# Renormalization theory for interacting crumpled manifolds 

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#### Abstract

We consider a continuous model of $D$-dimensional elastic (polymerized) manifold fluctuating in $d$-dimensional euclidean space, interacting with a single impurity via an attractive or repulsive $\delta$-potential (but without self-avoidance interactions). Except for $D=1$ (the polymer case), this model cannot be mapped onto a local field theory. We show that the use of intrinsic distance geometry allows for a rigorous construction of the high-temperature perturbative expansion and for analytic continuation in the manifold dimension $D$. We study the renormalization properties of the model for $0<D<2$, and show that for bulk space dimension $d$ smaller that the upper critical dimension $d^{\star}=2 D /(2-D)$, the perturbative expansion is ultraviolet finite, while ultraviolet divergences occur as poles at $d=d^{\star}$. The standard proof of perturbative renormalizability for local field theories (the Bogoliubov-Parasiuk-Hepp theorem) does not apply to this model. We prove perturbative renormalizability to all orders by constructing a subtraction operator $\mathbf{R}$ based on a generalization of the Zimmermann forests formalism, and which makes the theory finite at $d=d^{\star}$. This subtraction operation corresponds to a renormalization of the coupling constant of the model (strength of the interaction with the impurity). The existence of a Wilson function, of an $\epsilon$-expansion à la Wilson-Fisher around the critical dimension, of scaling laws for $d<d^{\star}$ in the repulsive case, and of non-trivial critical exponents of the delocalization transition for $d>d^{\star}$ in the attractive case, is thus established. To our knowledge, this study provides the first proof of renormalizability for a model of extended objects, and should be applicable to the study of self-avoidance interactions for random manifolds.


## 1. Introduction

One general problem arising in statistical physics is the understanding of the effect of interactions on the thermodynamical properties of extended fluctuating geometrical objects. These objects may be (one-dimensional) lines, like long linear macromolecules or polymers, (two-dimensional) surfaces, like membranes or interfaces, or even (three-dimensional) volumes, like gels. The interactions involve in general two-body attractive or repulsive forces, and one may in general reduce such problems into two different classes: (i) either one deals with self-interactions between distinct points of the same fluctuating object, or mutual

[^0]interactions between several fluctuating objects; (ii) or one deals with the interaction of a single freely fluctuating object with another non-fluctuating fixed object. Case (i) includes for instance self-avoiding polymers or membranes, polyelectrolytes and charged gels, as well as the description of intersections of random walks. Case (ii) includes the problems of binding/unbinding of a long molecule or a membrane on a wall, the wetting of an interface. One can also reduce to this class the problems of unbinding of two membranes or interfaces, and that of the steric repulsions between membranes in a lamellar phase.

Among the many different generic situations one can think of, one case is now well understood, namely that where the fluctuating objects are only onedimensional objects. Indeed, many problems in case (ii) can then be solved by simple analogy with quantum mechanics, i.e. by use of a diffusion equation. The situation is more complicated in case (i), a paradigm of which is the celebrated problem of self-avoiding polymers. Still in this case, the use of perturbative expansions and renormalization group techniques allows for explicit results on the thermodynamics of these objects. For instance, a self-avoiding polymer embedded in a $d$-dimensional external space can be described by the continuous Edwards hamiltonian [1,2]

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \int_{0}^{S} \mathrm{~d} s\left(\frac{\mathrm{~d} \boldsymbol{r}}{\mathrm{~d} s} \cdot \frac{\mathrm{~d} \boldsymbol{r}}{\mathrm{~d} s}\right)+\frac{b}{2} \int_{0}^{S} \mathrm{~d} s \int_{0}^{S} \mathrm{~d} s^{\prime} \delta^{d}\left(\boldsymbol{r}(s)-\boldsymbol{r}\left(s^{\prime}\right)\right) . \tag{1.1}
\end{equation*}
$$

This model can then be viewed as a one-dimensional field theory, with position field $\boldsymbol{r}(s)$ at abscissa $s$ along the chain of size $S$, and with a non-local interaction term. This field theory then has a formal perturbative expansion in $b$ : this point of view dates back the work of Fixman [3] and has been developed by des Cloizeaux [2,4]. The terms of this expansion are in general integrals over the internal coordinates $s$ of the interaction points and may diverge when these interaction points come close to each other $\left(\left|s-s^{\prime}\right| \rightarrow 0\right)$. The theory can then be regularized by analytic continuation in $d \geqslant 2$, and the natural expansion parameter is then $b S^{2-d / 2}$, hence large in the thermodynamic limit $S \rightarrow \infty$ for $d<4$. For dimensional reasons, the corresponding long-distance divergences are twinned with the short-distance divergences, and appear as poles in $d$ at $d=4$. Within a double expansion in $b$ and $\epsilon=4-d$, the structure of these poles is such that the theory is renormalizable for $\epsilon \geqslant 0$. This means that the poles at $\epsilon=0$ can actually be absorbed into redefinitions of the parameters of the model, and that a scaling limit is obtained for the thermodynamical properties of the polymer when $\epsilon \geqslant 0$. Still, a rigorous proof of renormalizability requires the use of the famous equivalence of the Edwards model with the $\mathrm{O}(n)$ model for $n=0$, that is a model with a $n$-component field $\boldsymbol{\Phi}(\boldsymbol{r})$ in the $d$-dimensional external space, as shown by de Gennes [5]. From this different point of view, which was the first to be developed in the 70's, the self-avoiding polymer problem is seen as a $d$-dimensional local field theory, that is a theory with local interactions, and
amenable to the standard renormalization group treatments for critical phenomena [6,7]. Again, this field theory can be studied via a perturbative expansion, the terms of which may diverge when two external interaction points $r$ and $r^{\prime}$ come close to each other $\left(\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right| \rightarrow 0\right)$. Now the general renormalization scheme for local field theories applies and ensures (perturbative) renormalizability, from which one deduces a posteriori the renormalizability of the direct approach "à la des Cloizeaux" [8-10]. This equivalence with a local field theory also holds for one-dimensional problems in case (ii), and methods of perturbative field theory can also be applied in this case. Although they are in general more complicated than the simple diffusion equation, they give comparable results (see ref. [11]).

Beside the perturbative framework, one should notice that rigorous non-perturbative results have been obtained for the Edwards model and related models: the mathematical construction of the measure on random paths associated with (1.1) [12]; the large distance behavior of intersection properties of independent random walks at $d=4$ [13]; the large distance behavior of weakly self-avoiding polymers at $d=4$ in constructive field theory [14]. These non-perturbative studies always corroborate the results of the perturbative renormalization group analysis.

The existence of an underlying local field theory in the external $d$-dimensional space, which is crucial to ensure renormalizability and allows for predictions from the perturbative expansion, is however directly related to the one-dimensional nature of the object. When we now consider a $D$-dimensional object with $D \neq 1$, embedded in $d$ dimensions, no such equivalence with a $d$-dimensional local field theory exists. Still, the approach "à la des Cloizeaux" can be generalized, by considering a $D$-dimensional field theory. For instance, the Edwards hamiltonian reads for a $D$-dimensional manifold with internal coordinate $x$ [15-17]

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \int \mathrm{~d}^{D} x\left(\nabla_{x} \boldsymbol{r} \cdot \nabla_{x} \boldsymbol{r}\right)+\frac{b}{2} \int \mathrm{~d}^{D} x \int \mathrm{~d}^{D} x^{\prime} \delta^{d}\left(\boldsymbol{r}(x)-\boldsymbol{r}\left(x^{\prime}\right)\right) \tag{1.2}
\end{equation*}
$$

This describes a polymerized or "tethered" manifold with a fixed internal metric (to be distinguished from the case of fluid membranes, with a fluctuating metric). The self-avoidance interaction term leads to a perturbative expansion in $b$, with poles in $\epsilon=4 D-d(2-D)$. This method has been used to first order in $\epsilon[16,17]$, and leads to first-order estimates of critical exponents [16-19], assuming that renormalizability holds and that a renormalization group equation can thus be used.

Two crucial questions remain however open, which show that new mathematical developments are required:
(I) A perturbative approach cannot be performed directly at $D$ larger or equal to 2 . Indeed, for $D \geqslant 2$ (and $d \geqslant 0$ ), $\epsilon$ is never small $(\epsilon \geqslant 8)$. The double expansion in $b$ and $\epsilon$ requires to consider the case of real non-integer $D$ (typically
$1 \leqslant D<2$ ). The term of order $N$ in the perturbative expansion being an integral over $2 N$ (resp. $N$ ) interaction points in case (i) (resp. case (ii)) in internal $D$ dimensional space, the meaning of these integrations for non integer $D$ has to be defined.
(II) Since, as a $D$-dimensional field theory, the theory is either non local (case (i)) or local (case (ii)) but with a singular potential with explodes at the origin $\boldsymbol{r}=0$ (typically $1 /|\boldsymbol{r}|^{\gamma}$ or $\delta^{d}(\boldsymbol{r})$ ), standard methods of local field theory do not apply. Since furthermore, as mentioned above, we cannot rely (as for $D=1$ ) on an equivalence with a $d$-dimensional local field theory, the question arises of the actual renormalizability of the theory, and in particular of the validity of the use of a (for instance first order) renormalization group equation to predict a scaling behavior.

Beyond the one-loop calculations of refs. [16-19] for the model of self-avoiding random manifold, which assume renormalizability, a next step in a general analysis of the problem of renormalization for interacting extended object with dimensionality $D \neq 1$ has been performed by one of the present authors in ref. [20]. In ref. [20] a model describing the simple avoidance interaction of a $D$ dimensional fluctuating manifold with a fixed euclidean element was considered. The leading UV divergences of the model were analyzed in perturbation theory and resummed, so that the consistency of a renormalization group equation at one loop was established for this model. A similar direct approach has been applied to the Edwards manifold model (1.2), and the one-loop renormalizability established [21].

The purpose of this paper is to present a general, mathematically rigorous, framework to study these questions, and to analyze the renormalizability of models of interacting objects to all orders in perturbation theory. In this paper, we shall discuss the simple model of ref. [20], of a $D$-dimensional fluctuating manifold interacting with a single fixed point (or more generally a fixed euclidean element), defined by the following hamiltonian:

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \int \mathrm{~d}^{D} x\left(\nabla_{x} \boldsymbol{r}(x) \cdot \nabla_{x} \boldsymbol{r}(x)\right)+b \int \mathrm{~d}^{D} x \delta^{d}(\boldsymbol{r}(x)) \tag{1.3}
\end{equation*}
$$

We prove perturbative renormalizability for this model, to all orders in perturbation theory, from the internal-space formulation of ref. [20]. For that purpose we rely on methods devised in perturbative field theory, in particular by Bergère and Lam, for renormalizing the Feynman amplitudes in the so-called $\alpha$-parameter or Schwinger representation. Indeed, our construction can be seen as a generalization of renormalization theory in Schwinger representation to the case of a $D$-dimensional $\alpha$-parameter space.

This paper is organized as follows.
In sect. 2 we present the model of a $D$-dimensional manifold interacting with a single fixed point, discuss its physical relevance for the problem of entropic
as long as the integration over the squared distances $a_{i j}$ is given by a measure, while in sect. 9 we show that this remains true in the general case where the measure term is a distribution. This ends the proof of the renormalizability of the model. The rest of sect. 9 is devoted to some physical consequences of this renormalizability property, such as the existence of a Wilson-Fisher $\epsilon$-expansion and of universal scaling behaviors.

In sect. 10 we summarize our work and discuss various prospects, in particular for the problem of self-avoiding random manifolds.

A lot of technical points are relegated into various appendices.
The reader not interested in the details of the proof of renormalizability may skip (at least in a first reading. . .) sections 5,7 and 8 .

## 2. The model

### 2.1. THE ACTION

We first define the model that we shall study and the formal structure of its perturbative expansion, without taking care of the possible infinities which may arise from short and/or large distance divergences. It is the purpose of next sections (in particular sect. 4) to define proper regularization schemes.

We start with the manifold hamiltonian [20]

$$
\begin{equation*}
\mathcal{H}=\int_{\mathcal{V}} \mathrm{d}^{D} x\left[\frac{1}{2} \boldsymbol{r}(x) \cdot(-\Delta)^{k / 2} \boldsymbol{r}(x)+b \delta^{d}(\boldsymbol{r}(x))\right] \tag{2.1}
\end{equation*}
$$

where $x$ labels the internal position in the $D$-dimensional manifold with volume $\mathcal{V}$ and $\boldsymbol{r}(x)$ is the corresponding position in the $d$-dimensional euclidean space. For the physical case $k=2$, the first term in (2.1) corresponds to the elastic energy of the gaussian manifold (the internal tension is set to unity). For reasons of mathematical convenience, which will be clear in the following, we shall consider in full generality the more general class of elastic hamiltonians with $k \geqslant 2$. This allows in particular to define in a proper way a consistent analytic continuation in the internal dimension $D$. The case $k=4$ corresponds to a manifold with vanishing tension but with bending rigidity. The absence from eq. (2.1) of a two-point self-avoidance interaction term (as compared to eq. (1.2)) means that we are dealing with a "phantom" manifold which can intersect itself freely. The second term in (2.1) corresponds to the interaction of the manifold with a fixed impurity, that is a single point in the external $d$-dimensional space, here at the origin $\boldsymbol{r}=\mathbf{0}$ (fig. 1). The coupling constant $b$ may be either positive (repulsive interaction) or negative (attractive interaction).

As mentioned in sect. 1, this model is interesting as a toy model for the more complex problem of self-avoiding manifolds. In both cases the interaction term is a singular $\delta$-function, and similar mathematical techniques can be used to

critical internal dimension, given by

$$
\begin{equation*}
D^{\star}=k \frac{d}{d+2} \tag{2.3}
\end{equation*}
$$

or equivalently if $d<d^{\star}$, where $d^{\star}$ is the critical embedding dimension

$$
\begin{equation*}
d^{\star}=\frac{2 D}{k-D} \tag{2.4}
\end{equation*}
$$

simply equal to the fractal dimension of the manifold. In particular, this model possesses an upper critical dimension $0<d^{\star}<\infty$ for a "membrane" dimension $0<D<k$. For the standard interface model ( $k=2$ ), we recover the conditions $0<D<2[16,17,20]$. The exponent $\nu$ plays the role of the size exponent of the elastic manifold. For fluctuating interfaces, that is "directed manifolds", it is also called in the literature the wandering exponent, and denoted by $\zeta$ [26]. $\nu$ has its natural range between 0 (collapsed manifold) and 1 (stretched manifold). This corresponds exactly to

$$
\begin{equation*}
k-2 \leqslant D \leqslant k \tag{2.5}
\end{equation*}
$$

or equivalently to the "physical" conditions $D \leqslant d^{\star} \leqslant \infty$.
In ref. [20] a dimensionless effective coupling constant $g$ was introduced which measures the effective strength of the interaction as a function of the length scale $X$ measuring the linear internal extent of the manifold, defined by $\mathcal{V}=X^{D}$. In the vicinity of the critical dimension ( $\epsilon \simeq 0$ ), and for the physical case $k=2$, a one-loop calculation [20] shows that this effective coupling constant obeys a renormalization group ( RG ) flow equation, which reads

$$
\begin{equation*}
X \frac{\partial g}{\partial X}=W(g)=\epsilon g-\frac{1}{2} S_{D} g^{2}+\mathcal{O}\left(g^{3}\right) \tag{2.6}
\end{equation*}
$$

with $S_{D}=2 \pi^{D / 2} / \Gamma(D / 2)$ the volume of the unit sphere in $\mathbb{R}^{D}$. Apart from the trivial $g=0$ solution, this flow equation has a fixed point solution at the non-trivial zero of the Wilson function $W(g)$

$$
\begin{equation*}
g^{\star}=\frac{2 \epsilon}{S_{D}}+\mathcal{O}\left(\epsilon^{2}\right) \tag{2.7}
\end{equation*}
$$

For large negative $g, W(g)$ behaves like

$$
\begin{equation*}
W(g) \simeq D g \log (-g) \tag{2.8}
\end{equation*}
$$

The physical consequences of these equations are the following:
(I) $\epsilon>0$ : This corresponds to $D>D^{\star}$ or $d<d^{\star}$. The RG flow has an infrared (IR) stable fixed point at $g^{\star}>0$ and an IR unstable (ultraviolet (UV) stable) fixed point at $g=0$, as depicted in fig. 2. For arbitrarily small negative $b$ (attractive interaction), $g$ is negative and flows to $(-\infty)$ at large length scale $X$; the manifold is localized (or pinned) at the origin $r=0$, and its average
( $D=1$ ), exact solutions corroborate this picture. Finally, let us mention the exact treatment of the renormalization group flow for small $b(b \simeq 0)$ for the problem of interface pinning of ref. [27]. This corresponds to the case $D=2$, $d=1$ and $k=2(\epsilon=2)$.

### 2.2. THE PARTITION FUNCTION

The partition function $\mathcal{Z}$ for the model is defined by

$$
\begin{equation*}
\mathcal{Z}=\int \mathcal{D}[\boldsymbol{r}] \exp (-\mathcal{H}) \tag{2.9}
\end{equation*}
$$

Its perturbative expansion in the coupling constant $b$ is

$$
\begin{equation*}
\mathcal{Z}=\sum_{N=0}^{\infty} \frac{(-b)^{N}}{N!} \mathcal{Z}_{N}, \tag{2.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{Z}_{N}=\left\langle\int_{V} \prod_{i=1}^{N} \mathrm{~d}^{D} x_{i} \delta^{d}\left(\boldsymbol{r}\left(x_{i}\right)\right)\right\rangle_{0} \tag{2.11}
\end{equation*}
$$

and $\langle\ldots\rangle_{0}$ is the average with respect to the gaussian measure

$$
\exp \left[-\int_{\mathcal{V}} \mathrm{d}^{D} x \frac{1}{2} r \cdot(-\Delta)^{k / 2} r\right]
$$

The evaluation of $\mathcal{Z}_{N}$ is best performed in Fourier space by introducing the vertex function

$$
\begin{equation*}
V(x, \boldsymbol{k})=\exp (i \boldsymbol{k} \cdot \boldsymbol{r}(x)), \tag{2.12}
\end{equation*}
$$

with $\boldsymbol{k}$ a $d$-dimensional vector, and by writing $\mathcal{Z}_{N}$ as

$$
\begin{equation*}
\mathcal{Z}_{N}=\left\langle\prod_{i=1}^{N} \int_{\mathcal{V}} \mathrm{d}^{D} x_{i} \int \frac{\mathrm{~d}^{d} \boldsymbol{k}_{i}}{(2 \pi)^{d}} V\left(x_{i}, \boldsymbol{k}_{i}\right)\right\rangle_{0} \tag{2.13}
\end{equation*}
$$

We compute the above functional average by taking care of the overall displacement of the manifold (zero-mode):

$$
\begin{equation*}
\boldsymbol{r}_{\mathrm{G}}=\frac{1}{\mathcal{V}} \int_{\mathcal{V}} \mathrm{d}^{D} x \boldsymbol{r}(x) \tag{2.14}
\end{equation*}
$$

We have explicitly

$$
\begin{align*}
\left\langle\prod_{i=1}^{N} V\left(x_{i}, \boldsymbol{k}_{i}\right)\right\rangle_{0} & =\int \mathrm{d}^{d} \boldsymbol{r}_{0} \int \mathcal{D}[\boldsymbol{r}(x)] \delta^{d}\left(\boldsymbol{r}_{\mathrm{G}}-\boldsymbol{r}_{0}\right) \\
& \times \exp \left[-\int_{\mathcal{V}} \mathrm{d}^{D} x \frac{1}{2} \boldsymbol{r} \cdot(-\Delta)^{k / 2} \boldsymbol{r}+i \sum_{i=1}^{N} \boldsymbol{k}_{i} \cdot \boldsymbol{r}\left(x_{i}\right)\right] \tag{2.15}
\end{align*}
$$

Performing the shift $\boldsymbol{r}=\boldsymbol{r}_{\mathrm{G}}+\tilde{r}$, we get

$$
\begin{equation*}
\int \mathrm{d}^{d} \boldsymbol{r}_{0} \int \mathcal{D}[\tilde{\boldsymbol{r}}(x)] \delta^{d}\left(\tilde{r}_{\mathrm{G}}\right) \exp \left[-\int_{\mathcal{V}} \mathrm{d}^{D} x \frac{1}{2} \tilde{\boldsymbol{r}} \cdot(-\Delta)^{k / 2} \tilde{\boldsymbol{r}}+i \sum_{i=1}^{N} \boldsymbol{k}_{i} \cdot\left(\tilde{\boldsymbol{r}}\left(x_{i}\right)+\boldsymbol{r}_{0}\right)\right] \tag{2.16}
\end{equation*}
$$

Integrating over the displacement $\boldsymbol{r}_{0}$, and performing the gaussian average, with normalization

$$
\begin{equation*}
\int \mathcal{D}[\tilde{\boldsymbol{r}}(x)] \delta^{d}\left(\tilde{\boldsymbol{r}}_{\mathrm{G}}\right) \exp \left[-\frac{1}{2} \int_{\mathcal{V}} \mathrm{d}^{D} x \tilde{\boldsymbol{r}} \cdot(-\Delta)^{k / 2} \tilde{\boldsymbol{r}}\right]=1 \tag{2.17}
\end{equation*}
$$

we finally get

$$
\begin{equation*}
\mathcal{Z}_{N}=\int \prod_{i=1}^{N} \frac{\mathrm{~d}^{D} x_{i} \mathrm{~d}^{d} \boldsymbol{k}_{i}}{(2 \pi)^{d}}(2 \pi)^{d} \delta^{d}\left(\sum_{i=1}^{N} \boldsymbol{k}_{i}\right) \exp \left[-\frac{1}{2} \sum_{i, j=1}^{N} \boldsymbol{k}_{i} \cdot \boldsymbol{k}_{j} G\left(x_{i}, x_{j}\right)\right], \tag{2.18}
\end{equation*}
$$

where $G(x, y)$ is the propagator, solution (in infinite flat $D$-dimensional space) of

$$
\begin{equation*}
\left(-\Delta_{x}\right)^{k / 2} G(x, y)=\delta^{D}(x-y), \tag{2.19}
\end{equation*}
$$

namely

$$
\begin{equation*}
G(x, y)=\frac{1}{2^{k} \pi^{D / 2}} \frac{\Gamma((D-k) / 2)}{\Gamma(k / 2)}|x-y|^{k-D} . \tag{2.20}
\end{equation*}
$$

This propagator, which is a Coulomb-like potential, will play a fundamental role in what follows. In the range of parameters (2.5), it vanishes at $|x-y|=0$.

The first term of the expansion of $\mathcal{Z}(N=0)$ is simply the (infinite) volume of external space

$$
\begin{equation*}
\mathcal{Z}_{0}=(2 \pi)^{d} \delta^{d}(\boldsymbol{k}=0)=\int \mathrm{d}^{d} r_{\mathrm{G}} \equiv V_{\mathbb{R}^{d}} \tag{2.21}
\end{equation*}
$$

But the next terms are finite. Indeed, for $N>0$ we can deal with the $\delta^{d}$ constraint in eq. (2.18) by setting $\boldsymbol{k}_{1}=-\sum_{i=2}^{N} \boldsymbol{k}_{\boldsymbol{i}}$. The integration over $\boldsymbol{k}$ becomes gaussian and leads for $N=1$ to

$$
\begin{equation*}
\mathcal{Z}_{1}=\int \mathrm{d}^{D} x_{1}=\mathcal{V} \tag{2.22}
\end{equation*}
$$

and for $N>1$ to the basic formula [20]

$$
\begin{equation*}
\mathcal{Z}_{N}=(2 \pi)^{-d(N-1) / 2} \int \prod_{i=1}^{N} \mathrm{~d}^{D} x_{i}\left(\operatorname{det}\left[\Pi_{i j}\right]_{2 \leqslant i, j \leqslant N}\right)^{-d / 2}, \tag{2.23}
\end{equation*}
$$

where $\Pi_{i j}(2 \leqslant i, j \leqslant N)$ is the $(N-1) \times(N-1)$ matrix

$$
\begin{equation*}
\Pi_{i j}=G\left(x_{i}, x_{j}\right)-G\left(x_{1}, x_{j}\right)-G\left(x_{i}, x_{1}\right)+G\left(x_{1}, x_{1}\right) \tag{2.24}
\end{equation*}
$$

Notice that $\Pi_{i j}$ is function of the point $x_{1}$ which acts as a reference point, and that $G\left(x_{1}, x_{1}\right)$ is actually equal to zero.

### 2.3. CORRELATION FUNCTIONS

Similarly, all expectation values of observables can be obtained from the partition functions with inserted vertex operators (2.12)

$$
\begin{equation*}
\mathcal{Z}^{(M)}\left(X_{a}, \boldsymbol{k}_{a}\right)=\mathcal{Z} \cdot\left\langle\prod_{a=1}^{M} V\left(X_{a}, \boldsymbol{k}_{a}\right)\right\rangle=\int \mathcal{D}[\boldsymbol{r}] \exp \left[-\mathcal{H}+\sum_{a=1}^{M} i \boldsymbol{k}_{a} \cdot \boldsymbol{r}\left(X_{a}\right)\right] . \tag{2.25}
\end{equation*}
$$

Each term of their perturbative expansion

$$
\begin{equation*}
\mathcal{Z}^{(M)}\left(X_{a}, \boldsymbol{k}_{a}\right)=\sum_{N=0}^{\infty} \frac{(-b)^{N}}{N!} \mathcal{Z}_{N}^{(M)}\left(X_{a}, \boldsymbol{k}_{a}\right) \tag{2.26}
\end{equation*}
$$

can be computed by the same techniques. The final result is for $N>1$

$$
\begin{align*}
\mathcal{Z}_{N}^{(M)}\left(X_{a}, \boldsymbol{k}_{a}\right)= & (2 \pi)^{-d(N-1) / 2} \int \prod_{i=1}^{N} \mathrm{~d}^{D} x_{i}\left(\operatorname{det}\left[\Pi_{i j}\right]_{2 \leqslant i, j \leqslant N}\right)^{-d / 2} \\
& \times \exp \left[-\frac{1}{2} \sum_{a, b=1}^{M} \boldsymbol{k}_{a} \cdot \boldsymbol{k}_{b} \Delta_{a b}\right] \tag{2.27}
\end{align*}
$$

where $\Delta_{a b}$ is a ratio of determinants:

$$
\Delta_{a b}=\frac{\operatorname{det}_{N}\left(\begin{array}{cc}
\Pi_{a b} & \Pi_{a j}  \tag{2.28}\\
\Pi_{i b} & \Pi_{i j}
\end{array}\right)}{\operatorname{det}_{N-1}\left(\Pi_{i j}\right)}
$$

with an obvious extension of the definition of the $\Pi$ matrix (2.24) to include external points (in particular $\Pi_{a b}=G\left(X_{a}, X_{b}\right)-G\left(x_{1}, X_{b}\right)-G\left(X_{a}, x_{1}\right)+$ $\left.G\left(x_{1}, x_{1}\right)\right)$. The cases $N=0$ and $N=1$ require a specific analysis. For $N=0$ we get simply

$$
\begin{equation*}
\mathcal{Z}_{0}^{(M)}\left(X_{a}, \boldsymbol{k}_{a}\right)=(2 \pi)^{d} \delta^{d}\left(\sum_{a=1}^{M} \boldsymbol{k}_{a}\right) \exp \left[-\frac{1}{2} \sum_{a, b=1}^{M} \boldsymbol{k}_{a} \cdot \boldsymbol{k}_{b} G\left(X_{a}, X_{b}\right)\right] \tag{2.29}
\end{equation*}
$$

and for $N=1$

$$
\begin{equation*}
\mathcal{Z}_{1}^{(M)}\left(X_{a}, \boldsymbol{k}_{a}\right)=\int \mathrm{d}^{D} x_{1} \exp \left[-\frac{1}{2} \sum_{a, b=1}^{M} \boldsymbol{k}_{a} \cdot \boldsymbol{k}_{b} \Pi_{a b}\right] \tag{2.30}
\end{equation*}
$$

(Notice in this last equation that $\Pi_{a b}$ actually depends on $x_{1}$ ).

### 2.4. MEAN SQUARED DISTANCES

From eq. (2.29) one can in particular derive the mean squared distance between any two points $x$ and $y$ for the free model $(b=0)$ :

$$
\begin{equation*}
\frac{1}{2 d}\left\langle(\boldsymbol{r}(x)-\boldsymbol{r}(y))^{2}\right\rangle_{0}=-G(x, y)=\frac{1}{4^{\nu}(4 \pi)^{D / 2}} \frac{\Gamma(1-\nu)}{\nu \Gamma(\nu+D / 2)}|x-y|^{2 \nu} \tag{2.31}
\end{equation*}
$$

which is IR- and UV-finite and positive for $0<\nu<1(k-2<D<k)$.

## 3. Analytic continuation in the internal dimension $D$

### 3.1. INDEPENDENT PARAMETERS: $D, \nu$ AND $\epsilon$

We now want to give a meaning to the above expressions for arbitrary real $D, d$ and $k$, so as to have a continuous approach to the "physical" elastic membrane problem $D=2$ and $k=2$. As is clear from (2.27), the general observables of the form (2.25) depend on the external dimension $d$ only through: (i) the external invariants $\boldsymbol{k}_{a} \cdot \boldsymbol{k}_{b}$, (ii) the exponent $-d / 2$ in (2.27). We can therefore, as usual in field theory, consider $d$ as a continuous parameter. The same is true for the exponent $k$ associated with the internal laplacian, which appears only as a parameter in the propagator (2.20). Since we shall be interested in the range $d$ close to $d^{\star}$, it is natural to substitute to the continuous parameters $d$ and $k$ the set of continuous parameters $\epsilon$ and $\nu$. Their relevant range is $\epsilon \simeq 0$ (where we expect a non-trivial universal fixed point) and $0<\nu<1$ (where the manifold is crumpled, that is neither collapsed nor stretched).

The analytic continuation in the internal dimension $D$ is a new feature of this model and requires a separate analysis, namely that of the signification of the measure $\prod_{i} \mathrm{~d}^{D} x_{i}$ for non-integer $D$. We now discuss equivalent geometric definitions of this measure, which have a natural extension to non-integer $D$.

### 3.2. DISTANCE GEOMETRY IN $D$ DIMENSIONS

We are looking at generalized integrals of the type

$$
\int \mathrm{d}^{D} x_{1} \ldots \mathrm{~d}^{D} x_{N} f\left(x_{1}, \ldots, x_{N}\right)
$$

where $f$ is invariant by rotation in $D$-dimensional space and thus depends only on the invariant scalar products


Fig. 3. Equivalent representations of the positions of a given set of $N$ interaction points (here $N=6$ ). The points are described (a) by their position $x_{i}$ in $\mathbb{R}^{D}$ or $\mathbb{R}^{N-1}$ or (b) by the set of their mutual squared distances $a_{i j}=\left(x_{i}-x_{j}\right)^{2}$ or (c) by their relative vector $y_{i}=x_{i+1}-x_{1}$ in $\mathbb{R}^{D}$ or $\mathbb{R}^{N-1}$ (relative to the point $x_{1}$ ) or (d) by the line vectors (labeled by $\alpha$ ) of an arbitrary spanning tree joining these points.
which form a symmetric matrix $\left[u_{i j}\right]$. For $D \geqslant N$ we can reduce the integration over the $x_{i}$ 's to an integral over the $u_{i j}$ 's of the form (see appendix A)

$$
\begin{equation*}
\int \prod_{i=1}^{N} \mathrm{~d}^{D} x_{i} f\left(u_{i j}\right)=\int_{\mathcal{U}_{N}} \prod_{i \leqslant j} \mathrm{~d} u_{i j} \sigma_{N}^{(D)}\left(\left[u_{i j}\right]\right) f\left(\left[u_{i j}\right]\right) \tag{3.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma_{N}^{(D)}\left(\left[u_{i j}\right]\right)=\frac{S_{D}}{2} \frac{S_{D-1}}{2} \ldots \frac{S_{D-N+1}}{2}\left(\operatorname{det}_{N}\left[u_{i j}\right]\right)^{(D-N-1) / 2} \tag{3.3}
\end{equation*}
$$

$S_{D}$ is the volume of the unit sphere in $\mathbb{R}^{D}, S_{D}=2 \pi^{D / 2} / \Gamma(D / 2)$. The domain of integration $\mathcal{U}_{N}$ for $u_{i j}$ is such that $u_{i j}$ is the actual scalar product of vectors in euclidean space, i.e. [ $u_{i j}$ ] is a positive matrix.

If moreover the integrand is translationally invariant in $D$-dimensional space, we can go to relative vectors $y_{i}=x_{i+1}-x_{1}(1 \leqslant i \leqslant N-1)$ and reduce by
one unit the number of points, i.e. use $\sigma_{N-1}^{(D)}\left(\left[y_{i} \cdot y_{j}\right]\right)$,

$$
\begin{array}{rl}
\prod_{i=1}^{N} \mathrm{~d}^{D} x_{i}=\mathrm{d}^{D} x_{1} \prod_{1 \leqslant i \leqslant j \leqslant N-1} & \mathrm{~d}\left(y_{i} \cdot y_{j}\right) \frac{S_{D}}{2} \frac{S_{D-1}}{2} \ldots \frac{S_{D-N+2}}{2} \\
& \times\left(\operatorname{det}_{N-1}\left[y_{i} \cdot y_{j}\right]\right)^{(D-N) / 2} \tag{3.4}
\end{array}
$$

This is equivalent to a measure expressed uniquely in terms of the complete set of $N(N-1) / 2$ squared distances (see fig. 3)

$$
\begin{equation*}
a_{i j}=\left(x_{i}-x_{j}\right)^{2} \tag{3.5}
\end{equation*}
$$

by simply rewriting $y_{i} \cdot y_{j}$ as

$$
\begin{align*}
y_{i-1} \cdot y_{j-1} & =D_{i j}(a) \\
D_{i j}(a) & \equiv \frac{1}{2}\left(a_{i 1}+a_{j 1}-a_{i j}\right), \quad 2 \leqslant i, j \leqslant N \tag{3.6}
\end{align*}
$$

Finally, after the simple change of variables (3.6) we arrive, for a translationally and rotationally invariant integrand, at an integral over distances

$$
\begin{equation*}
\int_{\mathbb{R}^{D}} \prod_{i=1}^{N} \mathrm{~d}^{D} x_{i} f\left(a_{i j}\right)=\mathcal{V} \int_{\mathcal{A}_{N}} \prod_{1 \leqslant i<j \leqslant N} \mathrm{~d} a_{i j} \mu_{N}^{(D)}\left(\left[a_{i j}\right]\right) f\left(\left[a_{i j}\right]\right) \tag{3.7}
\end{equation*}
$$

where

$$
\begin{align*}
\mu_{N}^{(D)}\left(\left[a_{i j}\right]\right)= & 2^{-(N-1)(N-2) / 2} \frac{S_{D}}{2} \frac{S_{D-1}}{2} \ldots \frac{S_{D-N+2}}{2} \\
& \times\left(\operatorname{det}_{N-1}\left[D_{i j}(a)\right]_{2 \leqslant i, j \leqslant N}\right)^{(D-N) / 2} \tag{3.8}
\end{align*}
$$

This last formula is valid for $D \geqslant N-1$. Indeed, $D=N-1$ is the smallest dimension for which $N$ linearly independent points can be embedded in euclidean space. The domain of integration $\mathcal{A}_{N}$ for $a_{i j}$ is then simply the set for which [ $D_{i j}(a)$ ] is a positive matrix.

In eq. (3.8) appears the important quantity

$$
\begin{equation*}
P_{N}(a) \equiv \operatorname{det}_{N-1}\left[D_{i j}(a)\right]=\operatorname{det}_{N-1}\left[y_{i} \cdot y_{j}\right] \tag{3.9}
\end{equation*}
$$

which is a homogeneous polynomial of degree $N-1$ in the $a_{i j} . P_{N}(a)$ is actually fully symmetric under permutations of the indices $i$ or $j$ in [ $a_{i j}$ ], as can be seen from its expression as a Cayley-Menger determinant * well-known in distance geometry [28],

$$
P_{N}(a)=\frac{(-1)^{N}}{2^{N-1}}\left|\begin{array}{ccccc}
0 & 1 & 1 & \ldots & 1  \tag{3.10}\\
1 & 0 & a_{12} & \ldots & a_{1 N} \\
1 & a_{21} & 0 & \ldots & a_{2 N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_{N 1} & a_{N 2} & \ldots & 0
\end{array}\right|
$$

[^1]We have for instance for $N=2$ and 3 points

$$
\begin{equation*}
P_{2}(a)=a_{12}, \quad P_{3}(a)=\frac{1}{4}\left(2 a_{12} a_{23}+2 a_{23} a_{31}+2 a_{31} a_{12}-a_{12}^{2}-a_{23}^{2}-a_{31}^{2}\right) \tag{3.11}
\end{equation*}
$$

The matrix [ $D_{i j}$ ] will be positive iff any bordered principal minor $P_{K}(a) \geqslant 0$ for any $K \leqslant N$ :

$$
P_{K}(a)=\frac{(1)^{K}}{2^{K-1}}\left|\begin{array}{ccccc}
0 & 1 & 1 & \ldots & 1  \tag{3.12}\\
1 & 0 & a_{12} & \ldots & a_{1 K} \\
1 & a_{21} & 0 & \ldots & a_{2 K} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_{K 1} & a_{K 2} & \ldots & 0
\end{array}\right| \geqslant 0
$$

For $K=2$, this is simply the positivity condition $a_{12} \geqslant 0$. For $K=3$, one recovers the familiar triangular inequality

$$
\begin{equation*}
\left(a_{12}-a_{13}-a_{23}\right)^{2} \leqslant 4 a_{13} a_{23} \Leftrightarrow\left|a_{13}^{1 / 2}-a_{23}^{1 / 2}\right| \leqslant a_{12}^{1 / 2} \leqslant a_{13}^{1 / 2}+a_{23}^{1 / 2} \tag{3.13}
\end{equation*}
$$

For $K>3$ one gets more general inequalities which are the necessary and sufficient conditions for the $a_{i j}$ to be realized as squared distances between $N$ points of the euclidean space $\mathbb{R}^{N-1}$. The volume $\mathcal{V}\left(x_{1}, \ldots, x_{K}\right)$ of the (possibly degenerate) parallelotope [30] ( $(K-1)$-dimensional parallelepiped) with vertices $x_{1}, x_{2}, \ldots, x_{K}$ is given by

$$
\begin{equation*}
\mathcal{V}^{2}\left(x_{1}, \ldots, x_{K}\right)=P_{K}(a) \tag{3.14}
\end{equation*}
$$

Thus $P_{K}(a)=0$ indicates that the first $K$ points are linearly dependent, i.e. can be embedded in $\mathbb{R}^{K-2}$.

For $D \leqslant N-2$, the expression (3.8) becomes singular due to the appearance of zeroes in the sphere volumes $S_{D-K+2}$ for $D+2 \leqslant K \leqslant N$ on the one hand, and due to divergences of the term $\left(P_{N}(a)\right)^{(D-N) / 2}$, which occur when $P_{N}(a)$ vanishes, that is on the boundary of the domain $\mathcal{A}_{N}$, on the other hand. Nevertheless $\mu_{N}^{(D)}(a)$ can now be considered as a distribution with a support in submanifolds of $\mathcal{A}_{N}$ of dimension $D(N-(D+1) / 2)$, which correspond to $D$-dimensional euclidean subspaces of $\mathbb{R}^{N-1}$. One therefore still reproduces the natural euclidean measure in $\mathbb{R}^{D}$, as can be shown by analytic continuation, which we now describe.

