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Large Order Behaviour of 2D Gravity Coupled to $d < 1$ Matter

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We discuss the large order behaviour and Borel summability of the topological expansion of models of 2D gravity coupled to general (p,q) conformal matter. In a previous work it was proven that at large order k the string susceptibility had a generic $a^k\Gamma(2k-\frac{1}{2})$ behaviour. Moreover the constant a, relevant for the problem of Borel summability, was determined for all one-matrix models. We here obtain a set of equations for this constant in the general (p,q) model. String equations can be derived from the construction of two differential operators P, Q satisfying canonical commutation relations $[P,Q]=1$. We show that the equation for a is determined by the form of the operators P, Q in the spherical or semiclassical limits. The results for the general one-matrix models are then easily recovered. Moreover, since for the (p,q) string models such $p=(2m+1)q\pm 1$ the semiclassical forms of P, Q are explicitly known, the large order behaviour is completely determined. This class contains all unitary $(q+1,q)$ models for which the answer is specially simple. As expected we find that the topological expansion for unitary models is not Borel summable.

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1. Introduction

We report here new results concerning the large order behaviour of the perturbation series of models [1–3] of 2D gravity coupled to $D < 1$ matter. Our motivation is to gather some information about non-perturbative features of quantum gravity and string theory studying the asymptotic behaviour of perturbation series. In particular we want to understand whether perturbation theory indeed provides a proper definition of the physical theory of interest, in more technical terms whether the perturbation series is Borel summable.

We first recall that the coupled differential equations for the partition function in the formulation of 2D quantum gravity coupled to arbitrary (p, q) minimal conformal matter can be derived from canonical commutation relations $[P, Q] = 1$ [4] where P, Q are two differential operators of degree p and q respectively:

$$P = \mathrm{d}^p - \tfrac{1}{2}\sum_{i=1}\{u_i(x), \mathrm{d}^{p-2i}\}, \qquad Q = \mathrm{d}^q - \tfrac{1}{2}\sum_{i=1}\{v_i(x), \mathrm{d}^{q-2i}\}, \tag{1.1}$$

and $u(x) = u_1(x)/p = v_1(x)/q$ is the specific heat or string susceptibility. Note that our normalization of $u(x)$ differs by a factor 2 from the most commonly used in this problem (this normalization corresponds in the one-matrix case to consider potentials which are not even). In this way the double pole of smallest residue of $u(x)$ has residue 1. Since the partition function F is given by $F''(x) = -u(x)$, e^F has then simple zeros.

When one of the operators is given it can be shown that the other operator can be taken of the form:

$$P = Q_+^{p/q}$$

where the subscript $+$ means that P is the sum of the terms of non-negative power in the formal expansion of $Q^{p/q}$ for d "large". In [5] (see also [6]), it was shown that the coupled differential equations also follow from an action principle. The basic action for a critical (p, q) model takes the general form

$$S = \int \mathrm{d}x \left(\mathrm{Res}\, Q^{p/q+1} + xu\right), \tag{1.2}$$

where Res denotes the residue (coefficient of d^{-1}) of its fractional powers.

In the simple one-matrix case the "string equation" for the specific heat $u(x)$ [1–3] reduces to:

$$(l + \tfrac{1}{2})R_l[u] = x\,, \tag{1.3}$$

where the R_l's are the usual KdV potentials [7]. Due to the elementary properties of the R_l's, the above equation follows as the variational derivative with respect to u of the action

$$S = \int \mathrm{d}x \left(R_{l+1}[u] + xu \right) \,. \tag{1.4}$$

In the following sections, we shall combine these properties with a direct analysis of the differential equations satisfied by the partition functions of the $d < 1$ models to determine the large order behaviour of the topological expansion of their solutions. Previous work [8] has allowed to determine that the topological expansion of the specific heat had the general property of behaving like $a^k \Gamma(2k - 1/2)$ for k, the order in the topological expansion, large. The constant a was determined as the solution of an explicit algebraic equation for the one-matrix model ($q = 2$) and in two examples the critical and tricritical Ising model ($(3,4)$ and $(4,5)$ models). The importance of an explicit determination of a relies on the following property: If a is real and positive the perturbative expansion is not Borel summable and does not determine a unique function. Moreover there are good reasons to expect the corresponding model to be actually unstable. Such a result was obtained for half of the one-matrix models (this includes pure gravity), and is expected for all unitary $(q+1, q)$ models. This latter property is derived here and the more general models $p = (2m+1)q \pm 1$ are explicitly discussed*.

