

# Large Random Matrices: Eigenvalue Distribution

B. Eynard

Service de Physique Théorique de Saclay  
F-91191 Gif-sur-Yvette Cedex, FRANCE  
Email: eynard@amoco.saclay.cea.fr

## Abstract:

A recursive method is derived to calculate all eigenvalue correlation functions of a random hermitian matrix in the large size limit, and after smoothing of the short scale oscillations. The property that the two-point function is universal, is recovered and the three and four-point functions are given explicitly. One observes that higher order correlation functions are linear combinations of universal functions with coefficients depending on an increasing number of parameters of the matrix distribution.

In a recent article [1], Brézin and Zee have calculated explicitly correlation functions of eigenvalues of a class of stochastic hermitian matrices of large size  $N$ . They have found that some statistical properties of the eigenvalues are universal in the large  $N$  limit, and can thus be obtained from the correlation functions of the gaussian model. More precisely, they have discovered that the two-point correlation function, after smoothing of the short scale oscillations, is universal while all other correlations vanish at the same order.

This property can be related to a renormalization group analysis [2,3] which has shown that the gaussian model is a stable fixed point in the large  $N$  limit.

Their analysis is based on the, by now standard, method of orthogonal polynomials. An essential ingredient in the final answer is a proposed ansatz for an asymptotic form of the orthogonal polynomials  $P_n$  in the limit  $N \rightarrow \infty$  and  $N - n$  finite. In ref. [1] the ansatz is verified in the case of even integrands, and only up to an unknown function.

Here, we propose a direct proof of the ansatz, using a saddle point method, which does not depend on the parity of the integrand, and which allows to determine the previously unknown function. Moreover, using a completely different approach [4], we present a recursive method to evaluate all smoothed correlation functions at leading order. We give the three and four-point functions explicitly.

## 1. Correlation functions of eigenvalues

Let us first explain the problem and recall the method used in ref. [1] to explicitly evaluate the eigenvalue correlation functions.

We consider  $N \times N$  hermitian matrices  $M$  with a probability distribution of the form:

$$\mathcal{P}(M) = \frac{1}{Z} e^{-N \text{tr} V(M)}, \quad (1.1)$$

where  $V(M)$  is a polynomial, and  $Z$  the normalization (i.e. the partition function). We want to derive the asymptotic form for  $N$  large of various eigenvalue correlation functions. All can be derived from the correlation functions of the operator  $\mathcal{O}(\lambda)$ :

$$\mathcal{O}(\lambda) = \frac{1}{N} \text{tr} \delta(\lambda - M) = \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \mu_i) \quad (1.2)$$

(the  $\mu_i$  being the eigenvalues of  $M$ ). Indeed, for any set of functions  $(f_1, \dots, f_k)$ , one has:

$$\frac{1}{N^k} \langle \text{tr} f_1(M) \dots \text{tr} f_k(M) \rangle = \int d\lambda_1 f_1(\lambda_1) \dots d\lambda_k f_k(\lambda_k) \langle \mathcal{O}(\lambda_1) \dots \mathcal{O}(\lambda_k) \rangle.$$

The correlation functions of the operator  $\mathcal{O}(\lambda)$  can in turn be expressed in terms of the partially integrated eigenvalue distributions like  $\rho(\lambda)$  the density of eigenvalues, which is the probability that  $\lambda$  belongs to the spectrum of  $M$ ,  $\rho_2(\lambda, \mu)$  the probability that  $\lambda$  and  $\mu$  are simultaneously eigenvalues of  $M$  and more generally  $\rho_n(\lambda_1, \dots, \lambda_n)$  the probability that  $\lambda_1 \dots \lambda_n$  are simultaneously eigenvalues of  $M$ . For  $\lambda_1 \neq \lambda_2 \dots \neq \lambda_n$  (else some additional contact terms have to be added) we find

$$\langle \mathcal{O}(\lambda_1) \mathcal{O}(\lambda_2) \dots \mathcal{O}(\lambda_n) \rangle = \frac{1}{N^n} \frac{N!}{(N-n)!} \rho_n(\lambda_1, \dots, \lambda_n).$$

Actually the interesting functions are not directly the  $\mathcal{O}(\lambda)$  correlation functions, but their connected parts. Indeed, at leading order, when  $N \rightarrow \infty$ , we have the factorization property

$$\langle \mathcal{O}(\lambda_1) \mathcal{O}(\lambda_2) \dots \mathcal{O}(\lambda_n) \rangle \sim \prod_{i=1, \dots, n} \langle \mathcal{O}(\lambda_i) \rangle,$$

and thus no new information can be obtained from the complete  $n$ -point function. The connected function which will be denoted

$$\langle \mathcal{O}(\lambda_1) \dots \mathcal{O}(\lambda_n) \rangle_{\text{conn}} = \mathcal{O}_n(\lambda_1, \dots, \lambda_n)$$

is only of order  $1/N^n$ , and thus is a subleading contribution. The method of orthogonal polynomials allows to determine directly all these connected functions from only one auxiliary function  $\kappa(\lambda, \mu) = \kappa(\mu, \lambda)$  (see ref. [1] or appendix 2 for details). For instance we have:

$$\begin{aligned} \mathcal{O}_1(\lambda) &= \rho(\lambda) = \kappa(\lambda, \lambda), \\ \mathcal{O}_2(\lambda, \mu) &= -\kappa^2(\lambda, \mu), \\ \mathcal{O}_3(\lambda_1, \lambda_2, \lambda_3) &= 2\kappa(\lambda_1, \lambda_2)\kappa(\lambda_2, \lambda_3)\kappa(\lambda_3, \lambda_1), \\ \mathcal{O}_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= -2\kappa(\lambda_1, \lambda_2)\kappa(\lambda_2, \lambda_3)\kappa(\lambda_3, \lambda_4)\kappa(\lambda_4, \lambda_1) \\ &\quad - 2\kappa(\lambda_1, \lambda_3)\kappa(\lambda_3, \lambda_2)\kappa(\lambda_2, \lambda_4)\kappa(\lambda_4, \lambda_1) \\ &\quad - 2\kappa(\lambda_1, \lambda_2)\kappa(\lambda_2, \lambda_4)\kappa(\lambda_4, \lambda_3)\kappa(\lambda_3, \lambda_1), \end{aligned}$$

