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Introduction to Conformal Invariance in Statistical Mechanics and to Random Surface Models

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0 Introduction

In the first part of these lectures I give a brief and somewhat superficial introduction to the technics of conformal invariance and to a few applications in statistical mechanics in two dimensions. My purpose is to introduce the basic ideas and some standard results for the students which are not familiar with the theory, and to introduce concepts and tools which will be useful for the other lecturers, rather than to give a complete and up to date review of the subject. There are now several very good and complete lecture notes, review papers and books available, including the seminal paper by Belavin, Polyakov and Zamolochikov (Belavin et al. (1984)), and the content of these lectures will be by no mean

original. Most of the material is inspired from the Les Houches '88 lectures by P. Ginsparg (Ginsparg (1989)) and by J. Cardy (Cardy (1989)), and from some lectures notes by P. Di Francesco, P. Mathieu and D. Senechal (Di Francesco et al. (1994)). More bibliography is provided by Cardy (1987), Dotsenko (1988), Christe and Henkel (1993), Itzykson et al. (1988),...

In the second part I discuss several problems in the statistical mechanics of two dimensional random surfaces and membranes. As an introduction, I present some basic facts about the statistical mechanics of one-dimensional objects and polymers, which are classical examples of objects with critical properties. Then I emphasize the special role of curvature energy and of the elastic energy associated with the internal structure of membranes, and the corresponding models of random surfaces. Finally, I discuss the specific problem of self-avoiding tethered surfaces, whose critical properties are still poorly understood, and for which the applicability of some basic technics of field theory, such as renormalization group calculations, has been understood only recently.

I am very grateful to the organizers (and especially H. Geyer and E. Pheiffer) and to all its participant for making this school so successful.

1 Introduction to Conformal Invariance in Statistical Mechanics

1.1 Renormalization Group, Scale and Conformal Invariance

• Critical Points and Renormalization Group

It is known that for a model of statistical mechanics at a second order critical point, some correlation lengths diverge, so that this model is scale invariant at large distances, and its large distance properties are related to those of a continuum Euclidean quantum field theory. The seminal example is the Ising model, corresponding to classical spins $\sigma_i = \pm 1$ on the sites i of a hypercubic lattice in D dimensions, with ferromagnetic interactions between spins pairs $< ij >$ of nearest-neighbouring sites. The energy of a spin configuration is

$$H[\sigma] \;=\; J \sum_{<ij>} (-\sigma_i \sigma_j) \;+\; H \sum_i \sigma_i \qquad 1.1.1$$

In the J-H half plane ($J > 0$ is the ferromagnetic coupling, H is the magnetic field), the model has a critical point at $J = J_c$, $H = 0$. It separates along the $H = 0$ line the ferromagnetic phase for $J < J_c$ from the paramagnetic phase for $J > J_c$.

Close to the critical point, the correlation length ξ is much larger than the microscopic lattice mesh a, and one can perform "block-spin" transformations by considering the average magnetisation $\phi(r)$ at the point r, defined on a squared block of sites of size ℓ around r, with $a < \ell < \xi$, as

$$\phi(r) \;=\; \ell^{-D} \sum_{i \in \text{block}} \sigma_i \qquad (1.1.2)$$

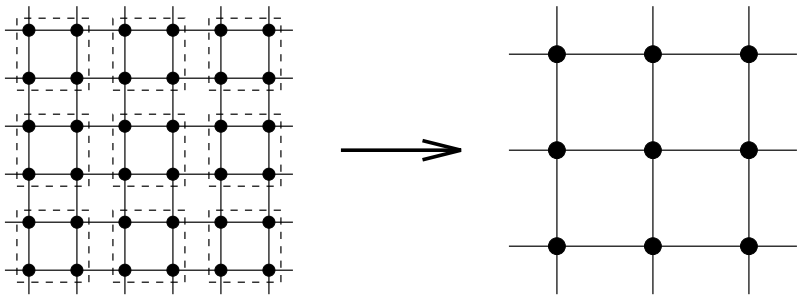

Fig. 1. Block spin transformation

The idea of renormalization group is to express the large distance physics in terms of an effective Hamiltonian $H_{\rm eff}[\phi]$ for the block-spin variable $\phi(r)$, valid at distances larger than ℓ. It is intuitively clear that in the limit $\ell \to \infty$ the variable ϕ will become a real continuous field, and that the effective Hamiltonian $H_{\rm eff}$ should involve *local* polynomials $\mathcal{O}_\alpha(r)$ in the field ϕ and its spacial derivatives $\nabla_r\phi$, $\nabla_{rr}\phi$, ... The variation of the rescaling factor $b = \ell/a$ defines a flow in the space of effective Hamiltonians $H_{\rm eff}$. The behavior of this flow for large b will be governed by RG fixed points, which describe the possible large distance physical properties of the system. For instance, when looking only at large distance properties, it is equivalent to consider the spin correlation functions

$$\langle \sigma_i \sigma_j \cdots \rangle \;=\; \sum_{[\sigma]} \sigma_i \sigma_j \cdots \mathrm{e}^{-H[\sigma]} \Big/ \sum_{[\sigma]} \mathrm{e}^{-H[\sigma]} \tag{1.1.3}$$

or the correlation functions for the effective field

$$\langle \phi(r_1)\phi(r_2)\cdots \rangle \;=\; \sum_{\phi(r)} \phi(r_1)\phi(r_2)\cdots \mathrm{e}^{-H_{\rm eff}[\phi]} \Big/ \sum_{\phi(r)} \mathrm{e}^{-H_{\rm eff}[\phi]} \tag{1.1.4}$$

This is the crude, but very general, picture of the renormalization group, which is expected to be valid under some general assumptions (locality of interactions, i.e. no long range forces, and homogeneity, i.e. position independent interactions). For a general presentation, I refer to Wilson and Kogut (1974), Wegner (1976), Brézin et al. (1976). To describe a system, one starts from the microscopic Hamiltonian H, which depends on a set of couplings κ_α (in practice only a finite number of them are non-zero, for instance J and H for the Ising model). Integrating out the short-distance (high momentum) degrees of fredom with wave-length $a < \ell < ba$, and rescaling the distances by $r \to r' = r/b$, one obtains a renormalized Hamiltonian $H'(b)$, defined in terms of renormalized couplings $\kappa'_\alpha(b)$. The mapping

$$\kappa_\alpha \;\stackrel{S}{\longrightarrow}\; \kappa'_\alpha(b) \tag{1.1.5}$$

defines the RG flow in the space of Hamiltonians $\{H_{\boldsymbol{\kappa}}\}$, labeled by the couplings $\boldsymbol{\kappa} = \{\kappa_\alpha\}$.

• RG Fixed Points and Scale Invariance

Critical points correspond to fixed points of the RG flow, $\boldsymbol{\kappa}^\star = \{\kappa^\star_\alpha\}$, such that $\kappa^{\star\prime}_\alpha(b) = \kappa^\star_\alpha$. In the neighborhood of a fixed point, it is possible to find new couplings g_i (that is a new coordinate system in the space of Hamiltonians) which linearize the RG flow. These couplings, such that $g_i = 0 \Leftrightarrow \kappa_\alpha = \kappa^\star_\alpha$, are called nonlinear scaling fields. Linearity means that they transform as

$$g_i'(b) = b^{y_i} g_i \ , \tag{1.1.6}$$

where y_i is called the scaling dimension of g_i. From locality, there are local operators $\varphi_i(r)$, conjugate to the g_i, such that the Hamiltonian can be written in terms of the scaling fields as

$$H \ = \ H^\star + \sum_i g_i \int d^d r \, \varphi_i(r) \ , \tag{1.1.7}$$

where $H^\star$ is the fixed point Hamiltonian. The scaling dimension of φ_i is $x_i = d - y_i$. Indeed the RG transformation implies that, at the fixed point, the correlation functions of these operators must scale with the distance as

$$\langle \varphi_1(br_1)\, \varphi_2(br_2) \cdots \rangle_\star \ = \ b^{-x_1}\, b^{-x_2} \cdots \langle \varphi_1(r_1)\, \varphi_2(r_2) \cdots \rangle_\star \tag{1.1.8}$$

Of course, this scaling behavior is valid only for distances $|r|$ much larger than the microscopic cut-off a. Moreover, in most cases it turns out that at the fixed point the correlation functions are also invariant under rotations and translations in the d-dimensional space. As a consequence one can associate to the fixed point a continuum Euclidean field theory, which describes the large distance (or continuum limit) of the corresponding discrete statistical system.

In this limit we choose a normalization for the operators φ such that the 2-points functions are

$$\langle \varphi(r_1) \varphi(r_2) \rangle \ = \ |r_1 - r_2|^{-2x} \tag{1.1.9}$$

If one moves away from the critical point in the so-called relevant directions, i.e. if the couplings g_i such that $y_i > 0$ are taken to be non-zero, the correlation length becomes finite, and the model is not critical any more. Still the approach to criticality can be described by a continuum Euclidean field theory, but this theory with the relevant g_i non-zero is no more scale invariant and has massive excitations.

• **The Operator Product Expansion**

In a local field theory, the product of two local operators has a short distance operator product expansion (OPE) in terms of local operators. This OPE has the form

$$\varphi_i(r_0 + \frac{r}{2})\varphi_j(r_0 - \frac{r}{2}) \;=\; \sum_k C_{ij}{}^k(r)\,\varphi_k(r_0) \tag{1.1.10}$$

The sum has to be considered as an asymptotic series. For a scale invariant theory (and considering for simplicity only scalar operators with no spin indices), the OPE takes a simpler form when expressed in the basis of scale invariant operators, since the OPE coefficients $C_{ij}{}^k(r)$ are scale invariant functions of r, with the simple dependence

$$C_{ij}{}^k(r) \;=\; C_{ij}{}^k\,|r|^{-x_i-x_j+x_k} \tag{1.1.11}$$

The dimensionless coefficients $C_{ij}{}^k$ and the coefficients for the 2-points functions[1] which are of the form

$$\langle\varphi_i(r)\varphi_j(r')\rangle \;=\; d_{ij}\,|r-r'|^{-x_i-x_j} \tag{1.1.12}$$

are sufficient to characterize the theory, since the OPE can be used to reduce any correlation function involving more than two operators $\langle\varphi_1(r_1)\varphi_2(r_2)\cdots\varphi_N(r_N)\rangle$ to 2-points functions. However a non-trivial scale invariant theory has an infinite set of scale invariant operators and the family of coefficients $C_{ij}{}^k$ is infinite.

• **Conformal Invariance**

Up to now I have considered only global scale invariance, and the effect of global dilations $\boldsymbol{r} \to \boldsymbol{r}' = b\boldsymbol{r}$ on the correlation functions of a statistical system at a critical point. What happens if one performs a *conformal transformation*, that is a change of coordinates $\boldsymbol{r} \to \boldsymbol{r}'$ which is a combination of the usual translations and rotations with *local dilations*, characterized by a space dependent dilation factor $b(\boldsymbol{r})$? It occurs that for several critical systems, to be discussed later, scale invariance is enlarged to the invariance under these conformal transformations. This implies, for instance, that there are local operators φ_i which transform under a conformal transformation such that their correlation functions obey

$$\langle\varphi_1(\boldsymbol{r}_1)\varphi_2(\boldsymbol{r}_2)\cdots\rangle \;=\; b(\boldsymbol{r}_1)^{-x_1}b(\boldsymbol{r}_2)^{-x_2}\cdots\langle\varphi_1(\boldsymbol{r}'_1)\varphi_2(\boldsymbol{r}'_2)\cdots\rangle \tag{1.1.13}$$

where the local scale factor is given by

$$b(\boldsymbol{r})^d \;=\; \det\left|\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{r}'}\right| \tag{1.1.14}$$

Locally, these conformal transformations are coordinate changes which preserve the angles, but not the distances. The situation is different if the dimension of space d is two or larger than two.

[1] strictly speaking, $d_{ij} = C_{ij}{}^0$ where $\varphi_0 = \mathbb{1}$ is the identity operator.

• If $d > 2$, conformal transformations are generated by
- translations $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{a}$
- rotations $\boldsymbol{r} \to O\boldsymbol{r}$
- global dilations $\boldsymbol{r} \to b\boldsymbol{r}$
- inversions $\boldsymbol{r} \to \boldsymbol{r}/|\boldsymbol{r}|^2$

Infinitesimal conformal transformation are of the form

$$\boldsymbol{r} \ \to \ \boldsymbol{r}' \ = \ \boldsymbol{r} \ + \ |\boldsymbol{r}|^2 \boldsymbol{\epsilon} \ - \ 2(\boldsymbol{\epsilon}\boldsymbol{r})\boldsymbol{r} \tag{1.1.15}$$

and have rescaling factor $b(\boldsymbol{r}) = 1 + 2(\boldsymbol{\epsilon}\boldsymbol{r})$. The group of conformal transformations is a non-compact finite dimensional group isomorphic to $\mathrm{O}(d+1,1)$. Conformal transformations do not act globally on the d-dimensional plane (since in general the center of the inversion is sent to ∞). However it is possible to define a global action of the conformal group on the d-dimensional sphere $\mathcal{S}_d$ with constant curvature.

• If $d = 2$, the group of *local* conformal transformations is much larger, and in fact infinite dimensional, since is contains all analytic transformations. Indeed, using complex coordinates $(z, \bar{z})$ instead of the Cartesian coordinates $\boldsymbol{r} = (r^1, r^2)$

$$z \ = \ r^1 + \mathrm{i}r^2 \ , \ \bar{z} \ = \ r^1 - \mathrm{i}r^2 \tag{1.1.16}$$

any analytic transformation

$$z \ \to \ z' \ = \ f(z) \qquad , \qquad \bar{z} \ \to \ \bar{z}' \ = \ \bar{f}(\bar{z}) \tag{1.1.17}$$

is conformal around the origin $z = 0$ (if $f'(0) \neq 0$). As for $d > 2$, no global conformal transformation maps the complex plane onto itself, and the group of (orientation preserving) global conformal transformations of the Riemann sphere $\mathcal{S}_2 = \mathbb{C} \cup \{\infty\}$ reduces to the finite dimensional group of Moebius transformations $z \to z' = (az+b)/(cz+d)$, i.e. to $\mathrm{SL}(2,\mathbb{C}) \sim \mathrm{SO}(3,1)$.

• Some Consequences of global conformal Invariance

If a theory is invariant under global conformal transformations, i.e. those generated by inversions, translations, rotations and dilations, this implies interesting constraints, in particular for correlation functions of operators which satisfy (1.1.13). Such operators are called *quasi-primary operators*.

For instance, since one can always find a conformal transformation which sends three points onto three other points, the 3-points function must have the form

$$\langle \varphi_i(\boldsymbol{r}_1)\varphi_j(\boldsymbol{r}_2)\varphi_k(\boldsymbol{r}_3)\rangle \ = \ \frac{d_{ijk}}{r_{12}^{x_i+x_j-x_k} r_{23}^{x_j+x_k-x_i} r_{31}^{x_k+x_i-x_j}} \quad ; \quad r_{ab} \ = \ |\boldsymbol{r}_a - \boldsymbol{r}_b| \tag{1.1.18}$$

Similarly, the 4-points function is a nontrivial function of the two conformally invariant cross-ratios

$$R \ = \ \frac{(\boldsymbol{r}_1 - \boldsymbol{r}_3)(\boldsymbol{r}_2 - \boldsymbol{r}_4)}{(\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{r}_3 - \boldsymbol{r}_4)} \quad , \quad R' \ = \ \frac{(\boldsymbol{r}_1 - \boldsymbol{r}_4)(\boldsymbol{r}_2 - \boldsymbol{r}_3)}{(\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{r}_4 - \boldsymbol{r}_3)} \tag{1.1.19}$$

of the general form

$$\langle \varphi_i(\boldsymbol{r}_1)\varphi_j(\boldsymbol{r}_2)\varphi_k(\boldsymbol{r}_3)\varphi_l(\boldsymbol{r}_4)\rangle \;=\; \frac{r_{13}^{x_j+x_l} r_{24}^{x_i+x_k}\; F(R,R')}{r_{12}^{x_i+x_j} r_{23}^{x_j+x_k} r_{34}^{x_k+x_l} r_{41}^{x_l+x_i}} \tag{1.1.20}$$

Moreover, for two quasi-primary fields with different dimensions, one can show that the two-points function vanishes

$$\langle \varphi_i(r_1)\varphi_j(r_2)\rangle \;=\; 0 \qquad , \qquad \text{if } \; x_i - x_j \neq \text{integer.} \tag{1.1.21}$$

This last restriction comes from the fact that when one performs a conformal change of coordinates for the theory described by the fixed point Hamiltonian $H^\star$, this adds gradients of operators, and in general mixes the two operators φ_i and φ_j (see Cardy (1989) for more details).

Finally, using the OPE and (1.1.13), one shows that for operators which do not mix, the coefficients d_{ijk} coincide with those of the OPE

$$d_{ijk} \;=\; C_{ij}{}^{k} \tag{1.1.22}$$

and in particular that the $C_{ij}{}^{k}$ are symmetric.

Several other applications of global conformal invariance are discussed in Cardy (1987) and Cardy (1989), in particular for systems in finite geometries. Let me just mention dynamical critical phenomena, surface critical behavior and boundary effects in finite geometry.

• Conformal Invariance and the Stress-Energy Tensor

Up to now we have considered scale and conformal transformations as changes of coordinates $\boldsymbol{r} \to \boldsymbol{r}'$ and the response of correlation functions to these change of coordinates of the positions of the fields for a theory defined on flat space. However, we have seen that in general it is possible to define conformal transformations only locally, or on a curved space such as the sphere. It is natural to chose the alternate point of view, and to consider the continuum theory as defined on a curved space, with a Riemannian metric $g_{\mu\nu}(\boldsymbol{r})$, and to study how the theory responds to changes of the metric. Indeed, a global dilation, i.e. a change of the distance between two points, may be performed either by changing the coordinates $\boldsymbol{r} \to b\boldsymbol{r}$, or by keeping the coordinates fixed and by changing the metric $g_{\mu\nu} \to b^2 g_{\mu\nu}$, since the element of distance is defined through

$$ds^2 \;=\; dr^\mu dr^\nu g_{\mu\nu}(\boldsymbol{r}) \tag{1.1.23}$$

.

Let us consider a Euclidean field theory on a curved space. The fields are assumed to be coupled to the background metric tensor $g_{\mu\nu}$ in such a way that reparametrization invariance is ensured. The stress-energy tensor is defined as the response to a local variation of the metric[2], by

[2] There are other definitions for the stress-energy tensor, related to change of coordinates, that I shall not discuss here.

$$T^{\mu\nu}(\boldsymbol{r}) \;=\; -\frac{2}{\sqrt{g}}\,\frac{\delta}{\delta g_{\mu\nu}(\boldsymbol{r})} \tag{1.1.24}$$

(see for instance Birrell and Davies (1982)). This means that the correlation function with one $T^{\mu\nu}$ is defined through

$$\langle\varphi_1(\boldsymbol{r}_1)\cdots\varphi_n(\boldsymbol{r}_n)T^{\mu\nu}(\boldsymbol{r})\rangle \;=\; -\frac{2}{\sqrt{g}}\,\frac{\delta}{\delta g_{\mu\nu}(\boldsymbol{r})}\,\langle\varphi_1(\boldsymbol{r}_1)\cdots\varphi_n(\boldsymbol{r}_n)\rangle \tag{1.1.25}$$

etc...