2. Large order behaviour of pure gravity

We first recall the derivation of the large behaviour of pure gravity, because it illustrates several features of the general analysis.

For pure gravity, the differential equation satisfied by $u(x)$ is

$$u^2(x) - \frac{1}{6} u''(x) = x \,. \tag{2.1}$$

If $u(x)$ has an asymptotic expansion for x large, it satisfies $u(x) = \pm\sqrt{x} + O\left(x^{-2}\right)$. The solution that corresponds to pure gravity has a x large expansion of the form

$$u(x) = x^{1/2} \Big(1 - \sum_{k=1} u_k \, x^{-5k/2} \Big) \,, \tag{2.2}$$

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* For a recent treatment of some standard features of divergent series, Borel summability, and summation methods, with physical applications, see, for example, pp. 840–842 of [9].

where the u_k are all positive.

To determine the large order behaviour of the expansion we first analyze the stability properties of the solution for x large. Let us set $u(x) \mapsto u(x)[1+\epsilon(x)]$ in eq. (2.1) and write the equation obtained by expressing that the term linear in ϵ vanishes:

$$\left(12u - \frac{u''}{u}\right)\epsilon - 2\frac{u'}{u}\epsilon' - \epsilon'' = 0\,. \tag{2.3}$$

One verifies that at leading order for x large only the leading order in eq. (2.2) is needed and u''/u is negligible. Eq. (2.3) can then easily be solved by the WKB method. We set

$$\epsilon'/\epsilon = r\sqrt{u} + bu'/u + O\left(u'^2/u^{5/2}\right),$$

and find $r = 2\sqrt{3}$ and $b = -5/4$. Replacing u by its asymptotic form $u \sim x^{1/2}$ and integrating we obtain:

$$\epsilon(x) \ \propto \ x^{-5/8}\,\mathrm{e}^{-\frac{8\sqrt{3}}{5}x^{5/4}}\,. \tag{2.4}$$

To leading order, the function ϵ is also proportional to the difference between any Borel sum of the series and the exact non-perturbative solution of the differential equation (up to even smaller exponential corrections corresponding to multi-instanton like effects). In terms of the expansion parameter (string loop coupling) $\kappa^2 = x^{-5/2}$, ϵ reads

$$\epsilon\big(x(\kappa)\big) \ \propto \ \kappa^{1/2}\,\mathrm{e}^{-\frac{8}{5}(\sqrt{3}/\kappa)}\,. \tag{2.5}$$

The above solution is valid for x large, i.e. κ small. The large order behaviour in (2.2) is then given by

$$u_k \underset{k\to\infty}{\propto} \int_0 \frac{\mathrm{d}\kappa}{\kappa^{2k+1}}\,\epsilon(\kappa) \ \propto \ \left(\frac{5}{8\sqrt{3}}\right)^{2k}\Gamma(2k-\tfrac{1}{2})\,. \tag{2.6}$$

(The constant of proportionality in the above cannot be determined by this method.) The asymptotic $\Gamma(2k-\frac{1}{2})$ behaviour is a slight refinement of the $(2k)!$ behaviour determined in [1,3,10].

The reality of r^2 has implied that all terms at large order have the same sign. This induces a singularity on the real positive axis in the Borel plane, obstruction to Borel summability.

In [11], it is confirmed that the exponential in (2.4) coincides with the action for a single eigenvalue climbing to the top of the barrier in the matrix model potential, allowing us to interpret the exponential piece of the solution to (2.1) as an instanton effect.

3. The general string equations

3.1. The general one-matrix problem

We now consider the string equation (1.3), $R_l[u] \propto x$. Substituting as before $u(x) \mapsto u(x)(1+\epsilon(x))$ we get a linear equation for ϵ. At leading order for x large we expect the equation to be again solved by the WKB ansatz $\epsilon'/\epsilon = ru^{1/2}$. It is then easy to verify that to obtain the leading large order behaviour of perturbation theory, it is only necessary to know the terms in $R_l[u]$ that contain at most one derivative of u factor. The next leading contribution is given by terms such as $u^{j-2}u^{(2l-2j-1)}u'$, i.e. with a single factor of u' as well.