and analogous expressions for larger values of  $n$ :

$$\mathcal{O}_n = (-1)^{n+1} \frac{1}{n} \sum_{\text{permutations } \sigma} \prod_{i=1}^n \kappa(\lambda_{\sigma_i}, \lambda_{\sigma_{i+1}}),$$

(each product of  $\kappa$  appears exactly  $n$  times in the sum and this counting factor is cancelled by the  $1/n$  factor).  $\mathcal{O}_n$  may also be written:

$$\mathcal{O}_n = (-1)^{n+1} \sum_{\text{cyclic permutations } \sigma} \prod_{i=1}^n \kappa(\lambda_i, \lambda_{\sigma_i})$$

The function  $\kappa$  is simply related to the polynomials  $P_n$  orthogonal with respect to the measure  $d\lambda e^{-NV(\lambda)}$ :

$$\kappa(\lambda, \mu) \propto \frac{1}{N} \frac{P_N(\lambda)P_{N-1}(\mu) - P_{N-1}(\lambda)P_N(\mu)}{\lambda - \mu} \exp[-(N/2)(V(\lambda) + V(\mu))]. \quad (1.3)$$

The asymptotic evaluation of correlation functions is reduced to an evaluation of the function  $\kappa(\lambda, \mu)$  and thus of the orthogonal polynomials  $P_n(\lambda)$  in the peculiar limit:  $N$  large,  $N - n$  finite, and  $\lambda \in [a, b]$  ( $[a, b]$  being the support of  $\rho(\lambda)$ ).

The ansatz proposed in ref. [1] in the case of even potentials  $V(M)$  (for which  $b = -a$ ) was:

$$P_n(\lambda) \propto e^{NV(\lambda)/2} \frac{1}{\sqrt{f(\lambda)}} \cos(N\zeta(\lambda) + (N-n)\varphi(\lambda) + \chi(\lambda)),$$

where

$$\lambda = a \cos \varphi,$$

$$f(\lambda) = a \sin \varphi,$$

$$\frac{d}{d\lambda} \zeta(\lambda) = -\pi \rho(\lambda),$$

$\chi(\lambda)$  remaining undetermined, except in the gaussian and quartic cases, for which:  $\chi(\lambda) = \varphi/2 - \pi/4$ .

We will prove below that this ansatz is still true for any  $V$ , and that  $\chi$  is always of the form  $\chi = \varphi/2 + \text{const}$ . The method which leads to the proof is also interesting in itself because it uses some general tools of the saddle point calculations of the one-matrix model [5,6].

## 2. Orthogonal Polynomials

Let us consider the set of polynomials  $P_n$  ( $n$  is the degree), orthogonal with respect to the following scalar product:

$$\langle P_n \cdot P_m \rangle = \delta_{nm} = \int d\lambda e^{-NV(\lambda)} P_n(\lambda) P_m(\lambda). \quad (2.1)$$

Remarkably enough an explicit expression of these orthogonal polynomials in terms of a hermitian matrix integral can be derived (see appendix 1):

$$P_n(\lambda) = Z_{n+1}^{-1/2} \int d^{n^2} M e^{-N \text{tr} V(M)} \det(\lambda - M), \quad (2.2)$$

where  $M$  is here a  $n \times n$  hermitian matrix, and the normalization  $Z_{n+1}$  the partition function corresponding to the  $n+1 \times n+1$  matrix integral. We will use this expression to evaluate  $P_n$  in the relevant limit, i.e  $N \gg 1$  and  $N-n = O(1)$  by the steepest descent method.

Let us first introduce the matrix integral:

$$Z(g, h, \lambda) = \int d^{n^2} M e^{-(n/g) \text{tr} \mathcal{V}(M)},$$

where

$$\mathcal{V}(z) = V(z) - h \ln(\lambda - z).$$

With these definitions  $P_n$  is proportional to  $Z(g=n/N, h=1/N, \lambda)$ , and thus we need  $Z$  or equivalently the free energy  $F_n = -n^{-2} \ln Z$  for  $h$  and  $g-1$  small.

As we are interested only in the  $\lambda$  dependance of  $F$ , let us differentiate  $F$  with respect to  $\lambda$ :

$$\frac{\partial F}{\partial \lambda} = -\frac{h}{g} \omega(z = \lambda, g, h, \lambda)$$

where  $\omega(z)$  is the resolvent:

$$\omega(z) = \frac{1}{n} \left\langle \text{tr} \frac{1}{z - M} \right\rangle. \quad (2.3)$$

Since we want the asymptotic expression of  $\omega(\lambda)n^2h/g$ , we need  $\omega$  up to the order  $1/n$ . It is known from the random matrix theory [7], that  $F_n$ , and also  $\omega$  have an expansion in powers of  $1/n^2$  which is the topological expansion. At order  $1/n$ , only the contribution of the sphere is required. We thus replace  $\omega$  by its dominant contribution obtained by the saddle point method.

