Random Matrices and Two-Dimensional Gravity[†]

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Abstract

A brief introduction to some problems in quantum gravity and string theories is given, in order to explain the recent interest for random two dimensional lattices and two-dimensional gravity. The connection between these problems, random matrix models and statistical mechanics of — and on — random lattices is discussed. Some properties of the critical points for the matrix models and the associated "double scaling limit" are studied, and their consequences for the non-perturbative behavior of string theories are outlined.

1 INTRODUCTION

Recently some unexpected and deep connections have been discovered between problems in pure mathematics (topology of the moduli space of Riemann surfaces), in mathematical physics (integrable hierarchies of partial differential equations), in high energy physics and quantum field theory (string theories and quantum gravity), and in statistical mechanics (statistics of eigenvalues of random matrices, statistics of random planar networks and of random surfaces). In these developments random matrix models, which are known already to play an important role in the theory of some disordered systems, and in that of quantum chaos, are prominent. In these lectures I shall try to give a brief introduction to some of these recent developments. Since many participants are probably not very familiar with string theory and quantum gravity, I shall first give a brief introduction to this subject, mostly to explain the motivations for studying matrix models. In the second lecture I shall discuss the connection between matrix models, statistics of random planar lattices, and statistical mechanical models on such lattices. In the third lecture I shall discuss the so-called "double scaling limit" for the matrix model and the connection with integrable systems.

2 A BRIEF INTRODUCTION TO STRING THEORY

2.1 Quantum Gravity

General Relativity is a geometric theory of gravitation [1]: the gravitational field is described by the curvature properties of space-time, that is by a pseudo-Riemannian metric $g^{(4)}_{\mu\nu}$ (i.e. with signature $(-+++)$) over 4-dimensional space-time, and its evolution equations (the Einstein equations) are differential equations which give the time evolution of the metric over 3-dimensional "space-like slices". They can be written as Euler-Lagrange equations for the Einstein-Hilbert action

$$S[g^{(4)}] \;=\; \frac{1}{16\pi G_{\mathrm{N}}} \int d^4x \sqrt{|g^{(4)}|} \left(-R^{(4)} + 2\Lambda\right) \tag{1}$$

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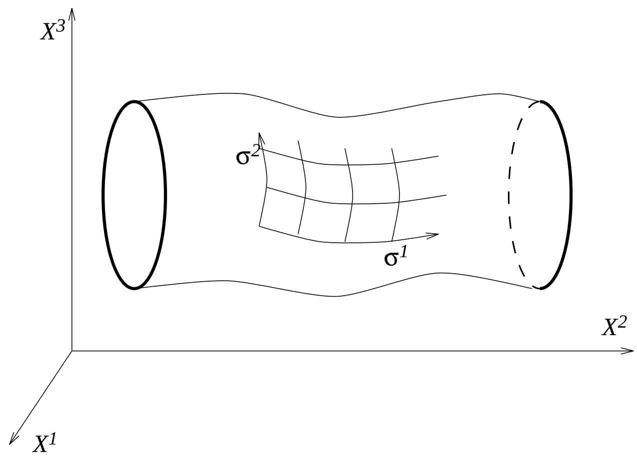


Figure 1: Schematic picture of a closed string.

with $R^{(4)}$ the scalar curvature, and $\mathrm{G_N}$ and Λ respectively the Newton's constant and the cosmological constant.

The construction of a consistent quantum theory of gravity is a notoriously difficult problem. For a recent review see for instance [2,3]. One approach consists in starting from the above formulation and to try to construct a "quantumgeometrodynamics" by using a Feynman path integral over histories of space-like 3-geometries. Wave functions and transition amplitudes are then formally defined in terms of functional integrals over 4-metrics $g^{(4)}(x)$ of the form

$$\int \mathcal{D}[g^{(4)}]\, \exp\left\{-\frac{\mathrm{i}}{\hbar}\, S[g^{(4)}]\right\} \tag{2}$$

Such integrals remain formal as long as one does not know how to define properly the measure and the integrals $\int \mathcal{D}[g^{(4)}]$ over 4-metrics. In quantum mechanics and in quantum field theory, one can usually define such integrals through discretization procedures, i.e. by using discrete time and space coordinates and then by taking a continuum limit. For quantum gravity such ideas of discretization of space-time lead in general to serious problems, since they will break reparametrization invariance and spoil the equivalence principle. Other features of the Einstein theory of gravitation, such as the unboundness of the Einstein-Hilbert action, also preclude a satisfactory treatment of this functional integral.

2.2 Critical Strings

Presently string theory is the most elaborated attempt to quantize gravity; it provides in addition a general framework for the unification of gravitation with the other fundamental interactions[4]. Although the physically interesting models must incorporate supersymmetry, I shall concentrate on the simple case of bosonic string models, and shall discuss the formulation usually denoted the Polyakov string model[5].

The string is initially defined as a 1-dimensional loop evolving with time in a D-dimensional "target space" with coordinates X^μ, $\mu = 1, \ldots, D$. This target space is embodied with a "classical background of fields", which will define its geometrical properties, and the way the string propagates, i.e. the string dynamics. These classical fields are bosonic (they have integer spin and obey the usual statistics). They consist of: a scalar "dilaton" $\Phi(X)$ (spin 0), a rank two metric tensor $G_{\mu\nu}(X)$ which corresponds to the gravitational field in GR (spin 2), plus a spin zero "tachyon" $T(X)$, an antisymmetric tensor field $H^{\mu\nu}(X)$, and higher spin ($S > 2$) fields that I shall not write or discuss here. As the string propagates it spans a 2-dimensional "world-sheet". Let us label the points of the word-sheet with two world-sheet coordinates σ^i, $i = 1, 2$. The dynamical degrees of freedom of the string, which have to be quantized, are the string coordinate $X^\mu(\sigma)$; but it is convenient to introduce an additional auxiliary field, which is a Riemannian metric tensor $g_{ij}(\sigma)$ *on the world-sheet*[b]. The dynamics of the string in the classical background is obtained from the 2-dimensional action

$$\begin{aligned} S[X] &= \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{|g|}\, g^{ij} \partial_i X^\mu \partial_j X^\nu \, G_{\mu\nu}(X) + \frac{1}{4\pi} \int d^2\sigma \sqrt{|g|}\, R^{(2)}\, \Phi(X) \\ &+ \int d^2\sigma \sqrt{|g|}\, T(X) + \cdots \end{aligned} \tag{3}$$

$R^{(2)}$ is the world-sheet scalar curvature (associated to the metric g_{ij}). The $\cdots$ represent additional coupling of the string to the background higher spin fields. The coupling constant α' is the inverse of the string tension, it has the dimension of $[length]^2$.

The quantization of the string degrees of freedom can be performed through Feynman path integration over the string coordinates X^μ and over the world-sheet metric g_{ij}. The classical string action $S[X]$ is invariant under local diffeomorphisms, i.e. changes of the metric and of the fields corresponding to coordinate changes $\sigma \to \sigma'(\sigma)$, but also under *conformal transformations* of the world-sheet metric $g_{ij}(\sigma) \to \mathrm{e}^{f(\sigma)} g_{ij}(\sigma)$, with $f(\sigma)$ an arbitrary function. It turns out that this last invariance is technically essential for a consistent quantization of the string[c], and that it is ensured only if the background fields $G_{\mu\nu}$, Φ, ... satisfy some special field equations (which corresponds to the vanishing of the beta-functions for the string couplings). To first orders in perturbation theory, i.e. in α', these equations are explicitly[d] [6]

$$\begin{aligned} \beta_{G_{\mu\nu}} &= R_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi + \cdots = 0 \\ \beta_\Phi &= (D-26)/3 + \alpha' \left(4(\nabla\Phi)^2 - 4\nabla^2\Phi + R \right) + \cdots = 0 \end{aligned} \tag{4}$$

($R_{\mu\nu} = R_{\mu\nu}(X)$ and $R = R(X)$ are the Ricci and scalar curvatures for the background metric $G_{\mu\nu}(X)$). One recovers the constraint that the dynamics of a single quantum bosonic string is consistent in an empty flat space-time ($R_{\mu\nu} = 0$ and $\Phi = 0$) only if the dimension of space-time is $D = 26$. Moreover, for the case of slowly varying background fields $G_{\mu\nu}$, Φ, ... in $D = 26$, one obtains consistency *classical equations* for the classical background fields. Viewing these fields as some "condensates of strings", these consistency equations can be considered as the classical equations of motion for the "string field" (I shall not try to define properly what it is here). It appears that these classical equations can be derived explicitly from an action principle in 26-dimensional space-time. The part of this action involving the metric

[b] for clarity I omit the subscript $g^{(2)}_{ij}$ for the 2-dimensional metric g_{ij}.