The stress-energy tensor allows to recover how the theory changes under coordinates transformations, since an infinitesimal change of coordinates

$$r^\mu \;\to\; r'^\mu = r^\mu + \epsilon^\mu(\boldsymbol{r}) \tag{1.1.26}$$

amounts to a change in the metric tensor

$$g_{\mu\nu}(\boldsymbol{r}) \;\to\; g'_{\mu\nu}(\boldsymbol{r}') \;=\; g_{\mu\nu}(\boldsymbol{r}') - D_\mu\epsilon_\nu(\boldsymbol{r}') - D_\nu\epsilon_\mu(\boldsymbol{r}') \tag{1.1.27}$$

with D_μ the covariant derivative. I do not discuss here the case of spinor fields, which requires the introduction of a local orthonormal frame e^a_μ (the vielbein).

The stress-energy tensor is clearly symmetric

$$T^{\mu\nu} \;=\; T^{\nu\mu} \tag{1.1.28}$$

Reparametrization invariance implies that under an arbitrary infinitesimal change of coordinates the vacuum energy does not change

$$\int d^d\boldsymbol{r}\,\sqrt{g}\,T^{\mu\nu}(D_\mu\epsilon_\nu + D_\nu\epsilon_\mu) \;=\; 0 \tag{1.1.29}$$

This leads to the conservation of the current

$$D_\mu T^{\mu\nu} \;=\; 0 \tag{1.1.30}$$

Scale transformations correspond to global dilations of the metric, while conformal transformation corresponds to *local* dilations of the metric

$$g_{\mu\nu}(\boldsymbol{r}) \;\to\; b(\boldsymbol{r})^2\,g_{\mu\nu}(\boldsymbol{r}) \tag{1.1.31}$$

called Weyl transformations. The conformal coordinate transformations discussed above are coordinate transformations which amount to a metric change of this form, using (1.1.27). This implies that a theory is conformally invariant if and only if the trace of the stress-energy tensor, Θ, vanishes

$$\Theta \;=\; T^\mu_{\ \mu} \;=\; 0 \tag{1.1.32}$$

The theory is scale invariant if the integral of Θ over the whole space vanishes, which is the case if Θ is a total derivative ($\Theta = D_\mu J^\mu$).

In general a scale invariant theory is not conformally invariant. The simplest example is the free bosonic scalar field. Classically it is given by a real scalar field ϕ, with Hamiltonian

$$H = \int d^d \boldsymbol{r}\, \sqrt{g}\, \frac{1}{2}\, g^{\mu\nu}\, \partial_\mu \phi \partial_\nu \phi \tag{1.1.33}$$

Its classical stress-energy tensor is

$$T_{\mu\nu} = -\partial_\mu \phi \partial_\nu \phi + \frac{1}{2}(\partial\phi)^2 g_{\mu\nu} \tag{1.1.34}$$

and is not traceless, since $\Theta = -\frac{2-d}{2}(\partial\phi)^2$ is non zero, except for $d = 2$. Using the classical equation of motion $\Delta\phi = 0$ one sees that the model is always scale invariant. However, by adding to the Hamiltonian a term which couples the field with the scalar curvature R, of the form $\int d^d \boldsymbol{r}\, R\, \phi^2$, one can change the coupling with the metric and "improve" the stress-energy tensor by a term of the form $(\partial_\mu \partial_\nu - g_{\mu\nu}\Delta)\phi^2$ which compensates the first one and makes the theory conformally invariant.

1.2 Conformal Invariance in Two Dimensions

• What is special in two Dimensions?

The two dimensional case is especially interesting for two reasons:

Firstly, the group of local conformal transformations is infinite dimensional. Therefore a conformally invariant theory will satisfy a infinite set of constraints (given by the conformal Ward identities), which will give much more information than if $d > 2$. In particular, as we shall see, there is an infinite discrete family of models such that all scaling dimensions and (at least in principle) all correlation functions can be calculated from these constraints of conformal invariance. Another consequence is that conformally invariant theories have an infinite family of conserved quantities (obtained from the currents generated by the stress-energy tensor), and there is a deep connection between conformal invariance and integrable systems in two dimensions.

Secondly, under the general condition of unitarity, scale invariance in two dimensions implies conformal invariance. Indeed, following Lüscher and Mack (1976), let us consider a theory which is local and invariant under translations, rotations and scale transformations in d dimensions. It has a stress-energy tensor $T^{\mu\nu}$ with dimension $x = d + 2$, and from the above symmetries the general form of the two points correlator in flat space is using Cartesian coordinates

$$\begin{aligned} \langle T^{\mu\nu}(\boldsymbol{r}) T^{\rho\sigma}(\boldsymbol{0}) \rangle &= A\, \frac{\delta^{\mu\nu}\delta^{\rho\sigma}}{|\boldsymbol{r}|^{2d}} + B\, \frac{\delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}}{|\boldsymbol{r}|^{2d}} \\ &\quad + C\, \frac{r^\mu r^\nu \delta^{\rho\sigma} + \delta^{\mu\nu} r^\rho r^\sigma}{|\boldsymbol{r}|^{2d+2}} + D\, \frac{r^\mu r^\nu r^\rho r^\sigma}{|\boldsymbol{r}|^{2d+4}} \end{aligned} \tag{1.2.1}$$

The conservation law (1.1.30) gives three linear relations between the coefficients A, B, C, D, and the 2-points function for the trace of the stress-energy tensor is finally

$$\langle \Theta(\boldsymbol{r}) \Theta(\boldsymbol{0}) \rangle = -B\, d(d-1)(d-2)\, \frac{1}{|\boldsymbol{r}|^{2d}} \tag{1.2.2}$$

Thus it is zero if $d = 2$, and from the vanishing of the 2-points function one shows that for unitary theories, the operator Θ itself vanishes.

• **Holomorphic Coordinates**

Since in two dimensions orientation preserving conformal transformations are analytic transformations, it is natural to use analytic coordinates $(z,\ \bar{z})$ (see e.g. Friedan (1984)). This means that we treat z and $\bar{z}$ as independent variables. The additional constraint $\bar{z} = z^*$ (where * denotes complex conjugation) enforces that we describe the Euclidean plane. But this formalism allows also to consider $1+1$ dimensional space with Lorentzian signature $(+,-)$, since with the constraint that z and $\bar{z}$ are real, z and $\bar{z}$ are nothing but the light-cone coordinates x^+ and x^-.

In the complex plane the length element is $ds^2 = dzd\bar{z}$, so that the metric tensor is

$$(\ g_{\mu\nu}\ ) \ = \ \begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} \ = \ \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.2.3}$$

The stress-energy tensor and its trace are

$$(\ T_{\mu\nu}\ ) \ = \ \begin{pmatrix} T_{zz} & T_{z\bar{z}} \\ T_{\bar{z}z} & T_{\bar{z}\bar{z}} \end{pmatrix} \qquad , \qquad \Theta \ = \ 2(T_{z\bar{z}} + T_{\bar{z}z}) \tag{1.2.4}$$

Symmetry and the current conservation imply that

$$T_{z\bar{z}} \ = \ T_{\bar{z}z} \qquad , \qquad \partial_{\bar{z}}T_{zz} + \partial_z T_{\bar{z}z} \ = \ \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}}T_{z\bar{z}} \ = \ 0 \tag{1.2.5}$$

A conformally invariant theory is characterized by $\Theta = 0$ and therefore $T_{\mu\nu}$ has only two non-zero components T_{zz} and $T_{\bar{z}\bar{z}}$, which depend only respectively of z and $\bar{z}$. They are denoted T and $\bar{T}$

$$T_{zz}(z,\bar{z}) \ = \ T(z) \qquad , \qquad T_{\bar{z}\bar{z}}(z,\bar{z}) \ = \ \bar{T}(\bar{z}) \qquad , \qquad T_{z\bar{z}} \ = \ T_{\bar{z}z} \ = \ 0 \tag{1.2.6}$$

T and $\bar{T}$ give the response to conformal change of coordinates

$$z \to z' = w(z) \qquad , \qquad \bar{z} \to \bar{z}' = \bar{w}(\bar{z}) \tag{1.2.7}$$

• **Primary Operators and conformal Ward Identities**

A conformally invariant theory is characterized by a set of local operators $\varphi_i(z,\bar{z})$ which transform under general *local* conformal change of coordinates (1.2.7) according to (1.1.13), namely in complex coordinates

$$\varphi(z,\bar{z}) \ \to \ \varphi'(z',\bar{z}') \ = \ w'(z)^h \bar{w}'(\bar{z})^{\bar{h}} \varphi(w,\bar{w}) \tag{1.2.8}$$

with $w' = \partial_z w$, $\bar{w}' = \partial_{\bar{z}}\bar{w}$. Since we treat z and $\bar{z}$ as independent coordinates, the operator φ is now characterized by two conformal weights $(h,\bar{h})$. Operators which obey (1.2.8) are called *primary operators*. Since dilations are given by $z \to bz$, $\bar{z} \to b\bar{z}$ and rotations by $z \to \mathrm{e}^{\mathrm{i}\alpha}z$, $\bar{z} \to \mathrm{e}^{-\mathrm{i}\alpha}\bar{z}$, the conformal weights give the scaling dimension x and the spin s of the operator through

$$x \ = \ h + \bar{h} \qquad , \qquad s \ = \ h - \bar{h} \tag{1.2.9}$$

Conformal invariance means that correlation functions of primary operators are invariant under conformal transformations, namely that

$$\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_n(z_n,\bar{z}_n)\rangle \;=\; \prod_{i=1}^{n}\left[w'(z_i)^{h_i}\bar{w}'(\bar{z}_i)^{\bar{h}_i}\right]\,\langle \varphi_1(w_1,\bar{w}_1)\cdots\varphi_n(w_n,\bar{w}_n)\rangle \tag{1.2.10}$$

The effect of an infinitesimal conformal transformation $w(z) = z + \epsilon(z)$ can be expressed in term of the stress-energy tensor. Considering for simplicity only the holomorphic part of the transformation law, i.e. a coordinate change $z \to z' = z + \epsilon$, $\bar{z} \to \bar{z}' = \bar{z}$, with ϵ vanishing at ∞ and analytic in a domain $\mathcal{D}$ which contains the operators φ_i, one gets the conformal Ward identity (CWI), which is in complex coordinates

$$\begin{aligned} 2\int d^2\boldsymbol{r}\,\partial_{\bar{z}}\epsilon(z,\bar{z})\langle T(z)\varphi(z_1,\bar{z}_1)\cdots\varphi(z_n,\bar{z}_n\rangle \;=& \\ \sum_{i=1}^{n}\,(h_i\epsilon'(z_i)+\epsilon(z_i)\partial_{z_i})\,\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_n(z_n,\bar{z}_n)\rangle & \end{aligned} \tag{1.2.11}$$

Using (1.2.5) and the analyticity in z of $T(z)$, one can write, by a careful treatment of the boundary conditions and an integration by part over $\bar{z}$, the l.h.s of (1.2.11) as a contour integral over a contour $\mathcal{C}$ (in $\mathcal{D}$) encircling the operators φ_i

$$\mathrm{i}\int_{\mathcal{C}} dz\,\epsilon(z)\,\langle T(z)\varphi_1(z_1,\bar{z}_1)\cdots\varphi_n(z_n,\bar{z}_n)\rangle \tag{1.2.12}$$

while the r.h.s of (1.2.11) can also be written as a contour integral

$$\sum_{i=1}^{n}\frac{1}{2\mathrm{i}\pi}\int_{\mathcal{C}} dz\;\epsilon(z)\;\left(\frac{h_i}{(z-z_i)^2}\;+\;\frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}\right)\langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_n(z_n,\bar{z}_n)\rangle \tag{1.2.13}$$

Since (1.2.11) is satisfied for an arbitrary ϵ, the conformal Ward identity can be rewritten in terms of the first terms of the operator product expansion of the stress-energy tensor[3] with primary operators

$$T(w)\varphi(z,\bar{z}) \;=\; \frac{h}{(w-z)^2}\,\varphi(z)\;+\;\frac{1}{(w-z)}\,\partial_z\varphi(z)\;+\;\cdots \tag{1.2.14}$$

where $\cdots$ represents terms analytic in w around $w = z$, which do not contribute to the Cauchy integral (1.2.13). There is of course a corresponding anti-analytic CWI involving $\bar{T}$ and the weight $\bar{h}$ of φ.

[3] the definition for T used in CFT differs from that given by (1.1.24) by a universal multiplicative factor

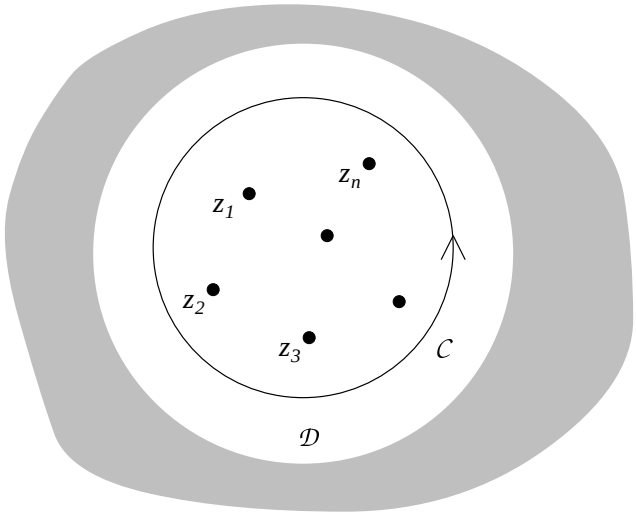

Fig. 2. Contour integration for the conformal Ward identity

• **The Virasoro Algebra and the central Charge**

Scale and rotationally invariant operators (characterized by their scaling dimension x and spin s) which are not primary operators are called *secondary operators*. Their transformation properties under conformal transformations is similarly encoded in their OPE with T and $\bar{T}$, which now can contain arbitrarily negative powers of the distance. The coefficients of this OPE are still local operators, denoted $L_n\varphi$, defined by

$$T(w)\varphi(z,\bar{z}) \;=\; \sum_{n=-\infty}^{\infty} (w-z)^{-2-n}\, L_n\varphi(z,\bar{z}) \tag{1.2.15}$$

From the action of global dilations, rotations and dilations, for any operator with weight $(h,\bar{h})$ one has

$$L_0\varphi \;=\; h\,\varphi \quad ; \qquad L_{-1}\varphi \;=\; \partial_z\varphi \tag{1.2.16}$$

Primary operators are such that

$$L_n\varphi \;=\; 0 \qquad \text{for } n>0 \tag{1.2.17}$$

The stress-energy tensor $T(z)$ is not a primary operator. From (1.1.24) its weights are $(2,0)$, and its OPE with itself has as most singular term $(w-z)^{-4}$. One can check that the term in $(w-z)^{-3}$ is absent, and therefore the OPE for T is of the form

$$T(w)T(z) \;=\; \frac{c/2}{(w-z)^4}\mathbb{1} + \frac{2}{(w-z)^2}T(z) + \frac{1}{(w-z)}\partial_z T(z) + \cdots \qquad (1.2.18)$$

The leading coefficient for the unity operator $\mathbb{1}$ contains a pure number c. This number is called the *central charge*. There is a similar OPE for $\bar{T}$, with c replaced by $\bar{c}$. The pair of central charges $(c, \bar{c})$ is characteristic of a given conformal field theory. For most interesting theories one has $c = \bar{c}$ real. This is the case for critical statistical systems with real Boltzman weights (even when they are not necessarily positive, or do not correspond to unitary theories), as well as for field theories which have no gravitational anomalies (invariance under real diffeomorphisms). Since conformal transformations w.r.t. z and $\bar{z}$ are formally decoupled, the OPE of T with $\bar{T}$ has no singular term.

One can consider the L_n as the generators of an algebra of transformations, acting on the space of all operators $\{\phi_i(z\bar{z})\}$. The product of two L_n is obtained by two successive application of the OPE of T, and their commutation relation can easily be computed using the contour integral representation of L_n

$$L_n\varphi(z,\bar{z}) \;=\; \frac{1}{2\mathrm{i}\pi}\int_{\mathcal{C}} dz_1\,(z_1-z)^{n+1}T(z_1)\varphi(z,\bar{z}) \qquad (1.2.19)$$

(where $\mathcal{C}$ encircles anticlockwise z).

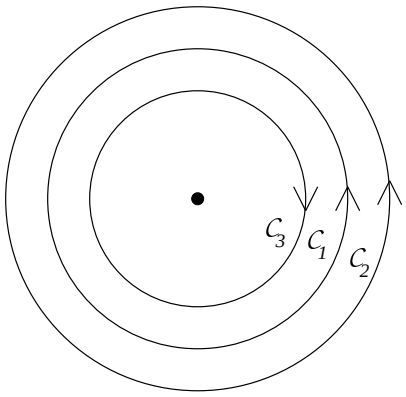

Fig. 3. Contour integrals for the commutation relation of two L's

Considering for simplicity that φ is located at the origin 0, and let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be the three contours depicted on Fig. 3, we write the commutator of two L's as

$$[L_m, L_n]\,\varphi \;=\; \int_{\mathcal{C}_2\cup\mathcal{C}_3}\frac{dz_2}{2\mathrm{i}\pi}\int_{\mathcal{C}_1}\frac{dz_1}{2\mathrm{i}\pi}\,z_1^{n+1}z_2^{m+1}\,T(z_1)T(z_2)\,\varphi \qquad (1.2.20)$$

The contours $\mathcal{C}_2 \cup \mathcal{C}_3$ can be deformed into a contour encircling z_1. Using (1.2.18) and the residue formula we get

$$\int_{\mathcal{C}_1} \frac{dz_1}{2\mathrm{i}\pi}\, z_1^{m+n-1} \left(z_1^3 \partial_{z_1} T(z_1) \,+\, 2(m+1) z_1^2 T(z_1) \,+\, \frac{c}{12} n(n^2-1)\mathbb{1} \right) \varphi \quad (1.2.21)$$

and with (1.2.19) we obtain the commutator

$$[L_m, L_n] \;=\; (m-n)L_{m+n} \,+\, \frac{c}{12}\, n(n^2-1)\, \delta_{n+m,0} \qquad (1.2.22)$$

This defines the famous Virasoro algebra, generated by the L_n's, which is usually denoted **Vir**. The $\bar{L}_n$ (defined similarly from $\bar{T}(\bar{z})$) satisfy a similar Virasoro commutation relation (characterized by the central charge $\bar{c}$), and generates the algebra $\overline{\mathbf{Vir}}$. The algebras **Vir** and $\overline{\mathbf{Vir}}$ commute.

For the generators of global conformal transformations L_{-1}, L_0, L_1 the contact term proportional to c vanishes and one recovers the commutation relations of the Lie algebra $sl(2,\mathbb{R})$.