With this approximation,  $\omega$  may be written:

$$\omega(z) = \frac{1}{n} \sum_{i=1, \dots, n} \frac{1}{z - \lambda_i}$$

where the  $\lambda_i$  are the eigenvalues verifying the saddle point equation:

$$\frac{1}{g} \mathcal{V}'(\lambda_i) = \frac{2}{n} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j},$$

i.e. they extremize the integrand of

$$Z(g, h, \lambda) \propto \int \prod_{i=1}^n e^{-N\mathcal{V}(\lambda_i)} d\lambda_i \prod_{i < j} (\lambda_i - \lambda_j)^2.$$

In the large  $n$  limit the  $\lambda_i$  are distributed along an interval  $[a, b]$  with a continuous density  $\rho(\lambda)$ . Then:

$$\omega(z) = \int_a^b d\mu \rho(\mu) \frac{1}{z - \mu} \quad (2.4)$$

and the saddle point equation becomes:

$$\frac{1}{g} \mathcal{V}'(\mu) = \omega(\mu + i0) + \omega(\mu - i0) \quad \text{for } \mu \in [a, b] \quad (2.5)$$

Note an important property of this equation: At  $a$  and  $b$  fixed it is linear and therefore the derivatives of  $\omega$  with respect to  $g$  or  $h$  will also satisfy a linear equation.

At leading order we introduce the resolvent  $\omega_0(z) = \omega(z, g = 1, h = 0)$ , and write:

$$g\omega(z, g, h, \lambda) = \omega_0(z) + (g - 1)\Omega_g(z, \lambda) + h\Omega_h(z, \lambda) + O(1/n^2),$$

where we have defined the two functions:

$$\Omega_g = \left. \frac{\partial g\omega}{\partial g} \right|_{g=0, h=1}, \quad \Omega_h = \left. \frac{\partial g\omega}{\partial h} \right|_{g=0, h=1}.$$

The function  $\omega_0(z)$  is the resolvent of the usual one-matrix model, and from eqs. (2.4) and (2.5) we obtain:

$$\omega_0(z \pm i0) = \frac{1}{2} V'(z) \mp i\pi\rho(z) \quad \text{for } z \in [a, b].$$

As we noted above, the two functions  $\Omega_g$  and  $\Omega_h$  obey linear equations, obtained by differentiation of the equation (2.5) satisfied by  $\omega$ . Following a method introduced in ref. [8] we can then easily determine them from their analyticity properties, and the boundary conditions.  $\Omega_g(z)$  verifies the linear equation:

$$\Omega_g(z + i0) + \Omega_g(z - i0) = 0$$

and behaves as  $1/z$  when  $z \rightarrow \infty$ , and as  $1/\sqrt{z-a}\sqrt{z-b}$  near the cut end-points  $a$  and  $b$ , because  $\omega$  behaves as  $\sqrt{z-a}\sqrt{z-b}$ . These conditions determine  $\Omega_g$  uniquely:

$$\Omega_g(z, \lambda) = \frac{1}{\sqrt{(z-a)(z-b)}}.$$

The same method applies to  $\Omega_h$  which satisfies

$$\Omega_h(z + i0) + \Omega_h(z - i0) = 1/(\lambda - z),$$

and behaves like  $\Omega_h \sim O(1/z^2)$  for  $z$  large,  $\Omega_h \sim 1/\sqrt{z-a}\sqrt{z-b}$  near  $a, b$ , and is regular near  $z = \lambda$ . It follows:

$$\Omega_h(z, \lambda) = \frac{1}{2} \frac{1}{\sqrt{(z-a)(z-b)}} \left( 1 - \frac{\sqrt{(z-a)(z-b)} - \sqrt{(\lambda-a)(\lambda-b)}}{z-\lambda} \right).$$

Then if we set  $z = \lambda$ :

$$\Omega_h(z = \lambda) = \frac{1}{2\sqrt{(\lambda-a)(\lambda-b)}} - \frac{1}{2} \frac{d}{d\lambda} \ln \sqrt{(\lambda-a)(\lambda-b)}.$$

We now have the necessary ingredients to determine  $\partial F/\partial\lambda$ :

$$-n^2 \frac{\partial F}{\partial \lambda} = N\omega_0(z = \lambda) + (n-N)\Omega_g(z = \lambda) + \Omega_h(z = \lambda) + O(1/N).$$

We still have to integrate all these terms with respect to  $\lambda$ . In order to integrate  $\omega_0 = V'/2 - i\pi\rho$ , we introduce a function  $\zeta(\lambda)$

$$\zeta(\lambda) = -\pi \int_a^\lambda d\lambda' \rho(\lambda').$$

We also need the integral of  $[(\lambda-a)(b-\lambda)]^{-1/2}$ . For this purpose, we parametrize  $\lambda = \frac{1}{2}(a+b) - \frac{1}{2}(b-a) \cos \varphi$ , so that the integral of  $[\frac{1}{2}(b-a) \sin \varphi]^{-1}$  is simply  $\varphi$ . Finally, the result takes the form:

$$\begin{aligned} -n^2 F(\lambda \pm i0) &= \frac{1}{2} NV(\lambda) \pm iN\zeta(\lambda) \mp i(N-n)\varphi \pm i\frac{1}{2}\varphi \\ &\quad - \frac{1}{2} \ln \sqrt{(\lambda-a)(b-\lambda)} + \text{const.} \end{aligned}$$