[c] I shall not discuss here the important point of the Fadeev-Popov ghost fields associated to diffeomorphism invariance

[d] I do not consider for simplicity the contribution of the tachyon field $T(X)$.

field $G_{\mu\nu}$ and the dilaton field Φ is

$$\mathcal{S}[G,\Phi] \;=\; \int d^{26}X\,\sqrt{|G|}\,\mathrm{e}^{-2\Phi}\,\left(R+4(\nabla\Phi)^2\,+\,\cdots\right) \tag{5}$$

where $\cdots$ represents $\mathcal{O}(\alpha')$ terms involving higher derivatives of the fields. The contributions of the tachyon and higher spin fields is not written. This implies that in the low energy limit ($\alpha' \to 0$), the dynamics of the string field reduces to that of a spin 2 graviton coupled to the dilaton scalar field, and thus it incorporates GR.

This relationship between the requirement of conformal invariance on the world-sheet for the string action and the classical field equations for the background fields in space-time is summarized below:

$$\begin{array}{c} \text{conformally invariant theory} \\ \text{on the world-sheet} \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} \text{classical solution of} \\ \text{string field theory} \end{array} \tag{6}$$

In fact it can be made more general, since one can consider other realizations of the bosonic string by considering any 2-dimensional conformal field theory (involving some fields $\Psi(\sigma)$), living on the world-sheet, and coupled to the 2-d metric g_{ij} in such a way that the resulting theory is conformally invariant. One can still consider such a model as corresponding to a classical solution of string field theory, but this classical solution does not have necessarily a simple geometric interpretation in term of a space-time target space.

2.3 Non-Critical Strings and Liouville Theory

As an example, if one tries to quantize the Polyakov string in a flat Euclidean space-time ($G_{\mu\nu} = \delta_{\mu\nu}$), but with dimension $D < 26$, and in a constant dilaton background $\Phi(X) = \mathrm{cst}$, the action seems no longer conformally invariant. However, if one gauges away diffeomorphism invariance (on the world-sheet) by choosing a conformal system of coordinates where the world-sheet metric is

$$g_{ij}(\sigma) \;=\; \hat{g}_{ij}(\sigma)\,\mathrm{e}^{\phi(\sigma)} \tag{7}$$

where $\hat{g}_{ij}$ is a *reference metric* on the world-sheet, fixed up to now, and if one treats the conformal factor $\phi(\sigma)$ of the metric as a quantum field (like the X^μ), the string can be quantized consistently, for instance by Feynman path integration over X and ϕ. One thus obtains an effective action for the field ϕ [5]. This effective action is the so-called Liouville action, and it is of the form

$$S_{\mathrm{L}}[\phi] \;=\; \frac{25-D}{48\pi}\int d^2\sigma\,\sqrt{\hat{g}}\,\left[\frac{1}{2}\partial_i\phi\hat{g}^{ij}\partial_j\phi+\hat{R}^{(2)}\phi\right] \tag{8}$$

If one considers the conformal field ϕ as an additional coordinate X^0, the total action for the non-critical string, $S[X]+S_{\mathrm{L}}[\phi]$, can be viewed as the action $S[\vec{\mathbf{X}}]$ for a critical string in a $D+1$ *dimensional* space-time, with coordinate.

$$\vec{\mathbf{X}} \;=\; (X^0,\,X^\mu) \quad ; \quad X^0 \;=\; \sqrt{25-D}\,\phi \tag{9}$$

in a flat background $(D+1)$-dimensional metric $G_{ab}(\vec{\mathbf{X}}) = \delta_{ab}$, but now in a non-constant dilaton background

$$\Phi(\vec{\mathbf{X}}) \;=\; \sqrt{25-D}\,X^0 \tag{10}$$

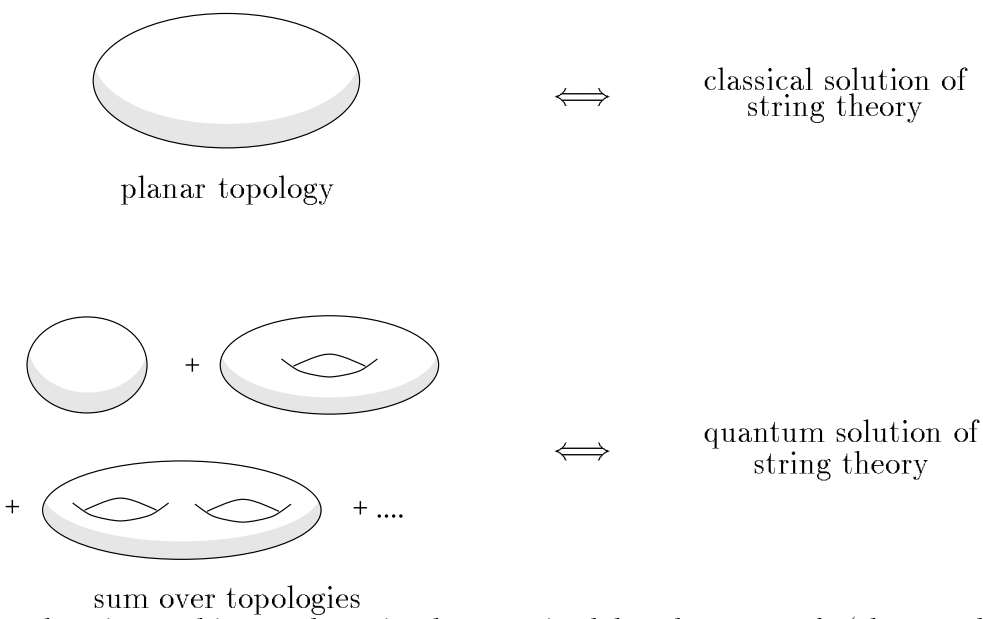
depending linearly on the additional coordinate. Thus the non-critical Polyakov string can be viewed equivalently:

- either as a quantum theory of Euclidean two-dimensional gravity on the world sheet (the metric field g_{ij} on the world-sheet has signature $(++)$), coupled to D free massless real scalar fields (the coordinates of the string X^μ, $\mu = 1,\cdots,D$):

- or as an ordinary bosonic string theory, in a flat $D+1$ dimensional space, but with a non-trivial dilaton background (depending linearly on one of the coordinates).

2.4 String interactions

The above considerations apply to the free string, i.e. to the case where the world-sheet is an infinite cylinder (which is topologically equivalent to the sphere $\mathcal{S}_2$). String interactions are described by the breaking and fusion of strings, and corresponds geometrically to world-sheets with higher topologies. In ordinary field theories (like QED), the quantum theory is obtained by summing over all Feynman diagrams (corresponding to all interaction processes between electrons and photons). Similarly, the quantum theory of strings is formally obtained by summing over the contribution of world-sheets with all topologies.



