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The $O(n)$ model on a random surface: critical points and large order behaviour

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In this article we report a preliminary investigation of the large N limit of a generalized one-matrix model which represents an $O(n)$ symmetric model on a random lattice. The model on a regular lattice is known to be critical only for $-2 \leq n \leq 2$. This is the situation we shall discuss also here, using steepest descent. We first determine the critical and multicritical points, recovering in particular results previously obtained by Kostov. We then calculate the scaling behaviour in the critical region when the cosmological constant is close to its critical value. Like for the multi-matrix models, all critical points can be classified in terms of two relatively prime integers p, q. In the parametrization $p = (2m+1)q \pm l$, m, l integers such that $0 < l < q$, the string susceptibility exponent is found to be $\gamma_{\text{string}} = -2l/(p+q-l)$. When $l = 1$ we find that all results agree with those of the corresponding (p, q) string models, otherwise they are different.

We finally explain how to derive the large order behaviour of the corresponding topological expansion in the double scaling limit.

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1. Introduction

We discuss in this article an $O(n)$ symmetric matrix model in the large N limit. The model has a direct interpretation in terms of a gas of loops [1], characterized by an $O(n)$ index, drawn on Feynman diagrams. The partition function is initially given by an integral over $n+1$ matrices, but the integral over n of them is gaussian. The gaussian integrations generate an unusual effective one-matrix model which, unlike the standard one-matrix model [2], cannot be solved by the standard orthogonal polynomial method [3].

The resulting model is solved instead, in the spherical limit, by the method of steepest descent. The saddle point equations generalize similar equations for the usual one-matrix model, but the method of solution involves new technical considerations. We show in particular that when n is of the form $n = -2\cos(\pi p/q)$, with p, q positive relatively prime integers, the trace of the resolvent of the matrix is the solution of an algebraic equation of degree q with polynomial coefficients.

As expected from the analysis of two dimensional regular lattices, non-trivial critical behaviour is only found for $-2 \le n \le 2$. We exhibit a class of critical and multicritical points, recovering in particular results previously derived by Kostov [1]. We mainly discuss the case $n = -2\cos(\pi p/q)$, p, q being two relatively prime integers, although some considerations also apply to arbitrary values of n. When p has the form $p = (2m+1)q \pm 1$ we find results consistent with the (p, q) models (in the CFT classification [4]) as previously obtained in the multi-matrix models [5]. This includes the unitary family. However in general the scaling properties are different (or at least the operator content). This is not totally surprising since in general these models cannot be directly related to multi-matrix models. There is however one exception, the two-matrix model representing the Ising model on a random triangular lattice. This model is identical to an $O(1)$ model with a special cubic potential. It will be investigated first in order to show that the results obtained by the method of orthogonal polynomials [6] are recovered at leading order.

We then discuss multicritical points in a general potential in the $O(1)$ model. We finally study the critical points of the general $O(n)$ models.

For the one-matrix model the large order behaviour of the topological expansion has first been determined by a direct analysis of the differential equations [7,8] satisfied by the scaling specific heat. It has been shown that perturbation series is divergent and for half of the models (this includes pure gravity) non-Borel summable. This property has later been related to the existence of non-trivial saddle points of the initial matrix integral in the large N limit [9] (the equivalent of the instanton solution of quantum mechanics and field theory). In the non-Borel summable case these saddle points are real, a property related to the unbounded nature of the integrand in the scaling limit or equivalently to the instability of the corresponding statistical models by creation of surfaces of higher topologies.

In the case of the multi-matrix model, the large order behaviour of the critical and tricritical Ising models has been completely determined, again by a direct analysis of the differential equations. For the general (p, q) model only the $2k!$ divergence of perturbation theory has been established. The application of the steepest descent method to the multi-matrix integrals instead is not easy, because the leading contributions cancel since now the square of the Vandermonde determinant is replaced by the product of two independent determinants involving the eigenvalues of the first and last matrices of the chain.

We show that the models considered here, because they can be solved by steepest descent, allow for a direct determination of the large order behaviour.

Since the discussion of the critical points involves many technical considerations, in this article we explain the methods, establish the scaling of the free energy, explicitly calculate the trace of the resolvent of the matrix in the scaling limit, and compare it to the resolvent of the (p, q) string models. In a next article we shall examine the critical behaviour in more details.

2. The critical Ising model

We first consider a special example of the two-matrix model with the partition function given by

$$Z = \int \mathrm{d}M_1 \mathrm{d}M_2 \, \mathrm{e}^{-(N/g)\mathrm{tr}\, U(M_1, M_2)}, \tag{2.1}$$

M_1, M_2 being two $N \times N$ hermitian matrices and the potential U having the form

$$U(M_1, M_2) = \tfrac{1}{2}(M_1^2 + M_2^2) - cM_1M_2 + (M_1^3 + M_2^3)/3, \quad 0 < c < 1. \tag{2.2}$$

This model can of course be exactly solved by the method of orthogonal polynomials [6]. We expect that it exhibits a double scaling limit representing the critical Ising model on a random surface. We solve it however here by a different method and only at leading order for N large. A comparison with the large N limit of the exact result provides a check for the new method of solution which we can then apply to the general $O(n)$ model.

We first transform integral (2.1), setting $A = (M_1 - M_2)/2$ and $S = M_1 + M_2 + 1 + c$. Substituting into the potential (2.2) we find

$$\operatorname{tr} U(M_1, M_2) = \operatorname{tr} [A^2 S + V(S)],$$

with

$$4V = S^3/3 - 2cS^2 + (3c-1)(1+c)S + \text{const.}. \tag{2.3}$$

Eventually the matrix A will be replaced by n copies, the Ising model being identified with the $O(1)$ case of the $O(n)$ model.

The gaussian integral over A can be performed and Z is now given by an integral over only one matrix S:

$$Z = \int \mathrm{d}S\, [\det(S \otimes 1)]^{-1/2} \mathrm{e}^{-(N/g) \operatorname{tr} V(S)}. \tag{2.4}$$

The integral over unitary matrices can still be performed, and the result is

$$Z = \int \Delta^2(\Lambda) \prod_{i=1}^{N} \mathrm{d}\lambda_i\, \mathrm{e}^{-(N/g)V(\lambda_i)} \prod_{i,j} (\lambda_i + \lambda_j)^{-1/2}\,, \tag{2.5}$$

where we denote by $\Delta(\Lambda)$ the usual Vandermonde determinant of the eigenvalues of S. In this form the model can no longer be solved by the method of orthogonal polynomials, but conversely it is much easier, as we shall show, to solve it, in the spherical limit, by the method of steepest descent and then to also find non-trivial saddle points.

2.1. The saddle point equation

Varying one eigenvalue of S we obtain the saddle point equation

$$\frac{2}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \frac{1}{N} \sum_{j} \frac{1}{\lambda_i + \lambda_j} + \frac{1}{g} V'(\lambda_i), \tag{2.6}$$

with

$$4V'(\lambda) = (\lambda - 2c)^2 - (c-1)^2 = (\lambda - 3c + 1)(\lambda - c - 1). \tag{2.7}$$

In addition to the repulsive potential between eigenvalues, already present in the one-matrix model, we have now an attractive and unbounded potential at $-\lambda_j$. A potential barrier must separate the eigenvalues from their reflected images and thus a neighbourhood of the origin must be free of eigenvalues, all eigenvalues being for g positive on the positive real axis.

It is convenient to rewrite equation (2.6) in terms of the trace $\omega_0(z)$ of the resolvent of the matrix S,

$$\omega_0(z) = \frac{1}{N} \mathrm{tr} \frac{1}{z-S} = \frac{1}{N} \sum_i \frac{1}{z-\lambda_i}. \tag{2.8}$$

In the large N limit the distribution of eigenvalues $\rho(\lambda) = N^{-1} \sum_i \delta(\lambda - \lambda_i)$ becomes a continuous function and equation (2.6) can be rewritten:

$$\omega_0(z+i0) + \omega_0(z-i0) = -\omega_0(-z) + V'(z)/g. \tag{2.9}$$

The distribution $\rho(\lambda)$ is then given by

$$\rho(\lambda) = -\frac{1}{2i\pi} \big(\omega(\lambda + i0) - \omega_0(\lambda - i0)\big).$$

A particular polynomial solution of the equation (2.9) is

$$\omega_r(z) = \frac{1}{3g} \big(2V'(z) - V'(-z)\big). \tag{2.10}$$

Setting

$$\omega_0(z) = \omega_r(z) + \frac{1}{12g} \omega(z), \tag{2.11}$$

we then look for a one-cut solution $\omega(z)$ of the homogeneous equation which, as a consequence of equation (2.7) and the definition of ω_0, for z large behaves like

$$\omega(z) = -4\big(2V'(z) - V'(-z)\big) + (12g)/z + O\left(z^{-2}\right) \sim -z^2. \tag{2.12}$$

A quadratic relation. We have now to solve the homogeneous equation

$$\omega(z+i0) + \omega(z-i0) = -\omega(-z), \tag{2.13}$$

with the hypothesis that $\omega(z)$ is a real analytic function with a unique cut on the real positive axis ($z = 0$ excluded).