Since  $P_n$  is a polynomial, we have  $P_n(\lambda) = \frac{1}{2} [P_n(\lambda + i0) + P_n(\lambda - i0)]$ , and therefore:

$$P_n(\lambda) = \sqrt{\frac{2}{\pi}} e^{NV(\lambda)/2} \frac{1}{\sqrt{f(\lambda)}} \cos [N\zeta(\lambda) - (N-n)\varphi(\lambda) + \frac{1}{2}\varphi(\lambda) + \text{const}], \quad (2.6)$$

where  $f(\lambda) = \sqrt{(\lambda - a)(b - \lambda)}$ . The constant factor  $\sqrt{2/\pi}$  is fixed by the condition that  $\kappa(\lambda, \lambda) = \rho(\lambda)$ . In the case of even potentials  $V$ , parity considerations imply that the arbitrary constant phase is  $\text{const} = -\pi/4$ . Indeed, we have  $\zeta(a) = 0$ ,  $\zeta(b) = -\pi$ ,  $\varphi(a) = 0$ ,  $\varphi(b) = \pi$ , (we have  $f(a) = f(b) = 0$ , but for  $\epsilon$  a small positive number :  $f(a+\epsilon)/f(b-\epsilon) = 1$ ) and thus:

$$P_n(b - \epsilon)/P_n(a + \epsilon) \sim (-1)^{n+1} \tan(\text{const}).$$

For general potentials the constant phase remains undetermined at this order, but we note that the general form of  $P_n$  does not depend on the parity properties of  $V$ .

### 3. Connected Correlation Functions

From this asymptotic expansion of  $P_n$  one can now derive the kernel  $\kappa(\lambda, \mu)$ , and then the connected correlation functions, in the large  $N$  limit. The authors of ref. [1] have calculated some correlation functions in two regimes: short range correlations  $(\lambda_i - \lambda_j) \sim 1/N$ , and mesoscopic correlations  $(\lambda_i - \lambda_j) \gg 1/N$ . Note that the polynomials  $P_N$  and  $P_{N-1}$  oscillate at a frequency of order  $N$  (which corresponds to the discrete spectrum of a matrix of size  $N$  finite), and therefore, all the correlation functions will present such oscillations.

In the short distance regime, these oscillations give the dominant behaviour, and eq. (2.6) leads to:

$$\kappa(\lambda, \mu) \sim \frac{\sin[2\pi N(\lambda - \mu)\rho(\lambda)]}{2\pi N(\lambda - \mu)}$$

and all connected correlation functions follow. Note that  $\kappa$  being of order  $1/N$ , the connected microscopic  $n$ -point function will be of order  $1/N^n$ . All this is studied in detail in ref. [1], and we will now concentrate our attention on the mesoscopic case.

In the regime  $\lambda_i - \lambda_j \gg 1/N$ , it is interesting to consider smoothed functions, defined by averaging the fast oscillations. For instance we find that:

$$[\mathcal{O}_2(\lambda, \mu)]_{\text{smooth}} = \frac{-1}{2N^2\pi^2} \frac{1}{(\lambda - \mu)^2} \frac{1 - \cos \varphi \cos \psi}{\sin \varphi \sin \psi}, \quad (3.1)$$

where

$$\lambda = \frac{a+b}{2} - \frac{b-a}{2} \cos \varphi, \quad \mu = \frac{a+b}{2} - \frac{b-a}{2} \cos \psi. \quad (3.2)$$

Brézin and Zee noted (ref. [1]) that the smoothed higher order  $n$ -points correlation functions vanished identically at the order  $1/N^n$  for  $n > 2$ . Indeed, we will prove below that they are of order  $1/N^{2(n-1)}$ , and give a recursive method to compute them with the help of loop-correlators.

#### 4. The $n$ -Loop Correlation Functions

Let us introduce the functions:

$$\omega_n(z_1, \dots, z_n) = N^{n-2} \left\langle \text{tr} \frac{1}{z_1 - M} \times \dots \times \text{tr} \frac{1}{z_n - M} \right\rangle_{\text{conn}} \quad (4.1)$$

They are related to the previous correlation functions by the relations:

$$\omega_n(z_1, \dots, z_n) = N^{2n-2} \int \prod_{i=1}^n \frac{d\lambda_i}{z_i - \lambda_i} \mathcal{O}_n(\lambda_1, \dots, \lambda_n) \quad (4.2)$$

$\omega_n$  is called the  $n$ -loop correlation function, because it is the Laplace-transform of the partition function of a discrete random surface limited by  $n$  loops (the  $z_i$  are conjugated to the lengths of the loops). This remark allows to understand the topological origin of the factor  $N^{2n-2}$ : indeed,

the Laplace-transform of the complete  $n$ -point correlation function which is of order 1, would be the partition function of every surface (not necessary connected) with  $n$  boundaries. Each surface contributes with a topological weight  $N^\chi$  where  $\chi$  is its Euler character. The leading term is the most disconnected one, with  $\chi = n$  (indeed, such a surface is made of  $n$  discs, each of them having  $\chi = 1$ ), while the connected term has  $\chi = 2 - n$  (it is a sphere ( $\chi = 2$ ) from which  $n$  discs have been removed). Therefore, the relative contribution of the connected part to the complete  $n$ -loop function is  $N^{2-2n}$ .

Remark that relation (4.2) can be inverted:  $\omega_n$  is analytical except when some of the  $z_i$  belong to the interval  $[a, b]$ .  $\mathcal{O}_n$  can then be expressed in terms of the differences of  $\omega_n$  between opposite sides of the cut. For instance:

$$\begin{aligned} \rho(\lambda) &= \frac{-1}{2i\pi} (\omega(\lambda + i0) - \omega(\lambda - i0)) \\ \mathcal{O}_2(\lambda, \mu) &= \frac{1}{(2i\pi N)^2} (\omega_2(\lambda + i0, \mu + i0) - \omega_2(\lambda + i0, \mu - i0) - \omega_2(\lambda - i0, \mu + i0) \\ &\quad + \omega_2(\lambda - i0, \mu - i0)). \end{aligned}$$

and for general  $n$ :

$$\mathcal{O}_n(\lambda_1, \dots, \lambda_n) = \frac{1}{N^{2n-2}} \left( \frac{-1}{2i\pi} \right)^n \sum_{\epsilon_i=\pm 1} (-1)^{(\epsilon_1+\dots+\epsilon_n)} \omega_n(\lambda_1 + \epsilon_1 i0, \dots, \lambda_n + \epsilon_n i0) \quad (4.3)$$

Note that these functions are directly the smoothed correlation functions, since we first compute  $\omega_n$  in the large  $N$  limit at complex arguments (which suppresses the oscillations), and then take the discontinuities along the cut.