(11)

For closed strings, this topology is characterized by the genus h (the number of handles), related to the Euler characteristics χ by the Gauss-Bonnet formula

$$2\,(1-h) \;=\; \chi \;=\; \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,R^{(2)} \tag{12}$$

From the action (3), one sees that the expectation value of the dilaton $\langle\Phi\rangle$ measures the strength of the string coupling g_S via $g_S = \exp(-\langle\Phi\rangle)$. This fact is the main motivation for the study of 2-dimensional quantum gravity models, and in particular for the search of solutions which takes into account all topologies of the 2D space-time, since they should correspond to full quantum solutions of string theory.

2.5 KPZ Scaling

Finally, let me briefly discuss an important result, known as Knizhnik-Polyakov-Zamolodchikov (KPZ) scaling, which relates the scaling properties of a conformal field theory (CFT) in flat 2D space to the scaling properties of this CFT coupled to 2D gravity[7,8]. We shall see that it allows to characterize the critical indices of some statistical mechanics models on a random lattice, knowing their critical indices on a fixed regular lattice. A CFT is characterized by its central charge c, and its primary fields, associated with the operators $\mathcal{O}_i$, with scaling dimensions Δ_i^0. In flat space the 2-points correlation functions scale as

$$\langle \mathcal{O}_i(\sigma)\mathcal{O}_j(\sigma')\rangle \;\propto\; \delta_{ij}\,|\sigma-\sigma'|^{-4\Delta_i^0} \tag{13}$$

Now, when this CFT is coupled to 2-dimensional gravity, the 2D metric fluctuates, as well as the geodesic distance between the two points. Nevertheless it makes sense to measure correlation functions of operators integrated over the whole space

$$O_i \;=\; \int d^2\sigma\,\sqrt{g}\,\mathcal{O}_i(\sigma) \tag{14}$$

KPZ have shown that, when averaged over all metrics *with fixed total area* $A = \int d^2\sigma\sqrt{g}$, such averages scale with A as

$$\langle O_1\cdots O_N\rangle \;\propto\; A^{\sum\limits_{i=1}^{N}(1-\Delta_i)} \tag{15}$$

where Δ_i is the scaling dimension of $\mathcal{O}_i$ "dressed by gravity". Similarly, the partition function $\mathcal{Z}_h(A)$ of a surface with fixed genus h (number of handle) and fixed area A (defined by functional integration over the 2D metric g_{ij} and over the CFT) can be shown to scale with A as

$$\mathcal{Z}_h(A) \;\propto\; A^{(2-\gamma_s)(1-h)-1} \tag{16}$$

with γ_s a non-trivial "string exponent", which depends on the CFT only through its central charge c, and is given by[e]

$$\gamma_s \;=\; \frac{1}{12}\left[(c-1)-\sqrt{(25-c)(1-c)}\right] \tag{17}$$

The dressed scaling dimension Δ_i is related to the "bare" dimension Δ_i^0 of $\mathcal{O}_i$ through

$$\Delta_i - \frac{\Delta_i(1+\Delta_i)}{1-\gamma_s} \;=\; \Delta_i^0 \tag{18}$$

The "classical limit", where the fluctuations of the metric are suppressed, corresponds to take $c\to-\infty$. Then $\gamma_s \simeq c/6 \to -\infty$ and $\Delta_i \to \Delta_i^0$. These results make sense only for $c\le 1$, and for $c>1$ give complex exponents. This "$c=1$" barrier is usually viewed as a transition to a strong coupling phase of 2D gravity for $c>1$. This phase, which is related to the existence of tachyonic excitation (with negative squared mass) for the bosonic string in space-time with dimension $D>2$, is still poorly understood.

3 RANDOM TRIANGULATIONS AND MATRIX MODELS

Some general review papers on the following two chapters are [9,10,11,12,13]

3.1 Random Triangulations

In order to make the functional integration over 2D Riemannian metrics g_{ij} and over topologies (characterized for compact orientable 2D surfaces only by the genus h) manageable, one can think of using a lattice discretization. Here we shall use abstract triangulations, i.e. abstract surfaces obtained by gluing pairwise triangles along edges in every possible ways, without reference to an embedding in some target space[14,15,16].

$$\sum_{\text{genus }h}\int \mathcal{D}[g_{ij}]_h \qquad\to\qquad \sum_{\text{triangulation}} \tag{19}$$

[e] at least for unitary CFT

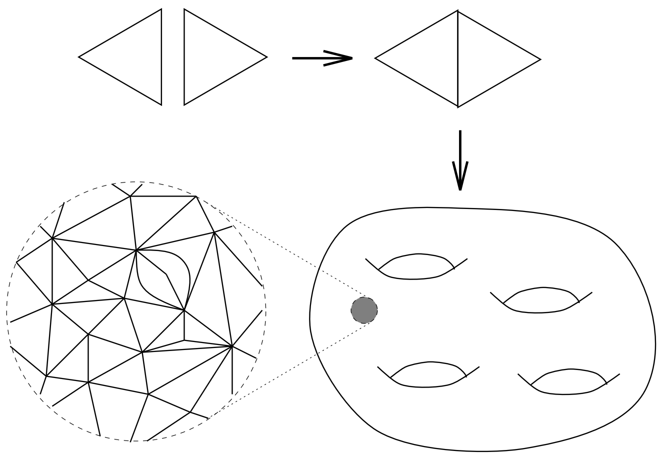


Figure 2: Triangulation of a surface

The topology of a triangulation with no free edges is uniquely characterized by its Euler characteristics χ, given by the Euler formula

$$\chi \;=\; 2\,(1-h) \;=\; \#_{\rm triangles} - \#_{\rm edges} + \#_{\rm vertices} \tag{20}$$

(plus its orientability). Assuming equilateral triangles with fixed edge length a (the lattice "cut-off"), the metric is specified on the triangulation. The scalar curvature R is concentrated on the vertices, and is proportional to the deficit angle $(6-c_v)\pi/3$ (where c_v is the coordination number of the vertex v). The partition function, which discretizes the functional integral $\int \mathcal{D}[g_{ij}]\exp(-S_{\rm EH})$ for 2D Euclidean gravity, is given by a formal sum over all triangulations $\mathcal{T}$

$$\mathcal{Z}(\lambda,\gamma) \;=\; \sum_{\substack{\text{triangulations}\\ \mathcal{T}}} \frac{1}{s(\mathcal{T})}\, \mathrm{e}^{-\lambda\#_{\rm triangles}-\gamma\,\chi} \tag{21}$$

with $s(\mathcal{T})$ the order of the symmetry group of $\mathcal{T}$ ($s=1$ for generic $\mathcal{T}$). λ plays the role of the cosmological constant, γ that of the inverse Newton's constant $G_{\rm N}^{-1}$. It can also be viewed as a discretized non-critical string in a "0-dimensional" space limited to a single point o, with no background metric, but with a dilaton value $\Phi(o) \propto \gamma$ and a tachyon $T(o) \propto \lambda$.

3.2 Random Matrix Models

Remarkably, one can construct a matrix integral which generates this sum over triangulations[19,20]. Let us consider the integral over $N \times N$ Hermitian matrices $M = (M_{ab})$ (with $M_{ab} = \overline{M}_{ba}$)

$$\mathcal{Z}_N(g) \;\equiv\; \int dM\, \mathrm{e}^{-N\,{\rm Tr}[V(M)]} \qquad ; \qquad V(M) \;=\; \frac{M^2}{2} - g\,\frac{M^3}{3} \tag{22}$$

with the flat measure

$$dM \;=\; \prod_{a<b} d{\rm Re}(M_{ab})\, d{\rm Im}(M_{ab}) \prod_a dM_{aa} \tag{23}$$

This integral is not convergent, and is just considered at that stage as a formal series in powers of g (whose individual terms are given by convergent integrals). We can use diagrammatic technics to write this expansion in g. If $\langle\cdots\rangle_0$ represents the Gaussian average with respect to the weight $\exp(-N\,{\rm Tr}(M^2/2))$, the pair average is represented

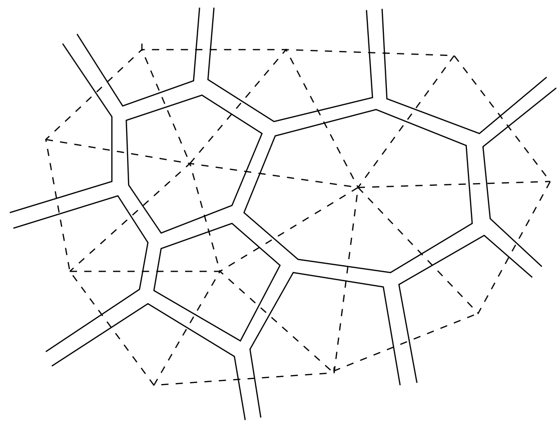
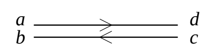


Figure 3: A trivalent fat diagram and the dual triangulation.