This equation has an equivalent form. Let us consider the even function

$$r(z) = \tfrac{1}{3} \left[\omega^2(z) + \omega^2(-z) + \omega(z)\omega(-z)\right], \tag{2.14}$$

and calculate its discontinuity across the cut on $z > 0$, where $\omega(-z)$ is regular:

$$\begin{aligned} r(z+i0) - r(z-i0) = \tfrac{1}{3} &\left[\omega(z+i0) - \omega(z-i0)\right] \\ &\times \left[\omega(z+i0) + \omega(z-i0) + \omega(-z)\right]. \end{aligned} \tag{2.15}$$

The r.h.s. vanishes and thus $r(z)$ is an even regular function, which taking into account the large z behaviour of $\omega(z)$ is a polynomial of degree 4. Moreover equation (2.15) is equivalent to equation (2.13).

Direct derivation. Note that equation (2.14) can be directly derived from equation (2.6) by multiplying (2.6) by $1/(z-\lambda_i)$ and summing over i. From the two identities

$$\frac{2}{N^2}\sum_{i\neq j}\frac{1}{z-\lambda_i}\frac{1}{\lambda_i-\lambda_j}=\omega_0^2(z)+\frac{1}{N}\omega_0'(z),$$
$$\frac{1}{N^2}\sum_{i,j}\left(\frac{1}{z-\lambda_i}-\frac{1}{z+\lambda_j}\right)\frac{1}{\lambda_i+\lambda_j}=-\omega_0(z)\omega_0(-z),$$

and neglecting a term of order $1/N$, we find for $\omega_0(z)$:

$$\omega_0^2(z)+\omega_0^2(-z)+\omega_0(z)\omega_0(-z)=\frac{1}{g}\big(V'(z)\omega_0(z)+V'(-z)\omega_0(-z)\big)+r_1(z),$$

where r_1 is a constant when $V'(z)$ is a polynomial of degree 2:

$$g\,r_1(z)=\int \mathrm{d}\lambda\,\rho(\lambda)\left[\frac{V'(\lambda)-V'(z)}{z-\lambda}-\frac{V'(\lambda)-V'(-z)}{z+\lambda}\right]=2c-\int \mathrm{d}\lambda\,\rho(\lambda)\lambda\,.$$

Shifting $\omega_0(z)$ we obtain an explicit expression for $r(z)$. We note in particular that a variation of g translates for ω only into a variation of r_1. Finally the same equation can also obtained from the loop equations (the equations of motion) in the large N limit.

2.2. General one-cut solution

A cubic algebraic equation. Equation (2.13) implies that the branch points are square root branch points because turning twice around them we return to the initial function. However in the second sheet we find also the cut of $\omega(-z)$. If we pass through this new cut we arrive in general onto a new sheet and so on. Here, due to the special coefficients, we find only three different sheets, which implies that $\omega(z)$ satisfies an algebraic equation of third degree. This equation can immediately be obtained by multiplying equation (2.14) by $\omega(z)-\omega(-z)$. We then find:

$$\omega^3(z)-3r(z)\omega(z)=\omega^3(-z)-3r(z)\omega(-z)=2s(z), \tag{2.16}$$

where $s(z)$ is an even function which is everywhere regular because the first expression is regular for $z<0$ and the second for $z>0$. It is therefore a polynomial.

General solution. The explicit solution of equation (2.16) can be written

$$\omega(z) = \mathrm{e}^{-2i\pi/3}\omega_+(z) + \mathrm{e}^{2i\pi/3}\omega_-(z), \tag{2.17a}$$

$$\omega_\pm(z) = \left(s(z) \pm \sqrt{\Delta(z)}\right)^{1/3}, \tag{2.17b}$$

where Δ is the discriminant of the equation

$$\Delta(z) = s^2(z) - r^3(z). \tag{2.18}$$

Because we look for solutions $\omega(z)$ which are not even in z, we verify that we can write $\sqrt{\Delta}$:

$$\sqrt{\Delta} = i3^{-3/2}\big(\omega(z) - \omega(-z)\big)\big(\omega^2(z) + \omega^2(-z) + \tfrac{5}{2}\omega(z)\omega(-z)\big).$$

This expression in particular shows that $\sqrt{\Delta}$ is an odd function. It follows that $\omega_-(z) = \omega_+(-z)$ and therefore

$$\omega_\pm(z) = \pm\frac{1}{i\sqrt{3}}\left[\mathrm{e}^{\mp 2i\pi/3}\omega(z) - \mathrm{e}^{\pm 2i\pi/3}\omega(-z)\right]. \tag{2.19}$$

Our normalizations are such that $\omega_\pm \sim z^2$.

The reciprocal formulae are also useful

$$r(z) = \omega_+(z)\omega_-(z), \qquad s(z) = \tfrac{1}{2}\left(\omega_+^3(z) + \omega_-^3(z)\right). \tag{2.20}$$

It follows:

$$\sqrt{\Delta} = \tfrac{1}{2}\left(\omega_+^3(z) - \omega_-^3(z)\right). \tag{2.21}$$

The expansion (2.12) of $\omega(z)$ for large z implies that $\sqrt{\Delta}$ behaves like z^5. For the general one-cut solution $\sqrt{\Delta}$ must then have the form

$$\sqrt{\Delta} = 12\sqrt{3}icz\left(z^2 - e^2\right)\left[\left(z^2 - a^2\right)\left(z^2 - b^2\right)\right]^{1/2},$$

where we have chosen $b \geq a \geq 0$. The condition that $\Delta(0) = 0$ implies the relation

$$s^2(0) = r^3(0). \tag{2.22}$$

The determination of $\sqrt{\Delta}$ has been chosen such that

$$\omega_\pm(z - i0) = \mathrm{e}^{\pm 2i\pi/3}\omega_\mp(z + i0), \quad \text{for } z \in [a, b].$$

The relation $\omega_+(z) = \omega_-(-z)$ then automatically implies that $\omega(z)$ has in the first sheet a cut only along $[a,b]$. Indeed

$$\begin{aligned} &\text{for } z>0 \quad \omega(z-i0) = \omega_+(z+i0) + \omega_-(z+i0), \\ &\text{for } z<0 \quad \omega(z-i0) = \omega(z+i0). \end{aligned}$$

Critical points. A critical point is generated by the confluence of two different zeros of Δ. Two situations can arise:

(i) $a = e$, this is the case of an ordinary critical point of the one-matrix model, the determinant coming from the integration over A playing no special role. It just modifies the form of the potential. From the point of view of the Ising model this is the low temperature phase where all Ising degrees of freedom are frozen.

(ii) $a = 0$, this is a new critical point specific to the structure of integral (2.5), and the only case we shall consider from now on. The condition $a = 0$ implies that $N < \mathrm{tr}\, A^2 >$, which characterizes the Ising spin fluctuations, diverges. Indeed $N < \mathrm{tr}\, A^2 >$ is proportional to $\sum_{i,j} 1/(\lambda_i + \lambda_j)$ and diverges only when some eigenvalue of S vanishes [1]. This argument is confirmed in the continuum limit by a determination of the scaling properties of $N < \mathrm{tr}\, A^2 >$.

Finally the critical Ising model in the continuum limit is reached when both confluences occur simultaneously, $a = e = 0$. Note, however, that in this limit the eigenvalue distribution approaches a singularity of the integrand. Therefore we can no longer be certain that the steepest descent method is valid beyond the leading order for N large. We have made a few checks but this point requires a more detailed investigation.

2.3. Simple critical point

General critical solutions. Solutions corresponding to these critical points will be characterized by the property that they have a cut only between the origin and $z = b > 0$. The functions $\omega_\pm$ corresponding to these solutions have the following general form:

$$\omega_\pm = \left(\sqrt{1 - b^2/z^2} \mp ib/z\right)^{1/3} \left(A(z)\sqrt{1 - b^2/z^2} \pm ibB(z)/z\right), \tag{2.23}$$

where A and B are even polynomials. The functions r, s, Δ have then indeed the expected analytic properties:

$$r(z) = z^{-2}\left[A^2(z)\left(z^2 - b^2\right) + b^2 B^2(z)\right], \tag{2.24a}$$

$$s(z) = 2z^{-4}\left[A^3\left(z^2 - b^2\right)^2 + 3\left(A^2 B - B^2 A\right) b^2 \left(z^2 - b^2\right) - b^4 B^3\right], \tag{2.24b}$$

$$\sqrt{\Delta} = ibz^{-4}\sqrt{z^2 - b^2}\left[b^2 B^2 (3A - B) + \left(z^2 - b^2\right) A^2 (3B - A)\right], \tag{2.24c}$$

provided $A(z)$ and $B(z)$ vanish at $z=0$. The degrees of the polynomials A, B are directly connected to the large z behaviour of $\omega_\pm$ and thus to the degree of the potential $V(S)$.

The cubic potential. In the case of the potential (2.3) $\omega_\pm \sim z^2$ for $z \to \infty$. These conditions imply $A(z) = z^2$ and $B(z) \propto z^2$, and therefore the general solution (2.23) reduces to

$$\omega_\pm = z^2 \left(\sqrt{1-b^2/z^2} \mp ib/z\right)^{1/3} \left(\sqrt{1-b^2/z^2} \pm ib\sigma/z\right). \tag{2.25}$$

The criticality condition $a = 0$ defines a line $g = g_c(c)$ in the g, c plane, and thus the two parameters b, σ are functions of c. Note that the double zeros $\pm e$ of Δ are given by $e^2 = b^2(\sigma-1)^3/(3\sigma-1)$. They vanish for $\sigma = 1$ which thus corresponds to the Ising model critical point, a case we examine later. We first discuss the generic case $\sigma \neq 1$.