In order to compute  $\omega_n$ , we consider the partition function:

$$Z = e^{N^2 F} = \int dM e^{-N \text{tr} V(M)}$$

with  $V(z) = \sum_{k=1}^{\infty} g_k z^k / k$ , and define the loop insertion operator:

$$\frac{\delta}{\delta V(z)} = - \sum_{k=1}^{\infty} \frac{k}{z^{k+1}} \frac{\partial}{\partial g_k}. \quad (4.4)$$

Note that with this definition

$$\frac{\delta V(z')}{\delta V(z)} = \frac{1}{z - z'} - \frac{1}{z}. \quad (4.5)$$

The  $\omega_n$  are obtained from the free energy  $F$  by the repeated action of this operator:

$$\omega_n(z_1, \dots, z_n) = \frac{\delta}{\delta V(z_1)} \cdots \frac{\delta}{\delta V(z_n)} F, \quad (4.6a)$$

$$= \frac{\delta}{\delta V(z_n)} \omega_{n-1}(z_1, \dots, z_{n-1}). \quad (4.6b)$$

It is not necessary to calculate the free energy  $F$ , since we already know the one-loop function  $\omega(z)$ . We have already emphasized that  $\omega(z)$  satisfies a linear equation (eq. (2.5)), and thus, all its derivatives satisfy the same linear equation, with a different l.h.s.. Then, analyticity properties, and boundary conditions determine the form of  $\omega_n$ .

The linear equation for  $\omega$  is:

$$\omega(\lambda + i0) + \omega(\lambda - i0) = V'(\lambda). \quad (4.7)$$

By a repeated action of the loop insertion operator, we obtain:

$$\omega_2(\lambda + i0, z) + \omega_2(\lambda - i0, z) = -\frac{1}{(z - \lambda)^2}, \quad (4.8)$$

and for  $n > 2$ :

$$\omega_n(\lambda + i0, z_2, \dots, z_n) + \omega_n(\lambda - i0, z_2, \dots, z_n) = 0. \quad (4.9)$$

The function  $\omega(z)$  has the form

$$\omega(z) = \frac{1}{2} \left( V'(z) - M(z) \sqrt{(z - a)(z - b)} \right)$$

where  $M(z)$  is a polynomial such that  $\omega(z) \sim 1/z$  when  $z \rightarrow \infty$ , and therefore:

$$M(z) = \left( \frac{V'(z)}{\sqrt{(z-a)(z-b)}} \right)_+.$$

Then

$$\rho(\lambda) = \frac{1}{2\pi} M(\lambda) \sqrt{(\lambda-a)(b-\lambda)} = \frac{1}{2\pi} M(\lambda) \frac{b-a}{2} \sin \varphi,$$

where we have used the parametrization (3.2).

The two-loop function is also completely determined by the linear equation (4.8) and boundary conditions:

$$\omega_2(x, y) = -\frac{1}{4} \frac{1}{(x-y)^2} \left( 2 + \frac{(x-y)^2 - (x-a)(x-b) - (y-a)(y-b)}{\sqrt{(x-a)(x-b)} \sqrt{(y-a)(y-b)}} \right)$$

and thus, in agreement with eq. (3.1):

$$\mathcal{O}_2(\lambda, \mu) = -\frac{1}{2N^2 \pi^2} \frac{1}{(\lambda-\mu)^2} \frac{1 - \cos \varphi \cos \psi}{\sin \varphi \sin \psi}.$$

The other loop functions all satisfy an homogeneous equation, and can be written:

$$\omega_n(\lambda_1, \dots, \lambda_n) = \left( \prod_{i=1}^n \frac{1}{\sin \varphi_i} \right)^{2n-3} W_n(\lambda_1, \dots, \lambda_n),$$

with now  $\lambda_n = \frac{a+b}{2} - \frac{b-a}{2} \cos \varphi_n$ , and where the  $W_n$  are some symmetric polynomials of degree less than  $2n-5$  in each  $\lambda_i$ , which are no longer determined by the boundary conditions. It is necessary to directly use the recursion relation (4.6b). Since  $\omega_2$  depends on the potential  $V(M)$  only through  $a$  and  $b$ , we need the actions of loop-insertion operator on  $a$  and  $b$ , for instance  $\delta a / \delta V(z)$ ,  $\delta^2 a / \delta V(z) \delta V(z')$ .... For this purpose, we introduce the following moments of the potential:

$$M_k = -\frac{1}{2i\pi} \oint dz \frac{1}{(z-a)^k} \frac{V'(z)}{\sqrt{(z-a)(z-b)}}$$

$$J_k = -\frac{1}{2i\pi} \oint dz \frac{1}{(z-b)^k} \frac{V'(z)}{\sqrt{(z-a)(z-b)}}$$

(the integration path turns clockwise around the cut  $[a, b]$ ).

They are such that:

$$M(z) = \sum_k M_{k+1} (z-a)^k = \sum_k J_{k+1} (z-b)^k$$

The  $M_k$  and  $J_k$  are linearly related to the coefficients of the potential  $V$ , and if  $V$  is a polynomial of degree  $v$ ,  $M$  is of degree  $v-2$ , and there are only  $v-1$  independent coefficient among the  $M_k$  and  $J_k$ . Note also, that if  $V$  is even, we have  $M_k = -(-1)^k J_k$ .

