\ Y) &= U(X)\ U(Y) \\
U\big(\overbrace{\frown\!\!\!\!\mathbf{X}\ \ \mathbf{Y}}\big) &= \overbrace{\mathbf{U(\,X\,\frown\!\!\!\!Y\ )}}\ .
\end{aligned}
\qquad (6.38)
$$

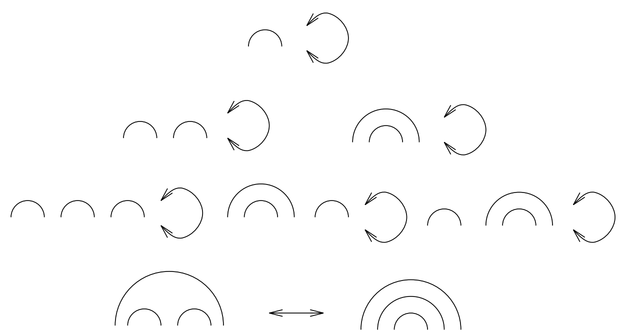


Fig. 22: The mapping U between arch configurations is indicated by arrows, for the cases of order 1, 2, 3. In these cases U appears to be involutive, but this is only an accident, as one readily checks for $n \geq 4$.

A few examples of the action of U are given in Fig.22. The coincidence between the number of connected components after closing a given configuration by $\mathcal{A}_1$, and the number of interior arches in its image by U needs only to be checked on a configuration with one exterior arch, because the second relation of (6.38) can be used to reduce any arch configuration into a succession of arch configurations with one exterior arch only, whereas the numbers of connected components in the closure by $\mathcal{A}_1$ and of interior arches both add up in a succession of arch configurations. Counting the number k of connected components in the lhs of the last line of (6.38), we find

$$
k = \#\text{c.c.}\big(\overbrace{\frown\!\!\!\!\mathbf{X}\ \ \mathbf{Y}}\big) = \#\text{c.c.}(X) + \#\text{c.c.}(\overbrace{\mathbf{Y}}) \qquad (6.39)
$$

On the other hand, the number of interior arches in the rhs is

$$
i = \#\text{int.}U(X) + \#\text{int.}U(\overbrace{\mathbf{Y}})\ . \qquad (6.40)
$$

Therefore, comparing (6.39) and (6.40), we get the desired result by recursion on the order of the arch configuration: the number of connected components in the closing by $\mathcal{A}_1$ is equal to the number of interior arches after the action of U.

Double arch. The repetition (say n times) of a double arch (rainbow of order 2) gives an arch configuration $\mathcal{A}_2$ of order $2n$. Let us compute the number $\bar{M}_{2n}^{(k)}(\mathcal{A}_2)$ of closings of this configuration with k connected components. As pointed out in the previous section, the loop equation method is parallel to the general recursion principle of Fig.6 for arch configurations. To further illustrate this similarity, let us compute $\bar{M}_{2n}^{(k)}(\mathcal{A}_2)$ in the language of arches, by deriving some recursion relations. When trying to apply the general process of Fig.6, we immediately see that the recursion will not involve closings of $\mathcal{A}_2$ only, but requires the introduction of some new arch configuration closings.

Let us introduce the notations

$$\begin{aligned} \lambda_{2n+1}^{(k)} &= \bar{M}_{2n+1}^{(k)}(\mathcal{A}_2 \frown \ ) \\ \mu_{2n}^{(k)} &= \bar{M}_{2n}^{(k)}(\mathcal{A}_2) \\ \nu_{2n+3}^{(k)} &= \bar{M}_{2n+3}^{(k)}(\frown \overbrace{\mathcal{A}_2} \frown) \ , \end{aligned} \tag{6.41}$$

for $n \geq 0$. The configuration $\mathcal{A}_2$ and its slight modifications listed in (6.41) form a system which is closed under the recursion principle of Fig.6.

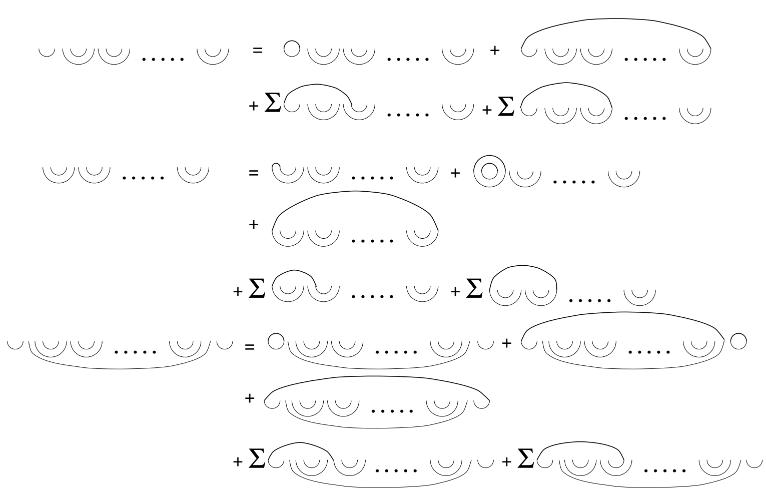


Fig. 23: The three recursion relations involving the numbers $\lambda_{2n+1}^{(k)}$, $\mu_{2n+2}^{(k)}$ and $\nu_{2n+3}^{(k)}$ respectively. The leftmost arch is indicated in thick solid line. The symbol Σ indicates a sum over the position of the right bridge of the leftmost arch.