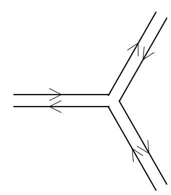
by a "fat" propagator

$$\langle M_{ab} M_{cd} \rangle_0 \;=\; \frac{1}{N}\, \delta_{ad}\delta_{bc} \qquad (24)$$



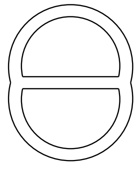
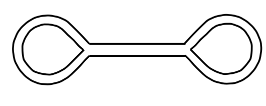
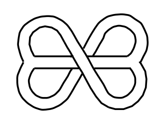
and the cubic term by the interaction vertex

$$N\, g\, \mathrm{Tr}(M^3) \;=\; N\, g\, M_{ab} M_{bc} M_{ca} \qquad (25)$$



The logarithm of $\mathcal{Z}$, $\mathcal{F}_N(g) = \ln[\mathcal{Z}_N(g)/\mathcal{Z}_N(0)]$, (which is the free energy of the matrix model) generates all connected vacuum diagram made of trivalent vertices and fat propagators

$$\mathcal{F}_N(g) \;=\; \frac{1}{2}\left(\frac{Ng}{3}\right)^2 \left\langle \left(\mathrm{Tr}(M^3)\right)^2 \right\rangle_{\substack{0 \\ \text{connected}}} + \cdots$$

=  +  +  + ...

$$= \quad \frac{1}{6} \quad N^2 g^2 \quad + \quad \frac{1}{2} \quad N^2 g^2 \quad + \quad \frac{1}{6} \quad N^0 g^2 \quad + \qquad (26)$$

It is easy to see that the arrows on the lines restrict the sum over "fat graphs" to *oriented* ones, and that the contribution of a graph $\mathcal{G}$ is equal to $s(\mathcal{G})^{-1}\, N^{\chi_{\mathcal{G}}} g^{\#_{\text{vertices}}}$ where $\chi_{\mathcal{G}} = \#_{\text{vertices}} - \#_{\text{propagators}} + \#_{\substack{\text{arrow} \\ \text{loops}}}$ is the Euler characteristics of the fat graph, and $s(\mathcal{G})$ the order of the symmetry group of the graph.

Trivalent fat graphs are dual to triangulations, and therefore the matrix model free energy $\mathcal{F}_N(g)$ is the generating functional for oriented triangulations, and can be identified with the partition function $\mathcal{Z}(\lambda, \gamma)$, provided that $g = \exp(-\lambda)$ and $N = \exp(-\gamma/4\pi)$. Moreover, one can reorganize the expansion as a topological expansion in powers of $1/N^2$, the term of order N^{2-2h} giving the sum over all triangulations with fixed genus h.

One can use the same trick to generate triangulations with boundaries, via correlation functions of the matrix model. For instance, the average

$$W(K) \;=\; \langle \mathrm{Tr}(M^K) \rangle \tag{27}$$

gives the sum over triangulations with one boundary consisting of a loop with K links. Instead of using the cubic potential, one can start from a more general polynomial potential of the form

$$V(M) \;=\; \frac{1}{2}M^2 - \frac{g_3}{3}M^3 - \frac{g_4}{4}M^4 - \cdots \tag{28}$$

which generates polyedral tessellations with triangles, squares, etc...

Finally, if instead of the Hermitian ensemble one uses the symmetric ensemble, the diagrammatic rules are the same, but the lines are unoriented, and one can see that one thus generates *unoriented* triangulations.

3.3 The Planar Limit

In the limit where the dimension N of the matrix becomes large, only planar diagrams, i.e. diagrams that can be drawn on a sphere, survive. This limit is of interest because: (a) it is a simple case where topology fluctuations are suppressed, and it corresponds to the classical limit of non-interacting strings; (b) the number of planar diagrams is much smaller than the total number of diagrams with a given number of vertices, and simplifications are expected to occur. Indeed, we shall see that this limit, albeit non-trivial, is exactly solvable.

3.4 Large N Eigenvalues Distribution

A standard method[21] to study statistics of large matrices is to diagonalize M, and to write it in terms of a unitary matrix U and a diagonal matrix Λ, with $M = U^\dagger \Lambda U$. Both the measure dM and the interaction $\mathrm{Tr}(V(M))$ are invariant under the action of the unitary group $\mathrm{SU}(N)$: $M \to V^\dagger M V$. One can integrate over the unitary part U of M, and the measure reduces to the measure over the diagonal part (the eigenvalues λ_i, $i = 1, \cdots, N$)

$$\int dM \;\equiv\; \mathrm{Vol}(\mathrm{SU}(N)) \int \prod_{i=1}^{N} d\lambda_i\, \Delta[\lambda]^2 \tag{29}$$

with $\Delta[\lambda]$ the Vandermonde determinant

$$\Delta[\lambda] \;=\; \prod_{i>j} (\lambda_i - \lambda_j) \;=\; \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-1} \end{vmatrix} \tag{30}$$

The partition function $\mathcal{Z}_N$ can be rewritten as the partition function for a system of N charges of the real line, subjected to a one-body potential $V(\lambda)$ and to a logarithmic 2-body potential[f]

$$\mathcal{Z}_N \;\propto\; \int \prod_{i=1}^{N} d\lambda_i\, \Delta[\lambda]^2\, \mathrm{e}^{-N\sum_{i=1}^{N} V(\lambda_i) + \sum_{i\neq j} \ln|\lambda_i - \lambda_j|} \tag{31}$$

The planar limit is the thermodynamic limit where the number of charges becomes infinite. When $N \to \infty$ the density of charge diverges, and the fluctuations

[f] viewed as the Coulomb electrostatic potential in the complex λ plane

become negligible, so that the problem reduces to the determination of the normalized eigenvalues density $\rho(\lambda)$, defined as $N\rho(\lambda)d\lambda = \#$ of eigenvalues in $[\lambda, \lambda + d\lambda]$. The density is subjected to the normalization condition

$$\int d\lambda\, \rho(\lambda) \;=\; 1 \qquad ; \qquad \rho(\lambda) \geq 0 \tag{32}$$

and is solution of the equilibrium condition for the eigenvalues

$$V'(\lambda) \;=\; \fint d\mu\, \frac{\rho(\mu)}{\lambda - \mu} \qquad \text{if } \rho(\lambda) \neq 0 \tag{33}$$

From ρ one computes easily observables in the planar limit. For instance the free energy for closed planar triangulations is

$$\mathcal{F} \;=\; \ln(\mathcal{Z}) \;=\; N^2 \int d\lambda\, \rho(\lambda) \left\{ \int d\mu\, \rho(\mu)\, \ln|\lambda - \mu| \;-\; V(\lambda) \right\} \tag{34}$$

and the generating function for open planar triangulations is given by

$$\langle \mathrm{Tr}(M^K) \rangle \;=\; N \int d\lambda\, \lambda^K\, \rho(\lambda) \tag{35}$$

For the Gaussian potential $V = \lambda^2/2$ one recovers the celebrated Wigner semicircle distribution $\rho(\lambda) = (4\pi)^{-1}\sqrt{2 - \lambda^2}$ on the interval $-\sqrt{2} < \lambda < \sqrt{2}$. For a generic polynomial $V(\lambda)$, the solution takes the form $\rho(\lambda) = Q(\lambda)\sqrt{P(\lambda)}$, with P and Q some polynomials, and has support on one (or several) intervals along the real line.