### 3.3. ANALYTIC CONTINUATIONS IN $D$

3.3.1. Distance geometry for non-integer $D$. The first way to define integrals of the form $\int \mathrm{d}^{D} x_{1} \ldots \mathrm{~d}^{D} x_{N} f\left(x_{1}, \ldots, x_{N}\right)$ for non integer $D$ is to start from (3.2) and (3.3) or equivalently from (3.7) and (3.8). The measures (3.3) and (3.8) now involve $D$ as a simple parameter and therefore provide a natural
squared distances $a_{i j}$ (in $\mathcal{A}_{N}$ ), we can use (3.4) back to reexpress the measure over the scalar products $\mathrm{d}\left(y_{i} \cdot y_{j}\right)$ as a measure over $N$ points in $\mathbb{R}^{N-1}$

$$
\begin{equation*}
\prod_{i=1}^{N-1} \mathrm{~d}^{N-1} y_{i}=\prod_{1 \leqslant i \leqslant j \leqslant N-1} \mathrm{~d}\left(y_{i} \cdot y_{j}\right) \frac{S_{N-1}}{2} \frac{S_{N-2}}{2} \ldots \frac{S_{1}}{2}\left(\operatorname{det}_{N-1}\left[y_{i} \cdot y_{j}\right]\right)^{-1 / 2} \tag{3.20}
\end{equation*}
$$

Thus we can implement the analytic continuation in $D$ by modifying the euclidean measure in $\mathbb{R}^{N-1}$ by a suitable analytic measure term:

Analytic continuation can thus be summarized in the following compact formula, which is a formal rewriting of (3.21):

$$
\begin{align*}
& \prod_{i=1}^{N-1} \mathrm{~d}^{D} y_{i}=\prod_{i=1}^{N-1} \mathrm{~d}^{N-1} y_{i} \Omega(D, N)\left(\mathcal{V}\left(0, y_{1}, \ldots, y_{N-1}\right)\right)^{D-N+1} \\
& \Omega(D, N)=\frac{\operatorname{Vol}(\mathrm{SO}(D))}{\operatorname{Vol}(\operatorname{SO}(D-N+1)) \operatorname{Vol}(\operatorname{SO}(N-1))} \tag{3.22}
\end{align*}
$$

where $\operatorname{Vol}(\mathrm{SO}(D))$ is the volume of the special orthogonal group in $D$ dimensions:

$$
\begin{equation*}
\operatorname{Vol}(\mathrm{SO}(D))=\frac{S_{D}}{2} \frac{S_{D-1}}{2} \ldots \frac{S_{1}}{2} \tag{3.23}
\end{equation*}
$$

When $M$ external points $X_{a}$ are present (that is points over which we do not integrate), eq. (3.22) has to be replaced by the more general formula

$$
\begin{align*}
\prod_{i=1}^{N} \mathrm{~d}^{D} x_{i}= & \prod_{i=1}^{N} \mathrm{~d}^{M+N-1} x_{i} \Omega(D, M, N) \\
& \times\left(\frac{\mathcal{V}\left(x_{1}, x_{2}, \ldots, x_{N}, X_{1}, \ldots, X_{M}\right)}{\mathcal{V}\left(X_{1}, \ldots, X_{M}\right)}\right)^{D-M-N+1} \\
\Omega(D, M, N)= & \frac{\operatorname{Vol}(\operatorname{SO}(D-M+1))}{\operatorname{Vol}(\operatorname{SO}(D-M-N+1)) \operatorname{Vol}(\operatorname{SO}(N))} . \tag{3.24}
\end{align*}
$$

3.3.3. Spherical coordinates. A third (equivalent) way to perform an analytic continuation in $D$ is the use of spherical coordinates. We first consider again the case of $N$ points in $\mathbb{R}^{D}$ with $D$ integer and $D \geqslant N-1$. We take $x_{1}$ as the center of the spherical coordinates, and describe the $N-1$ other points by their relative coordinate, as before

$$
\begin{equation*}
y_{i}=x_{i+1}-x_{1} \quad i=1, \ldots, N-1 \tag{3.25}
\end{equation*}
$$

Introducing generalized spherical coordinates for the $y_{i}$, we write

$$
\begin{align*}
y_{i, 1} & =\left|y_{i}\right| \cos \theta_{i, 1} \\
y_{i, 2} & =\left|y_{i}\right| \sin \theta_{i, 1} \cos \theta_{i, 2} \\
& \vdots \\
y_{i, D-1} & =\left|y_{i}\right| \sin \theta_{i, 1} \sin \theta_{i, 2} \ldots \sin \theta_{i, D-2} \cos \theta_{i, D-1}, \\
y_{i, D} & =\left|y_{i}\right| \sin \theta_{i, 1} \sin \theta_{i, 2} \ldots \sin \theta_{i, D-2} \sin \theta_{i, D-1}, \tag{3.26}
\end{align*}
$$

where $\theta_{i, n} \in[0, \pi]$ for $1 \leqslant n \leqslant D-2$ and $\theta_{i, D-1} \in[0,2 \pi)$. The corresponding measure is given by

$$
\begin{equation*}
\mathrm{d}^{D} y_{i}=\left|y_{i}\right|^{D-1} \mathrm{~d}\left|y_{i}\right| \prod_{n=1}^{D-1}\left(\sin \theta_{i, n}\right)^{D-1-n} \mathrm{~d} \theta_{i, n} \tag{3.27}
\end{equation*}
$$

For rotationally invariant integrands, we can furthermore fix successively

$$
\begin{equation*}
\theta_{i, n}=0, \quad n \geqslant i . \tag{3.28}
\end{equation*}
$$

Taking care of the successive rotational symmetries, we arrive at

$$
\begin{equation*}
\prod_{i=1}^{N-1} \mathrm{~d}^{D} y_{i}=S_{D} S_{D-1} \ldots S_{D-N+2} \prod_{i=1}^{N-1}\left|y_{i}\right|^{D-1} \mathrm{~d}\left|y_{i}\right| \prod_{i=2}^{N-1} \prod_{n=1}^{i-1}\left(\sin \theta_{i, n}\right)^{D-1-n} \mathrm{~d} \theta_{i, n} \tag{3.29}
\end{equation*}
$$

with all the $\theta_{i, n}$ now integrated from 0 to $\pi$. In (3.29), $D$ again appears only as a parameter. This therefore provides another natural path to analytic continuation in $D$. Indeed, possible singularities at integer $D$ arise from the negative powers of the $\sin \theta_{i, n}$, which diverge at $\theta_{i, n}=0$ or $\pi$. It is clear from the spherical coordinates representation (3.26) that when some of the $\theta$ 's are equal to 0 or $\pi$ the vectors $y_{i}$ are not linearly independent and the $N$ points $x_{i}$ are in a euclidean subspace with dimension smaller than $N-1$. Away from integer values of $D$ (with $0<D<N-1$ ), these divergences can be treated by the standard finite part prescription (independently for each $\theta_{i, n}$ ). To prove that for integer $D$, (3.29) remains a distribution and can be rewritten as a finite measure localized on some subspace (corresponding to spherical coordinates in some $D$-dimensional submanifold) requires a more elaborate discussion, not presented here.

This analytic continuation (3.29) is totally equivalent to the analytic continuation (3.22), as can be seen by going back as before to coordinates in $\mathbb{R}^{N-1}$. Using (3.29), we have formally

$$
\begin{equation*}
\left.\prod_{i=1}^{N-1} \mathrm{~d}^{D} y_{i}=\prod_{i=1}^{N-1} \frac{\mathrm{~d}^{N-1} y_{i} \frac{S_{D} S_{D-1} \ldots S_{D-N+2}}{S_{N-1} S_{N-2} \ldots S_{1}}}{\sum_{i=1}^{N-1}} \prod_{i=1}^{N}\left|y_{i}\right| \times \prod_{i=2}^{N-1} \prod_{n=1}^{i-1} \sin \theta_{i, n}\right]_{(3.30)}^{D-N+1} \tag{3.30}
\end{equation*}
$$

where the $\theta_{i, n}$ 's are spherical angles in $\mathbb{R}^{N-1}$. We read on this equation the angular
representation of the squared parallelotope volume [28]

$$
\begin{align*}
P_{N}(a)=\operatorname{det}\left[y_{i} \cdot y_{j}\right]_{1 \leqslant i, j \leqslant N-1} & =\prod_{i=1}^{N-1}\left|y_{i}\right|^{2} \times \prod_{i=2}^{N-1} \prod_{n=1}^{i-1} \sin ^{2} \theta_{i, n} \\
& =\mathcal{V}^{2}\left(0, y_{1}, \ldots, y_{N-1}\right) \tag{3.31}
\end{align*}
$$

and eq. (3.30) is therefore identical to (3.22).
Finally, when $M$ external points are present, (3.29) has to be replaced by

$$
\begin{gather*}
\prod_{i=1}^{N} \mathrm{~d}^{D} x_{i}=S_{D-M+1} S_{D-M} \ldots S_{D-M-N+2} \prod_{i=1}^{N}\left|x_{i}\right|^{D-1} \mathrm{~d}\left|x_{i}\right| \\
 \tag{3.32}\\
\times \prod_{i=1}^{N} \prod_{n=1}^{M+i-2}\left(\sin \theta_{i, n}\right)^{D-1-n} \mathrm{~d} \theta_{i, n}
\end{gather*}
$$

where the $\theta_{i, n}$ 's are the $M+i-2$ successive relative spherical angles for $x_{i}$ necessary to assign position to the vector $x_{i}-X_{1}$ with respect to the $M-1$ external vectors $X_{2}-X_{1}, \ldots, X_{M}-X_{1}$ and to the $i-1$ internal vectors $x_{j}-X_{1}$ for $j<i$, in a reference frame where $X_{1}$ is at the origin.

### 3.4. FACTORIZATION

Of course, for integer $D$, the measure $\prod_{i} \mathrm{~d}^{D} y_{i}$ is naturally factorized, when applied to a product of functions of independent variables:

$$
\begin{align*}
\int \prod_{k=1}^{P+Q} \mathrm{~d}^{D} y_{k} f & \left(\left\{y_{k ; k=1, P}\right\}\right) g\left(\left\{y_{k ; k=P+1, P+Q\}}\right)\right. \\
& =\int \prod_{i=1}^{P} \mathrm{~d}^{D} y_{i} f\left(\left\{y_{i}\right\}\right) \cdot \int \prod_{j=P+1}^{P+Q} \mathrm{~d}^{D} y_{j} g\left(\left\{y_{j}\right\}\right) \tag{3.33}
\end{align*}
$$

This important factorization property becomes non trivial when extended to arbitrary $D$, as can be seen from (3.21). Still, if we consider the scalar product matrix $\left[u_{i j}\right]_{1 \leqslant i, j \leqslant P+Q}$ and denote by $[u]_{P}$ (respectively $[u]_{Q}$ ) the submatrix [ $\left.u_{i j}\right]_{1 \leqslant i, j \leqslant P}$ (respectively $\left[u_{i j}\right]_{P+1 \leqslant i, j \leqslant P+Q}$ ), one has (see appendix B)

$$
\begin{align*}
\int_{\mathcal{U}_{P+Q}} \mathrm{~d}[u] \sigma_{P+Q}^{(D)}([u]) f\left([u]_{P}\right) g\left([u]_{Q}\right) & =\int_{\mathcal{U}_{P}} \mathrm{~d}[u]_{P} \sigma_{P}^{(D)}\left([u]_{P}\right) f\left([u]_{P}\right) \\
& \times \int_{\mathcal{U}_{Q}} \mathrm{~d}[u]_{Q} \sigma_{Q}^{(D)}\left([u]_{Q}\right) g\left([u]_{Q}\right) \tag{3.34}
\end{align*}
$$

which means that the integration over the mixed scalar products $u_{i j}, 1 \leqslant i \leqslant$ $P<j \leqslant P+Q$ can be performed and amounts to factorize $\sigma_{P+Q}^{(D)}$ into $\sigma_{P}^{(D)} \sigma_{Q}^{(D)}$.

The factorization property of the measure is thus preserved under analytic continuation in $D$.

### 3.5. THE INTERACTION AS A CAYLEY-MENGER DETERMINANT

The $N$-point interaction term $\left(\operatorname{det}_{N-1}\left[\Pi_{i j}\right]\right)^{-d / 2}$ depends explicitly on $D$ through the occurrence of the Green function (2.31) and is readily analytically continued to non-integer $D$. Let us recall that we consider $D, \nu$ and $\epsilon$ as the three independent parameters of the model, so that $d$ itself is a function of $D$ given by $d=(D-\epsilon) / \nu$. From a distinct, geometrical point of view, it is particularly interesting to notice that the interaction term also involves a determinant of the Cayley-Menger type with $a_{i j}$ replaced by its power $\left(a_{i j}\right)^{\nu}$

$$
P_{N}\left(a^{\nu}\right) \equiv \frac{(-1)^{N}}{2^{N-1}}\left|\begin{array}{ccccc}
0 & 1 & 1 & \ldots & 1  \tag{3.35}\\
1 & 0 & a_{12}^{\nu} & \ldots & a_{1 N}^{\nu} \\
1 & a_{21}^{\nu} & 0 & \ldots & a_{2 N}^{\nu} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_{N 1}^{\nu} & a_{N 2}^{\nu} & \ldots & 0
\end{array}\right|
$$

Indeed, from definition (2.24) and from (2.31), we have

$$
\begin{equation*}
\Pi_{i j}=A_{D}(\nu) D_{i j}\left(a^{\nu}\right) \tag{3.36}
\end{equation*}
$$

with

$$
\begin{equation*}
D_{i j}\left(a^{\nu}\right)=\frac{1}{2}\left(a_{i 1}^{\nu}+a_{j 1}^{\nu}-a_{i j}^{\nu}\right) \tag{3.37}
\end{equation*}
$$

and the factor

$$
\begin{equation*}
A_{D}(\nu)=\frac{2}{4^{\nu}(4 \pi)^{D / 2}} \frac{\Gamma(1-\nu)}{\nu \Gamma(\nu+D / 2)}, \tag{3.38}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\operatorname{det}_{N-1}\left[\Pi_{i j}\right]=\left[A_{D}(\nu)\right]^{N-1} P_{N}\left(a^{\nu}\right) \tag{3.39}
\end{equation*}
$$

Finally we have the compact formula, analytic in $D, \epsilon$ and $\nu$, for the term of order $N$ of the partition function (2.10)

$$
\begin{align*}
\mathcal{Z}_{N}= & \mathcal{V}\left(2 \pi A_{D}(\nu)\right)^{-d(N-1) / 2} \prod_{K=2}^{N}\left(\frac{S_{D-K+2}}{2^{K-1}}\right) \\
& \times \int_{\mathcal{A}_{N}} \prod_{1 \leqslant i<j \leqslant N} \mathrm{~d} a_{i j}\left[P_{N}(a)\right]^{(D-N) / 2}\left[P_{N}\left(a^{\nu}\right)\right]^{-d / 2} \tag{3.40}
\end{align*}
$$

with again $d=(D-\epsilon) / \nu$.

### 3.6. ANALYTIC EXPRESSION OF $\mathcal{Z}_{N}$ IN CARTESIAN COORDINATES

An immediate corollary of the above formalism is the following alternative formula for $\mathcal{Z}_{N}$, now in cartesian coordinates in $\mathbb{R}^{N-1}$, which provides an equivalent definition of the analytic continuation of $\mathcal{Z}_{N}$ :

$$
\begin{align*}
\mathcal{Z}_{N}= & (2 \pi)^{-d(N-1) / 2} \mathcal{V} \int \prod_{i=1}^{N-1} \frac{\mathrm{~d}^{N-1} y_{i} \frac{S_{D} \ldots S_{D-N+2}}{S_{N}-1 \cdots S_{I}}}{} \\
& \times\left(\operatorname{det}\left[y_{i} \cdot y_{j}\right]_{1 \leqslant i, j \leqslant N-1}\right)^{(D-N+1) / 2}\left(\operatorname{det}\left[\Pi_{i j}\right]_{2 \leqslant i, j \leqslant N}\right)^{-d / 2} . \tag{3.41}
\end{align*}
$$

### 3.7. DETERMINANT ATTACHED TO TREES

In the following we shall find it useful to express both the measure and the interaction contributions in terms of more general variables $\lambda_{\alpha}$ obtained from the positions $x_{i}$ and attached to arbitrary oriented trees. A spanning tree is a connected graph whose vertices are the previous $N$ points $x_{i}$, and without loops (see fig. 3d). This graph therefore has $N-1$ internal lines labeled by $\alpha=1, \ldots, N-1$ for which one also specifies an orientation. An oriented tree is characterized by its $N \times(N-1)$ incidence matrix [ $\epsilon_{i \alpha}$ ] defined by $\epsilon_{i \alpha}=1$ if the line $\alpha$ is incident to $i$ and points toward $i, \epsilon_{i \alpha}=-1$ if $\alpha$ is incident to $i$ and points outward $i, \epsilon_{i \alpha}=0$ otherwise. One has

$$
\begin{equation*}
\sum_{i=1}^{N} \epsilon_{i \alpha}=0 \tag{3.42}
\end{equation*}
$$

For each line $\alpha$ of the tree we define the line vector (or edge vector) $\lambda_{\alpha}$ in $\mathbb{R}^{N-1}$ by

$$
\begin{equation*}
\lambda_{\alpha}=\sum_{i=1}^{N} \epsilon_{i \alpha} x_{i}=\sum_{i=1}^{N-1} \epsilon_{i+1 \alpha} y_{i}, \tag{3.43}
\end{equation*}
$$

where the $y_{i}$ 's have been defined in (3.25).
Expression for the measure. Since the jacobian of the linear transformation (3.43) from the $y_{i}$ 's to the $\lambda_{\alpha}$ 's is

$$
\left|\operatorname{det}^{\prime}[\epsilon]\right|=\left|\operatorname{det}\left[\epsilon_{i \alpha}\right]_{\substack{1 \leqslant \alpha \leqslant N-1}}^{\substack{1 \leqslant i \leqslant N}}\right|=1
$$

and

$$
\operatorname{det}\left[\lambda_{\alpha} \cdot \lambda_{\beta}\right]=\left(\operatorname{det}^{\prime}[\epsilon]\right)^{2} \operatorname{det}\left[y_{i} \cdot y_{j}\right]=\operatorname{det}\left[y_{i} \cdot y_{j}\right]
$$

one has directly from (3.21)

$$
\begin{align*}
\prod_{i=1}^{N-1} \mathrm{~d}^{D} y_{i} & =\prod_{\alpha=1}^{N-1} \mathrm{~d}^{D} \lambda_{\alpha} \\
& \equiv \prod_{\alpha=1}^{N-1} \mathrm{~d}^{N-1} \lambda_{\alpha} \frac{S_{D} S_{D-1} \ldots S_{D-N+2}}{S_{N-1} S_{N-2} \ldots S_{1}}\left[\operatorname{det}\left[\lambda_{\alpha} \cdot \lambda_{\beta}\right]_{1 \leqslant \alpha, \beta \leqslant N-1}\right]^{(D-N+1) / 2} \tag{3.44}
\end{align*}
$$

This also means that one can replace in (3.4) the integration over the matrix elements $u_{i j}=y_{i} \cdot y_{j}$ by an integration over matrix elements $u_{\alpha \beta}=\lambda_{\alpha} \cdot \lambda_{\beta}$ associated with an arbitrary tree.
Expression for the interaction. We now derive the expression of the determinant $P_{N}\left(a^{\nu}\right)$ which enters the interaction factor in terms of the $\lambda_{\alpha}$ 's. Eq. (2.23) was actually a particular representation of the interaction, associated with a particular choice of a tree, namely the star centered at $x_{1}$ and lines pointing toward the other points. This can be seen in our choice $\boldsymbol{k}_{1}=-\sum_{i=2}^{N} \boldsymbol{k}_{i}$ to account for the $\delta^{d}\left(\sum_{i} \boldsymbol{k}_{i}\right)$ constraint in the momentum integral (2.18). We can generalize this construction to an arbitrary oriented tree $\mathbf{T}$ by writing $\boldsymbol{k}_{i}$ as

$$
\begin{equation*}
\boldsymbol{k}_{i}=-\sum_{\alpha=1}^{N-1} \epsilon_{i \alpha} \boldsymbol{q}_{\alpha} \tag{3.45}
\end{equation*}
$$

These vectors $\boldsymbol{q}_{\alpha}$ can be seen as flowing along the lines of the tree while the vectors $\boldsymbol{k}_{i}$ can be thought of as being injected at the nodes of the tree. Eq. (3.45) expresses the momentum conservation at the nodes and moreover, together with (3.42), ensures $\sum_{i} \boldsymbol{k}_{i}=\mathbf{0}$ for any set of $\boldsymbol{q}_{\alpha}$ 's. Using then

$$
\begin{equation*}
\prod_{i=1}^{N} \mathrm{~d}^{d} \boldsymbol{k}_{i} \delta^{d}\left(\sum_{i=1}^{N} \boldsymbol{k}_{i}\right)=\prod_{\alpha=1}^{N-1} \mathrm{~d}^{d} \boldsymbol{q}_{\alpha} \tag{3.46}
\end{equation*}
$$

we get for the interaction term (2.18)

$$
\begin{align*}
\mathcal{Z}_{N} & =\mathcal{V} \int \prod_{\alpha=1}^{N-1} \frac{\mathrm{~d}^{D} \lambda_{\alpha} \mathrm{d}^{d} \boldsymbol{q}_{\alpha}}{(2 \pi)^{d}} \exp \left[-\frac{1}{2} \sum_{\alpha, \beta=1}^{N-1} \boldsymbol{q}_{\alpha} \cdot \boldsymbol{q}_{\beta} \Pi_{\alpha \beta}^{\mathrm{T}}\right] \\
& =(2 \pi)^{-d(N-1) / 2} \mathcal{V} \int \prod_{\alpha=1}^{N-1} \mathrm{~d}^{D} \lambda_{\alpha}\left(\operatorname{det}\left[\Pi_{\alpha \beta}^{\mathrm{T}}\right]_{1 \leqslant \alpha, \beta \leqslant N-1}\right)^{-d / 2} \tag{3.47}
\end{align*}
$$

where we take advantage of (3.44) and define a new matrix $\Pi^{\mathrm{T}}$ attached to the tree T :

$$
\begin{equation*}
\Pi_{\alpha \beta}^{\mathrm{T}}=\sum_{i, j=1}^{N} \epsilon_{i \alpha} G\left(x_{i}, x_{j}\right) \epsilon_{j \beta} \tag{3.48}
\end{equation*}
$$

Indeed $\operatorname{det}\left[\Pi_{\alpha \beta}^{\mathbf{T}}\right]$ is independent of the choice of the tree $\mathbf{T}$.
In terms of pairs of oriented lines $\alpha, \beta$ of the tree, with extremities ( $i_{\alpha}, i^{\prime}{ }_{\alpha}$ ) and ( $i_{\beta}, i_{\beta}^{\prime}$ ) respectively, the matrix element $\Pi_{\alpha \beta}^{\mathrm{T}}$ is associated with the quadrilateral $\left(i_{\alpha}, i_{\alpha}^{\prime} ; i_{\beta}, i_{\beta}^{\prime}\right)$

$$
\begin{equation*}
\Pi_{\alpha \beta}^{\mathrm{T}}=G\left(x_{i_{\alpha}}, x_{i_{\beta} \beta}\right)+G\left(x_{i^{\prime} \alpha}, x_{i^{\prime} \beta}\right)-G\left(x_{i_{\alpha}}, x_{i^{\prime} \beta}\right)-G\left(x_{i^{\prime} \alpha}, x_{i_{\beta}}\right) . \tag{3.49}
\end{equation*}
$$

It can be viewed as an interaction potential between two dipoles $\lambda_{\alpha}$ and $\lambda_{\beta}$ and has the following pictorial representation:


Expression for correlation functions. For correlation functions $\mathcal{Z}^{(M)}\left(X_{a}, \boldsymbol{k}_{a}\right)$ (2.25) one can generalize the above construction simply (i) by considering the spanning star tree $\mathrm{T}_{\mathrm{ex}}$ with line vectors $\Lambda_{a}=X_{a}-X_{1}(a>1)$ for the external points, (ii) by choosing an arbitrary tree $\mathrm{T}_{\text {in }}$ with line vectors $\lambda_{\alpha}$ for the internal points, and (iii) by attaching these two trees by a line vector $\Lambda_{1}$ joining the external point $X_{1}$ to an arbitrary internal point. In this way, we obtain a larger tree $\mathbf{T}$ to which we can associate a generalized form of (2.27):

$$
\begin{align*}
& \mathcal{Z}_{N}^{(M)}\left(X_{a}, \boldsymbol{k}_{a}\right)=(2 \pi)^{-d(N-1) / 2} \int \mathrm{~d}^{D} \Lambda_{1} \prod_{\alpha=1}^{N-1} \mathrm{~d}^{D} \lambda_{\alpha}\left(\operatorname{det}\left[\Pi_{\alpha \beta}^{\mathrm{T}}\right]_{1 \leqslant \alpha, \beta \leqslant N-1}\right)^{-d / 2} \\
& \times \exp \left[-\frac{1}{2} \sum_{a, b=1}^{M} \boldsymbol{k}_{a} \cdot \boldsymbol{k}_{b} \Delta_{a b}\right]  \tag{3.51}\\
& \Delta_{a b}=\frac{\operatorname{det}_{N}\binom{\Pi_{a b}^{\mathrm{T}} \Pi_{a \beta}^{\mathrm{T}}}{\Pi_{\alpha b}^{\mathrm{T}} \Pi_{\alpha \beta}^{\mathrm{T}}}}{\operatorname{det}_{N-1}\left(\Pi_{\alpha \beta}^{\mathrm{T}}\right)} \tag{3.52}
\end{align*}
$$

As discussed above, the determinants in (3.52) are independent of the tree $T$ chosen. In (3.51), the integral over the $\lambda_{\alpha}$ 's and $\Lambda_{1}$ has to be understood, for
real $D$, as

$$
\begin{align*}
& \int \mathrm{d}^{D} \Lambda_{1} \prod_{\alpha=1}^{N-1} \mathrm{~d}^{D} \lambda_{\alpha}=\int \mathrm{d}^{M+N-1} \Lambda_{1} \prod_{\alpha=1}^{N-1} \mathrm{~d}^{M+} \lambda_{\alpha} \frac{S_{D-M+1} \ldots S_{D-M-N+2}}{S_{N} \ldots S_{1}} \\
& \times\left\{\frac{\left[\operatorname{det}_{N+M-1}\left(\begin{array}{ll}
\Lambda_{a} \cdot \Lambda_{b} & \Lambda_{a} \cdot \lambda_{\beta} \\
\lambda_{\alpha} \cdot \Lambda_{b} & \lambda_{\alpha} \cdot \lambda_{\beta}
\end{array}\right)\right]}{\left[\operatorname{det}_{M-1}\left(\Lambda_{a} \cdot \Lambda_{b}\right)_{2 \leqslant a, b \leqslant M}\right]}\right\}^{(D-M-N+1) / 2} \tag{3.53}
\end{align*}
$$

and (3.51) is a function of the invariants $a_{a b}=\left(X_{a}-X_{b}\right)^{2}$, which are quadratic forms in terms of the line vectors $\Lambda_{a}$.

### 3.8. THE LIMIT $D=1$ AND THE SCHWINGER REPRESENTATION

As an example, for a manifold with internal dimension $D=1$, one can recover the standard Schwinger representation ${ }^{\star}$ of an interacting field theory with interaction term $(\boldsymbol{\Phi})^{2}(\mathbf{0})$ (see subsect. 6.1 for further details), here in direct correspondence with the continuous Edwards-like model for a polymer interacting with a single fixed point at the origin. Choosing $D=1$ and $k=2$ in (2.1) corresponding to the gaussian weight of a brownian chain, one has $\nu=\frac{1}{2}$ and the propagator along the chain

$$
\begin{equation*}
G(x, y)=-\frac{1}{2}|x-y| . \tag{3.54}
\end{equation*}
$$

Furthermore, for the perturbative order $N$, the measure term (3.8) reconstructs in the limit $D=1$ (like in eq. (3.19)) the measure over all relative distances of $N$ ordered points along the chain, as well as all their permutations. For a given permutation $x_{i_{1}} \leqslant \ldots \leqslant x_{i_{N}}$, the measure term is simply

$$
\begin{equation*}
\prod_{\alpha=1}^{N-1} \mathrm{~d} a_{i_{\alpha} i_{\alpha+1}}^{1 / 2} \tag{3.55}
\end{equation*}
$$

Choosing as a particular tree T the successive oriented links $\left(i_{\alpha}, i^{\prime}{ }_{\alpha}\right)=\left(i_{\alpha}, i_{\alpha+1}\right)$ the matrix $\Pi^{\mathrm{T}}$ (3.49) is diagonal

$$
\begin{equation*}
\Pi_{\alpha \beta}^{\mathrm{T}}=s_{\alpha} \delta_{\alpha \beta} \quad \text { with } \quad s_{\alpha}=a_{i_{a} i_{\alpha+1}}^{1 / 2}=x_{i_{\alpha+1}}-x_{i_{\alpha}} \tag{3.56}
\end{equation*}
$$

The $s_{\alpha}$ are nothing but the usual Schwinger parameters (proper time) for the propagator lines $\alpha$, or in polymer theory the lengths of the successive polymer segments. The interaction gives for the partition function a term of the form

$$
\begin{equation*}
\mathcal{Z}_{N}=\int \prod_{\alpha=1}^{N-1} \mathrm{~d} s_{\alpha}(\ldots)\left(\prod_{\alpha=1}^{N-1} s_{\alpha}\right)^{-d / 2} \tag{3.57}
\end{equation*}
$$

[^2]

Fig. 4. The daisy diagram corresponding to the term (3.57)
which is nothing but the Schwinger representation for the "daisy" diagram in $d$ dimensions (fig. 4).

## 4. Ultraviolet and infrared properties of the integrand

### 4.1. EXISTENCE AND POSITIVENESS OF THE INTEGRAND

The rules that we have proposed above for defining the perturbative expansion of the model in non-integer dimension $D$ remain formal. Indeed, we have not shown yet that the integrands do exist and that the integrals are convergent (for $D$ large enough), and define an analytic function in $D$. Let us concentrate on the $N$ th term for the partition function, $\mathcal{Z}_{N}$, which is explicited by the integral (3.40) in terms of distance variables $a_{i j}$, by the integral (3.41) in terms of cartesian coordinates in $\mathbb{R}^{N-1}$ or by the integral (3.47) in terms of tree variables $\lambda_{\alpha}$. We shall furthermore assume in the following sections that $D \geqslant N-1$, that is $D$ large enough for $\mu_{N}^{(D)}$ to be a measure density (similarly, for $\mathcal{Z}_{N}^{(M)}$, we shall assume $D \geqslant N+M-1$ ). We shall discuss in sect. 9 how our results can be extended to smaller $D$.

Schoenberg's theorem. First, in view of the formula (3.40), the positiveness of the Cayley-Menger determinant $P_{N}\left(a^{\nu}\right)(3.35)$ has to be ensured inside the domain of integration $\mathcal{A}_{N}$ for the variables $a_{i j}$. For $0<\nu \leqslant 1$, this actually is just a consequence of a remarkable theorem in distance geometry due to Schoenberg [31].

Theorem 4.1. If we change the metric of the euclidean space $\mathbb{R}^{m}$ from the euclidean distance $d(x, y)=|x-y|$ to the new distance

$$
\begin{equation*}
\tilde{d}(x, y)=(d(x, y))^{\nu} \quad 0<\nu \leqslant 1, \tag{4.1}
\end{equation*}
$$

the new metric space $\mathbb{R}_{(\nu)}^{m}$ thus arising may be embedded isometrically in the Hilbert space $\mathbb{R}^{\infty}$ with the $L^{2}$-norm.

A practical (equivalent) statement is that any set of $N$ distinct points of $\mathbb{R}_{(\nu)}^{m}$ can be embedded in the euclidean space $\mathbb{R}^{N-1}$. In our language, this means that, if the $a_{i j}$ are actual squared distances of $N$ points in $\mathbb{R}^{N-1}$, then $a_{i j}^{\nu}$ with $0<\nu \leqslant 1$ can also be realized as actual squared distances between $N$ transformed points in $\mathbb{R}^{N-1}$. An immediate consequence is that $P_{N}\left(a^{\nu}\right) \geqslant 0$, as well as all the lower rank polynomials $P_{K}\left(a^{\nu}\right) \geqslant 0$.

We moreover have the useful refined result for $0<\nu<1$ [31]:
Theorem 4.2. If $x_{1}, \ldots, x_{N}$ are $N$ distinct points in $\mathbb{R}^{m}$, and $\left[a_{i j}\right]$ the corresponding squared distance matrix, the matrix $D_{i j}\left(a^{\nu}\right)=\frac{1}{2}\left(a_{i 1}^{\nu}+a_{j 1}^{\nu}-a_{i j}^{\nu}\right)$, ( $0<\nu<1$ ), is positive definite.

The positiveness is a consequence of theorem 4.1. The novelty here concerns the definiteness and states that the determinant $P_{N}\left(a^{\nu}\right)$ vanishes if and only if two points at least coincide, that is $a_{i j}=0$ for some $i \neq j$. Notice that this property does not hold for the case $\nu=1$ for which we already know that $P_{N}(a)$ vanishes as soon the $a_{i j}$ can be realized as distances between $N$ points in $\mathbb{R}^{K}$ for $K \leqslant N-2$, which can be obtained with none of the $a_{i j}(i \neq j)$ vanishing.

### 4.2. SHORT-DISTANCE DIVERGENCES

The above result ensures that for $0<\nu<1$ the only possible divergences in eq. (3.40) occur when some distances $a_{i j}$ go to 0 (UV divergences) or $\infty$ (IR divergences). Let us first discuss the UV behavior.

If one scales the distances by a global factor $\rho$,

$$
\begin{equation*}
a_{i j} \longrightarrow \rho^{2} a_{i j} \tag{4.2}
\end{equation*}
$$

the measure term in eq. (3.40) is scaled according to

$$
\begin{equation*}
\prod_{1 \leqslant i<j \leqslant N} \mathrm{~d} a_{i j}\left[P_{N}(a)\right]^{(D-N) / 2} \longrightarrow \rho^{D(N-1)} \prod_{1 \leqslant i<j \leqslant N} \mathrm{~d} a_{i j}\left[P_{N}(a)\right]^{(D-N) / 2} \tag{4.3}
\end{equation*}
$$

while the interaction term scales as

$$
\begin{equation*}
\left[P_{N}\left(a^{\nu}\right)\right]^{-d / 2} \longrightarrow \rho^{-(N-1) \nu d}\left[P_{N}\left(a^{\nu}\right)\right]^{-d / 2} \tag{4.4}
\end{equation*}
$$

We therefore obtain a global scaling factor $\rho^{(N-1)(D-\nu d)}=\rho^{(N-1) \epsilon}$. This means that the contribution to $\mathcal{Z}_{N}$ of the region of $\mathcal{A}_{N}$ such that all squared distances $a_{i j} \leqslant \rho^{2}$ is of order $\rho^{(N-1) \epsilon}$, indicating that $\mathcal{Z}_{N}$ is superficially UV convergent for $\epsilon>0$, but divergent for $\epsilon \leqslant 0$.

Similarly, we expect that when the squared distances between some subset of $P$ points are $\leqslant \rho^{2}$, we get a contribution of order $\rho^{(P-1) \epsilon}$ to $\mathcal{Z}_{N}$. This is indeed what occurs, due to the following crucial factorization property of the interaction term.


Fig. 5. Schematic picture of the short-distance factorization of the interaction term relative to some set $\mathcal{G}$ of $N$ interaction points (here $N=10$ ). When the points of a subset $\mathcal{P}$ of $\mathcal{G}$ are contracted toward one of its point $x_{1}$, the interaction term factorizes into the product of the interaction term relative to $\mathcal{P}$ and the interaction term relative to $\overline{\mathcal{P}}=(\mathcal{G} \backslash \mathcal{P}) \cup\left\{x_{1}\right\}$.

## Theorem 4.3. Short-distance factorization of the interaction term.

Consider the subset $\mathcal{P}$ of (for instance) the first $P$ interacting points (considered as embedded in $\mathbb{R}^{N-1}$ ) $x_{1}, \ldots, x_{P}$ and let us contract it toward one of its points, which we choose to be $x_{1}$. We set

$$
x_{k}(\rho)=\left\{\begin{array}{cc}
x_{1}+\rho\left(x_{k}-x_{1}\right) & \text { if } 1 \leqslant k \leqslant P  \tag{4.5}\\
x_{k} & \text { if } P<k \leqslant N
\end{array} .\right.
$$

This defines a mapping in distance variables

$$
a_{i j}(\rho)= \begin{cases}\rho^{2} a_{i j} & \text { if } 1 \leqslant i \leqslant j \leqslant P \\ a_{1 j}-\rho\left(a_{1 i}+a_{1 j}-a_{i j}\right)+\rho^{2}\left(a_{1 i}\right) & \text { if } 1 \leqslant i \leqslant P<j \leqslant N \\ a_{i j} & \text { if } P<i \leqslant j \leqslant N\end{cases}
$$

Then, in the limit $\rho \rightarrow 0$, the determinant of the matrix $D_{i j}\left(a^{\nu}\right)$ (3.37) factorizes as (fig. 5)

$$
\begin{align*}
\operatorname{det}_{N-1}\left[D_{i j}\left(a^{\nu}(\rho)\right)\right]= & \rho^{2 \nu(P-1)} \operatorname{det}_{P-1}\left[D_{i j}\left(a^{\nu}\right)\right]_{2 \leqslant i, j \leqslant P} \\
& \times \operatorname{det}_{N-P}\left[D_{i j}\left(a^{\nu}\right)\right]_{P+1 \leqslant i, j \leqslant N}\left\{1+\mathcal{O}\left(\rho^{2 \delta}\right)\right\} \tag{4.6}
\end{align*}
$$

with

$$
\begin{equation*}
\delta=\min (\nu, 1-\nu)>0 \tag{4.7}
\end{equation*}
$$

Proof: The matrix $D_{i j}$ transforms under a contraction according to
$D_{i j}\left(a^{\nu}(\rho)\right)= \begin{cases}\rho^{2 \nu} D_{i j}\left(a^{\nu}\right) & \text { if } 1 \leqslant i \leqslant j \leqslant P \\ \frac{1}{2}\left\{\rho^{2 \nu} a_{1 i}^{\nu}+a_{1 j}^{\nu}\right. & \\ \left.-\left[a_{1 j}-\rho\left(a_{1 i}+a_{1 j}-a_{i j}\right)+\rho^{2}\left(a_{1 i}\right)\right]^{\nu}\right\} & \text { if } 1 \leqslant i \leqslant P<j \leqslant N \\ D_{i j}\left(a^{\nu}\right) & \text { if } P<i \leqslant j \leqslant N\end{cases}$

For small $\rho$, the mixed term $D_{i j}, i \leqslant P<j$, has the expansion

$$
\begin{align*}
D_{i j}\left(a^{\nu}(\rho)\right) & =\rho^{2 \nu} a_{1 i}^{\nu}+\rho \nu a_{1 j}^{\nu-1}\left(a_{1 i}+a_{1 j}-a_{i j}\right)+\mathcal{O}\left(\rho^{2}\right) \\
& =\rho^{\nu} \mathcal{O}\left(\rho^{\delta}\right) \tag{4.9}
\end{align*}
$$

since the leading term is $\propto \rho^{2 \nu}$ or $\propto \rho$, depending on whether $\nu$ is greater or less than $1 / 2$. Thus we can write the matrix $D_{i j}\left(a^{\nu}(\rho)\right)$ in blocks associated respectively with the subsets $\mathcal{P}$ and $\overline{\mathcal{P}}=\left\{x_{1}\right\} \cup\left\{x_{P+1}, \ldots, x_{N}\right\}$

$$
D\left(a^{\nu}(\rho)\right)=\left(\begin{array}{cc}
2^{2} \ldots, P & P_{+11 \ldots, N} \\
\rho_{\mathcal{P}}\left(a^{\nu}\right) & \rho^{\nu} \mathcal{O}\left(\rho^{\delta}\right)  \tag{4.10}\\
\rho^{\nu} \mathcal{O}\left(\rho^{\delta}\right) & D_{\overline{\mathcal{P}}}\left(a^{\nu}\right)
\end{array}\right)
$$

Hence*

$$
\operatorname{det}\left(D\left(a^{\nu}(\rho)\right)\right)=\rho^{2 \nu(P-1)}\left[\operatorname{det}\left(D_{\mathcal{P}}\left(a^{\nu}\right)\right) \operatorname{det}\left(D_{\overline{\mathcal{P}}}\left(a^{\nu}\right)\right)+\mathcal{O}\left(\rho^{2 \delta}\right)\right]
$$

Furthermore, from Schoenberg's theorem, if $\operatorname{det}\left(D_{\mathcal{P}}\left(a^{\nu}\right)\right)$ or $\operatorname{det}\left(D_{\overline{\mathcal{P}}}\left(a^{\nu}\right)\right)$ vanishes, some subset of points $x_{k}(\rho)$ coincides for any $\rho$ and so $\operatorname{det}\left(D\left(a^{\nu}(\rho)\right)\right)$ also vanishes. The equivalence in eq. (4.6) and the theorem follow.