To relate the parameters b, σ with the parameters c, g which characterize the special potential (2.3) of the Ising model we can compare the expansions of $\omega_+(z)$ for z large up to order $1/z$. Using equations (2.12) and (2.17a) we find at the critical point $b(3\sigma-1) = 12\sqrt{3}c$, $b^2(3\sigma-5)/9 = (3c-1)(1+c)$, and

$$12g_c\sqrt{3} = b^3\left(\frac{19}{54} - \frac{\sigma}{6}\right). \tag{2.26}$$

The conditions $0 \le c$ and $g > 0$ imply $1/3 \le \sigma \le 19/9$. The limit $\sigma \to -\infty$, which cannot be reached in the Ising model, corresponds to a quadratic potential $V(S)$. For $c = 0$, i.e. $\sigma = 1/3$ we know the critical value of g from the solution of the one-matrix model, $g_c = 1/(12\sqrt{3})$, value which agrees with the result (2.26).

Critical behaviour. The critical behaviour is obtained from the small z behaviour of $\omega_\pm$. For $\sigma \neq 1$,

$$\omega(z) \propto z^{2/3}.$$

For $\sigma = 1$ the behaviour is different and is discussed in the section 2.5. The amplitude in front of $z^{2/3}$, which is needed below to determine the normalization of $\omega(z)$ in the scaling limit, depends on the determinations. We therefore choose to let z approach the origin on the negative imaginary axis: $z = -i\lambda$, $\lambda \to 0_+$, a choice we shall keep throughout the article. Then

$$\omega_+(-i\lambda) \sim -2^{1/3}(1-\sigma)b^{4/3}\lambda^{2/3}, \tag{2.27a}$$

$$\omega_-(-i\lambda) \sim -2^{-1/3}(\sigma+1)b^{2/3}\lambda^{4/3}. \tag{2.27b}$$

2.4. The scaling limit

We now calculate $\omega(z)$ near the critical point, when the cosmological constant is close to its critical value, i.e. for

$$|x = 1 - g/g_c| \ll 1\,,$$

in the scaling limit. Scaling functions are solutions of equation (2.13) with a cut for $z \ge a > 0$. The scaling functions $\omega_\pm$ have the general form

$$\omega_{\mathrm{sc},\pm} = \left(\sqrt{a^2 - z^2} \mp iz\right)^{1/3} \left(C(z)\sqrt{a^2 - z^2} \pm izD(z)\right), \tag{2.28}$$

where D, C are two even polynomials whose form is fixed by comparing the large z behaviour of ω_{sc} with the small z expansion of the critical functions.

Equations (2.27) yield the large z behaviour of ω_{sc} for $\lambda \to +\infty$ and thus the degrees of the polynomials C, D. Here C and D must be equal constants and thus:

$$\omega_{\mathrm{sc},\pm} = -2^{-1/3} b^{4/3} (1 - \sigma) \left(\sqrt{a^2 - z^2} \pm iz\right)^{2/3}. \tag{2.29}$$

It remains to obtain the relation between a and x. One can for this purpose directly calculate the variation of the polynomials r, s at leading order in x. However we use here a different argument which can easily be generalized to more complicate cases.

An auxiliary function. We consider the function $\Omega(z)$:

$$\Omega(z) = \frac{\partial}{\partial g}\big(g\omega_0(z)\big). \tag{2.30}$$

We note that (2.11) implies

$$\Omega(z) = \frac{1}{12}\frac{\partial \omega(z)}{\partial g}.$$

It thus satisfies the homogeneous equation (2.13). It is then convenient to introduce the corresponding two functions $\Omega_\pm$:

$$\Omega_\pm = \pm\frac{1}{i\sqrt{3}}\left[\mathrm{e}^{\mp 2i\pi/3}\Omega(z) - \mathrm{e}^{\pm 2i\pi/3}\Omega(-z)\right],$$

and therefore

$$\Omega(z) = \mathrm{e}^{-2i\pi/3}\Omega_+(z) + \mathrm{e}^{2i\pi/3}\Omega_-(z). \tag{2.31}$$

Since Ω is the derivative of a function which has a singularity of the form $(z - z_0)^{1/2}$, $z_0 = \pm a, \pm b$, it can have a stronger singularity of $(z - z_0)^{-1/2}$ type. Moreover from the definition of ω_0 we infer the behaviour of Ω for z large, $\Omega(z) \sim 1/z$. These conditions determine $\Omega(z)$ uniquely as a function only of the location a, b of the singularities, a property we shall use later. At the critical point $\Omega_\pm$ have the form (2.23), where however now $A(z)$ can have poles at $z = \pm b$ and $A(z)$ and $B(z)$ no longer necessarily vanish at $z = 0$. The large z behaviour of $\Omega(z)$ implies

$$B(z) \underset{z\to\infty}{\sim} 1/(b\sqrt{3}), \quad A(z) = \widetilde{A}(z)/(z^2 - b^2) \text{ with } \widetilde{A}(z) = O(1).$$

The leading correction to the large z behaviour of ω_{sc} is found from equation (2.29) to be of order $z^{-2/3}$. This determines the singularity at $z = 0$ and implies $A(0) = B(0)$. The unique solution is thus

$$\Omega_\pm = \pm \frac{i}{\sqrt{3} z \sqrt{1 - b^2/z^2}} \left(\sqrt{1 - b^2/z^2} \pm ib/z\right)^{2/3}. \tag{2.32}$$

Therefore

$$\omega_\pm(z, g) = \omega_\pm(z, g_c) \mp x g_c 4i\sqrt{3} \frac{1}{z\sqrt{1 - b^2/z^2}} \left(\sqrt{1 - b^2/z^2} \pm ib/z\right)^{2/3} + O\left(x^2\right). \tag{2.33}$$

By identifying the large z expansion of ω_{sc} with the small z expansion of (2.33) we find the relation between the parameter a and x. For example for $z = -i\lambda$ and $\lambda \to 0_+$ we derive from the expansion (2.33):

$$\begin{aligned} \omega_+(-i\lambda) &\sim -2^{1/3}(1 - \sigma) b^{4/3} \lambda^{2/3} (1 + O(x)), \\ \omega_-(-i\lambda) &\sim -x g_c 3^{1/2} 2^{8/3} b^{-1/3} \lambda^{-2/3}, \end{aligned}$$

while the scaling function for $\lambda \to +\infty$

$$\omega_{\text{sc},-} = -\tfrac{1}{2}(1 - \sigma) b^{4/3} a^{4/3} \lambda^{-2/3} + O\left(\lambda^{-4/3}\right).$$

It follows

$$a \sim b\sqrt{2} \left(\frac{19/81 - \sigma/9}{1 - \sigma}\right)^{3/4} x^{3/4}. \tag{2.35}$$

Note finally that, taking into account the phase factors, ω_{sc} can then also be written

$$\omega_{\text{sc}}(z) = 2^{-1/3} b^{4/3} (1 - \sigma) \left[\left(-z + \sqrt{z^2 - a^2}\right)^{2/3} + \left(-z - \sqrt{z^2 - a^2}\right)^{2/3}\right]. \tag{2.36}$$

This expression can be conveniently parametrized setting

$$z = -a\cos\varphi, \quad \omega_{\text{sc}}(z) = 2^{2/3} a^{2/3} b^{4/3} (1 - \sigma) \cos(2\varphi/3). \tag{2.37}$$

We have assumed $\varphi \in [0, \pi]$ for $z \in [-a, +a]$.

Interpretation: the resolvent in the $(p, 3)$ *model.* We recall that pure gravity is a $(2, 3)$ model in the CFT classification, which can be constructed as the solution of the canonical commutation relation $[P, Q] = 1$ where P, Q are two differential operators of order $2, 3$ respectively [10].

In the corresponding one-matrix model the saddle point equations have a natural interpretation in terms of the resolvent of the operator of order two. However, as we show in section 4, the trace of the resolvent of the other differential operator is also solution of an algebraic equation, of third degree. Solving the equation explicitly, we obtain an expression proportional to expression (2.36). Moreover we verify that the scaling relation (2.35) is consistent with the behaviour expected for the specific heat $u(x)$ in pure gravity.

Our results are thus consistent with the hypothesis that the matrix S becomes, in the scaling limit, the differential operator of order 3, $\mathrm{d}^3 - (3/4)\{u, \mathrm{d}\}$ of pure gravity.

2.5. The critical Ising model

The line of pure gravity reaches the Ising critical point when three zeros of Δ coincide: $a = e = 0$. This situation is found for $\sigma = 1$ and thus $c = (-1 + 2\sqrt{7})/27$, $g_c = 5 \times 2^{-2} 3^{-9/2} b^3 = 10c^3$ or $b = 6\sqrt{3}c$.