More precisely, we find the three recursion relations, graphically depicted in Fig.23

$$\begin{aligned}
\lambda_{2n+1}^{(k)} &= \mu_{2n}^{(k-1)} + \mu_{2n}^{(k)} + \sum_{j=0}^{n-1}\sum_{i=1}^{k} \lambda_{2j+1}^{(i)}\lambda_{2n-2j-1}^{(k+1-i)} \\
&+ \sum_{j=1}^{n-1}\sum_{i=1}^{k-1} \mu_{2j}^{(i)}\mu_{2n-2j}^{(k-i)} \\
\mu_{2n+2}^{(k)} &= \lambda_{2n+1}^{(k)} + \mu_{2n}^{(k-2)} + \nu_{2n+1}^{(k)} \\
&+ \sum_{j=0}^{n-1}\sum_{i=1}^{k} \lambda_{2j+1}^{(i)}\mu_{2n-2j}^{(k+1-i)} + \sum_{j=1}^{n}\sum_{i=1}^{k-1} \mu_{2j}^{(i)}\nu_{2n+1-2j}^{(k-i)} \\
\nu_{2n+3}^{(k)} &= \mu_{2n+2}^{(k-1)} + \lambda_{2n+1}^{(k-1)} + \mu_{2n+2}^{(k)} \\
&+ \sum_{j=0}^{n}\sum_{i=1}^{k} \nu_{2j+1}^{(i)}\lambda_{2n+1-2j}^{(k+1-i)} + \sum_{j=1}^{n}\sum_{i=1}^{k} \mu_{2j}^{(i)}\mu_{2n+2-2j}^{(k+1-i)} \ ,
\end{aligned} \tag{6.42}$$

valid for $n \geq 0$ and $k \geq 1$. In terms of the generating functions

$$\begin{aligned}
\lambda(z,x) &= \sum_{n\geq 0}\sum_{k=1}^{2n+1} \lambda_{2n+1}^{(k)} x^k z^{2n+1} = zx + \ldots \\
\mu(z,x) &= \sum_{n\geq 0}\sum_{k=1}^{2n+2} \mu_{2n+2}^{(k)} x^k z^{2n+2} = z^2(x+x^2) + \ldots \\
\nu(z,x) &= \sum_{n\geq 0}\sum_{k=1}^{2n+3} \nu_{2n+3}^{(k)} x^k z^{2n+3} = z^3(x+3x^2+x^3) + \ldots
\end{aligned} \tag{6.43}$$

we find the following system of algebraic equations

$$\begin{aligned}
\lambda(z,x)\left(\frac{1}{z} - \frac{\lambda(z,x)}{x}\right) &= (1+\mu(z,x))(x+\mu(z,x)) \\
(\mu(z,x)+x)\left(\frac{1}{z} - \frac{\lambda(z,x)}{x}\right) &= (1+\mu(z,x))(\nu(z,x)+zx^2) + \frac{x}{z} \\
(\nu(z,x)+zx^2)\left(\frac{1}{z} - \frac{\lambda(z,x)}{x}\right) &= (x+\mu(z,x))\left(x+\frac{\mu(z,x)}{x}\right) .
\end{aligned} \tag{6.44}$$

Combining these three equations, we easily eliminate the functions λ and ν and get a polynomial equation of order 4 for $\mu(z,x)$

$$z^2(1+\mu)(x+\mu)(x+x^2+2\mu)^2 - x\mu(x+x^2+\mu) = 0 \ , \tag{6.45}$$

which, together with the initial condition $\mu(z,x) = z^2(x+x^2)+\ldots$, determines the numbers $\mu_{2n}^{(k)}$ completely. The functions λ and ν are simply expressed in terms of μ thanks to (6.44). For instance,

$$\lambda(z,x) = \frac{x\mu(z,x)}{z(x+x^2+2\mu(z,x))} \ , \tag{6.46}$$

in the sense of formal series expansion in powers of z.