3.5 Critical Point and the Scaling Limit

For the cubic potential the explicit solution is analytic in g, and is of the form

$$\rho(\lambda) \;=\; \sqrt{(a - \lambda)(b - \lambda)}(c - \lambda) \qquad : \qquad a(\lambda) < b(\lambda) < c(\lambda) \tag{36}$$

as long as g is smaller than some critical value g_c. At this critical point $b(g_c) = c(g_c)$. For $g > g_c$, ρ becomes complex valued. The physical significance of this singularity is the following: The potential V has a local minimum at $\lambda = 0$, but is unbounded from below as $\lambda \to \infty$. As long as $g < g_c$, the minimum at the origin is deep enough to keep the eigenvalues concentrated on a single interval $[a, b]$. The density still vanishes like a square root at the endpoints ($\rho(\lambda) \propto |b - \lambda|^{1/2}$). The partition function and other observables are, in the large N limit, analytic in g at the origin, with a finite radius of convergence in g.

At g_c the density behaves like $\rho(\lambda) \propto |b_c - \lambda|^{3/2}$. and the partition function $\mathcal{Z}(g)$ has an algebraic singularity

$$\mathcal{Z}(g) \;\simeq\; \text{regular part} \;+\; (g_c - g)^{5/2} \;+\; \ldots \tag{37}$$

which reflects the fact that the potential well is not deep enough to prevent the fall of the eigenvalues in the infinite well at $+\infty$.

The properties of this critical point are to a large extent universal. For instance they do not depend on the exact form of the potential, provided that it has the global features of the cubic potential. This has two important consequences:

Firstly, the fact that the series expansion in powers of g for the observables has a finite radius of convergence (equal to g_c) in the planar limit implies that the number of planar triangulations with a fixed number of triangles (area A) grows with A like C^A (with $C = 1/g_c$). This number grows much more slowly than the total number of triangulations, with any topology, which behaves like $A!$. Moreover, this behavior holds not only for planar triangulations, but for triangulations with *fixed* arbitrary topology. Indeed, the sum over triangulations with genus h is given by the term of order $N^{2(1-h)}$ of the large N expansion, and one can show that all the terms of this expansion, considered as functions of g, have a singularity at the same g_c.

Secondly, since the critical point governs the statistical properties of *large* triangulations, one can associate to g_c a universal *continuum limit*, as in the theory of ordinary critical phenomena. For that purpose, we remind that if we consider that the edges have a length a (lattice cut-off), the physical area $A_{\rm phys}$ of the surface and the physical length $L_{\rm phys}$ of a boundary are

$$A_{\rm phys} = a^2 \#_{\rm triangles} \quad ; \quad L_{\rm phys} = a \#_{\rm edges} \tag{38}$$

The continuum limit consists in taking the small cut-off limit $a \to 0$ while physical quantities such as $A_{\rm phys}$ and $L_{\rm phys}$ are kept finite. In this limit observables like the singular part of the partition function scales, and are dominated by the end-point behavior (at $\lambda_e = b(g_c)$) of the eigenvalue density ρ. One can show that this behavior becomes universal, when expressed in terms of rescaled coupling constant t_0 and eigenvalue coordinate p

$$g_c - g \quad \propto \quad a^2 t_0 \tag{39}$$

$$\lambda - \lambda_e \quad \propto \quad a\, p \tag{40}$$

and that for instance the density scales as $\rho \simeq (p - t_0^{1/2})(p + 2t_0^{1/2})^{1/2}$.

The remarkable – and at that stage somehow miraculous – fact is that this continuous limit coincides with that of pure two-dimensional gravity, constructed for instance by the Liouville theory. This can be checked on the quantities which can be explicitly calculated in both approaches. The simplest example is that of the string exponent γ_s. From the matrix model calculations one can check that the term of order h of the free energy $\mathcal{F}$ scales with t_0 like $t_0^{(1-h)5/2}$. On the other hand the KPZ scaling predicts that it should scale as the power $(1-h)(2-\gamma_s)$ of the cosmological constant. Thus t_0 can be identified with the cosmological constant and the value for the string exponent is

$$\gamma_s = -1/2 \tag{41}$$

It coincides with that obtained by KPZ for pure gravity (with central charge $c = 0$). Other explicit checks involve dimensionless ratios of correlations functions.

3.6 The O(n) *Model on a Random Lattice*

Let me present now an example of matrix model which is a realization of a statistical model on a random triangulation, namely the so-called O(n) (or loop gas) model[23,22]. This example is both important and instructive since the O(n) models on a fixed regular two dimensional lattice are known to have critical points, which in the continuum limit correspond to some CFT. It allows to apply the discretization via random triangulations to models of 2D gravity coupled to non-trivial CFT's.

On a regular 2D honeycomb lattice the O(n) model is defined as follows: The configurations are *non-intersecting* loops on the lattice, and the Boltzmann weight W depends simply on the number of closed loops, n_ℓ, and on the total length (number of links) of the loops l_ℓ

$$W \; = \; z^{l_\ell}\, n^{n_\ell} \tag{42}$$

There is a critical value z_c for the link fugacity z, which separates a dilute phase for $z < z_c$ from a dense phase for $z > z_c$. For loop fugacity $-2 \leq n \leq 2$, this transition is continuous, and the critical point corresponds to a CFT with central charge given by

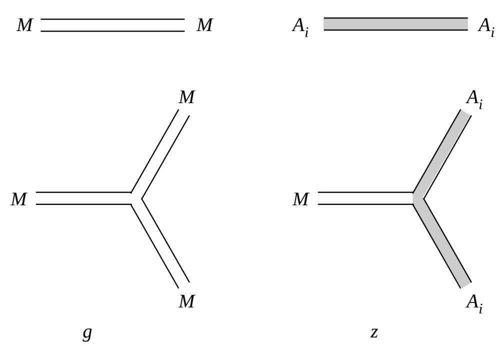
$$c = 1 - \frac{6}{p(p+1)} \qquad n = 2\cos(\pi/p) \tag{43}$$

$n = 2$ gives the XY model, with the KT transition characterized by $c = 1$. $n = 1$ gives the Ising model ($c = 1/2$), $n = 0$ gives the self-avoiding walk model (polymer in good solvent) with $c = 0$. In fact the dense phase is also associated with some CFT.

This model can be extended to the case of loops on a fluctuating trivalent lattice. In this model both lattice configurations and loops configurations fluctuate, i.e. the geometric disorder given by lattice fluctuations is *annealed*, not quenched. A multi-matrix model representation of this model can be written. It involves the $N \times N$ Hermitian matrix M, plus n matrices A_i, $i = 1, \ldots n$, and the Boltzman weight is taken to be $\exp(N\,\mathrm{Tr} S(M, A_i))$, with

$$S(M, A_i) = \frac{1}{2} M^2 - \frac{g}{3} M^3 + [1 - zM][A_1^2 + \cdots + A_n^2] \tag{44}$$

One has now two kinds of propagator and two kinds of trivalent vertices



(45)

and the $\langle AA \rangle$ propagators generate the closed loops on the random lattice. Integrating over the A_i and diagonalizing M one obtains an eigenvalue integral representation which can be analytically continued to non-integer n.

$$\mathcal{Z} = \int \prod_{i=1}^{N} d\lambda_i\, \Delta[\lambda_i]^2 \prod_{i,j=1}^{N} [1 - z(\lambda_i + \lambda_j)]^{-n/2}\, \mathrm{e}^{-N \sum_{i=1}^{N} (\frac{1}{2}\lambda_i^2 - \frac{g}{3}\lambda_i^3)} \tag{46}$$

Using this representation one can show that the model has a critical point in the (g, z) plane, which corresponds to the critical O(n) model coupled to 2D gravity. In particular the associated string exponent is found to be

$$\gamma_s = -\frac{1}{p} \tag{47}$$

in agreement with KPZ scaling. Here also, the critical point corresponds to a singularity in the end-point distribution for the eigenvalues of M.