At leading order only the terms in which the derivatives act on ϵ are relevant and thus ϵ satisfies an equation of the form

$$0 = \sum_{j=1}^{l} A_{lj}\, u^{j-1}\epsilon^{(2l-2j)} \ . \tag{3.1}$$

The WKB ansatz leads to an $(l-1)^{\mathrm{st}}$ order equation for the constant r^2

$$0 = A_l(r) \equiv \sum_{j=1}^{l} A_{lj}\, r^{2l-2j} \, . \tag{3.2}$$

From the properties of the R_l's one can derive an explicit expression for the polynomials $A_l(r)$

$$A_l(r) \propto \frac{1}{r}\left(r^2-8\right)_+^{l-1/2} = \frac{\Gamma(l+1/2)}{\Gamma(l)\Gamma(1/2)}\int_0^1 \frac{\mathrm{d}s}{\sqrt{s}}\left(r^2(1-s)-8\right)^{l-1} , \tag{3.3}$$

where the subscript $+$ again means the polynomial part of the large r expansion. The function $(z^2-1)_+^{l+1/2}$ is also proportional to $C_{2l+1}^{-l}(z)$ where C_{2l+1}^{ν} is a Gegenbauer polynomial defined by analytic continuation in ν [3]. Note that the number of zeros is exactly the same as the number of operators in a $(p = 2l-1, 2)$ minimal conformal model [12]. This is a property we shall meet again in the general case. Actually in the one-matrix case there is a natural explanation for it. The steepest descent analysis shows that the number of different instantons is related to the degree of the minimal potential corresponding to a critical point. This degree in turn is also related to the number of relevant perturbations *.

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* We thank F. David for this remark.

For l even, eq. (3.2) is an odd–order equation that will have at least one real solution for r^2, positive as is obvious from the integral representation (3.3). The series therefore cannot be Borel summable.

For l odd, on the other hand, the equation (3.2) for r^2 has no real solutions and therefore we expect the solution of the differential equation to be determined by the perturbative expansion.

Actually there exists a direct correspondence between the property of Borel summability and the existence of the original integral. It has been noted [13] that according to whether l is odd or even, the original minimal matrix integral is well-defined or not because the integrand goes to zero in the first case while in the latter case it blows up for M large. A direct calculation, using steepest descent, of the instanton action [8] confirms that when the potential is unbounded from below the instanton action is indeed real and the series therefore non-Borel summable.

The subleading terms in $R_l[u]$ are immediately deduced from the leading terms by noting that since $R_l[u]$ is derived from an action, eq. (1.4), the operator acting on ϵ is hermitian. Therefore the operator $u^{j-1}\mathrm{d}^{2l-2j}$ should be replaced by the symmetrized form $\frac{1}{2}\{u^{j-1}, \mathrm{d}^{2l-2j}\}$, correcting (3.1) to

$$0 = \sum_{j=1}^{l} A_{lj}\left(u^{j-1}\epsilon^{(2l-2j)} + \tfrac{1}{2}(2l-2j)(j+1)u^{j-2}\, u'\, \epsilon^{(2l-2j-1)}\right) . \tag{3.4}$$

To characterize more precisely the large order behaviour, to next order we set $\epsilon'/\epsilon = ru^{1/2} + bu'/u$, from which it follows, to the same order, that

$$\frac{\epsilon^{(k)}}{\epsilon} = r^k u^{k/2} + r^{k-1} u^{(k-3)/2}\, u'\, k\Big(b + \tfrac{1}{4}(k-1)\Big) . \tag{3.5}$$

Substituting into (3.4), we find $b = -(2l+1)/4$, independent of r. Then

$$\epsilon(x) \;\propto\; x^{-(2l+1)/4l}\, \mathrm{e}^{-\frac{2l}{2l+1} r x^{(2l+1)/2l}} , \tag{3.6}$$

generalizing (2.4). In terms of the expansion parameter $\kappa = x^{-(2l+1)/2l}$, we find

$$u_k \underset{k\to\infty}{\propto} \int_0 \frac{\mathrm{d}\kappa}{\kappa^{2k+1}} \epsilon\big(x(\kappa)\big) \;\propto\; \left(\frac{2l+1}{2lr}\right)^{2k} \Gamma(2k - \tfrac{1}{2}) . \tag{3.7}$$

The $\Gamma(2k - \frac{1}{2})$ factor in (2.6) is general, and is related to the property that the original equations descended from an action principle.