- **Radial Quantization: from Operators to States**

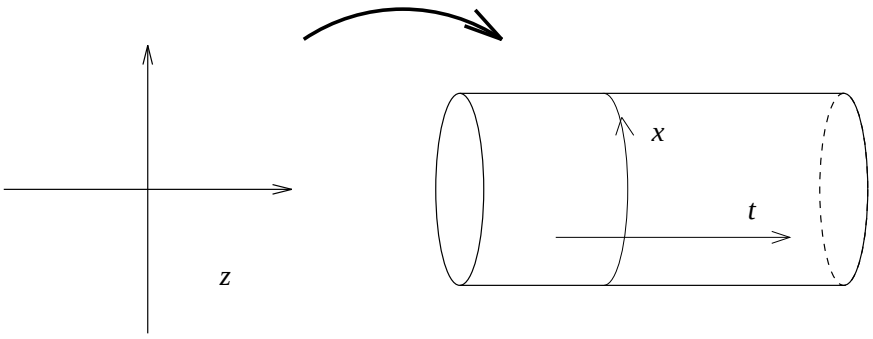

Fig. 4. Mapping the plane on the cylinder

We have considered the L_n's as operators acting on the fields φ's located at some point. This operator definition can be made more precise through radial quantization. To each operator $\varphi(0)$ located at the origin we associate a "in-state" $|\varphi\rangle$, and consider the space generated by these in-states. The scalar product on this vector space is defined through the coefficient of the 2-points function

$$\langle \varphi_2 | \varphi_1 \rangle \;=\; \lim_{\xi \to \infty} \xi^{2h_2} \bar{\xi}^{2\bar{h}_2} \langle \varphi_2(\xi, \bar{\xi}) \varphi_1(0) \rangle \qquad (1.2.23)$$

Going to radial coordinates through the change of coordinates

$$z \;=\; \mathrm{e}^{t+\mathrm{i}x} \qquad , \qquad t \in]-\infty, \infty[\ ,\ x \in [0, 2\pi] \qquad (1.2.24)$$

which maps the punctured plane $\mathbb{C}\backslash\{0\}$ onto the infinite cylinder $\mathbb{R} \times \mathrm{T}$ (t is "time" and x periodic space), to each operator $\varphi(z, \bar{z})$ located on the unit circle

at $z = \mathrm{e}^{\mathrm{i}x}$ is associated an operator $\varphi(x)$ (located on the $t = 0$ time slice). This operator acts on the in-states, and is defined through its matrix elements

$$\langle\varphi_2|\varphi(x)|\varphi_1\rangle \;=\; \lim_{\xi\to\infty} \xi^{2h_2}\bar{\xi}^{2\bar{h}_2}\,\langle\varphi_2(\xi,\bar{\xi})\varphi(z,\bar{z})\varphi_1(0)\rangle \tag{1.2.25}$$

The L_n are thus defined by the expansion of T

$$L_n \;=\; \int_0^{2\pi}\frac{dx}{2\pi}\,T(z)\,\mathrm{e}^{2\mathrm{i}(n+2)x} \qquad\Longleftrightarrow\qquad T(z) \;=\; \sum_{n=-\infty}^{\infty} L_n\,z^{-n-2} \tag{1.2.26}$$

and their adjoints are simply $L_n^\dagger = L_{-n}$. The vacuum state $|0\rangle$ in the Hilbert space is (for unitary theories) associated to the identity operator located at the origin $|0\rangle = |\mathbb{1}\rangle$ and is $sl(2,\mathbb{R})$ invariant. Eq. (1.2.25) leads to the standard definition of the state $|\varphi\rangle$ as obtained by applying the operator φ to the vacuum state $|0\rangle$: $|\varphi\rangle = \varphi|0\rangle$. In the following I shall often refer equivalently to the operators φ (often also denoted fields, since they correspond to quantum field operators) or to the states $|\varphi\rangle$.

- **Secondary Operators and conformal Families**

To each primary field φ is associated a highest-weight state (h.w.s.) $|\varphi\rangle$, which is such that $L_n|\varphi\rangle = \bar{L}_n|\varphi\rangle = 0$ for $n > 0$. Applying successive L_n's and $\bar{L}_n$'s with $n < 0$ to $|\varphi\rangle$, one generates secondary fields which are called the descendents of φ. The set of such fields is often denoted the *conformal family* of φ, $[\varphi]$. Since the L_n and $\bar{L}_n$ commute let us consider only the descendents obtained by the action of **Vir**. Using the Virasoro commutation relations a secondary field can always be written as $L_{-1}^{p_1}L_{-2}^{p_2}\cdots L_{-n}^{p_n}\varphi$ with $p_i \geq 0$. Indeed, the other fields, involving for instance the action of some L_n with $n > 0$ can always be reduced to this form. The number $l = p_1 + 2p_2 + \cdots + np_n$ is called the *level* of the descendent. If the primary field has dimension h, its descendents at level l have dimension $h + l$.

fields				level	weight
φ				0	h
$L_{-1}\varphi$				1	$h+1$
$L_{-2}\varphi$	$L_{-1}^2\varphi$			2	$h+2$
$L_{-3}\varphi$	$L_{-1}L_{-2}\varphi$	$L_{-1}^3\varphi$		3	$h+3$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$

(1.2.27)

However, not all secondary fields obtained in this way are independent. Let us for instance consider the family of the identity $\mathbb{1}$. We have

$$L_{-1}\mathbb{1} \;=\; \partial_z\mathbb{1} \;=\; 0 \quad,\quad L_{-2}\mathbb{1} \;=\; T(z) \quad,\quad L_{-1}^2\mathbb{1} \;=\; 0 \quad,\quad \ldots \tag{1.2.28}$$

In this simple example, some secondary fields vanish identically, and one sees also why the stress-energy tensor is not a primary operator. T is only a quasi-primary operator, namely, it is invariant under global conformal transformations. Since T is the generator of infinitesimal conformal transformations, under such

a transformation $z \to z' = z + \epsilon(z)$, the variation of T itself is given by (1.2.12), which gives, using the OPE (1.2.18), a contour integral around z

$$\begin{aligned} \delta T(z) &= \oint \frac{dw}{2\mathrm{i}\pi}\, \epsilon(w)\, T(w)T(z) \\ &= \frac{c}{12}\epsilon'''(z)\mathbb{1} + 2\epsilon'(z)T(z) + \epsilon(z)\partial_z T(z) \end{aligned} \tag{1.2.29}$$

This infinitesimal variation can be integrated. The result shows how T is transformed under a general conformal transformation $z \to z' = w(z)$

$$\begin{aligned} z &\longrightarrow z' = w(z) \\ T(z) &\longrightarrow T'(z') = (w'(z))^2\, T(z) + \frac{c}{12}\{w,z\} \\ \{w,z\} &= \frac{w'''w' - \frac{3}{2}w''^2}{w'^2} \end{aligned} \tag{1.2.30}$$

$\{w,z\}$ is the *Schwartzian derivative* of w. It vanishes for $sl(2,\mathbb{C})$ transformations $w(z) = \frac{az+b}{cz+d}$, and $T(z)$ is indeed a quasiprimary with weight $(2,0)$.

1.3 The Trace Anomaly and finite Size Effects

- **The central Charge and the Trace Anomaly**

Let us return to the physical significance of the central charge c and to the central extension term in the Virasoro commutation relation. We have seen that for a conformally invariant theory the trace of the stress-energy tensor $\Theta = T^{\mu}_{\ \mu} = 4T_{\bar{z}z}$ must vanish. However, this is true only in flat space, and the occurrence of c implies the existence of contact terms. Indeed, the correlator

$$\langle T(z)T(0)\rangle = \frac{c}{2}\frac{1}{z^4} \tag{1.3.1}$$

is singular at $z = 0$. Let us regularize it as (for instance)

$$\langle T(z)T(z')\rangle_{\mathrm{reg.}} = \frac{c}{2}\frac{\bar{z}^4}{(z\bar{z}+a^2)^4} \tag{1.3.2}$$

with a a small short distance regulator. We can assume that reparametrization invariance is not broken by this regulator, so that the conservation of the current $4\partial_{\bar{z}}T + \partial_z\Theta = 0$ still holds. From this relation, the regularized form (1.3.2) allows to compute the correlator

$$\langle \Theta(z)T(0)\rangle_{\mathrm{reg.}} = 2c\frac{a^2\bar{z}^2}{(z\bar{z}+a^2)^4} \tag{1.3.3}$$

and repeating this argument we obtain

$$\langle \Theta(z)\Theta(0)\rangle_{\mathrm{reg.}} = -\frac{c}{3}\partial_z\partial_{\bar{z}}\frac{a^2}{(z\bar{z}+a^2)^2} \tag{1.3.4}$$

In the continuum limit $a \to 0$ the r.h.s. becomes a distribution

$$\langle \Theta(z)\Theta(0)\rangle \;=\; -4\pi\frac{c}{3}\,\partial_z\partial_{\bar z}\,\delta^2(z,\bar z) \;=\; -c\frac{\pi}{3}\,\Delta_x\delta^2(\boldsymbol{x}) \tag{1.3.5}$$

If we now compute the variation of the v.e.v of Θ under a conformal change of the *metric* around the flat space metric $g^0_{\mu\nu}$

$$\delta g_{\mu\nu} \;=\; \epsilon(z,\bar z)\, g^0_{\mu\nu} \tag{1.3.6}$$

we find

$$\begin{aligned}\delta\langle\Theta(0)\rangle \;&=\; -\frac{1}{2\pi}\int d^2\boldsymbol{x}\,\epsilon(x,\bar z)\,\langle\Theta(0)\Theta(z,\bar z)\rangle \\ &=\; \frac{c}{12}\,\Delta_x\epsilon(0) \;=\; -\frac{c}{12}\,\delta R(0)\end{aligned} \tag{1.3.7}$$

where $\delta R(0)$ is the variation of the *scalar curvature* R induced by the conformal change of metric $\delta g_{\mu\nu}$. This result is in a covariant form and it can be integrated for non-infinitesimal variations of the metric. The final result is the so-called *trace anomaly formula*. It states that on an arbitrary curved metric background, the v.e.v. of the trace of the stress energy tensor of a conformally invariant theory with central charge c does not vanish, but is a local quantity proportional to the scalar curvature

$$\langle\Theta\rangle \;=\; -\frac{c}{12}\,R \tag{1.3.8}$$

• **Finite Size Scaling at Criticality**

A first important application of conformal invariance is the study of finite size effects at criticality (Cardy (1986a–c)). Let us consider a critical system on a finite closed two dimensional surface Σ. The simplest example, easy to implement via periodic boundary conditions, is the two dimensional torus, but one can think of other curved surfaces like the sphere or higher genus surfaces. If the system is critical, it is described by a conformal theory with central charge c, and there are only two scales in the system: the short distance cut-off $a = \Lambda^{-1}$ and the size of the surface $L \sim \sqrt{\text{Area}}$. The free energy scales with L as

$$F(L) \;=\; A\,a^{-2}L^2 \,+\, C\,\ln(La^{-1}) \,+\, \cdots \tag{1.3.9}$$

where A is a non-universal constant (which depends for instance of the lattice discretization chosen of the system, of the location on the critical surface in the space of couplings, etc...). The dimensionless constant C is universal, and turns out to depend only on the central charge c and on the topology of the surface Σ

$$C \;=\; L\frac{\partial F}{\partial L} \;=\; -\frac{c}{6}\,\chi \qquad , \qquad \chi = \text{ Euler characteristics of } \Sigma \tag{1.3.10}$$

Indeed, a change in the size L amounts to a global conformal transformation (change of the scale), and therefore the variation of the vacuum energy is proportional to the expectation value of Θ integrated over the whole surface. With the correct normalization one has

$$L\frac{\partial F}{\partial L} \;=\; \frac{1}{2\pi}\int_\Sigma \langle\Theta\rangle \;=\; -\frac{c}{24\pi}\int_\Sigma R \;=\; -\frac{c}{6}\chi \tag{1.3.11}$$

This relation is still valid for critical systems on an open surface with a smooth boundary (with no cusps), if the boundary conditions do not break conformal invariance. The free energy now has an area and a perimeter term, plus the same universal logarithmic correction

$$F(L) \;=\; Aa^{-2}L^2 \,+\, Ba^{-1}L \,-\, \frac{c}{6}\chi\ln(La^{-1}) \,+\, \cdots \tag{1.3.12}$$

Another important physical application is that of critical systems defined on the infinite cylinder $\mathbb{R}\times \mathrm{T}$, since this is related to the study of statistical systems through transfer matrix methods. Let us consider a conformal theory on the cylinder with infinite length and perimeter ℓ. One can map the punctured plane (with coordinates z) onto the cylinder (with coordinates $w = t + \mathrm{i}x$) by the analytic mapping

$$z \;\longrightarrow\; w \,=\, t + \mathrm{i}x \,=\, \frac{\ell}{2\pi}\ln(z) \tag{1.3.13}$$

From (1.2.30) the stress-energy tensor on the cylinder is related to that on the plane by

$$T(w)_{\rm cylinder} \;=\; \left(\frac{2\pi}{\ell}\right)^2\left[z^2 T(z)_{\rm plane} - \frac{c}{24}\right] \tag{1.3.14}$$

The generator of "time" translations $t \to t+t_0$ on the cylinder is the Hamiltonian H. It is related to the generators L_0 and $\bar L_0$ of dilations on the plane, defined by

$$L_0 \;=\; \frac{1}{2\mathrm{i}\pi}\oint dz\, zT(z)_{\rm plane} \qquad , \qquad \bar L_0 \;=\; -\frac{1}{2\mathrm{i}\pi}\oint d\bar z\, \bar z\bar T(\bar z)_{\rm plane} \tag{1.3.15}$$

through

$$\begin{aligned} H \;&=\; \frac{1}{2\pi}\int_0^\ell dx\, \left(T_{\rm cylinder} + \bar T_{\rm cylinder}\right) \\ &=\; \frac{2\pi}{\ell}\left(L_0 - \bar L_0\right) \,+\, \frac{c\pi}{6L} \end{aligned} \tag{1.3.16}$$

Similarly, the generator of space translation $x \to x + x_0$ is the momentum operator P and is given by

$$P \;=\; \frac{2\pi}{\ell}\left(L_0 - \bar L_0\right) \tag{1.3.17}$$

If we know the spectrum of the theory on the plane, i.e. the conformal weights of its primary operators, we know its spectrum on the cylinder. In particular, the ground state energy is given by the vacuum state $|0\rangle$ on the plane, which is such that $L_0|0\rangle >= \bar L_0|0\rangle = 0$, and one deduces that the free energy per unit of length on the cylinder with perimeter ℓ and length T scales in the large length limit $T \to \infty$ as

$$E_0 \;=\; \frac{F_{\rm cylinder}}{T} \;=\; -\,\frac{\pi c}{6\,\ell} \tag{1.3.18}$$

As for the total free energy, there is a non-universal divergent bulk contribution of the form $cst\Lambda^2\ell$, that we have subtracted. Similarly, the energy E and the momentum p of the eigenstates of the Hamiltonian H on the cylinder are related to the scaling dimensions x and to the spin s of the states on the plane by

$$E \;=\; E_0 \,+\, \frac{2\pi}{\ell}\, x \qquad ; \qquad p \;=\; \frac{2\pi}{\ell}\, s \tag{1.3.19}$$

Starting from the half plane, a similar analysis allows to compute the free energy per unit of length on a strip with width ℓ, with conformally invariant boundary conditions, with result

$$E_0 \;=\; -\frac{\pi c}{24\,\ell} \tag{1.3.20}$$

1.4 Minimal Models and the FQS Classification

• Unitary Representations of the Virasoro Algebra

Most (but not all) interesting models of statistical mechanics correspond at their critical points to *unitary* quantum field theories in the continuum limit. This is in particular the case for lattice models with a Hermitian transfer matrix. Notable examples of models which do not correspond to unitary theories are polymers and percolation models, and the Lee-Yang problem. The constraint of unitarity leads to very strong and interesting conditions for conformally invariant two dimensional theories. We have seen that the Virasoro algebras **Vir** and $\overline{\mathbf{Vir}}$ act on the vector space of states of the theory. The fact that the theory is unitary means that the scalar product of states is positive define, i.e. that the space of states is a Hilbert space. This implies constraints on the possible values for the central charge c and the weights $(h, \bar{h})$ of the primary fields, as shown by Friedan, Qiu and Shenker (Friedan et al. (1984) and Friedan et al. (1986)).

Let us consider a highest weight state $|h\rangle$ with weight h. It is such that $L_n|h\rangle = 0$ for $n > 0$ and $L_0|h\rangle = h|h\rangle$. Its descendents states are obtained by successively applying some L_n's, $n < 0$. Descendents at level N can be written as linear combinations of vectors of the form

$$L_{-n_1} \cdots L_{-n_k}|h\rangle \quad , \quad n_1 \leq \cdots \leq n_k \quad , \quad \sum_i n_i \;=\; N \tag{1.4.1}$$

The vector space generated by all such states (considered as independent) at level N is a finite dimensional space $\mathcal{V}_N$ with dimension $P(N)$. Using the Virasoro commutation relations, starting from the identity for the norm of $|h\rangle$, $\langle h|h\rangle = \big\||h\rangle\big\|^2 = 1$, one computes the scalar product $\langle a|b\rangle$ for each states in $\mathcal{V}_N$, which depends on c and h. For instance, at level 1 we have

$$\big\|L_{-1}|h\rangle\big\|^2 \;=\; \langle h|L_1 L_{-1}|h\rangle \;=\; \langle h|L_{-1}L_1 + 2L_0|h\rangle \;=\; 2h\langle h|h\rangle \;=\; 2h \tag{1.4.2}$$

and at level 2

$$\begin{aligned} \langle h|L_2\, L_{-2}|h\rangle \;=\; 4h - \frac{c}{3} \quad , \quad \langle h|L_1^2\, L_{-1}^2|h\rangle \;=\; 4h(2h+1) \\ \langle h|L_2\, L_{-1}^2|h\rangle \;=\; 6h \end{aligned} \tag{1.4.3}$$

However, for large enough N, some states $|a\rangle$ in $\mathcal{V}_N$ may have zero norm (they are then called *null-states*), or a negative norm. In order to still have a Hilbert space with positive definite scalar product for the physical states, one must fulfill the conditions that: (a) there are no negative-norm states in $\mathcal{V}_N$, (b) null-states in $\mathcal{V}_N$ correspond to a vanishing linear combination of states in $\mathcal{H}$, and in particular descendants of null-states must also be null-states, and the scalar product between null-states must vanish. With these unitarity constraints on the $\mathcal{V}_N$'s, the physical Hilbert space can simply be written as $\mathcal{H} = \mathcal{V}/\mathcal{V}_{\rm null}$.