Finally, let me mention that other classes of multi-matrix models have been solved (see for instance [11] for a review). In particular, the so-called chain matrix models describe critical theories of the A series (in the ADE classification of RCFT) coupled to gravity. Up to now, all solved models corresponds to $c \leq 1$ theories, for which KPZ scaling is valid.

4 THE DOUBLE SCALING LIMIT

Up to now I discussed the planar limit, which corresponds to the classical limit of string field theory. Can one obtain for these models results on the *sum over all topologies*, which should correspond to a full quantum solution of strings? We shall see that this is indeed possible on the example of the one matrix model ($c = 0$ string)[24,25,26]. This follows from the fact that, from KPZ scaling, order by order in the topological expansion, the singular part of the free energy $\mathcal{F}$ scales at the critical point g_c as

$$\mathcal{F}_{\text{sing}}(g,N) \;=\; -\,N^2\,(g_c-g)^{5/2} \;+\; N^0\,(g_c-g)^0 \;+\; N^{-2}\,(g_c-g)^{-5/2} \;+\; \cdots \quad (48)$$

Therefore, in the limit where

$$N \to \infty \quad , \quad g \to g_c \quad , \quad (g_c-g)N^{4/5} = t \;\; \text{fixed} \quad (49)$$

the singular part of $\mathcal{F}$ at g_c becomes a function of the single scaling variable t, rather than a function of the two variables g and N,

$$\mathcal{F}_{\text{sing}}(g,N) \;\to\; F(t) \quad (50)$$

This function contains all the information about the sum over topologies in the continuum limit. This limit is called the "double scaling limit" (DS limit). Similarly, for loop observables we expect that

$$W(K,N,g) \;\to\; W(\ell,t) \qquad \ell \;=\; K\,N^{2/5} \;\text{rescaled length} \quad (51)$$

In fact, for some simple models such as the one matrix model ($c = 0$), the DS limit is also solvable, and reveals very interesting structures.

4.1 The DS Limit and Universal Statistics of Eigenvalues

From the point of view of the eigenvalue integral representation of the one matrix model, the DS limit is one particular example of limits where *universal statistics of eigenvalues*, independent of the specific potential V, emerge.

The simplest, and most known example, is bulk statistics. Let us consider a small interval $I = [\lambda_0 - a, \lambda_0 + a]$ inside the support of the eigenvalues density at $N \to \infty$. Since the density $\rho(\lambda) \sim \mathcal{O}(1)$ the average number of eigenvalues in I is of order aN. If we take the limit $N \to \infty$, aN fixed, and rescale the eigenvalue coordinate inside I, $\lambda = \lambda_0 + ap$, correlation functions such as the probability density $\mathcal{P}(\lambda_1, \ldots, \lambda_p)$ to have a first eigenvalue at λ_1, …, a $p-$th eigenvalue at λ_p, should take a universal form $\mathcal{P}(p_1, \ldots, p_p)$, independent of the specific form of the potential and of the point λ_0.

Another example is the end-point statistics. For a generic potential, the eigenvalue density vanishes at the end point λ_e like $\rho(\lambda) \sim [\lambda_e - \lambda]^{1/2}$, and the number of eigenvalues in the interval $I = [\lambda_e - a, \lambda_e + a]$ scales as $Na^{3/2}$. As the potential is changed (for instance by varying g around some g_0) the end-point varies linearly with g since $d\lambda_e/dg \neq 0$. Thus we expect to obtain universal correlations, with generic dependence with respect to variations of the potential, in the limit $N \to \infty$, $aN^{3/2}$ fixed, $a \sim \lambda - \lambda_e(g_0) \sim g - g_0$.

The DS limit corresponds to *critical end-point statistic*. At the critical end-point λ_c, $\rho \sim |\lambda_c - \lambda|^{3/2}$ and the number of eigenvalues scales with the size of the interval around λ_c as $Na^{5/2}$, while for $g \simeq g_c$, $\lambda_e(g) - \lambda_c \sim (g_c - g)^{1/2}$. Hence we expect universal behavior in the scaling limit $\lambda-\lambda_c \sim (g_c-g)^{1/2} \sim N^{-5/2}$, which is precisely the DS limit.

4.2 The Orthogonal Polynomial Method

A very powerful method to study the matrix model is the orthogonal polynomial method (see for instance [21]). Let us discuss it in the case of the one matrix model with an *even* potential $V(\lambda)$[g]. One introduces orthonormal polynomials with respect to the measure $\exp(-N\,V(\lambda))$

$$\pi_i(\lambda) \;=\; h_i^{-1/2}\lambda^i \,+\, \mathcal{O}(\lambda^{i-1}) \qquad ; \qquad \int d\lambda\, \mathrm{e}^{-N\,V(\lambda)}\pi_i(\lambda)\pi_j(\lambda) \;=\; \delta_{ij} \tag{52}$$

The $N\times N$ Vandermonde determinant can be written as

$$\Delta[\lambda] \;=\; \det\left[\lambda_i^{\,j-1}\right] \;=\; \left(\prod_{i=1}^{N} h_i\right)^{1/2} \det\left[\pi_j(\lambda_i)\right] \tag{53}$$

Using the orthogonality of the π_i's, the partition function of the matrix model can be shown to reduce to

$$\begin{aligned} \mathcal{Z} \;&=\; \int\prod_{i=1}^{N} d\lambda_i\, \mathrm{e}^{-N\,V(\lambda_i)}\,\Delta[\lambda]^2 \;=\; N!\prod_{i=0}^{N-1} h_i \\ &=\; n!\,h_0^{\,N}\prod_{i=0}^{N-1} R_i^{\,N-i} \qquad \text{with } R_i \;=\; \frac{h_i}{h_{i-1}} \end{aligned} \tag{54}$$

The calculation of $\mathcal{Z}$ reduces to that of the R_i coefficients, which can be computed recursively. Let us consider the even quartic potential

$$V(\lambda) \;=\; \frac{1}{g}\left(\lambda^2-\frac{\lambda^4}{12}\right) \tag{55}$$

which has (both for positive and negative λ) the same features than the cubic potential. The R_i's satisfy the *recursion relation*

$$g\,\frac{i}{N} \;=\; 2R_i \,-\, \frac{1}{3}R_i(R_{i-1}+R_i+R_{i+1}) \tag{56}$$

whose proof is standard, and relies on the orthogonality of the π_i's, on the identity $\lambda\pi_i=\sqrt{R_{i+1}}\pi_{i+1}+\sqrt{R_i}\pi_{i-1}$, and on the identity $\int d\lambda\,\frac{d}{d\lambda}[\exp(-NV(\lambda))\pi_i(\lambda)\pi_j(\lambda)] = 0$.

In the large N limit this model has a critical point g_c, which belongs to the universality class of the cubic potential. At that point, the DS limit can be obtained explicitly. In this limit, $N\to\infty$ and $g\to g_c$ so that

$$g_c-g \;=\; N^{-4/5}t_0 \tag{57}$$

and one can check that only the *last* R_i's, with $i\simeq N$, are important. This suggests the following ansatz

$$1-\frac{i}{N} \;=\; N^{2/5}\,x \qquad , \qquad R_i \;=\; R_c \,-\, N^{-4/5}u(x,t_0) \,+\, \cdots \tag{58}$$

In the $N\to\infty$ limit, x becomes a continuous parameter, and the difference recursion equation for R_i becomes a second order differential equation for the function $u(x)$, while the relation $\mathcal{F}_{\mathrm{sing}} \simeq \sum\limits_{i=0}^{N-1}(N-i)\ln(R_i)$ becomes the following integral

$$\mathcal{F} \;=\; -\int_0^\infty dx\,x\,u(x,t_0) \tag{59}$$

[g] The case of a general V is slightly more complicated, but similar.