It is interesting to check the validity of the result (6.44) by taking $x=1$, in which case the numbers $\sum_{1\le k\le 2n}\mu_{2n}^{(k)}=c_{2n}$ simply count the total number of arch configurations of order $2n$, namely

$$\mu(z,1) \;=\; \sum_{n\ge 1} c_{2n}\; z^{2n} \;=\; C_+(z)-1\;, \tag{6.47}$$

where C_+ is the even part of the Catalan generating function (3.3)

$$C_+(z) \;=\; \frac{\sqrt{1+4z}-\sqrt{1-4z}}{4z}\;. \tag{6.48}$$

The relation (6.44) reduces for $x=1$ to

$$4z^2C_+(z)^4 \;=\; C_+(z)^2-1\;, \tag{6.49}$$

in agreement with (6.48). For $x=-1$, we find that $\mu(z,-1)=0$, which translates into the sum rule

$$\sum_{k=1}^{2n}(-1)^k\;\mu_{2n}^{(k)} \;=\; 0\;. \tag{6.50}$$

Let us now use the equation (6.44) to derive the leading behavior of the average number of connected components over closings of $\mathcal{A}_2$. This number reads

$$\langle \mathrm{comp}_2\rangle_{2n} \;=\; \frac{\dot\mu_{2n}}{c_{2n}} \;=\; \frac{\sum_{k=1}^{2n} k\;\mu_{2n}^{(k)}}{c_{2n}}\;. \tag{6.51}$$

We extract from (6.44) the generating function $\dot\mu(z)=\partial_x\mu(z,x)|_{x=1}$ of $\dot\mu_{2n}$ by differentiating it wrt x and then setting $x=1$

$$\dot\mu(z) \;=\; \frac{(C_+(z)+4)(C_+(z)-1)-16z^2C_+(z)^3}{C_+(z)(16z^2C_+(z)^2-2)}\;. \tag{6.52}$$

From the study of the limit $z\to 1/4^-$ in this last expression, we find the asymptotics of $\dot\mu_{2n}$ namely

$$\begin{aligned}\dot\mu(z) \;&\sim\; \frac{\sqrt{2}-1}{2\sqrt{1-(4z)^2}}\\ \Rightarrow\;\dot\mu_{2n} \;&\sim\; \frac{\sqrt{2}-1}{\sqrt{2}}\,\frac{4^{2n}}{\sqrt{\pi}(2n)^{\frac{1}{2}}}\;.\end{aligned} \tag{6.53}$$

Dividing this by the large n asymptotics of $c_{2n}\;\sim 4^{2n}/(\sqrt{\pi}(2n)^{3/2})$, we finally get the leading behavior of the average number of connected components in closings of $\mathcal{A}_2$

$$\langle \mathrm{comp}_2\rangle_{2n} \;=\; 2n\big(1-\frac{1}{\sqrt{2}}\big)\;. \tag{6.54}$$

This is to be compared with the analogous expression (6.37) for $\mathcal{A}_1$ closings, which gives asymptotically $\langle \text{comp}_1 \rangle_n \sim n/2$. There are much less connected components per bridge in average closings of $\mathcal{A}_2$ than $\mathcal{A}_1$.

The connected numbers $\mu_{2n}^{(1)} = \bar{M}_{2n}(\mathcal{A}_2)$ are generated by the coefficient of x in the series expansion of $\mu(z,x)$, namely

$$\mu(z,x) \ = \ x\mu(z) + O(x^2) \ = \ x \sum_{n=1}^{\infty} \bar{M}_{2n}(\mathcal{A}_2) \ z^{2n} + O(x^2) \ , \tag{6.55}$$

hence as a consequence of (6.45), the series $\mu(z)$ satisfies an algebraic equation of order 2

$$z^2(1+2\mu(z))^2 - \mu(z) \ = \ 0 \ , \tag{6.56}$$

which we can rewrite as

$$2z^2(2\mu(z)+1)^2 = (2\mu(z)+1) - 1 \ . \tag{6.57}$$

We recognize the second degree equation for the generating function for Catalan numbers (3.4), upon the substitution $2\mu(z)+1 = C(2z^2)$. We deduce that

$$\mu_{2n}^{(1)} \ = \ \bar{M}_{2n}(\mathcal{A}_2) \ = \ 2^{n-1} c_n \qquad \forall \, n \geq 1 \ . \tag{6.58}$$

Analogously, we find that

$$\begin{aligned} \lambda_{2n+1}^{(1)} \ &= \ 2^n c_n \\ \nu_{2n+1}^{(1)} \ &= \ 2^{n-1}(c_{n+1} - c_n) \ . \end{aligned} \tag{6.59}$$

The $\lambda_{2n+1}^{(1)}$ connected closings of $\mathcal{A}_2$ completed by a single arch are obtained, starting from the $\mathcal{A}_2$ connected closings of order $(2n)$, by the process (I) of Fig.10. Indeed, the resulting meander is of order $(2n+1)$ and has a single lower exterior arch surrounding $\mathcal{A}_2$. By cyclicity along the infinite river, this arch can be sent to the right of $\mathcal{A}_2$, thus forming one of the $\lambda_{2n+1}^{(1)}$ meanders. Therefore, $\lambda_{2n+1}^{(1)}$ is equal to the total number of exterior arches in connected closings of $\mathcal{A}_2$ (each exterior arch gives rise to one action by (I)). The average number of exterior arches in connected closings of $\mathcal{A}_2$ reads therefore

$$\langle \text{ext}_2 \rangle_{2n} \ = \ \frac{\lambda_{2n+1}^{(1)}}{\mu_{2n}^{(1)}} \ = \ 2, \tag{6.60}$$

independent of n. On the other hand, due to the asymptotic behaviour of the Catalan numbers ($c_n \sim 4^n$, $n \to \infty$), we find that for large n