Such constraints strongly restrict the allowed values of c and h for unitary CFT. The general analysis requires to consider the so-called Kač determinant, which is the determinant of the $P(N) \times P(N)$ matrix $\mathcal{M}_N = \{\langle a_i | a_j \rangle\}$ of scalar products for the elements $\{|a_i\rangle,\ i = 1, P(N)\}$ of a basis of $\mathcal{V}_N$ at level N. This determinant can be computed explicitly, as a function of c and h, and is given by the Kač formula (Kač (1979), Thorn (1984))

$$\begin{aligned} \det\big[\mathcal{M}_N\big](c,h) \ &\propto \prod_{\substack{p,q\in\mathbb{N}^+ \\ pq\leq N}} [h - h_{p,q}(c)]^{P(n-pq)} \\ h_{p,q}(c) \ &= \ \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \quad , \quad c \ = \ 1 - \frac{6}{m(m+1)} \end{aligned} \tag{1.4.4}$$

Unitarity implies that the Kač determinant must be > 0 (if there are no null-state at level N) or zero (if there are null-states) and cannot be negative. A careful analysis leads to the following conclusions:

- If $c > 1$ and $h > 0$, the matrix $\mathcal{M}_N$ is positive definite for any N. Therefore there are in general no null-state and the above procedure leads to a unitary representation of **Vir**.
- If $c = 1$ and $h > 0$, $\det(\mathcal{M}_N) \geq 0$ in general. There are null-states but one can still obtain unitary representations of **Vir**.
- If $c < 1$, the analysis is more subtle, and has been performed by Friedan, Qiu and Shenker. The conclusion is that there are unitary representations of **Vir** only for the discrete values of $0 < c < 1$ given by

$$c \ = \ 1 - \frac{6}{m(m+1)} \qquad , \qquad m = 3, 4, \cdots \text{ integer} \tag{1.4.5}$$

For these values of c a CFT has only a *finite* number of primary field. Such models are called *minimal models*. The dimensions h of these fields can take only a finite set of values, corresponding to the possible zeroes in h of the Kač determinant. These values form the so-called Kač table, with entries (p,q)

$$h \ = \ h_{pq}(c) \ = \ \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \quad , \quad 1 \leq q \leq p \leq m-1 \tag{1.4.6}$$

A state with dimension h_{pq} has null-states as descendants starting at level $N_{pq} = pq$.

Of course in the theory one must consider both the holomorphic and antiholomorphic sectors, and the whole Hilbert space is obtained from tensor products of highest-weight states with weights $(h_{pq},\ \bar{h}_{\bar{p}\bar{q}})$ in the Kač table. Additional constraints have to be considered in order to see which states are really in the Hilbert space, and to see if there are different minimal models with the same central charge c but with different operator contents. In fact it appears that it is sufficient to consider the constraint which comes from modular invariance, i.e. to consider the theory on a torus, and to ensure that it is invariant under global diffeomorphisms not reducible to identity — the so-called modular transformations (Cardy (1986b).

• **Example: the $m = 3$ minimal Model, free Fermions and the Ising Model**

As an example, let me discuss the simplest case $m = 3$. It corresponds to $c = \frac{1}{2}$ and there are three elements in the Kač table

$$h_{11} = 0 \qquad h_{12} = \frac{1}{16} \qquad h_{22} = \frac{1}{2} \tag{1.4.7}$$

associated to the primary fields $\varphi_{11} = \mathbb{1}$, φ_{12} and φ_{22}, and to the corresponding states $|0\rangle$ (the vacuum), $|\frac{1}{2}\rangle = \varphi_{22}|0\rangle$ and $|\frac{1}{16}\rangle = \varphi_{12}|0\rangle$. The null-states at level 1 and 2 are

$$\begin{aligned} L_{-1}\,|0\rangle &= 0 \\ \left(L_{-2} - \frac{3}{4}L_{-1}^2\right)|\frac{1}{2}\rangle &= 0 \\ \left(L_{-2} - \frac{4}{3}L_{-1}^2\right)|\frac{1}{16}\rangle &= 0 \end{aligned} \tag{1.4.8}$$

In the full theory, obtained by summing the holomorphic and antiholomorphic sectors, there are also three primary fields obtained by direct product (modular invariance forbids other combinations in that case), listed below with their scaling dimension x and their spin s

$$\begin{array}{lccc} \text{operator} & (h,\bar{h}) & x & s \\ \phi_{11} = \varphi_{11}\bar{\varphi}_{11} & (0,0) & 0 & 0 \\ \phi_{21} = \varphi_{21}\bar{\varphi}_{21} & (\frac{1}{2},\frac{1}{2}) & 1 & 0 \\ \phi_{12} = \varphi_{12}\bar{\varphi}_{12} & (\frac{1}{16},\frac{1}{16}) & \frac{1}{8} & 0 \end{array} \tag{1.4.9}$$

The $m = 3$ model can be identified with a theory of massless free fermions. One starts with two fermionic field $\psi(z,\bar{z})$, $\bar{\psi}(z,\bar{z})$ which obey Fermi statistics

$$\psi\psi = \bar{\psi}\bar{\psi} = 0 \qquad , \qquad \psi\bar{\psi} = -\bar{\psi}\psi \tag{1.4.10}$$

The action is

$$H = \frac{1}{2\pi}\int d^2\boldsymbol{x}\ \left(\psi\partial_{\bar{z}}\psi + \bar{\psi}\partial_z\bar{\psi}\right) \tag{1.4.11}$$

The theory is conformally invariant. The components of the stress-energy tensor are

$$T(z) = \frac{1}{2}\psi\partial_z\psi \qquad , \qquad \bar{T}(\bar{z}) = \frac{1}{2}\bar{\psi}\partial_{\bar{z}}\bar{\psi} \tag{1.4.12}$$

In the quantum theory the correlation functions are

$$\langle\psi(z,\bar{z})\psi(w,\bar{w})\rangle = -\frac{1}{z-w} \qquad , \qquad \langle\bar{\psi}(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle = -\frac{1}{\bar{z}-\bar{w}} \tag{1.4.13}$$

and the stress-energy tensor $T(z)$ is defined from (1.4.11) by the normal product subtraction prescription

$$T(z) = \frac{1}{2} :\psi\partial_z\psi: = \frac{1}{2}\left[\psi\partial_z\psi - \langle\psi\partial_z\psi\rangle\mathbb{1}\right] \tag{1.4.14}$$

Using these formulae and Wick's theorem it is easy to compute the OPE of T with itself and with ψ. The central charge of the massless free fermion is $c = \bar{c} = \frac{1}{2}$, and the operators ψ and $\bar{\psi}$ have respectively weights $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Thus ψ and $\bar{\psi}$ are identified with the primary fields φ_{21} and $\bar{\varphi}_{21}$ of the $m = 3$ model.

The identification of the φ_{12} operator is more subtle. It corresponds in fact to the so-called *twist operator* σ, defined as follows. Let us introduce a "twist defect" at the origin, by enforcing that in the Grassmann functional integral over φ, φ must change sign when one performs a rotation around the origin (anti-periodic boundary conditions – a.b.c.). To be consistent one must introduce a defect line going from 0 to ∞ and another σ at ∞. This defines the spin-operator σ through its correlation functions

$$\langle\sigma(0)\sigma(\infty)\psi(z_1,\bar{z}_1)\cdots\rangle = \langle\psi(z_1,\bar{z}_1)\cdots\rangle_{\text{a.b.c.}} \tag{1.4.15}$$

The scaling dimension of σ can be calculated by considering the $\langle\psi\psi\sigma\sigma\rangle$ 4-points function. From conformal invariance, the anti-periodic boundary condition (which implies that there must be square-root cuts at $z = 0$ and $w = 0$) and the symmetry properties of ψ and σ, it must be proportional to

$$\langle\psi(z,\bar{z})\psi(w,\bar{w})\sigma(0)\sigma(\infty)\rangle = \frac{C}{2}\frac{-1}{z-w}\left(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}\right) \tag{1.4.16}$$

with the constant $C = \langle\sigma(0)\sigma(\infty)\rangle$. Using the explicit form of the stress-energy tensor $T(z)$ one obtains

$$\langle T(z)\sigma(0)\sigma(\infty)\rangle = \frac{1}{16z^2}\langle\sigma(0)\sigma(\infty)\rangle \tag{1.4.17}$$

and from (1.2.14) the weight of σ is $\frac{1}{16}$. This allows to identify the spin operator σ with the primary operator φ_{12}. To be complete, there is a non-local "disorder operator" μ, dual of σ and with the same weight, defined through its OPE $\psi\sigma \propto \mu$.

It is well know from Onsager's exact solution of the 2-d Ising model that at the critical point the Ising model is equivalent to a model of massless free

fermions, with the spin operator σ_i corresponding to the operator $\sigma(z,\bar{z})$ and the energy operator $\sum_{<ij>} \sigma_i\sigma_j$ corresponding to the energy operator $\epsilon = \psi\bar{\psi}$ (see Lieb et al. (1964)). This allows to recover from conformal invariance the critical exponents of the 2-d Ising model, since one has

$$\begin{aligned} \langle\sigma(\boldsymbol{x})\sigma(\boldsymbol{0})\rangle &= |\boldsymbol{x}|^{-1/4} \propto |\boldsymbol{x}|^{-(d-2+\eta)} \qquad \eta = \frac{1}{4} \\ \langle\epsilon(\boldsymbol{x})\epsilon(\boldsymbol{0})\rangle &= |\boldsymbol{x}|^{-2} \propto |\boldsymbol{x}|^{-2(d-1/\nu)} \qquad \nu = 1 \end{aligned} \tag{1.4.18}$$

• **Correlation Functions and Crossing Symmetry**

Let us return to the general case and consider the correlation functions in a conformally invariant theory, as done in Belavin et al. (1984). Correlation functions involving secondary operators can be obtained from correlation functions of primary operators, by repeated use of the conformal Ward identities. As an example, let us consider a correlation function with one secondary $L_{-n}\varphi_1(z_1,\bar{z}_1)$ only, and several primaries $\varphi_2(z_2,\bar{z}_2)$, $\cdots$, $\varphi_k(z_k,\bar{z}_k)$. We have, using contour deformations and the Cauchy formula, with $\mathcal{C}_j$ small clockwise contours around z_i,

$$\begin{aligned} \langle L_{-n}\varphi_1(z_1,\bar{z}_1)\cdots\rangle &= \frac{1}{2\mathrm{i}\pi}\oint_{\mathcal{C}_1} dz\,(z-z_1)^{-n+1}\,\langle T(z)\varphi_1(z_1,\bar{z}_1)\cdots\rangle \\ &= -\sum_{j=2}^{k}\frac{1}{2\mathrm{i}\pi}\oint_{\mathcal{C}_j} dz\,(z-z_1)^{-n+1}\left(\frac{h_j}{(z-z_j)^2}+\frac{1}{(z-z_j)}\frac{\partial}{\partial z_j}\right)\langle\varphi_1(z_1,\bar{z}_1)\cdots\rangle \\ &= \sum_{j=2}^{k}\left(\frac{(1-n)h_j}{(z_i-z_j)^n}-\frac{1}{(z_1-z_j)^{n-1}}\frac{\partial}{\partial z_j}\right)\langle\varphi_1(z_1,\bar{z}_1)\cdots\rangle \end{aligned} \tag{1.4.19}$$

Thus, in general, correlation functions for secondaries can be written as the result of differential operators acting on the correlation function for primaries. For unitary theories, the existence of null states implies that correlation functions of primaries satisfy non-trivial differential equations, which are very useful to compute them.

Using the OPE expansion, N-points correlation functions can be calculated from the 3-points correlation functions. One proceeds as follows. From the 3-points functions for primary fields

$$\langle\phi_i\phi_j\phi_k\rangle \propto C_{ij}{}^k \tag{1.4.20}$$

using the CWI one can compute the 3-points functions for all the descendents of these fields

$$\langle\{\phi_i\}\{\phi_j\}\{\phi_k\}\rangle \propto C_{ij}{}^k\,\beta_{ij}^k\{\cdots\}\bar{\beta}_{ij}^k\{\cdots\} \tag{1.4.21}$$

where the coefficients $\beta_{ij}^k\{\,\}$ depend only on the central charge c, on the conformal weights h_i, h_j and h_k of the primaries, and of the descendent indices $\{\cdots\}$.

Applying the OPE to the 4-points correlation functions for 4 primaries one obtains

$$\langle \phi_i \phi_j \phi_k \phi_m \rangle \ \propto \ \sum_{\{\phi_m\}} C_{ij}{}^m \, C_{mk}{}^l \, \beta_{ij}^m\{\,\} \, \beta_{kl}^m\{\,\} \, \bar{\beta}_{ij}^m\{\,\} \, \bar{\beta}_{kl}^m\{\,\} \, \langle \{\phi_m\}\{\phi_m\}\rangle \quad (1.4.22)$$

This amounts to sum over the intermediate $\{m\}$ states in the s-channel.

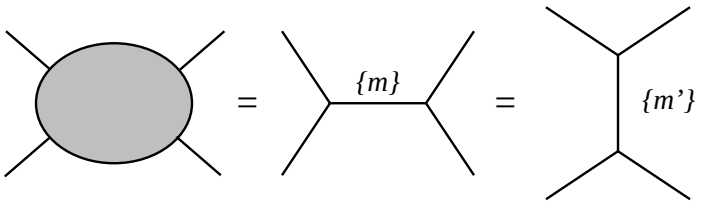

Fig. 5. Decomposition of the 4-points function in the s and in the t channels.

Thus the 4-points functions can be expressed in terms of the OPE coefficients $C_{ij}{}^k$ and of some functions $\mathcal{F}_{ij}^{kl}(m)$ and $\bar{\mathcal{F}}_{ij}^{kl}(m)$, called the conformal blocks, which involve sum over the descendents $\{m\}$ of the state m of the β's and the $\bar{\beta}$'s. These functions depends only on the weights of the states (i,j,k,l,m) and of c (and of course of the locations of the operators $(z_i,\ldots,z_m)$.

It is of course equivalent to decompose a 4-points function in terms of 3-points functions in the s-channel or in the t-channel, as on Fig. (5). This property is called crossing symmetry (it is known as duality in string theories), and is equivalent to the associativity property for the algebra of the operators of the theory. Crossing symmetry imposes strong constraints on the coefficients $C_{ij}{}^k$ of the OPE, which are not fixed yet from conformal invariance. In particular, if a CFT has a *finite* number of primary fields, it fixes the central charge to be of the form

$$c \ = \ 1 \ - \ \frac{6(m-m')^2}{mm'} \qquad , \qquad (m,m') \ = \ 1 \qquad (1.4.23)$$

with m and m' coprime integers. The models with a finite numbers of primaries are called minimal models. The unitary models with $c<1$ of FQS are a subclass of them.

The N-points functions can (at least in principle) be calculated along the same lines. No additional constraints are required, since crossing symmetry ensures that all decompositions are equivalent.

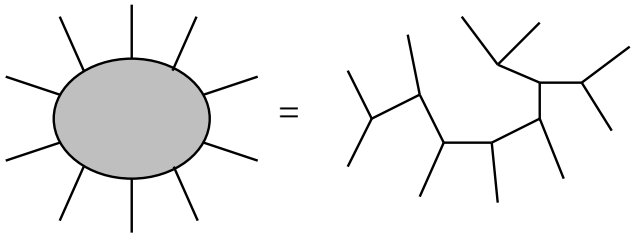

Fig. 6. Decomposition of a N-points functions in terms of 3-points functions

- **Other Examples: the Models for $m = 4$ and $m = 5$**

Finally, let me briefly discuss the unitary models corresponding to the next cases:

For $m = 4$ one has $c = \frac{7}{10}$. The possible dimensions h_{pq} in the Kač table are

$$\begin{array}{cccc} 3 & & & \frac{1}{10} \\ 2 & & \frac{3}{80} & \frac{3}{5} \\ 1 & 0 & \frac{7}{16} & \frac{3}{2} \\ q/_p & 1 & 2 & 3 \end{array} \tag{1.4.24}$$

As for the $m = 3$ case, the primary operators are obtained by products of holomorphic and antiholomorphic primaries and are spinless operators $\phi_{pq} = \varphi_{pq}\bar{\varphi}_{pq}$. This theory can be identified with the continuum limit of the tricritical Ising model, i.e. with the Φ^6 Landau-Ginsburg-Wilson theory. The operators content of the Kač table is nothing but

$$\begin{array}{ccc} & & \Phi^2 \\ & \Phi & \Phi^4 \\ \Phi^0 & \Phi^3 & \Phi^6 \end{array} \tag{1.4.25}$$

From the equation of motions $\Phi^5 \propto \Delta\Phi$ and is a secondary operator. All these operators but Φ^6 have scaling dimension $x < 1$ and correspond to relevant perturbations, while Φ^6 is an irrelevant operator.

The case $m = 5$ is even more interesting. Indeed, it turns out that there are two inequivalent unitary theories with $c = \frac{4}{5}$. The Kač table is now

$$\begin{array}{cccc} & & & \frac{1}{80} \\ & & \frac{1}{15} & \frac{2}{3} \\ & \frac{1}{40} & \frac{21}{40} & \frac{13}{8} \\ 0 & \frac{2}{5} & \frac{7}{5} & 3 \end{array} \tag{1.4.26}$$

The operator content of the first theory is similar to that of the $m = 4$ case. All operators of the form $\phi = \varphi_{pq}\bar{\varphi}_{pq}$ are present and are spinless. This theory is identified with the tetra-critical Ising model, i.e. the Φ^8 LGW theory. The series of theories constructed in that way for general m are similarly identified with the $(m-1)$-critical Ising models, and form the so-called A-series (Huse (1984), Zamolodchikov (1986a)).

In the second theory, which also fulfills the fusion rules and the requirements of modular invariance, not all operators are allowed. In fact the operator content is:

$$\begin{aligned} &6\ s=0 \text{ operators: } \left\{ \begin{array}{lll} \varphi_{11}\bar{\varphi}_{11} & \varphi_{12}\bar{\varphi}_{12} & \varphi_{13}\bar{\varphi}_{13} \\ \varphi_{14}\bar{\varphi}_{14} & \varphi_{33}\bar{\varphi}_{33} & \varphi_{34}\bar{\varphi}_{34} \end{array} \right. \\ &1\ s=1 \text{ operator } \varphi_{12}\bar{\varphi}_{13} \quad \text{and its } s=-1 \text{ conjugate } \varphi_{13}\bar{\varphi}_{12} \\ &1\ s=3 \text{ operator } \varphi_{11}\bar{\varphi}_{14} \quad \text{and its } s=-3 \text{ conjugate } \varphi_{14}\bar{\varphi}_{11} \end{aligned} \tag{1.4.27}$$

This theory can be identified with the critical 3-states Potts model, which has a $\mathbb{Z}_3$ symmetry (Dotsenko (1984)). It is the first element (for $m = 5$) of another series of unitary CFT, the so-called D-series.