The function ω_+ then becomes

$$\omega_+ = z^2 \left[(1 - b^2/z^2)^{1/2} + ib/z\right]^{2/3}. \tag{2.38}$$

For z small for example $z = -i\lambda$, $\lambda \to 0_+$, we find that ω_- gives the leading contribution to the two first terms of small x expansion:

$$\omega_- = -(2b)^{2/3}\lambda^{4/3} - x g_c 3^{1/2} 2^{8/3} b^{-1/3} \lambda^{-2/3} + O\left(x^2\right). \tag{2.39}$$

This implies that $\omega_\pm(z)$ in the scaling limit, $g - g_c$ small, have the form

$$\omega_{\mathrm{sc},\pm} = \vartheta_0 \left(\sqrt{a^2 - z^2} \mp iz\right)^{4/3}.$$

Again ϑ_0 and a can be determined by expanding the scaling form for z large $z = -i\lambda$, $\lambda \to +\infty$:

$$\omega_{\mathrm{sc},-}(z) = \vartheta_0 (2\lambda)^{4/3} \left(1 + \frac{a^2}{3\lambda^2}\right) + O\left(\lambda^{-4/3}\right).$$

Comparing with the expansion (2.39) we conclude

$$\vartheta_0 = -\,(b/2)^{2/3}\,, \tag{2.40a}$$

$$a/b = 5^{1/2} 3^{-3/2} x^{1/2}. \tag{2.40b}$$

As we show in section 4 these results agree, up to the normalizations of x and ω, with the scaling of the resolvent of the differential operator $P = \mathrm{d}^3 - (3/2)\{\mathrm{d}, u(x)\}$ of the $(3,4)$ model in the spherical limit. We thus find complete consistency with the established results of the critical Ising model in the spherical limit.

Note that, here again, it is convenient to use the parametrization (2.37):

$$\omega_{\mathrm{sc}}(z) = \vartheta_0 \left[\left(-z + \sqrt{z^2 - a^2}\right)^{4/3} + \left(-z - \sqrt{z^2 - a^2}\right)^{4/3} \right].$$

and thus

$$z = -a\cos\varphi, \ \Rightarrow\ \omega_{\mathrm{sc}}(z) = -2^{1/3} b^{2/3} a^{4/3} \cos(4\varphi/3)\,. \tag{2.41}$$

2.6. The singular free energy

We show now how one can explicitly calculate the singular part of the free energy $F = \ln Z$ and thus verify its universal character, when compared to other quantities.

We start from

$$\begin{aligned} g^2 \frac{\partial F}{\partial g} &= N \langle \operatorname{tr} V(S) \rangle \sim N^2 \int \mathrm{d}s\, \rho(s) V(s) \\ &= \frac{N^2}{2i\pi} \oint \mathrm{d}z\, \omega_0(z) V(z)\,, \end{aligned} \tag{2.42}$$

where the contour in the last integral encloses the cut of $\omega_0(z)$.

We then multiply by g and differentiate again

$$\frac{\partial}{\partial g}\left(g^3 \frac{\partial F}{\partial g} \right) = N^2 g f(g), \tag{2.43a}$$

$$f(g) = \frac{1}{2i\pi g} \oint \mathrm{d}z\, V(z) \Omega(z)\,. \tag{2.43b}$$

where we have introduced the function $\Omega(z)$ defined by equation (2.30). Using then the decomposition (2.31) into (2.43) we find that we can write

$$f = \frac{1}{2i\pi g} \oint \mathrm{d}z\, \left[\mathrm{e}^{-2i\pi/3} V(z) - \mathrm{e}^{2i\pi/3} V(-z) \right] \Omega_+(z). \tag{2.44}$$

The singular part of the free energy is thus related to the singular part of $\Omega(z)$ for a small which we determine now.

Expansion of $\Omega(z)$ *for* a^2 *small.* We recall that $\Omega(z)$ satisfies the homogeneous equation (2.13). Moreover $\Omega(z)$ has a cut along $[a,b]$ and behaves for large argument z like $1/z$. It has singularities of the form $(a-z)^{-1/2}$ and $(z-b)^{1/2}$ at $z=\pm a, \pm b$. It can be directly determined by solving an equation of the form (2.16)

$$\Omega^3(z) - 3r(z)\Omega(z) - 2s(z) = 0\,. \tag{2.45}$$

The functions r,s which appear here can have simple poles at $z=a,b$. This implies that they can be parametrized as

$$r(z) = \frac{1}{3}\frac{z^2+\tau^2 d}{(z^2-a^2)(z^2-b^2)}, \qquad s(z) = \frac{2}{3\sqrt{3}}\frac{b\tau^3}{(z^2-a^2)(z^2-b^2)}.$$

The discriminant Δ of the equation $\Delta = s^2 - r^3$ has as a numerator a polynomial of degree six which must be a perfect square, condition which determines the parameters d and τ. However, it is more transparent to use a different method.

The critical function ($a=0$) is given by equation (2.32):

$$\Omega_\pm = \pm\frac{i}{\sqrt{3}z\sqrt{1-b^2/z^2}}\left(\sqrt{1-b^2/z^2} \pm ib/z\right)^{2/3}.$$

The most singular, for z small, of the two functions behaves like $z^{-2/3}$. This determines the corresponding scaling function. Indeed it has the general form (2.28), C can have poles at $z=\pm a$, and C,D behave like $1/z^2$ for z large. Thus

$$\Omega_{\mathrm{sc},\pm} = \frac{\vartheta_0}{\sqrt{a^2-z^2}}\left(\sqrt{a^2-z^2} \mp iz\right)^{1/3},$$

with

$$\vartheta_0 = 2^{1/3}3^{-1/2}b^{-1/3}.$$

Expanding Ω_{sc} at next order for z large we get a correction of order $z^{-4/3}$. More precisely for $\lambda\to+\infty$

$$\Omega_{\mathrm{sc},+}(-i\lambda) \sim 3^{-1/2}a^{2/3}b^{-1/3}\lambda^{-4/3}.$$

This next to leading contribution must also be the small z behaviour of the correction to the critical function we are looking for. This correction on the other hand is at most of order $1/z^2$ for z large. These conditions determine it entirely and we get

$$\Omega_+(z,a) - \Omega_+(z,0) \sim -(a/b)^{2/3} \frac{2^{-1/3}3^{-1/2}b}{z^2\sqrt{1-b^2/z^2}} \left(\sqrt{1-b^2/z^2}+ib/z\right)^{-1/3}. \tag{2.46}$$

Note that the expression in the r.h.s. has a singularity at $z=b$ only of strength $(z-b)^{-1/2}$ while $(z-b)^{-3/2}$ could have been expected. Actually it can be verified that the location of the singularity at $z=b$ varies by a term of order a^2 at least and thus the corresponding contribution to Ω is even smaller.

The free energy. We note the identity

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\sqrt{1-b^2/z^2}+ib/z\right)^{-1/3} = \frac{ib}{3}\frac{1}{z^2\sqrt{1-b^2/z^2}}\left(\sqrt{1-b^2/z^2}+ib/z\right)^{-1/3}.$$

Introducing the expansion (2.46) into equation (2.44), using this identity and integrating by parts we find

$$\begin{aligned} f(a) = f(0) - \frac{i2^{-1/3}3^{1/2}(a/b)^{2/3}}{2i\pi g}\oint \mathrm{d}z\, \left[\mathrm{e}^{-2i\pi/3}V'(z)+\mathrm{e}^{2i\pi/3}V'(-z)\right] \\ \times \left(\sqrt{1-b^2/z^2}+ib/z\right)^{-1/3}. \end{aligned}$$

We now use the identity

$$\mathrm{e}^{-2i\pi/3}V'(z)+\mathrm{e}^{2i\pi/3}V'(-z) = ig\sqrt{3}\left[\mathrm{e}^{2i\pi/3}\omega_0(z)-\mathrm{e}^{-2i\pi/3}\omega_0(-z)\right]-\frac{1}{4}\omega_- \,,$$

consequence of the definitions (2.10),(2.11), in the integral.

We need ω_- only at leading order. From (2.38) we deduce

$$\omega_- = z^2\left(\sqrt{1-b^2/z^2}+ib/z\right)^{-2/3}.$$

We then see that ω_- gives no contribution to the integral. We calculate the contribution of $\omega_0(z)$ by taking the residue at infinity. We need only the large z behaviour of $\omega_0(z)$ at leading order.

$$ig\sqrt{3}\left[\mathrm{e}^{2i\pi/3}\omega_0(z)-\mathrm{e}^{-2i\pi/3}\omega_0(-z)\right] \sim -\frac{ig\sqrt{3}}{z}+O\left(z^{-2}\right).$$

We thus obtain

$$\oint \frac{\mathrm{d}z}{2i\pi g}\left[\mathrm{e}^{-2i\pi/3}V'(z)+\mathrm{e}^{2i\pi/3}V'(-z)\right]\left(\sqrt{1-b^2/z^2}+ib/z\right)^{-1/3}=-i\sqrt{3}\,.$$

This completes the calculation of second derivative of the singular free energy F_{sg} and we find:

$$g_c^2F''_{\mathrm{sg}}(g)\sim F''(x)\sim\frac{N^2}{2i\pi g_c}\oint \mathrm{d}z\,V(z)\Omega_{\mathrm{sg}}=-3\times 2^{-1/3}N^2(a/b)^{2/3}, \tag{2.47}$$

with $x=1-g/g_c$. In the case of the $(3,2)$ ordinary critical point this yields

$$F''(x)=-3(19/81-\sigma/9)^{1/2}(1-\sigma)^{-1/2}N^2x^{1/2}.$$

In particular for $c=0$ we recover the expected result, i.e. twice the value of the corresponding one-matrix model.