The consequences of this theorem are twofold. First, as expected, when a subset $\mathcal{P}$ of $P$ points coalesces to a single point $p$, this gives a divergence in $\mathcal{Z}_{N}$, as well as in any correlation function $\mathcal{Z}_{N}^{(M)}$, since from (2.27) the same interaction determinant $(\operatorname{det}(\Pi))^{-d / 2}$ is present. Second, this divergence is formally equal to the global divergence of the partition function amplitude $\mathcal{Z}_{P}$ for the $P$ contracted points times the amplitude obtained by replacing those points by the single contraction point $p, \mathcal{Z}_{N-P+1}$. This is a key point for ensuring renormalizability, since this shows that short-distance divergences can be absorbed

* This follows for instance from $\operatorname{det}\left(\begin{array}{cc}A & B \\ B^{\mathrm{t}} & C\end{array}\right)=\operatorname{det}(A) \operatorname{det}(C) \operatorname{det}\left(1-A^{-1} B C^{-1} B^{\mathrm{t}}\right)$ for invertible matrices $A$ and $C$.
into an effective interaction term, thanks to a short-distance operator product expansion for "interaction operators"

$$
\begin{equation*}
\prod_{i \in \mathcal{P}} \delta^{d}\left(\boldsymbol{r}\left(x_{i}\right)\right) \stackrel{\forall i, x_{i} \rightarrow x_{p}}{\sim}|\operatorname{size}(\mathcal{P})|^{-d \nu(P-1)} \delta^{d}\left(\boldsymbol{r}\left(x_{p}\right)\right), \tag{4.11}
\end{equation*}
$$

where $\operatorname{size}(\mathcal{P})$ is a "typical distance" between the points $x_{i}$ of $\mathcal{P}$ in $D$-dimensional space (which depends on the precise way the limit $x_{i} \rightarrow x_{p}$ is taken).

It is the purpose of the next sections to give a precise meaning to these assertions, to provide rigorous arguments, and to discuss their consequences for the physics of the model.

One can regularize those short-distance divergences and make the integrals (3.40), (3.51) UV-finite by changing the short-distance behavior of the propagator $G(x, y)$. However, it is both convenient and natural to use dimensional regularization, that is to consider the amplitudes as analytic functions of the parameters $D$ (the dimension of internal space), $\nu$ (the scaling dimension of the field $r$ ), and $\epsilon$ (the scaling dimension of the interaction). As we shall argue below, for fixed $D$ and $0<\nu<1$, the amplitudes are expected to be UV-finite, and therefore analytic functions of $\epsilon$, in the half-plane $\operatorname{Re}(\epsilon)>0$. Because of the short-distance behavior of its integrand, $\mathcal{Z}_{N}$ will exhibit poles at $\epsilon=0$. For instance, the singular contribution to the integral (3.40) arising from the integration over the global dilation parameter of the $N$-interaction point set gives a single pole $\propto 1 / \epsilon$. More generally, we expect that multiple poles in $1 / \epsilon^{k}$ ( $1 \leqslant k \leqslant N-1$ ) will occur at $\epsilon=0$, corresponding to the dominant singularities appearing when $k$ successive subsets of interaction points coalesce [20]. Apart from these poles at $\epsilon=0$, subdominant divergences will be shown to give poles in the $\epsilon$ plane for $\operatorname{Re}(\epsilon) \leqslant-\delta /(N-1)$. In field theory, the factorization property of the integrand under partial contractions of subdiagrams determines the pole structure of the resulting Feynman amplitude and is the key point that ensures renormalizability. Here, although the interacting manifold model is not mapped onto a standard field theory, a similar pole structure of $\mathcal{Z}_{N}$ will be found, due to the factorization property of the interaction term that we just discussed.

### 4.3. IR REGULARIZATION

By similar power counting arguments (i.e. dimensional analysis), it is expected that the integrals will diverge for large distances $a_{i j} \rightarrow \infty$ (when $\operatorname{Re}(\epsilon) \geqslant 0$ ). As usual in field theory, we shall deal with this problem by introducing an infrared regulator, and by showing that such a regulator does not change the short-distance properties and the renormalization of the model.

The simplest kind of regulator is to work in a finite $D$-dimensional space, i.e. to consider a "membrane" of finite size. This is in fact what is usually done for the continuous polymer Edwards model. Indeed, the polymer is taken to have
a finite total "length" $S$, which amounts to constrain the length variables $S_{\alpha}$ in (3.57) by a measure term $\left(S-\sum_{\alpha} s_{\alpha}\right) \theta\left(S-\sum_{\alpha} s_{\alpha}\right)$.

In our case, our formulation of the model in non-integer dimension relies on the invariance of the observables under euclidean motions in $\mathbb{R}^{D}$. A simple way to keep a similar symmetry over a finite manifold is to start from the $D$-dimensional hypersphere $\mathcal{S}_{D}$ with radius $R$ and volume $\mathcal{V}_{\mathcal{S}_{D}}=S_{D+1} R^{D}$, so that the group of invariance is now $\mathrm{SO}(D+1)$. One can easily generalize the concept of distance geometry on $\mathcal{S}_{D}$, and its analytic continuation for non-integer $D$. Indeed, we can embed the sphere into $\mathbb{R}^{D+1}$, and write the integral of a $\operatorname{SO}(D+1)$ invariant function of $N$ variables as an integral over scalar products $u_{i j}=x_{i} \cdot x_{j}$ :

$$
\int \prod_{i=1}^{N} \mathrm{~d}^{D+1} x_{i} \delta\left(\left|x_{i}\right|-R\right) f\left(u_{i j}\right)=\int_{\mathcal{U}_{N}(R)} \prod_{i<j} \mathrm{~d} u_{i j} \sigma_{N}^{(D)}\left(\left[u_{i j}\right], R\right) f\left(\left[u_{i j}\right]\right)
$$

with $u_{i j}=R^{2}$ if $i=j$, and the measure

$$
\begin{equation*}
\sigma_{N}^{(D)}\left(\left[u_{i j}\right], R\right)=S_{D+1} \ldots S_{D-N+2} R^{N}\left(\operatorname{det}_{N}\left[u_{i j}\right]\right)^{(D-N) / 2} \tag{4.13}
\end{equation*}
$$

$\mathcal{U}_{N}(R)$ being the domain of $u_{i j}(i<j)$ where the matrix $\left[u_{i j}\right.$ ] is positive with all the $u_{i i}$ set equal to $R^{2}$. Equivalently we can express the integral (4.12) in terms of squared distances $a_{i j}=2\left(R^{2}-u_{i j}\right)$ in $(D+1)$-dimensional space (this defines the so-called cord distance on $\mathcal{S}_{D}$ which differs from the geodesic distance):

$$
\int \prod_{i=1}^{N} \mathrm{~d}^{D+1} x_{i} \delta\left(\left|x_{i}\right|-R\right) f\left(a_{i j}\right)=\mathcal{V}_{\mathcal{S}_{D}} \int_{\mathcal{A}_{N}(R)} \prod_{i<j} \mathrm{~d} a_{i j} \mu_{N}^{(D)}\left(\left[a_{i j}\right], R\right) f\left(\left[a_{i j}\right]\right)
$$

with the measure

$$
\begin{equation*}
\mu_{N}^{(D)}\left(\left[a_{i j}\right], R\right)=2^{-N(N-1) / 2} S_{D} \ldots S_{D-N+2}\left(\frac{1}{R^{2}} \operatorname{det}_{N}\left[R^{2}-\frac{1}{2} a_{i j}\right]\right)^{(D-N) / 2} \tag{4.15}
\end{equation*}
$$

and $\mathcal{A}_{N}(R)$ the domain of $a_{i j}$ where the matrix [ $R^{2}-\frac{1}{2} a_{i j}$ ] is positive. In particular, the positiveness of the $2 \times 2$ minors ensures for any two points the diameter inequality $a_{i j} \leqslant 4 R^{2}$. Hence, $\mathcal{A}_{N}(R)$ is a bounded subset of $\mathbb{R}^{N(N-1) / 2}$.

One can check the identity

$$
\begin{equation*}
\operatorname{det}_{N}\left(\left[R^{2}-\frac{1}{2} a_{i j}\right]\right)=R^{2} \operatorname{det}_{N-1}\left(\left[D_{i j}(a)\right]\right)+\operatorname{det}_{N}\left(\left[-\frac{1}{2} a_{i j}\right]\right) \tag{4.16}
\end{equation*}
$$

where $D_{i j}(a)$ is defined in (3.6) (indeed the $N-2$ highest degree terms in the polynomial expansion in $R^{2}$ of the l.h.s of (4.16) vanish identically!). This implies that in the thermodynamic limit $R \rightarrow \infty$ one recovers the measure (3.8) in euclidean (infinite flat) space. Conversely, for a finite $R$, formula (4.16) shows that, at short distances, the measure is dominated by the first term of the r.h.s, i.e. the euclidean one, while the second term, which is one degree higher
in $a_{i j}$, becomes relevant for distances of order $R$ only, hence providing an IR regulator.

It remains to write the expression for the interaction term. In fact, the latter is the same as in (2.23), with the matrix $\Pi_{i j}(2.24)$, or more generally the tree ma$\operatorname{trix} \Pi_{\alpha \beta}^{\mathbf{T}}$ (3.48), involving the massless propagator $G(x, y)=\left[(-\Delta)^{k / 2}\right]^{-1}(x, y)$ now on $\mathcal{S}_{D}$. There is however no general simple analytic expression for $G\left(x_{i}, x_{j}\right)$ as a function of the distance variable $a_{i j}$ defined above for general $D$ and $k$. For definiteness, another simple possibility then consists in keeping a propagator on the sphere of the form (2.31)

$$
\begin{equation*}
-G\left(x_{i}, x_{j}\right)=\frac{1}{4^{\nu}(4 \pi)^{D / 2}} \frac{\Gamma(1-\nu)}{\nu \Gamma(\nu+D / 2)}\left|a_{i j}\right|^{\nu} \tag{4.17}
\end{equation*}
$$

This amounts to modifying the "elastic" term of the hamiltonian (2.1) by finitevolume corrections

$$
\begin{align*}
\boldsymbol{r}(x) \cdot(-\Delta)^{k / 2} \boldsymbol{r}(x) \rightarrow \boldsymbol{r}(x) \cdot & {\left[(-\Delta)^{k / 2}+\operatorname{cst} R^{-2}(-\Delta)^{(k-2) / 2}\right) } \\
& \left.+\operatorname{cst} R^{-4}(-\Delta)^{(k-4) / 2}+\ldots\right] \boldsymbol{r}(x) \tag{4.18}
\end{align*}
$$

which change its long-distance behavior (IR regulator), but not its short-distance behavior. In particular, Schoenberg's theorem 4.2, which is readily satisfied by the propagator $G$ given by (4.17), is expected to remain valid for the exact massless propagator on the sphere. The corrections in (4.18) vanish in the limit $R \rightarrow \infty$. In the following, we will keep in mind that the model is defined with the measure (4.15) and the propagator (4.17). However, since we shall be concerned with the UV renormalization of the model, we shall use formally the simpler euclidean $(R \rightarrow \infty)$ limit (3.8) of (4.15). As discussed above, they actually share the same short-distance properties.

## 5. Absolute convergence for $\epsilon=\boldsymbol{D}-\boldsymbol{v} \boldsymbol{d}>0$

In this section, we want to prove that
Theorem 5.1. For $\epsilon>0$ (i.e. $d<d^{\star}$ ), the integrals $\mathcal{Z}_{N}$ and $\mathcal{Z}_{N}^{(M)}$ are absolutely (UV) convergent.

As in field theory, this actually is a consequence of (i) the superficial convergence of $\mathcal{Z}_{P}$ for any $P \leqslant N$ and (ii) the basic factorization property (4.6), and generalizations thereof. Since the formalism developed above can be thought of as a natural generalization of the Schwinger representation of Feynman integrals, the proof of absolute convergence will be inspired by the standard method based on decomposition into Hepp sectors [32]. As discussed just above, we shall always assume the (implicit) presence of an IR regulator.

(a)


(b)

Fig. 6. (a) An example of construction of the ordered tree $\mathbf{T}=\left(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right)$ for a set of interaction points with $\left|\lambda_{1}\right| \leqslant\left|\lambda_{2}\right| \leqslant\left|\lambda_{3}\right| \leqslant\left|\lambda_{4}\right|$. This tree defines the generalized Hepp sector $\mathcal{H}^{\mathbf{T}}$ to which this set of points belongs. (b) Moving the point $x_{2}$ toward the point $x_{1}$ results in a change of generalized Hepp sector.

### 5.1. GENERALIZED HEPP SECTORS

We start with formula (3.41) and partition the domain of integration for the $y_{i}$ 's into generalized Hepp sectors as follows (fig. 6). Let us consider the $N$ points in $\mathbb{R}^{N-1}$ with cartesian coordinates $0, y_{1}, \ldots, y_{N-1}$. We first singularize the pair of points having the minimum mutual distance, and define $\lambda_{1}$ as the vector in $\mathbb{R}^{N-1}$ joining these two points, with an arbitrary orientation. We define $\lambda_{2}$ in a similar way, as the vector associated with the minimal distance among all the remaining mutual distances. $\lambda_{2}$ can (i) either share one of its extremities with $\lambda_{1}$, or (ii) be disjoint. At the next step, we define $\lambda_{3}$ as the vector associated with the minimal distance among all the remaining ones and such that ( $\lambda_{1}, \lambda_{2}, \lambda_{3}$ ) do not form a closed loop (this may occur only in case (i)). We iterate this construction, by requiring at each step that no loop ever appears, up to the emergence of the last vector $\lambda_{N-1}$. We thus have constructed an oriented ordered tree T with line vectors ( $\lambda_{1}, \ldots, \lambda_{N-1}$ ), which spans the $N$ points and is such that (fig. 6)

$$
\begin{equation*}
\left|\lambda_{1}\right| \leqslant\left|\lambda_{2}\right| \leqslant \ldots \leqslant\left|\lambda_{N-1}\right| . \tag{5.1}
\end{equation*}
$$

We shall denote $\mathbf{T}=\left(\lambda_{1}, \ldots, \lambda_{N-1}\right)$ although the tree $\mathbf{T}$ is not strictly speaking characterized by the line vectors $\lambda_{\alpha}$ but only by the incidence matrix $\epsilon_{i \alpha}$ of the linear transformation from the $x_{i}$ 's (or $y_{i}$ 's) to the $\lambda_{\alpha}$ 's. With any ordered tree
its orientation. It is clear that $\mathbb{R}^{N-1}=\bigcup_{T} \mathcal{H}^{\mathrm{T}}$.
In a given sector $\mathcal{H}^{\mathrm{T}}$, we make a change of variables from the $y_{i}$ 's to the $\lambda_{\alpha}$ 's associated with the ordered tree $\mathbf{T}$ (with an arbitrary choice of orientation) and, in particular, use $\Pi_{\alpha \beta}^{\mathbf{T}}$ to evaluate the interaction term. We parametrize the $\lambda_{\alpha}$ 's by their spherical coordinates in $\mathbb{R}^{N-1}$, namely by their modules $\left|\lambda_{\alpha}\right|$ and relative angles $\theta_{\alpha, 1}, \ldots \theta_{\alpha, \alpha-1}$ as in eq. (3.26) and (3.28). The variables $\left|\lambda_{\alpha}\right|$ will play the role of the Schwinger parameters $s_{\alpha}$ in field theory. Since $\left|\lambda_{1}\right| \leqslant\left|\lambda_{2}\right| \leqslant \ldots \leqslant$ $\left|\lambda_{N-1}\right|$, it is natural to rewrite the $|\lambda|$ 's as

$$
\begin{align*}
\left|\lambda_{1}\right| & =\beta_{1} \beta_{2} \ldots \beta_{N-1} \\
\left|\lambda_{2}\right| & =\beta_{2} \ldots \beta_{N-1} \\
& \vdots  \tag{5.2}\\
\left|\lambda_{N-1}\right| & =\beta_{N-1},
\end{align*}
$$

with $0 \leqslant \beta_{\alpha} \leqslant 1$ for $1 \leqslant \alpha \leqslant N-2$ and $0 \leqslant \beta_{N-1}<\infty$ (in the euclidean version of the problem, thus without IR regulator). The domain of integration $\mathcal{D}^{\mathbf{T}}$ for the $\beta$ and $\theta$ variables which reconstructs the domain $\mathcal{H}^{\mathbf{T}}$ for the $y_{i}$ 's in $\mathbb{R}^{N-1}$, depends on the topology of the ordered tree. For instance, the value $\beta_{\alpha}=1$ can in general be reached inside the sector only for some domain of the angle $\theta$ between $\lambda_{\alpha}$ and $\lambda_{\alpha+1}$. Still, the domain $\mathcal{D}^{\mathrm{T}}$ has the following general structure:

$$
\begin{array}{cll}
0 \leqslant \theta_{\alpha, n} \leqslant \pi & & 1 \leqslant n<\alpha \leqslant N-1 \\
\beta_{\alpha}^{\min }\left(\mathbf{T} ; \beta_{\gamma: \gamma<\alpha} ; \theta^{\prime} \mathbf{s}\right) \leqslant \beta_{\alpha} \leqslant \beta_{\alpha}^{\max }\left(\mathbf{T} ; \beta_{\gamma: \gamma<\alpha} ; \theta^{\prime} \mathbf{s}\right) & 1 \leqslant \alpha \leqslant N-2 \\
0 \leqslant \beta_{N-1} & \tag{5.3}
\end{array}
$$

where $\beta_{\alpha}^{\text {min }}\left(\mathbf{T} ; \beta^{\prime}\right.$ 's; $\theta$ 's) and $\beta_{\alpha}^{\text {max }}(\mathbf{T} ; \beta$ 's; $\theta$ 's) are (positive and possibly vanishing) functions of the $\theta$ 's and of the $\beta_{\gamma}$ 's for $\gamma<\alpha$. The inequality $\beta_{\alpha}^{\min }>\beta_{\alpha}^{\max }$ for some $\theta$ 's and $\beta_{\gamma: \gamma<\alpha}$ would indicate that such a partial configuration of $\theta$ 's and $\beta_{\gamma}$ 's always lies outside the given sector. The only important properties of $\mathcal{D}^{\mathbf{T}}$ that we shall use are:
(i) $\mathcal{D}^{\mathbf{T}}$ is by definition bounded, if one excepts the variable $\beta_{N-1}$, since by construction $\beta_{\alpha}^{\text {max }}\left(\mathbf{T} ; \beta^{\prime} \mathrm{s} ; \theta^{\prime} \mathrm{s}\right) \leqslant 1$. The variable $\beta_{N-1}$ itself stays bounded due to the implicit presence of an IR regulator;
(ii) $\operatorname{det}\left(\left[\Pi_{\alpha \beta}^{\mathrm{T}}\right]\right)$, when expressed in terms of the $\beta$ 's and the $\theta$ 's, is a continuous function of these variables and vanishes in $\mathcal{D}^{\mathbf{T}}$ if and only if one at least of the $\beta$ 's vanishes. Indeed, from Schoenberg's theorem, $\operatorname{det}\left(\left[\Pi_{\alpha \beta}^{\mathbf{T}}\right]\right)=0$ iff two points coincide, that is if their mutual distance is zero. Since this distance is by construction larger than or equal to $\left|\lambda_{1}\right|$ in the sector, this implies $\left|\lambda_{1}\right|=0$, or equivalently $\beta_{1} \beta_{2} \ldots \beta_{N-1}=0$.

### 5.2. ABSOLUTE CONVERGENCE

It is enough to prove the absolute convergence in each Hepp sector $\mathcal{H}^{\mathbf{T}}$. Omitting global factors in (3.41) we consider the integral

$$
\begin{gather*}
\int_{\mathcal{H}^{\mathrm{T}}} \prod_{i=1}^{N-1} \mathrm{~d}^{N-1} y_{i}\left(\operatorname{det}\left[y_{i} \cdot y_{j}\right]\right)^{(D-N+1) / 2}\left(\operatorname{det}\left[\Pi_{i j}\right]\right)^{-d / 2} \\
=\int_{\mathcal{D}^{\mathrm{T}}} \prod_{\alpha=1}^{N-1}\left(\beta_{\alpha}\right)^{\alpha D-1} \mathrm{~d} \beta_{\alpha} \prod_{\alpha=2}^{N-1} \prod_{n=1}^{\alpha-1}\left(\sin \left(\theta_{\alpha, n}\right)\right)^{D-1-n} \mathrm{~d} \theta_{\alpha, n} \\
\times\left(\operatorname{det}\left[\Pi_{\alpha \beta}^{\mathrm{T}}\left(\beta^{\prime} \mathrm{s}, \theta \prime \mathrm{~s}\right)\right]\right)^{-d / 2} \tag{5.4}
\end{gather*}
$$

As already mentioned, we shall limit ourselves to the case $D \geqslant N-1$. We shall discuss in sect. 9 how our results can then be extended to $D<N-1$. The product of sinuses in (5.4) is thus a bounded function on $\mathcal{D}^{\mathbf{T}}$. Possible ultraviolet divergences may only arise from the vanishing of $\operatorname{det}\left[\Pi_{\alpha \beta}\right]$, that is when some $\beta$ 's vanish. For $\epsilon>0\left(d<d^{\star}=D / \nu\right)$, it is sufficient to show that, on $\mathcal{D}^{\mathbf{T}}$,

$$
\begin{equation*}
\prod_{\alpha=1}^{N-1}\left(\beta_{\alpha}\right)^{\alpha D}\left(\operatorname{det}\left[\Pi_{\alpha \beta}^{\mathrm{T}}\right]\right)^{-d / 2}=\mathcal{O}\left(\prod_{\alpha=1}^{N-1}\left(\beta_{\alpha}\right)^{\alpha \epsilon}\right) \tag{5.5}
\end{equation*}
$$

As is clear from its definition, $\Pi_{\alpha \beta}^{\mathbf{T}}$ vanishes when $\lambda_{\alpha}$ and/or $\lambda_{\beta}$ vanish. The key point is that while $\Pi_{\alpha \alpha}^{\mathrm{T}}=A_{D}(\nu)\left|\lambda_{\alpha}\right|^{2 \nu}, \Pi_{\alpha \beta}^{\mathrm{T}}$ vanishes more rapidly than $\left|\lambda_{\alpha}\right|^{\nu}\left|\lambda_{\beta}\right|^{\nu}$ if $\alpha \neq \beta$ (see appendix C ). This property is best expressed by introducing the "normalized" matrix

$$
\begin{equation*}
Y_{\alpha \beta}^{\mathbf{T}} \equiv \frac{1}{A_{D}(\nu)\left|\lambda_{\alpha \beta}^{\mathrm{T}}\right|^{\nu}\left|\lambda_{\beta}\right|^{\nu}} \tag{5.6}
\end{equation*}
$$

(such that $Y_{\alpha \alpha}^{\mathbf{T}}=1$ ).
In terms of the $\beta$ 's, we can write

$$
\begin{align*}
& \Pi_{\alpha \alpha}^{\mathrm{T}}=A_{D}(\nu) \beta_{\alpha}^{2 \nu} \beta_{\alpha+1}^{2 \nu} \ldots \beta_{N-1}^{2 \nu} Y_{\alpha \alpha}^{\mathrm{T}} \\
& \Pi_{\alpha \beta}^{\mathrm{T}}=A_{D}(\nu) \beta_{\alpha}^{\nu} \ldots \beta_{\beta-1}^{\nu} \beta_{\beta}^{2 \nu} \ldots \beta_{N-1}^{2 \nu} Y_{\alpha \beta}^{\mathbf{T}}(\beta \text { 's, } \theta \text { 's }) \quad(\alpha<\beta) \tag{5.7}
\end{align*}
$$

leading to the identity

$$
\begin{equation*}
\operatorname{det}_{N-1}\left(\left[\Pi_{\alpha \beta}^{\mathbf{T}}\right]\right)=\left(A_{D}(\nu)\right)^{N-1} \beta_{1}^{2 \nu} \ldots \beta_{N-1}^{2 \nu(N-1)} \operatorname{det}_{N-1}\left(\left[Y_{\alpha \beta}^{\mathbf{T}}\right]\right) \tag{5.8}
\end{equation*}
$$

This amounts to factorizing out the maximal powers of $\beta$ 's. In particular, $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ is independent of $\beta_{N-1}$. In order to obtain (5.5), one has to show that on $\mathcal{D}^{\mathbf{T}}$ the positive quantity $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ in (5.8) cannot vanish and is actually bounded from below by a strictly positive number. This property is proven in appendix C. Indeed, if $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ were to vanish, $\operatorname{det}\left(\Pi^{\mathbf{T}}\right)$ would also vanish and, from Schoenberg's theorem, some subset of the $\beta$ 's must vanish. This corresponds to contract successively some subsets of points (by a contracting scale factor $\beta$ )
to single points. A generalization of the factorization property (4.6) (see appendix C ) shows that, in such a limit, the determinant $\operatorname{det}\left(\Pi^{\mathbf{T}}\right)$ factorizes into a product of similar determinants associated with subtrees of T . The normalized determinant $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ then becomes exactly equal to a product of normalized subdeterminants, each of them corresponding to a subtree of $\mathbf{T}$. In the sector, these subtrees have no coinciding (with vanishing distance) points and therefore their determinants do not vanish. Thus, $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ does not vanish even in this limit where some $\beta$ 's tend to zero.
From the above results, the quantity $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ in (5.8), seen as a function of $\beta_{\gamma}$ ( $1 \leqslant \gamma \leqslant N-2$ ) and of the $\theta$ 's, is a continuous positive non-vanishing function on the compact restriction of $\mathcal{D}^{\mathrm{T}}$ obtained by omitting the (here dummy) variable $\beta_{N-1}$. Therefore it admits a strictly positive lower bound on $\mathcal{D}^{\mathrm{T}}$ and thus (since $d>0$ )

$$
\begin{equation*}
\left(\operatorname{det}_{N-1}\left[\Pi_{\alpha \beta}^{\mathrm{T}}\right]\right)^{-d / 2}<\operatorname{cst} \cdot \beta_{1}^{-d \nu} \ldots \beta_{N-1}^{-(N-1) d \nu}, \tag{5.9}
\end{equation*}
$$

which is equivalent to (5.5). The convergence of the integral (5.4) in the Hepp sector $\mathcal{H}^{\mathrm{T}}$ for $\epsilon=D-\nu d>0$ follows.
We thus have proven the convergence of the generic perturbative term $\mathcal{Z}_{N}$ of the partition function $\mathcal{Z}$ (for $D \geqslant N-1$ ). Similarly, the perturbative terms $\mathcal{Z}_{N}^{(M)}$ (eq. (2.27)) of the vertex operators $\mathcal{Z}^{(M)}$ (eq. (2.25)) can be shown to be UV convergent for $\epsilon>0$ and $D$ large enough ( $D \geqslant N+M-1$ ). This follows from the same decomposition into Hepp sectors and the use of (3.51). The proof is then exactly the same up to the following modifications:
(I) The measure term in (5.4) is replaced by a measure similar to (3.32) for tree variables. The difference between this measure and that of (5.4) concerns only angular terms, which are bounded functions on $\mathcal{D}^{\mathrm{T}}$ (provided now that $D \geqslant N+M-1)$.
(II) The exponential term, depending of the external momenta, has for argument a negative quadratic form $-\frac{1}{2} \sum_{a, b} \boldsymbol{k}_{a} \cdot \boldsymbol{k}_{b} \Delta_{a b}$, and is therefore bounded between 0 and 1 .
The above proof therefore carries over to this generalized case.

## 6. The subtraction operation $\mathbf{R}$

### 6.1. RENORMALIZATION: INTRODUCTORY REMARKS

The purpose of renormalization is to show that the short-distance divergences that occur at $\epsilon=0$ can be absorbed into a redefinition of the coupling constants of the model. If true, this property allows us (i) to give a meaning to the theory at $\epsilon=0$, and (ii) to write a renormalization group equation and deduce the scaling behavior of the model for $\epsilon \lessgtr 0$. From the analysis of divergences performed in sects. 4 and 5 , we expect that the correlation functions can be made finite by a

The second, perturbative approach à la Bogoliubov-Parasiuk-Hepp-Zimmermann ( $\mathrm{BPHZ} \mathrm{)} \mathrm{[22]} \mathrm{consists} \mathrm{in} \mathrm{working} \mathrm{in} \mathrm{perturbation} \mathrm{theory} \mathrm{and} \mathrm{in} \mathrm{construct-}$ ing, directly or by a recursive process, a subtraction operation on the Feynman amplitudes of the theory, which makes all the terms of perturbation theory finite and well defined through convergent integrals; then one shows that this operation corresponds, in the field theory language, to a renormalization of the action by local counterterms, and that it preserves the equation of motions of the theory and the Ward identities associated with its symmetries. From the statistical mechanics point of view, this amounts to a change of variables from microscopic to effective coupling constants. Renormalization group equations and scaling behaviors are then derived from the renormalized theory. This BPHZ formulation of renormalization has a simple and general perturbative formulation for theories in non integer dimensions $D$, since there are now well defined recipes of "dimensional regularization" which allow us to construct Feynman amplitudes for non-integer $D$, and to study their properties, either in the real space representation, or in the momentum space representation, or in the so-called Schwinger parametric $\alpha$-representation. The BPHZ subtraction operation can then easily be extended to the case of non-integer space dimensions, at least in momentum space or in the $\alpha$-representation.

For our model the action (2.1) can also be seen also as that of a local field theory in $D$-dimensional space for a scalar $d$-component $r$-field

$$
\begin{equation*}
\mathcal{H}=\int_{\mathcal{V}} \mathrm{d}^{D} x\left[\frac{1}{2} \boldsymbol{r}(x) \cdot(-\Delta)^{k / 2} \boldsymbol{r}(x)+b \delta^{d}(\boldsymbol{r}(x))\right] \tag{6.4}
\end{equation*}
$$

but the interaction $\delta$-term is singular and non-polynomial, which makes the perturbative expansion very different from that of the ordinary case, since it does not involve usual Feynman diagrams. Furthermore, the dimension of the interaction term depends explicitly on the number of components of the field, here $d$.

In principle, nothing prevents the application of a renormalization program $\grave{a}$ $l a$ Wilson in the physical case of objects described by (6.4) with integer dimension ( $D=1,2$ ). Some preliminary rigorous results have indeed been obtained (for the case $D=2, d=1, k=2$ ) in ref. [27]. However, it is probably impossible to study by such methods the renormalizability of the model at (or near) its critical dimension $D^{\star}$ (eq. (2.3)), since the latter is in general non-integer (even for integer $d$ ), and between 0 and 2 (for the elastic membrane $k=2$ case). The so-called "functional renormalization", which is an approximate renormalization group scheme, has also been applied to the study of the specific case $d=1$ in ref. [36]. Such schemes are well defined by analytic continuation at non-integer $D$ but are only approximate and have no rigorous status.

On the other hand, in sect. 3, we constructed a perturbation theory for the model in non-integer dimension $D$, via distance geometry, which corresponds
to a dimensional regularization scheme in (internal) real or position space. In sects. 4 and 5 , we have shown that the structure of the UV divergences of the amplitudes (poles in $\epsilon$ ) is quite similar to that of Feynman amplitudes of ordinary local field theories. It is the purpose of the rest of this article to show that it is possible to develop a BPHZ-like formalism to prove renormalizability of this model. In this Section we shall propose a subtraction operation, which will appear to be a generalization of the BPHZ subtraction operation for ordinary Feynman integrals, with a similar structure in term of the so-called "Zimmermann forests". This subtraction operator, which in our case acts directly on the integrands of interaction terms like (3.40) and (3.47), involving positions $x_{i}$ or squared distances $a_{i j}$, will be shown to make the integrals UV-finite (for $\epsilon=0$ ), and to correspond to a renormalization of the coupling constant $b$. This will ensure (in perturbation theory) the renormalizability of the model, the validity of renormalization group equations, and of an $\epsilon$-expansion about the critical dimension.

Another fundamental structure underlies our approach, since the position variables $x_{i}$, (or the $a_{i j}$ 's in distance geometry) can be thought of as a $D$-dimensional generalization of the Feynman $\alpha$-parameters in the Schwinger representation. In field theory, this representation consists in writing the propagators in terms of an auxiliary $\alpha$ parameter via a Laplace transform of the free field propagator (in momentum space)

$$
\begin{equation*}
\frac{1}{p^{2}+m^{2}}=\int_{0}^{\infty} \mathrm{d} \alpha \mathrm{e}^{-\alpha\left(p^{2}+m^{2}\right)} \tag{6.5}
\end{equation*}
$$

and in writing all the Feynman amplitudes as multiple integrals over these $\alpha$ variables. As we have seen for our model (6.4) in sect. 3, the integrals giving the perturbative terms (3.40) of the partition function have a form generalizing that of a Feynman amplitude in $\alpha$-representation. Indeed, the subtraction operation and the mathematical techniques that we shall use to prove renormalizability are in fact extensions of techniques developed by Bergère and Lam in ref. [23] to study the renormalization of local field theories precisely in the $\alpha$-representation.

This analogy of the internal position $D$-space representation of a statistical mechanics model with the $\alpha$-representation of a local field theory is not surprising. Indeed, for $D=1$, it is well known that the Edwards model for self-avoiding polymer (1.1) embedded in $d$ dimensions can be formulated as a local $\Phi^{4}$ theory in $d$-dimensional space, with hamiltonian (6.3) (with $D$ now formally replaced by $d$ ), in the limit where the number of components of the field $\boldsymbol{\Phi}, n$, goes to zero (this is the well-known de Gennes equivalence). The length $S$ of the polymer is conjugate, via a Laplace transform, to the squared mass $m^{2}$ of the corresponding $n \rightarrow 0$ field theory. Similarly, for our model (6.4) (and for $k=2$ ), in the case $D=1$ (polymer interacting with an impurity), the same mapping allows to write it as a $n \rightarrow 0$ field theory in the external $d$-dimensional space $\mathbb{R}^{d}$, with
hamiltonian

$$
\begin{equation*}
\mathcal{H}[\boldsymbol{\Phi}]=\int_{\mathbb{R}^{d}} \mathrm{~d}^{d} \boldsymbol{r}\left(\frac{1}{4}(\partial \boldsymbol{\Phi})^{2}+\frac{m^{2}}{2} \boldsymbol{\Phi}^{2}\right)+b \boldsymbol{\Phi}^{2}(\boldsymbol{r}=\mathbf{0}) \tag{6.6}
\end{equation*}
$$

The interaction between the polymer and the impurity located at the origin is represented by the last term in the r.h.s. of (6.6), which is a singular mass term located at the origin. Here also, the length of the polymer $S$ (which corresponds to the radius $R$ of the manifold in the case $D=1$ ) is conjugate to the squared mass $m^{2}$ of the field $\boldsymbol{\Phi}$ in (6.6). The diagrams associated with (6.6) are "daisy diagrams" identical to those of fig. 4), with the $\alpha$-parameters for the propagators identified with the internal relative distances in (3.57) $s_{\beta}=\left|x_{i_{\beta+1}}-x_{i_{\beta}}\right|$ between successive vertices $i_{\beta}$ and $i_{\beta+1}$ in the internal one-dimensional manifold, i.e. the polymer itself (see eq. (3.56)).

Thus, it will appear that our BPHZ renormalization scheme in position space for the theory (2.1) defined in $\mathbb{R}^{D}$ is a generalization to continuous values of $D$ of the ordinary BPHZ renormalization in $\alpha$-representation of the theory (6.6) defined in $\mathbb{R}^{d}$, with $\alpha$ viewed as a $D=1$ relative position ${ }^{*}$. Finally let us stress that this remarkable mathematical analogy makes us hope that in a similar way, it will be possible to develop renormalization techniques in position space for the non-local theory (1.2) (which describes a self-avoiding $D$-dimensional manifold), which would reduce for $D=1$ to the ordinary renormalization theory for tha Efwnede modal ffomemintad aithar ne adimant mannmolimotion in ln dan

sns
with $b$. given implicitly by the equation

$$
\begin{equation*}
b=b_{\bullet}+\frac{1}{\mathcal{V}_{\mathcal{S}_{D}}} \sum_{N=2}^{\infty} \frac{\left(-b_{\bullet}\right)^{N}}{N!} \mathcal{Z}_{N} \tag{6.13}
\end{equation*}
$$

However, this subtraction is not sufficient to make $\mathcal{Z}^{(M)}$ finite (in terms of $b_{\bullet}$ ), since it does not deal with sub-divergences inside the subsets $\mathcal{P}$. As in standard renormalization theory, one deals with that problem by repeating this subtraction operation inside these subsets, that is by subtracting from each $I_{\mathcal{P}_{k}}$ the divergent parts associated with families of mutually disjoint subsets in $\mathcal{P}_{k}$, and iterating the process. One thus obtains at a given order $N$ a subtraction operation expressed in terms of the sets $\mathcal{F}=\left\{\mathcal{P}_{k}\right\}$ of mutually disjoint or strictly included subsets $\mathcal{P}_{k}$ of $\mathcal{G}$. In analogy with renormalization theory in field theory, such a set $\mathcal{F}$ will be called a forest ${ }^{\star}$ of $\mathcal{G}$.

In addition, for a given forest $\mathcal{F}$, at each subtraction step, that is for each subset $\mathcal{P}_{k}$ of $\mathcal{F}$, we have to specify a root $p_{k}$ of $\mathcal{P}_{k}$, toward which we contract $\mathcal{P}_{k}$, in order to calculate the associated counterterm. It is quite clear that, after integration over the position variables, the result of the subtraction operation does not depend on the specific choice of roots. However, it is natural to choose for each forest a set of roots in a way which is consistent with the geometrical picture of the subtraction operation as successive contractions of subsets toward their root. This leads to the notion of a compatibly rooted forest, which will be discussed below.

After these somehow heuristic considerations, let us give the precise definition of the subtraction operation that we shall use.

Let us consider a set $\mathcal{G}$ of $N$ abstract points, that we call vertices.
Definition 6.1. A rooted subset of $\mathcal{G}$ is a couple ( $\mathcal{P}, p$ ) of a subset $\mathcal{P}$ of $\mathcal{G}$ and of a vertex $p$ which belongs to $\mathcal{P}$, that we call the root of $\mathcal{P}$.

Definition 6.2. A forest $\mathcal{F}$ of $\mathcal{G}$ is a set of subsets $\mathcal{P}_{i}$ of $\mathcal{G}$ such that
(i) two elements of $\mathcal{F}$ are disjoint or strictly included into one another, i.e.

$$
\mathcal{P}_{i} \neq \mathcal{P}_{j} \quad \text { if } i \neq j
$$

and

$$
\mathcal{P}_{i} \cap \mathcal{P}_{j}=\mathcal{P}_{i}, \text { or } \mathcal{P}_{j}, \text { or } \emptyset, \quad \forall i, j
$$

(ii) all elements of $\mathcal{F}$ have at least two elements, i.e.

$$
\operatorname{Card}\left(\mathcal{P}_{i}\right)=\left|\mathcal{P}_{i}\right|>1
$$

Let us note that, by convention, the empty set $\varnothing$ is a forest.

[^3]Definition 6.3. A rooted forest $\mathcal{F}_{\oplus}$ is a set of rooted subsets $\left(\mathcal{P}_{i}, p_{i}\right)$ of $\mathcal{G}$ such that $\left\{\mathcal{P}_{i}\right\}$ is a forest.

Definition 6.4. A compatibly rooted forest is a rooted forest such that, if, for some $i, j, \mathcal{P}_{i} \supset \mathcal{P}_{j}$ and $p_{i} \in \mathcal{P}_{j}$, then $p_{i}=p_{j}$.

Definition 6.5. Finally with any rooted forest $\mathcal{F}_{\oplus}$ we associate its compatibly rooted forest ${ }^{\mathrm{c}} \mathcal{F}_{\oplus}$ by simply changing its roots according to the following recursion:
(i) First, replace the root $p_{i}$ of each $\mathcal{P}_{i}$ of the forest by the root $p_{j}$ of the smallest subset $\mathcal{P}_{j}$ of the forest such that $p_{i} \in \mathcal{P}_{j}$ ( $p_{j}$ may coincide with $p_{i}$ ). One thus obtain a new rooted forest.
(ii) Then, repeat this process for the new forest. One can easily show that after a finite number of iterations ( $\leqslant \operatorname{Card}(\mathcal{F})$ ), this process will leave the roots unchanged, so that one obtains a compatibly rooted forest ${ }^{\mathrm{c}} \mathcal{F}_{\oplus}$.