1.5 Massive Theories, RG Flows and Zamolodchikov's c-Theorem

• Off-critical Models and Renormalization Group Flows

Off the critical point, the fixed point Hamiltonian is perturbed by relevant operators. The theory becomes massive and is no more conformally invariant, so that the trace of the stress-energy tensor is not zero $\Theta \neq 0$. Let us consider the case of a theory perturbed only by one relevant primary operator Φ with scaling dimension $h < 1$, with Hamiltonian

$$H(\lambda) \;=\; H^* \;-\; \lambda \int d^2x\, \Phi(z,\bar{z}) \tag{1.5.1}$$

λ is the coupling constant, with dimension $y = 2 - 2h > 0$. Following Cardy (1988), let us compute Θ to first order in λ. The zz component of $T_{\mu\nu}$, T, is given by

$$\langle T(z,\bar{z})\rangle_\lambda \;=\; \langle T(z)\rangle_0 \;+\; \lambda \int d^2z_1\, \langle T(z)\Phi(z_1,\bar{z}_1)\rangle_0 \;+\; \cdots \tag{1.5.2}$$

The integration over z_1 has potential short distance divergences as $z_1 \to z$. Let us compute its $\bar{z}$ derivative. Using the OPE of T with Φ, given by (1.2.14), the logarithmically divergent potential term proportional to Φ vanishes by angular integration and one ends with a finite contribution

$$\langle \partial_{\bar{z}} T(z,\bar{z})\rangle \;=\; -\lambda(1-h)\,\pi\,\langle \partial_z \Phi(z,\bar{z})\rangle_0 \tag{1.5.3}$$

Using the conservation of the current (1.2.5), $4\partial_{\bar{z}}T + \partial_z\Theta = 0$, this implies

$$\langle \Theta(z,\bar{z}) \rangle \;=\; -4\pi\,\lambda\,(1-h)\,\langle \Phi(z,\bar{z}) \rangle_0 \;+\; \mathcal{O}(\lambda^2) \tag{1.5.4}$$

As for Eq. (1.3.8) valid for a CFT on a curved space, this result is a *trace anomaly* equation. It states that if scale invariance is anomalous, Θ is proportional to a combination of the local operators of the theory. Here we have only one relevant operator, and (1.5.4) defines (to first order in λ) the so-called β-function

$$\Theta(z,\bar{z}) \;=\; -\,2\pi\,\beta(\lambda)\,\Phi(z,\bar{z}) \qquad , \qquad \beta(\lambda) \;=\; 2(1-h)\lambda \tag{1.5.5}$$

At this order the β function is just λ times the scaling dimension $y = (2-2h)$ of λ.

Let us now consider a theory perturbed by several relevant operators

$$H[\lambda^i] \;=\; H^* \;-\; \sum_i \lambda^i \int d^2x\, \Phi_i(z,\bar{z}) \tag{1.5.6}$$

with scaling dimension $x_i = 2h_i$ close to two (this means that the operators are almost marginal), and let us compute to second order the derivative of T

$$\partial_{\bar{z}} \langle T \rangle \;=\; \lambda^i\,\partial_z \langle T\Phi_i \rangle_0 \;+\; \frac{1}{2}\,\lambda^i\lambda^j\,\partial_z \langle T\Phi_i\Phi_j \rangle_0 \;+\; \cdots \tag{1.5.7}$$

A new singularity occurs when T, Φ_i and Φ_j coalesce, but one can use the OPE $\Phi_i\Phi_j \sim C_{ij}{}^k\Phi_k$ to evaluate it and one finds a finite contribution to Θ of the form

$$\Theta \;=\; -\,2\pi\,\beta^i\,\Phi_i \tag{1.5.8}$$

with the β functions given at order two by

$$\beta^i \;=\; (2-x_i)\,\lambda^i \;-\; \pi\,C_{jk}{}^i\,\lambda^j\lambda^k \;+\; \cdots \tag{1.5.9}$$

We have seen that Θ measures the response of the theory to a local change of scale (see for instance Eq. (1.3.11)). The local form of the scale anomaly (1.5.8) implies that a global change of scale L for the off-critical model (on flat space) amounts to a change of the coupling constants $\lambda^i \to \lambda^i(L)$ governed by the renormalization group flow equations

$$L\frac{\partial}{\partial L}\lambda^i \;=\; -\beta^i \tag{1.5.10}$$

- **Zamolodchikov's c-Theorem**

A. Zamolodchikov showed a very interesting result for the renormalization group flows which holds for unitary theories in two dimensions, and which gives another physical interpretation for the central charge c (Zamolodchikov (1986b)). This c-theorem states that:

For a non-critical theory depending on some coupling constants λ^i, there is a function $c(\lambda)$ such that

- c is stationary at the fixed points $\lambda^\star$ of the RG flows, i.e. $\nabla c(\lambda^\star) = 0$;

- at the fixed points $c(\lambda^\star)$ is equal to the central charge $c^\star$ of the associated conformal field theory;
- for unitary theories, c is decreasing along the IR RG flows, i.e. $L\frac{\partial}{\partial L}c \leq 0$.

This theorem implies, for instance, that if in the space of couplings there is a RG trajectory which flows from an UV fixed point $\lambda^\star_{\rm uv}$ to an IR fixed point $\lambda^\star_{\rm ir}$, then one must have $c_{\rm uv} > c_{\rm ir}$. It has been suggested by several authors that this irreversibility of the RG flow is somewhat similar to the flow of entropy in thermodynamics. The c-function defined in this way is a measure of the "number of local degrees of freedom" of a theory, and it can only decrease as the theory is probed at larger and larger length scales L.

The theorem implies also that the RG flow can be written as a gradient flow, and that it does not have limit cycles or strange attractors (at least for unitary models).

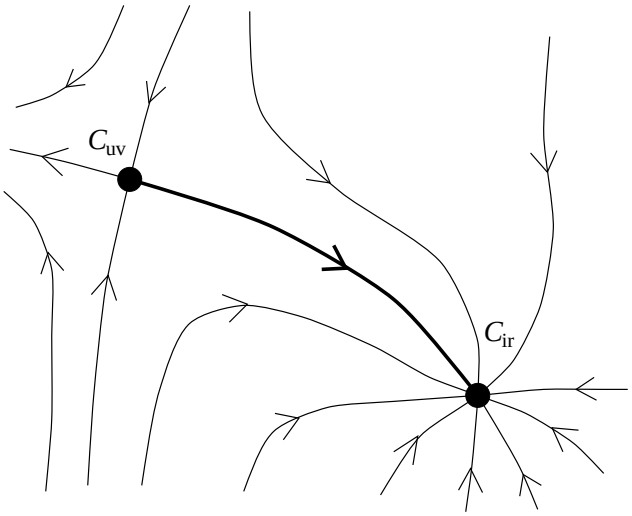

Fig. 7. RG flow from an UV to a IR fixed point

Rather than presenting the original derivation of the c-theorem (Zamolodchikov (1986b)), let me briefly discuss it along the lines of Cappelli et al. (1991). For a conformally invariant theory the two-points correlation of the trace of the stress-energy tensor is given by (1.3.5)

$$\langle \Theta(z)\Theta(0)\rangle \;=\; -4\pi\frac{c}{3}\,\partial_z\partial_{\bar z}\,\delta^2(z,\bar z) \tag{1.5.11}$$

Which means that we can write c as (for instance)

$$c \;=\; \ell^2\,\frac{3}{4\pi}\int d^2\boldsymbol{x}\,{\rm e}^{-\boldsymbol{x}^2\ell^{-2}}\,\langle\boldsymbol{\Theta}(\boldsymbol{x})\boldsymbol{\Theta}(\boldsymbol{0})\rangle \tag{1.5.12}$$

With ℓ some reference length scale (the value for c given by (1.5.12) is of course independent of ℓ). This expression can be generalized to theories which are not conformally invariant, and provides a definition of a c-function. One can use the spectral representation of the 2-points function

$$\langle \Theta(\boldsymbol{x})\Theta(0)\rangle \;=\; \frac{\pi}{3}\int_0^\infty d\mu\,\rho(\mu)\int \frac{d^2\boldsymbol{p}}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{x}}\,\frac{\boldsymbol{p}^4}{\boldsymbol{p}^2+\mu^2} \tag{1.5.13}$$

with $\rho(\mu)$ the spectral density function, which is positive, $\rho(\mu)\geq 0$, for unitary theories, and write c given by (1.5.12) as

$$c \;=\; \int_0^\infty d\mu\,\rho(\mu)\,f(\mu) \tag{1.5.14}$$

with $f(\mu)$ a positive decreasing smearing function[4] (such that $\mu\frac{\partial}{\partial\mu}f\leq 0$). The c-function defined by (1.5.12) is a pure number, but it depends explicitly of the length scale ℓ. Its evolution under the RG flow is therefore by definition

$$-\beta^i\frac{\partial c}{\partial\lambda^i} \;=\; \ell\frac{dc}{d\ell} \;=\; \int_0^\infty d\mu\,\rho(\mu)\,f'(\mu) \;\leq\; 0 \tag{1.5.15}$$

This formulation of the c-theorem has been the starting point for attempts to generalize it to higher dimensions $D>2$ (Cappelli et al. (1991), Cappelli et al. (1992)) (for another approach see Cardy (1988), Osborn (1991)), but no fully complete formulation has been found yet.

2 Introduction to Statistical Mechanics of Random Surfaces

2.1 Polymers and self-avoiding Walks

In this lecture I shall briefly discuss polymers and statistical properties of random walks. A general reference on this subject is the book by des Cloizeaux and Jannink (1990).

- **Gaussian random Walks**

Idealy, a polymer in a good solvent may be considered as a very long chain of N elements (monomers) with flexible connections. This chain fluctuates freely due to thermal fluctuations. If one considers a "phantom chain", without interactions between monomers far apart along the chain (such as steric self-avoidance interactions), it is known that the average distance between the two end-points of the chain, R, (as well as the average distance between two random elements

[4] For our choice of the c-function given by (1.5.12) one has explicitly $f(\mu)=8\int_0^\infty dt\,\mathrm{e}^{-t\mu^2}(1+4t)^{-3}$

of the chain, called the gyration radius R_G), scales with the length of the chain $L = Na$ (a being the monomer length and N the number of monomers) as

$$\langle R^2 \rangle \ \propto \ La \ \propto \ N^{2\nu} \qquad \nu = \frac{1}{2} \tag{2.1.1}$$

This is a consequence of the fact that the large distance properties of such a long chains can be described by a random walk (or Wiener process). More precisely, in the large length limit, the polymer is considered as a continuous chain of length L, its elements are labeled by a continuous coordinate $s \in [0, L]$, and their position in the physical d-dimensional space $\mathbb{R}$ by the vector $\boldsymbol{r}(s)$. The Hamiltonian (i.e. the free energy for a configuration $\boldsymbol{r}$) is

$$S_0[\boldsymbol{r}] \ = \ \int_0^L ds \, \frac{1}{2} \left(\dot{\boldsymbol{r}}(s)\right)^2 \qquad , \qquad \dot{\boldsymbol{r}} \ = \ \frac{d\boldsymbol{r}}{ds} \tag{2.1.2}$$

and the partition function is given by the sum over configurations with the usual Boltzmann weight

$$Z \ = \ \int \mathcal{D}[\boldsymbol{r}] \, \mathrm{e}^{-\beta S_0(r)} \tag{2.1.3}$$

We have included in β the line tension σ, so that $\beta = \sigma / \mathrm{k_B} T$. The line tension σ is of the order of the inverse of the short distance cut-off a. The Hamiltonian (2.1.2) is the universal effective Hamiltonian for a free random chain. This universality can be viewed as a consequence of the central limit theorem. For similar reasons of universality, that I shall not discuss at this stage, this continuous model, invariant under rotations in the d-dimensional space, describes also the statistics of long chains defined on discrete regular lattices, such as the hypercubic lattice $\mathbb{Z}^d$, which are not rotationally invariant.

From this model it is easy to recover (2.1.1). Indeed, the average squared distance between two points at distance s on the chain is given by

$$\langle [\boldsymbol{r}(s) - \boldsymbol{r}(0)]^2 \rangle \ = \ d|s| \tag{2.1.4}$$

Physical chains have also some finite bending rigidity. A model for such semi-rigid chains is obtained by adding a term of the form $\int ds \frac{\kappa}{2} \left(\ddot{\boldsymbol{r}}\right)^2$ with κ the rigidity modulus. κ has the dimension $[\kappa] = [length][energy]$ and the chain is characterized by a persistence length $\ell_p \propto \kappa / \mathrm{k_B} T$, beyond which the rigidity becomes negligible. Thus, semi-rigid long chains with length $L \gg \ell_p$ can be considered as ideal Gaussian chains with effective line tension $\sigma = \mathrm{k_B} T / \ell_p$.

• Self-avoiding random Walks

For obvious physical reasons, two elements of a real polymer cannot occupy the same location in physical space, and in fact cannot come too close to each others. This means that excluded volume effects must be taken into account. Without discussing the physics of the short range interactions between monomers, one expects on general grounds that this will induce some swelling of the chain, which

will span more space than if it was a phantom non-self-avoiding chain. In fact, if self-avoidance is important, the SAW (self-avoiding walk) will be characterized by a non-trivial scaling exponent ν larger than that of the free random walk

$$\langle R_G^2 \rangle \propto L^{2\nu} \quad ; \quad \frac{1}{2} < \nu < 1 \tag{2.1.5}$$

A continuum description of the SAW is provided by the Edwards model (Edwards (1965)). It is obtained by starting from the free random walk model, and by assuming that self-avoidance can be reduced to a weak repulsive two-body interaction. The Hamiltonian for the model of a single chain of length L is now

$$S[r] = \int_0^L ds \, \frac{1}{2}(\dot{\boldsymbol{r}})^2 + \frac{b}{2} \int_0^L ds \int_0^L ds' \, \delta^d(\boldsymbol{r}(s) - \boldsymbol{r}(s')) \tag{2.1.6}$$

with $b > 0$ the self-avoidance interaction coupling constant. As we shall see, this model describes at large distance, for space dimension $d < 4$, the physics of "real" self-avoiding chains, which differs from this ideal model by the fact that self-avoidance is strong (the interaction energy is infinite for contact points), and that this interaction has a finite range a, typically of the order of the monomer size or of the short distance cut-off (the lattice mesh for a model of SAW on a lattice).

• **The SAW and the $n = 0$ LGW Model**

A quite remarkable feature of the Edwards model is that it is in fact equivalent to the well-known Landau-Ginsburg-Wilson $(\boldsymbol{\Phi}^2)^2$ vector model, when the number of components n goes to zero (de Gennes (1972)). This is the so called de Gennes trick, which allows to treat SAW by the standard methods of renormalization group for critical phenomena.

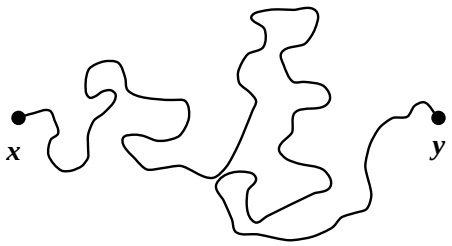

Fig. 8. A random walk with fixed end-points.

Let me give a brief derivation of this equivalence. Let us start from the partition function $G_0(\boldsymbol{x}, \boldsymbol{y}; L)$ of a random walk with length L and with fixed extremities (in d-dimensional space) at points $\boldsymbol{x}$, $\boldsymbol{y}$. It can be rewritten as the

matrix element of the heat-kernel in d dimensions (see for instance A. Jevicki lectures in this volume)

$$G_0(\boldsymbol{x},\boldsymbol{y};L) \;=\; \int_{\substack{r(0)=x\\ r(L)=y}} \mathcal{D}[\boldsymbol{r}]\,\mathrm{e}^{-\beta S_0[r]} \;=\; \langle \boldsymbol{x}|\mathrm{e}^{\frac{L}{2}\Delta}|\boldsymbol{y}\rangle \tag{2.1.7}$$

where Δ is the d-dimensional Laplacian operator. It is therefore the inverse Laplace transform of the standard propagator of a free real scalar field Φ with mass $2m$.

$$\begin{aligned} G_0(\boldsymbol{x},\boldsymbol{y};2m) \;&=\; \int_0^\infty dL\,\mathrm{e}^{-mL}\,G_0(\boldsymbol{x},\boldsymbol{y};L) \\ &=\; 2\,\langle \boldsymbol{x}|\frac{1}{-\Delta+2m}|\boldsymbol{y}\rangle \;=\; 2\,\langle \Phi(\boldsymbol{x})\Phi(\boldsymbol{y})\rangle_0 \\ &=\; 2\,\frac{1}{\mathcal{Z}}\int \mathcal{D}[\Phi]\,\Phi(\boldsymbol{x})\Phi(\boldsymbol{y})\,\mathrm{e}^{-\int d^dz\left[\frac{1}{2}(\partial\Phi)^2+m\Phi^2\right]} \end{aligned} \tag{2.1.8}$$

If we now consider the similar partition function for the Edwards SAW

$$G(\boldsymbol{x},\boldsymbol{y};L) \;=\; \int_{\substack{r(0)=x\\ r(L)=y}} \mathcal{D}[\boldsymbol{r}]\,\mathrm{e}^{-S[r]} \tag{2.1.9}$$

the interaction term can be rewritten as

$$\int_0^L ds\int_0^L ds'\,\delta^d(\boldsymbol{r}(s)-\boldsymbol{r}(s')) \;=\; \int_{\mathbb{R}^d} d^d\boldsymbol{z}\,\left[\int_0^L ds\,\delta^d(\boldsymbol{r}(s)-\boldsymbol{z})\right]^2 \tag{2.1.10}$$

Introducing an auxiliary $\lambda(\boldsymbol{z})$ field (in d-dimensional space) and performing a Hubbard-Stratanovich transformation, we rewrite (2.1.9) as

$$\int \mathcal{D}[\lambda(\boldsymbol{z})]\,\mathrm{e}^{-\frac{1}{2b}\int d^dz\,\lambda^2(z)}\int_{\substack{r(0)=x\\ r(L)=y}} \mathcal{D}[\boldsymbol{r}]\,\mathrm{e}^{-\int_0^L ds\left[\frac{1}{2}(\dot r)^2+\mathrm{i}\lambda(r(s))\right]} \tag{2.1.11}$$

Now, making use of (2.1.8) (for instance by first expanding in $\mathrm{i}\lambda$, then resumming the exponential), we obtain

$$\int \mathcal{D}[\lambda(\boldsymbol{z})]\,\mathrm{e}^{-\frac{1}{2b}\int d^dz\,\lambda^2(z)}\;\langle \boldsymbol{x}|\mathrm{e}^{-L(-\frac{\Delta}{2}+\mathrm{i}\lambda)}|\boldsymbol{y}\rangle \tag{2.1.12}$$

and equivalently, by taking the Laplace transform

$$\begin{aligned} G(\boldsymbol{x},\boldsymbol{y};2m) \;&=\; \int_0^\infty dL\,\mathrm{e}^{-mL}\,G(\boldsymbol{x},\boldsymbol{y};L) \\ &=\; \int \mathcal{D}[\lambda(\boldsymbol{z})]\,\mathrm{e}^{-\frac{1}{2b}\int d^dz\,\lambda^2(z)}\;2\,\langle \boldsymbol{x}|\frac{1}{-\Delta+2\mathrm{i}\lambda+2m}|\boldsymbol{y}\rangle \end{aligned} \tag{2.1.13}$$

One can view this last term as the quenched average of the propagator of a massive particle (with mass $2m$), in a random Gaussian potential $\mathrm{i}\lambda(\boldsymbol{z})$. To perform the average over λ, one uses the replica trick, by introducing n copies Φ_α, $\alpha=1,n$ of the scalar real field Φ introduced in (2.1.8), denoted by $\boldsymbol{\Phi}=(\Phi_\alpha)$.