For the critical Ising model instead we find

$$F''(x)=-(5/2)^{1/3}N^2x^{1/3}.$$

2.7. Large order behaviour

We now look for a non minimal saddle point of the integrand (2.5). We know from our experience of the one-matrix model, that the second lowest saddle point is obtained when we move only one eigenvalue from the edge of the distribution ρ, to the nearest stationary position. The variation of the action Σ is then:

$$\delta\Sigma=\int_a^{\lambda_f}\mathrm{d}\lambda\frac{\partial\Sigma}{\partial\lambda},$$

where Σ is given by

$$\frac{\partial\Sigma}{\partial\lambda}=N\left(\frac{V'(\lambda)}{g}-\frac{2}{N}\sum_j\frac{1}{\lambda-\lambda_j}+\frac{1}{N}\sum_j\frac{1}{\lambda+\lambda_j}\right).$$

In the continuum limit

$$\begin{aligned}\frac{\partial\Sigma}{\partial\lambda}&=N\left(\frac{V'(\lambda)}{g}-2\omega_0(\lambda)-\omega_0(-\lambda)\right)\\&=-\frac{N}{12g}\big(2\omega(\lambda)+\omega(-\lambda)\big).\end{aligned} \tag{2.48}$$

We are looking for a zero of $\partial\Sigma/\partial\lambda$ in the interval $(0, a)$ and thus λ_f is small. We can thus replace $\omega(z)$ by its scaling form. We then find the variation δF of the free energy in the scaling limit. For the generic critical point, using (2.37) and (2.35) we find

$$\delta\Sigma \propto a^{5/3} \propto x^{5/4}, \Rightarrow \delta F \propto \exp\left(-\text{const. } x^{5/4}\right).$$

in agreement with the result found in pure gravity. For the Ising critical point, using (2.41) and (2.40b) we obtain

$$\delta\Sigma \propto a^{7/3} \propto x^{7/6}, \Rightarrow \delta F \propto \exp\left(-\text{const. } x^{7/6}\right).$$

in agreement with the known result for the Ising model.

2.8. Multicritical points

With a general potential $V(S)$ higher order critical points can be generated. If the polynomials A, B, in expression (2.23) behave like: $A(z) \propto B(z) \propto z^{2m+2}$ for z small, $\omega(z) \propto z^{(6m+2)/3}$. The minimal example corresponds to

$$\begin{aligned} \omega_\pm &= \mp i(z/b)^{2m+1}\left(\sqrt{1-b^2/z^2} \pm ib/z\right)^{-1/3}, \\ r(z) &= (z/b)^{4m+2}, \quad s(z) = (z/b)^{6m+2}. \end{aligned}$$

If in addition $A(z)/B(z) \to 1$ then $\omega(z) \propto z^{(6m+4)/3}$. The minimal model is

$$\begin{aligned} \omega_\pm &= (z/b)^{2m+2}\left(\sqrt{1-b^2/z^2} \pm ib/z\right)^{2/3}, \\ r(z) &= (z/b)^{4m+4}, \quad s(z) = (z/b)^{6m+6} - 2(z/b)^{6m+4}. \end{aligned}$$

(Note the change of normalization compared to the $m = 0$ case.)

Scaling region. It will be convenient in what follows to classify the critical models when $\omega(z)$ behaves for $z \to 0$ like

$$\omega(z) \propto z^{p/q},$$

in terms of two relatively prime integers p, q, in analogy with the CFT classification (see section 4). For all the models considered up to now $q = 3$, while p belongs to one of the two sets $p = 6m + 2, 6m + 4$.

To determine the functions in the scaling limit in terms of the deviation x from the critical cosmological constant we now use the same method as in the case of the simple critical points $m = 0$. We first expand $\omega(z)$ at first order in x and then look for the leading term for $z \to 0$. The two cases have to be discussed separately.

First case $p = 6m + 2$. At leading order for $z = -i\lambda$, $\lambda \to 0_+$ we find

$$\omega_+(z) \sim (-1)^{m+1} 2^{1/3} (\lambda/b)^{2m+2/3}.$$

Therefore the scaling function $\omega_{\rm sc}$ has the form (2.28) with C, D polynomials of degree $2m$.

It is convenient to normalize the potential in such a way that the variation $\delta\omega$ at leading order in x and for z large is

$$\delta\omega_\pm \sim \mp ixb/z.$$

Then $\delta\omega_\pm$, which is independent of the potential and thus proportional to $\Omega_\pm$ of equation (2.32), is

$$\delta\omega_\pm = \mp \frac{ixb}{z\sqrt{1 - b^2/z^2}} \left(\sqrt{1 - b^2/z^2} \pm ib/z\right)^{2/3}. \tag{2.49}$$

Expanding $\delta\omega_-$ for $\lambda \to 0_+$ we find

$$\delta\omega_-(-i\lambda) \sim -x 2^{2/3} b^{2/3} \lambda^{-2/3}.$$

This behaviour of $\delta\omega$ determines completely the scaling function. It can be expressed in terms of the function $\vartheta(z)$:

$$\vartheta(z) = \int_{-iz}^{\sqrt{a^2 - z^2}} \mathrm{d}t \, (t + iz)^{m-1/3} (t - iz)^m \,. \tag{2.50}$$

Calculating the integral we first verify that the function $\vartheta(z)$ has a form (2.28):

$$\vartheta(z) = \left(\sqrt{a^2 - z^2} + iz\right)^{-1/3} \left(C(z)\sqrt{a^2 - z^2} + izD(z)\right).$$

For $\lambda \to +\infty$ we find

$$\vartheta(-i\lambda) \sim (-1)^m B(m + 1, m + 2/3)(2\lambda)^{2m+2/3},$$
$$\vartheta(i\lambda) \sim \frac{a^{2m+4/3}(2\lambda)^{-2/3}}{m + 2/3},$$

with

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

It follows

$$\omega_{\mathrm{sc},\pm}(z) = \vartheta_0 \vartheta(\pm z), \tag{2.51}$$

with

$$\begin{aligned}\vartheta_0 &= -\frac{2^{-2m-1/3} b^{-2m-2/3}}{B(m+1, m+2/3)},\\ a/b &= 2\left[2^{1/3}(m+2/3)B(m+1,m+2/3)x\right]^{1/(2m+4/3)}.\end{aligned}$$

Using equation (2.47) we conclude that the singular part of the free energy then scales like

$$F''(x) \propto x^{1/(3m+2)}.$$

As we show in section 4 all these results are consistent with the behaviour found in the $(6m+2,3)$ string model. Moreover equation (2.48) yields then the expected scaling for the "instanton" action $\delta\Sigma \propto x^{1+1/(6m+4)}$.

Case $p = 6m+4$. In the $\lambda \to 0_+$ limit ω_- gives the leading contributions:

$$\omega_- = (-1)^{m+1} 2^{2/3} (\lambda/b)^{2m+4/3} - x 2^{2/3} (b/\lambda)^{2/3} + O\left(\lambda^{-4/3}\right).$$

We introduce now the function $\vartheta(z)$:

$$\vartheta(z) = \int_{-iz}^{\sqrt{a^2-z^2}} \mathrm{d}t\, (t+iz)^{m+1/3} (t-iz)^m \,. \tag{2.52}$$

For $\lambda \to +\infty$ we find:

$$\vartheta(-i\lambda) = (-1)^m B(m+1, m+4/3)(2\lambda)^{2m+4/3} + \frac{2^{-2/3} a^{2m+2}}{m+1} \lambda^{-2/3} + O\left(\lambda^{-4/3}\right).$$

In the scaling limit we conclude that $\omega(z)$ must have the form (2.51) with now

$$\omega_\pm(z) = \vartheta_0 \vartheta(\mp z),$$

with

$$\begin{aligned}\vartheta_0 &= -\frac{b^{-2m-4/3} 2^{-2m-2/3}}{B(m+1, m+4/3)},\\ \frac{a}{b} &= 2\left[(m+1)B(m+1,m+4/3)x\right]^{1/(2m+2)}.\end{aligned}$$

It follows that the specific heat $F''(x) \propto x^{1/(3m+3)}$. Again these results are consistent with a $(6m+4,3)$ model (see section 4). Agreement is also found with the scaling form of the instanton action $\delta\Sigma \propto x^{1+1/(6m+6)}$.

3. The general $O(n)$ model

We now generalize the previous model by replacing the matrix A by a set of n matrices A_i:

$$Z = \int \mathrm{d}S\, \mathrm{d}A_1 \dots \mathrm{d}A_n\, \mathrm{e}^{-(N/g)\mathrm{tr}\,[S(A_1^2 + \cdots + A_n^2) + V(S)]}, \tag{3.1}$$

$V(S)$ being now also a general polynomial potential.

The model has then an $O(n)$ symmetry. The quantity $F = \ln Z$ can be interpreted as the free energy of a gas of loops, each indexed by an integer i, $i = 1, ..., n$, drawn on a random lattice of the form of a Feynman diagram.

The corresponding model on regular lattices can become critical only for $-2 \le n \le 2$. We shall verify that here that the integral is even defined only for $n \le 2$. It is thus convenient to set $n = -2\cos\theta$. Although we could restrict ourselves to the interval $0 \le \theta \le \pi$, it is convenient for book-keeping purpose, to consider all positive values of θ. Note that the case $n = 0, \theta = (2m+1)\pi/2$ reduces to the standard one-matrix model. For $n = 1$ we can assign $\theta = p\pi/3$ to the multicritical models when $\omega(z) \propto z^{p/3}$; the critical Ising model thus corresponds to $\theta = 4\pi/3$.