$$M_{2n}(\mathcal{A}_2) \sim (2\sqrt{2})^{2n} . \tag{6.61}$$

More cases. The triple arch case $\mathcal{A}_3$ (repetition n times of the rainbow $\mathcal{R}_3$ of order 3) involves the introduction of 9 slightly different arch configurations. We will not give the details of the solution here, and simply give the result: the generating function for the number of connected closings of $\mathcal{A}_3$,

$$\theta(z) = \sum_{n \geq 1} \bar{M}_{3n}(\mathcal{A}_3)\, z^{3n} , \tag{6.62}$$

satisfies the polynomial equation of order 4

$$2z^3(1+\theta(z))(1+3\theta(z))^3 - \theta(z)(1+2\theta(z))^2 = 0 . \tag{6.63}$$

The asymptotic behaviour of the numbers $\bar{M}_{3n}(\mathcal{A}_3)$ is found by solving the coupled system of equations

$$\begin{aligned} 2z^3(1+\theta)(1+3\theta)^3 - \theta(1+2\theta)^2 &= 0 \\ \frac{\partial}{\partial\theta}\big(2z^3(1+\theta)(1+3\theta)^3 - \theta(1+2\theta)^2\big) &= 0 . \end{aligned} \tag{6.64}$$

The solution (θ^*, z^*) is such that

$$\bar{M}_{3n}(\mathcal{A}_3) \sim (z^*)^{-3n} = (2.842923..)^{3n} . \tag{6.65}$$

The above behaviour is to be compared with that of connected closings of single arch repetitions $\sim 1^n$ and double arch repetitions $\sim (2\sqrt{2})^{2n} = (2.828..)^{2n}$. We see that very little entropy has been gained when replacing double arches by triple arches. Intuitively, these numbers should form an increasing sequence (with increasing order of the elementary raibow which is repeated), which may very well converge to the number $\bar{R}$, governing the large n asymptotics of semi–meander numbers.

Another case of interest is the repetition $\mathcal{B}_p$ (n times) of the arch configuration of order p obtained as a repetition of $(p-1)$ single arches of order 1 below a single arch, i.e. corresponding to the permutation $\mu(1) = 2p$, $\mu(2i) = 2i+1$, $i = 1, 2, ..., p-1$. The generating function

$$\beta(z) = \sum_{n=1}^{\infty} \bar{M}_{np}(\mathcal{B}_p)\, z^{np} \tag{6.66}$$

for the numbers of connected closings of $\mathcal{B}_p$ can be shown to satisfy a polynomial equation of order p

$$z^p(1+p\beta(z))^p - \beta(z) \;=\; 0 \; . \tag{6.67}$$

This leads to the large n behaviour

$$\bar{M}_{np}(\mathcal{B}_p) \;\sim\; \left(p^{\frac{p+1}{p}}(p-1)^{\frac{1-p}{p}}\right)^{np} \; . \tag{6.68}$$

For $p=2$, we recover the result (6.61) for double arch repetitions.

Note that, in view of the above cases, an empiric rule seems to emerge, indicating that, for fixed order, the number of connected closings of a motive is increased by nesting the arches rather than juxtaposing them. If this is true, it means that the number of closings of a given motive is maximal with the rainbow (semi–meander). We have checked this conjecture numerically up to order 12.

7. Irreducible meanders, exact results

A multi–component meander is said to be k–**reducible** if one (proper) subset of its connected components can be detached from it by cutting the river k times between the bridges. A multi–component meander is said to be k–**irreducible** if it is *not* k–reducible, i.e. if no (proper) subset of its connected components can be detached from it by cutting the river k times between the bridges.

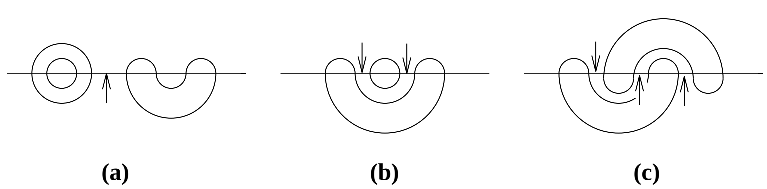


Fig. 24: Reducibility and irreducibility for meanders. The cuts reducing the meanders are indicated by arrows. The meander (a) is 1–reducible, i.e. the succession of two meanders along the river. The meander (b) is 1–irreducible and 2–reducible. The meander (c) is 1–and 2–irreducible and 3–reducible.

In Fig.24, we give a few examples to illustrate the notion of reducibility and irreducibility of meanders. The same definition applies to the semi–meanders in the formulation with a semi–infinite river.