The cut end-points  $a$  and  $b$  depend on  $V$  through the conditions that (see ref. [7]):

$$\begin{aligned} \frac{1}{2i\pi} \oint dz \frac{V'(z)}{\sqrt{(z-a)(z-b)}} &= 0, \\ \frac{1}{2i\pi} \oint dz \frac{zV'(z)}{\sqrt{(z-a)(z-b)}} &= 2. \end{aligned}$$

It follows (using (4.5) and performing the contour integrals):

$$\begin{aligned} \frac{\delta a}{\delta V(z)} &= \frac{1}{M_1} \frac{1}{z-a} \frac{1}{\sqrt{(z-a)(z-b)}}, \\ \frac{\delta b}{\delta V(z)} &= \frac{1}{J_1} \frac{1}{z-b} \frac{1}{\sqrt{(z-a)(z-b)}}. \end{aligned}$$

In order to determine the higher order derivatives of  $a$  and  $b$ , we need to differentiate the coefficients  $M_k$  and  $J_k$ :

$$\begin{aligned} \frac{\delta M_k}{\delta V(z)} &= \frac{2k+1}{2} \frac{\delta a}{\delta V(z)} \left( M_{k+1} - \frac{M_1}{(z-a)^k} \right) \\ &\quad + \frac{1}{2} \frac{\delta b}{\delta V(z)} \left( \frac{J_1}{(b-a)^k} - \frac{J_1}{(z-a)^k} - \sum_{l=0}^{k-1} \frac{M_{l+1}}{(b-a)^{k-l}} \right) \end{aligned}$$

and analogous formulae are obtained for the  $J_k$  by the exchange  $a \leftrightarrow b$ .

With these tools, we can now determine the  $\omega_n$  recursively. Let us for instance write  $\omega_3$ :

$$\begin{aligned} \omega_3(x, y, z) &= \frac{a-b}{8 \left( \sqrt{(x-a)(x-b)} \sqrt{(y-a)(y-b)} \sqrt{(z-a)(z-b)} \right)^3} \\ &\quad \times \left( \frac{1}{M_1} (x-b)(y-b)(z-b) - \frac{1}{J_1} (x-a)(y-a)(z-a) \right) \end{aligned}$$

For  $\omega_4$  we first define the polynomials

$$Q(x_i, a) = \prod_{i=1,4} (x_i - a).$$

With this notation

$$\begin{aligned}\omega_4(x_i) = & \frac{-1}{16 [Q(x_i, a)Q(x_i, b)]^{3/2}} \left( -3(b-a)\frac{M_2}{M_1^3}Q(x_i, b) - 3(a-b)\frac{J_2}{J_1^3}Q(x_i, a) \right. \\ & + 3\frac{1}{M_1^2}Q(x_i, b) \left[ (b-a) \left( \sum_{i=1}^4 \frac{1}{x_i - a} \right) - 1 \right] \\ & + 3\frac{1}{J_1^2}Q(x_i, a) \left[ (a-b) \left( \sum_{i=1}^4 \frac{1}{x_i - b} \right) - 1 \right] \\ & \left. + \frac{1}{M_1 J_1} [(x_1 - a)(x_2 - a)(x_3 - b)(x_4 - b) + 5 \text{ terms}] \right),\end{aligned}$$

where the last additional terms symmetrize in the four variables. The connected functions  $\mathcal{O}_n$ , are then obtained by (4.3) and (4.9), and they are simply given by:

$$\mathcal{O}_n(\lambda_1, \dots, \lambda_n) = \frac{1}{N^{2n-2}} \left( \frac{-1}{i\pi} \right)^n \omega_n(\lambda_1 + i0, \dots, \lambda_n + i0).$$

*Universality.* We observe that the only universal features of  $\mathcal{O}_1(\lambda) = \rho(\lambda)$  are the square-root singularities at the edge of the distribution, otherwise the function is potential dependent. Instead the two-point correlation function has a universal form, and can thus be calculated from the gaussian model.

The  $n$ -point smoothed correlation functions with  $n \geq 3$  can be calculated recursively by a systematic method described above. The main property is that they consist in a sum of a finite number of universal functions and involve only the first  $n - 2$  moments of the potential. That means for instance, that two potentials  $V$  and  $V^*$  induce the same three-point function as soon as they yield the same  $M_1$  and  $J_1$  coefficients, but they don't need to be identical. However, the  $n$ -point correlation function is no longer given by a gaussian model, since it is of order  $1/N^{2n-2}$ . Perturbative corrections to the gaussian model have to be considered.

Note that the determination of correlation functions allows an evaluation of the moments  $M_k$ , in a case where the potential  $V$  is unknown. Let us emphasize that these moments play an important role in the study of critical points. Since

$$\rho(\lambda) = \frac{1}{2\pi} M(\lambda) \sqrt{(\lambda - a)(b - \lambda)}$$

we see that if some moments vanish, the behaviour of  $\rho$  near the end-points is no longer a square-root. When the  $m$  first  $M_k$  vanish, one finds  $\rho \sim (\lambda - a)^{m+1/2}$  which corresponds to the  $m^{\text{th}}$  multicritical point of the one-matrix model of 2D gravity. [6,7].

## 5. Conclusions

In this article we have recovered, by a completely different method, the results of ref. [1]concerning the two-point eigenvalue correlation function of a random hermitian matrix in the limit in which the size  $N$  of the matrix becomes large. Our method has allowed us to generalize the results to new distributions. In addition, we have established a recursion relation between successive  $n$ -point correlation functions at leading order for  $N$  large.

Brézin and Zee in ref. [1]have shown that the two-point function is universal, and therefore identical to the function of the gaussian matrix model. The gaussian model is the fixed point of a renormalization group [2,3] and a direct RG analysis should be performed to put this result in perspective.