3.2. The general (p, q) model

In the case of the general (p, q) model (eq. (1.2)) there results a system of coupled linear differential equations for the variations $\epsilon_{u_i}(x)$ associated with the functions $u_i(x)$. At leading order we set $\epsilon'_{u_i}/\epsilon_{u_i} = ru^{1/2}$. We obtain, taking into account the leading relations between the u_i, a linear system for each of the ϵ_{u_i}'s multiplied by a power of u determined by the grading. Imposing again the vanishing of the determinant of the linear system gives an equation for the coefficient r (and to leading order all functions ϵ_{u_i} are thus proportional up to a power of u determined by the grading).

To determine more precisely the behaviour of $\epsilon_u = \epsilon_{u_1}$ we have to consider subleading terms. As in the one-matrix case they can be determined by a hermiticity argument. Since the equations for u_i derive from an action (1.2), the linear equations for ϵ_{u_i} define a hermitian operator. This property leads to a universal $\Gamma(k - \frac{1}{2})$ behaviour for all the (p, q) models.

We recall finally that the $(2k)!$ large order behaviour is also the generic behaviour [14] for $d = 1$ models [14,15].

4. The $p = (2m + 1)q \pm 1$ models in the spherical limit

In the analysis of the large behaviour the knowledge of the solutions of the string equation in the spherical limit was required. Actually we shall prove in next section that the knowledge of the differential operators P, Q in the same limit is sufficient. For $q = 2$ and $q = 3$ the form of the operator Q and P is known. In the general case $q \geq 4$ the explicit functional form of the operator Q in the spherical limit depends on the (p, q) models. In particular in the spherical or semiclassical limit the operator Q takes the form

$$Q(\mathrm{d}, u) = \sum_{i=0} q_i u^i(x) \mathrm{d}^{q-2i}, \tag{4.1}$$

but the coefficients q_i in equation (4.1) are in general p-dependent. Note that, in this limit, the order between the operators $u(x)$ and d is irrelevant.

However in [16] it has been shown, using the string actions (1.2) [5] in the semiclassical limit, that when $p = (2m + 1)q \pm 1$, the semiclassical form of the operator Q is m-independent and P and Q can be determined explicitly. This property can be recovered by a direct method. If we also set:

$$P(\mathrm{d}, u) = \sum_{i=0} p_i u^i(x) \mathrm{d}^{p-2i}, \tag{4.2}$$

we obtain the semiclassical limit of the equation $[P, Q] = 1$:

$$u' \left(\frac{\partial P}{\partial \mathrm{d}} \frac{\partial Q}{\partial u} - \frac{\partial Q}{\partial \mathrm{d}} \frac{\partial P}{\partial u} \right) = 1 \,. \tag{4.3}$$

We now use the homogeneity property of P, Q:

$$P(\mathrm{d}, u) = u^{p/2} P \left(\mathrm{d}u^{-1/2}, 1 \right), \qquad Q(\mathrm{d}, u) = u^{q/2} Q \left(\mathrm{d}u^{-1/2}, 1 \right).$$

From now on we call $P(z)$, $Q(z)$ the two polynomials $P(z = \mathrm{d}u^{-1/2}, 1)$, $Q(z = \mathrm{d}u^{-1/2}, 1)$. They thus satisfy the differential equation:

$$qP'(z)Q(z) - pP(z)Q'(z) = 2pq \,, \tag{4.4}$$

while as expected the equation for $u(x)$ yields $u^{(p+q-1)/2} \propto x$. When one of the polynomials is known the other is obtained by integrating the equation. In the special case $p = (2m+1)q \pm 1$ the polynomials $Q(z)$ are Tchebychev's polynomials. Setting $z = 2\cos\theta$ one finds that $Q(z) = 2\cos q\theta$ and

$$P = 2pQ^{p/q}(z) \int_0^z Q^{-1-p/q}(t)\mathrm{d}t \propto 2 \sum_{l=0}^{m} \binom{p/q}{l} \cos(p - 2lq)\theta \,,$$

satisfy the equation.

5. Instantons: A more direct method

5.1. Instantons in the one-matrix model revisited

Before discussing the general unitary models let us return to the one-matrix model for which the result is exactly known. From the analysis of the corresponding non-linear differential equations we have learned that if we call ϵ the variation of the specific heat $u(x)$ then it has for x large the asymptotic form:

$$\epsilon'/\epsilon \sim r\sqrt{u}, \tag{5.1}$$

where r is constant which is determined by an algebraic equation. Since the variation of u can be neglected at leading order, we can rescale d, i.e. set u to 1. Equation (5.1) can then be written as a commutation relation

$$\mathrm{d}\epsilon = \epsilon(\mathrm{d} + r) \;\Rightarrow\; f(\mathrm{d})\epsilon = \epsilon f(\mathrm{d} + r). \tag{5.2}$$