7.1. 1–irreducible meanders and semi–meanders

A meander is 1–irreducible if it is not the succession along the river of at least two meanders. We can enumerate all the meanders by their growing number of 1–irreducible components. Denoting by $P_n^{(k)}$ the total number of 1–irreducible meanders of order n with k connected components, we compute

$$M_n^{(k)} = \sum_{\substack{n_1+n_2+\ldots=n \\ k_1+k_2+\ldots=k}} P_{n_1}^{(k_1)} P_{n_2}^{(k_2)} \ldots \ , \tag{7.1}$$

hence the generating functions

$$\begin{aligned} M(z,x) &= \sum_{n\geq k\geq 1} M_n^{(k)} \ z^{2n} \ x^k \\ P(z,x) &= \sum_{n\geq k\geq 1} P_n^{(k)} \ z^{2n} \ x^k \end{aligned} \tag{7.2}$$

satisfy the equation

$$M(z,x) = P(z,x) + P(z,x)^2 + P(z,x)^3 + \ldots = \frac{P(z,x)}{1-P(z,x)} \ . \tag{7.3}$$

An analogous reasoning for semi–meanders leads to the relation

$$\frac{\bar{P}(z,x)}{1-P(z,x)} = \bar{M}(z,x) \ , \tag{7.4}$$

between the generating functions

$$\begin{aligned} \bar{M}(z,x) &= \sum_{n\geq k\geq 1} \bar{M}_n^{(k)} \ z^{2n} \ x^k \\ \bar{P}(z,x) &= \sum_{n\geq k\geq 1} \bar{P}_n^{(k)} \ z^{2n} \ x^k \end{aligned} \tag{7.5}$$

of respectively semi–meander and 1–irreducible semi–meander numbers of order n with k connected components, and $P(z,x)$ defined in (7.2).

7.2. 2–irreducible meanders and semi–meanders

The 2–irreducible meanders have been studied in [1] extensively (under the name of *irreducible meanders*). They use the following equivalent characterization of 2–irreducible meanders: for any subset of connected components of a 2–irreducible meander, its set of bridges is not consecutive. Indeed, if it were not the case, one could cut the river before the first bridge and after the last one and detach the corresponding piece. One easily checks on Fig.24 (c) that the two sets of bridges of the two connected components of the meander are intertwined.

For completeness, we reproduce here their computation of the numbers q_n of 2–irreducible meanders of order n, by slightly generalizing their argument to include the numbers $Q_n^{(k)}$ of 2–irreducible meanders of order n with k connected components. The idea is to enumerate the $M_n^{(k)}$ meanders by focussing on their leftmost 2–irreducible component, namely the largest 2–irreducible subset of its connected components, containing the leftmost bridge.

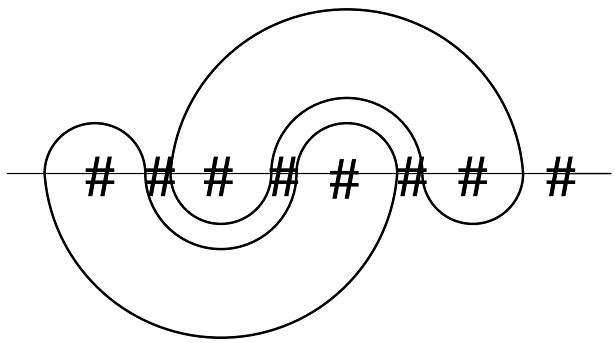


Fig. 25: A general meander. The leftmost 2–irreducible component is represented in thick solid lines. The positions marked by # can be decorated with any meanders to get the most general meander with this leftmost 2–irreducible component.

In Fig.25, the leftmost 2–irreducible piece of a meander is depicted in thick solid line. Suppose this piece is of order p and has l connected components. The most general meander having this leftmost 2–irreducible piece can be obtained by decorating any segment of river between two consecutive bridges with arbitrary meanders of respective orders n_1, n_2, ... n_{2p}, $p + n_1 + ... + n_{2p} = n$ (there are $2p$ positions, indicated by the symbols # in Fig.25, for these possible decorations), and with respective numbers of connected components k_1, k_2, ... k_{2p}, with $l + \sum k_i = k$. We get the relation

$$M_n^{(k)} = \sum_{\substack{p+n_1+...+n_{2p}=n \\ l+k_1+...+k_{2p}=k}} Q_p^{(l)} \, M_{n_1}^{(k_1)} \, M_{n_2}^{(k_2)} \, ... \, M_{n_{2p}}^{(k_{2p})} \, , \tag{7.6}$$

with the convention that $M_0^{(0)} = 1$. In terms of the generating functions $M(z, x)$ of eq.(7.2) and

$$Q(z, x) = \sum_{n \geq k \geq 1} Q_n^{(k)} \, z^{2n} \, x^k \, , \tag{7.7}$$

the relation (7.6) reads

$$M(z,x) \;=\; Q(z(1+M(z,x)),x) \;. \tag{7.8}$$

In the special case $x=1$, if we denote by

$$\begin{aligned} B(z) &= 1+M(z,1) = \sum_{n=0}^{\infty}(c_n)^2 z^{2n} \\ q(z) &= 1+Q(z,1) = \sum_{n=0}^{\infty} q_n \; z^{2n} \;, \end{aligned} \tag{7.9}$$