Of course, a forest $\mathcal{F}_{\oplus}$ is compatibly rooted iff ${ }^{\mathrm{c}} \mathcal{F}_{\oplus}=\mathcal{F}_{\oplus}$.
Dilation operation. For a rooted subset ( $\mathcal{P}, p$ ), we define the dilation operation $\mathcal{D}_{(\mathcal{P}, p)}^{\rho}$ as the transformation acting on the positions of the vertices according to (as in (4.5))

$$
\mathcal{D}_{(\mathcal{P}, p)}^{\rho}: \quad x_{i} \rightarrow x_{i}(\rho)= \begin{cases}x_{p}+\rho\left(x_{i}-x_{p}\right) & \text { if } i \in \mathcal{P}  \tag{6.14}\\ x_{i} & \text { if } i \notin \mathcal{P}\end{cases}
$$

or equivalently in distance space, according to

$$
\mathcal{D}_{(\mathcal{P}, p)}^{\rho}: \quad a_{i j} \rightarrow a_{i j}(\rho)= \begin{cases}\rho^{2} a_{i j} & \text { if } i \in \mathcal{P}, j \in \mathcal{P}  \tag{6.15}\\ a_{p j}-\rho\left(a_{p i}+a_{p j}-a_{i j}\right)+\rho^{2} a_{p i} & \text { if } i \in \mathcal{P}, j \notin \mathcal{P} \\ a_{i j} & \text { if } i \notin \mathcal{P}, j \notin \mathcal{P}\end{cases}
$$

More generally, for a function $I$, expressed as a function of the positions $x_{i}$ or the distances $a_{i j}$, we denote by $\mathcal{D}_{(\mathcal{P}, p)}^{\rho} I$ the value of this function at the positions (or distances) modified according to (6.14) (or (6.15)).

Taylor operator. We then define the "Taylor" operator $\tau_{(\mathcal{P}, p)}$ acting on functions $I$ by

$$
\begin{equation*}
\tau_{(\mathcal{P}, p)} I=\lim _{\rho \rightarrow 0} \rho^{d \nu(|\mathcal{P}|-1)} \mathcal{D}_{(\mathcal{P}, p)}^{\rho} I \tag{6.16}
\end{equation*}
$$

The functions that we shall consider are the integrands in (2.27) and (2.23), which are of the form

$$
\begin{align*}
I_{\mathcal{G}}\left(x_{i}, X_{a}\right) & =\left(\operatorname{det}\left[\Pi\left(x_{i, i \in \mathcal{G}}\right)\right]\right)^{-d / 2} \exp \left[-\frac{1}{2} \sum_{a, b} \boldsymbol{k}_{a} \cdot \boldsymbol{k}_{b} \Delta_{a b}\left(x_{i, i \in \mathcal{G}}, X_{a}\right)\right] \\
I_{\mathcal{G}}\left(x_{i}\right) & =\left(\operatorname{det}\left[\Pi\left(x_{i, i \in \mathcal{G}}\right)\right]\right)^{-d / 2} \tag{6.17}
\end{align*}
$$

where the $\Pi$ and $\Delta$ matrices, defined in (2.24), (3.49) and (2.28), (3.52), are functions of the positions of the internal vertices $i$ in $\mathcal{G}$ and external vertices $a$.

On such functions, the effect of $\tau_{(\mathcal{P}, p)}$ is to keep the most singular term in $\rho$ when performing the dilation $\mathcal{D}_{(\mathcal{P}, p)}^{\rho}$. For instance one operator $\tau_{(\mathcal{P}, p)}$ factorizes $I_{\mathcal{G}}$ into

$$
\begin{equation*}
\tau_{(\mathcal{P}, p)} I_{\mathcal{G}}\left(x_{i}, X_{a}\right)=I_{\mathcal{P}}\left(x_{i}\right) I_{\mathcal{G} / p \mathcal{P}}\left(x_{i}, X_{a}\right) \tag{6.18}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{G} / p \mathcal{P} \equiv \mathcal{G} \backslash(\mathcal{P} \backslash\{p\}) \tag{6.19}
\end{equation*}
$$

is the reduced set obtained by contracting $\mathcal{P}$ into $p$ ( $\backslash$ is the usual subtraction of sets). This operation can be repeated for rooted subsets which form a compatibly rooted forest, and the result does not depend on the order of the $\tau$ operators in this case (commutativity). The result is a product of integrands $I\left(x_{i}\right)$ of reduced internal subsets, times the integrand $I\left(x_{i}, X_{a}\right)$ of the set $\mathcal{G}$ reduced by all elements of the forest.

The subtraction operator. With these notations, we define the subtraction operation $\mathbf{R}$ as a sum of subtractions for all forests. For a given forest $\mathcal{F}$, subtractions associated with different roots give different results on the integrand. We shall sum over the subtractions for all compatibly rooted forests $\mathcal{F}_{\oplus^{c}}$, with some weight factor $W\left(\mathcal{F}_{\oplus^{c}}\right)$ associated with the (compatible) rooting of $\mathcal{F}$. In order to ensure the finiteness of the subtracted integrals, the weights $W\left(\mathcal{F}_{\oplus^{c}}\right)$ must be such that the sum of the $W$ 's for all rooted forests which correspond to the same unrooted forest $\mathcal{F}$ gives 1 . A convenient choice of weight factor $W\left(\mathcal{F}_{\oplus \mathrm{c}}\right)$ for $\mathcal{F}_{\oplus^{c}}$ is to make it proportional to the number of different (not necessarily compatibly-) rooted forests $\mathcal{F}_{\oplus}$ which give $\mathcal{F}_{\oplus^{c}}$ by the compatibilization operation ${ }^{\text {c }}$ (i.e. ${ }^{\mathrm{c}} \mathcal{F}_{\oplus}=\mathcal{F}_{\oplus^{c}}$ ). Our final definition for $\mathbf{R}$ is therefore expressed as a sum over all rooted forests, or equivalently as a sum over all compatibly rooted forests. It reads ${ }^{\star}$

$$
\begin{align*}
\mathbf{R} & =\sum_{\mathcal{F}_{\oplus}}\left[\prod_{(\mathcal{P}, p) \notin \in^{\mathcal{c}} \mathcal{F}_{\oplus}} \frac{1}{|\mathcal{P}|}\left(-\tau_{(\mathcal{P}, p)}\right)\right] \\
& =\sum_{\mathcal{F}_{\oplus^{\mathrm{c}}}} W\left(\mathcal{F}_{\oplus^{\mathrm{c}}}\right)\left[\prod_{(\mathcal{P}, p) \in \mathcal{F}_{\oplus^{\mathrm{c}}}}\left(-\tau_{(\mathcal{P}, p)}\right)\right] . \tag{6.20}
\end{align*}
$$

The weight factors are given explicitly by a product over all different roots $p$ of $\mathcal{F}_{\oplus^{\mathrm{c}}}$

$$
\begin{equation*}
W\left(\mathcal{F}_{\oplus^{\mathrm{c}}}\right)=\prod_{\substack{p \text { root } \\ \text { of } \mathcal{F}_{\oplus \mathrm{P}^{\mathrm{c}}}}} \frac{1}{\left|\mathcal{P}_{p}\right|}, \tag{6.21}
\end{equation*}
$$

where $\mathcal{P}_{p}$ is the largest subset of the forest whose root is $p$.

* In this equation, $\mathcal{F}_{\Theta^{c}}$ denotes an arbitrary compatibly rooted forest, while ${ }^{c} \mathcal{F}_{\oplus}$ denotes the compatibly rooted forest obtained from the (non necessarily compatibly) rooted forest $\mathcal{F}_{\oplus}$ by the compatibilization procedure of definition 6.5 .

Subtracted amplitudes and renormalization. We now restrict ourselves to the case of amplitudes defined in a finite volume, by using the IR regulator introduced in subsect. 4.3 ( $D$-dimensional sphere), that is by defining the integration over the positions of the vertices by (4.14) and (4.15). The subtracted correlation functions at order $N$ are simply defined by applying the subtraction operator $\mathbf{R}$ to the integrand of (6.7)

$$
\begin{equation*}
\mathcal{Z}_{N}^{\mathbf{R}^{(M)}}\left(X_{a}\right) \equiv \int \prod_{i \in \mathcal{G}} \mathrm{~d}^{D} x_{i} \mathbf{R}\left[I_{\mathcal{G}}\left(x_{i}, X_{a}\right)\right] \tag{6.22}
\end{equation*}
$$

Let us note that, since the integrand for the partition function is homogeneous under global rescaling, one has

$$
\begin{equation*}
\mathbf{R}\left[I_{\mathcal{G}}\left(x_{i}\right)\right]=0 \tag{6.23}
\end{equation*}
$$

(as soon as $|\mathcal{G}| \geqslant 2$, of course). This means that with our choice of subtraction, for $N \geqslant 2$, in the absence of external correlation points,

$$
\begin{equation*}
\mathcal{Z}_{N}^{\mathbf{R}}=0, \quad N \geqslant 2 \tag{6.24}
\end{equation*}
$$

The purpose of the next sections is to prove that this subtraction operation makes all correlation functions finite, as summarized in the following theorem:

Theorem 6.1. For $0<\nu<1$, the renormalized integral (6.22) is convergent for $\epsilon=0$ and defines a finite function $\mathcal{Z}_{N}^{\mathbf{R}_{N}^{(M)}}\left(X_{a}\right)$ for $D \geqslant N+M-1$.

The renormalized correlation functions are defined by their perturbative expansion in powers of a renormalized coupling constant $b_{\mathbf{R}}$

$$
\begin{equation*}
\mathcal{Z}^{\mathbf{R}^{(M)}}\left(X_{a} ; b_{\mathrm{R}}\right)=\sum_{N=0}^{\infty} \frac{\left(-b_{\mathrm{R}}\right)^{N}}{N!} \mathcal{Z}_{N}^{\mathbf{R}^{(M)}}\left(X_{a}\right) \tag{6.25}
\end{equation*}
$$

As discussed above, the forest structure of the subtraction operation $\mathbf{R}$ ensures that for $\epsilon>0$, there exists a renormalized coupling constant $b_{\mathrm{R}}(b)$ such that the renormalized correlation functions $\mathcal{Z}^{\mathbf{R}^{(M)}}\left(X_{a} ; b_{\mathrm{R}}\right)$ are equal to the original "bare" correlation functions $\mathcal{Z}^{(M)}\left(X_{a} ; b\right)$ for the model (2.1).

The relation between $b$ and $b_{\mathrm{R}}$ can be obtained directly from the identity of the partition functions

$$
\begin{equation*}
\mathcal{Z}(b)=\mathcal{Z}^{\mathbf{R}}\left(b_{\mathbf{R}}\right) \tag{6.26}
\end{equation*}
$$

From (6.24) we have $\mathcal{Z}^{\mathbf{R}}\left(b_{\mathrm{R}}\right) \equiv(2 \pi)^{d} \delta^{d}(\boldsymbol{k}=\mathbf{0})-b_{\mathrm{R}} \mathcal{V}_{\mathcal{S}_{D}} \equiv V_{\mathbb{R}^{d}}-b_{\mathrm{R}} \mathcal{V}_{\mathcal{S}_{D}}$ and therefore, equating to $\mathcal{Z}(b)$, we get

$$
\begin{equation*}
b_{\mathrm{R}}=-\frac{1}{\mathcal{V}_{\mathcal{S}_{D}}}\left(\mathcal{Z}-\overline{V_{\mathbb{R}^{d}}}\right) \tag{6.27}
\end{equation*}
$$

or the explicit series expansion in $b$ :

$$
\begin{equation*}
b_{\mathrm{R}}=b-\frac{1}{\mathcal{V}_{S_{D}}} \sum_{N \geqslant 2} \frac{(-b)^{N}}{N!} \mathcal{Z}_{N} \tag{6.28}
\end{equation*}
$$

Notice that the fully renormalized coupling constant $b_{\mathrm{R}}$ satisfies the indentity

$$
b=b_{\mathrm{R}}+\frac{1}{\mathcal{V}_{\mathcal{S}_{D}}} \sum_{N \geqslant 2} \frac{(-b)^{N}}{N!} \mathcal{Z}_{N}
$$

while the former partially renormalized coupling constant $b$. (built so as to absorb the superficial divergences) satisfies the truncated equation (6.13),

$$
b=b_{\bullet}+\frac{1}{\mathcal{V}_{\mathcal{S}_{D}}} \sum_{N \geqslant 2} \frac{\left(-b_{\bullet}\right)^{N}}{N!} \mathcal{Z}_{N},
$$

obtained from the equation for $b_{\mathbf{R}}$ mentioned just above, by replacing ( $-b$ ) by $\left(-b_{0}\right)$ in the r.h.s.

Eq. (6.27) shows that, in this scheme, renormalization simply amounts to a change of variable from the microscopic $b$ to an effective coupling constant $b_{\mathrm{R}}$, directly proportional to the connected partition function of the manifold interacting with a point. This scheme is precisely that used in ref. [20], and generalizes that of the "direct renormalization method" [4] for the polymer Edwards model.

Let us stress that $b_{\mathrm{R}}$ as defined above is not dimensionless. The corresponding dimensionless coupling constant can be conveniently chosen as

$$
\begin{equation*}
g=\left(2 \pi A_{D}(\nu)\right)^{-d / 2} b_{\mathrm{R}} \nu_{\mathcal{S}_{D}}^{1-\nu d / D} \tag{6.29}
\end{equation*}
$$

for which the Wilson function (2.6) has been calculated explicitly at one loop [20]. In this subtraction scheme, the subtraction scale $\mu$ of the general equation (6.1) is fixed by the $D$-dimensional volume (which fixes the IR cut-off) $\mu \sim$ $\left(\mathcal{V}_{\mathcal{S}_{D}}\right)^{-1 / D}$. In these notations, this precisely corresponds to $\hat{b}_{\mathrm{R}}=b_{\mathrm{R}}\left(\mathcal{V}_{\mathcal{S}_{D}}\right)^{\epsilon / D}$ and

$$
Z^{-1}\left(\hat{b}_{\mathrm{R}}, \epsilon\right) \equiv \frac{b_{\mathrm{R}}}{b}=1+\frac{1}{\mathcal{V}_{\mathcal{S}_{D}}} \sum_{N \geqslant 2}(-b)^{N-1} \frac{\mathcal{Z}_{N}}{N!},
$$

where $b$ is an implicit function of $b_{\mathrm{R}}$, thus $\hat{b}_{\mathrm{R}}$. Of course, other subtraction schemes can be chosen where the subtraction scale $\mu$ is not related to the volume of internal $D$-dimensional space. They are needed in order to define the theory (e.g. the normalized correlation functions) in the infinite volume limit.

## 7. Reorganization of the counterterms

### 7.1. FORMULATION OF THE SUBTRACTION OPERATION IN TERMS OF NESTS

As we shall see later, it will be more convenient in the proof of the finiteness of the renormalized amplitudes to express the subtraction operation $\mathbf{R}$ in term of nested subdiagrams. In the formalism of BPHZ renormalization in the Schwinger representation in field theory, a subdiagram is a set of lines (propagators) of a


Fig. 7. A subdiagram.


Fig. 8. A complete diagram, with connected components $\mathcal{P}_{i}$.

Feynman graph (and has in general many connected components). A nest is then a family of subdiagrams $P_{k}$ which are nested, that is included into one another (for any $k \neq l, P_{k} \subset P_{l}$ or $P_{l} \subset P_{k}$ ).

In our case we shall introduce a different notion of diagram, now in terms of vertices, rather than lines. Indeed, we have seen that the natural generalization of Schwinger parameters $s_{\alpha}$ is given by the larger set of all mutual distances $a_{i j}$ between points on the manifold. In terms of links, we thus would have to deal with the large number of (interdependent) mutual distances, which are constrained by triangular inequalities. Therefore, we prefer to define diagrams in terms of vertices. Denoting again by $\mathcal{G}$ a set of $N$ vertices, a diagram of $\mathcal{G}$ will now be a collection of disjoint vertex-subsets of $\mathcal{G}$. Each of these subsets of vertices can be thought of as a connected set (which stands for the the complete set of its pairwise mutual distances in the link representation). These ideas will be embodied in the following definitions.

We recall that a partition $P$ of a set $\mathcal{S}$ is a set of mutually disjoint non empty subsets $\mathcal{S}_{i}$ of $\mathcal{S}$, whose union is $\mathcal{S}$ itself.

Definition 7.1. (See figs. 7 and 8.) We shall call a subdiagram (respectively complete diagram) of $\mathcal{G}$ any partition $P$ of some subset $\mathcal{S}$ of $\mathcal{G}$ (respectively of $\mathcal{G}$ itself). The generic word diagram will be used in both cases.

The elements of this partition $P$ are called the connected components of the diagram $P$.


Fig. 9. The complete diagram with connected components $\mathcal{P}_{i}$ (dashed line) is contained in the complete diagram with connected components $\mathcal{Q}_{j}$ (full line).


Fig. 10. The intersection diagram (dark-grey diagram) of two diagrams (grey and white diagrams).

Definition 7.2. (See fig. 9.) A diagram $P$ is contained in a diagram $Q$ if any connected component of $P$ is included in one of the connected components of $Q$. This will be denoted $P \prec Q^{*}$. This defines a partial ordering among the diagrams of $\mathcal{G}$.

Definition 7.3. (See fig. 10.) We define the intersection of two diagrams $P$ and $Q$ as the maximal diagram which is contained in both $P$ and $Q$ (it is unique), and denote it by $P \wedge Q$. Its connected components are nothing but the (non-empty) intersections of a connected component of $P$ and one of $Q$.

Definition 7.4. (See fig. 11.) We define the union of two diagrams $P$ and $Q$ as the minimal diagram which contains both $P$ and $Q$ (it is also unique), and denote it by $P \vee Q$. Let us note that the connected components of $P \vee Q$ are unions of connected components of $P$ and $Q$, but in general not simply the union of one connected component of $P$ and of one of $Q$. Notice that the union and the intersection of complete diagrams of $\mathcal{G}$ are complete. The maximal complete diagram of $\mathcal{G}$ is $G=\{\mathcal{G}\}$. We shall denote by $G_{\odot}$ the (unique) minimal complete

[^4]

Fig. 11. The union diagram (dark-grey diagram) of two diagrams (grey and white diagrams).


Fig. 12. The minimal complete diagram $G_{\odot}$.


Fig. 13. The subtraction diagram (dark-grey diagram) of a diagram (grey diagram) from another diagram (white diagram).
diagram of $\mathcal{G}$. Its connected components are the $N$ single vertex subsets of $\mathcal{G}$ (see fig. 12). For any complete diagram $P$, we have $G_{\odot} \prec P \prec G$.

Definition 7.5 (See fig. 13.) We define the subtraction of a diagram $P$ from a diagram $Q$ as the (unique) maximal diagram contained in $Q$ and whose intersection with $P$ is empty, and denote it by $Q \backslash P$.

The usual properties of commutativity and associativity are satisfied by $\wedge$ and $V$. However these operations are not distributive with respect to one another.


Fig. 14. A rooted subdiagram. The roots are specified by squares.


Fig. 15. A complete rooted diagram. Its elements are rooted subsets $\left(\mathcal{P}_{i}, p_{i}\right)$.

They still satisfy the weaker relations

$$
\begin{align*}
& P \wedge(Q \vee R) \succ(P \wedge Q) \vee(P \wedge R) \\
& P \vee(Q \wedge R) \prec(P \vee Q) \wedge(P \vee R) \tag{7.1}
\end{align*}
$$

Definition 7.6. (See fig. 14.)
A rooted diagram $P_{\oplus}$ is a family $\left\{\left(\mathcal{P}_{1}, p_{1}\right), \ldots,\left(\mathcal{P}_{k}, p_{k}\right)\right\}$ of rooted subsets ( $\mathcal{P}_{i}, p_{i}$ ) of $\mathcal{G}$ such that $P=\left\{\mathcal{P}_{1}, \ldots, \mathcal{P}_{k}\right\}$ is a diagram of $\mathcal{G}$.

We call

$$
\begin{equation*}
P=\operatorname{comp}\left(P_{\oplus}\right)=\left\{\mathcal{P}_{1}, \ldots, \mathcal{P}_{k}\right\} \tag{7.2}
\end{equation*}
$$

the component diagram of $P_{\oplus}$, and

$$
\begin{equation*}
\wp=\operatorname{root}\left(P_{\oplus}\right)=\left\{\left\{p_{1}\right\}, \ldots,\left\{p_{k}\right\}\right\} \tag{7.3}
\end{equation*}
$$

the root diagram of $P_{\oplus}$. We shall use for a rooted diagram the equivalent notations:

$$
\begin{equation*}
P_{\oplus} \equiv\left(\operatorname{comp}\left(P_{\oplus}\right), \operatorname{root}\left(P_{\oplus}\right)\right) \equiv(P, \wp) \tag{7.4}
\end{equation*}
$$

Definition 7.7. (See fig. 15.) A complete rooted diagram is a rooted diagram such that its component diagram is complete.


$$
\mathcal{I}_{J+1, k}
$$

Fig. 16. Two successive complete rooted diagrams $T_{J \oplus}$, with connected components $\tau_{J, j}$ (dashed lines) and $T_{J+1 \oplus}$ with connected components $\tau_{J+1, k}$ (full lines) of a rooted nest. The roots of these two diagrams are not compatible.


Fig. 17. The two successive diagrams of fig. 16, with compatible roots. The roots $w_{J+1, k}$ have been obtained from the roots of fig. 16 by the construction of definition 7.11 .

Definition 7.8. A nest $\mathcal{N}$ is a set of $T+1$ complete diagrams $\left\{T_{0}, T_{1}, \ldots, T_{T}\right\}$ such that

$$
\begin{align*}
& T_{0}=G_{\odot}, \\
& T_{0} \prec T_{1} \prec T_{2} \prec \ldots \prec T_{T} . \tag{7.5}
\end{align*}
$$

Definition 7.9. (See fig. 16.) A rooted nest $\mathcal{N}_{\oplus}$ is a set of complete rooted diagrams $\left\{T_{0 \oplus}, T_{1 \oplus}, \ldots, T_{T \oplus}\right\}$ such that the associated component diagrams form a nest

$$
\begin{equation*}
\operatorname{comp}\left(T_{0 \oplus}\right) \prec \operatorname{comp}\left(T_{1 \oplus}\right) \prec \ldots \prec \operatorname{comp}\left(T_{T \oplus}\right) \tag{7.6}
\end{equation*}
$$

Definition 7.10. (See fig. 17.) A rooted nest is said to be compatibly rooted if we have moreover

$$
\begin{equation*}
\operatorname{root}\left(T_{0 \oplus}\right) \succ \operatorname{root}\left(T_{1 \oplus}\right) \succ \ldots \succ \operatorname{root}\left(T_{T \oplus}\right) \tag{7.7}
\end{equation*}
$$

(Notice that $\operatorname{root}\left(T_{0 \oplus}\right)=G_{\odot}$.) At level $J$, the generic element of the rooted
nest $\mathcal{N}_{\oplus}$ reads explicitly

$$
\begin{equation*}
T_{J \oplus}=\left\{\left(\mathcal{T}_{J, j}, t_{J, j}\right), j=1, \ldots, \operatorname{Card}\left(T_{J}\right)\right\} \tag{7.8}
\end{equation*}
$$

Eq. (7.7) means that when we consider two successive rooted complete diagrams of the rooted nest, $T_{J \oplus}$ and $T_{J+1 \oplus}$, if we consider a connected component $\mathcal{T}_{J+1, k}$ of $T_{J+1}$ and its root $t_{J+1, k}$, this root must coincide with the root $t_{J, j}$ of the connected component $\mathcal{T}_{J, j}$ of $T_{J}$ to which $t_{J+1, k}$ belongs (since $T_{J}$ is complete, $t_{J+1, k}$ belongs necessarily to some connected component of $T_{J}$ ). This property then implies by recursion that, at each level $L \leqslant J, t_{J+1, k}$ coincides with the root $t_{L, l}$ of the connected component $\mathcal{T}_{L, l}$ of $T_{L}$ to which it belongs.

Definition 7.11. With any rooted nest $\mathcal{N}_{\ominus}$ with elements given by (7.8), we associate the compatibly rooted nest

$$
\begin{align*}
{ }^{c} \mathcal{N}_{\oplus} & =\left\{{ }^{\mathrm{c}} T_{J \oplus}\right\}, \\
{ }^{\mathrm{c}} T_{J_{\oplus}} & =\left\{\left(\mathcal{T}_{J, j}, w_{J, j}\right), j=1, \ldots, \operatorname{Card}\left(T_{J}\right)\right\}, \tag{7.9}
\end{align*}
$$

with the same connected components $\mathcal{T}_{J, j}$ at each level $J$, and whose roots $w_{J, j}$ are obtained from the roots $t_{J, j}$ by the following recursion:
(i) at level 0 , the roots of $T_{0}$ are fixed since root $\left(T_{0 \oplus}\right)=G_{\odot}$;
(ii) at level 1 , we identify $w_{1, j}$ with the original root $t_{1, j}$, that is set $w_{1, j}=t_{1, j}$ for all $j=1, \ldots, \operatorname{Card}\left(T_{1}\right)$;
(iii) at level $J+1$ and for each connected component $\mathcal{T}_{J+1, k}$, we look for the component $\mathcal{T}_{J, j(k)}$ of the complete diagram $T_{J}$ at the preceding level $J$ to which the original root $t_{J+1, k}$ belongs. The root $w_{J, j(k)}$ has already been constructed at level $J$ and we make the roots compatible between level $J$ and $J+1$ by substituting to the original root $t_{J+1, k}$ the root $w_{J+1, k}=w_{J, j(k)}$ (Notice that, since $T_{J} \prec T_{J+1}, \mathcal{T}_{J, j(k)} \subset \mathcal{T}_{J+1, k}$ and therefore $\left.w_{J, j(k)} \in \mathcal{T}_{J+1, k}\right)$.

By construction, the rooted nest ${ }^{c} \mathcal{N}_{\oplus}$ is compatibly rooted. Of course, a rooted nest $\mathcal{N}_{\oplus}$ is compatibly rooted if and only if $\mathcal{N}_{\oplus}=\mathcal{N}_{\oplus}$, and in this case, $w_{J, j} \equiv t_{J, j}$ for all $J$ and $j$.

With a rooted diagram $T_{\oplus}$, we associate the Taylor operator $\tau_{T_{\oplus}}$ defined simply as the product of the Taylor operators $\tau_{(\tau, t)}$ of its rooted connected components:

$$
\begin{equation*}
\tau_{T_{\oplus}}=\prod_{\left(\mathcal{T}_{j}, t_{j}\right) \in T_{\oplus}} \tau_{\left(\mathcal{T}_{j}, t_{j}\right)} \tag{7.10}
\end{equation*}
$$

with the convention $\tau_{(\mathcal{T}, t)}=\mathbb{I I}$ if $|\mathcal{T}|=1$ (i.e. $\mathcal{T}=\{t\}$ ), which in particular implies that $\tau_{T_{0 \oplus}}=\mathbb{I}$ for $T_{0}=G_{\odot}$. We denote by $\left\|T_{\oplus}\right\|$ the product of the cardinals of the connected components $\mathcal{T}_{j}$ of the diagram $\operatorname{comp}\left(T_{\oplus}\right)$

$$
\begin{equation*}
\left\|T_{\oplus}\right\|=\prod_{\mathcal{T}_{j} \in \operatorname{comp}\left(T_{\oplus}\right)}\left|\mathcal{T}_{j}\right| . \tag{7.11}
\end{equation*}
$$

elements of the forest.

### 7.2. SECTORS

Definition 7.12: Saturated nest. A saturated nest $\mathcal{S}$ of $\mathcal{G}$ is a nest with $N=$ $\operatorname{Card}(\mathcal{G})$ (distinct) elements ${ }^{*}$, which we call $R^{0}, \ldots, R^{N-1}$.

The cardinal of a saturated nest is therefore maximal. A saturated nest is actually constructed from $G_{\odot}$ (the complete diagram made of $N$ single point connected components) by fusing recursively at each level $R^{I}$ exactly two connected components of the preceding level $R^{I-1}$ until $G=\{\mathcal{G}\}$ is obtained. A saturated nest is therefore characterized as follows:
(i) its minimal diagram is $R^{0}=G_{\odot}$,
(ii) its maximal diagram is $R^{N-1}=G=\{\mathcal{G}\}$,
(iii) $\operatorname{Card}\left(R^{I+1}\right)=\operatorname{Card}\left(R^{I}\right)-1$ for all $I=0, \ldots, N-1$

Saturated nest associated with ordered trees. The notion of saturated nest occurs naturally when spanning integration points by trees, as was done formally in subsect. 3.7. Indeed, let us consider a tree $\mathbf{T}=\left(\lambda_{\alpha} ; \alpha=1, \ldots, N-1\right)$, considered as ordered by increasing values of $\alpha$ (this order will actually correspond to increasing mutual distances, in a generalized sense to be made precise below). Such an ordered tree $\mathbf{T}$ generates naturally a saturated nest $\mathcal{S}(\mathbf{T})$ as follows (see fig. 18a):
(i) $R^{0}=G_{\odot}$;
(ii) at level $I(1 \leqslant I \leqslant N-1)$, we consider the line $\alpha=I$ with end points $i_{\alpha}, i^{\prime}{ }_{\alpha}$ and set $R^{I}=R^{I-1} \vee\left\{\left\{i_{\alpha}, i^{\prime}{ }_{\alpha}\right\}\right\}$, which corresponds to the fusion of the connected component of $R^{I-1}$ containing $i_{\alpha}$ with that containing $i^{\prime}{ }_{\alpha}$.

Of course, different trees $\mathbf{T}$ can yield the same $\mathcal{S}(\mathbf{T})$. This allows us to classify trees into equivalence classes, by regrouping all the trees $\mathbf{T}$ such that $\mathcal{S}(\mathbf{T})=\mathcal{S}$ for any given saturated nest $\mathcal{S}$. If two ordered trees $\mathbf{T}=\left(\lambda_{\alpha} ; \alpha=1, \ldots, N-1\right)$ and $\mathbf{T}^{\prime}=\left(\lambda_{\alpha}^{\prime} ; \alpha=1, \ldots, N-1\right)$ are equivalent, then the transformation from $\lambda$ to $\lambda^{\prime}$ is such that

$$
\begin{equation*}
\lambda_{\alpha}= \pm \lambda_{\alpha}^{\prime}+\sum_{\gamma<\alpha} c_{\alpha}^{\gamma} \lambda_{\gamma}^{\prime}, \quad c_{\alpha}^{\gamma}=0, \pm 1 \tag{7.14}
\end{equation*}
$$

where $c_{\alpha}^{\gamma}$ are coefficients equal to 0 or $\pm 1$ (which are in general further constrained so that $\mathbf{T}$ and $\mathbf{T}^{\prime}$ actually span the same set of integration points).

Oriented ordered tree associated with a compatibly rooted saturated nest. Conversely, if the saturated nest $\mathcal{S}$ is compatibly rooted, there is a natural way to associate with $\mathcal{S}_{\oplus}$ an oriented ordered tree $\mathbf{T}\left(\mathcal{S}_{\oplus}\right)$ (see fig. 18b). Indeed, by definition, a saturated nest $\mathcal{S}=\left\{R^{I}\right\}$ is constructed by fusing recursively at each level $R^{I}$ exactly two connected components $\mathcal{R}^{I-1, k}$ and $\mathcal{R}^{I-1, k^{\prime}}$ of the preceding level $R^{I-1}$. Denoting by $i_{I}$ and $i^{\prime}{ }_{I}$ their respective roots in $R_{\oplus}^{I-1}$, one of these

[^5]

Fig. 18. (a) Saturated nest associated with an ordered tree $T=\left(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right)$. The nest is made of four diagrams. Each diagram is represented by the contour of its connected components with at least two vertices (the diagrams 1,3 and 4 have only one such connected component, the diagram 2 has two such connected components). (b) Oriented ordered tree associated with a compatibly rooted saturated nest. We have first assigned compatible roots to the saturated nest of (a) (here the diagrams 3 and 4, and the connected component on the right of the diagram 2 have the same root) and then constructed the oriented ordered tree from these roots.
roots, say $i_{I}$, is the root of $\mathcal{R}^{I-1, k} \cup \mathcal{R}^{I-1, k^{\prime}}$ in $R_{\oplus}^{I}$, since the rooting is compatible. In this case the other root $i^{\prime}{ }_{I}$ can no longer be the root of any connected component of the diagrams $R^{I^{\prime}}$ for $I^{\prime} \geqslant I$. Therefore, if we define by $\lambda_{I}=x_{i^{\prime} I}-x_{i_{I}}$ the oriented line vector joining the positions of the roots $i_{I}$ and $i^{\prime}{ }_{I}$, the set of $\lambda_{I}$ for $I=1, \ldots, N-1$ defines an oriented ordered (by $I$ ) tree, which we denote by $\mathbf{T}\left(\mathcal{S}_{\oplus}\right)$. Of course, we have by construction $\mathcal{S}\left(\mathbf{T}\left(\mathcal{S}_{\oplus}\right)\right)=\mathcal{S}$. Moreover, one can easily check that the tree $\mathbf{T}\left(\mathcal{S}_{\oplus}\right)$ has the following property: for any $I$ and $I^{\prime}$, the path on the tree joining the two origins $x_{I}$ and $x_{I^{\prime}}$ of the vectors $\lambda_{I}$ and $\lambda_{I^{\prime}}$ passes only through vectors $\lambda_{K}$ for $K>\min \left(I, I^{\prime}\right)$.

Although this construction does not play any role in the present sect. 7 , it will turn out to be useful in sect. 8.

Definition 7.13: Extended Hepp sectors. Now we want to associate with an unrooted saturated nest $\mathcal{S}$ an extended Hepp sector, defined from the Hepp sectors attached to ordered trees constructed in subsect. 5.1 .

If we consider as in subsect. 5.1 the $N$ points as being embedded in $\mathbb{R}^{N-1}$ with cartesian coordinates $0, y_{1}, \ldots, y_{N-1}$, and denote as before $\mathcal{H}^{\mathbf{T}}$ the domain of the $y_{i}$ 's defining the Hepp sector attached to the ordered tree* $T$, we define the Hepp sector $\mathcal{H}^{\mathcal{S}}$ as the union of all Hepp sectors attached to all ordered trees $\mathbf{T}$

[^6]such that $\mathcal{S}(\mathrm{T})=\mathcal{S}$, that is the domain of the $y_{i}$ 's given by
\[

$$
\begin{equation*}
\mathcal{H}^{\mathcal{S}}=\bigcup_{\mathrm{T}: \mathcal{S}(\mathbf{T})=\mathcal{S}} \mathcal{H}^{\mathrm{T}} \tag{7.15}
\end{equation*}
$$

\]

This extended Hepp sector is best described by the vectors $\lambda_{\alpha}$ associated with a given (arbitrary) tree $\mathbf{T}$ such that $\mathcal{S}(\mathbf{T})=\mathcal{S}$. Let us stress that now the $\lambda_{\alpha}$ 's are no longer successive minimal distances when the $y_{i}$ 's move everywhere inside $\mathcal{H}^{\mathcal{S}}$, but are so only for $y_{i}$ 's inside the subset $\mathcal{H}^{\mathbf{T}}$ of $\mathcal{H}^{S}$. In particular, the inequalities $\left|\lambda_{\alpha}\right| \leqslant\left|\lambda_{\alpha+1}\right|$ of (5.1) are not necessarily satisfied inside $\mathcal{H}^{\mathcal{S}}$. Still, for $y_{i}$ 's inside $\mathcal{H}^{\mathcal{S}}$, one can find a tree $\mathrm{T}^{0}$ such that $\mathcal{S}\left(\mathbf{T}^{0}\right)=\mathcal{S}(\mathbf{T})$ and $\left\{y_{i}\right\} \in \mathcal{H}^{\mathbf{T}^{0}}$. The $\lambda_{\alpha}^{0}$ associated with $\mathbf{T}^{0}$ satisfy for this set of $y_{i}^{\prime}$ 's the inequalities $\left|\lambda_{1}^{0}\right| \leqslant \ldots \leqslant\left|\lambda_{N-1}^{0}\right|$. By construction, one has inside $\mathcal{H}^{\mathbf{T}^{0}}$ at each level $\alpha$ : $\left|\lambda_{\alpha}^{0}\right| \leqslant\left|\lambda_{\alpha}\right|$ and, as in (7.14), a relation between the $\lambda_{\alpha}$ 's and the $\lambda_{\alpha}^{0}$ 's of the form $\lambda_{\alpha}= \pm \lambda_{\alpha}^{0}+\sum_{\gamma<\alpha} c_{\alpha}^{\gamma} \lambda_{\gamma}^{0}$ with coefficients $c_{\alpha}^{\gamma}$ equal to 0 or $\pm 1$. We can thus write

$$
\begin{align*}
\left|\lambda_{\alpha}\right|= & \left| \pm \lambda_{\alpha}^{0}+\sum_{\gamma<\alpha} c_{\alpha}^{\gamma} \lambda_{\gamma}^{0}\right| \\
& \leqslant\left|\lambda_{\alpha}^{0}\right|+\sum_{\gamma<\alpha}\left|c_{\alpha}^{\gamma}\right|\left|\lambda_{\gamma}^{0}\right| \\
& \leqslant\left(1+\sum_{\gamma<\alpha}\left|c_{\alpha}^{\gamma}\right|\right)\left|\lambda_{\alpha}^{0}\right| \\
& \leqslant \alpha\left|\lambda_{\alpha}^{0}\right| . \tag{7.16}
\end{align*}
$$

We thus have the set of inequalities

$$
\begin{equation*}
\left|\lambda_{\alpha}^{0}\right| \leqslant\left|\lambda_{\alpha}\right| \leqslant \alpha\left|\lambda_{\alpha}^{0}\right| \tag{7.17}
\end{equation*}
$$

which, together with $\left|\lambda_{\alpha}^{0}\right| \leqslant\left|\lambda_{\alpha+1}^{0}\right|$ implies

$$
\begin{equation*}
\frac{\left|\lambda_{\alpha}\right|}{\left|\lambda_{\alpha+1}\right|} \leqslant \alpha . \tag{7.18}
\end{equation*}
$$

This is an example of constraints satisfied by all tree variables compatible with the nest $\mathcal{S}$ in the extended sector $\mathcal{H}^{\mathcal{S}}$, which is a relaxed extension of (5.1). Another consequence of (7.17) is that if $\mathbf{T}$ and $\mathbf{T}^{\prime}$ are two trees such that $\mathcal{S}(\mathbf{T})=$ $\mathcal{S}\left(\mathbf{T}^{\prime}\right)=\mathcal{S}$, then inside $\mathcal{H}^{\mathcal{S}}$ the corresponding line vectors satisfy

$$
\begin{align*}
\frac{1}{\alpha} \leqslant \frac{\left|\lambda_{\alpha}\right|}{\left|\lambda^{\prime}\right|} & \leqslant \alpha \\
\frac{\left|\lambda_{\alpha}\right|}{\left|\lambda^{\prime} \alpha^{\prime}\right|} & \leqslant \alpha \quad \text { for } \quad \alpha^{\prime}>\alpha \tag{7.19}
\end{align*}
$$

These bounds will be useful in sect. 8 .
The corresponding extended Hepp sector $\mathcal{A}_{N}^{\mathcal{S}}$ in the space $\mathcal{A}_{N}$ of mutual squared distances $a_{i j}$ between vertices (see sect. 3.2) can be described simply, without reference to ordered trees. Given a saturated nest $\mathcal{S}=\left\{R^{0}, \ldots, R^{N-1}\right\}$,
let us consider, for a given diagram $R^{I}$, the smallest squared distance between vertices which belong to two different connected components of the diagram $R^{I}$ (minimal squared distance between connected components):

$$
a_{\min }\left(R^{I}\right)=\min _{\mathcal{R}^{l, k} \neq \mathcal{R}^{l, l} \in R^{I}}\left(\min _{i \in \mathcal{R}^{l, k}, j \in \mathcal{R}^{I, l}}\left(a_{i j}\right)\right)
$$

For the minimal diagram $R^{0}=G_{\odot}$ one has obviously

$$
a_{\min }\left(G_{\odot}\right)=\min _{i \neq j}\left(a_{i j}\right)
$$

and by convention for the maximal diagram $G=\{\mathcal{G}\}$ (which has only one connected component) we set $a_{\text {min }}(G)=\infty$. One can check that one has always, for any saturated nest, $a_{\min }\left(R^{0}\right) \leqslant a_{\min }\left(R^{1}\right) \leqslant \ldots \leqslant a_{\min }\left(R^{N-2}\right)<a_{\min }\left(R^{N-1}\right)$.