Then the operators P, Q are simply

$$Q = \mathrm{d}^2 - 2\,, \qquad P \equiv P_{2l+1}(\mathrm{d}) = \left(d^2 - 2\right)_+^{l+1/2}.$$

The equation for ϵ is obtained by expanding at first order in ϵ the commutation relation $[P, Q] = 1$. Setting:

$$\delta P = \{\epsilon, R(\mathrm{d})\} \equiv \sum_{k=0} R_k \{\epsilon, d^{2l-1-2k}\},$$

we find:

$$[\{\epsilon, R(\mathrm{d})\}, \mathrm{d}^2 - 2] + [P, -2\epsilon] = 0\,.$$

Using the commutation relation (5.2) to commute ϵ to the left we find the equation:

$$-\left(2r\mathrm{d} + r^2\right)\left(R(\mathrm{d}) + R(\mathrm{d}+r)\right) - 2\big(P_{2l+1}(\mathrm{d}+r) - P_{2l+1}(\mathrm{d})\big) = 0\,.$$

The first term vanishes for $\mathrm{d} = -r/2$, which must thus be a zero of the second term. Taking into account the parity of P_{2l+1} we obtain

$$P_{2l+1}(r/2) = 0\,, \tag{5.3}$$

in agreement with the direct calculation. The polynomial $R(\mathrm{d})$ is then determined by division.

5.2. General (p, q) problem

In the general (p, q) case, in the same classical limit, and after the same rescaling we have:

$$Q = Q(\mathrm{d}), \quad P = P(\mathrm{d}) = Q_+^{p/q}(\mathrm{d}), \quad \delta Q = \{S(\mathrm{d}), \epsilon\}, \quad \delta P = \{R(\mathrm{d}), \epsilon\},$$

where P, Q are polynomials of degrees p, q respectively, and R, S of degrees $p-2$, $q-2$ and same parity as P, Q.

The equation for ϵ then leads to

$$\begin{aligned} &[P, \delta Q] + [\delta P, Q] = 0 \\ \Leftrightarrow\ &\big(P(\mathrm{d}+r) - P(\mathrm{d})\big)\big(S(\mathrm{d}) + S(\mathrm{d}+r)\big) \\ &-\big(Q(\mathrm{d}+r) - Q(\mathrm{d})\big)\big(R(\mathrm{d}) + R(\mathrm{d}+r)\big) = 0\,. \end{aligned}$$

The polynomials $P(\mathrm{d}+r)-P(\mathrm{d})$ has a degree $p-1$ in d, while R has only a degree $p-2$. An equivalent property is true for Q,S. The polynomials $P(\mathrm{d}+r)-P(\mathrm{d})$ and $Q(\mathrm{d}+r)-Q(\mathrm{d})$ must thus have at least one common root. Note that the first polynomial has $p-1$ roots and the second $q-1$. Moreover this roots are symmetric in the exchange $\mathrm{d}\mapsto -r-\mathrm{d}$. Therefore expressing the existence of a common root leads to $(p-1)(q-1)$ values of r, up to the symmetry. Note that the number of zeros is again exactly the same as the number of relevant operators in a (p,q) minimal conformal model [12] of gravitationally dressed weights

$$d_{m,n}=\frac{p+q-|pn-qm|}{p+q-1}\ , \tag{5.4}$$

with $1\leq n\leq q-1$, $1\leq m\leq p-1$ with the symmetry $d_{m,n}=d_{q-n,p-m}$. The explanation of this relation is probably again that the number of different instanton actions is related to the degree of the minimal potential needed to generate a critical point in the multi-matrix model, and thus to the number of different relevant operators. Also we note that we are studying a general deformation of a critical solution and therefore the appearance in some form of the relevant operators should be expected.

This condition determines the possible values of r when the polynomials P and Q, i.e. the differential operators are known in the classical limit. Examples are provided by the models $p=(2m+1)q\pm 1$ where integral representations for these polynomials have been found. The simplest examples are provided by the $q+1,q$ models, i.e. the unitary models which we examine below.