then (7.8) reduces to

$$B(z) \;=\; q(zB(z)) \;. \tag{7.10}$$

The radius of convergence of the series $B(z)$ is $1/4$, due to the asymptotics of c_n, hence that of $q(z)$ is

$$\begin{aligned} z^* &= \frac{1}{4}B(\frac{1}{4}) = \frac{1}{4}\sum_{n\geq 0} c_n^2 \; 4^{-2n} \\ &= \frac{1}{4}C(\frac{x}{4})C(\frac{1}{4x})\big|_{x^0} \\ &= \frac{1}{4}\oint \frac{dx}{2i\pi x}C(\frac{x}{4})C(\frac{1}{4x}) \\ &= \oint \frac{dx}{2i\pi x}(1-\sqrt{1-x})(1-\sqrt{1-1/x}) \\ &= \oint \frac{dx}{2i\pi x}(-1+\sqrt{2-x-1/x}) \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta(-1+2\sin\frac{\theta}{2}) \\ &= \frac{4-\pi}{\pi} \;. \end{aligned} \tag{7.11}$$

Hence a leading behaviour for large n [1]

$$q_n \;\sim\; \left(\frac{\pi}{4-\pi}\right)^{2n} \;. \tag{7.12}$$

This gives an upper bound on the leading behaviour of M_n

$$R \;\leq\; \left(\frac{\pi}{4-\pi}\right)^2 \;=\; 13.3923... \tag{7.13}$$

A refined study of the equation (7.10) also yields the subleading power correction

$$q_n \sim \frac{1}{n^3}\left(\frac{\pi}{4-\pi}\right)^{2n} \quad . \tag{7.14}$$

Let us extend these considerations to the semi–meander case. If $\bar{Q}_n^{(k)}$ denotes the number of 2–irreducible semi–meanders of order n with k connected components, let us enumerate the $\bar{M}_n^{(k)}$ meanders of order n with k connected components, focussing on their leftmost 2–irreducible piece, say of order p with l connected components.

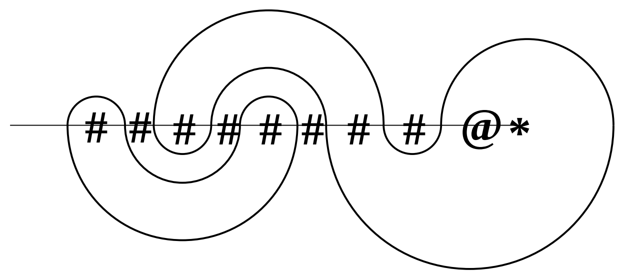


Fig. 26: A general semi–meander. The leftmost 2–irreducible component is represented in thick solid lines. The positions marked by # can be decorated with any meanders to get the most general semi–meander with this leftmost 2–irreducible component. The last position, indicated by @, can be decorated with any semi–meander, possibly winding around the source of the river ($*$).

The most general semi–meander with given leftmost 2–irreducible component (as indicated by thick solid lines in Fig.26) is obtained by decorating the segments of river between any two consecutive bridges (there are $(p-1)$ such positions) with meanders of respective orders n_1, n_2,... n_{p-1}, with respectively k_1, ...,k_{p-1} connected components, and the segment of river between the source and the first bridge by any semi–meander of order q, with r connected components, such that $p+q+2\sum n_i = n$ and $l+r+\sum k_i = k$. This amounts to the following relation

$$\bar{M}_n^{(k)} = \sum_{\substack{p+q+2(n_1+\cdots+n_{p-1})=n \\ l+r+k_1+\cdots+k_{p-1}=k}} \bar{Q}_p^{(l)} M_{n_1}^{(k_1)} \ldots M_{n_{p-1}}^{(k_{p-1})} \bar{M}_q^{(r)} \ , \tag{7.15}$$

where the sum extends over non–negative values of n_i and k_i, with the convention that $M_0^{(0)} = 1 = \bar{Q}_0^{(0)} = \bar{M}_0^{(0)}$. This translates into a relation between the generating functions

$$\begin{aligned} \bar{M}(z,x) &= \sum_{n\geq k\geq 1} \bar{M}_n^{(k)}\ z^n\ x^k \\ \bar{Q}(z,x) &= \sum_{n\geq k\geq 1} \bar{Q}_n^{(k)}\ z^n\ x^k \end{aligned} \tag{7.16}$$

and $M(z,x)$ of eq.(7.2) (note that in (7.16) we have given a weight z per bridge in the *semi–infinite river* framework, while there would be twice that number of bridges in the *rainbow closing* framework)

$$\begin{aligned} \bar{M}(z,x) &= \bar{Q}(z(1+M(z,x)),x)\ \frac{1+\bar{M}(z,x)}{1+M(z,x)} \\ \Rightarrow \frac{\bar{M}(z,x)}{1+\bar{M}(z,x)}(1+M(z,x)) &= \bar{Q}(z(1+M(z,x)),x)\ . \end{aligned} \tag{7.17}$$