The extended Hepp sector $\mathcal{A}_{N}^{\mathcal{S}}$ associated with the saturated nest $\mathcal{S}$ is the subset of $\mathcal{A}_{N}$ such that

$$
\begin{equation*}
a_{\min }\left(R^{0}\right)<a_{\min }\left(R^{1}\right)<\ldots<a_{\min }\left(R^{N-2}\right)<a_{\min }\left(R^{N-1}\right) . \tag{7.20}
\end{equation*}
$$

One can check that the sectors associated with two different saturated nests are $\operatorname{disjoint} \mathcal{A}_{N}^{\mathcal{S}} \cap \mathcal{A}_{N}^{\mathcal{S}^{\prime}}=\emptyset$, and that $\mathcal{A}_{N}$ is the union of the closure of sectors over all saturated nests

$$
\mathcal{A}_{N}=\bigcup_{\mathcal{S} \text { saturated }} \overline{\mathcal{A}_{N}^{\mathcal{S}}}
$$

### 7.3. EQUIVALENCE CLASSES OF NESTS: AN EXAMPLE

In order to prove the finiteness of subtracted correlation functions $\mathcal{Z}_{N}^{\mathbf{R}}{ }_{N}^{(M)}$ in (6.22) when $\epsilon=0$, we shall proceed in a way similar to what was done in sect. 5, by decomposing the domain of integration over positions into extended Hepp sectors and prove that the integration of $\mathbf{R}\left[I_{\mathcal{G}}\left(x_{i}, X_{a}\right)\right]$ inside each extended Hepp sector yields a finite result.

We have seen that UV divergences arise generally when successive subsets of points coalesce. Inside the Hepp sector $\mathcal{H}^{\mathcal{S}}$, these successions must be compatible with the nested structure of $\mathcal{S}$. From (7.12) the subtracted integrand is a sum of contributions associated with (rooted) nests $\mathcal{N}_{\oplus}$, and many contributions (for different nests) give the same divergences inside $\mathcal{H}^{\mathcal{S}}$. The general strategy to prove that the subtracted integrand $\mathbf{R}\left[I_{\mathcal{G}}\left(x_{i}, X_{a}\right)\right]$ is convergent inside the sector $\mathcal{H}^{\mathcal{S}}$ is to regroup the nests giving the same UV divergences into equivalence classes, and to show that all divergences cancel within each equivalence class.

Let us first consider the simple example of a sector associated with a saturated nest $\mathcal{S}$ such that, at some level $I_{0}$, the diagram $R \equiv R^{I_{0}}$ has one and only one connected component $\mathcal{R}$ with $|\mathcal{R}|>1$ and let us focus on the behavior of the subtracted integrand when the points of $\mathcal{R}$ coalesce. More precisely, let us consider the contribution in $\mathbf{R}$ of a rooted nest $\mathcal{N}_{\oplus}$ with one single rooted diagram
$T_{\oplus}$ where $T_{\oplus}$ also has one and only one element ( $\left.\mathcal{T}, w\right)$ with $|\mathcal{T}|>1$ (notice that the nest $\mathcal{N}_{\oplus}$ is automatically compatible). The corresponding contribution is (up to a factor $-1 /|T|$ ):

$$
\begin{equation*}
\tau_{(\mathcal{T}, w)} I_{\mathcal{G}}\left(x_{i}, X_{a}\right)=I_{\mathcal{T}}\left(x_{i}\right) I_{\mathcal{G} / \omega \tau}\left(x_{i}, X_{a}\right), \tag{7.21}
\end{equation*}
$$

where we used as before in (6.19) the short-hand notation $\mathcal{G} / w \mathcal{T} \equiv \mathcal{G} \backslash(\mathcal{T} \backslash\{w\})$ which simply corresponds to replacing $\mathcal{T}$ in $\mathcal{G}$ by its single vertex $w$. We now ask which are the nests whose contribution leads to the same UV behavior when the points of $\mathcal{R}$ coalesce, that is when the positions $x_{i}$ for $i \in \mathcal{R}$ tend altogether to an arbitrary position $x_{0}$ : we shall denote this limit by $\mathcal{R} \rightarrow 0$. In this limit, the first term $I_{\mathcal{T}}\left(x_{i}\right)$ in the r.h.s. of eq. (7.21) factorizes into $I_{\mathcal{R} \cap \mathcal{T}}\left(x_{i}\right) I_{\mathcal{T} / 0(\mathcal{R} \cap \mathcal{T})}\left(x_{i}\right)$, where the notation "/o" means that the vertices of $\mathcal{R} \cap \mathcal{T}$ have been replaced by a single contraction vertex 0 with position $x_{0}$. The factorization of the second term $I_{\mathcal{G} / \omega} \tau\left(x_{i}, X_{a}\right)$ depends on whether or not the point $w$ belongs to $\mathcal{R}$.

Case (a): $w \in \mathcal{R}$.
If $w \in \mathcal{R}$, then we get $I_{\mathcal{G} /_{w} \mathcal{T}}\left(x_{i}, X_{a}\right) \rightarrow I_{\mathcal{R} h_{w}(\mathcal{R} \cap \mathcal{T})}\left(x_{i}\right) I_{\mathcal{G} /(\mathcal{R} \cup \mathcal{T})}\left(x_{i}, X_{a}\right)$. The contribution of $T_{\oplus}$ (7.21) thus behaves as

$$
\begin{gather*}
\tau_{(\mathcal{T}, w)} I_{\mathcal{G}}\left(x_{i}, X_{a}\right) \stackrel{\mathcal{R} \rightarrow 0}{\sim} I_{\mathcal{R} \cap \mathcal{T}}\left(x_{i}\right) I_{\mathcal{T} / o(\mathcal{R} \cap \mathcal{T})}\left(x_{i}\right) I_{(\mathcal{R} \cup \mathcal{T}) / w \mathcal{T}}\left(x_{i}\right) \\
\times I_{\mathcal{G} / o(\mathcal{R} \cup \mathcal{T})}\left(x_{i}, X_{a}\right), \tag{7.22}
\end{gather*}
$$

where we used the fact that $\mathcal{R} / \omega(\mathcal{R} \cap \mathcal{T})=(\mathcal{R} \cup \mathcal{T}) /{ }_{w} \mathcal{T}$. In view of (7.22), let us now consider the product of Taylor operators associated with the larger rooted nest $\tilde{\mathcal{N}}_{\oplus}$ defined as (see fig. 19)

$$
\begin{equation*}
\tilde{\mathcal{N}}_{\oplus}=\{\{(\mathcal{R} \cap \mathcal{T}, w)\},\{(\mathcal{T}, w)\},\{(\mathcal{R} \cup \mathcal{T}, \bullet)\}\} \tag{7.23}
\end{equation*}
$$

with "•" standing for an arbitrary compatible root*. This new nest can be seen as resulting from the superposition of the two nests $\mathcal{N}_{\oplus}$ and $\mathcal{S}$ at level $I_{0}$. Applying the corresponding three $\tau$ 's on the amplitude $I_{\mathcal{G}}$ one obtains

$$
\begin{align*}
\prod_{\tilde{T}_{\oplus} \in \tilde{\mathcal{N}}_{\oplus}} \tau_{\tilde{T}_{\oplus}}\left[I_{\mathcal{G}}\left(x_{i}, X_{a}\right)\right] \propto & I_{\mathcal{R} \cap \mathcal{T}}\left(x_{i}\right) I_{\mathcal{T} h w}(\mathcal{R} \cap \mathcal{T}) \\
& \times x_{\mathcal{G} / \odot(\mathcal{R} \cup \mathcal{T})}\left(x_{i}, X_{a}\right) \tag{7.24}
\end{align*}
$$

In the same limit when all points in $\mathcal{R}$ coalesce to the single point $0, w$ and the compatible root - are replaced by 0 since they both belong to $\mathcal{R}$, and (7.24) is equal to the r.h.s. of (7.22).

[^7]
(a)

(b)

Fig. 19. Schematic picture of the rooted nest $\tilde{\mathcal{N}}_{\oplus}$ when the root $w$ of $\mathcal{T}$ (a) belongs to $\mathcal{R}$, or (b) does not belong to $\mathcal{R}$.

Case (b): $w \notin \mathcal{R}$.
If $w \notin \mathcal{R}$, then we get $I_{\mathcal{G} h_{w} \tau}\left(x_{i}, X_{a}\right) \rightarrow I_{(\mathcal{R} \backslash \mathcal{T})}\left(x_{i}\right) I_{\left(\mathcal{G} h_{w} \tau\right) /_{(\mathcal{R} \backslash \mathcal{T})}}\left(x_{i}, X_{a}\right)$ and the contribution of $T_{\oplus}$ (7.21) behaves as

$$
\begin{align*}
\tau_{(\mathcal{T}, w)} I_{\mathcal{G}}\left(x_{i}, X_{a}\right) \stackrel{\mathcal{R} \rightarrow 0}{\sim} & I_{\mathcal{R} \cap \mathcal{T}}\left(x_{i}\right) I_{\mathcal{T} / b(\mathcal{R} \cap \mathcal{T})}\left(x_{i}\right) I_{\mathcal{R} \backslash \mathcal{T}}\left(x_{i}\right) \\
& \times I_{\left(\mathcal{G} \not W_{w}\right) / o(\mathcal{R} \backslash \mathcal{T})}\left(x_{i}, X_{a}\right) . \tag{7.25}
\end{align*}
$$

The larger rooted nest $\tilde{\mathcal{N}}_{\oplus}$ which gives a similar contribution when $\mathcal{R} \rightarrow 0$ is now defined as (fig. 19)

$$
\begin{equation*}
\tilde{\mathcal{N}}_{\oplus}=\{\{(\mathcal{R} \cap \mathcal{T}, \bullet)\},\{(\mathcal{T}, w)\},\{(\mathcal{R} \backslash \mathcal{T}, \bullet),(\mathcal{T}, w)\}\} . \tag{7.26}
\end{equation*}
$$

Notice that the largest element of $\tilde{\mathcal{N}}_{\oplus}$ is now a diagram with two connected components $\mathcal{R} \backslash \mathcal{T}$ and $\mathcal{T}$. The two cases (a) and (b) can be unified in a single formula. If we denote $T_{\oplus}$ by $(T, \omega)$ where $\omega=\operatorname{root}\left(T_{\oplus}\right)=\{\{w\}\}$, the nest $\tilde{\mathcal{N}}_{\oplus}$ can be written in both cases as

$$
\begin{equation*}
\tilde{\mathcal{N}}_{\oplus}=\left\{(R \wedge T, \bullet),(T, \omega),\left(R \vee_{\omega} T, \bullet\right)\right\} \tag{7.27}
\end{equation*}
$$

where we introduce the union operation $V_{\omega}$ of an unrooted diagram $R$ and a


Fig. 20. The unrooted complete diagram $R \vee_{\omega} T$ (thick full lines) obtained from the unrooted complete diagram $R$ (dashed lines) and the complete rooted diagram $T$ (thin full lines). The diagram $R \vee_{\omega} T$ is obtained by fusing each connected component of $T$ to the connected component of $R$ to which its root belongs, and cutting it out from all the other connected components of $R$.
rooted diagram $(T, \omega)$

$$
\begin{equation*}
R \vee_{\omega} T \equiv[R \backslash(T \backslash \omega)] \vee T=\left[R \backslash\left\{\operatorname{comp}\left(T_{\oplus}\right) \backslash \operatorname{root}\left(T_{\oplus}\right)\right\}\right] \vee \operatorname{comp}\left(T_{\oplus}\right) \tag{7.28}
\end{equation*}
$$

where " $\backslash$ " is the subtraction operation acting on diagrams as in definition 7.5 in sect. 7. The result of this operation is an unrooted diagram equal to $\{\mathcal{R} \cup \mathcal{T}\}$ if the root $w$ of $T$ belongs to the connected component $\mathcal{R}$ of $R$, and equal to $\{(\mathcal{R} \backslash \mathcal{T}, \mathcal{T})\}$ if $w$ does not belong to $\mathcal{R}$ :

$$
R \vee_{\omega} T=\left\{\begin{array}{cl}
\{\mathcal{R} \cup \mathcal{T}\} & \text { if } w \in \mathcal{R}  \tag{7.29}\\
\{\mathcal{R} \backslash \mathcal{T}, \mathcal{T}\} & \text { if } w \notin \mathcal{R}
\end{array}\right.
$$

The operation $R \vee_{\omega} T$ thus consists in a fusion operation of $\mathcal{R}$ and $\mathcal{T}$ into $\mathcal{R} \cup \mathcal{T}$, followed by a cutting out of $\mathcal{T}$ from $\mathcal{R} \cup \mathcal{T}$ if the root $w$ is not shared by $\mathcal{R}$. The above expression for $R \vee_{\omega} T$ can be applied to the more general case when $R=\left\{\mathcal{R}^{i}\right\}$ and $T_{\oplus}=(T, \omega)=\left(\left\{\mathcal{T}_{j}\right\},\left\{w_{j}\right\}\right)$ have more than one connected component, with the result that each connected component $\mathcal{T}_{j}$ of $T$ is fused to the connected component $\mathcal{R}^{i}$ of $R$ which contains its root $w_{j}$, and cut out from all the other connected components of $R$ which it intersects (see fig. 20). Note that the operation $V_{\omega}$ crucially depends on the position of the roots of the diagram $T$ on the right with respect to the connected components of the diagram $R$ on the left, but that these roots are not retained as roots of the resulting diagram $R \vee_{\omega} T$ which by definition is unrooted. The product of Taylor operators associated with the nest $\tilde{\mathcal{N}}_{\oplus}$ as given by (7.27) still corresponds in this case to the combined result of the Taylor operation $\tau_{T_{\oplus}}$ followed by the coalescence of the $\operatorname{Card}(R)$ connected components of $R$ toward arbitrary points.

Finally, we return to the original question of finding the nests $\mathcal{N}^{\prime} \oplus$ which give the same UV behavior as $T_{\oplus}$ when components of $R$ coalesce. These are the rooted nests which build the same factorized integrand (7.22) or (7.25) (possibly generalized to several connected components). They are characterized
by $\mathcal{N}_{\oplus} \subset \mathcal{N}^{\prime}{ }_{\oplus} \subset \tilde{\mathcal{N}}_{\oplus}$. We therefore get the four nests

$$
\begin{align*}
\mathcal{N}_{\oplus} & =\{(T, \omega)\} \\
\mathcal{N}_{\oplus 2} & =\{(R \wedge T, \bullet),(T, \omega)\} \\
\mathcal{N}_{\oplus 3} & =\left\{(T, \omega),\left(R \vee_{\omega} T, \bullet\right)\right\} \\
\tilde{\mathcal{N}}_{\oplus} & =\left\{(R \wedge T, \bullet),(T, \omega),\left(R \vee_{\omega} T, \bullet\right)\right\} \tag{7.30}
\end{align*}
$$

One can check (see appendix $D$ ) that the ( -1 ) and symmetry factors associated with these four nests sum up to give zero exactly (this includes a sum over the unspecified compatible roots $\bullet$ ). As a consequence, the divergences induced in the contributions of the four nests above by the coalescence of the points in the subset $\mathcal{R}$ cancel exactly. This property can be generalized to nests $\mathcal{N}_{\oplus}$ with an arbitrary number of diagrams as well as to successive coalescences associated with a saturated nest $\mathcal{S}$. Indeed, from the nest $\mathcal{N}_{\oplus}$, we can build a family of nests $\mathcal{N}^{\prime} \oplus$ giving the same divergences when points coalesce successively according to the nested structure of $\mathcal{S}$; we then can check that these divergences cancel exactly within the obtained family. The details of this construction will be discussed in subsect. 7.4.

### 7.4. EQUIVALENCE CLASSES OF NESTS: GENERAL CONSTRUCTION

In this section, we present a general procedure for classifying nests according to the diverging behavior of the associated counterterm in a given sector. Our construction is inspired by a construction by Bergère and Lam in ref. [23] in the context of local field theories in the Schwinger representation. Extensive modifications are however necessary in order to make this construction applicable in our context.

We denote by $\mathcal{S}=\left\{R^{0}, R^{1} \ldots, R^{N-1}\right\}$ a saturated nest of $\mathcal{G}$, which will be kept fixed throughout this section. We are going to regroup all rooted nests into equivalence classes, associated with $\mathcal{S}$.

Tableau construction. From now on and until the end of the article, the only rooted nests which we shall consider will be compatibly rooted nests.

Let us thus consider an arbitrary compatibly rooted nest $\mathcal{N}_{\oplus}=\left\{T_{J \oplus} ; J=\right.$ $0, \ldots, T\}$ where $T_{J \oplus}=\left(T_{J}, \omega_{J}\right)$. For this compatibly rooted nest, we define the (unrooted) complete diagram

$$
\begin{equation*}
R_{J}^{I} \equiv R^{I} \vee_{\omega_{J}} T_{J} \equiv\left(R^{I} \backslash\left(T_{J} \backslash \omega_{J}\right)\right) \vee T_{J} \tag{7.31}
\end{equation*}
$$

and build the tableau

where by convention $T_{T+1} \equiv G=\{\mathcal{G}\}$. Notice that for $R^{0}=G_{\odot}$, we have $R_{J}^{0}=R_{J}^{0} \wedge T_{J+1}=T_{J}$. Hence the first column $T_{J}=R_{J}^{0} \wedge T_{J+1}$ of the tableau can be seen as being built from $R^{0}$, with the same structure as the other columns. Notice also that since $R^{N-1}=G, R_{J}^{N-1}=G$ for any $J$, hence $R_{J}^{N-1} \wedge T_{J+1}=$ $T_{J+1}$. Therefore the last element of a given line of the tableau is identical to the first element of the following line. Finally, since $R^{I} \prec R^{I+1}$, then $R_{J}^{I} \prec R_{J}^{I+1}$ and

$$
\begin{equation*}
R_{J}^{I} \wedge T_{J+1} \prec R_{J}^{I+1} \wedge T_{J+1} \tag{7.33}
\end{equation*}
$$

Therefore, reading the tableau in the natural order, i.e. reading successive lines from the left to the right, we get a totally nested structure, which defines an unrooted nest $\tilde{\mathcal{N}}$. This nest $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}\right)$ depends on both the sector nest $\mathcal{S}$ and the subtraction nest $\mathcal{N}_{\oplus}$. By construction, $\tilde{\mathcal{N}}$ contains all the diagrams of $\mathcal{N}$. Of course, it may happen that two successive elements of the tableau are identical (this is for instance the case for the last element of a line and the first element of the next line), hence the tableau contains redundant information.

The nest $\tilde{\mathcal{N}}$ is a generalization of the one constructed in the previous section (Eq. (7.27)). Indeed, if we consider the nest $\mathcal{N}_{\oplus}=\left\{\left(G_{\odot}, G_{\odot}\right),(T, \omega)\right\}$ and set $R^{I_{0}}=R$ at level $I_{0}$ of the nest $\mathcal{S}$, we obtain in this case the simple tableau

$$
\begin{array}{lll}
T_{0}=G_{\odot} & \ldots & R_{0}^{I_{0}} \wedge T_{1}=\left(R \vee_{G_{\odot}} G_{\odot}\right) \wedge T=R \wedge T \ldots \\
T_{1}=T & \ldots & R_{1}^{I_{0}} \wedge T_{2}=\left(R \vee_{\omega} T\right) \wedge G=R \vee_{\omega} T \quad \ldots \tag{7.34}
\end{array}
$$

where only columns 1 and $I_{0}$ are specified. The general construction (7.32) therefore reproduces in this simple case exactly the largest nest $\tilde{\mathcal{N}}$ (here unrooted) of (7.30).

Reduction of the tableau. Going back to the general case, we are now interested in finding the smallest rooted nest $\mathcal{N}_{\oplus}^{0}$ which, under a construction similar to (7.32), gives the same nest $\tilde{\mathcal{N}}$ (that is $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right)=\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}\right)$ ). More precisely, we must remove from $\mathcal{N}_{\oplus}$ the diagrams $T_{J}$ which are not necessary to build $\tilde{\mathcal{N}}$. Since $T_{J}$ is involved in the construction of the two lines $J-1$ and $J$, removing $T_{J}$ from the nest $\mathcal{N}_{\oplus}$ amounts to replace these two lines by a single
line, which will be built directly from $T_{J-1}$ and $T_{J+1}$. In this process, $N$ diagrams will be lost. Therefore, removing $T_{J}$ will be possible if the tableau contains $N$ redundant diagrams, which happens when at least $N+1$ successive diagrams of the two lines $J-1$ and $J$ are identical. This implies that there exists an $I_{0}$ such that the two vertically adjacent elements of the column $I_{0}$ coincide at levels $J-1$ and $J$ :

$$
\begin{equation*}
R_{J-1}^{I_{0}} \wedge T_{J}=R_{J}^{I_{0}} \wedge T_{J+1} \tag{7.35}
\end{equation*}
$$

that is, on the tableau


Then, by the inclusion property (7.33), all the diagrams of $\tilde{\mathcal{N}}$ between $R_{J_{-1}}^{I_{0}} \wedge T_{J}$ and $R_{J}^{I_{0}} \wedge T_{J+1}$ are identical, hence equal to $T_{J}$ itself. We thus do not loose any information by replacing the two lines $J-1$ and $J$ by the single line

$$
\begin{align*}
T_{J-1}, R_{J-1}^{1} \wedge T_{J}, \ldots, & R_{J-1}^{I_{0}} \wedge T_{J} \\
& =T_{J} \\
& =R_{J}^{I_{0}} \wedge T_{J+1}, \ldots, R_{J}^{N-1} \wedge T_{J+1} \tag{7.37}
\end{align*}
$$

The important point is that this new line is precisely the one which would have been constructed directly by (7.32), when applied to the nest

$$
\mathcal{N}_{\oplus}^{\prime}=\left(T_{0 \oplus}, T_{1 \oplus}, \ldots, T_{J-1 \oplus}, T_{J+1 \oplus}, \ldots, T_{T \oplus}\right)
$$

obtained from $\mathcal{N}_{\oplus}$ by removing $T_{J_{\oplus}}$ (notice that the induced rooting of this nest remains compatible). Indeed, the construction (7.32) for $\mathcal{N}^{\prime} \oplus$ simply corresponds to suppressing the $J$-line and to substituting to the ( $J-1$ )-line the new line, constructed from $T_{J-1 \oplus}$ and $T_{J+1 \oplus}$ :

$$
\begin{equation*}
T_{J-1}, \quad R_{J-1}^{1} \wedge T_{J+1}, \quad \ldots, \quad R_{J-1}^{I_{0}} \wedge T_{J+1}, \quad \ldots, \quad R_{J-1}^{N-1} \wedge T_{J+1} \tag{7.38}
\end{equation*}
$$

the other lines remaining unchanged. It is the purpose of appendix $E$ to establish in detail the statement, on which all our construction will rely, that the lines (7.37) and (7.38) are actually identical when (7.35) is satisfied. As a consequence, the nests $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}\right)$ and $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}^{\prime} \oplus\right)$ are equal. In particular, we note
that $T_{J}$, while absent from $\mathcal{N}^{\prime}{ }_{\oplus}$, is still present in $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}^{\prime}{ }_{\oplus}\right)$ since

$$
\begin{equation*}
T_{J}=R_{J-1}^{I_{0}} \wedge T_{J+1} \tag{7.39}
\end{equation*}
$$

The "suppression" of line $J$ from (7.36) when (7.35) is satisfied, consistent with the construction of $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}^{\prime} \oplus\right)$, can be visualized as follows:

where the double and triple lines represent successively nested (in general distinct) diagrams, while the single line represents a series of identical diagrams.

We therefore have at our disposal a reduction procedure, which allows for the substitution to the nest $\mathcal{N}_{\oplus}$ of the reduced nest $\mathcal{N}^{\prime}{ }_{\oplus}$, with one diagram less, which still generates the same nest $\tilde{\mathcal{N}}$. This process can be iterated to suppress all the diagrams $T_{J}$ of the original nest $\mathcal{N}$ which are such that they satisfy the coincidence property (7.35) for at least one $I_{0}\left(1 \leqslant I_{0} \leqslant N-1\right)$. When two successive lines possess this coincidence property, for some $I_{0}$ and $I_{1}$, the reduction is associative, that is its result is independent of the order of the operations, as represented on the following picture:

$$
\begin{aligned}
& \overline{=} I_{0}:::: I_{1} \ldots \\
& =I_{0} \longrightarrow \rightarrow \ldots \ldots \ldots . I_{1} \bar{\Longrightarrow} \\
& \text { _ } I_{0}:::: I_{1} \ldots . \quad \rightarrow \overline{\overline{I_{0}}:::: I_{1} \overline{\overline{ }}} \\
& \ldots \ldots \ldots . . I_{1} \equiv \rightarrow=I_{0} \\
& -I_{0}:::: I_{1} \bar{\Longrightarrow}
\end{aligned}
$$

Notice furthermore that a configuration like

$$
J-1 \Longrightarrow I_{0}
$$


which would cause obstruction to associativity, is actually forbidden since it would imply $T_{J}=T_{J+1}$, which is ruled out by definition. Notice finally that the "suppression" of a line $J$ does not create new coincidences (that is coincidences which did not exist before suppression ). Indeed, the only pairs of vertical neighbors which are modified by the suppression are those of the lines $J-2$ and $J-1$
for $I>I_{0}$ on the one hand, and those of the lines $J$ and $J+1$ for $I<I_{0}$ on the other hand, as can be seen on the following picture:


A new coincidence would imply $T_{J-1}=T_{J}$ in the first (upper right) case, and $T_{J}=T_{J+1}$ in the second (lower left) case, and is thus impossible. Therefore, after "suppression" of all the lines of the original nest which present a vertical coincidence with the preceding line, we end up with a tableau which no longer contains any pair of coinciding vertical neighbors. We denote by $\mathcal{N}_{\oplus}^{0}$ the nest resulting from this reduction procedure, that is the subset of $\mathcal{N}_{\oplus}$ made of the diagrams $T_{J \oplus}$ for values of $J$ corresponding to lines which remain after reduction.
Equivalence classes of nests. The above reduction allows to assign to any compatibly rooted nest $\mathcal{N}_{\oplus}$ a unique minimal nest $\mathcal{N}_{\oplus}^{0}$, which is a subset of the original nest $\mathcal{N}_{\oplus}$ (and in particular whose compatible rooting is the restriction of the original rooting of $\mathcal{N}_{\oplus}$ to $\mathcal{N}^{0}$, such that $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right)=\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}\right)$, and whose tableau (7.32) is "minimal", i.e. has no vertically adjacent coinciding elements${ }^{\star}$.
We define the equivalence class $\mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)$ of a minimal (with respect to $\mathcal{S}$ ) nest $\mathcal{N}_{\oplus}^{0}$ as the set of all compatibly rooted nests $\mathcal{N}_{\oplus}$ which lead by reduction of their $\mathcal{S}$-tableau to that minimal nest $\mathcal{N}_{\oplus}^{0}$ :

$$
\mathcal{N}_{\oplus} \in \mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right) \quad \Longleftrightarrow \quad \mathcal{N}_{\oplus}^{\text {tableau }} \tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}\right)^{\text {reduction }} \mathcal{N}_{\oplus}^{0} .
$$

Of course, if $\mathcal{N}_{\oplus}^{0}$ is minimal with respect to $\mathcal{S}$, one has $\mathcal{N}_{\oplus}^{0} \in \mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)$. For any $\mathcal{N}_{\oplus} \in \mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)$, one has $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}\right)=\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right)$.
We have the following characterization, for any compatibly rooted nest $\mathcal{N}_{\oplus}$ (with $\mathcal{N}$ the corresponding unrooted nest):
Theorem 7.1. Characterization of $\mathcal{C}_{S}\left(\mathcal{N}_{\oplus}^{0}\right)$.

$$
\begin{equation*}
\mathcal{N}_{\oplus} \in \mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right) \quad \Longleftrightarrow \quad \text { (a) } \mathcal{N}_{\oplus}^{0} \subset \mathcal{N}_{\oplus} \text { and (b) } \mathcal{N} \subset \tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right) \tag{7.40}
\end{equation*}
$$

A nest of the equivalence class $\mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)$ is thus constituted of all the diagrams of $\mathcal{N}^{0}$ plus some of the diagrams of $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right)$ not in $\mathcal{N}^{0}$. Its rooting is constrained to be both compatible and such that its restriction to $\mathcal{N}^{0}$ is the rooting of $\mathcal{N}_{\oplus}^{0}$. Conversely, one builds all the elements of $\mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)$ by completing $\mathcal{N}_{\oplus}^{0}$ by an arbitrary number of diagrams of $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right) \backslash \mathcal{N}^{0}$ (that is diagrams of $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right)$

[^8]not in $\mathcal{N}^{0}$ ), and assigning to these extra elements any roots compatible with the roots of $\mathcal{N}_{\oplus}^{0}$. The direct implication $(\Rightarrow)$ is immediate since
(i) the reduced rooted nest is always a subset of the original rooted nest, hence (a);
(ii) any diagram of $T_{J}$ of $\mathcal{N}_{\oplus}$ belongs to $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}\right)$ and the reduction process is defined so as to leave $\tilde{\mathcal{N}}$ invariant. Thus $T_{J} \in \tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right)$, hence (b).

The reverse implication $(\Leftarrow)$ is not immediate and is proven in appendix $F$.
Notice finally that the diagram $G=\{\mathcal{G}\}$ is always a diagram of $\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right)$ since the last element ( $I=N-1$ ) of the last line ( $J=T$ ) of the tableau of any nest is always equal to $G$. As a consequence, $G$ is never a diagram of $\mathcal{N}_{\oplus}^{0}$ since it can be rebuilt from $\mathcal{N}_{\oplus}^{0}$ by the tableau construction. Actually, if a nest contains the diagram $G$, the line of its tableau built from $G$ has all its elements equal to $G$, while the preceding line has its last element equal to $G$; this leads to the coincidence property for these two lines for $I_{0}=N-1$, indicating that $G$ is to be suppressed in the construction of $\mathcal{N}_{\oplus}^{0}$. Therefore, for any minimal nest $\mathcal{N}_{\oplus}^{0}$, one has $G \in \tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right) \backslash \mathcal{N}^{0}$.

### 7.5. FACTORIZATION OF THE R OPERATOR INSIDE AN EQUIVALENCE CLASS

As we have seen before, the reason for classifying nests into equivalence classes was to regroup nests whose diverging contributions in a given sector $\mathcal{S}$ in the $\mathbf{R}$ operator (7.12) cancel exactly. Given a sector nest $\mathcal{S}$, it is therefore natural to rewrite the $\mathbf{R}$ operator, which is a sum over all compatibly rooted nests, as a sum of reduced operators $\mathbf{R}_{\mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)}$, each of them involving all the nests in the equivalence class $\mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)$ of a minimal (with respect to $\mathcal{S}$ ) nest $\mathcal{N}_{\oplus}^{0}$. This reads

$$
\begin{equation*}
\mathbf{R}=\sum_{\substack{\mathcal{N}_{\oplus}^{0} \text { minimal } \\ \text { w.r.t. } \mathcal{S}}} \mathbf{R}_{\mathcal{C}_{S}\left(\mathcal{N}_{\oplus}^{0}\right)}, \quad \mathbf{R}_{\mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)}=-\sum_{\mathcal{N}_{\oplus} \in \mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)} W\left(\mathcal{N}_{\oplus}\right) \prod_{T_{\oplus} \in \mathcal{N}_{\oplus}}\left(-\tau_{T_{\oplus}}\right) . \tag{7.41}
\end{equation*}
$$

Each operator $\mathbf{R}_{\mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)}$, can then be rewritten as a sum of factorized contributions associated with different rootings of the elements of the equivalence class, as will now be explained.

We will need a lemma about partial sums over compatible rootings of nests. Let us consider a nest $\mathcal{M}=\left\{T_{J} ; J=1, \ldots, T\right\}$. We denote by $\oplus_{\mathcal{M}}$ a compatible rooting of $\mathcal{M}$, that is simply the specification for each diagram $T_{J}$ of $\mathcal{M}$ of a root diagram $\omega_{J}$ such that $\mathcal{M}_{\oplus \mathcal{M}} \equiv\left\{\left(T_{J}, \omega_{J}\right) ; J=1, \ldots, T\right\}$ is a compatibly rooted nest.

Lemma 7.1. Given a compatibly rooted nest $\mathcal{N}_{\oplus}$ and an unrooted nest $\mathcal{M}$ such that $\mathcal{N} \subset \mathcal{M}$ (that is all the diagrams of $\mathcal{N}$ are diagrams of $\mathcal{M}$ ), we can consider all the compatible rootings $\oplus \mathcal{M}$ of $\mathcal{M}$ such that $\mathcal{N}_{\oplus} \subset \mathcal{M}_{\oplus \mathcal{M}}$, that is the
compatible rootings of $\mathcal{M}$ whose restriction to $\mathcal{N}$ is the rooting in $\mathcal{N}_{\oplus}$; we then have the useful sum rule for the weights (7.13):

$$
\begin{equation*}
\sum_{\oplus_{\mathcal{M}}: \mathcal{N}_{\oplus} \subset \mathcal{M}_{\oplus \mathcal{M}}} W\left(\mathcal{M}_{\oplus \mathcal{M}}\right)=W\left(\mathcal{N}_{\oplus}\right) \tag{7.42}
\end{equation*}
$$

This lemma is proven in appendix $G$.
We can now use this property in the case of an arbitrary nest $\mathcal{N}_{\oplus} \in \mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)$ if we choose

$$
\begin{equation*}
\mathcal{M}=\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{N}_{\oplus}^{0}\right) \tag{7.43}
\end{equation*}
$$

since, from (7.40), we have $\mathcal{N} \subset \mathcal{M}$. Inserting (7.42) in the formula (7.41) for $\mathbf{R}_{\mathcal{C}_{s}\left(\mathcal{N}_{\oplus}^{0}\right)}$, we get

$$
\begin{align*}
& \mathbf{R}_{\mathcal{C}_{S}\left(\mathcal{N}_{\oplus}^{0}\right)}=-\sum_{\mathcal{N}_{\oplus} \in \mathcal{C}_{S}\left(\mathcal{N}_{\oplus}^{0}\right)} \sum_{\substack{\oplus \\
\mathcal{N}_{\oplus} \subset \mathcal{M}_{\oplus}}} W\left(\mathcal{M}_{\oplus}\right) \prod_{T_{\oplus} \in \mathcal{N}_{\oplus}}\left(-\tau_{T_{\oplus}}\right) \\
& =-\sum_{\substack{\oplus \\
\mathcal{N}_{\oplus}^{0} \subset \mathcal{M}_{\oplus}:}} W\left(\mathcal{M}_{\oplus}\right) \sum_{\substack{\mathcal{N}_{\oplus}: \\
\mathcal{N}_{\oplus}^{0} \subset \mathcal{N}_{\oplus} \subset \mathcal{M}_{\oplus}}} \prod_{\substack{T_{\oplus} \in \mathcal{N}_{\oplus}}}\left(-\tau_{T_{\oplus}}\right) \\
& =-\sum_{\substack{\oplus \mathcal{M}^{\prime} \\
\mathcal{N}_{\oplus}^{0} \subset \mathcal{M}_{\oplus}}} W\left(\mathcal{M}_{\oplus}\right) \prod_{T_{\oplus}^{0} \in \mathcal{N}_{\oplus}^{0}}\left(-\tau_{T_{\oplus}^{0}}\right) \prod_{T_{\oplus} \in\left(\mathcal{M}_{\oplus} \backslash \mathcal{N}_{\oplus}^{0}\right)}\left(1-\tau_{T_{\oplus}}\right), \tag{7.44}
\end{align*}
$$

where $\mathcal{M}_{\oplus}$ stands for $\mathcal{M}_{\oplus \mathcal{M}}$. In the second equation, we used the characterization (7.40) of $\mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)$. The sum rule (7.42) allows us to reconstruct all possible rootings of the nests $\mathcal{N}$ in $\mathcal{C}_{\mathcal{S}}\left(\mathcal{N}_{\oplus}^{0}\right)$ with the appropriate weight, by first fixing the roots of $\mathcal{M}_{\oplus}$ by a compatible extension of the roots of $\mathcal{N}_{\oplus}^{0}$, and then restricting these roots of $\mathcal{M}_{\oplus}$ to all intermediate subnests $\mathcal{N}$ between $\mathcal{N}^{0}$ and $\mathcal{M}$ (notice that a given rooting of such a nest $\mathcal{N}$ can come from different rootings of $\mathcal{M}$ ). In the last equation, we used the fact that the set of rooted nests $\mathcal{N}_{\oplus}$ such that $\mathcal{N}_{\oplus}^{0} \subset \mathcal{N}_{\oplus} \subset \mathcal{M}_{\oplus}$ is built by taking necessarily, on the one hand all the diagrams $T_{\oplus}^{0}$ of the minimal nest $\mathcal{N}_{\oplus}^{0}$ and, for each diagram $T_{\oplus}$ of $\mathcal{M}_{\oplus} \backslash \mathcal{N}_{\oplus}^{0}$ on the other hand, deciding whether to take it or not, hence choosing 1 or $-\tau_{T_{\oplus}}$ in the expansion of the product of Taylor operators.

Notice finally that the compatibly rooted nests $\mathcal{M}_{\oplus}$ involved in (7.44) can actually be characterized independently of the minimal nest $\mathcal{N}_{\oplus}^{0}$ from which they are built, by the property

$$
\begin{equation*}
\tilde{\mathcal{N}}\left(\mathcal{S}, \mathcal{M}_{\oplus}\right)=\mathcal{M} \tag{7.45}
\end{equation*}
$$

A compatibly rooted nest satisfying (7.45) will be called maximal with respect to $\mathcal{S}$. With this definition, the equations (7.41) and (7.44) can be replaced by
the single equation

$$
\begin{equation*}
\mathbf{R}=\sum_{\substack{\mathcal{M}_{\oplus} \text { maximal } \\ \text { w.r.t. } \mathcal{S}}} W\left(\mathcal{M}_{\oplus}\right) \mathbf{R}_{\mathcal{M}_{\oplus}} \tag{7.46}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathbf{R}_{\mathcal{M}_{\oplus}}=-\prod_{T_{\oplus}^{0} \in \mathcal{N}_{\oplus}^{0}}\left(-\tau_{T_{\oplus}^{0}}\right) \prod_{T_{\oplus} \in\left(\mathcal{M}_{\oplus} \backslash \mathcal{N}_{\oplus}^{0}\right)}\left(1-\tau_{T_{\oplus}}\right) \tag{7.47}
\end{equation*}
$$

where $\mathcal{N}_{\oplus}^{0}$ is now the minimal nest obtained by reducing the tableau of the maximal nest $\mathcal{M}_{\oplus}$.

## 8. Proof of UV convergence

We are now in a position to prove the finiteness of subtracted correlation functions $\mathcal{Z}^{\mathbf{R}_{N}^{(M)}}$ in (6.22) when $\epsilon=0$. Our strategy is the following:
(I) First we partition the domain of integration over positions into extended Hepp sectors (as defined in subsect. 7.2), each of them being characterized by a saturated nest $\mathcal{S}$.
(II) In each sector $\mathcal{S}$, we reorganize the $\mathbf{R}$ operator by use of (7.46) as a sum of operators $\mathbf{R}_{\mathcal{M}_{\oplus}}$ associated with the different nests $\mathcal{M}_{\oplus}$ maximal with respect to $\mathcal{S}$.
(III) At the end, one can write $\mathcal{Z}_{N}^{\mathbf{R}^{(M)}}$ as

$$
\begin{equation*}
\mathcal{Z}_{N}^{\mathbf{R}^{(M)}}\left(X_{a}\right)=\sum_{\mathcal{S}} \sum_{\substack{\mathcal{M}_{\oplus}^{\text {maximal }} \\ \text { w.i.s. }}} W\left(\mathcal{M}_{\oplus}\right) \int_{\mathcal{H}^{s}} \prod_{i \in \mathcal{G}} \mathrm{~d}^{D} x_{i} \mathbf{R}_{\mathcal{M}_{\oplus}}\left[I_{\mathcal{G}}\left(x_{i}, X_{a}\right)\right] \tag{8.1}
\end{equation*}
$$

It is therefore sufficient to prove the finiteness of the integral

$$
\begin{equation*}
\int_{\mathcal{H}^{\mathcal{S}}} \prod_{i \in \mathcal{G}} \mathrm{~d}^{D} x_{i} \mathbf{R}_{\mathcal{M}_{\oplus}}\left[I_{\mathcal{G}}\left(x_{i}, X_{a}\right)\right] \tag{8.2}
\end{equation*}
$$

where we integrate over the domain $\mathcal{H}^{\mathcal{S}}$ defined by (7.15) * with the measure (3.24), and where $\mathcal{M}_{\oplus}$ is any nest maximal with respect to $\mathcal{S}$.
(IV) Using the factorized form (7.47) for $\mathbf{R}_{\mathcal{M}_{\oplus}}$, we first apply the Taylor operators $\tau_{T_{J}^{0}}$ associated with diagrams of the minimal nest $\mathcal{N}_{\oplus}^{0}$. This results in factorizing $I_{\mathcal{G}}\left(x_{i}, X_{a}\right)$ into a product of amplitudes $I_{\tilde{T}}=\prod_{\tilde{T} \in \tilde{T}} I_{\tilde{T}}$ for suitable reduced diagrams $\tilde{T}$ made of subsets $\tilde{\mathcal{T}}$ of $\mathcal{G}$.
(V) We show that the product of the remaining ( $1-\tau_{T_{\oplus}}$ ) operators acts independently on each subdiagram $\tilde{T}$, leading to a subtracted integrand for $\tilde{T}$.
(VI) We show that this subtracted integrand, once integrated over points in the Hepp sector $\mathcal{H}^{\mathcal{S}}$, yields a finite result when $\epsilon=0$.