The explicit calculation shows that the differential equation for u is the Painlevé I equation

$$-\frac{1}{6}\frac{\partial^2 u}{\partial x^2} + u^2 = x + t_0 \tag{60}$$

so that u is a function of the single variable $u(x+t_0)$. This implies that u is the second derivative of the free energy $u(t_0) = \mathcal{F}''(t_0)$, hence its name *string susceptibility*.

4.3 Connection with K.d.V. Hierarchy

The occurrence of the Painlevé I equation is in fact a consequence of the so-called *canonical commutation relations*, which are realized in the matrix model[27]. Out of the polynomials π_i one constructs the orthonormal functions Ψ_i

$$\Psi_i(\lambda) = \pi_i(\lambda)\,\mathrm{e}^{-\frac{N}{2}V(\lambda)} \tag{61}$$

The multiplication operator Q and derivation operator P take in this basis the form of (infinite dimensional) real matrices with only a finite number of non-zero subdiagonals (Jacobi property)

$$\lambda\,\Psi_i = Q\,\Psi_i = \sum_j Q_{ij}\,\Psi_j \qquad Q_{ij} = 0 \ \text{ if } \ |i-j| \geq 2 \tag{62}$$

$$\frac{\partial}{\partial\lambda}\Psi_i = P\,\Psi_i = \sum_j P_{ij}\,\Psi_j \qquad P_{ij} = 0 \ \text{ if } \ |i-j| \geq \text{degree}(V) \tag{63}$$

P and Q satisfy obviously the Heisenberg algebra commutation relation

$$[P,Q] = PQ - QP = 1 \tag{64}$$

In the DS limit the index i becomes the continuous parameter x and $\psi_i(\lambda) = \Psi(i,\lambda)$ becomes a continuous function $\Psi(x,p)$, while (thanks to the Jacobi property) the operators Q and P become differential operators with respect to the variable x of degree at most 2 for Q and degree(V) for P. In our case one can show that P is antisymmetric and of degree 3, so that Q and P are of the general form

$$Q = d_x^2 - 2u(x) \quad , \quad P = \frac{1}{3}\,d_x^3 + d_x v(x) + v(x) d_x \quad , \quad d_x = \frac{\partial}{\partial x} \tag{65}$$

The functions u and v are then fixed by the commutation relation $[P,Q] = 1$, which implies that $v = -\frac{1}{2}u$ (up to an irrelevant constant shift for u) and that $1 = -\frac{1}{6}u''' + 2uu'$, which integrated yields Painlevé I.

The canonical commutation relation implies that the solution of the matrix model in the DS limit is connected to the Korteweg-de Vries (KdV) hierarchy[28]. This hierarchy of PDE describes how the string susceptibility u and the free energy $\mathcal{F}$ change with the potential V in the vicinity of the critical point, and allows to compute correlation functions. Moreover it allows to make contact between the matrix model approach to 2D gravity and other approaches, such as "topological 2D gravity"[29], where similar hierarchies of equations appear, and it leads to a better understanding of the symmetries of these models. A detailed discussion of these issues is beyond the scope of these lectures.

4.4 Some Properties of the Painlevé I Equation

Let us come back to the properties of the solution of the PI equation. The susceptibility $u(t) = -\mathcal{F}''(t)$ with $t \simeq (g_c - g)N^{4/5}$ should behave in the large t limit like $\sqrt{t}$, since this corresponds to the large N behavior $\mathcal{F} \sim N^2 \sim t^{5/2}$ of the free energy. With

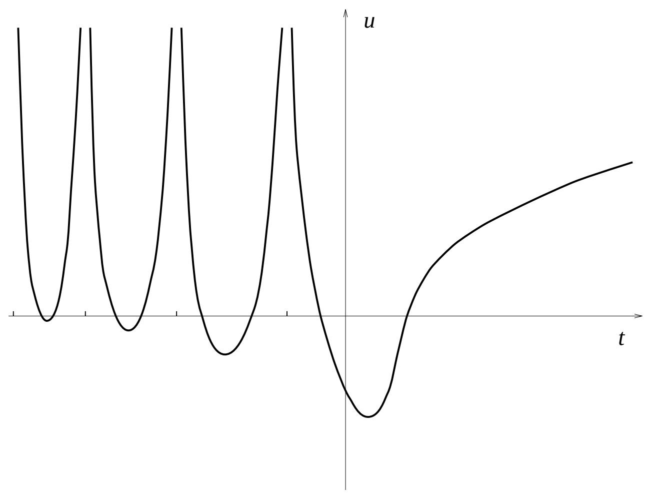


Figure 4: A real solution of the Painlevé equation

this asymptotics, the Painlevé equation fixes all the terms of the large t expansion of u

$$u \;=\; t^{1/2} - \frac{1}{48}\, t^{-2} - \frac{49}{4608}\, t^{-9/2} - \cdots - u_k\, t^{\frac{1-5k}{2}} - \cdots \tag{66}$$

This expansion corresponds to the topological expansion of the sum over closed surfaces with two marked point. Thus one has obtained at once the whole topological expansion through the DS limit!

However, it appears that this expansion is not convergent. Indeed, one can show that the term of order k, corresponding to the sum over surfaces with k handles, grows like

$$u_k \;\sim\; (2k)! \tag{67}$$

and all u_k have the same sign. So the series is not summable (even in the Borel sense) for arbitrary small t, and it does not fix uniquely the solution of the equation.

The Painlevé I equation has the Painlevé property: its solutions, extended to functions of a complex variable, have only poles as moveable (i.e. dependent of the initial conditions) singularities. In our case, the analytic structure of a solution $u(t)$, with the physical conditions that $u(t) \sim t^{1/2}$ as $t \to +\infty$ and that u is real for real t, is the following (see for instance [30]):

- u has only one essential singularity at ∞, and has as moveable singularities double poles with residue 1. In fact at each pole, $u(t) \simeq (t-t_0)^{-2} + \mathcal{O}((t-t_0)^2)$.

- The poles are asymptotically (i.e. for large $|t|$) located in only three fifth of the complex plane, while u is analytic, and behaves like $t^{1/2}$ for $-2\pi/5 < \mathrm{Arg}(t) < 2\pi/5$. In particular there is always an infinite number of double poles along the negative real axis.

- The large t asymptotic expansion of u contains, in addition to analytic terms, exponentially small non-perturbative terms

$$u(t) \;=\; t^{1/2} - t^{-2} - t^{-9/2} - \cdots \;+\; \mathbf{C}\, t^{-1/8}\, \mathrm{e}^{-\frac{8\sqrt{3}}{5} t^{5/4}} (1 + t^{-5/2} + \cdots) \;+\; \cdots \tag{68}$$

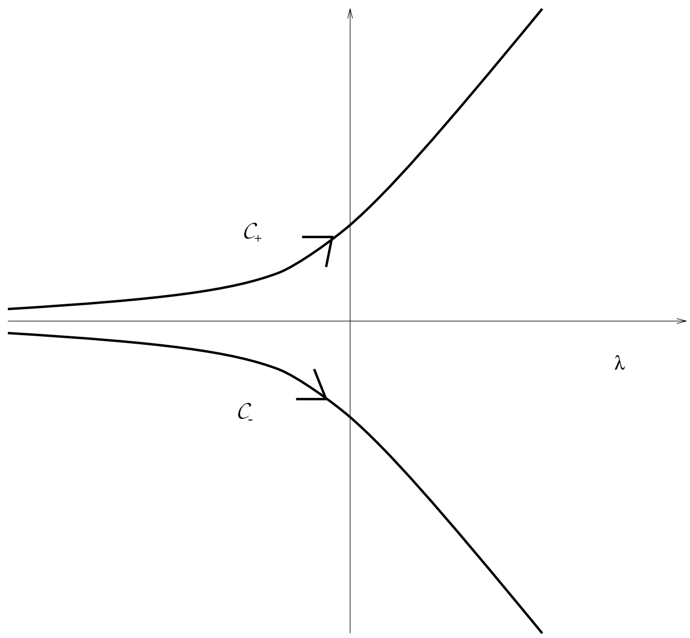


Figure 5: The two integration contours in the eigenvalues complex plane.