In the same way, Brézin and Zee have shown that higher correlation functions vanish at leading order. The contributions we have calculated here should be considered as corrections to the leading scaling behaviour. The explicit expressions we have obtained show that they depend now on successive moments of the potential, indicating an implicit classification of the deviations from the gaussian model in terms of their irrelevance for  $N$  large. Here also it would be interesting to confirm the qualitative aspects of these results by a direct RG analysis.

## Appendix 1. Orthogonal polynomials $P_n$ : An explicit expression

Let us show that the orthogonal polynomials  $P_n$  defined by the orthogonality condition:

$$\langle P_n \cdot P_m \rangle = \delta_{nm} = \int d\lambda e^{-NV(\lambda)} P_n(\lambda) P_m(\lambda).$$

are given, up to a normalization, by equation (2.2):

$$P_n(\lambda) \propto \int d^{n^2} M e^{-N \text{tr} V(M)} \det(\lambda - M).$$

First, this integral clearly yields a polynomial of degree  $n$  in  $\lambda$ . Let us then verify the orthogonality property: after integration over the unitary group, the integration measure  $\int dM$  reduces to an integration over the eigenvalues of  $M$ , and the Jacobian of this transformation is a square Vandermonde determinant:

$$P_n(\lambda) \propto \int \prod_{i=1,\dots,n} d\lambda_i e^{-NV(\lambda_i)} (\lambda - \lambda_i) \Delta^2(\lambda_1, \dots, \lambda_n),$$

where

$$\Delta(\lambda_1, \dots, \lambda_n) = \prod_{i < j} (\lambda_i - \lambda_j).$$

In this form we recognize a more classical expression [9]. Then, setting  $\lambda = \lambda_0$ :

$$\langle P_n(\lambda_0) \cdot \lambda_0^m \rangle \propto \int \prod_{i=0, \dots, n} d\lambda_i e^{-NV(\lambda_i)} \Delta(\lambda_0, \lambda_1, \dots, \lambda_n) \Delta(\lambda_1, \dots, \lambda_n) \lambda_0^m.$$

The first Vandermonde is completely antisymmetric in the  $n+1$  variables, we can therefore antisymmetrize the factor  $\Delta(\lambda_1, \dots, \lambda_n) \lambda_0^m$ , the result is zero if  $m < n$  because the only polynomial completely antisymmetric and of degree less than  $n-1$  in  $\lambda_0$  is zero. Thus

$$\langle P_n \cdot \lambda^m \rangle = 0.$$

it proves that  $P_n$  is orthogonal to any  $P_m$  with  $m < n$ , and that  $\langle P_n \cdot P_m \rangle = 0$  as soon as  $m \neq n$ .

Remark that a similar integral representation can also be found for multi-matrix models. Consider two families of orthogonal polynomials  $P_n$  and  $Q_n$ , such that:

$$\langle P_n(\lambda) \cdot Q_m(\mu) \rangle = \int \int d\lambda d\mu e^{-N(V(\lambda) + U(\mu) - c\lambda\mu)} P_n(\lambda) Q_m(\mu) \propto \delta_{nm}.$$

Then, we have:

$$P_n(\lambda) = \int \int dM_1 dM_2 e^{-N\text{Tr}[V(M_1) + U(M_2) - cM_1 M_2]} \det(\lambda - M_1)$$

where  $M_1$  and  $M_2$  are hermitian  $n \times n$ . Similarly:

$$Q_m(\mu) = \int \int dM_1 dM_2 e^{-N\text{Tr}[V(M_1) + U(M_2) - cM_1 M_2]} \det(\mu - M_2).$$

## Appendix 2. Connected correlation functions and the kernel $\kappa(\lambda, \mu)$

We again consider the matrix distribution (1.1):

$$\mathcal{P}(M) = \frac{1}{Z} e^{-N\text{Tr} V(M)}.$$

The corresponding measure can be rewritten in terms of the eigenvalues  $\lambda_i$  of  $M$  and a unitary transformation  $U$  which diagonalizes  $M$ :

$$\mathcal{P}(M) dM = Z^{-1} dU \prod_{i=1 \dots N} d\lambda_i e^{-NV(\lambda_i)} \Delta^2(\lambda_1, \dots, \lambda_N)$$

( $\Delta = \prod_{i < j} (\lambda_i - \lambda_j)$  being the Vandermonde determinant). Therefore, the probability that the eigenvalues of  $M$  are  $\lambda_1, \dots, \lambda_N$  is:

$$\rho_N(\lambda_1, \dots, \lambda_N) \propto \prod_{i=1 \dots N} e^{-NV(\lambda_i)} \Delta^2(\lambda_1, \dots, \lambda_N),$$

and the correlation functions are obtained by partially integrating over some eigenvalues:

$$\begin{aligned} \rho_1(\lambda_1) &= \int \prod_{i=2, \dots, N} d\lambda_i \rho_N(\lambda_1, \dots, \lambda_N), \\ \rho_2(\lambda_1, \lambda_2) &= \int \prod_{i=3, \dots, N} d\lambda_i \rho_N(\lambda_1, \dots, \lambda_N) \end{aligned}$$

... and so on.