The integral over the matrices A_i's is still gaussian and can be performed. We can then parametrize S in terms of a unitary transformation and its eigenvalues λ_i. After the integration over unitary transformations, the integral (3.1) becomes:

$$\begin{aligned} Z &= \int \Delta^2(\Lambda) \prod_{i,j} (\lambda_i + \lambda_j)^{-n/2} \prod_i \mathrm{d}\lambda_i \mathrm{e}^{-(N/g)V(\lambda_i)} \\ &= \int \mathrm{d}\lambda\, \mathrm{e}^{-N\,\Sigma[\lambda]}, \end{aligned}$$

with the effective action Σ

$$\Sigma = \sum_i \frac{1}{g} V(\lambda_i) - \frac{1}{N} \sum_{i \ne j} \ln|\lambda_i - \lambda_j| + \frac{n}{2N} \sum_{i,j} \ln(\lambda_i + \lambda_j).$$

3.1. The saddle-point equation

In the planar limit $N \to \infty$, Z can be calculated by the steepest descent method. The saddle point equation is:

$$\frac{\partial \Sigma}{\partial \lambda_i} = 0 = \frac{1}{g} V'(\lambda_i) - \frac{2}{N} \sum_{j \ne i} \frac{1}{\lambda_i - \lambda_j} + \frac{n}{N} \sum_j \frac{1}{\lambda_i + \lambda_j}. \tag{3.2}$$

We again introduce the density of eigenvalues $\rho(\lambda) = \frac{1}{N}\sum_i \delta(\lambda - \lambda_i)$, and its Hilbert's transform, the trace of the resolvent:

$$\omega_0(z) = \frac{1}{N}\sum_i \frac{1}{z-\lambda_i} = \int \mathrm{d}\lambda \frac{\rho(\lambda)}{z-\lambda}.$$

In the large N limit $\rho(\lambda)$ becomes a continuous function and ω_0 a function analytic except when z belongs to the spectrum of S, that is with a cut on a segment $[a, b]$ of the real positive axis.

Equation (3.2) may be written in term of ω_0:

$$\omega_0(\lambda + i0) + \omega_0(\lambda - i0) + n\omega_0(-\lambda) = \frac{1}{g}V'(\lambda), \qquad (\lambda \in [a, b]). \tag{3.3}$$

This linear equation has a polynomial solution:

$$\omega_r(z) = \frac{1}{g}\frac{1}{4-n^2}\big(2V'(z) - nV'(-z)\big). \tag{3.4}$$

Note that the cases $n = \pm 2$ are special and must be examined separately. The function $\omega(z)$, defined by

$$\omega_0 = \omega_r + \omega/g, \tag{3.5}$$

then satisfies the homogeneous equation:

$$\omega(\lambda + i0) + \omega(\lambda - i0) + n\omega(-\lambda) = 0\,. \tag{3.6}$$

Since $\omega_0(z)$ behaves as $1/z$ for z large, $\omega(z)$ has the large z expansion:

$$\omega(z) = -\frac{4}{\sqrt{2-n}}\big(2V'(z) - nV'(-z)\big) + \frac{g}{z} + O\left(z^{-2}\right)\,. \tag{3.7}$$

A quadratic relation. Let us introduce the following function:

$$r(z) = \omega^2(z) + \omega^2(-z) + n\omega(z)\omega(-z). \tag{3.8}$$

We verify, as in the case of the Ising model, that the discontinuity on the cut of $r(z)$ vanishes as a consequence of equation (3.6):

$$\begin{aligned} r(z+i0) - r(z-i0) = {} & [\omega(z+i0) - \omega(z-i0)] \\ & \times [\omega(z+i0) + \omega(z-i0) + n\omega(-z)] = 0\,. \end{aligned}$$

Therefore r is an even function, analytic in the whole complex plane. The behaviour of ω for z large, then implies that r is a polynomial. Again equation (3.8) can be directly derived from the saddle point equation or the loop equation. An expression of $r(z)$ in terms of the potential follows.

3.2. The general solution

We introduce two auxiliary functions:

$$\begin{cases} \omega_+(z) = \dfrac{i}{2\sin\theta}\left[e^{i\theta/2}\omega(z) - e^{-i\theta/2}\omega(-z)\right] \\ \omega_-(z) = -\dfrac{i}{2\sin\theta}\left[e^{-i\theta/2}\omega(z) - e^{i\theta/2}\omega(-z)\right] \end{cases} \tag{3.9}$$

such that $\omega_+(-z) = \omega_-(z)$. Then

$$\omega_+(z)\omega_-(z) = r(z).$$

Conversely $\omega(z)$ is given in terms of ω_+, ω_- by

$$\omega(z) = -\left[e^{i\theta/2}\omega_+(z) + e^{-i\theta/2}\omega_-(z)\right]. \tag{3.10}$$

Equation (3.6) then is equivalent to the simple relations

$$\omega_\pm(z - i0) = e^{\pm i\theta}\omega_\mp(z + i0), \tag{3.11}$$

which, as in Ising case, imply that $\omega(z)$ has cuts only on the positive axis.

If we now choose θ such that $e^{iq\theta} = \pm 1$, q being a positive integer, then $\omega(z)$ satisfies an algebraic equation of degree q. We have to examine the two signs separately.

Case $e^{iq\theta} = 1$. This implies $\theta = \pi p/q$, where p is an even integer. The function $s(z)$,

$$s(z) = \tfrac{1}{2}(\omega_+^q + \omega_-^q), \tag{3.12}$$

has no discontinuity on the cut, and is analytic in the whole complex plane. It is therefore an even polynomial of a degree determined by the degree of the potential . We have thus the two following algebraic equations:

$$\omega_+\omega_- = r(z), \qquad \omega_+^q + \omega_-^q = 2s(z). \tag{3.13}$$

The solution of these equations is then $\omega_\pm^q = s \pm \sqrt{\Delta}$, with

$$\Delta = s^2 - r^q, \qquad \sqrt{\Delta} = \tfrac{1}{2}\left(\omega_+^q - \omega_-^q\right). \tag{3.14}$$

Note that $\sqrt{\Delta}$ is thus an odd function. The function ω is then given by equation (3.10).

Case $e^{iq\theta} = -1$. This implies $\theta = \pi p/q$, where p is now an odd integer. The expressions are quite similar but the role of $s(z)$ and $\sqrt{\Delta}$ are formally exchanged. It is now the function $s(z)$,

$$s(z) = \frac{1}{2i}(\omega_+^q - \omega_-^q), \tag{3.15}$$

which has no discontinuity on the cut, and is analytic in the whole complex plane. It is therefore an odd polynomial. We have thus the two following algebraic equations:

$$\omega_+\omega_- = r(z), \qquad \omega_+^q - \omega_-^q = 2is(z)\,. \tag{3.16}$$

The solution of these equations is $\omega_\pm^q = \sqrt{\Delta} \pm is$, with

$$\Delta = r^q - s^2\,, \qquad \sqrt{\Delta} = \tfrac{1}{2}\left(\omega_+^q + \omega_-^q\right).$$

Here instead $\sqrt{\Delta}$ is even. The function ω is still given by equation (3.10).

One-cut solution. We still have to determine the coefficients of the polynomials r, s, Δ. They can be found from the additional condition that ω has only one cut $[a, b]$ on the positive real axis. We can see from (3.10) that the singularities of ω are the single roots of Δ. We demand that except for a and b (and $-a$,$-b$ by parity), all the roots of Δ are double, in such a way that Δ can be written:

$$\Delta = -(z^2 - a^2)(z^2 - b^2)M^2(z),$$

where $M(z)$ is an odd or even polynomial depending on the different cases. Due to the special form of the conditions (3.11) all one-cut solutions $\omega_\pm$ can be factorized:

$$\omega_\pm = \Omega_\pm(z)\left[\pm zA(z) + B(z)\sqrt{(z^2 - a^2)(z^2 - b^2)}\right],$$

where A and B are even functions, rational fractions in general because $\Omega\pm(z)$ may have zeros, and the function Ω, which has only singularities at $\pm a, \pm b$, is a "minimal" solution of:

$$\Omega_+(z) = \Omega_-(-z), \qquad \Omega_\pm(z - i0) = -e^{\pm i\theta}\Omega_\mp(z + i0).$$

This factorization property is a consequence of the algebraic equation satisfied by $\omega(z)$ and may not necessarily hold when θ/π is not rational.

3.3. Critical points

We again consider only critical points for which $a = 0$. The function $\omega(z)$ at a critical point has then a cut for $0 \le z \le b$. The general form of such a solution is analogous to the form (2.23) of the case $n = 1$:

$$\omega_\pm(z) = \left(\sqrt{1 - b^2/z^2} \pm ib/z\right)^{-l/q} \left[A(z)\sqrt{1 - b^2/z^2} \pm ibB(z)/z\right], \qquad (3.17)$$

where l, q are relatively prime integers with $0 < l < q$ and A, B are polynomials which can be chosen even without loss of generality. Indeed the situation A, B odd is equivalent to A, B even with the change $l \mapsto q - l$.

One immediately verifies that $r(z), s(z)$ and $\Delta(z)$ are polynomials of a form consistent with a one-cut solution provided A and B vanish at $z = 0$.