In the special case $x=1$, with the generating function

$$\bar{q}(z) = 1+\bar{Q}(z,1) = \sum_{n\geq 0} \bar{q}_n\ z^n, \tag{7.18}$$

where $\bar{q}_n = \sum_{k=1}^{n} \bar{Q}_n^{(k)}$ is the total number of 2–irreducible multi–component semi–meanders of order n, and with $C(z)=1+\bar{M}(z,1)$, $B(z)=1+M(z,1)$ defined respectively in (3.3) (7.9), the relation (7.17) reduces to

$$1+zB(z)C(z) = \bar{q}(zB(z))\ , \tag{7.19}$$

where we used the fact that $(C(z)-1)/C(z) = zC(z)$ (3.4). Reasoning as above, we find the convergence radius of the series $\bar{q}(z)$, $z^* = B(1/4)/4 = (4-\pi)/\pi$. Hence we have the asymptotics

$$\bar{q}_n \sim \left(\frac{\pi}{4-\pi}\right)^n\ . \tag{7.20}$$

This implies the upper bound on the leading behaviour of $\bar{M}_n$

$$\bar{R} \leq \frac{\pi}{4-\pi} = 3.659... \tag{7.21}$$

This upper bound is below the mean field estimate (4.46) which therefore cannot be exact and needs to be improved. Again, a more refined study of the relation (7.19) gives the subleading power law

$$\begin{aligned} \bar{q}_n &\sim \frac{1}{n^{\frac{3}{2}}}\left(\frac{\pi}{4-\pi}\right)^n \\ &\sim \sqrt{q_n}. \end{aligned} \tag{7.22}$$

8. Conclusion

In this paper we have studied the statistics of arch configurations, meanders and semi–meanders. This study emphasizes the role of exterior arches, instrumental in the recursive generation of both semi–meanders and arch configurations, and whose average number is directly linked to the corresponding entropies. A complete solution of these problems however requires the knowledge of the correlation between these exterior arches and those of a given depth (see for instance eqs.(4.11)(4.12)). Discarding these correlations leads to a rough mean field result.

Using the alternative formulation (5.17)-(5.20) in the framework of random matrix models, we are left with the computation of particular Gaussian matrix averages of traces of words. Equation (5.31) provides a recursive way of computing any such average. As a by–product, we have derived an exact formula (5.43) expressing the semi–meander numbers in terms of the characters of the symmetric group.

Finally, we have presented the exact solutions for simpler meandric problems such as the closing of particular lower arch configurations (made of rereated motives), as well as the 2–irreducible meanders and semi–meanders.

In conclusion, we are still lacking of an efficient treatment of the main recursion relation for semi–meanders. Beyond the entropy problem, it would also be interesting to determine values of critical exponents in this problem, such as the exponent governing the subleading large order behaviour of the meander and semi–meander numbers (2.11). The latter are nothing but the usual α and γ configuration exponents of polymer chains. From this point of view, a fundamental issue is the determination of the universality class of the (self–avoiding) chain folding problem.

Acknowledgements

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Appendix A. Combinatorial identities for arch statistics

Many combinatorial identities can be proved directly by identifying the result as a certain coefficient, say of x^n, in a (Laurent) series expansion of some function $f(x)$ around $x=0$. In the following, we use the notation

$$f(x)\big|_{x^n} \equiv \oint \frac{dx}{2i\pi}\frac{f(x)}{x^{n+1}} \tag{A.1}$$

for the coefficient of x^n in the small x expansion of the function f.

Proof of eq.(3.12). We want to prove the identity (3.12), which may be rewritten as

$$\begin{aligned}\sum_{k=1}^{n}\frac{k}{2n-k}\binom{2n-k}{n}\binom{k}{l} &= \sum_{m=0}^{n-1}\frac{n-m}{n}\binom{n+m-1}{m}\binom{n-m}{l} \\ &= \frac{2l+1}{2n+1}\binom{2n+1}{n-l}\,,\end{aligned}\tag{A.2}$$

where we changed the summation variable to $m=n-k$. The above sum over m can be expressed as

$$\sum_{\substack{r,m\geq 0\\ r+m=n-l}}\binom{n+m-1}{m}\times\frac{r+l}{n}\binom{r+l}{r}\,,\tag{A.3}$$

where we introduced a second summation over $r=n-l-m$. Using the infinite series expansion

$$\frac{1}{(1-x)^{\alpha}} = \sum_{m=0}^{\infty}\binom{m+\alpha-1}{m}x^m\,,\tag{A.4}$$

we identify the expression (A.3) as the coefficient of x^{n-l} of some series

$$\begin{aligned}&\frac{1}{(1-x)^n}\times\frac{1}{n}(l+x\frac{d}{dx})\frac{1}{(1-x)^{l+1}}\bigg|_{x^{n-l}} \\ &= \frac{l}{n(1-x)^{n+l+1}}\bigg|_{x^{n-l}}+\frac{l+1}{n(1-x)^{n+l+2}}\bigg|_{x^{n-l-1}} \\ &= \frac{l}{n}\binom{n+l+n-l}{n-l}+\frac{l+1}{n}\binom{n+l+1+n-l-1}{n-l-1} \\ &= \frac{1}{n}(l(n+l+1)+(l+1)(n-l))\frac{(2n)!}{(n+l+1)!(n-l)!} \\ &= \frac{2l+1}{2n+1}\binom{2n+1}{n-l}\,.\end{aligned}\tag{A.5}$$

This proves the identity (3.12).