The Vandermonde determinant  $\Delta$  can be written as:

$$\Delta(\lambda_i) = \det \lambda_i^{j-1},$$

and thus, after some linear combinations of columns of the matrix:

$$\Delta = \det \Pi_{j-1}(\lambda_i), \quad \Pi_n(\lambda) = \lambda^n + O(\lambda^{n-1}),$$

identity true for any set of polynomials  $\{\Pi_n\}$  normalized as above. In order to perform the  $\lambda$  integrations, we choose  $\Pi_n \propto P_n$ ,  $P_n$  being the orthogonal polynomials (2.1).  $\Delta^2$  is the product of two such determinants, therefore it is the determinant of a matrix product:

$$\Delta^2 \propto \det K(\lambda_i, \lambda_j) \tag{2.1a}$$

$$K(\lambda_i, \lambda_j) = \sum_{k=0}^{N-1} P_k(\lambda_i) P_k(\lambda_j). \tag{2.1b}$$

The proportionality constant in (2.1a) is here irrelevant because the eigenvalue distribution is normalized. The Darboux–Christoffel formula (appendix 3) tells that:

$$K(\lambda, \mu) = \alpha \frac{P_N(\lambda)P_{N-1}(\mu) - P_N(\mu)P_{N-1}(\lambda)}{\lambda - \mu}$$

where  $\alpha$  is a normalization constant depending on  $N$ , and  $\alpha = (a - b)/4$  when  $N \rightarrow \infty$ . The important properties of  $K(\lambda, \mu)$  are:

$$\begin{aligned} \int d\nu e^{-NV(\nu)} K(\lambda, \nu) &= 1, \quad \int d\nu e^{-NV(\nu)} K(\nu, \nu) = N, \\ \int d\nu e^{-NV(\nu)} K(\lambda, \nu) K(\nu, \mu) &= K(\lambda, \mu). \end{aligned}$$

An explicit expression for  $\rho_n$

$$\rho_n(\lambda_1, \dots, \lambda_n) = \frac{1}{N!} \int \prod_{i=n+1, \dots, N} d\lambda_i \prod_{i=1, \dots, N} e^{-NV(\lambda_i)} \det K(\lambda_i, \lambda_j),$$

can be obtained by successively integrating over eigenvalues. Using the rules of  $K$  integration it is easy to prove by induction

$$\rho_n(\lambda_1, \lambda_2, \dots, \lambda_n) = \frac{N^n(N-n)!}{N!} \det \kappa(\lambda_i, \lambda_j),$$

where we have introduced the reduced function

$$\kappa(\lambda, \mu) = \frac{1}{N} e^{-(N/2)[V(\lambda) + V(\mu)]} K(\lambda, \mu).$$

Therefore, when the  $\lambda_i$  are all distinct we have:

$$\langle \mathcal{O}(\lambda_1) \dots \mathcal{O}(\lambda_n) \rangle = \det \kappa(\lambda_i, \lambda_j) = \sum_{\sigma} (-1)^{\sigma} \prod_i \kappa(\lambda_i, \lambda_{\sigma_i})$$

The connected function will involve only the sum over the permutations  $\sigma$  such that  $\prod_i \kappa(\lambda_i, \lambda_{\sigma_i})$  cannot be split in the product of two cyclic products, i.e. only cyclic permutations will contribute to the connected function. This intuitive result is a classical combinatorial identity. The connected function can thus be written:

$$\begin{aligned} \langle \mathcal{O}(\lambda_1) \dots \mathcal{O}(\lambda_n) \rangle_{\text{conn}} &= (-1)^{n+1} \frac{1}{n} \sum_{\text{permutations } \sigma} \prod_i \kappa(\lambda_{\sigma_i}, \lambda_{\sigma_{i+1}}), \\ &= (-1)^{n+1} [\kappa(\lambda_1, \lambda_2) \kappa(\lambda_2, \lambda_3) \dots \kappa(\lambda_n, \lambda_1) + \dots], \end{aligned}$$

where the additional terms in the r.h.s. symmetrize the expression. Or in a more compact way:

$$\langle \mathcal{O}(\lambda_1) \dots \mathcal{O}(\lambda_n) \rangle_{\text{conn}} = (-1)^{n+1} \sum_{\text{cyclic permutations } \sigma} \prod_{i=1}^n \kappa(\lambda_i, \lambda_{\sigma_i})$$

### Appendix 3. Derivation of the Darboux–Christoffel formula

The polynomial  $\lambda P_n(\lambda)$  can be expanded on the basis of the  $P_m$  with  $m \leq n+1$ :

$$\lambda P_n(\lambda) = \sum_{m=n-1}^{n+1} Q_{nm} P_m(\lambda).$$

The orthogonality condition (2.1) implies that the matrix  $Q$  is symmetric:

$$Q_{nm} = \langle \lambda P_n \cdot P_m \rangle = \langle P_n \cdot \lambda P_m \rangle = Q_{mn}.$$

The polynomial  $(\lambda - \mu)K(\lambda, \mu)$  can thus be written:

$$(\lambda - \mu)K(\lambda, \mu) = \sum_{k=0}^{N-1} \left( \sum_{i=0}^N Q_{ki} P_i(\lambda) P_k(\mu) - \sum_{j=0}^N Q_{kj} P_k(\lambda) P_j(\mu) \right).$$

All the terms cancel, except the upper-bounds:

$$(\lambda - \mu)K(\lambda, \mu) = Q_{N,N-1} (P_N(\lambda)P_{N-1}(\mu) - P_N(\mu)P_{N-1}(\lambda)),$$

therefore  $\alpha = Q_{N,N-1}$ . In the large  $N$  limit, it is possible to calculate  $\alpha$ . The simplest way of doing this is to calculate  $\lambda P_{N-1}$  from expression (2.6). Since

$$\left( \frac{a+b}{2} - \frac{b-a}{2} \cos \varphi \right) \cos(\psi - \varphi) = \frac{a+b}{2} \cos(\psi - \varphi) - \frac{b-a}{4} (\cos \psi + \cos(\psi - 2\varphi)),$$

we have:

$$\lambda P_{N-1} = \frac{a+b}{2} P_{N-1} - \frac{b-a}{4} (P_N + P_{N-2})$$

and therefore  $\alpha = (a-b)/4$ .

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