A minimal realization of a critical point with polynomial potentials is:

$$\omega_\pm(z) = \mp i(z/b)^{2m+1} \left(\sqrt{1 - b^2/z^2} \pm ib/z\right)^{-l/q}, \qquad (3.18a)$$

$$\omega_\pm(z) = (z/b)^{2m+2} \left(\sqrt{1 - b^2/z^2} \pm ib/z\right)^{1-l/q}. \qquad (3.18b)$$

Then in both cases

$$\omega_+(z - i0) = \mathrm{e}^{i\pi(1-l/q)}\omega_-(z + i0) \ \Rightarrow \ \theta = \pi(1 - l/q). \qquad (3.19)$$

The two cases $q - l$ even and odd correspond to the two situations $\mathrm{e}^{iq\theta} = \pm 1$.

For $z \to 0$ we find that

$$\omega(z) \propto z^{p/q}$$

where $p = (2m + 1)q - l$ in the case (a) and $p = (2m + 1)q + l$ in the case (b). Note that the values of p are such that m can also be defined as the integer part of $p/2q$ since $p/(2q) - 1 < m < p/(2q)$. Finally we see that for book-keeping purpose it is convenient to assign the angle $\theta = \pi p/q$ to the critical point characterized by the integers (p, q).

3.4. Scaling region

We want now to derive $\omega(z)$ in the scaling region when the variable $x = 1 - g/g_c$ which characterizes the deviation of the cosmological constant from its critical value is small. Functions which satisfy equation (3.6) and are singular only at $z = \pm a$ and $z = \infty$ have the general form:

$$\omega_{\mathrm{sc},\pm} = \left(\sqrt{a^2 - z^2} \pm iz\right)^{-l/q} \left[C(z)\sqrt{a^2 - z^2} \pm izD(z)\right], \tag{3.20}$$

where again C and D are even polynomials. A comparison between the large z behaviour of ω_{sc} and the small z behaviour of ω at the critical point yields the degrees of C and D. This determines them completely only for the minimal critical points $m = 0$. We then find

$$\omega_{\mathrm{sc},\pm} \propto \left(\sqrt{a^2 - z^2} \pm iz\right)^{p/q}.$$

To obtain the relation between x and a and completely determine the form of the polynomials C, D for multicritical points ($m > 0$) we then proceed as in the $q = 3$ case and calculate the deviation from the critical form at leading order for x small.

Deviation from the critical form at leading order. We now calculate the deviation from the critical form at leading order in the variable $x = 1 - g/g_c$ which characterizes the deviation of the cosmological constant from its critical value. We normalize the potential in such a way that for z large the variation $\delta\omega_\pm$ is:

$$\delta\omega_\pm \sim \mp ixb/z\,.$$

As in the case of the Ising model we introduce the function $\Omega(z)$:

$$\Omega(z) = \frac{\partial}{\partial g}\big(g\omega_0(z)\big).$$

Equation (3.5) then implies

$$\Omega(z) = \frac{\partial\omega(z)}{\partial g}. \tag{3.21}$$

It thus satisfies the homogeneous equation (3.6). We also introduce the decomposition

$$\Omega_\pm(z) = \pm\frac{i}{2\sin\theta}\left[\mathrm{e}^{\pm i\theta/2}\Omega(z) - \mathrm{e}^{\mp i\theta/2}\Omega(-z)\right] \tag{3.22a}$$

$$\Omega(z) = -\left[\mathrm{e}^{i\theta/2}\Omega_+(z) + \mathrm{e}^{-i\theta/2}\Omega_-(z)\right]. \tag{3.22b}$$

From the definition of ω_0 we infer the behaviour of Ω for z large, $\Omega(z) \sim 1/z$. Moreover, since Ω is the derivative of a function which has singularities of the form $(z - z_0)^{1/2}$, $z_0 = \pm a, \pm b$, it can have a stronger singularity of $(z - z_0)^{-1/2}$ type. These conditions determine $\Omega(z)$ uniquely as a function of the location of the singularities, as in the Ising case. The variation $\delta\omega$ is proportional to Ω at the critical point. At the critical point Ω has the form (3.17). To obtain its complete form we need its small z behaviour. This behaviour must be consistent with the leading correction to the large z behaviour of $\omega_{\rm sc}$. Since Ω is independent of the potential we can compare it to the form of $\omega_{\rm sc}$ in the case (a) for $m=0$. The leading correction to $\omega_{\rm sc}$ is then of order $z^{l/q-1}$. The unique solution is then

$$\delta\omega_\pm = \mp \frac{ixb}{z\sqrt{1-b^2/z^2}}\left(\sqrt{1-b^2/z^2}\pm ib/z\right)^{1-l/q}.$$

With this information we can now explicitly calculate the scaling functions for all critical points.

Scaling function. From the preceding analysis we conclude that for z large $\omega_{\rm sc}$ satisfies

$$\omega_{\rm sc}(z) - \text{const.}\ z^{p/q} = O\left(xz^{l/q-1}\right). \tag{3.23}$$

It is easy to verify that this fixes the polynomials C, D. We now show that the scaling function $\omega_{\rm sc}$ can then be expressed in terms of the function $\vartheta(z)$ given by the integral representation

$$\vartheta(z) = \int_{-iz}^{\sqrt{a^2-z^2}} \mathrm{d}t\,(t+iz)^{p/q-m-1}(t-iz)^m\,, \tag{3.24}$$

the proof relying on a verification of condition (3.23). Calculating the integral (3.24) we first verify that the function $\vartheta(z)$ has a form consistent with expression (3.20):

$$\vartheta(z) = \left(\sqrt{a^2-z^2}+iz\right)^{\pm l/q}\left(C(z)\sqrt{a^2-z^2}+izD(z)\right),$$

where C, D are two even polynomials of degree $2m$ and $\pm l/q = p/q-2m-1$. Moreover if we introduce the parametrization of case (a), $p=(2m+1)q-l$, we find

$$\vartheta(l,z) = \int_{-iz}^{\sqrt{a^2-z^2}} \mathrm{d}t\,(t+iz)^{(p-q-l)/(2q)}(t-iz)^{(p-q+l)/(2q)}.$$

It follows

$$\vartheta(l,z)-\vartheta(-l,-z) = (2z)^{p/q}\mathrm{e}^{i\pi(q-l)/(2q)}\sigma_{pq},$$

where we have set

$$\sigma_{pq} = B(m+1,p/q-m) = \frac{\Gamma[(p+q+l)/(2q)]\Gamma[(p+q-l)/(2q)]}{\Gamma[(p+q)/q]}. \tag{3.25}$$

We then expand $\vartheta(\pm z)$ for z large.

Large z expansion. For $z = -i\lambda$ large we find

$$\begin{aligned}\vartheta(-i\lambda) &= (-1)^m (2\lambda)^{p/q} \frac{\Gamma(m+1)\Gamma(p/q-m)}{\Gamma(p/q+1)} + (2\lambda)^{p/q-2m-2} \frac{a^{2m+2}}{m+1} \\ &\quad + O\left(\lambda^{p/q-2m-4}\right), \qquad (3.26a) \\ \vartheta(i\lambda) &\sim \frac{(2\lambda)^{2m-p/q} a^{2p/q-2m}}{p/q-m}. \qquad (3.26b)\end{aligned}$$

Case (a). If $p = (2m+1)q - l$, and thus $p/q < 2m+1$, $\vartheta(-z)$ is asymptotically larger than the correction to $\vartheta(z)$. Moreover $2m - p/q = l/q - 1$. The solution is then

$$\omega_{\mathrm{sc},\pm}(z) = \vartheta_0 \vartheta(\pm z). \qquad (3.27)$$

Moreover comparing the expansion (3.26) with the expansions of the critical functions $\omega_\pm$ and $\delta\omega_\pm$:

$$\begin{aligned}\omega_+(-i\lambda) &\sim (-1)^{m+1} 2^{l/q} (\lambda/b)^{p/q}, \\ \delta\omega_-(-i\lambda) &\sim -x 2^{1-l/q} (\lambda/b)^{l/q-1},\end{aligned}$$

we obtain the normalization constant ϑ_0 and the relation between a and x

$$\begin{aligned}\vartheta_0 &= -2^{(l-p)/q} b^{-p/q} \sigma_{pq}^{-1}, \\ \left(\frac{a}{b}\right)^{(p+q-l)/q} &= 2^{(p+q-3l)/q} [(p+q-l)/q] \sigma_{pq} x,\end{aligned}$$

where we have used the definition (3.25).

Case (b). If $p = (2m+1)q + l$, and thus $p/q > 2m+1$, the correction to $\vartheta(z)$ is asymptotically larger than the correction to $\vartheta(-z)$. Moreover $p/q - 2m - 2 = l/q - 1$. The solution is then

$$\omega_{\mathrm{sc},\pm}(z) = \vartheta_0 \vartheta(\mp z), \qquad (3.28)$$

with

$$\begin{aligned}\vartheta_0 &= -2^{(q-l-p)/q} b^{-p/q} \sigma_{pq}^{-1}, \\ \left(\frac{a}{b}\right)^{(p+q-l)/q} &= 2^{(p-l)/q} [(p+q-l)/q] \sigma_{pq} x\,.\end{aligned}$$

Note that in the set of variables p, q, l the behaviour of a takes the same form in both cases.