Proof of the identity (3.17). We want to prove the identity (3.17), which reads

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n}\binom{k}{l}\binom{n}{k}\binom{n}{k-1} \\ &\qquad = \frac{1}{n}\binom{n}{l}\sum_{k=1}^{n}\binom{n-l}{k-l}\binom{n}{k-1} \\ &\qquad = \frac{1}{n}\binom{n}{l}\binom{2n-l}{n-l+1}\,.\end{aligned}\tag{A.6}$$

The sum over k in the second line of (A.6) is nothing but the coefficient of x^{l-1} of some Laurent series, identified as

$$\begin{aligned} \sum_{k=1}^{n} \binom{n-l}{k-l}\binom{n}{k-1} &= (1+x)^{n-l}(1+\frac{1}{x})^n \Big|_{x^{1-l}} \\ &= (1+x)^{2n-l}\Big|_{x^{n-l+1}} = \binom{2n-l}{n-l+1} \ . \end{aligned} \tag{A.7}$$

This completes the proof of the identity (3.17).

Appendix B. Stability of the mean field fixed points for semi–meanders.

The mean field equations for arch numbers in connected semi–meanders read

$$\begin{aligned} \frac{A_1(n+1,0)}{A_1(n,0)}\langle A(\mathcal{M},j)\rangle_{n+1,1} &= \langle A(\mathcal{M},j+1)\rangle_{n,1} + \langle [A(\mathcal{M})-1]\rangle_{n,1}\langle A(\mathcal{M},j-1)\rangle_{n,1} \\ \frac{A_1(n+1,0)}{A_1(n,0)}\langle A(\mathcal{M},1)\rangle_{n+1,1} &= \langle A(\mathcal{M},2)+2A(\mathcal{M},1)\rangle_{n,1} \\ \frac{A_1(n+1,0)}{A_1(n,0)} &= \langle A(\mathcal{M},1)\rangle_{n,1} \ . \end{aligned} \tag{B.1}$$

To linearize this system around the fixed point $a_1(j)$ of eq.(4.23), for fixed $\bar{R} > 3$, we write

$$\langle A(\mathcal{M},j)\rangle_{n,1} = \epsilon_{n,j} + \frac{\bar{R}}{\bar{R}-2}(1+(\bar{R}-3)(\bar{R}-1)^{j-1}) \ . \tag{B.2}$$

Up to first order terms in $\epsilon_{n,j}$, we find

$$\begin{aligned} \epsilon_{n+1,j} &= \frac{1}{\bar{R}}\epsilon_{n,j+1} + \frac{\bar{R}-1}{\bar{R}}\epsilon_{n,j-1} - (\bar{R}-3)(\bar{R}-1)^{j-2}\epsilon_{n,1} \\ \epsilon_{n+1,1} &= \frac{1}{\bar{R}}\epsilon_{n,2} - \frac{\bar{R}-2}{\bar{R}}\epsilon_{n,1} \ . \end{aligned} \tag{B.3}$$

Let $M(N)$ be the $N \times N$ matrix associated to this linear system, when restricted to values of $j = 1, 2, ..., N$, in which case (B.3) can be rewritten as $E_{n+1} = M(N)E_n$, in terms of the vectors $E_n = (\epsilon_{n,1}, \epsilon_{n,2}, ..., \epsilon_{n,N})$. Let us prove that some eigenvalues of $M(N)$ are larger than 1, hence that the system is unstable, for N large enough. Using the fact that the largest eigenvalue $\lambda_{\max}(N)$ of $M(N)$ satisfies

$$\lambda_{\max}(N) = \sup \frac{x^t M x}{x^t x} \ , \tag{B.4}$$

for any non–vanishing vectors x, we find a minorant of $\lambda_{\max}(N)$ by evaluating $x^t Mx/x^t x$ for some particular vector x. With the choice $x = (-1, 1, 1, ..., 1)$, we find that

$$\begin{aligned} \lambda_{\max}(N) &\geq \frac{1}{N}\left(\frac{\bar{R}-2}{\bar{R}} + (N-1)(\frac{1}{\bar{R}} + \frac{\bar{R}-1}{\bar{R}}) + (\bar{R}-3)\sum_{j=2}^{N}(\bar{R}-1)^{j-2}\right) \\ &= \frac{\bar{R}-2}{N\bar{R}} + \frac{N-1}{N} + \frac{\bar{R}-3}{\bar{R}-2}((\bar{R}-1)^{N-1} - 1) \ . \end{aligned} \qquad (B.5)$$

We see that for N large enough, this lower bound gets larger than 1, provided $\bar{R} > 3$. This completes the proof of the unstability of the fixed point $a_1(j)$ for all $\bar{R} > 3$. Hence the limit can only be the fixed point at $\bar{R} = 3$.

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