[^9]Furthermore, from the compatibility requirement for $\mathcal{M}_{\oplus}$, the root in $\mathcal{M}_{\oplus}$ of any connected component of the diagram $T_{J-1}^{l}$ automatically belongs to the corresponding reduced connected component of the reduced diagram $\tilde{T}_{J-1}^{I}$. Therefore, the rooting of $\mathcal{M}_{\oplus}$ naturally induces a rooting for the diagrams $\tilde{T}_{J-1}^{I}$. We denote by $w_{J-1, j}^{I, i}$ the root of $\tilde{\mathcal{T}}_{J-1, j}^{I, i}$ and by $\tilde{T}_{J-1 \oplus}^{I}$ the set of all ( $\tilde{\mathcal{T}}_{J-1, j}^{I, i}, w_{J-1, j}^{I, i}$ ) for varying $i$ and $j$.

Let us for a while concentrate on what happens inside some given subset $\tilde{T}_{J, j}$ which we shall assume to have at least two vertices $\left(\operatorname{Card}\left(\tilde{\mathcal{T}}_{J, j}\right)>1\right)$. We can consider the family of different (and non empty) rooted subsets ( $\tilde{\mathcal{T}}_{J-1, j}^{I, i}, w_{J-1, j}^{I, i}$ ) for all $i=1, \ldots, \operatorname{Card}\left(R^{I}\right)$ (with $J$ and $j$ fixed) as a complete rooted diagram $\tilde{T}_{J-1, j \oplus}^{I}$ of the subset $\tilde{\mathcal{T}}_{J, j}$ in which we are now working. From (8.8), this is nothing but the restriction of the diagram $R^{I}$ to this subset $\tilde{\mathcal{T}}_{J, j}$, together with a set of roots. The family of distinct $\tilde{T}_{J-1, j \oplus}^{I}$ for varying $I$ forms a compatibly rooted and saturated nest, $\mathcal{S}_{J, j \oplus}$, of $\tilde{\mathcal{T}}_{J, j}$, which is nothing but the restriction of the saturated nest $\mathcal{S}$ to $\tilde{\mathcal{T}}_{J, j}$, with a given rooting. We define

$$
\begin{equation*}
\overline{\operatorname{Ind}}(J, j)=\left\{I \geqslant 1: \tilde{T}_{J-1, j}^{I} \neq \tilde{T}_{J-1, j}^{I-1}\right\} \tag{8.9}
\end{equation*}
$$

as the set of indices $I$ (of the sector $\mathcal{S}$ ) such that inside $\tilde{\mathcal{T}}_{J, j}$, a new element $\tilde{T}$ appears at level $I$ in the saturated nest $\mathcal{S}_{J, j}$.

We now again consider the whole diagram $\tilde{T}_{J}$ and define, in a way similar to (8.9):

$$
\begin{equation*}
\overline{\operatorname{Ind}}(J)=\left\{I \geqslant 1: \tilde{T}_{J-1}^{I} \neq \tilde{T}_{J-1}^{I-1}\right\}=\left\{I \geqslant 1: T_{J-1}^{I} \neq T_{J-1}^{I-1}\right\} \tag{8.10}
\end{equation*}
$$

as the set of $I$ 's such that a new diagram appears in the tableau at level $I$ between $T_{J-1}^{0}$ and $T_{J}^{0}$. Of course, if $\tilde{T}_{J-1}^{I} \neq \tilde{T}_{J-1}^{I-1}$, there exists at least one $j$ such that $\tilde{T}_{J-1, j}^{I} \neq \tilde{T}_{J-1, j}^{I-1}$ and thus

$$
\begin{equation*}
\overline{\operatorname{Ind}}(J)=\bigcup_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)} \overline{\operatorname{Ind}}(J, j) \tag{8.11}
\end{equation*}
$$

We moreover denote

$$
\begin{equation*}
I^{\min }(J)=\min (\overline{\operatorname{Ind}}(J)), \quad I^{\max }(J)=\max (\overline{\operatorname{Ind}}(J)) \tag{8.12}
\end{equation*}
$$

with the property that

$$
\begin{equation*}
I^{\max }(J)=\min \left\{I: \tilde{T}_{J-1}^{I}=\tilde{T}_{J}\right\}=\min \left\{I: T_{J-1}^{I}=T_{J}^{0}\right\} \tag{8.13}
\end{equation*}
$$

is the index $I$ such that $T_{J}^{0}$ appears at first in the tableau. We set

$$
\begin{equation*}
\operatorname{Ind}(J)=\overline{\operatorname{Ind}}(J) \backslash\left\{I^{\max }(J)\right\} \tag{8.14}
\end{equation*}
$$

(which may be empty). Finally, we define

$$
\begin{equation*}
\overline{\mathrm{Ind}}=\bigcup_{J=1}^{T+1} \overline{\operatorname{Ind}}(J) \tag{8.15}
\end{equation*}
$$

With these notations, the $(1-\tau)$ operators in (7.44) act independently on each amplitude $I_{\tilde{T}_{J}}$. The operator ( $1-\tau_{T_{K \oplus}^{\prime}}$ ) acts on $I_{\tilde{T}_{J}}$ only if $K=J-1$, and results in this case in $\left(1-\tau_{\tilde{T}_{J-1 \oplus}^{\prime}}\right)\left[I_{\tilde{T}_{J}}\right]$. We thus can express $\mathbf{R}_{\mathcal{M}_{\oplus}}\left[I_{\mathcal{G}}\right]$ as a product of subtracted amplitudes for each reduced diagram $\tilde{T}_{J}$. The subtracted amplitude for $\tilde{T}_{J}$ is obtained by the successive action on $I_{\tilde{T}_{J}}$ of a ( $1-\tau_{\tilde{T}_{J-1 \oplus}^{I}}$ ) operator for each $I \in \operatorname{Ind}(J)$. The case $J=T+1$ is special since, since in addition to the $\left(1-\tau_{\tilde{T}_{T_{\oplus}^{\prime}}}\right)$ operator for each $I \in \operatorname{Ind}(T+1)$, a ( $1-\tau$ ) operator is also associated with $\tilde{T}_{T \oplus}^{I^{\max }(T+1) \star}{ }^{*}$. The factorization of $\mathbf{R}_{\mathcal{M}_{\oplus}}\left[I_{\mathcal{G}}\right]$ is then expressed in the following equation:

$$
\begin{align*}
\mathbf{R}_{\mathcal{M}_{\oplus}}\left[I_{\mathcal{G}}\left(x_{i}, X_{a}\right)\right]= & \prod_{I \in \operatorname{Ind}(T+1)}\left(1-\tau_{\tilde{T}_{T \oplus}^{I}}\right) I_{\tilde{T}_{T+1}}\left(x_{i}, X_{a}\right) \\
& \times \prod_{J=1}^{T}\left[\prod_{I \in \operatorname{Ind}(J)}\left(1-\tau_{\tilde{T}_{J-1 \oplus}^{J}}\right)\left[I_{\tilde{T}_{J}}\left(x_{i}\right)\right]\right] . \tag{8.16}
\end{align*}
$$

We recall that

$$
\begin{align*}
& \tau_{\tilde{T}_{J-1 \oplus}^{\prime}}=\prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)} \tau_{\tilde{T}_{J-1, j \oplus}^{\prime}} \\
&=\prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)}  \tag{8.17}\\
& \prod_{\left(\tilde{T}_{J-1, j}^{l}, w_{J-1, j}^{l, i}\right) \in \tilde{T}_{J-1, j \oplus}^{\prime}} \tau_{\left(\tilde{\mathcal{T}}_{J-1, j}^{I, i}, w_{j-1, j}^{I, j}\right)} .
\end{align*}
$$

We have thus achieved point $(\mathrm{V})$. It remains to show that the subtractions associated with the ( $1-\tau$ )'s are sufficient to make (8.16) integrable in the sector $\mathcal{S}$.

### 8.2. APPROPRIATE TREE VARIABLES

In sect. 5, in order to prove the convergence of the original (unsubtracted) integral in some Hepp sector (for $\epsilon>0$ ), we found useful to express the measure in terms of tree variables for some specific tree (which defined the sector). Those tree variables are no longer adapted to deal with the subtracted amplitude $\mathbf{R}_{\mathcal{M}_{\oplus}}\left[I_{\mathcal{G}}\left(x_{i}, X_{a}\right)\right]$ since they do not take into account the factorization (8.16) of $\mathbf{R}_{\mathcal{M}_{\oplus}}\left[I_{\mathcal{G}}\left(x_{i}, X_{a}\right)\right]$. Instead, we shall look for tree variables associated with a tree that, inside each subset $\tilde{\mathcal{T}}_{J, j}$, forms a subtree compatible with the sector. The basic idea is that, since the nest $\mathcal{S}_{J, j \oplus}$, which has been defined above as the restriction of the sector nest $\mathcal{S}$ to $\tilde{\mathcal{T}}_{J, j}$, is both saturated in $\tilde{\mathcal{T}}_{J, j}$ and rooted, it naturally defines a unique oriented ordered tree $\mathbf{T}_{J, j}$ spanning the vertices

[^10]

Fig. 21. Appropriate tree variables. At level $J$, inside a connected component $\tilde{T}_{J, j}$ of $\tilde{T}_{J}$ (dashed circles), we build an oriented ordered tree with line vectors $\lambda_{J, j}^{I}$. As shown in the framed box, this tree is built in a way similar to what was done in fig. (18) (b), now from the rooted saturated nest $\mathcal{S}_{J, j \oplus}$. This nest is here made of the three diagrams $\tilde{T}_{J-1, j}^{I_{1}}, \tilde{T}_{J-1, j}^{I_{2}}$ and $\tilde{T}_{J-1, j}^{I_{3}}$ (i.e. $\overline{\operatorname{Ind}}(J, j)=\left\{I_{1}, I_{2}, I_{3}\right\}$ ) whose roots are represented by the dashed squares. At level $J+1$, the connected components of $\tilde{T}_{J}$ are fully contracted toward their roots (big black dots), which are the vertices of $\tilde{\mathcal{T}}_{J+1, k}$. An oriented ordered tree with line vectors $\lambda_{J+1, k}^{I^{\prime}}$ is then built inside $\tilde{T}_{J+1, k}$. The trees at levels $J$ and $J+1$ can be fused into a single oriented (but only partially ordered) larger tree contributing to (8.19).
of $\tilde{\mathcal{T}}_{J, j}$, as discussed in sect. 7.2. The corresponding line vectors are naturally ordered by increasing values of $I$ in $\overline{\operatorname{Ind}}(J, j)$ and denoted by

$$
\begin{equation*}
\lambda_{J, j}^{I} ; \quad I \in \overline{\operatorname{Ind}}(J, j) \tag{8.18}
\end{equation*}
$$

From the nested structure of $\mathcal{N}_{\oplus}^{0}$, we deduce that the union of the trees $\mathbf{T}_{J, j}$ for varying $J$ and $j$ (including $J=T+1$ ) forms a tree of the set $\mathcal{G}$ :

$$
\begin{equation*}
\mathbf{T}=\bigcup_{J, j} \mathbf{T}_{J, j}=\left(\lambda_{J, j}^{I} ; \quad J=1, \ldots, T+1 ; j=1, \ldots, \operatorname{Card}\left(T_{J}^{0}\right) ; I \in \overline{\overline{\operatorname{Ind}}}(J, j)\right) \tag{8.19}
\end{equation*}
$$

(see fig. 21)). In particular, this tree has $N-1$ line vectors. We can therefore use the corresponding tree variables $\lambda_{J, j}^{I}$ as integration variables, instead of the

Eq. (8.24) defines $\beta^{I}$ for $I \in \overline{\text { Ind. We shall also use the convention }}$

$$
\begin{equation*}
\beta^{I}=1 \quad \text { if } \quad I \notin \overline{\text { Ind. }} \tag{8.26}
\end{equation*}
$$

In order to compare $\lambda_{J, j}^{I}$ to its representative $\lambda^{I}$, we define

$$
\begin{equation*}
\chi_{J, j}^{I}=\frac{\left|\lambda_{J, j}^{I}\right|}{\left|\lambda^{I}\right|}, \tag{8.27}
\end{equation*}
$$

with of course $\chi_{J, j}^{I}=1$ if $\lambda_{J, j}^{I}$ has been chosen as a representative. We thus have

$$
\begin{equation*}
\left|\lambda_{J, j}^{I}\right|=\chi_{J, j}^{I} \beta^{I} \beta^{I+1} \ldots \beta^{N-1} \tag{8.28}
\end{equation*}
$$

and the bounds

$$
\begin{equation*}
\frac{1}{I} \leqslant \chi_{J, j}^{I} \leqslant I \tag{8.29}
\end{equation*}
$$

Finally, in addition to their moduli, the line vectors $\lambda_{J, j}^{I}$ are characterized by a set of relative angles $\theta_{J, j}^{I, n}$ labeled by some extra index $n$. These angles can be constructed in different ways, corresponding in particular to different orderings of the tree T. In any case, as in subsect. 5.2 , these angle variables do not actually play any role in the proof of the finiteness of the integral (8.2). Therefore, we shall not make their construction more explicit.

In terms of the $|\lambda|$ - and $\theta$-variables, the measure term can be written, up to a global numerical factor, as in e.g. (3.32), as

$$
\begin{equation*}
\prod_{J=1}^{T+1} \prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)} \prod_{I \in \operatorname{Ind}(J, j)}\left[\mathrm{d}\left|\lambda_{J, j}^{I}\right|\left|\lambda_{J, j}^{I}\right|^{D-1}\left(\prod_{n}\left(\sin \theta_{J, j}^{I, n}\right)^{p(D, n)} \mathrm{d} \theta_{J, j}^{I, n}\right)\right] \tag{8.30}
\end{equation*}
$$

where $p(D, n)$ is some positive number (when $D \geqslant N+M-1$ ).

### 8.3. SUBTRACTED INTEGRAND

The tree variables of the preceding section, together with the factorization (8.16) allow us to work separately inside each reduced diagram $\tilde{T}_{J}$. Indeed, the amplitude $I_{\tilde{T}_{J}}$ for $J \leqslant T$ is a function of the variables $\lambda_{J, j}^{I}$ for the same $J$ only, with $I \in \overline{\operatorname{Ind}}(J, j)$ (the case $J=T+1$ which is special since it also involves the set of external points, will be discussed separately). Going back to the definition of the amplitude $I_{\tilde{T}_{J}}$, we can write (8.2) in a form where the measure and the integrand are factorized simultaneously. For each $\tilde{T}_{J}$ (with $J \leqslant T$ ), we get

$$
\begin{align*}
& \prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)} \prod_{I \in \underline{\operatorname{Ind}(J, j)}} \mathrm{d}\left|\lambda_{J, j}^{I}\right|\left|\lambda_{J, j}^{I}\right|^{D-1}\left(\prod_{n} \sin \left(\theta_{J, j}^{I, n}\right)^{p(D, n)}\right) \\
& \times \prod_{I \in \operatorname{Ind}(J)}\left(1-\tau_{\tilde{T}_{J-I \oplus}^{I}}\right)\left[\prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)} \operatorname{det}\left(\Pi^{\mathrm{T}_{J, j}}\right)\right]^{-d / 2} \tag{8.31}
\end{align*}
$$

where $\Pi^{\mathrm{T}_{J, j}}$ is the matrix defined by (3.49), (3.50) for the subtree $\mathrm{T}_{J, j}$. Its elements $\Pi_{I, I^{\prime}}^{\mathrm{T}_{J, j}}$ are labeled by elements $I, I^{\prime}$ of $\overline{\operatorname{Ind}}(J, j)$. As in the convergence proof of sect. 5 , we introduce the normalized matrix

$$
\begin{equation*}
Y_{I, I^{\prime}}^{\mathbf{T}_{J, j}} \equiv \frac{1}{A_{D}(\nu)} \frac{\Pi_{I, I^{\prime}}^{\mathbf{T}_{J, j}}}{\left|\lambda_{J, j}^{I}\right|^{\mid}\left|\lambda_{J, j}^{I}\right|^{\nu}} \tag{8.32}
\end{equation*}
$$

which, inserted in eq. (8.31), gives

$$
\begin{align*}
& \prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)} \prod_{I \in \operatorname{Ind}(J, j)} \frac{\mathrm{d}\left|\lambda_{J, j}^{I}\right|}{\left|\lambda_{J, j}^{I}\right| 1-\epsilon}\left(\prod_{n} \sin \left(\theta_{J, j}^{I, n}\right)^{p(D, n)}\right) \\
& \times \prod_{I \in \operatorname{Ind}(J)}\left(1-\tau_{\tilde{T}_{J-1 \oplus}^{\prime}}^{0}\right)\left[\prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)} \operatorname{det}\left(Y^{\mathbf{T}_{J, j}}\right)\right]^{-d / 2} \tag{8.33}
\end{align*}
$$

Since through (8.32) we have extracted the most singular factor of the $\Pi$ matrices under rescalings $\mathcal{D}^{\rho}$ (see eq. (6.14)), the Taylor operators $\tau^{0}$ appearing in (8.33) are now defined as

$$
\begin{equation*}
\tau^{0}=\lim _{\rho \rightarrow 0} \mathcal{D}^{\rho} \tag{8.34}
\end{equation*}
$$

The properties of $\operatorname{det}\left(Y^{\mathrm{T}_{J, j}}\right)$ are the same as those mentioned in subsect. 5.2. In particular, $\operatorname{det}\left(Y^{\mathbf{T}_{J, j}}\right)$ is a function of the ratios of $\lambda_{J, j}^{I}$ for successive $I$ 's in $\overline{\text { Ind }}(J, j)$, which play the role of the $\beta_{\alpha}$ variables of sect. 5 , and are now products of the $\beta$ and $\chi$ variables defined above.

Then $\operatorname{det}\left(Y^{\mathbf{T}_{J, j}}\right)$ is a bounded function of the $\beta^{I}$ and $\chi_{J, j}^{I}$ variables on the domain $\mathcal{H}^{\mathcal{S}}$, and is equal to 1 when all the $\beta^{I}$ 's are set to zero.

Due to our choice for the $\lambda_{J, j}^{I}$ variables, the action of $\tau_{\tilde{T}_{J-1 \oplus}^{I}}^{0}$ on

$$
\prod_{j=1}^{\operatorname{Card}\left(T_{j}^{0}\right)}\left(\operatorname{det}\left(Y^{\mathbf{T}_{J, j}}\right)\right)^{-d / 2}
$$

simply corresponds to set $\beta^{I}=0$ in all the $Y^{\mathbf{T}_{J, j}}$ for different $j$ (see eqs. (8.21) and (8.22)). Therefore

$$
\left(1-\tau_{\tilde{T}_{J-1 \oplus}^{\prime}}^{0}\right)\left[\prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)}\left(\operatorname{det}\left(Y^{\mathbf{T}_{J, j}}\right)\right)^{-d / 2}\right]
$$

vanishes when $\beta^{I} \rightarrow 0$. This is the key property which will ensure the finiteness of the subtracted integrals. First we have to generalize this fact to all the $\beta$ variables. This is contained in the following stronger property, as shown in appendix H :

Proposition

$$
\begin{equation*}
\prod_{I \in \operatorname{Ind}(J)}\left(1-\tau_{\tilde{T}_{J-1 \oplus}^{\prime}}^{0}\right)\left[\prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)}\left(\operatorname{det}\left(Y^{\mathbf{T}_{J, j}}\right)\right)^{-d / 2}\right]=\mathcal{O}\left(\prod_{I^{\min }(J) \leqslant I<I_{\max }(J)}\left(\beta^{I}\right)^{\delta}\right) \tag{8.35}
\end{equation*}
$$

with $\delta=\min (\nu, 1-\nu)$ as in (4.7).
The above discussion holds for $J \leqslant T$ only. The case $J=T+1$ (and $j=1$ ) requires a separate analysis. We then have

$$
\begin{equation*}
I_{\tilde{T}_{T+1 \oplus}}=\left(\operatorname{det}\left(Y^{\mathbf{T}_{T+1,1}}\right)\right)^{-d / 2} \exp \left[-\frac{1}{2} \sum_{a, b} \boldsymbol{k}_{a} \cdot \boldsymbol{k}_{b} \Delta_{a b}\right] \tag{8.36}
\end{equation*}
$$

and a property similar to (8.35):

$$
\begin{gather*}
\prod_{I \in \overline{\operatorname{Ind}}(T+1)}\left(1-\tau_{\tilde{T}_{T \oplus}^{\prime}}^{0}\right)\left[\left(\operatorname{det}\left(Y^{\mathbf{T}_{T+1,1}}\right)\right)^{-d / 2} \exp \left[-\frac{1}{2} \sum_{a, b} \boldsymbol{k}_{a} \cdot \boldsymbol{k}_{b} \Delta_{a b}\right]\right] \\
=\mathcal{O}\left(\prod_{I \geqslant I^{\min }(T+1)}\left(\beta^{I}\right)^{\delta}\right) \tag{8.37}
\end{gather*}
$$

### 8.4. PROOF OF FINITENESS

From the above discussion, we arrive at the following form for (8.2) at $\epsilon=0$ :

$$
\begin{align*}
\int_{\mathcal{D}^{s}} & \prod_{J=1}^{T+1} \prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)}\left[\prod_{I \in \operatorname{Ind}(J, j)}\left(\prod_{n}\left(\sin \theta_{J, j}^{I, n}\right)^{p(D, n)} \mathrm{d} \theta_{J, j}^{I, n}\right) \prod_{I \in \operatorname{Ind}(J, j)}^{\prime} \frac{\mathrm{d} \chi_{J, j}^{I}}{\chi_{J, j}^{I}}\right] \\
& \times \prod_{I \in \operatorname{Ind}} \frac{\mathrm{~d} \beta^{I}}{\beta^{I}} \mathcal{O}\left(\prod_{I \geqslant I^{\min }(T+1)}\left(\beta^{I}\right)^{\delta}\right) \prod_{J=1}^{T} \mathcal{O}\left(\prod_{I^{\min }(J) \leqslant I<I^{\max }(J)}\left(\beta^{I}\right)^{\delta}\right), \tag{8.38}
\end{align*}
$$

where $\Pi^{\prime}$ means that we omit the values of $I$ such that $\lambda_{J, j}^{I}$ is a representative, and where the domain of integration $\mathcal{D}^{\mathcal{S}}$ reproduces the domain of integration $\mathcal{H}^{\mathcal{S}}$ for the relative positions of internal points. Inside $\mathcal{D}^{S}$, the variables $\chi_{J, j}^{I}$ are bounded from below according to (8.29). Therefore, the integration over these $\chi_{J, j}^{I}$ variables and the integration over the $\theta_{J, j}^{I, n}$ variables do not produce any divergence. For the integral to be convergent, it is actually sufficient that, for each $I \in \overline{\operatorname{Ind}}$, at least one $\left(\beta^{I}\right)^{\delta}$ is present in the product of $\mathcal{O}$ 's appearing in (8.38), thus making the integration over $\beta^{I}$ UV convergent. This will be true if

$$
\begin{equation*}
\overline{\operatorname{Ind}} \subset\left\{\bigcup_{J=1}^{T}\left[I^{\min }(J), I^{\max }(J)\right)\right\} \cup\left[I^{\min }(T+1), N-1\right] . \tag{8.39}
\end{equation*}
$$

one sees that the problem of UV convergence (which comes from the small $\beta_{\alpha}$ behavior) is completely decoupled from the problem of analytic continuation of the measure in $D$ (which comes from the behavior of the integral when $\theta_{\alpha, n} \rightarrow$ 0 or $\pi$ for $n>D$ ). As already discussed in subsect. 3.3, an explicit representation of the analytically continued amplitude can be written, for non-integer $D$, by subtracting the divergent powers of $\theta$ and $\pi-\theta$ (this is the standard finite part prescription). The resulting integration over the $\theta$ 's are convergent, for fixed non-zero $\beta$ 's. From the explicit form of the matrix $Y_{\alpha \beta}^{\mathrm{T}}$, one can check that the subtractions in $\theta$ do not introduce dangerous negative powers of the $\beta$ 's (at least in the sector $\mathcal{H}^{\mathbf{T}}$, i.e. $\mathcal{D}^{\mathbf{T}}$ ), so that the power counting argument in the $\beta$ 's stays valid. Finally one can check that (as already done in subsect. 3.3), the poles that occur at integer $D$ are cancelled by the corresponding zeroes of the global factor $S_{D} S_{D-1} \ldots S_{D-N+2}$ in the measure (3.29), so that the unsubtracted amplitude $\mathcal{Z}_{N}$ is finite for any $D>0$ and $\epsilon>0$.

The same argument can be applied to the subtracted amplitude at $\epsilon=0$. Starting from the expression (8.38) for the part associated with the maximal nest $\mathcal{M}_{\oplus}$ of the subtracted amplitude in an extended Hepp sector, some of the $p(D, n)$ exponents become negative for $D<N+M-1$, and the integration over the corresponding angular variables $\theta_{J, j}^{I, n}$ requires a finite part subtraction prescription. Again, one can argue that these subtractions do not interfere with the power counting in $\beta$ 's and $\chi$ 's, and that the small $\beta$ estimates (8.35) and (8.37) remain valid for the $\theta$-subtracted integrands.

Finally, one can extend this analysis to small negative $\epsilon$, and show that for a subtracted amplitude of order $N$, no UV divergences occur as long as $\operatorname{Re}(\epsilon)>$ $-\delta /(N-1)$, with $\delta=\min (\nu, 1-\nu)$, as in (4.7). Indeed, for $\epsilon \neq 0$, we must modify (8.38) by inserting in the integrand

$$
\begin{equation*}
\prod_{J=1}^{T+1} \prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)} \prod_{I \in \overline{\operatorname{Ind}}(J, j)}^{\prime}\left(\chi_{J, j}^{I}\right)^{\epsilon} \times \prod_{I \in \overline{\operatorname{Ind}}}\left(\beta^{I}\right)^{n(I) \epsilon} \tag{9.2}
\end{equation*}
$$

where $n(I)$ is the number of line vectors $\lambda_{J, j}^{I^{\prime}}$ with an index $I^{\prime} \leqslant I$. One has clearly $n(I) \leqslant I \leqslant N-1$. Since the subtracted interaction term is (from (8.38) and (8.39)) $\mathcal{O}\left(\prod_{I \in \overline{\text { Ind }}}\left(\beta^{I}\right)^{\delta}\right)$, the convergence at small $\beta$ 's is guaranteed for $\operatorname{Re}(\epsilon)>-\delta /(N-1)$.

Finally, we have not discussed the problem of the convergence or summability of the perturbative series for our model. Since the model is expected to make sense for both $b>0$ and $b<0$ (with a finite free energy proportional to the internal volume in the latter case), we expect that the radius of convergence of these series will be non-zero, and in fact infinite for the unrenormalized series (which exists for $\epsilon>0$, thus defining entire functions of $b$ ).

### 9.2. UNIVERSAL SCALING PROPERTIES OF THE MANIFOLD

In this subsection, we shall derive some physical implications of the existence of a renormalized theory, well defined at $\epsilon=0$. We shall consider here explicitly the case of elastic membranes with $k=2$ in (2.1).

The main result of the preceding sections is that the subtracted amplitudes (6.22) for the correlation functions remain finite at $\epsilon=0$. In terms of these, the full correlation functions

$$
\begin{align*}
\mathcal{Z}^{(M)}\left(X_{a}, \boldsymbol{k}_{a} ; b\right) & =\mathcal{Z}^{\mathbf{R}^{(M)}}\left(X_{a}, \boldsymbol{k}_{a} ; b_{\mathbf{R}}\right) \\
& =\sum_{N=0}^{\infty} \frac{\left(-b_{\mathrm{R}}\right)^{N}}{N!} \mathcal{Z}_{N}^{\mathbf{R}^{(M)}}\left(X_{a}, \boldsymbol{k}_{a}\right) \tag{9.3}
\end{align*}
$$

have a series expansion in terms of the effective excluded volume parameter:

$$
\begin{equation*}
b_{\mathrm{R}}=\frac{1}{\mathcal{V}_{\mathcal{S}_{D}}}\left(V_{\mathbb{R}^{d}}-\mathcal{Z}\right), \tag{9.4}
\end{equation*}
$$

which represents the resummed one-point interaction of the manifold with the impurity. As functions of $b_{\mathrm{R}}$ and $\epsilon$, these correlation functions thus stay finite at $\epsilon=0$.

Existence of a Wilson function. Our renormalization operation involves a peculiar renormalized coupling constant $b_{\mathrm{R}}$ (9.4), which is a function:

$$
\begin{equation*}
b_{\mathrm{R}} \equiv b_{\mathrm{R}}(b, X ; \epsilon) \tag{9.5}
\end{equation*}
$$

where $X$ is the internal linear size of the manifold, defined by

$$
\begin{equation*}
\mathcal{V}_{\mathcal{S}_{D}} \equiv X^{D} \tag{9.6}
\end{equation*}
$$

As usual, since the renormalization operator $\mathbf{R}$ deals only with local counterterms, other choices of the renormalized coupling constant are possible, keeping the correlation functions finite as in (9.3). In particular, the theory describing the manifold of a given size $X$ remains finite when expressed in terms of the parameter

$$
\begin{equation*}
b_{\mathrm{R}}(\lambda) \equiv b_{\mathrm{R}}(b, \lambda X ; \epsilon) \tag{9.7}
\end{equation*}
$$

which corresponds to the renormalized coupling constant of a (reference) manifold with different size $\lambda X$. In particular, the original $b_{\mathrm{R}}(b, X ; \epsilon)$ itself can be expressed in terms of $b_{\mathrm{R}}(\lambda)$ (and $\lambda$ ):

$$
\begin{equation*}
b_{\mathrm{R}}(b, X ; \epsilon)=B_{\mathrm{R}}\left(b_{\mathrm{R}}(\lambda), \lambda, X ; \epsilon\right), \tag{9.8}
\end{equation*}
$$

where $B_{\mathrm{R}}$ stays finite at $\epsilon=0$. This information is best expressed by writing

$$
\begin{equation*}
0=\lambda \frac{\mathrm{d}}{\mathrm{~d} \lambda} b_{\mathrm{R}}(b, X ; \epsilon)=\left.\lambda \frac{\mathrm{d}}{\mathrm{~d} \lambda} b_{\mathrm{R}}(\lambda) \frac{\partial}{\partial b_{\mathrm{R}}}\right|_{\lambda, X} B_{\mathrm{R}}+\left.\lambda \frac{\partial}{\partial \lambda}\right|_{b_{\mathrm{R}}(\lambda), X} B_{\mathrm{R}}, \tag{9.9}
\end{equation*}
$$

from which we deduce that the quantity $\lambda(\mathrm{d} / \mathrm{d} \lambda) b_{\mathrm{R}}(\lambda)$ remains finite at $\epsilon=0$ when expressed in terms of $b_{\mathrm{R}}(\lambda), X$ and $\lambda$. This ensures in particular the finiteness at $\epsilon=0$ of the Wilson function

$$
\begin{equation*}
\left.\left.X \frac{\partial}{\partial X}\right|_{b} b_{\mathbf{R}} \equiv \lambda \frac{\mathrm{d}}{\mathrm{~d} \lambda} b_{\mathrm{R}}(\lambda)\right|_{\lambda=1} . \tag{9.10}
\end{equation*}
$$

As in (6.29), it is convenient to introduce the dimensionless coupling constants

$$
\begin{align*}
g & \equiv\left(2 \pi A_{D}(\nu)\right)^{-d / 2} b_{\mathrm{R}} X^{\epsilon}, \\
z & \equiv\left(2 \pi A_{D}(\nu)\right)^{-d / 2} b X^{\epsilon} \tag{9.11}
\end{align*}
$$

with $A_{D}(\nu)=\left(S_{D}(2-D) / 2\right)^{-1}$ for $k=2$. The associated Wilson function then does not depend on $X$ explicitly and reads

$$
\begin{equation*}
\left.W(g, \epsilon) \equiv X \frac{\partial}{\partial X}\right|_{b} g=\epsilon z \frac{\mathrm{~d} g}{\mathrm{~d} z} \tag{9.12}
\end{equation*}
$$

It is finite at $\epsilon=0$, to all orders in $g$, and has the first order expansion (2.6):

$$
\begin{equation*}
W(g)=\epsilon g-\frac{1}{2} S_{D} g^{2}+\mathcal{O}\left(g^{3}, g^{2} \epsilon\right) \tag{9.13}
\end{equation*}
$$

with a fixed point at

$$
\begin{equation*}
g^{\star}=\frac{2 \epsilon}{S_{D}}+\mathcal{O}\left(\epsilon^{2}\right) \tag{9.14}
\end{equation*}
$$

Universality for the excluded volume and the osmotic pressure. Let us consider the quantity

$$
\begin{equation*}
\mathbb{A}=V_{\mathbb{R}^{d}}-\mathcal{Z}=b_{\mathrm{R}} \mathcal{V}_{\mathcal{S}_{D}} \tag{9.15}
\end{equation*}
$$

which has the dimension of a $d$-volume. For $b>0$ (repulsive interaction) it is positive and represents an effective hard-sphere like excluded volume for the manifold around the impurity.

According to the definition (9.11) of $g$, we have explicitly

$$
\begin{equation*}
\mathbb{A}=g\left(2 \pi A_{D}(\nu)\right)^{d / 2} \nu_{S_{D}}^{d \nu / D} \tag{9.16}
\end{equation*}
$$

The internal volume of the manifold, $\mathcal{V}_{\mathcal{S}_{D}}$, is not directly observable, but, according to (2.31) and (3.38), it is related to the geometrical extension of the membrane in bulk $d$-dimensional space, when no impurity is present ( $b=0$ ). This extension can be measured, for instance, by the radius of gyration $\mathbb{R}_{\mathrm{G}}$ of the noninteracting manifold, defined as

$$
\begin{align*}
\mathbb{R}_{\mathrm{G}}^{2} & \equiv \frac{1}{2 \mathcal{V}_{S_{D}}^{2}}\left\langle\int_{\mathcal{S}_{D}} \mathrm{~d}^{D} x \int_{\mathcal{S}_{D}} \mathrm{~d}^{D} y[\boldsymbol{r}(x)-\boldsymbol{r}(y)]^{2}\right\rangle_{0} \\
& =\operatorname{Tr}^{\prime}\left(\frac{1}{-\Delta}\right) \tag{9.17}
\end{align*}
$$

Owing to the internal spherical symmetry of the manifold, $\mathcal{Z}^{\circ}$ is independent of $X_{1}$ and actually equals

$$
\begin{equation*}
\mathcal{Z}^{\circ}=\int \mathcal{D}[\boldsymbol{r}] \exp (-\mathcal{H}) \frac{1}{\mathcal{V}_{\mathcal{S}_{D}}} \int_{\mathcal{S}_{D}} \mathrm{~d}^{D} x \delta^{d}(\boldsymbol{r}(x)) \tag{9.24}
\end{equation*}
$$

From (2.9), one has clearly

$$
\begin{equation*}
\mathcal{Z}^{\diamond}=-\left.\frac{1}{\mathcal{V}_{S_{D}}} \frac{\partial}{\partial b}\right|_{X} \mathcal{Z}(b, X)=\left.\frac{\partial b_{\mathrm{R}}}{\partial b}\right|_{X} \tag{9.25}
\end{equation*}
$$

Notice that, while the unrestricted partition function $\mathcal{Z}$ has the dimension of a $d$-volume, the pinned-manifold partition function $\mathcal{Z}^{\circ}$ is dimensionless and is thus a function $\mathcal{Z}^{\circ}(z ; \epsilon)$ of $z$ (and $\epsilon$ ) only. According to (9.11) and (9.12), we have

$$
\begin{equation*}
\mathcal{Z}^{\diamond}=\left.\frac{\partial}{\partial b}\right|_{X} b_{\mathrm{R}}=\frac{\mathrm{d} g}{\mathrm{~d} z}=\frac{1}{\epsilon z} W(g(z) ; \epsilon) \tag{9.26}
\end{equation*}
$$

Notice that $\mathcal{Z}^{\circ}$ itself is not renormalized, i.e. not finite at $\epsilon=0$ as a function of $g$, but that $\epsilon z \mathcal{Z}^{\diamond}=W(g, \epsilon)$ is renormalized. When the size $X$ becomes large (for $\epsilon$ and $b$ positive) $z$ becomes large and $g(z)$ tends to its limit $g^{\star}$, the Wilson function vanishing as

$$
\begin{equation*}
W(g(z) ; \epsilon)=\left(g(z)-g^{\star}\right) W^{\prime}\left(g^{\star}\right)+\ldots, \tag{9.27}
\end{equation*}
$$

with

$$
\begin{equation*}
g(z)-g^{\star} \sim \operatorname{const} z^{W^{\prime}\left(g^{\star}\right) / \epsilon} \tag{9.28}
\end{equation*}
$$

(Notice that $W^{\prime}\left(g^{\star}\right)<0$; see fig. 2.) This finally leads to the scaling law for $\mathcal{Z}^{\circ}$ :

$$
\begin{equation*}
\mathcal{Z}^{\diamond} \sim \text { const } z^{-1+W^{\prime}\left(g^{\star}\right) / \epsilon} \sim \text { const }\left(b^{1 / \epsilon} X\right)^{W^{\prime}\left(g^{\star}\right)-\epsilon} \tag{9.29}
\end{equation*}
$$

At first order in $\epsilon, W^{\prime}\left(g^{\star}\right)=-\epsilon+\mathcal{O}\left(\epsilon^{2}\right)$, whence

$$
\begin{equation*}
\mathcal{Z}^{\circ} \sim \text { const }\left(b^{1 / \epsilon} X\right)^{-2 \epsilon} \tag{9.30}
\end{equation*}
$$

Universal $1 / r$ repulsion law. The pinned-manifold partition function $\mathcal{Z}^{\circ}$ is a particular case of a more general restricted partition function to which we now turn. We introduce

$$
\begin{equation*}
\mathcal{Z}^{\circ}\left(X_{1}, \boldsymbol{r} ; X, b ; \epsilon\right)=\int \mathcal{D}[\boldsymbol{r}] \exp (-\mathcal{H}) \delta^{d}\left(\boldsymbol{r}\left(X_{\mathbf{1}}\right)-\boldsymbol{r}\right) \tag{9.31}
\end{equation*}
$$

which describes the partition function of a manifold held by one of its points at the position $r$ relative to the origin. It is the Fourier transform of the one-point correlation function (2.25) for $M=1$, i.e.

$$
\begin{equation*}
\mathcal{Z}^{\diamond}\left(X_{1}, \boldsymbol{r} ; X, b ; \epsilon\right)=\int \mathrm{d}^{d} \boldsymbol{k}_{1} \exp \left(-i \boldsymbol{k}_{1} \cdot \boldsymbol{r}\right) \mathcal{Z}^{(1)}\left(X_{1}, \boldsymbol{k}_{1} ; X, b ; \epsilon\right) \tag{9.32}
\end{equation*}
$$

As above, for a closed manifold, $\mathcal{Z}^{\circ}\left(X_{1}, r ; X, b ; \epsilon\right)$ is actually independent of $X_{1}$ and equal to

$$
\begin{equation*}
\mathcal{Z}^{\diamond}(\boldsymbol{r} ; X, b ; \epsilon)=\int \mathcal{D}[\boldsymbol{r}] \exp (-\mathcal{H}) \frac{1}{\mathcal{V}_{\mathcal{S}_{D}}} \int_{\mathcal{S}_{D}} \mathrm{~d}^{D} x \delta^{d}(\boldsymbol{r}(x)-\boldsymbol{r}) \tag{9.33}
\end{equation*}
$$

The relations of this partition function to the former ones are

$$
\begin{align*}
\mathcal{Z}^{\diamond}(\mathbf{0}) & =\mathcal{Z}^{\diamond} \\
\int_{\mathbb{R}^{d}} \mathrm{~d}^{d} \boldsymbol{r} \mathcal{Z}^{\diamond}(\boldsymbol{r}) & =\mathcal{Z} \tag{9.34}
\end{align*}
$$

By rotational symmetry, the quantity $\mathcal{Z}^{\circ}$ depends only on $r \equiv|r|$. It is furthermore dimensionless, and thus can be written as a function of $z$ and $r / X^{\nu}$ (and $\epsilon)$,

$$
\begin{equation*}
\mathcal{Z}^{\diamond}(\boldsymbol{r} ; X, b ; \epsilon) \equiv \mathcal{Z}^{\diamond}\left[r / X^{\nu}, z ; \epsilon\right] \tag{9.35}
\end{equation*}
$$

As we have seen for $\mathcal{Z}^{\diamond}(9.26), \mathcal{Z}^{\diamond}\left[r / X^{\nu}, z ; \epsilon\right]$ is not exactly renormalized, when expressed in terms of $g$, but $\epsilon z \mathcal{Z}^{\diamond}\left[r / X^{\nu}, z ; \epsilon\right]$ is. It is interesting to consider the limit when the interaction parameter $b$ goes to infinity, while keeping the size $X$ of the manifold finite. We expect $\mathcal{Z}^{\diamond}\left[r / X^{\nu}, z ; \epsilon\right]$ to reach a finite limit

$$
\begin{equation*}
\mathcal{Z}_{\infty}^{\diamond}\left[r / X^{\nu} ; \epsilon\right] \equiv \lim _{z \rightarrow \infty} \mathcal{Z}^{\diamond}\left[r / X^{\nu}, z ; \epsilon\right] \tag{9.36}
\end{equation*}
$$

According to (9.34) and (9.4), we have

$$
\begin{equation*}
\int_{\mathbb{R}^{d}} \mathrm{~d}^{d} r\left(\mathcal{Z}^{\circ}\left[r / X^{\nu}, z ; \epsilon\right]-1\right)=-b_{\mathrm{R}} \mathcal{V}_{\mathcal{S}_{D}}=-g\left(2 \pi A_{D}(\nu)\right)^{d / 2} X^{\nu d} \tag{9.37}
\end{equation*}
$$

In the limit $z \rightarrow \infty, g$ tends to $g^{\star}$, and we therefore have

$$
\begin{equation*}
\int_{\mathbb{R}^{d}} \mathrm{~d}^{d} u\left(\mathcal{Z}_{\infty}^{\diamond}[u ; \epsilon]-1\right)=-g^{\star}\left(2 \pi A_{D}(\nu)\right)^{d / 2} \tag{9.38}
\end{equation*}
$$

which is consistent with the assumption that the limit in (9.36) actually exists.
In the scaling regime $r / X^{\nu} \ll 1$, we expect the marked point to be strongly repelled from the origin, and thus $\mathcal{Z}_{\infty}^{\circ}$ to vanish as a power law:

$$
\begin{equation*}
\mathcal{Z}_{\infty}^{\diamond}\left[r / X^{\nu} ; \epsilon\right] \sim \operatorname{const}\left(\frac{r}{X^{\nu}}\right)^{\theta} \tag{9.39}
\end{equation*}
$$

This vanishing of $\mathcal{Z}^{\diamond}\left[r / X^{\nu}, z ; \epsilon\right]$ in the successive limits $z \rightarrow \infty$ and $r \rightarrow 0$ is consistent with that obtained in the reversed double limit $r=0$, and $z \rightarrow \infty$, which corresponds to the vanishing of $\mathcal{Z}^{\circ}$ at infinite $z$ according to (9.29).