The $n \to 0$ limit takes care of the normalization factor $1/\mathcal{Z}$ in the functional integration over the Φ's in (2.1.13), so that

$$\langle \boldsymbol{x}| \frac{1}{-\Delta + 2\mathrm{i}\lambda + 2m} |\boldsymbol{y}\rangle \;=\; \lim_{n\to 0} \int \mathcal{D}[\boldsymbol{\Phi}]\, \Phi_1(\boldsymbol{x})\Phi_1(\boldsymbol{y})\, \mathrm{e}^{-\int d^d z \left[\frac{1}{2}(\partial \boldsymbol{\Phi})^2 + (\mathrm{i}\lambda + m)\boldsymbol{\Phi}^2\right]} \tag{2.1.14}$$

We can now perform explicitly the integration over the auxiliary field λ. We end up with the representation of the Laplace transform of the partition function (2.1.7) for a SAW with fixed extremities as the 2-points correlation function of a scalar LGW theory in the limit of $n \to 0$ components.

$$G(\boldsymbol{x}, \boldsymbol{y}; 2m) \;=\; \lim_{n\to 0} \int \mathcal{D}[\boldsymbol{\Phi}]\, \Phi_1(\boldsymbol{x})\Phi_1(\boldsymbol{y})\, \mathrm{e}^{-\int d^d z \left[\frac{1}{2}(\partial \boldsymbol{\Phi})^2 + m\boldsymbol{\Phi}^2 + \frac{b}{2}(\boldsymbol{\Phi}^2)^2\right]} \tag{2.1.15}$$

The length of the chain L is conjugate to the mass m of the field. This implies that the limit of long chains $L \to \infty$ corresponds to the massless limit $m \to 0$ of the LGW theory. The weak self-avoidance coupling b is the coupling constant of the theory. Since the critical limit is amenable by renormalization group methods, this allows to study long polymers by field theoretical methods, and in particular to study the renormalization group flow for the coupling b.

Dimensional analysis, both on the original Edwards model or on the LGW models, shows that if one rescales the length of the chain by a factor B, $L \to L' = BL$ ($m \to m' = B^{-1}m$), this amounts to rescale the coupling constant $b \to B^{\epsilon/2} b$, with $\epsilon = 4 - d$. This implies that if the dimension of space d is larger than 4, since $\epsilon < 0$ the effective coupling b' goes to 0 for large L and self-avoidance becomes irrelevant. Thus one should recover the Gaussian exponent $\nu_0 = 1/2$ of free chains. If $d < 4$, $\epsilon > 0$ and self-avoidance is relevant. Considering that the dimension of space can be extended to non-integer values (this makes sense, at least in perturbative calculations when one performs an expansion in powers of b), the RG flow for the model and the critical exponents can be studied via the famous Wilson-Fisher ϵ-expansion. One finds that the effective coupling constant $b(L)$ flows toward an attractive fixed point $b^\star = \mathcal{O}(\epsilon)$ for large L. To first order in ϵ one finds $\nu = 1/2 + \epsilon/16 + \cdots$, to be compared with the value for $d = 3$ ($\epsilon = 1$), $\nu = .588\cdots$. Many other quantities can be obtained by these methods: various critical exponents, physical quantities such as virial coefficients, results for systems with several polymers, branched polymers, etc...

Let me mention two other facts. The equivalence between the SAW and the $n \to 0$ limit of some models with a O(n) symmetry holds also for models on lattices. This equivalence, both for discrete lattice models and for continuous field theories is especially fruitful for SAW in two dimensions ($d = 2$). In this case one can derive exact results for many critical exponents (in particular the famous Flory exponent $\nu = 3/4$) (see in particular Cardy and Hamber (1980), Nienhuis (1982,1984), Saleur (1986), Duplantier and Saleur (1987)). The methods of conformal field theories are especially useful, since the single SAW model is in fact described by a non-unitary CFT with central charge $c = 0$.

2.2 Random Surfaces and Membranes

• Ideal random Surfaces

In this lecture I shall discuss some models of "ideal" random surfaces. Many recent theoretical works on random surfaces are motivated by the study of self-assembling amphiphilic systems and of membranes. I refer in particular to the Jerusalem '87 Lectures Notes (Nelson et al. (1989)) and to the recent review by Gompper and Schick (Gompper and Schick (1994)) for more details. By ideal, I mean that I shall be interested in the statistics of large two-dimensional films, such that: (i) their thickness e is much smaller than their internal extent L, (ii) they are flexible, i.e. with a small bending elasticity (see later), (iii) they are tensionless, i.e. they are not submitted to some external stress which makes them flat.

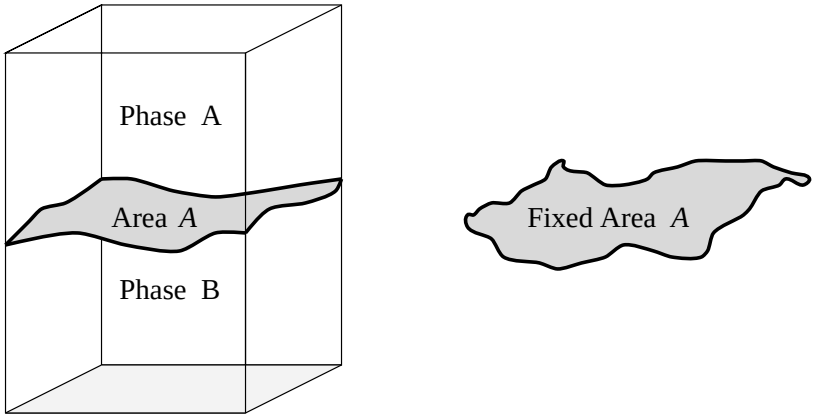

Fig. 9. An interface (fluctuating area) versus an isolated free membrane (fixed area)

Let me first discuss more precisely what tensionless means. If one considers the surface formed by the interface between two phases A and B of a binary mixture (for instance), the free energy F of a configuration is proportional to the total area A of the interface:

$$F = \sigma A \tag{2.2.1}$$

where the free energy per unit of area σ is nothing but the surface tension. Under thermal fluctuations the area A of the system will fluctuate, and the system will minimize its free energy by choosing a configuration with minimal interface area (provided the other external constraints are satisfied). On the contrary, if one considers a free isolated film consisting of a fixed number of elements (molecules) N, its total area A will be proportional to N and will not fluctuate (or will have only very small fluctuations due to internal elasticity). One can show that in this situation, where the requirement of minimizing the total area is not present,

one may consider (except in some situations when the surface is crumpled) that the two-dimensional film is tensionless. In this case, the terms involving the curvature properties of the surface, i.e. bending elasticity, have to be considered in the free energy. They turn out to be essential in order to understand the statistical properties of such tensionless flexible surfaces.

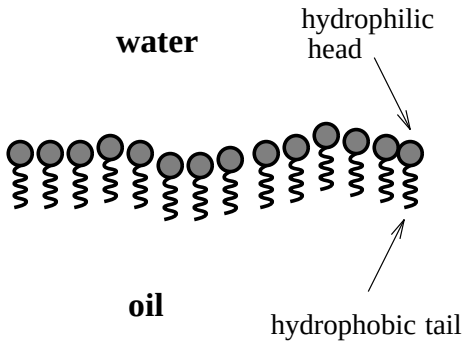

Fig. 10. Schematic description of a monolayer

Examples of tensionless films are provided by monolayers and bilayers made of amphiphilic molecules (see Figs.10-11). For instance monolayers may form at the interface between water and oil in ternary systems and can span an enormous internal area in microemulsions. Bilayers can form very large spherical vesicules, with size as large as 1 μm, to be compared with the width of the bilayer (typically a few 10 Å's. At that stage it is important to stress that in the most interesting cases (at least for the purpose of these lectures) the internal organization of these films is not fixed, but they form a two-dimensional fluid. I shall not discuss here the physics and the chemistry of such systems, and I refer to Nelson et al. (1989) and Gompper and Schick (1994) for more detailed discussions. Another example, with less numerous clean experimental realizations yet, is provided by polymerized membranes, made out of elements (monomers), linked together by strong (covalent) bonds and which could form large two-dimensional structures. Such systems, if they are flexible enough while keeping their internal two-dimensional structure fixed, correspond to flexible 2d crystalline surfaces.

- **Curvature and Rigidity**

Let us consider a two dimensional surface embedded in three dimensional space. Locally its curvature properties are encoded into its two curvature radii R_1 and R_2, or equivalently into the principal curvatures $K_i = 1/R_i$, $i = 1, 2$. From these curvature radii one gets the mean curvature

$$H = \frac{1}{R_1} + \frac{1}{R_2} \tag{2.2.2}$$

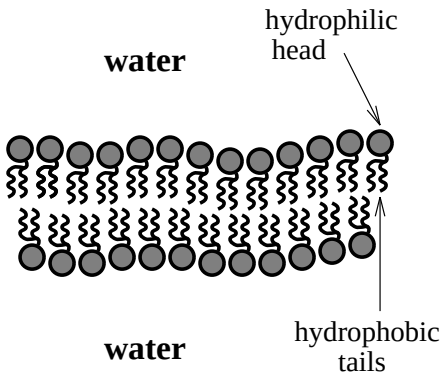

Fig. 11. Schematic description of a bilayer

and the Gaussian curvature

$$K \;=\; \frac{2}{R_1 R_2} \tag{2.2.3}$$

The mathematical definition of H and K can be found in standard textbooks. Let me just recall that the mean curvature H depends on the extrinsic properties of the surface, i.e. of the way it is curved in 3d space. In particular, H depends on the orientation we have chosen on the surface. On the contrary, the Gaussian curvature turns out to depend only on the intrinsic geometrical properties of the surface.

More precisely, if $\boldsymbol{\sigma} = (\sigma^i,\ i = 1,2)$ are local coordinates on the surface, and if the vector $\boldsymbol{r}(\boldsymbol{\sigma})$ denotes the position in 3d space of the point of the surface with coordinates $\boldsymbol{\sigma}$, the first fundamental form (or extrinsic metric) is defined as the 2×2 matrix $g_{ij} = \partial_i \boldsymbol{r} \partial_j \boldsymbol{r}$ ($\partial_i = \partial/\partial\sigma_i$). D_i denoting the covariant derivative for the metric g_{ij}, the extrinsic curvature is $\boldsymbol{K}_{ij} = D_i D_j \boldsymbol{r}$ and the second fundamental form is $K_{ij} = \boldsymbol{n} \cdot \boldsymbol{K}_{ij}$ with $\boldsymbol{n}$ a unit normal vector to the surface. Then $H = \mathrm{Tr}(K_{ij})$ and $K = 2\det(K_{ij})$. These definitions have natural extensions for surfaces in $d > 3$ dimensional space.

Finally, let me recall the well-known Gauss-Bonnet theorem. It states that for a closed surface with no boundary, the integral over the whole surface of the Gaussian curvature is a topological invariant, proportional to the Euler characteristics χ

$$\begin{aligned} \int d^2S\, K \;&=\; 4\pi\chi \;=\; 8\pi(1-h) \\ h \;&=\; \text{genus} = \text{number of handles} \end{aligned} \tag{2.2.4}$$

• **Fluid Surfaces and the Helfrich Hamiltonian**

One expects that the large distance properties of a fluid surface depend only on its geometrical shape, and not on its internal structure, since the fluid character of the surface implies that internal fluctuations have fast equilibration time, and are averaged much more quickly than shape fluctuations. Then it is natural to write the free energy F for such a membrane in terms of geometrical quantities only, and to expand it in powers of the curvature and its derivatives . One thus obtain the free energy for a closed surface (first written by Canham (1970) and Helfrich (1973))

$$F = \int d^2S \left[\frac{P}{3} \boldsymbol{n} \cdot \boldsymbol{r} + \tau + \frac{\kappa}{2}(H - H_0)^2 + \frac{\bar{\kappa}}{2} K \right] \qquad (2.2.5)$$

The first term is proportional to the (algebraic) 3d volume V enclosed by the surface, and P is the pressure difference between the interior and the exterior (for vesicles this is the physical osmotic pressure). The second term is just the total area A, and τ is the surface tension. This term enters if the surface is in contact with a reservoir of molecules, so that the total area is free to fluctuate, τ is then the chemical potential per unit of area. The last two terms are the curvature elastic energies. H_0 is the spontaneous curvature, which may be present for asymmetric monolayers. For symmetric bilayers it vanishes by in/out symmetry. The third term is the extrinsic curvature energy, and κ is the bending rigidity modulus. The last term is the Gaussian curvature energy, and $\bar{\kappa}$ the Gaussian rigidity modulus. For a surface with fixed topology this last term is a constant and is not relevant. It plays an important role when one considers fluctuations of topology.

The reason why one does not retain terms involving higher derivatives or higher powers of the curvature is their scaling dimension. In (2.2.5) all the terms with positive dimensions in terms of [length] have been taken into account. All other terms are small if one rescales by a factor $\lambda \gg 1$ the size of the surface, and are irrelevant for large distance physics. They play an important role for small vesicles, or dense assemblies of surfaces such that the width e is not small with respect to typical radii of curvature and typical distances between surfaces.

The mechanical problems implied by the free energy (2.2.5) are already highly non-trivial. For instance, the problem of finding the equilibrium shapes for a closed spherical vesicle, which minimize its free energy with the physical constraints that V, A and the area difference δA are fixed (these are the geometrical parameters conjugate to P, τ and H_0) is a very rich one, and leads to a very interesting morphological phase diagram (see e.g. Berndl et al. (1990), Miao et al. (1991)).

• **Thermal Fluctuations**

Let us concentrate on the effect of thermal fluctuations, and consider an infinite flat tensionless fluid membrane (this means that we set P, τ and H_0 to zero).

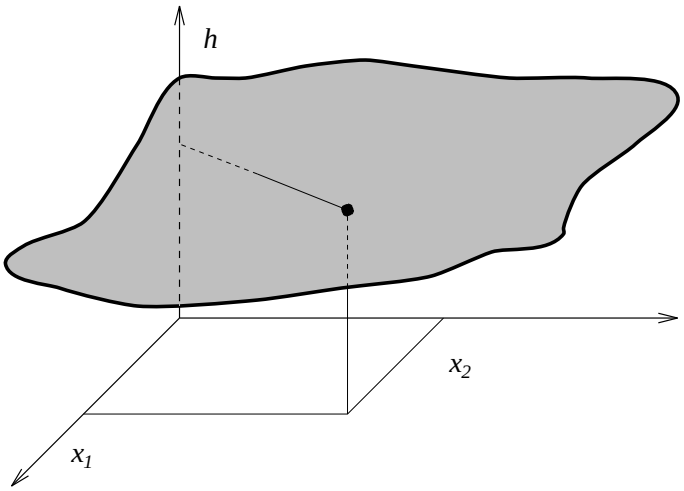

Fig. 12. The Monge representation for small undulations of a fluid membrane.

The shape undulations of the membrane can be represented (using the Monge representation) by the height function $h(\boldsymbol{x})$ above the rest plane (this amounts to use a coordinate system on the surface such that the position vector $\boldsymbol{x} \equiv \boldsymbol{r}$ is $x^1(\boldsymbol{\sigma}) = \sigma^1$, $x^2(\boldsymbol{\sigma}) = \sigma^2$, $x^3(\boldsymbol{\sigma}) = h$). In the linearized approximation (small undulations), one keeps only the terms quadratic in h and the free energy is

$$F_{\text{lin.}} \;=\; \int d^2S \,\Big[\tau \,+\, \frac{\tau}{2}(\nabla h)^2 \,+\, \frac{\kappa}{2}(\Delta h)^2 \,+\, \cdots\Big] \tag{2.2.6}$$

The excursion of the surface is measured by the large $\boldsymbol{x}$ behavior of $\langle \big(h(\boldsymbol{x}) - h(\boldsymbol{0})\big)^2 \rangle$. If the tension is positive, the $(\nabla h)^2$ term dominates and since the $\boldsymbol{x}$ reference space is two dimensional, this gives a logarithmic behavior

$$\langle \big(h(\boldsymbol{x}) - h(\boldsymbol{0})\big)^2 \rangle \;\propto\; \tau^{-1} \ln(|\boldsymbol{x}|/a) \tag{2.2.7}$$

characteristic of a rough but planar interface. If the tension vanishes we get a quadratic growth

$$\langle \big(h(\boldsymbol{x}) - h(\boldsymbol{0})\big)^2 \rangle \;\propto\; \kappa^{-1} \,|\boldsymbol{x}|^2 \tag{2.2.8}$$

Thus the transverse extension $|h|$ grows linearly with the size $|x|$ of the surface, as expected from the fact that κ is dimensionless. This implies that the linear approximation breaks down at large distances and that one must consider mode

coupling between undulations (de Gennes and Taupin (1982), Helfrich (1985)). For instance, if one consider the normal/normal correlator, this approximation gives

$$\langle \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{0}) \rangle \;=\; 1 \,-\, \kappa^{-1} \ln(|x|/a) \,+\, \cdots \tag{2.2.9}$$

which is clearly not valid for $\ln(|\boldsymbol{x}|a^{-1}) \gg \kappa$.

Non-linear mode coupling comes from the higher order terms in the Hamiltonian (Helfrich (1985)). Typically, the free energy density expands as

$$\sqrt{g} H^2 \;=\; 1 \,+\, \frac{1}{2} (\partial_i \partial_i h)(\partial_j \partial_j h) \,+\, (\partial.\partial.h)(\partial.\partial.h)(\partial.h)(\partial.h) \,+\, \cdots \tag{2.2.10}$$

(the dots represents various contracted space indices $i = 1, 2$). In the perturbative expansion in powers of κ^{-1} one encounters short distance divergences which can be recast into a renormalization of the effective parameters of the model, here the rigidity κ. For instance, the calculation at one loop of the two-points correlator (2.2.8) gives, in momentum space (Peliti and Leibler (1985))

$$\langle \hat{h}(\boldsymbol{p}) \hat{h}(-\boldsymbol{p}) \rangle \;=\; \kappa^{-1} |\boldsymbol{p}|^4 \,-\, \kappa^{-2} \frac{d}{4\pi} |\boldsymbol{p}|^4 \ln(|\boldsymbol{p}|a) \,+\, \cdots \tag{2.2.11}$$

Here a is the short-distance cut-off, and we give the result for the general case of a 2D fluid surface in d-dimensional space (Polyakov (1986)). This logarithmic term can be recast[5] into an effective, momentum dependent, rigidity $\kappa_{\rm eff}(\boldsymbol{p})$, given to first order by

$$\kappa_{\rm eff}(\boldsymbol{p}) \;=\; \kappa \,+\, \frac{d}{4\pi} \ln(|\boldsymbol{p}|a) \,+\, \cdots \tag{2.2.12}$$

What is the physical significance of this result? As the momentum scale $\boldsymbol{p}$ gets smaller, that is as the length scale ℓ gets larger, the effective rigidity becomes smaller. This means that a fluid membrane is more flexible at large distance. If one extends this one loop calculation up to a momentum scale where $\kappa_{\rm eff}(\boldsymbol{p}) \sim 1$, there is a typical macroscopic length scale

$$\ell_p \;\propto\; a \; {\rm e}^{\frac{4\pi}{d}\kappa} \tag{2.2.13}$$

beyond which the effect of the rigidity becomes negligible, and such that the membrane is expected to be crumpled. As for the case of polymers, this length scale is called the *persistence length*. But for 2D membranes the persistence length grows exponentially with the rigidity, while for 1D polymers it grows only linearly. The nature of the crumpled phase cannot be inferred from this one loop calculation alone. However, large d calculations show that the normal-normal correlation function (2.2.9) has a exponential decay at large distances, with a finite correlation length equal to this persistence length, as expected in a crumpled phase.