- The coefficient **C** of the leading non-perturbative term labels a one-parameter family of solutions, with the same asymptotic expansion, but with different position for the double poles.

4.5 Some Nonperturbative Issues

If one views $t^{-5/2}$ as a "handle fugacity" for the surface, it has been suggested that the first singularity on the real axis describes a transition between a low fugacity phase with a finite average number of handles and a high fugacity phase where the number of handles is infinite. However, since the position of the poles of u are not universal, it is difficult to make this picture consistent. Another problem is to understand if there is a physical interpretation (in term of fluctuations of surfaces, or in term of string theory) of the non-perturbative parameter which governs the position of the singularity.

In fact, from the point of view of the matrix model, these effects can be understood in terms of integration contours in the matrix integral. Up to now, this integral has only been considered at a formal level, as a generating functional for triangulations, and for deriving the recursion relations. For the cubic potential, the eigenvalue integral is divergent when integrated along the real axis, since the potential is unbounded from below as $\lambda \to +\infty$. This divergence is essential for obtaining the critical point, where ($c = 0$) pure 2D gravity is recovered, and cannot simply be eliminated by a change in the potential without eliminating the critical point itself. However, a simple mathematical trick allows to recover a convergent integral[31]. It consists in integrating the λ_i's (the eigenvalues) along *complex contours.* For the cubic potential, there are two independent contours, $\mathcal{C}_\pm$, which are depicted on Fig. 5, which go from $-\infty$ to $+\infty\, e^{\pm i\pi/3}$. The most general way to define the integral is to take a linear combination

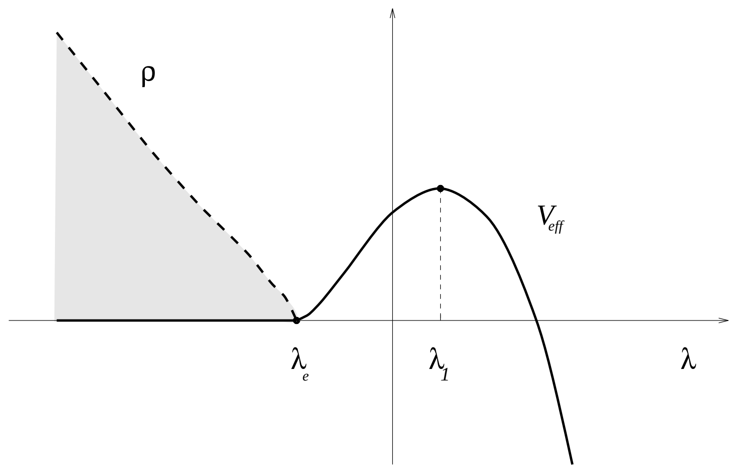


Figure 6: The effective potential V_{eff} and the eigenvalue density ρ u=in the vicinity of the end point.

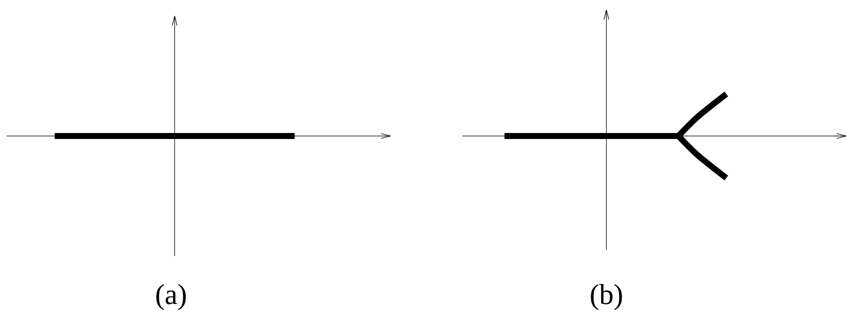


Figure 7: supports of eigenvalues for $g < g_c$ (a) and $g > g_c$ (b)

of these contours

$$\int d\lambda \;=\; c_+ \int_{\mathcal{C}_+} d\lambda \;+\; c_- \int_{\mathcal{C}_-} d\lambda \tag{69}$$

The large N eigenvalue distribution can be estimated in that case[32]. Below the critical point, i.e. for $g < g_c$, one recovers the solution which has been discussed above (provided that $c_+ + c_- \neq 0$). The support of eigenvalues is a real interval, and the eigenvalue density depends analytically on the coupling constant g. Within this "mean-field" picture, one can compute the effective potential $V_{\text{eff}}(\lambda)$ seen by a single eigenvalue (it is the sum of the one body potential V and of the logarithmic Coulomb potential created by the $N-1$ other charges). The schematic form of V_{eff} in the vicinity of the end-point where the singularity occurs at g_c is depicted on Fig. 6. It is constant on the support (since this is the equilibrium position for the eigenvalue), has a small positive "wall" (which forbids the eigenvalues to fall to infinity for $g < g_c$), and goes to $-\infty$ as $\lambda \to +\infty$. The potential difference between the top of the wall and the "sea of eigenvalues" is found to be (with a proper normalization in term of the rescaled coupling constant $t \propto (g_c - g)N^{4/5}$)

$$\Delta\, V_{\text{eff}} \;=\; \frac{8\sqrt{3}}{5}\, t^{5/4} \tag{70}$$

This is precisely the exponential factor in Eq. 68. This allows to give a semiclassical picture (decay of eigenvalues from the metastable well to infinity) to the non-perturbative terms in the large t expansion of the free energy.

Beyond the critical point, i.e. for $g > g_c$, something peculiar occurs. The support of eigenvalues is now complex, and has the shape of a "fork" with two branches at the endpoint. Moreover, the distribution is found to depend *non-analytically* of the coupling constant g (i.e. both of g and $\bar{g}$). This non-analyticity is in fact related to the existence of an infinite number of poles[h] (in the variable t) of the susceptibility $u = \mathcal{F}''$ for large negative t. The up and down branches contribute to the eigenvalues integrals with the phase factors c_+ and c_- respectively, and the non-perturbative coefficient **C** in eq. 68 is proportional to the "phase" $\theta = \mathrm{i}\frac{c_+ - c_-}{c_+ + c_-}$.

Thus, the matrix model with complex integration contours gives consistent solutions of the Painlevé I string equation. The non-perturbative parameter θ is related to the choice of contours, and for each θ one can construct loop observables $W(\ell)$ which satisfy the KdV flow equations.

From the point of view of string theory, many unsatisfactory points remains: The interpretation of these solutions and of the non-perturbative parameter is still unclear. The existence of poles in the t plane for the string susceptibility u, and for the loop amplitudes, makes problematic some fundamental properties, such as unitarity and positivity, for the corresponding string theory. Finally, the one matrix model corresponds to a string theory in a $0+1$ dimensional universe (one "time" and no space). For more complicated models, such as the $\mathrm{O}(n)$ model described above, which corresponds to $D+1$ dimensional strings (with $D = c$ the central charge of the matter sector), the number of non-perturbative parameters is in fact larger, and becomes infinite as $c \to 1$ (see [10]).

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