The contact exponent $\theta$ can be obtained as follows. For finite $b$ and large $X$, we expect a universal $X$-dependence of $\mathcal{Z}^{\circ}\left[r / X^{\nu}, z ; \epsilon\right]$, irrespective of the particular value given to $r$. This dependence is in particular known exactly when $r=0$, according to (9.29). It must also be the same for $r \neq 0$ fixed and $b \rightarrow \infty$, that is a behavior which is given by (9.39). This leads to identifying the contact exponent with

$$
\begin{equation*}
\theta=\frac{\epsilon-W^{\prime}\left(g^{\star}\right)}{\nu} \tag{9.40}
\end{equation*}
$$

Notice that the argument above, intuitively clear on physical grounds, is usually mathematically justified in field theory from the existence of a short-distance
operator product expansion. A rigorous proof of the existence of such a shortdistance expansion in our case is beyond the scope of this paper. The repeated appearance of $W^{\prime}\left(g^{\star}\right)$ in (9.29) and (9.40) suggests that all scaling behaviors in this theory are controlled by a single scaling anomalous dimension, i.e. the universal slope of the Wilson function at the fixed point.

Eq. (9.39) allows us to derive a universal expression for the repulsive force exerted by the impurity on the membrane,

$$
\begin{equation*}
\boldsymbol{f}(\boldsymbol{r}) / k_{\mathrm{B}} T=\nabla_{\boldsymbol{r}} \log \mathcal{Z}^{\circ}(\boldsymbol{r})=\theta \frac{\boldsymbol{r}}{\boldsymbol{r}^{2}} . \tag{9.41}
\end{equation*}
$$

According to the discussion above, this force law is valid in the scaling regime $b^{-\nu / \epsilon} \ll r \ll X^{\nu}$, where $b^{-1 / \epsilon}$ plays the same physical role as an ultraviolet cut-off for internal distances.

Scaling laws for the delocalization transition. Finally, we have seen in subsect. 2.1 that for $d>d^{\star}$ (that is $\epsilon<0$ ), the non-trivial fixed point $g^{\star}$ is now negative and IR repulsive, and corresponds to a delocalization transition with non-trivial critical exponents, for a particular negative critical value $b^{\star}$ of the bare coupling constant $b$. In the localized phase ( $b<b^{\star}$ ), the correlation functions such as $\langle\boldsymbol{r}(x) r(y)\rangle$ and the associated correlation length $\xi_{\| \|}$(in the internal $D$-dimensional space) should be finite, as well as the average distance $r=\langle | \boldsymbol{r}| \rangle$ of the manifold to the attractive impurity. At the transition these quantities should diverge as

$$
\begin{equation*}
\xi_{\|} \propto\left(b^{\star}-b\right)^{-\nu_{\|}}, \quad r \propto\left(b^{\star}-b\right)^{-\nu_{\perp}} \tag{9.42}
\end{equation*}
$$

Standard arguments lead to

$$
\begin{equation*}
\nu_{\|}=\frac{1}{W^{\prime}\left(g^{\star}\right)}=-\frac{1}{\epsilon}+\ldots \tag{9.43}
\end{equation*}
$$

and

$$
\begin{equation*}
\nu_{\perp}=\nu_{\|} \nu \tag{9.44}
\end{equation*}
$$

Indeed, $r$ has no anomalous dimension and therefore, $r$ scales as $\xi_{\|}{ }^{\nu}$ with $\nu=$ (2-D)/2 from (2.2).

## 10. Conclusion

### 10.1. SUMMARY

In this last section, we would like to summarize the main steps of our construction and outline the main ingredients which ensure the renormalizability of the theory. We then discuss some possible extensions of our results.

Existence of a perturbative expansion analytically continued in $D$.
(I) The first ingredient is the existence, for integer dimension $D$ of the manifold, of a formal perturbative expansion for the model. The diagrams present an
(V) Factorization of the measure: this property, obviously satisfied for integer $D$, is preserved by the analytic continuation of the measure to non integer $D$. It allows us to integrate separately the factorized determinants which are to be subtracted from the original amplitude, and thus to interpret them as counterterms: the subtraction operation is then a simple reexpression of the partition function (or correlation functions) in terms of an effective (renormalized) coupling constant.

Points (III) and (IV) are properties of the interaction determinants themselves, while point (V) is a general property of the measure.

### 10.2. PROSPECTS

Let us finally discuss possible outcomes of our results. As already discussed, the model (2.1) of a manifold interacting with a single point serves indeed as a laboratory for studying the renormalizability of more general models of interacting crumpled manifolds. A prominent model of this class is of course the Edwards model (1.2) of a self-avoiding manifold interacting via a short range two-body pseudopotential. Its perturbative expansion is similar in structure to the one studied here. We indeed believe that the mathematical techniques developed in this article can be applied and generalized to the Edwards model, and provide both conceptually and practically a framework for a similar proof of its renormalizability.

When reviewing the general scheme above, we note that point (I) is already known for the self-avoiding model [21]. Points (II) and (V) are actually valid for any manifold hamiltonian. The specificity of a given model is actually encoded in its interaction determinants, for which properties similar to those of (III) and (IV) have to be analyzed in each case, and established in order to eventually build a subtraction procedure and prove renormalizability [38].

This scheme should be directly applicable to a series of manifold theories with interactions, such as many-body or long-range interactions ... These models generalize to arbitrary internal dimension $D$ models of interacting polymers ( $D=1$ ). All the latter models are known to be equivalent to some $n$-component field theories in the limit $n=0$, with standard Feynman diagram expansions. When extended to manifolds of arbitrary internal dimension, these models become theories with a single diagram to each order in perturbation (a property which is shared with string theories, although in our case the manifold has a fixed internal metric). Interestingly enough, the topological complexity of the usual Feynman diagrams is encoded in the $D$-measure on the manifold, and arises in the limit $D=1$ from the ordering constraints along the one-dimensional (polymer) line. More generally, it would be interesting to try and express field theories with an arbitrary number $n$ of components as $D=1$ limits of "manifolds" string-like models, yet to be invented.

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## Appendix A. From vectors to scalar products

In this appendix we derive (3.2) (3.3). First we insert the relation $u_{i j}=x_{i} \cdot x_{j}$ in the l.h.s. of (3.2)

$$
\int \prod_{i=1}^{N} \mathrm{~d}^{D} x_{i} f\left(x_{i} \cdot x_{j}\right)=\int \prod_{i \leqslant j} \mathrm{~d} u_{i j} \int \prod_{i} \mathrm{~d}^{D} x_{i} \prod_{i \leqslant j} \delta\left(u_{i j}-x_{i} \cdot x_{j}\right) f\left(\left[u_{i j}\right]\right)
$$

Second we use the fact that the function

$$
\begin{equation*}
\sigma_{N}^{(D)}\left(u_{i j}\right)=\int \prod_{i} \mathrm{~d}^{D} x_{i} \prod_{i \leqslant j} \delta\left(u_{i j}-x_{i} \cdot x_{j}\right) \tag{A.2}
\end{equation*}
$$

is invariant under $\mathrm{SO}(N)$ rotations $R\left(u \rightarrow R^{\mathfrak{t}} u R\right)$ to diagonalize $u_{i j}$ and express (A.2) in terms of the $N$ eigenvalues $\lambda_{i}, i=1, \ldots, N$, of $u_{i j}$

$$
\begin{equation*}
\sigma_{N}^{(D)}\left(u_{i j}\right)=\int \prod_{i} \mathrm{~d}^{D} x_{i} \prod_{i \leqslant j} \delta\left(\lambda_{i} \delta_{i j}-x_{i} \cdot x_{j}\right) \tag{A.3}
\end{equation*}
$$

Third we perform the change of variables $x_{i} \rightarrow \sqrt{\lambda_{i}} x_{i}$ and get

$$
\begin{equation*}
\sigma_{N}^{(D)}\left(u_{i j}\right)=\prod_{i=1}^{N} \lambda_{i}^{(D-N-1) / 2} \int \prod_{i} \mathrm{~d}^{D} x_{i} \prod_{i \leqslant j} \delta\left(\delta_{i j}-x_{i} \cdot x_{j}\right) \tag{A.4}
\end{equation*}
$$

The remaining integral over the $x_{i}$ 's gives the volume of $\mathrm{SO}(D) / \mathrm{SO}(D-N)$ and we obtain finally (3.3)

$$
\begin{align*}
\sigma_{N}^{(D)}\left(u_{i j}\right) & =\left(\prod_{i} \lambda_{i}\right)^{(D-N-1) / 2} \cdot \frac{\operatorname{Vol}(\operatorname{SO}(D))}{\operatorname{Vol}(\operatorname{SO}(D-N))} \\
& =\left(\operatorname{det}\left[u_{i j}\right]\right)^{(D-N-1) / 2} \frac{S_{D}}{2} \ldots \frac{S_{D-N+1}}{2} \tag{A.5}
\end{align*}
$$

## Appendix B. Factorization of the measure

To prove (3.34) let us decompose the $N \times N$ symmetric positive definite scalar product matrix $[u]_{N}$ into blocks of size $P$ and $Q(P+Q=N)$ :

$$
[u]_{N}=\left(\begin{array}{cc}
{[u]_{P}} & {[v]}  \tag{B.1}\\
{[v]^{t}} & {[u]_{Q}}
\end{array}\right) .
$$

Eq. (3.34) is equivalent to the fact that, given the positive definite matrices $[u]_{P}$ and $[u]_{Q}$, when integrating over all $P \times Q$ matrices $[v]$ such that $[u]_{N}$
(defined by (B.1)) is positive definite, we have for arbitrary non-integer $D$

$$
\begin{equation*}
\int \mathrm{d}[v] \sigma_{N}^{(D)}\left([u]_{N}\right)=\sigma_{P}^{(D)}\left([u]_{P}\right) \sigma_{Q}^{(D)}\left([u]_{Q}\right) \tag{B.2}
\end{equation*}
$$

Since $[u]_{P}$ and $[u]_{Q}$ are positive definite we can take their square root $[u]_{P}^{1 / 2}$ and $[u]_{Q}^{1 / 2}$ and write $\operatorname{det}\left([u]_{N}\right)$ in the expression (3.3) for $\sigma_{N}^{(D)}$ as

$$
\begin{equation*}
\operatorname{det}[u]_{N}=\operatorname{det}\left([u]_{P}\right) \operatorname{det}\left([u]_{Q}\right) \operatorname{det}\left(\mathbf{1}-[u]_{P}^{-1 / 2}[v][u]_{Q}^{-1}[v]^{\mathrm{t}}[u]_{P}^{-1 / 2}\right) \tag{B.3}
\end{equation*}
$$

Now, one can perform the change of variable $[v] \rightarrow[u]_{P}^{1 / 2}[v][u]_{Q}^{1 / 2}$ which induces a jacobian $J=\operatorname{det}\left([u]_{P}\right)^{Q / 2} \operatorname{det}\left([u]_{Q}\right)^{P / 2}$ in (B.2). We thus obtain finally that the l.h.s. of (B.2) is equal to the r.h.s. of (B.2), up to a constant $C$ which depends on $D, P$ and $Q$, but not on $[u]_{P}$ and $[u]_{Q}$, and which is given by

$$
\begin{align*}
C= & \frac{\operatorname{Vol}(\mathrm{SO}(D))}{\operatorname{Vol}(\mathrm{SO}(D-N))} \frac{\operatorname{Vol}(\mathrm{SO}(D-P))}{\operatorname{Vol}(\mathrm{SO}(P))} \frac{\operatorname{Vol}(\mathrm{SO}(D-Q))}{\operatorname{Vol}(\mathrm{SO}(Q))} \\
& \times \int \mathrm{d}[v]\left(\operatorname{det}\left(\mathbf{1}-[v][v]^{\mathrm{t}}\right)\right)^{(D-N-1) / 2} . \tag{B.4}
\end{align*}
$$

(The domain of integration for $[v]$ is now such that

$$
\left(\begin{array}{cc}
\mathbf{1}_{P} & {[v]}  \tag{B.5}\\
{[v]^{\mathrm{t}}} & \mathbf{1}_{Q}
\end{array}\right)
$$

is positive definite)
It remains to prove that $C=1$. This can be done in a simple way by proving that the factorization identity (3.34) holds for some particular function $f([u])$. As an example we can take the exponential

$$
\begin{equation*}
f\left([u]_{N}\right)=\exp \left[-\operatorname{tr}[u]_{N}\right], \tag{B.6}
\end{equation*}
$$

since we can easily calculate explicitly (see below)

$$
\begin{equation*}
I_{N}=\int_{\mathcal{U}_{N}} \mathrm{~d}[u]_{N} \sigma_{N}^{(D)}\left([u]_{N}\right) \exp \left[-\operatorname{tr}\left([u]_{N}\right)\right]=(\pi)^{N D / 2} \tag{B.7}
\end{equation*}
$$

and therefore factorization holds in this case since

$$
\begin{equation*}
f\left([u]_{N}\right)=f\left([u]_{P}\right) f\left([u]_{Q}\right) \quad \text { and } \quad I_{N}=I_{P} I_{Q} \tag{B.8}
\end{equation*}
$$

The direct computation of $I_{N}$ (eq. (B.7)) for any $D$ proceeds as follows. The set $\mathcal{U}_{N}$ is the set of symmetric positive matrices. By $\operatorname{SO}(N)$ orthogonal transformations, it can be reduced to the set of diagonal matrices with positive eigenvalues $\lambda_{i}(i=1, \ldots, N)$, with the new measure

$$
\begin{equation*}
\mathrm{d}[u]_{N}=\operatorname{Vol}(\operatorname{SO}(N)) \frac{1}{N!} \prod_{i=1}^{N} \mathrm{~d} \lambda_{i} \Delta(\lambda), \tag{B.9}
\end{equation*}
$$



Fig. C.1. The quadrilateral picturing the matrix element $\Pi_{\alpha \beta}^{\mathbf{T}}$ and its "basis" vector $R_{\alpha \beta}$.
for some $P, 2 \leqslant P \leqslant N$ and with a contraction factor $\rho, 0 \leqslant \rho \leqslant 1$. Under this rescaling, $R_{\alpha \beta}$ becomes

$$
\begin{align*}
R_{\alpha \beta}(\rho) & =\sum_{\gamma=P}^{N-1} c_{\alpha \beta}^{\gamma} \lambda_{\gamma}+\rho \sum_{\gamma=1}^{P-1} c_{\alpha \beta}^{\gamma} \lambda_{\gamma} \\
& =R_{\alpha \beta}^{0}+\rho R_{\alpha \beta}^{1} \tag{C.5}
\end{align*}
$$

We therefore have two possibilities:
(a) $R_{\alpha \beta}^{0}=0$. This means that $R_{\alpha \beta}$ is formed only of vectors $\lambda_{\gamma}$ with $\gamma \leqslant P-1$, which are all contracted, hence $R_{\alpha \beta}$ itself is contracted. By definition, this is also the case when $R_{\alpha \beta}$ is 0 , that is when $x_{i_{\alpha}}=x_{i_{\beta}}$.
(b) $R_{\alpha \beta}^{0} \neq 0$. This occurs when $R_{\alpha \beta}$ is spanned by at least one $\lambda_{\gamma}$ which is not contracted, that is with $\gamma \geqslant P$.

This allows us to classify the $\lambda$ 's into subtrees as follows (see fig. C.2):
(i) We regroup the $\lambda_{\alpha}$ 's with $\alpha \leqslant P-1$ (i.e. corresponding to contracted lines) into equivalence classes by deciding that $\lambda_{\alpha}$ and $\lambda_{\beta}$ are equivalent if $R_{\alpha \beta}^{0}=0$. The equivalence classes $\mathbf{T}_{1}, \ldots, \mathbf{T}_{m-1}$ (with $2 \leqslant m \leqslant P$ depending on $\mathbf{T}$ ) correspond to the $m-1$ distinct connected subtrees which build the subset of the contracted lines. Case (a) above thus corresponds to $\lambda_{\alpha}$ and $\lambda_{\beta}$ in the same equivalence class, that is in the same connected subtree of contracted lines. Case (b) corresponds to $\lambda_{\alpha}$ and $\lambda_{\beta}$ in two distinct equivalence classes, that is in two distinct connected subtrees of contracted lines.
(ii) We regroup the $\lambda_{\alpha}$ 's with $\alpha \geqslant P$ into a single connected tree $\mathbf{T}_{m}$ obtained by setting $\lambda_{\beta}=0$ for $\beta \leqslant P-1$ in the original tree $\mathbf{T}$.

We will now show that for $\rho \rightarrow 0$

$$
\begin{equation*}
\operatorname{det}\left(\left[\Pi^{\mathbf{T}}(\rho)\right]\right)=\rho^{2 \nu(P-1)} \prod_{i=1}^{m} \operatorname{det}\left(\left[\Pi^{\mathbf{T}_{i}}\right]\right)\left\{1+\mathcal{O}\left(\rho^{2 \delta}\right)\right\} \tag{C.6}
\end{equation*}
$$

Let us consider two lines $\lambda_{\alpha}$ and $\lambda_{\beta}$.
Case 1: $\alpha \leqslant P-1, \beta \leqslant P-1$.
(a) $R_{\alpha \beta}^{0}=0$.


Fig. C.2. Classification of the line vectors of the tree $\mathbf{T}$ into subtrees $\mathbf{T}_{i}$. The dashed lines in (a) correspond to contracting branches of the tree $T$, and are organized into two connected subtrees $\mathrm{T}_{1}$ and $\mathrm{T}_{2}$ in (b). The full lines in (a) correspond to non-contracting branches and are organized into a single connected subtree $\mathbf{T}_{3}$ in (b), by fully contracting the dashed lines in (a).

This case corresponds to two $\lambda$ 's in the same contracting connected subtree $\mathbf{T}_{i}$ for some $i \leqslant m-1$. In eq. (C.1), $\lambda_{\alpha}, \lambda_{\beta}$ and $R_{\alpha \beta}$ all get a factor $\rho$; hence

$$
\begin{equation*}
\Pi_{\alpha \beta}^{\mathbf{T}}(p)=\rho^{2 \nu} \Pi_{\alpha \beta}^{\mathbf{T}} \tag{C.7}
\end{equation*}
$$

It is furthermore clear that $R_{\alpha \beta}$ is spanned only by $\lambda$ 's in $\mathbf{T}_{i}$, hence

$$
\begin{equation*}
\Pi_{\alpha \beta}^{\mathbf{T}}(\rho)=\rho^{2 \nu} \Pi_{\alpha \beta}^{\mathbf{T}_{i}} \tag{C.8}
\end{equation*}
$$

(b) $R_{\alpha \beta}^{0} \neq 0$.

This case corresponds to two $\lambda$ 's in two distinct contracting connected subtrees $\mathbf{T}_{i_{1}}$ and $\mathbf{T}_{i_{2}}$. Since $R_{\alpha \beta}$ does not contract to zero, we can formally expand (C.1) in powers of $\lambda_{\alpha}$ and $\lambda_{\beta}$. The matrix element $\Pi_{\alpha \beta}^{\mathrm{T}}$ is by definition the interaction between two dipoles $\lambda_{\alpha}, \lambda_{\beta}$ separated by $R_{\alpha \beta}$. It is therefore clear that the first term in the multipolar expansion is of order

$$
\begin{equation*}
\Pi_{\alpha \beta}^{\mathbf{T}} \propto\left|R_{\alpha \beta}\right|^{2 \nu-2} \lambda_{\alpha} \cdot \lambda_{\beta}+\ldots \tag{C.9}
\end{equation*}
$$

Therefore, expanding in $\rho$ yields immediately

$$
\begin{align*}
\Pi_{\alpha \beta}^{\mathbf{T}}(\rho) & \propto \rho^{2}\left|R_{\alpha \beta}^{0}\right|^{2 \nu-2} \lambda_{\alpha} \cdot \lambda_{\beta}+\ldots \\
& =\mathcal{O}\left(\rho^{2}\right) \\
& =\rho^{2 \nu} \mathcal{O}\left(\rho^{2 \delta}\right) \tag{C.10}
\end{align*}
$$

(see eq. (4.7)). As we shall see below, this element, which mixes several subtrees $\mathbf{T}_{i}$, is vanishing sufficiently fast so as to disappear in the limit $\rho \rightarrow 0$.

Case 2: $\alpha \leqslant P-1, \beta \geqslant P$.
In this case, we have

$$
\begin{align*}
\Pi_{\alpha \beta}^{\mathrm{T}}(\rho) \propto & \left|R_{\alpha \beta}(\rho)+\lambda_{\beta}-\rho \lambda_{\alpha}\right|^{2 \nu}  \tag{1}\\
& -\left|R_{\alpha \beta}(\rho)+\lambda_{\beta}\right|^{2 \nu}  \tag{2}\\
& -\left|R_{\alpha \beta}(\rho)-\rho \lambda_{\alpha}\right|^{2 \nu}  \tag{3}\\
& +\left|R_{\alpha \beta}(\rho)\right|^{2 \nu} \tag{4}
\end{align*}
$$

(a) $R_{\alpha \beta}^{0}=0$.

Substituting $R_{\alpha \beta}(\rho)=\rho R_{\alpha \beta}^{1}$ in (C.11), the last two terms (3) and (4) are homogeneous to $\rho^{2 \nu}$, while the expansion of (1) - (2) in powers of $\rho$ gives a leading term linear in $\rho$. On the whole, we can write

$$
\begin{equation*}
\Pi_{\alpha \beta}^{\mathrm{T}}=\rho^{\nu} \mathcal{O}\left(\rho^{\delta}\right) \tag{C.12}
\end{equation*}
$$

(b) $R_{\alpha \beta}^{0} \neq 0$.

This time, the expansion of (1) - (2) on the one hand, and $-(3)+(4)$ on the other hand, in formal powers of $\rho \lambda_{\alpha}$ leads immediately to a matrix element of order $\rho$, hence

$$
\begin{equation*}
\Pi_{\alpha \beta}^{\mathrm{T}}=\mathcal{O}(\rho)=\rho^{\nu} \mathcal{O}\left(\rho^{\delta}\right) \tag{C.13}
\end{equation*}
$$

Case 3: $\alpha \geqslant P, \beta \geqslant P$.
In this case, $\lambda_{\alpha}$ and $\lambda_{\beta}$ are not contracted and belong to $\mathrm{T}_{m}$. In the limit $\rho \rightarrow 0$, $R_{\alpha \beta}$ is simply replaced by $R_{\alpha \beta}^{0}$. Whatever the value of $R_{\alpha \beta}^{0}$, this corresponds precisely to

$$
\begin{align*}
\Pi_{\alpha \beta}^{\mathbf{T}}(\rho) & =\Pi_{\alpha \beta}^{\mathbf{T}_{m}}+\rho^{\nu} \mathcal{O}\left(\rho^{\delta}\right) \\
& =\Pi_{\alpha \beta}^{\mathbf{T}_{m}}+\mathcal{O}\left(\rho^{2 \delta}\right) \tag{C.14}
\end{align*}
$$

We can summarize all these cases by writing the synoptic table
and we can now let $\rho \rightarrow 0$ and get

$$
\begin{equation*}
\operatorname{det}\left(\left[Y^{\mathbf{T}}(\rho \rightarrow 0)\right]\right)=\prod_{i=1}^{m} \operatorname{det}\left(\left[Y^{\mathbf{T}_{i}}\right]\right) \tag{C.17}
\end{equation*}
$$

which means that, in this limit, the tree has been disconnected into several components on which its determinant is exactly factorized.

Let us now turn to the variables $\beta$ 's defined in (5.2). Notice that due to the rescaling (5.6), $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ is actually independent of the global scale factor $\beta_{N-1}{ }^{\star}$. Each variable $\beta_{\gamma}$ can be associated with a contracting factor $\rho=\beta_{\gamma}$. Therefore, once expressed in term of the $\beta$ 's, $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ is such that, if we let one $\beta$ tend to zero (say $\beta_{\gamma}$ ), keeping the others non zero, we have

$$
\begin{align*}
& \operatorname{det}\left(Y^{\mathbf{T}}\left(\beta_{1}, \ldots, \beta_{\gamma-1}, \beta_{\gamma} \rightarrow 0, \beta_{\gamma+1}, \ldots, \beta_{N-2} ; \theta^{\mathbf{T}}\right)\right) \\
& \quad=\prod_{i=1}^{m-1} \operatorname{det}\left(Y^{\mathbf{T}_{i}}\left(\beta_{1}, \ldots, \beta_{\gamma-1} ; \theta^{\mathbf{T}_{i}}\right)\right) \times \operatorname{det}\left(Y^{\mathbf{T}_{m}}\left(\beta_{\gamma+1}, \ldots, \beta_{N-2} ; \theta^{\mathbf{T}_{m}}\right)\right) \tag{C.18}
\end{align*}
$$

where the $m-1$ first determinants in the r.h.s. involve $\beta_{\alpha}$ with $\alpha<\gamma$ only, while the last determinant involves $\beta_{\alpha}$ with $\alpha>\gamma$ only. The angular parameter set $\theta^{\mathbf{T}}$ associated with $\mathbf{T}$ is left untouched by the rescaling, but simply decomposed into subsets $\theta^{\mathbf{T}_{i}}$ associated with the line vectors of the distinct subtrees $\mathbf{T}_{i}$ (see fig. C.2). We are now interested in values of $\beta$ 's and $\theta$ 's varying inside the domain $\mathcal{D}^{\mathbf{T}}$ and look at the possible zeroes of $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ inside $\mathcal{D}^{\mathbf{T}}$. We already know that such zeroes can be reached only when one $\beta$ at least goes to zero. We thus fix all the variables $\theta$, and all the variables $\beta$ non zero except for one of them, $\beta_{\gamma}$. The quantity $\beta_{\gamma}^{\min }$ in (5.3) is therefore fixed, either strictly positive or zero. If it is strictly positive, this means that $\beta_{\gamma}$ cannot reach 0 within the domain $\mathcal{D}^{\mathbf{T}}$ for this particular configuration of the other variables. This happens when the tree $\mathbf{T}_{m}$, obtained by fully contracting the lines $\lambda_{1}, \ldots, \lambda_{\gamma}$ of $\mathbf{T}$, is not compatible with the definition of the sector $\mathcal{D}^{\mathbf{T}}$. The only relevant case is therefore $\beta_{\gamma}^{\min }=0$. When $\beta_{\gamma} \rightarrow 0$, we can use eq. (C.18). The trees $\mathbf{T}_{i}, 1 \leqslant i \leqslant m-1$, were already subtrees of $\mathbf{T}$, hence the associated determinants $\operatorname{det}\left(Y^{\mathbf{T}_{i}}\right)$, which involve only non vanishing $\beta$ 's, do not vanish. The new tree $\mathrm{T}_{m}$, which appears in the contraction process, is now compatible with the sector, which again implies that no fortuitous coincidence of its vertices can occur, and $\operatorname{det}\left(Y^{\mathbf{T}_{m}}\right)$ itself cannot vanish. Thus $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ cannot vanish in this limit $\beta_{\gamma} \rightarrow 0$. This process can be iterated on the remaining determinants in (C.18) for successive $\beta$ 's going to zero. This shows that $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ does not vanish for any number of $\beta$ 's going to zero. Hence we reach the important result that $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ cannot vanish inside the

[^11]whole sector $\mathcal{D}^{\mathbf{T}}$. Since $\mathcal{D}^{\mathbf{T}}$ is bounded (excluding the variable $\beta_{N-1}$ which does not enter in $\left.\operatorname{det}\left(Y^{\mathbf{T}}\right)\right), \operatorname{det}\left(Y^{\mathbf{T}}\right)$ is moreover bounded from below by a strictly positive number.

## Appendix D. Example of cancellation of symmetry factors

Let us consider as in (7.30) the four compatible nests:

$$
\begin{align*}
\mathcal{N}_{\oplus} & =\{(T, \omega)\} \\
\mathcal{N}_{\oplus 2} & =\{(R \wedge T, \bullet),(T, \omega)\} \\
\mathcal{N}_{\oplus 3} & =\left\{(T, \omega),\left(R \vee_{\omega} T, \bullet\right)\right\} \\
\tilde{\mathcal{N}}_{\oplus} & =\left\{(R \wedge T, \bullet),(T, \omega),\left(R \vee_{\omega} T, \bullet\right)\right\} \tag{D.1}
\end{align*}
$$

where $R=\{\mathcal{R}\}, T=\{\mathcal{T}\}$ and $\omega=\{\{w\}\}$ with $w \in \mathcal{T}$. We want to show that the sum of the $(-1)$ and symmetry factors associated with these nests (taking into account the degeneracy coming from the unspecified compatible roots •) is equal to 0 . We recall that with a compatible nest $\mathcal{N}^{\prime}{ }_{\oplus}$ is associated the factor in front of the associated Taylor operators (here we forget about the first diagram $T_{0}=\left(G_{\odot}, G_{\odot}\right)$ implicit in all the nests of (D.1), and the corresponding global $(-1)$ factor):

$$
\begin{equation*}
(-1)^{\operatorname{Card}\left(\mathcal{N}^{\prime} \oplus\right)} W\left(\mathcal{N}_{\oplus}^{\prime}\right)=(-1)^{\operatorname{Card}\left(\mathcal{N}^{\prime} \oplus\right)} \prod_{\substack{w^{\prime} \text { root } \\ \text { of } \mathcal{N}^{\prime} \oplus}} \frac{1}{\left|\mathcal{T}_{w^{\prime}}\right|} \tag{D.2}
\end{equation*}
$$

with $\mathcal{I}_{w^{\prime}}$ being the largest connected component (among all connected components of all diagrams of $\mathcal{N}^{\prime} \oplus$ ) whose root is $w^{\prime}$.

The factor associated with $\mathcal{N}_{\oplus}$ in (D.1) is thus $(-)|\mathcal{T}|^{-1}$. Let us now discuss the three remaining nests in (D.1).

Case (a): $w \in \mathcal{R}$ (see fig. 19)
The root of the connected component $\mathcal{R} \cap \mathcal{T}$ of $R \wedge T$ must be equal to $w$. The factor associated with $\mathcal{N}_{\oplus 2}$ is then $|\mathcal{T}|^{-1}$. In $\mathcal{N}_{\oplus 3}$, the root of the connected component $\mathcal{R} \cup \mathcal{T}$ of $R \vee_{\omega} T$ is either equal to $w$, or belongs to $\mathcal{R} \backslash \mathcal{T}$. The factor associated with $\mathcal{N}_{\oplus 3}$ is therefore $|\mathcal{R} \cup \mathcal{T}|^{-1}$ in the first case, and $|\mathcal{T}|^{-1} \cdot|\mathcal{R} \cup \mathcal{T}|^{-1}$ in the second case, with degeneracy $|\mathcal{R} \backslash \mathcal{T}|$. Hence, the global factor associated with $\mathcal{N}_{\oplus 3}$ and its possible rootings is $|\mathcal{R} \cup \mathcal{T}|^{-1}+|\mathcal{T}|^{-1} \cdot|\mathcal{R} \backslash \mathcal{T}| \cdot|\mathcal{R} \cup \mathcal{T}|^{-1}$ which, using $|\mathcal{R} \backslash \mathcal{T}|+|\mathcal{R} \cap \mathcal{T}|=|\mathcal{R} \cup \mathcal{T}|$, is nothing but $|\mathcal{T}|^{-1}$. The factor associated with $\tilde{\mathcal{N}}_{\oplus}$ is similarly equal to

$$
(-)\left[\frac{1}{|\mathcal{R} \cup \mathcal{T}|}+\frac{1}{|T|} \frac{|\mathcal{R} \backslash \mathcal{T}|}{|\mathcal{R} \cup \mathcal{T}|}\right]=-\frac{1}{|\mathcal{T}|} .
$$

By summing up all these factors for all elements of (D.1), we get zero as expected.

Case (b): $w \notin \mathcal{R}$ (see fig. (19))
The root of the connected component $\mathcal{R} \cap \mathcal{T}$ of $R \wedge T$ can now be any vertex of $\mathcal{R} \cap \mathcal{T}$. The factor associated with $\mathcal{N}_{\oplus 2}$ is in this case $|\mathcal{R} \cap \mathcal{T}| \cdot|\mathcal{R} \cap \mathcal{T}|^{-1} \cdot|\mathcal{T}|^{-1}=$ $|\mathcal{T}|^{-1}$, since $\mathcal{N}_{\oplus 2}$ has now two distinct roots. In $\mathcal{N}_{\oplus 3}$, the two roots of the two connected components $\mathcal{T}$ and $\mathcal{R} \backslash \mathcal{T}$ of $R \vee_{\omega} T$ are respectively $w$ and any vertex in $\mathcal{R} \backslash \mathcal{T}$. The factor associated with $\mathcal{N}_{\oplus 3}$ is then $|\mathcal{T}|^{-1}|\mathcal{R} \backslash \mathcal{T} \| \mathcal{R} \backslash \mathcal{T}|^{-1}=|\mathcal{T}|^{-1}$, while the factor associated with $\tilde{\mathcal{N}}_{\oplus}$ is

$$
(-) \frac{|\mathcal{R} \cap \mathcal{T}|}{|\mathcal{R} \cap \mathcal{T}|} \frac{1}{|\mathcal{T}|} \frac{|\mathcal{R} \backslash \mathcal{T}|}{|\mathcal{R} \backslash \mathcal{T}|}=(-) \frac{1}{|\mathcal{T}|} .
$$

Here too the sum of these factors gives zero as expected.

## Appendix E. "Suppression" of a reducible line from the tableau nest

We assume here that the coincidence (7.35) holds in the tableau (7.36). We therefore have as a starting point the set of identities:

$$
\begin{array}{ll}
R_{J-1}^{I} \wedge T_{J}=T_{J} & I \geqslant I_{0} \\
T_{J}=R_{J}^{I} \wedge T_{J+1} & I \leqslant I_{0} \tag{E.1}
\end{array}
$$

We want to prove that the lines (7.37) and (7.38) are then identical, i.e. that $T_{J}$ can be skipped in the construction of the tableau. We thus have to prove the two following sets of identities:
(1) For $I \geqslant I_{0}$

$$
\begin{equation*}
R_{J}^{I} \wedge T_{J+1}=R_{J-1}^{I} \wedge T_{J+1} \tag{E.2}
\end{equation*}
$$

(2) For $I \leqslant I_{0}$

$$
\begin{equation*}
R_{J-1}^{I} \wedge T_{J+1}=R_{J-1}^{I} \wedge T_{J} \tag{E.3}
\end{equation*}
$$

These two sets of identities are consequences of the stronger equality

$$
\begin{equation*}
R_{J}^{I}=R_{J-1}^{I}, \quad \forall I \geqslant I_{0} \tag{E.4}
\end{equation*}
$$

Indeed eq. (E.4) clearly implies (E.2) for case (1). Furthermore, for case (2), we make the following argument: We use (E.1) to write $T_{J}$ as

$$
\begin{equation*}
T_{J}=R_{J}^{I_{0}} \wedge T_{J+1} \tag{E.5}
\end{equation*}
$$

Hence, we have

$$
\begin{align*}
R_{J-1}^{I} \wedge T_{J} & =\left(R_{J-1}^{I} \wedge R_{J}^{I_{0}}\right) \wedge T_{J+1} \\
& =R_{J-1}^{I} \wedge T_{J+1}, \quad \text { q.e.d. } \tag{E.6}
\end{align*}
$$

where we have made use of $R_{J-1}^{I} \prec R_{J-1}^{I_{0}}$ since $I \leqslant I_{0}$, together with $R_{J-1}^{I_{0}}=R_{J}^{I_{0}}$ as a particular case of (E.4). We are thus left with proving (E.4).
notations, we have

$$
\begin{equation*}
\mathcal{R}_{J-1}^{I, i}=\left(\mathcal{R}^{I, i} \cup A_{J-1}^{I, i}\right) \backslash\left(\mathcal{R}^{I, i} \cap B_{J-1}^{I, i}\right) \tag{E.10}
\end{equation*}
$$

and a similar equation for the connected components $\mathcal{R}_{J}^{I, i}$ of $R_{J}^{I}$. Therefore, to prove $R_{J-1}^{I}=R_{J}^{I}$, it is enough to prove

$$
\begin{align*}
A_{J}^{I, i} & =A_{J-1}^{I, i} \\
B_{J}^{I, i} & =B_{J-1}^{I, i} \tag{E.11}
\end{align*}
$$

The main ingredient comes from the property in (E.1):

$$
\begin{equation*}
T_{J}=R_{J-1}^{I} \wedge T_{J}, \quad \forall I \geqslant I_{0} \tag{E.12}
\end{equation*}
$$

which implies

$$
\begin{equation*}
T_{J} \prec R_{J-1}^{I} \tag{E.13}
\end{equation*}
$$

This means that any connected component $\tau_{J, j}$ of $T_{J}$ which intersects a connected component $\mathcal{R}_{J-1}^{I, i}$ is actually entirely included in the latter.