[5] In usual cases, one must consider both the 2-points and the 4-points functions to obtain the coupling constant renormalization. This is unnecessary here because there is no wave-function renormalization of the $\boldsymbol{r}$ field, thanks to rotational invariance.

• **Polymerized Surfaces**

Let us now consider the behavior of 2D "flexible crystal". The difference between a liquid and a crystal is that the later has shear modulus, so that longitudinal deformations cost shear elastic energy. This implies that one cannot take into account only the transverse undulations of a polymererized membrane to describe its elastic properties, as first pointed out by Nelson and Peliti (1987). In the small deformation approximation, one can describe the position of the surface as in the theory of elasticity for thin plates. Let the rest conformation of the surface (at zero temperature) be $\boldsymbol{r}_{\rm rest}(\boldsymbol{x}) = (x^1, x^2, 0)$. Small displacements are labeled by the longitudinal displacement $\boldsymbol{u} = (u^1, u^2)$ and by the transverse undulation h, so that the configuration of the deformed surface is

$$\boldsymbol{r}(\boldsymbol{x}) \;=\; \big(x^1 + u^1(\boldsymbol{x}), x^2 + u^2(\boldsymbol{x}), h(\boldsymbol{x})\big) \tag{2.2.14}$$

As for fluid membranes the curvature energy is proportional to the squared mean curvature, where the mean curvature is given (in the small deformation approximation) by $H = \Delta h$, but one must take also into account the internal elastic energy associated to longitudinal deformations of the membrane. The stress tensor is related to the rank 2 strain tensor u_{ij}, given by

$$u_{ij}(\boldsymbol{x}) \;=\; \frac{1}{2}\left(\partial_i u^j + \partial_j u^i + \partial_i h \partial_j h\right) \tag{2.2.15}$$

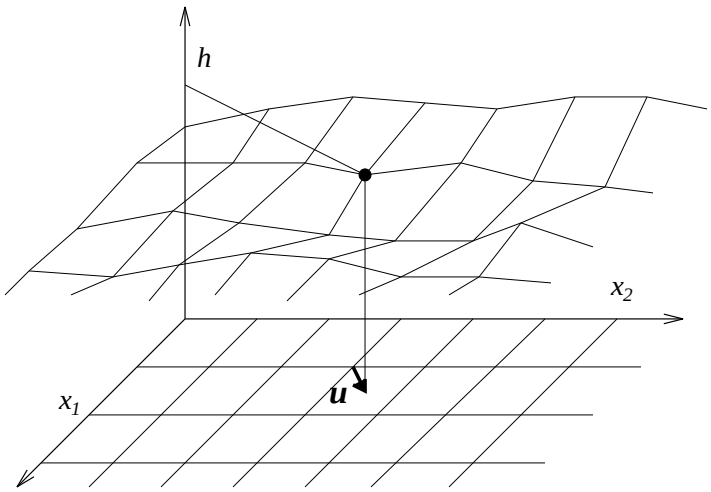

Fig. 13. Deformations for a polymerized membrane

The total elastic energy, taking into account both the external curvature elasticity and the internal elasticity, is in this approximation

$$E \;=\; \int d^2\boldsymbol{x}\;\left[\frac{\kappa}{2}(\Delta h)^2 \,+\, \mu(\tilde{u}_{ij})^2 \,+\, \frac{K}{2}(u_{ii})^2\right] \tag{2.2.16}$$

where $\tilde{u}_{ij} = u_{ij} - \frac{1}{2}\delta_{ij}\,u_{kk}$ is the traceless part of the strain tensor. κ is as above the bending elasticity modulus, μ is the shear modulus, non-zero for 2D solids, and K is the compressibility modulus.

One can estimate the importance of internal elastic forces by fixing the transverse deformation of the membrane, given by h, and by letting the membrane adjust its internal configuration, given by $\boldsymbol{u}$, in order to minimize its total elastic free energy E. This energy (2.2.16) is at most quadratic in $\boldsymbol{u}$, and the minimization can be performed exactly. The result is a long distance effective interaction between the Gaussian curvature $K(\boldsymbol{x}) = -\Delta(\partial_i h \partial_i h) + \partial_i\partial_j(\partial_i h \partial_j h)$, proportional to the Young elastic modulus $K_0 = \mu K/(\mu + K)$, which is > 0 as long as the shear modulus μ is non-zero,

$$E_{\min}[h] \;=\; \int d^2\boldsymbol{x}\,\frac{\kappa}{2}(\Delta h)^2 \,+\, K_0 \int d^2\boldsymbol{x} \int d^2\boldsymbol{y}\,K(\boldsymbol{x})\langle \boldsymbol{x}|\Delta^{-2}|\boldsymbol{y}\rangle K(\boldsymbol{y}) \tag{2.2.17}$$

The kernel $\langle \boldsymbol{x}|\Delta^{-2}|\boldsymbol{y}\rangle \propto |\boldsymbol{x}-\boldsymbol{y}|^2 \ln(|\boldsymbol{x}-\boldsymbol{y}|)$ is increasing with the distance. The physical interpretation of this effect is in fact quite simple. Transverse undulations in one direction make the surface more difficult to bend in the orthogonal direction, as for a corrugated iron roof.

From this effective interaction mediated by transverse phonons, one expects a strong increase of the effective rigidity $\kappa_{\rm eff}$ with the length-scale ℓ, implying that a flexible crystalline surface is *flat*, not crumpled. The first approximate calculation of this effect by Nelson and Peliti (1987) predicts a linear increase: $\kappa_{\rm eff}(\ell) \propto \ell$. Subsequent renormalization group calculations based on a $1/d$ expansion (David and Guitter (1988)) predict a slower, but still net increase of the rigidity, as $\kappa_{\rm eff}(\ell) \propto \ell^{\eta}$, with $0 < \eta < 1$.

The small undulations approximation is valid if the microscopic rigidity is already large ($\kappa \gg 1$). For flexible crystalline membranes ($\kappa \sim 1$) it is better to describe the membrane without breaking the O(d) rotational invariance in external space, that is by treating on the same footing transverse deformations (undulations) and longitudinal deformations (phonons). Then the position of the membrane will be given by the d-components vector $\boldsymbol{r}(\boldsymbol{x})$, and the strain tensor is related to the first fundamental form (induced metric) $g_{ij} = \partial_i \boldsymbol{r}\partial_j \boldsymbol{r}$. The natural Landau-Ginsburg form for the free energy is then

$$H[\boldsymbol{r}] \;=\; \int d^2\boldsymbol{x}\,\left[\frac{\kappa}{2}(\Delta \boldsymbol{r})^2 \,+\, t g_{ii} \,+\, \mu(\tilde{g}_{ij})^2 \,+\, \frac{K}{2}\,(g_{ii})^2\right] \tag{2.2.18}$$

with κ, t, μ and K the effective elasticity moduli (which depend on the temperature and on the microscopic elastic moduli), and $\tilde{g}_{ij}$ the traceless part of g_{ij} (Paczuski et al. (1988)). The natural geometrical order parameter is the extension factor ζ

$$\langle g_{ij}\rangle \;=\; \delta_{ij}\,\zeta^2 \tag{2.2.19}$$

In the mean field approximation, if $t < 0$ then ζ is non-zero, and the membrane is flat. In this phase the O(d) rotational symmetry is spontaneously broken,

and the large distance properties of the membrane are described by the small undulation approximation model of plates discussed above. However as $t \to 0_-$, ζ vanishes as $\sqrt{|t|}$ and for $t > 0$ ζ is zero. The $t > 0$ phase corresponds to a low rigidity (or high temperature) *crumpled phase*, separated from the flat phase by a *crumpling transition* at $t = 0$. In the crumpled phase, the O(d) rotational symmetry is not broken, and the large distance properties of the membrane are obtained by keeping the most relevant term in (2.2.19), that is by the simple Gaussian model

$$H_{\rm eff}[\boldsymbol{r}] \;=\; t \int d^2\boldsymbol{x}\; (\partial_i \boldsymbol{r} \partial_i \boldsymbol{r}) \tag{2.2.20}$$

The existence of this crumpling transition and of a low temperature flat phase is a crucial specific feature of crystalline membranes that distinguishes them from fluid membranes, which are always crumpled.

One can extend these considerations to "membranes" with higher internal dimension $D > 2$. In fact the mean field study of the LGW Hamiltonian (2.2.18) is valid only above some (internal) upper critical dimension $D_{\rm uc}$. $D_{\rm uc}$ turns out to be 4, as for ordinary critical phenomena. For $D < 4$, and in particular to study the physical case $D = 2$, $d = 3$, one must perform RG calculations via either a $D = 4 - \epsilon$ expansion, of via large d expansions. Another method is of course provided by numerical simulations. These studies show that the flat phase and the crumpling transition still exist for $D = 2$. The nature of the crumpling transition (first or second order) is not completely settled, but most numerical studies lead to a continuous second order crumpling transition.

Let me stress however that up to now I discussed only the role of interactions within the membrane which come from local couplings between elements of the membrane, and which can be described by local terms in the free energy (curvature, strain tensor, ...). Steric interactions do not belong to this class. We have seen that steric interactions change qualitatively the behavior of long one dimensional chains in three dimensional space. I shall discuss in the last lecture steric interactions for crystalline surfaces. We shall see that self-avoidance may induce effects that are even stronger for membranes than for polymers, and that this problem raises very interesting theoretical and mathematical issues.

2.3 Self-avoiding polymerized Surfaces

Does the crumpled phase exist for "real" self-avoiding crystalline membranes? The theory predicts a low temperature flat phase, where the fractal dimension of the membrane is 2. In this phase, since the membrane is flat with only small undulations (but with a non-trivial excitation spectrum), self-avoidance is irrelevant. At high temperature the membrane is crumpled, and self-avoidance is expected to be very relevant. In fact both numerical simulations (Abraham and Nelson (1990), Abraham et al. (1989), Grest and Murat (1990)) and some experiments suggest that self-avoidance can make the membrane flat, and that the crumpled phase and the crumpling transition may disappear completely (at least

in 3 dimensions)! In order to understand these issues, one must first construct theoretical models of self-avoiding crystalline membranes.

• **The SAM Edwards Model**

A simple model, first studied thoroughly by Nelson & Kardar (1987) and by Aronovitz & Lubensky (1987), is a simple extension of the Edwards' model of polymers. We have argued above that in the high temperature crumpled phase, a phantom crystalline membrane is described by the simple Gaussian model (2.2.20). The simplest way to take into account self-avoidance is to add a repulsive two-body interaction, as done for polymers in (2.1.6). Let the Hamiltonian for a D-dimensional membrane (self-avoiding manifold Edwards' model) be

$$S[\boldsymbol{r}] \;=\; \int d^D\boldsymbol{x}\,\frac{1}{2}(\nabla\boldsymbol{r})^2 \;+\; \frac{b}{2}\int d^D\boldsymbol{x}\int d^D\boldsymbol{y}\;\delta^d(\boldsymbol{r}(\boldsymbol{x})-\boldsymbol{r}(\boldsymbol{y})) \qquad (2.3.1)$$

with $b>0$ the weak self-avoidance coupling constant.

For $b=0$ one recovers the Gaussian model. The canonical dimension of $\boldsymbol{r}$ is $[\boldsymbol{r}]=[\boldsymbol{x}]^{\nu_0}$, with $\nu_0=(2-D)/2=1/d_H$, d_H being the fractal dimension of the free Gaussian surface. The dimension of b in now $[b]=[x]^{2D-d\nu_0}$. As we have done for polymers, let us play our favorite game and consider the general case where both the dimension d of space and now the dimension D of the membrane vary continuously and are non-integer. We expect two different cases:

- If $2D-d\nu_0<0$ self-avoidance is irrelevant at large distances. A self-avoiding membrane will have similar scaling properties than the Gaussian membrane, and it will be characterized by the same ν exponent defined through (2.1.1) $\nu=\nu_0=(2-D)/2$
- If $2D-d\nu_0>0$ self-avoidance is relevant at large distances. A self-avoiding membrane exhibits critical swelling and is characterized by an exponent ν such that $\nu_0<\nu\leq 1$. If $\nu<1$ the membrane is in the swollen crumpled phase, while if $\nu=1$ it is flat.

The physical point $(d=3, D=2)$ lies deep inside the region where self-avoidance is relevant. It can be studied by approximation schemes, andby renormalization group methods. A first approximation consists in a simple extension of the Flory theory for polymers. It amounts to assume that $\boldsymbol{r}$ gets an anomalous scaling dimension ν such that the Gaussian elastic term and the self-avoidance interaction term in (2.3.1) scale in the same way, and it leads to $\nu_{\rm Flory}=(2+D)/(2+d)$. Another more elaborate approximation is the so-called Gaussian variational method, where one approximates the Hamiltonian (2.3.1) by the best non-local Hamiltonian $S_{\rm var}[\boldsymbol{r}]$ which is quadratic in $\boldsymbol{r}$. It leads (Guitter and Palmeri (1992)) to $\nu_{\rm var}=2D/d$. For 2D membranes ($D=2$) both methods predict that self-avoidance is relevant for any finite space dimension $d<\infty$. However the Flory theory predicts that the flat phase $\nu=1$ occurs for $d\leq 2$, and that physical self-avoiding membranes are crumpled[6], while the

[6] neglecting bending rigidity, which can also make the membrane flat

variational approximation predicts that the flat phase occurs for $\nu \leq 4$ and that physical membranes are flat.

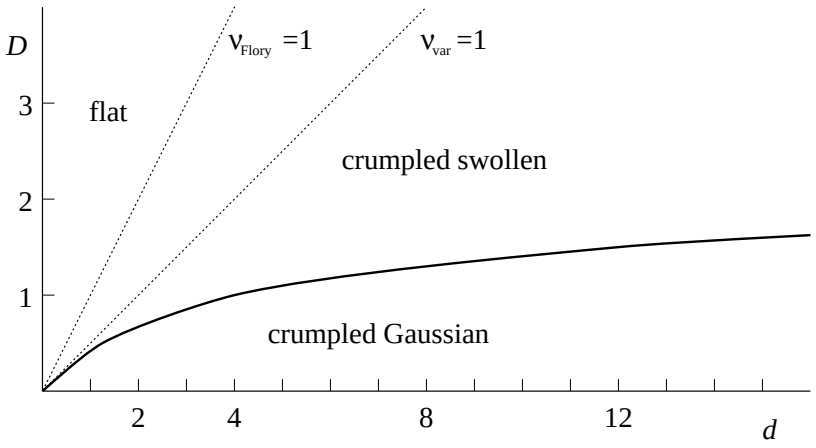

Fig. 14. The phases for flexible polymerized membranes in the (d, D) plane

Renormalization group methods, inspired by the so-called direct renormalization method for polymers (des Cloizeaux (1981)), have been proposed and applied to one loop calculations for self-avoiding membranes (Aronowitz and Lubensky (1987), Kardar and Nelson (1987)). They lead to a correction for the exponent ν, proportional to the parameter ϵ, which is the engineering dimension of the self-avoidance coupling constant b, $\epsilon = 2D - d(2 - D)/2$. Similar corrections can be estimated for several other exponents which characterize closed or open self-avoiding membranes (Duplantier (1987), Duplantier et al. (1990)). It is important to understand the applicability of these technics, and to see if they can be extended into a systematic ϵ-expansion, for at least two reasons. Firstly, more controlled calculational schemes are needed in order to understand whether self-avoiding membranes are crumpled or flat in 3 dimensions. Secondly, the study of the mathematical consistency of these calculation raises non-trivial and very interesting issues in quantum field theory and in renormalization group theory.

As discussed before, the validity of the renormalization group methods for the Edwards' model of polymers relies in fact on the equivalence of the model with some local field theory in d-dimensional space. Direct calculational schemes, such as the direct renormalization method, are often more convenient. However, at each steps they are in one-to-one correspondence with the renormalization steps for the O(n) model, and it is in fact the general results of renormalization theory for local quantum field theories that ensure their validity (Benamou and Mahoux (1986)).

Such an equivalence between a D-dimensional membrane model and a d-dimensional formulation as a local theory in external space is still missing if $D \neq 1$ (if it exists it should be a string-like theory of extended objects). Therefore, the only realistic approach seems to study the D-dimensional model (2.3.1) directly, but this raises at least two problems: (*i*) the interaction term is a singular δ distribution, which leads to several technicalities; (*ii*) the interaction term is *non-local*, since it involves the position field $\boldsymbol{r}$ at two different points on the membrane, $\boldsymbol{x}$ and $\boldsymbol{y}$, which may be arbitrarily far apart. This last point is the most interesting. Indeed, renormalization theory has been developed for models with short range interactions, which can be reduced in the continuum limit to *local* theories. Non-local models are in general out of scope of renormalization theory. I shall now present recent work with E. Guitter and B. Duplantier which shows that indeed renormalization group theory can be applied to this specific class of models.

• **Renormalization for the LGW Model**

Before discussing the non-local model (2.3.1), I shall recall briefly how renormalization group theory applies to a local field theory and to critical phenomena. We start from the well-know Landau-Ginsburg-Wilson critical (massless) Φ^4 Hamiltonian

$$S[\Phi] \;=\; \int d^d\boldsymbol{x}\; \left[(\nabla\Phi)^2 + b\Phi^4\right] \tag{2.3.2}$$

in dimension $d = 4-\epsilon$, $\epsilon > 0$. Φ is a 1-component real scalar field[7]. We are interested in its perturbative expansion in b around the Gaussian theory $b = 0$. The dimension of the free field Φ is $(1-d)/2 = -1+\epsilon/2$, and the free propagator scales as

$$\langle\Phi(\boldsymbol{x})\Phi(\boldsymbol{y})\rangle_0 \;\propto\; |\boldsymbol{x}-\boldsymbol{y}|^{-2+\epsilon} \tag{2.3.3}$$

The dimensions of the relevant operators for $\epsilon = 0$ are respectively (I consider only the even sector under $\Phi \to -\Phi$)

operator	dimension
$\mathbb{1}$	0
Φ^2	$-2+\epsilon$
Φ^4	$-4+2\epsilon$
$(\nabla\Phi)^2$	$-4+\epsilon$

(2.3.4)

The term of order b^n of the perturbative expansion involves the vacuum expectation value of n Φ^4 operators integrated over space

$$\int d^d\boldsymbol{x}_1 \cdots \int d^d\boldsymbol{x}_n \,\langle\cdots\Phi^4(\boldsymbol{x}_1)\cdots\Phi^4(\boldsymbol{x}_n)\cdots\rangle_0 \tag{2.3.5}$$

[7] One can apply the same considerations to the $n = 0$ O(n) model for polymers, only symmetry combinatoric factors differ from the $n = 1$ model.

which can be expressed as a sum of Feynman diagrams using Wick's theorem. These integrals have short distance singularities when some Φ^4 operators come close to each other. These singularities are encoded in the Wilson short distance operator product expansion (OPE) for the free theory. At first order, it is sufficient to look at the 2-points divergence. When two Φ^4 operators coalesce, their OPE is

$$\begin{aligned}\Phi^4(\boldsymbol{x}/2)\Phi^4(-\boldsymbol{x}/2) \;=\;& c_0\,|\boldsymbol{x}|^{-8+4\epsilon}\,\mathbb{1}(\mathbf{0}) \\ &+\; c_1\,|\boldsymbol{x}|^{-6+3\epsilon}\,\Phi^2(\mathbf{0}) \\ &+\; c_2\,|\boldsymbol{x}|^{-4+3\epsilon}\,(\nabla\Phi)^2(\mathbf{0}) \\ &+\; c_4\,|\boldsymbol{x}|^{-4+2\epsilon}\,\Phi^4(\mathbf{0}) \\ &+\; \text{less singular terms as } \boldsymbol{x}\to 0\end{aligned} \tag{2.3.6}$$

with c's the OPE coefficients. On Fig. 15 I have represented the corresponding divergent Feynman diagram which contribute to (2.3.6).