Let us finally verify that we can then indeed find the polynomials R,S. We call α the common root and assume first that $\alpha\neq -r/2$. Then the parity properties imply that $-r-\alpha$ is also a common root. Setting then

$$\begin{aligned}\big(P(\mathrm{d}+r)-P(\mathrm{d})\big)&=(\mathrm{d}-\alpha)(\mathrm{d}+r+\alpha)\tilde{P}(\mathrm{d}),\\ \big(Q(\mathrm{d}+r)-Q(\mathrm{d})\big)&=(\mathrm{d}-\alpha)(\mathrm{d}+r+\alpha)\tilde{Q}(\mathrm{d}),\end{aligned}$$

we find that R and S are solutions of:

$$R(\mathrm{d})+R(\mathrm{d}+r)=(\mathrm{d}+r/2)\tilde{P}(\mathrm{d}),\quad S(\mathrm{d})+S(\mathrm{d}+r)=(\mathrm{d}+r/2)\tilde{Q}(\mathrm{d}).$$

Note that these equations satisfy both the degree and parity requirement.

If $\alpha=-r/2$ the situation is even simpler

$$\begin{aligned}R(\mathrm{d})+R(\mathrm{d}+r)&=\big(P(\mathrm{d}+r)-P(\mathrm{d})\big)/(\mathrm{d}+r/2)\\ S(\mathrm{d})+S(\mathrm{d}+r)&=\big(Q(\mathrm{d}+r)-Q(\mathrm{d})\big)/(\mathrm{d}+r/2).\end{aligned}$$

5.3. The unitary models

We have shown that the differential operators P, Q may be written in the classical limit as:

$$P = 2T_p(\mathrm{d}/2)\,, \qquad Q = 2T_q(\mathrm{d}/2),$$

where T_p is the p-th Tchebychev's polynomial:

$$T_p(\cos\varphi) = \cos p\varphi.$$

As explained above, taking into account the degrees of the polynomials R and S, we conclude that the polynomials $T_q((r+\mathrm{d})/2) - T_q(\mathrm{d}/2)$ and $T_p((r+\mathrm{d})/2) - T_p(\mathrm{d}/2)$ must have a common root $\alpha = 2\cos\varphi_0$. Let also set $\alpha + r = 2\cos\psi_0$. We have

$$\cos p\psi_0 = \cos p\varphi_0 \quad \text{and} \quad \cos q\psi_0 = \cos q\varphi_0.$$

The solution is :

$$\psi_0 = \pm\varphi_0 + \frac{2m\pi}{p} = \mp\varphi_0 + \frac{2n\pi}{q}$$

Since $r = 2\cos\psi_0 - 2\cos\varphi_0$, excluding the solutions $r = 0$ which is not acceptable, we have the different solutions:

$$r = \pm 4\sin m\pi/p \sin n\pi/q, \quad 0 < 2m \le p\,, \quad 0 < 2n \le q\,.$$

It is easy to verify that these results agree with the explicit solutions of the $(2,3)$, $(4,3)$ and $(4,5)$ models. They show that, as expected, all unitary models lead to non-Borel summable topological expansions because all terms of the series have the same sign. These models thus suffer from the same disease as the pure gravity model. Note finally that indeed the number of different values of r is the same as the number of operators in the minimal (p,q) conformal model.

5.4. The $p = (2m+1)q \pm 1$ models

For $m \neq 0$ r is solution of more complicated algebraic equations. In the notation of previous subsection we still have:

$$\psi_0 = \pm\varphi_0 + \frac{2n\pi}{q}.$$

Let us set

$$\alpha = \tfrac{1}{2}(\psi_0 + \varphi_0), \qquad \beta = \tfrac{1}{2}(\psi_0 - \varphi_0),$$

then, making a choice of signs

$$\beta = \frac{n\pi}{q}, \qquad r = 4\sin\alpha\sin\beta = 4\sin\alpha\sin(n\pi/q),$$

where

$$\sum_{l=0}^{m} \binom{p/q}{l} \sin[(p-2ql)\alpha]\sin[(p-2ql)\beta] = 0\,.$$

We note that $\sin[(p-2ql)\beta] = \sin(n\pi p/q)$ which can be factorized. We thus find an equation for α:

$$A(\alpha) \equiv \sum_{l=0}^{m} \binom{p/q}{l} \sin[(p-2ql)\alpha] = 0\,.$$

This function satisfies the differential equation

$$pA(\alpha)(\cos q\alpha)' - qA'(\alpha)\cos q\alpha = K(p,q)\cos\alpha\,,$$

where K is a constant. This equation implies that $A(\pi/2q)A(3\pi/2q) \leq 0$ and thus $A(\alpha)$ vanishes at least once in the interval $(0,\pi)$. We conclude that for all these models the topological expansion is not Borel summable.

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