We first prove $A_{J-1}^{I, i} \subset A_{J}^{I, i}$. Let us consider a connected component $\mathcal{T}_{J-1, k}$ of $T_{J-1}$, such that $w_{J-1, k} \in \mathcal{R}^{I, i}$. From the nest property, this connected component is included in a connected component $\mathcal{T}_{J, j}$ of $T_{J}$. By definition, $\mathcal{T}_{J-1, k} \subset \mathcal{R}_{J-1}^{I, i}$ and therefore $\mathcal{T}_{J, j}$ intersects $\mathcal{R}_{J-1}^{I, i}$. From (E.13), $\mathcal{T}_{J, j}$ is necessarily included in $\mathcal{R}_{J-1}^{I, i}$ and in particular its root $w_{J, j}$ belongs to $\mathcal{R}_{J-1}^{I, i}$, thus to $\mathcal{R}^{I, i} \cup A_{J-1}^{I, i}$. One has either $w_{J, j} \in \mathcal{R}^{I, i}$, or $w_{J, j} \in \mathcal{T}_{J-1, l}$ for some connected component $\mathcal{T}_{J-1, l}$ (with $l \neq k$ in general) of $T_{J-1}$ such that $w_{J-1, l} \in \mathcal{R}^{I, i}$. In the latter case, from the compatibility condition for the roots, we have $w_{J, j}=w_{J-1, l} \in \mathcal{R}^{I, i}$. Therefore, in any case, $w_{J, j} \in \mathcal{R}^{I, i}$ and $\mathcal{T}_{J, j} \subset A_{J}^{I, i}$. This implies $\mathcal{T}_{J-1, k} \subset A_{J}^{I, i}$, which leads to

$$
\begin{equation*}
A_{J-1}^{I, i} \subset A_{J}^{I, i} \tag{E.14}
\end{equation*}
$$

We now prove $B_{J-1}^{I, i} \subset B_{J}^{I, i}$ :
We can use the fact that $R^{I}$ is a complete diagram, thus each root $w_{J-1, k}$ of a connected component $\mathcal{T}_{J-1, k}$ belongs to one and only one connected component $\mathcal{R}^{I, i}$ of $R^{I}$. The set $B_{J-1}^{I, i}$ can therefore be expressed as

$$
\begin{equation*}
B_{J-1}^{I, i}=\bigcup_{i^{\prime} \neq i} A_{J-1}^{I, i^{\prime}} \tag{E.15}
\end{equation*}
$$

A similar equation holds for $B_{J}^{I, i}$. Making use of (E.14) for each $i^{\prime}$ in the r.h.s. of (E.15), we directly arrive at

$$
\begin{equation*}
B_{J-1}^{I, i} \subset B_{J}^{I, i} \tag{E.16}
\end{equation*}
$$

The inclusion properties (E.14) and (E.16), together with the fact that $A_{j-1}^{I, i}$ and $B_{J-1}^{I, i}$ on the one hand, and $A_{J}^{I, i}$ and $B_{J}^{I, i}$ on the other hand, are pairs of complementary sets of $\mathcal{G}$, imply (E.11), hence (E.4).

## Appendix F. Addition of reducible lines in the tableau nest

We want to prove first that, if we consider a compatibly rooted nest $\mathcal{N}_{\oplus}=$ $\left\{T_{0 \oplus}, \ldots, T_{T \oplus}\right\}$ and build the larger nest

$$
\mathcal{N}_{\oplus}^{\prime}=\left\{T_{0 \oplus}, \ldots, T_{J-1 \oplus}, T_{J-1 \oplus}^{I_{0}}, T_{J \oplus}, \ldots, T_{T \oplus}\right\}
$$

by inserting between the levels $J-1$ and $J$ of $\mathcal{N}_{\oplus}$ an extra rooted diagram $T_{J-1 \oplus}^{I_{0}}=\left(T_{J-1}^{I_{0}}, \omega_{J-1}^{I_{0}}\right)$ with

$$
\begin{equation*}
T_{J-1}^{I_{0}}=R_{J-1}^{I_{0}} \wedge T_{J} \equiv T^{\prime} \tag{F.1}
\end{equation*}
$$

and $\omega_{J-1}^{I_{0}} \equiv \omega^{\prime}$ an arbitrary set of roots compatible with the rooting of $\mathcal{N}_{\oplus}$ (making $\mathcal{N}_{\oplus}^{\prime}$ compatibly rooted), then the tableau of $\mathcal{N}_{\oplus}^{\prime}$ can be reduced to that of $\mathcal{N}_{\oplus}$. For convenience, we denote $T_{J-1_{\oplus}}^{I_{0}}$ by $T_{\oplus}^{\prime}=\left(T^{\prime}, \omega^{\prime}\right)$. More precisely, the tableau built from $\mathcal{N}_{\oplus}^{\prime}$ is

$$
\begin{array}{rlllll}
T_{J-1} & R_{J-1}^{1} \wedge T^{\prime} & \ldots & R_{J-1}^{I_{0}} \wedge T^{\prime} & \ldots & R_{J-1}^{N-1} \wedge T^{\prime} \\
T^{\prime} & R^{\prime 1} \wedge T_{J} & \ldots & R^{I_{0}} \wedge T_{J} \ldots & R^{\prime N-1} \wedge T_{J} \\
T_{J} & & & &
\end{array}
$$

where

$$
\begin{equation*}
R^{\prime I} \equiv R^{I} \vee_{\omega^{\prime}} T^{\prime} \tag{F.3}
\end{equation*}
$$

We want to prove that this tableau has the coincidence property for $I=I_{0}$ :

$$
\begin{equation*}
R_{J-1}^{I_{0}} \wedge T^{\prime}=R^{I_{0}} \wedge T_{J} \tag{F.4}
\end{equation*}
$$

and therefore can be reduced to the tableau of $\mathcal{N}_{\oplus}$. From the definition of $T^{\prime}$, the term on the l.h.s. of (F.4) is nothing but $R_{J-1}^{I_{0}} \wedge T_{J}$ and the coincidence property is equivalent to

$$
\begin{equation*}
R_{J-1}^{I_{0}} \wedge T_{J}=R^{I_{0}} \wedge T_{J} \tag{F.5}
\end{equation*}
$$

This last equation is actually a consequence of the stronger identity

$$
\begin{equation*}
R_{J-1}^{I_{0}}=R^{\prime I_{0}} \tag{F.6}
\end{equation*}
$$

which we prove now.
Proof of (F.6): As in appendix E, we consider a typical connected component $\mathcal{R}_{J-1}^{I_{0}, i_{0}}$ of $R_{J-1}^{I_{0}}$, defined by

$$
\begin{equation*}
\mathcal{R}_{J-1}^{I_{0}, i_{0}}=\left[\mathcal{R}^{I_{0}, i_{0}} \bigcup_{w_{J-1, k} \in \mathcal{R}^{I_{0}, i_{0}}} \mathcal{T}_{J-1, k}\right] \backslash\left(\bigcup_{w_{J-1, l} \notin \mathcal{R}^{I_{0}, i_{0}}} \mathcal{T}_{J-1, l}\right), \tag{F.7}
\end{equation*}
$$

or by the equivalent equation

$$
\begin{equation*}
\mathcal{R}_{J-1}^{I_{0}, i_{0}}=\left(\mathcal{R}^{I_{0}, i_{0}} \cup A_{J-1}^{I_{0}, i_{0}}\right) \backslash\left(\mathcal{R}^{I_{0}, i_{0}} \cap B_{J-1}^{I_{0}, i_{0}}\right) \tag{F.8}
\end{equation*}
$$

where

$$
\begin{align*}
& A_{J-1}^{I_{0}, i_{0}}=\bigcup_{w_{J-1, k} \in \mathcal{R}^{I_{0}, i_{0}}} T_{J-1, k},  \tag{F.9}\\
& B_{J-1}^{I_{0}, i_{0}}=\bigcup_{w_{J-1, l} \notin \mathcal{R}_{0}^{I_{0}, i_{0}}} \mathcal{T}_{J-1, l} . \tag{F.10}
\end{align*}
$$

The sets $A_{J-1}^{I_{0}, i_{0}}$ and $B_{J-1}^{I_{0}, i_{0}}$ are complementary subsets of $\mathcal{G}$ and, as in appendix E,

$$
\begin{equation*}
B_{J-1}^{I_{J}, i_{0}}=\bigcup_{i \neq i_{0}} A_{J-1}^{I, i} \tag{F.11}
\end{equation*}
$$

We then can write for $R^{\prime I_{0}}$ an equation similar to (F.8) with $A_{J-1}^{I_{0}, i_{0}}$ and $B_{J-1}^{I_{0}, i_{0}}$ replaced by

$$
\begin{align*}
&{A^{\prime} I_{0}, i_{0}}=\bigcup_{w_{k}^{\prime i} \in \mathcal{R}^{I_{0}, i_{0}}} \mathcal{T}_{k}^{\prime i},  \tag{F.12}\\
&{B^{\prime} I_{0}, i_{0}}= \bigcup_{w_{i}^{\prime} \notin \mathcal{R}^{I_{0}, i_{0}}} \mathcal{T}^{\prime}{ }_{l}^{i} \tag{F.13}
\end{align*}
$$

which are complementary subsets of $\mathcal{G}$ and satisfy an equation similar to (F.11). In eqs. (F.12) and (F.13), $\mathcal{T}^{\prime}{ }_{k}$ is the generic connected component of $T^{\prime}$ given by

$$
\begin{equation*}
\mathcal{T}_{k}^{\prime i}=\mathcal{R}_{J-1}^{I_{0}, i} \cap \mathcal{T}_{J, k} \tag{F.14}
\end{equation*}
$$

and $w_{k}^{\prime i}$ is its root.
In order to prove (F.6), it is sufficient to prove that $A_{J-1}^{I_{0}, i_{0}} \subset A^{\prime I_{0}, i_{0}}$. Indeed, from (F.11) and the similar equation for $B^{\prime I_{0}, i_{0}}$, this inclusion will imply $B_{J-1}^{I_{0}, i_{0}} \subset$ $B^{\prime I_{0}, i_{0}}$. From the complementarity property of $A_{J-1}^{I_{0}, i_{0}}$ and $B_{J-1}^{I_{0}, i_{0}}$ on the one hand, and that of $A^{\prime I_{0}, i_{0}}$ and $B^{\prime I_{0}, i_{0}}$ on the other hand, the two equalities

$$
\begin{align*}
& A_{J-1}^{I_{0}, i_{0}}=A^{\prime I_{0}, i_{0}}  \tag{F.15}\\
& B_{J-1}^{I_{0}, i_{0}}=B^{\prime I_{0}, i_{0}} \tag{F.16}
\end{align*}
$$

follow, leading to (F.6).
We are thus left with proving $A_{J-1}^{I_{0}, i_{0}} \subset A^{I_{0}, i_{0}}$ : Let us consider a connected component $\mathcal{T}_{J-1, k}$ of $T_{J-1}$, such that $w_{J-1, k} \in \mathcal{R}^{I_{0}, i_{0}}$. From the nest property, this connected component is included in a connected component $\mathcal{T}_{J, j}$ of $T_{J}$. By definition, $\mathcal{T}_{J-1, k} \subset \mathcal{R}_{J-1}^{I_{0}, i_{0}}$ and therefore $\mathcal{T}_{J-1, k} \subset \mathcal{R}_{J-1}^{I_{0}, i_{0}} \cap \mathcal{T}_{J, j} \equiv \mathcal{T}_{j}^{t_{0}, l_{j}}$. The root $w_{j}^{\prime i_{0}}$ of $\mathcal{T}_{j}^{\prime i_{0}}$ belongs to $\mathcal{R}_{J-1}^{I_{0}, i_{0}}$, thus to $\mathcal{R}^{I_{0}, i_{0}} \cup A_{J-1}^{I_{0}, i_{0}}$. One has either $w_{j}^{\prime i_{0}} \in \mathcal{R}^{I_{0}, i_{0}}$ or $w_{j}^{\prime i_{0}} \in \mathcal{T}_{J-1, l}$ for some connected component $\mathcal{T}_{J-1, l}$ (with $l \neq k$ in general)
compatible rootings of $\mathcal{N}$ can be built in that way. Given such a rooting $\oplus_{\mathcal{N}}$, the number of distinct orderings $\sigma$ which build $\oplus_{\mathcal{N}}$ is

$$
\begin{equation*}
K\left(\mathcal{N}_{\oplus_{\mathcal{N}}}\right) \equiv \operatorname{Card}\left(\left\{\sigma: \forall(\mathcal{T}, w) \text { rooted connected comp. of } \mathcal{N}_{\oplus_{\mathcal{N}}}, \mathcal{T} \xrightarrow{\sigma} w\right\}\right) \tag{G.3}
\end{equation*}
$$

It is simply related to the weight $W\left(\mathcal{N}_{\oplus_{\mathcal{N}}}\right)$ by

$$
\begin{equation*}
\frac{K\left(\mathcal{N}_{\oplus_{\mathcal{N}}}\right)}{N!}=W\left(\mathcal{N}_{\oplus_{\mathcal{N}}}\right)=\prod_{w} \frac{1}{\left|\mathcal{T}_{w}\right|} \tag{G.4}
\end{equation*}
$$

Indeed, given a subset $\mathcal{P}$ of $\mathcal{G}$ and a vertex $p$ in $\mathcal{P}$, the number of $\sigma$ 's which assign $p$ to $\mathcal{P}$ is $N!/|\mathcal{P}|$ (the probability for $p$ to be the first vertex of $\mathcal{P}$ to appear in the sequence $\sigma(1), \ldots, \sigma(N)$ is $1 /|\mathcal{P}|)$. A compatible rooting $\oplus_{\mathcal{N}}$ of $\mathcal{N}$ is entirely known once one specifies for each vertex $w$ the largest connected component of $\mathcal{N}, \mathcal{T}_{w}$, which has $w$ as its root. The above argument can then be extended to all these largest connected components of $\mathcal{N}$ containing the roots of $\oplus_{\mathcal{N}}$, and leads to (G.4).

The proof of (7.42) is then straightforward. Indeed, the r.h.s. of (7.42) is simply $1 /(N$ !) times

$$
\begin{equation*}
\operatorname{Card}\left(\left\{\sigma: \forall(\mathcal{T}, w) \text { rooted connected comp. of } \mathcal{N}_{\oplus}, \mathcal{T} \xrightarrow{\sigma} w\right\}\right), \tag{G.5}
\end{equation*}
$$

while each term of the sum in the l.h.s of (7.42) is $1 /(N!)$ times

$$
\begin{align*}
\operatorname{Card}(\{\sigma: & \forall(\mathcal{T}, w) \text { rooted connected comp. of } \mathcal{N}_{\oplus}, \mathcal{T} \stackrel{\sigma}{\longrightarrow} w \\
& \left.\left.\forall(\mathcal{T}, w) \text { rooted connected comp. of } \mathcal{M}_{\oplus \mathcal{M}} \operatorname{not} \operatorname{in} \mathcal{N}_{\oplus}, \mathcal{T} \xrightarrow{\sigma} w\right\}\right) \tag{G.6}
\end{align*}
$$

The sum over $\oplus_{\mathcal{M}}$ in (7.42) relaxes the second constraint on $\sigma$ in (G.6), and reproduces (G.5). Hence (7.42) follows.

## Appendix H. Estimates of subtracted integrands in a Hepp sector

In this appendix we prove (8.35) and (8.37). We shall proceed in three steps:
(I) We first analyze the properties of the elements of the matrix $Y^{\mathbf{T}_{J, j}}$ in terms of the $\beta^{I}$ variables.
(II) We then write an integral representation of the $(1-\tau)$ operators appearing in the l.h.s. of (8.35) or (8.37).
(III) We finally show (8.35) and (8.37).
(I) Properties of $Y^{\mathrm{T}_{J, j}}$. In this subsection, we shall work separately inside each connected component $\tilde{\mathcal{T}}_{J, j}$ of $\tilde{T}_{J}$. As explained in subsect. 8.2 , the line vectors $\lambda_{J, j}^{I}$ of the oriented ordered tree $\mathbf{T}_{J, j}$ spanning $\tilde{\mathcal{T}}_{J, j}$ are uniquely labeled by $I \in \overline{\operatorname{Ind}}(J, j)$. From now on, we shall suppress the indices ( $J, j$ ) and thus
denote $\lambda_{J, j}^{I}$ by $\lambda^{I}$. A typical element of the matrix $Y^{\mathbf{T}_{J, j}}$ reads

$$
\begin{align*}
Y_{K L}^{\mathbf{T}_{J j}}=\frac{-1}{2\left|\lambda^{K}\right|^{\nu}\left|\lambda^{L}\right|^{\nu}} & \left\{\left|R^{K L}+\lambda^{L}-\lambda^{K}\right|^{2 \nu}-\left|R^{K L}+\lambda^{L}\right|^{2 \nu}\right. \\
& \left.-\left|R^{K L}-\lambda^{K}\right|^{2 \nu}+\left|R^{K L}\right|^{2 \nu}\right\}, \tag{H.1}
\end{align*}
$$

where $R^{K L}$ is the "basis" of the quadrilateral

$$
\begin{equation*}
R^{K L}=x_{i_{L}}-x_{i_{K}} \tag{H.2}
\end{equation*}
$$

with $i_{K}$ and $i_{L}$ being the origins of $\lambda^{K}$ and $\lambda^{L}$. The vector $R^{K L}$ is a linear combination of the $\lambda^{I}$ 's joining $x_{i_{K}}$ and $x_{i_{L}}$, and since the tree $\mathrm{T}_{J, j}$ has been built from the rooted sector $\mathcal{S}_{J, j \oplus}$, this linear combination involves only $\lambda^{I}$ 's for $I>\min (K, L)$ (see subsect. 7.2):

$$
\begin{equation*}
R^{K L}=\sum_{M>\min (K, L)} c_{M}^{K L} \lambda^{M} \tag{H.3}
\end{equation*}
$$

with $c_{M}^{K L}=0, \pm 1$.
Proposition 1. $\operatorname{det}\left(Y^{\mathbf{T}_{s, j}}\right)$ is a positive, non-vanishing continuous function on the compact domain $\mathcal{H}^{\mathcal{S}}$, and is therefore bounded from below on $\mathcal{H}^{\mathcal{S}}$ by a strictly positive number. In particular, the matrix $Y^{\mathrm{T}_{J, j}}$ is invertible.

Proposition 2. $\quad Y_{K L}^{\mathrm{T}_{J j}}$, as a function of the $\beta, \chi$ and $\theta$ variables, depends on the $\beta^{I}$ 's for $I$ in some subset $\mathcal{J}_{J, j}(K, L)$ only, defined as

$$
\begin{equation*}
\mathcal{J}_{J, j}(K, L)=\left\{I: \min (K, L) \leqslant I<\max \left(K, L, \max \left(M: c_{M}^{K L} \neq 0\right)\right)\right\} \tag{H.4}
\end{equation*}
$$

with the convention that $\max \left(K, L, \max \left(M: c_{M}^{K L} \neq 0\right)\right)=\max (K, L)$, if all the $c_{M}^{K L}$ are zero (that is if $R^{K L}=0$ ).

Proposition 3. Inside the sector $\mathcal{H}^{\mathcal{S}}$,

$$
\begin{equation*}
Y_{K L}^{\mathbf{T}_{J j}}=\mathcal{O}\left(\prod_{I \in \mathcal{J}_{J, j}(K, L)}\left(\beta^{I}\right)^{\delta}\right) \tag{H.5}
\end{equation*}
$$

Proposition 4. The matrix $Y^{\mathrm{T}_{J, j}}$ is positive, and bounded from below by a strictly positive constant. By this we mean that there exists a strictly positive number $C$ such that ( $Y^{\mathrm{T}_{J, j}}-C \mathbb{I I}$ ) is a positive matrix on $\mathcal{H}^{\mathcal{S}}$.

Proposition 1 has already been proven in appendix $C$, in the restricted case of a generalized Hepp sector $\mathcal{H}^{\mathbf{T}}$ attached to some tree T. The proof can be carried over to the whole extended Hepp sector $\mathcal{H}^{\mathcal{S}}$ attached to the nest $\mathcal{S}=\mathcal{S}(\mathrm{T})$. Indeed, the spirit of the proof is that $Y^{\mathbf{T}}$ depends only on ratios of successive $\lambda$ 's ( $\beta$ variables); from the bounds on those ratios inside $\mathcal{H}^{\mathbf{T}}$, we deduce that if some points coincide then one of these ratios at least must vanish, and $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ factorizes and remains strictly positive. Since from Schoenberg's theorem this is the only case when $\operatorname{det}\left(Y^{\mathbf{T}}\right)$ might have vanished, we deduce that it actually
never vanishes, and remains positive inside $\mathcal{H}^{\mathbf{T}}$. Inside $\mathcal{H}^{\mathcal{S}}$, we have weaker bounds on the ratios of $\lambda$ 's but one can check that this does not alter the proof.

To prove propositions 2 and 3, we first consider the trivial case $K=L$. In this case $\mathcal{J}_{J, j}(K, K)=\varnothing$ but then $Y_{(K, K)}^{\mathbf{T}_{J, j}}=1$, which satisfies these propositions.
We can therefore assume that $K<L$. Four distinct situations may occur:
(a) $R^{K L}=0$ : then $\mathcal{J}_{J, j}(K, L)=\{I: K \leqslant I<L\}$;

If $R^{K, L} \neq 0$, we denote by

$$
\begin{equation*}
P=\max \left(M: c_{M}^{K L} \neq 0\right) \tag{H.6}
\end{equation*}
$$

(b) If $P>L$, then $\mathcal{J}_{J, j}(K, L)=\{I: K \leqslant I<P\}$;
(c) If $K<P<L$, then $\mathcal{J}_{J, j}(K, L)=\{I: K \leqslant I<L\}$;
(d) If $P=L$, then $\mathcal{J}_{J, j}(K, L)=\{I: K \leqslant I<L\}$.

We shall use the property that, if $A>B$ and $A>C$, then the quantity

is bounded (in module) from above inside $\mathcal{H}^{\mathcal{S}}$. This follows from the fact that, inside the sector $\mathcal{H}^{\mathcal{S}}$, the ratios $\left|\lambda^{B}\right| /\left|\lambda^{A}\right|,\left|\lambda^{C}\right| /\left|\lambda^{A}\right|$ and $\left|\lambda^{A}\right| /\left|\lambda^{A}+\sum_{B} \pm \lambda^{B}\right|$ are bounded. An upper bound on (H.7) can then easily be obtained by use of the mean value theorem.

By a simple generalization of this property, one can show that, if $A>B, A>C$ and $A>D$, then the quantity

$$
\begin{align*}
& {\left[\left|\lambda^{A}+\left(\sum_{B} \pm \lambda^{B}\right) \pm \lambda^{C} \pm \lambda^{D}\right|^{2 \nu}-\left|\lambda^{A}+\left(\sum_{B} \pm \lambda^{B}\right) \pm \lambda^{C}\right|^{2 \nu}\right.} \\
& \left.-\left|\lambda^{A}+\left(\sum_{B} \pm \lambda^{B}\right) \pm \lambda^{D}\right|^{2 \nu}+\left|\lambda^{A}+\left(\sum_{B} \pm \lambda^{B}\right)\right|^{2 \nu}\right]  \tag{H.8}\\
& \left|\lambda^{A}\right|^{2 \nu-2}\left|\lambda^{C}\right|\left|\lambda^{D}\right|
\end{align*}
$$

is also bounded (in module) from above inside $\mathcal{H}^{\mathcal{S}}$.
Let us now consider cases (a)-(d) above.
Case (a). We can write

$$
\begin{align*}
Y_{K L}^{\mathbf{T}_{J j}} & =-\frac{1}{2}\left\{\left(\frac{\left|\lambda^{K}\right|}{\left|\lambda^{L}\right|}\right)^{1-\nu} \frac{\left|\lambda^{L}-\lambda^{K}\right|^{2 \nu}-\left|\lambda^{L}\right|^{2 \nu}}{\left|\lambda^{L}\right|^{2 \nu-1}\left|\lambda^{K}\right|}-\left(\frac{\left|\lambda^{K}\right|}{\left|\lambda^{L}\right|}\right)^{\nu}\right\} \\
& =\mathcal{O}\left\{\left(\frac{\left|\lambda^{K}\right|}{\left|\lambda^{L}\right|}\right)^{1-\nu}\right\}+\mathcal{O}\left\{\left(\frac{\left|\lambda^{K}\right|}{\left|\lambda^{L}\right|}\right)^{\nu}\right\} \\
& =\mathcal{O}\left(\prod_{K \leqslant I<L}\left(\beta^{I}\right)^{\delta}\right) \tag{H.9}
\end{align*}
$$

Case (c): We now write

$$
\begin{align*}
Y_{K L}^{\mathbf{T}_{J j}}= & \frac{1}{2}\left(\frac{\left|\lambda^{K}\right|}{\left|\lambda^{P}\right|}\right)^{1-\nu}\left(\frac{\left|\lambda^{P}\right|}{\left|\lambda^{L}\right|}\right)^{\nu} \\
& \times\left\{\frac{\left|\lambda^{P}+\sum c_{M}^{K L} \lambda^{M}-\lambda^{K}\right|^{2 \nu}-\left|\lambda^{P}+\sum c_{M L}^{K L M}\right|^{2 \nu}}{\left|\lambda^{P}\right|^{2 \nu-1}\left|\lambda^{K}\right|}\right\} \\
& -\frac{1}{2}\left(\frac{\left|\lambda^{K}\right|}{\left|\lambda^{L}\right|}\right)^{1-\nu} \\
& \times\left\{\frac{\left|\lambda^{L}+\lambda^{P}+\sum c_{M}^{K L} \lambda^{M}-\lambda^{K}\right|^{2 \nu}-\left|\lambda^{L}+\lambda^{P}+\sum c_{M}^{K L} \lambda^{M}\right|^{2 \nu}}{\left|\lambda^{L}\right|^{2 \nu-1}\left|\lambda^{K}\right|}\right\} \\
= & \mathcal{O}\left\{\left(\frac{\left|\lambda^{K}\right|}{\left|\lambda^{P}\right|}\right)^{1-\nu}\left(\frac{\left|\lambda^{P}\right|}{\left|\lambda^{L}\right|}\right)^{\nu}\right\}+\mathcal{O}\left\{\left(\frac{\left|\lambda^{K}\right|}{\left|\lambda^{L}\right|}\right)^{1-\nu}\right\} \\
= & \mathcal{O}\left\{\prod_{K \leqslant I<P}\left(\beta^{I}\right)^{1-\nu} \prod_{P \leqslant I<L}\left(\beta^{I}\right)^{\nu}\right\}+\mathcal{O}\left\{\prod_{P \leqslant L}\left(\beta^{I}\right)^{1-\nu}\right\} \\
= & \mathcal{O}\left(\prod_{K \leqslant I<L}\left(\beta^{I}\right)^{\delta}\right), \tag{H.12}
\end{align*}
$$

which proves proposition 2. Here again, we can write $Y_{K L}^{\mathbf{T}_{J j}}$ as a function of the ratios $\left|\lambda^{K}\right| /\left|\lambda^{L}\right|,\left|\lambda^{P}\right| /\left|\lambda^{L}\right|$ and $\left|\lambda^{M}\right| /\left|\lambda^{L}\right|$. Since $K<M<P<L$, we deduce proposition 3.

Case (d): In this case $R^{K L}=-\lambda^{L}+\sum c_{M}^{K L} \lambda^{M}$ and the propositions can be obtained from case (c) by simply interchanging $R^{K L}$ and $R^{K L}+\lambda^{L}$. This achieves the proof of propositions 2 and 3.

Finally, proposition 4 is a consequence of propositions 1 and 3. Indeed, from proposition 3 and the fact that the $\beta^{\prime}$ 's are bounded from above inside $\mathcal{H}^{\mathcal{S}}$, we obtain a uniform upper bound for $\left|Y_{K L}^{\mathbf{T}_{J, j}}\right|$ inside $\mathcal{H}^{\mathcal{S}}$. This upper bound, together with the lower bound of proposition 1 on $\operatorname{det}\left(Y^{\mathbf{T}_{J, j}}\right)$, gives a uniform upper bound for the modules $\left|\left(Y^{\mathbf{T}_{J, j}}\right)_{K L}^{-1}\right|$ of the elements of the inverse matrix. This implies that $\left(Y^{\mathbf{T}_{J, j}}\right)^{-1}$ is bounded from above by a positive number $C^{-1}$ (that is $\left(Y^{\mathbf{T}}{ }_{f, j}\right)^{-1}-C^{-1}$ II is a negative matrix), and, since $\left(Y^{\mathbf{T}_{J, j}}\right)^{-1}$ is a positive matrix, that $Y^{\mathbf{T}_{J, j}}$ is bounded from below by the strictly positive number $C$.
(II) Integral representation of $(1-\tau)$. From now on, we shall work inside the whole diagram $\tilde{T}_{J}$ for fixed $J$ and treat in parallel its distinct connected components $\tilde{\mathcal{T}}_{J, j}$ for varying $j$. This is achieved by introducing the block diagonal
matrix:

$$
Y^{J}=\left[\begin{array}{cccc}
\left.\begin{array}{|c|cc}
Y^{\mathbf{T}_{J, 1}} & 0 & \cdots \\
\hline Y^{\mathbf{T}_{J, 2}} & \cdots & 0 \\
0 & & \cdots \\
\vdots & \vdots & \ddots
\end{array}\right]  \tag{H.13}\\
0 & 0 & \cdots & \vdots \\
0 \mathbf{T}_{l, \max } \\
\hline
\end{array}\right.
$$

with $j^{\max }=\operatorname{Card}\left(T_{J}^{0}\right)$. This matrix is such that (for $\left.J \leqslant T\right)$ :

$$
\begin{equation*}
I_{\tilde{T}_{J}}=\left(\operatorname{det}\left(Y^{J}\right)\right)^{-d / 2} \tag{H.14}
\end{equation*}
$$

Now we must consider the action of $\left(1-\tau_{\bar{T}_{J-1 \oplus}^{\prime}}^{0}\right)$ on $I_{T_{j}}$. For our particular choice of tree variables, the action of $\tau_{T_{J-1 \oplus}^{\prime}}^{0}$ simply corresponds to set $\beta^{I}=0$ in the matrix $Y^{J}$. From the propositions 2 and 3 of the preceding subsection, we know that an element $Y_{K L}^{\mathrm{T}_{J, j}}$ of the matrix $Y^{J}$ either is independent of $\beta^{I}$ (if $I \notin \mathcal{J}_{J, j}(K, L)$ for this value of $j$ ), or vanishes with $\beta^{I}$ at least as $\left(\beta^{I}\right)^{\delta}$ (if $I \in \mathcal{J}_{J, j}(K, L)$ ). Therefore the action of $\tau_{T_{J-1 \oplus}^{\prime}}^{0}$ simply corresponds to set to zero those elements of $Y^{J}$ which depend on $\beta^{I}$, leaving the other elements unchanged. In particular, this action is non trivial (i.e. non reduced to the identity) when

$$
\begin{equation*}
I \in \mathcal{J}_{J} \equiv \bigcup_{j, K, L} \mathcal{J}_{J, j}(K, L) \tag{H.15}
\end{equation*}
$$

Conversely, if $I \notin \mathcal{J}_{J}$, then $\left(1-\tau_{\tilde{T}_{J-\mid \oplus}^{\prime}}^{0}\right)\left[I_{\tilde{T}_{J}}\right]=0$. To perform the action of $\tau_{\bar{T}_{J-t \oplus}^{J}}^{0}$, it is convenient to introduce an extra variable $t^{I}$ which multiplies the elements $Y_{K L}^{\mathbf{T}_{J j}}$ such that $I \in \mathcal{J}_{J, j}(K, L)$. We thus define

$$
\begin{equation*}
Y_{K L}^{\mathbf{T}_{J, j}}(\{t\}) \equiv\left(\prod_{I \in \mathcal{J}_{J, j}(K, L)} t^{I}\right) Y_{K L}^{\mathbf{T}_{J, j}} \tag{H.16}
\end{equation*}
$$

and obtain a matrix $Y^{J}(\{t\})$ which is a function of the $t^{I}$ 's for $I \in \mathcal{J}_{J}$. The action of $\tau_{T_{J-1 \oplus}^{\prime}}^{0}$ then corresponds to set $t^{I}=0$ (and set the other $t^{I}$ 's equal to 1). We then have the following integral representation of a ( $1-\tau_{\tilde{T}_{J-\mid \oplus}^{J}}$ ) operator with $I \in \mathcal{J}_{J}$ :

$$
\left(1-\tau_{\tilde{T}_{J-1 \oplus}^{\prime}}^{0}\right)\left[\prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)}\left(\operatorname{det}\left(Y^{\mathbf{T}_{J, j}}\right)\right)^{-d / 2}\right]=\int_{0}^{1} \mathrm{~d} t^{I} \frac{\partial}{\partial t^{I}}\left[\operatorname{det}\left(Y^{J}(\{t\})\right)\right]_{(\mathrm{H} .17}^{-d / 2}
$$

Now we must apply a product of such Taylor operators for all the $I \in \operatorname{Ind}(J)$. We can use the fact that ( $1-\tau$ ) is a projector, and can thus be applied several times to the same diagram. Since all the reduced diagrams $\tilde{T}_{J-1}^{I}$ for

$$
\begin{equation*}
I \in \overline{\mathcal{J}_{J}} \equiv\left\{I: I^{\min }(J) \leqslant I<I^{\max }(J)\right\} \tag{H.18}
\end{equation*}
$$

are equal to some $\tilde{T}_{J-1}^{I}$ for $I \in \operatorname{Ind}(J)$, we have

$$
\begin{equation*}
\prod_{I \in \operatorname{Ind}(J)}\left(1-\tau_{\tilde{T}_{J-1 \oplus}^{\prime}}^{0}\right)=\prod_{I \in \overline{\mathcal{J}_{J}}}\left(1-\tau_{\tilde{T}_{J-1 \oplus}^{J}}^{0}\right) \tag{H.19}
\end{equation*}
$$

From their definitions and (8.11) and (8.12), it is clear that $\mathcal{J}_{J} \subset \overline{\mathcal{J}_{J}}$. If $\mathcal{J}_{J} \varsubsetneqq \overline{\mathcal{J}_{J}}$, then the above product of $(1-\tau)$, when acting on $I_{\tilde{T}_{J}}$, gives 0 , as a consequence of the discussion above. Eq. (8.35) is then obviously satisfied. We can therefore assume that $\mathcal{J}_{J}=\overline{\mathcal{J}_{J}}$. We then write the l.h.s of (8.35) as

$$
\begin{align*}
\prod_{I \in \operatorname{Ind}(J)}(1 & \left.-\tau_{\tilde{T}_{J-1 \oplus}^{I}}^{0}\right)\left[\prod_{j=1}^{\operatorname{Card}\left(T_{J}^{0}\right)}\left(\operatorname{det}\left(Y^{\mathbf{T}_{J, j}}\right)\right)^{-d / 2}\right] \\
& =\int_{0}^{1} \prod_{I \in \mathcal{J}_{J}} \mathrm{~d} t^{I} \frac{\partial}{\partial t^{I}}\left[\operatorname{det}\left(Y^{J}(\{t\})\right)\right]^{-d / 2} \tag{H.20}
\end{align*}
$$

with $\mathcal{J}_{J}=\overline{\mathcal{J}_{J}}=\left\{I: I^{\min }(J) \leqslant I<I^{\max }(J)\right\}$.
(III) Proof of estimates (8.35) and (8.37). First we use the fact that the property 4 of the matrix $Y^{\mathbf{T}_{J, j}}$ extends to the matrix $Y^{J}(\{t\})$. Specifically we have

Proposition 5: The matrix $Y^{J}(\{t\})$ is positive and bounded from below (i.e. $Y^{J}(\{t\})-C$ II $>0$ for some positive $\left.C\right)$ for all $0 \leqslant t^{I} \leqslant 1, I \in \mathcal{J}_{J}$.

Indeed, this property holds when each $t^{I}$ equals 0 or 1 . In this case, each block $Y^{\mathbf{T}_{J, j}}$ of the matrix is "factorized" into a product of sub-blocks $Y^{\mathbf{T}}$ for subtrees $\mathbf{T}$ (see appendix C) compatible with the sector. Each of these sub-matrices $Y^{\mathbf{T}}$ then satisfies proposition 4 , as well as the matrix $Y^{J}$. To complete the proof of proposition 5, we use the fact that the matrix $Y^{J}(\{t\})$ is a linear function of each $t^{I}$, and that it is thus sufficient to have a lower bound at each corner of the hypercube $0 \leqslant t^{I} \leqslant 1\left(I \in \mathcal{J}_{J}\right)$ to have this bound inside the whole hypercube.

A direct consequence of proposition 5 is that $Y^{J}(\{t\})$ is invertible, and that $\left(Y^{J}(\{t\})\right)^{-1}$ is positive and bounded from above uniformly in the sector. In particular, the module of all the elements $\left(Y^{J}(\{t\})\right)_{K L}^{-1}$ is also bounded from above. Another consequence of proposition 5 is that $\operatorname{det}\left(Y^{J}(\{t\})\right)$ is uniformly bounded from below by a strictly positive number.

Finally, if $\mathcal{E}$ is some subset of $\mathcal{J}_{J}$, it is clear from proposition 3 and the definition (H.16) of $Y^{J}(\{t\})$ that, in the considered Hepp sector, the partial set-

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[^1]:    * This determinant appears, in a different disguise, in a letter by Descartes to the Princess Elisabeth of Bohemia (1643), as quoted by Coxeter in ref. [29].

[^2]:    * In the context of polymers, it is also known as the Fixman representation [3].

[^3]:    * In renormalization theory, a forest is a family of diagrams $\mathcal{P}_{k}$ such that for any $k \neq l$ one has either $\mathcal{P}_{k} \subset \mathcal{P}_{l}$, or $\mathcal{P}_{l} \subset \mathcal{P}_{k}$, or $\mathcal{P}_{k} \cap \mathcal{P}_{l}=\varnothing$.

[^4]:    * Let us stress that $P \prec Q$ does not mean that $P$, considered as a set (whose elements are subsets of $\mathcal{G}$ ), is included in $Q$. Still if $P \subset Q$, then $P \prec Q$.

[^5]:    * We use superscripts here in $R^{I}$ rather than subscripts as before in $T_{J}$ for future convenience.

[^6]:    * We recall that the domain $\mathcal{H}^{\mathbf{T}}$ corresponds to the domain where the $\lambda_{\alpha}$ 's obtained from the $y_{i}$ 's by eq. (3.43) are actual successive minimal distances, and in particular satisfy $\left|\lambda_{1}\right| \leqslant \ldots \leqslant\left|\lambda_{N-1}\right|$.

[^7]:    * This root is either $w$ or some vertex of $\mathcal{R} \backslash \mathcal{T}$. We use here the convention that a diagram is explicited by keeping only each of its connected components having more that one element. For instance, $\{(\mathcal{T}, w)\}$ is a short-hand notation for $\left(G_{\odot} \vee\{\mathcal{T}\}, \omega\right)$ which means that the diagram must be completed by the set of all remaining isolated points not already in $\mathcal{T}$, while $\omega$ consists of the root $w$ plus these isolated points. Similarly, Eq. (7.23) is a short-hand notation for $\tilde{\mathcal{N}}_{\oplus}=\left\{\left(G_{\odot}, G_{\odot}\right),\left(G_{\odot} \vee\{\mathcal{R} \cap \mathcal{T}\}, \omega\right),\left(G_{\odot} \vee\{\mathcal{T}\}, \omega\right),\left(G_{\odot} \vee\{\mathcal{R} \cup \mathcal{T}\}, \bullet\right)\right\}$.

[^8]:    * In general, this tableau still contains series of identical successive elements, but not more than $N$ successive elements can be identical.

[^9]:    ${ }^{*}$ More precisely, we integrate over the $x_{i}$ 's such that the $y_{i}$ 's defined by $y_{i}=x_{i+1}-x_{1}$ are in $\mathcal{H}^{\mathcal{S}}$, since a Hepp sector is actually defined in terms of relative positions.

[^10]:    * Notice that $\tilde{T}_{T+1}$ has only one connected component.

[^11]:    * This homogeneity property holds only for the choice (4.17) for the propagator, even on the sphere. Otherwise, both $\beta_{N-1}$ and the IR regulator $R$ would appear and lead to a slightly more complicated discussion.