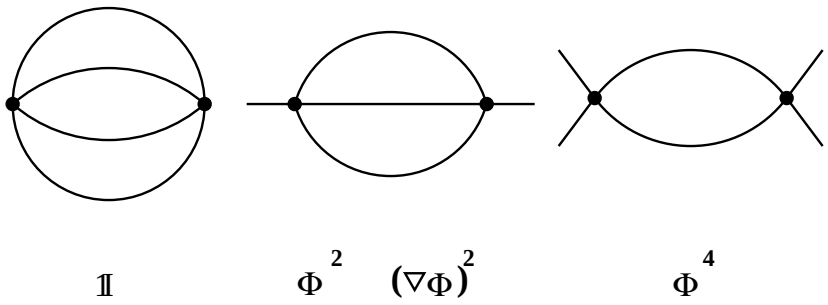

Fig. 15. Divergent Φ^4 diagrams and the corresponding operators

The renormalization group transformation can be obtained easily from (2.3.6). The free energy F (vacuum energy) of the interacting model is up to order b^2

$$F_\Lambda(b) \;=\; \ln(Z) \;=\; F(0) - b\int d^d\boldsymbol{x}\,\langle\Phi^4(\boldsymbol{x})\rangle + \frac{b^2}{2}\int_{|\boldsymbol{x}-\boldsymbol{y}|>\Lambda^{-1}} d^d\boldsymbol{x}\,d^d\boldsymbol{y}\,\langle\Phi^4(\boldsymbol{x})\Phi^4(\boldsymbol{y})\rangle + \cdots \tag{2.3.7}$$

where Λ is the UV cutoff. Performing an infinitesimal RG transformation, i.e integrating out the fluctuations of the field with wave-length ℓ such that $\Lambda^{-1} < \ell < \Lambda'^{-1} = \Lambda^{-1}(1+\delta)$, amounts to integrate in (2.3.7) $\boldsymbol{y}$ in the domain $\Lambda^{-1} < |\boldsymbol{y}-\boldsymbol{x}| < \Lambda'^{-1} = \Lambda^{-1}(1+\delta)$. Using the OPE (2.3.6), and forgetting the contribution of the relevant operators $\mathbb{1}$ and Φ^2 (for simplicity), we see that this amounts to replace the original Hamiltonian $S_\Lambda = S$ given by (2.3.2) by an effective Hamiltonian $S_{\Lambda'}$

$$S_\Lambda \;=\; \int (\nabla\Phi)^2 + b\Phi^4 \;\to\; S_{\Lambda'} \;=\; \int (\nabla\Phi)^2 + (b - b^2\delta\Lambda^{-\epsilon}c_4 S_d)\Phi^4 \qquad (2.3.8)$$

(S_d is the volume of the unit $(d-1)$-dimensional sphere). Rescaling the distances $\boldsymbol{x}$ and the field Φ as

$$\boldsymbol{x} \;=\; (1+\delta)\boldsymbol{x}' \qquad ; \qquad \Phi = (1+\delta(-1+\epsilon/2))\Phi' \qquad (2.3.9)$$

we obtain the renormalized Hamiltonian $S^R_{\Lambda'}$ and the renormalized coupling constant b^R

$$\begin{aligned} S^R_{\lambda'} \;&=\; \int d^d\boldsymbol{x}'\,(\nabla\Phi')^2 + b^R\Phi'^4 \\ b^R \;&=\; b \,+\, \delta(b\epsilon - b^2\Lambda^\epsilon c_4 S_d + \cdots) \end{aligned} \qquad (2.3.10)$$

Iterating this infinitesimal RG transformation one obtains the RG flow for the dimensionless effective coupling constant $b_{\text{eff}} = b^R\Lambda^{-\epsilon}$ as a function of the scale factor S

$$S\frac{db(S)}{dS} \;=\; W(b) \;=\; \epsilon b \,-\, c_4 S_d b^2 \,+\, \cdots \qquad (2.3.11)$$

The OPE coefficient c_4 is positive. Thus for $d<4$, $\epsilon>0$ and $b(S)$ flows for large scales ($S\to\infty$) towards a non-trivial infra-red fixed point $b^\star = \epsilon(c_4 S_d)^{-1} + \cdots$ which describes the non-trivial large distance properties of the massless theory.

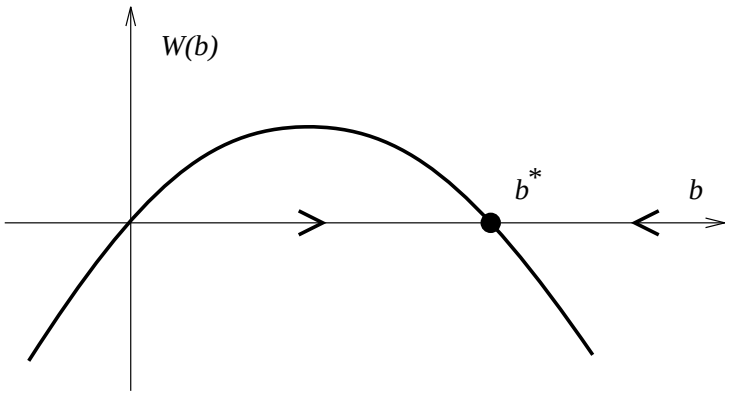

Fig. 16. The Wilson function and the flow of the coupling constant for $d<4$

Renormalization theory shows that these considerations can be extended to higher orders of perturbation theory. Short distance singularities occurs when an arbitrary number of Φ^4 operators coalesce into a single point (this corresponds to higher loop divergent diagrams), but also when subsets of operators coalesce faster than the whole set (this corresponds to divergent subdiagrams), when sub-subsets coalesce inside subsets, etc... These singularities can be treated in an iterative way by the renormalization group. The most dangerous singularities

can be absorbed into a renormalization of the action (2.3.2), which takes the general form:

$$S^R[\Phi] \;=\; Z\int d^d\boldsymbol{x}\,(\nabla\Phi_R)^2 \;+\; b_R\mu^\epsilon Z_b\int d^d\boldsymbol{x}\,\Phi_R^4 \qquad (2.3.12)$$

(Z and Z_b are the wave-function and coupling constant renormalization, Φ_R and b_R the renormalized field and coupling constant, μ the renormalization momentum scale, and we have not written the relevant mass counterterm, proportional to Φ_R^2). The RG flow equation follows from the existence of this renormalized Hamiltonian. It can be extended to all orders in perturbation theory.

If we return to the case of polymers, and to the $n=0$ LGW model, a similar calculation can be done. Only the value of the c_4 coefficient differs, but it is still positive. Identifying the scale factor S with the actual scale length L of the polymer, (2.3.11) means that the effective steric repulsion strength $b_{\rm eff}(L)$ flows towards a fixed value $b^\star$ for large length L. This fixed point governs the scaling properties of long chains, and the exponent ν can be extracted from these calculations. This fixed point $b^\star$ is related to a measurable physical quantity, the second virial coefficient g of polymers.

• Renormalization for the SAM Model

One can a priori construct a perturbative expansion for the SAM model defined by the Hamiltonian (2.3.1). The interaction term is now bi-local in the field $\boldsymbol{r}$. It can be written as a Fourier transform (in $\boldsymbol{r}$ space) of the product of two local operators

$$\delta^d(\boldsymbol{r}(\boldsymbol{x})-\boldsymbol{r}(\boldsymbol{y})) \;=\; \int\frac{d^d\boldsymbol{k}}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}k r(x)}\,\mathrm{e}^{-\mathrm{i}k r(y)} \qquad (2.3.13)$$

The term of order b^n of the perturbative expansion now involves the expectation value (in the free Gaussian theory) of a product of $2n$ such exponential operators. This can be calculated, using the well known formula for the v.e.v. of exponential of free fields

$$\langle\prod_{a=1}^{2n}\mathrm{e}^{\mathrm{i}k_a r(x_a)}\rangle_0 \;=\; \mathrm{e}^{-\frac{1}{2}\sum\limits_{ab}k_a\langle r(x_a)\,r(x_b)\rangle_0 k_b} \qquad (2.3.14)$$

In our case the dimension of the membrane is $0<D<2$ and the free propagator is

$$\langle r^\mu(\boldsymbol{x}_a)r^\nu(\boldsymbol{x}_b)\rangle_0 \;=\; -|\boldsymbol{x}_a-\boldsymbol{x}_b|^{2-D}\,\frac{\Gamma(D/2)}{2(2-D)\pi^{D/2}} \qquad (2.3.15)$$

There is an infra-red singular part in the propagator, proportional to L^{2-d} with L the size of the membrane, that has been subtracted out. This is justified since the IR divergences cancel for physical quantities, and this subtraction explains why the propagator is negative.

The physical quantities are obtained by integrating over the positions $\boldsymbol{x}$ and the momenta $\boldsymbol{k}$ expressions of the form (2.3.14). One must remember that all the $\boldsymbol{k}$'s are not independent, since from (2.3.13) they are grouped in n "dipoles"

such that $\boldsymbol{k} + \boldsymbol{k}' = 0$. The integration over the $\boldsymbol{k}$'s is Gaussian and can be performed explicitly. One obtains for the terms of order n an integral over $2n$ points $\boldsymbol{x}_a$ of a function of the relative distances between the points $|\boldsymbol{x}_a - \boldsymbol{x}_b|$. The mathematical framework to define properly these integrals over a space with non-integer dimension D is known as dimensional regularization, and it is detailed for this problem in David et al. (1993). At this level of presentation, it is sufficient to assume that the naive rules of integration and of power counting apply.

For instance the first term for the free energy is given by

$$F(b) \;=\; F(0) \;-\; \frac{b}{2} \int d^D\boldsymbol{x} \int d^D\boldsymbol{y}\, \langle \delta^d(\boldsymbol{r}(\boldsymbol{x}) - \boldsymbol{r}(\boldsymbol{y}))\rangle_0 \;+\; \mathcal{O}(b^2) \qquad (2.3.16)$$

with

$$\langle \delta^d(\boldsymbol{r}(\boldsymbol{x}) - \boldsymbol{r}(\boldsymbol{y}))\rangle_0 \;\sim\; \int d^d\boldsymbol{k}\; \mathrm{e}^{k^2\langle r(x)r(y)\rangle_0} \;\sim\; |\boldsymbol{x} - \boldsymbol{y}|^{-d(2-D)/2} \qquad (2.3.17)$$

The integrand in (2.3.17) has a short-distance singularity as $|\boldsymbol{x} - \boldsymbol{y}| \to 0$. The crucial property is that this singular behavior can be encoded in an operator product expansion, as in the case of the local LGW theory. Writing the operator $\delta^d(\boldsymbol{r}(\boldsymbol{x}) - \boldsymbol{r}(\boldsymbol{y}))$ as a *bilocal* operator $\Delta(\boldsymbol{x}, \boldsymbol{y})$, one can show that as $\boldsymbol{x} \to \boldsymbol{y}$, Δ can be expanded into *local* operators

$$\begin{aligned} \Delta(\boldsymbol{x}/2, -\boldsymbol{x}/2) \;=\;& c_0\, |\boldsymbol{x}|^{-d(2-D)/2}\, \mathbb{1}(\boldsymbol{0}) \\ &+\; c_1\, |\boldsymbol{x}|^{2-(d+2)(2-D)/2} (\nabla \boldsymbol{r})^2(\boldsymbol{0}) \\ &+\; \text{less singular terms as } \boldsymbol{x} \to 0 \end{aligned} \qquad (2.3.18)$$

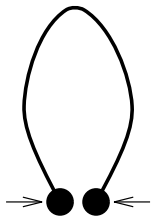

Fig. 17. A local divergence proportional to local operators

Now general terms have additional short distance singularities, which are not proportional to local operators. The simplest one occurs at order b^2, when at least two Δ operators coalesce as shown on Fig. 18. One can show that the corresponding singularity is proportional to the operator Δ itself, and that it can be written as

$$\Delta(\boldsymbol{x}/2, \boldsymbol{z}+\boldsymbol{y}/2)\Delta(-\boldsymbol{x}/2, \boldsymbol{z}-\boldsymbol{y}/2) \;=\; d_0(|\boldsymbol{x}|,|\boldsymbol{y}|)\,\Delta(\boldsymbol{0},\boldsymbol{z}) \;+\; \begin{array}{l}\text{less singular terms}\\ \text{as } \boldsymbol{x} \text{ and } \boldsymbol{y}\to 0\end{array} \tag{2.3.19}$$

where the coefficient d_0 is an homogeneous function of $|\boldsymbol{x}|$ and $|\boldsymbol{y}|$ with degree $-d(2-D)/2$ (as expected from power counting). This is in fact the first term of a *multilocal operator product expansion* (MOPE) for the product of the two bilocal operators when $\boldsymbol{x}$, $\boldsymbol{y} \to 0$ while $\boldsymbol{z}$ stays finite. This MOPE generates an infinite family of bi-local operators, with general form

$$\left(\frac{\partial}{\partial \boldsymbol{r}}\right)^q \delta^d(\boldsymbol{r}(\boldsymbol{x})-\boldsymbol{r}(\boldsymbol{y}))\,A(\boldsymbol{x})\,B(\boldsymbol{y}) \tag{2.3.20}$$

where A and B are any local operators involving $\boldsymbol{r}$ (in fact only products of derivatives of $\boldsymbol{r}$, as a consequence of the invariance under translations in $\boldsymbol{r}$ space $\boldsymbol{r} \to \boldsymbol{r}+\boldsymbol{r}_0$). These operators can be considered as local convolutions in $\boldsymbol{r}$ space of local vertex operators of the form $A(\boldsymbol{x})\mathrm{e}^{\mathrm{i}k\boldsymbol{r}(\boldsymbol{x})}$.

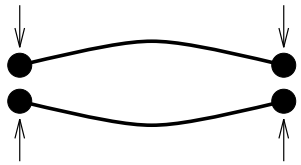

Fig. 18. A divergence proportional to bi-local operators

This argument can be generalized to multilocal operators involving more than two points. For instance, three bi-local operators can generate a tri-local operator, as shown on Fig. 18.

The most general m-local operator generated by the MOPE is of the form

$$\int d^d\boldsymbol{r} \prod_{a=1}^{m} \left(\frac{\partial}{\partial \boldsymbol{r}}\right)^{\boldsymbol{q}_a} \delta^d(\boldsymbol{r}(\boldsymbol{x}_a)-\boldsymbol{r})\,A_a(\boldsymbol{x}_a) \tag{2.3.21}$$

and this infinite family of multilocal operators generates a closed algebra under the MOPE. Moreover, it is possible to show that the usual steps of renormalization theory can be extended for the theories involving these multi-local operators, and that one can derive RG flow equations for the model of self-avoiding random surfaces. Indeed, at the critical dimension ($\epsilon = 2D - d(2-D)/2 = 0$), power counting shows that the short distance divergences can only be proportional to the three operators present in (2.3.18-19), namely:

- two local operators: $\mathbb{1}$ and $(\nabla \boldsymbol{r})^2$
- one bi-local operator: $\Delta = \delta^d(\boldsymbol{r}(\boldsymbol{x})-\boldsymbol{r}(\boldsymbol{y}))$

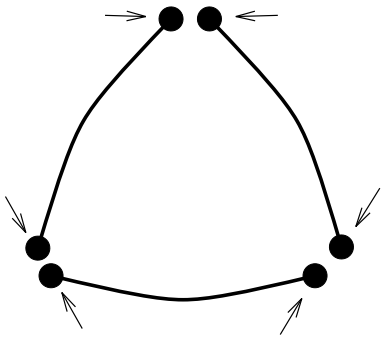

Fig. 19. A divergence proportional to tri-local operators

One can thus write a renormalized Hamiltonian similar to that of the LGW model (2.3.12), in terms of the renormalized fields $\boldsymbol{r}_R$ and of the renormalized coupling constant b_R

$$S^R[\boldsymbol{r}] \;=\; Z \int d^D\boldsymbol{x}\, \frac{1}{2}(\nabla \boldsymbol{r}_R)^2 \;+\; \frac{b_R}{2}\mu^\epsilon Z_b \int d^D\boldsymbol{x} \int d^D\boldsymbol{y}\; \delta^d(\boldsymbol{r}_R(\boldsymbol{x}) - \boldsymbol{r}_R(\boldsymbol{y})) \tag{2.3.22}$$

The renormalization factors Z and Z_b can be obtained (at first order in b_R) from the MOPE coefficients c_1 and d_0 in (2.3.18) and (2.3.19) respectively, by an argument similar to that used for the LGW model. Using the fact that the self-avoiding manifold model depends only (for $\epsilon > 0$) of the "bare variables" $\boldsymbol{r} = Z^{1/2}\boldsymbol{r}_R$ and $b = b_R\mu^\epsilon Z_b Z^{d/2}$, one can derive renormalization group equations. They involve the Wilson function $W(b_R) = \mu\frac{\partial}{\partial\mu} b_R\big|_b$ which gives the RG flow for the effective coupling constant, and the anomalous dimension $\nu(b_R) = (2 - D)/2 - \frac{1}{2}\frac{\partial}{\partial\mu}\ln Z\big|_b$ of the field $\boldsymbol{r}$. The Wilson function is found to have the same structure than for 1D polymers

$$W(b) \;=\; \epsilon b \;-\; \mathbf{c} b^2 \;+\; \mathcal{O}(b^2) \tag{2.3.23}$$

with $\mathbf{c}$ a positive number depending on the dimension D of the membrane. It has an IR fixed point at $b^\star = \epsilon/\mathbf{c} + \cdots$. The anomalous exponent of SA surfaces in the swollen crumpled phase is given by $\nu = 1/d_H = \nu(b^\star)$.

This renormalization group formulation provides a general framework to study and compute explicitly the scaling properties of self-avoiding polymerized surfaces. Other critical exponents, such as configuration exponents for closed and open surfaces, can be extracted. I refer to Duplantier (1987) and David et al. (1994) for details. Let me just mention that the range of validity of the Flory theory and of the variational approximation scheme can be better understood in this framework. The results of the Flory theory are derived by assuming that the elastic and self-avoidance terms in the Hamiltonian scale in the same way,

i.e. that the renormalization factors Z and Z_b are equal. However this is not true even to first order in ϵ, and it is not clear under which condition this approximation is valid (except in the very special case of 2d polymers, i.e. $D = 1$, $d = 2$). The variational approximation uses Gaussian trial Hamiltonians, and one can show that it becomes exact if only the Gaussian elastic term is renormalized, that is if the self-avoidance coupling constant b is not renormalized ($Z_b = 1$, $Z \neq 1$). Some arguments indicate that this becomes exact in the limit $d \to \infty$.

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