3.5. The singular free energy

We can find the singular part of the free energy, using the same method as in the Ising model case. We have shown that

$$\frac{\partial}{\partial g}\left(g^3\frac{\partial F}{\partial g}\right)=\frac{N^2}{2i\pi}\oint V(z)\Omega(z)\,\mathrm{d}z\,, \tag{3.29}$$

where $\Omega(z)$ is the function (3.21). Using the decomposition (3.22b) we can rewrite equation (3.29)

$$\frac{\partial}{\partial g}\left(g^3\frac{\partial F}{\partial g}\right)=-\frac{N^2}{2i\pi}\oint\Omega_+(z)\left(\mathrm{e}^{i\theta/2}V(z)-\mathrm{e}^{-i\theta/2}V(-z)\right)\mathrm{d}z\,. \tag{3.30}$$

The critical function for $a=0$ is

$$\Omega_\pm=\pm\frac{i}{2\sin(\theta/2)z\sqrt{1-b^2/z^2}}\left(\sqrt{1-b^2/z^2}\pm ib/z\right)^{1-l/q}. \tag{3.31}$$

The scaling region. Let us now consider the case $a\neq 0$ but small. For z small the function (3.31) behaves like $z^{l/q-1}$. This, together with the other properties, determines the scaling form of $\Omega(z)$

$$\Omega_{\mathrm{sc},\pm}(z)=\frac{\vartheta_0}{\sqrt{a^2-z^2}}\left(\sqrt{a^2-z^2}\mp iz\right)^{l/q},\qquad \vartheta_0=\frac{2^{-2l/q}b^{-l/q}}{\sin(\theta/2)}.$$

Conversely, as in the $q=3$ case, the next to leading term in the large z expansion of Ω provides the additional information needed to completely determine the first correction to the critical function for a small. This correction behaves like $z^{-1-l/q}$, therefore

$$\Omega_+(a)-\Omega_+(a=0)\propto\frac{1}{z^2\sqrt{1-b^2/z^2}}\left(\sqrt{1-b^2/z^2}+ib/z\right)^{-l/q}.$$

The leading correction to $\Omega_{\mathrm{sc},+}(z)$ is:

$$\Omega_{\mathrm{sc},+}(-i\lambda)\sim\vartheta_0 2^{-l/q}a^{2l/q}\lambda^{-1-l/q}.$$

We thus have:

$$\Omega_+(a)-\Omega_+(0)\sim-\frac{2^{-4l/q}}{\sin(\theta/2)}\left(\frac{a}{b}\right)^{2l/q}\frac{b}{z^2\sqrt{1-b^2/z^2}}\left(\sqrt{1-b^2/z^2}+ib/z\right)^{-l/q}.$$

The identity

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\sqrt{1-b^2/z^2}+ib/z\right)^{-l/q}=ib\frac{l}{q}\frac{1}{z^2\sqrt{1-b^2/z^2}}\left(\sqrt{1-b^2/z^2}+ib/z\right)^{-l/q},$$

allows to cast this expression into the form

$$\Omega_+(a) - \Omega_+(0) = -i\frac{q}{l\sin(\theta/2)}\left(\frac{a}{4b}\right)^{2l/q}\frac{\mathrm{d}}{\mathrm{d}z}\left(\sqrt{1-b^2/z^2}+ib/z\right)^{-l/q}.$$

With this expression we can integrate by parts integral (3.30). The second derivative of the singular part of the free energy is then given by

$$\begin{aligned}g_c^2 F''_{\mathrm{sg}} = {} & \frac{N^2}{2i\pi g_c}\frac{iq}{l\sin\theta/2}\left(\frac{a}{4b}\right)^{2l/q}\oint \mathrm{d}z\left(\sqrt{1-b^2/z^2}+ib/z\right)^{-l/q} \\ & \times\left(\mathrm{e}^{i\theta/2}V'(z)+\mathrm{e}^{-i\theta/2}V'(-z)\right).\end{aligned}$$

We then use the identity

$$\begin{aligned}\mathrm{e}^{i\theta/2}V'(z)+\mathrm{e}^{-i\theta/2}V'(-z) = {} & 2ig\sin\theta\left(\mathrm{e}^{-i\theta/2}\,\omega_0(z)-\mathrm{e}^{i\theta/2}\omega_0(-z)\right) \\ & + 4\sin^2\theta\,\omega_-(z).\end{aligned}$$

We need ω_- only at leading order as given by expressions (3.18). We see that the contribution to the integral coming from ω_- vanishes. The contribution due to ω_0 can be calculated by taking the residue at infinity. Then only the leading behaviour of ω_0 for z large is relevant:

$$2ig\sin\theta\left(\mathrm{e}^{-i\theta/2}\omega_0(z)-\mathrm{e}^{i\theta/2}\omega_0(-z)\right)\sim\frac{4ig\sin\theta\cos(\theta/2)}{z}.$$

In terms of the variable $x = 1 - g/g_c$ we finally obtain:

$$g_c^2\frac{\mathrm{d}^2F_{\mathrm{sg}}}{(\mathrm{d}g)^2} = F''_{\mathrm{sg}}(x) = -N^2(q/l)(2-n)2^{1-4l/q}\left(\frac{a}{b}\right)^{2l/q}.$$

If we set $q = 3$ and $l = 1$ we find $3\times 2^{-1/3}\,(a/b)^{2/3}$, in agreement with the Ising model result.

We derive from this expression the scaling of the free energy for all critical points:

$$F''_{\mathrm{sg}}(x)\propto x^{2l/(p+q-l)},\ \Rightarrow\ \gamma_{\mathrm{string}} = -\frac{2l}{p+q-l}. \tag{3.32}$$

3.6. Large order behaviour

The calculation of section 2.7 can easily be generalized. The variation of the action is:

$$\delta\Sigma = \int_a^{\lambda_f} \mathrm{d}\lambda \frac{\partial \Sigma}{\partial \lambda}$$

and we have:

$$\begin{aligned}\frac{\partial \Sigma}{\partial \lambda} &= N\left(\frac{V'(\lambda)}{g} - \frac{2}{N}\sum_j \frac{1}{\lambda-\lambda_j} + \frac{n}{N}\sum_j \frac{1}{\lambda+\lambda_j}\right), \\ &= N\left(\frac{V'(\lambda)}{g} - 2\omega_0(\lambda) - n\omega_0(-\lambda)\right) \\ &= -N\left[2\omega(\lambda) + n\omega(-\lambda)\right]/g.\end{aligned}$$

In the critical region g close to g_c, the argument λ in the integral remains of order a. We conclude that $\delta\Sigma$ scales like

$$\delta\Sigma \propto N x^{(p+q)/(p+q-l)},$$

result consistent with the scaling of the free energy. Thus the variation δF of the free energy has the form

$$\delta F \propto \exp\left(-\text{const.}\ x^{(p+q)/(p+q-l)}\right), \tag{3.33}$$

result consistent with the $2k!$ behaviour at large orders of the topological expansion also found in other matrix models.

4. The resolvent in (p,q) string models

We have solved the saddle point equations of the $O(n)$ model by transforming them into an algebraic equation for the trace of the resolvent of some matrix in the large N limit. It is therefore interesting, for comparison purpose, to discuss the form of the resolvent in multimatrix models. The string equations of the (p,q) models can be generated by constructing a representation of the canonical commutation relations in terms of two differential operators P, Q of order p, q respectively (p, q are relatively prime and not both odd):

$$\begin{aligned}P &= \mathrm{d}^p - (p/4)\{\mathrm{d}^{p-2}, u(x)\} + \sum_{i=2}\{v_i(x), \mathrm{d}^{p-2i}\}, \\ Q &= \mathrm{d}^q - (q/4)\{\mathrm{d}^{q-2}, u(x)\} + \sum_{i=2}\{w_i(x), \mathrm{d}^{q-2i}\},\end{aligned}$$

where d means d/dx and $\{.,.\}$ means anticommutator. Moreover $u(x)$ is the specific heat, the second derivative of the free energy.

Let us consider the trace of the resolvent of one of these operators, for example Q:

$$\omega(z) = \mathrm{tr}\,(z-Q)^{-1}.$$

In the spherical, i.e. the semiclassical limit, because the non commutation between space and derivative can be neglected, $\omega(z)$ is given by

$$\omega(z,x) = -\int^{x} \mathrm{d}x' \int \frac{\mathrm{d}y}{(iy)^q - (q/2)u(x')(iy)^{q-2} + 2\sum_{i=2} w_i(x')(iy)^{q-2i} - z}. \tag{4.1}$$

The integral over y just selects a residue.

The scaling relations in the (p,q) model, in the same limit, are

$$w_i(x) \propto u^i(x), \qquad u(x) \sim x^{2/(p+q-1)}.$$

Thus $\omega(z,x)$ has the scaling form:

$$\omega(z,x) = z^{p/q}\omega\left(1, xz^{-(p+q-1)/q}\right).$$

Moreover in the critical limit, i.e. for z large, it behaves like

$$\omega(z,x) \propto z^{p/q} + xz^{1/q-1} + o\left(z^{1/q-1}\right).$$

Note immediately that these scaling properties are consistent with those of the $O(n)$ model studied above if and only if $l=1$, i.e. p has the special form $p=(2m+1)q\pm 1$.

We have shown that in the case of the $O(n)$ models $\omega(z)$ satisfies, in the large N limit, an algebraic equation of a special type. We show now that in this special class of (p,q) models the resolvent satisfies the same equations.

It is well known that in the case of the one-matrix model, which corresponds to $q=2$, $\omega(z)$ satisfies a second degree equation. More precisely, setting $p=2m+1$, one finds

$$\omega(z) = \frac{m+1}{\pi}\int_{-z}^{u(x)} \frac{\mathrm{d}u'\,u'^m}{\sqrt{z+u'}},$$

and thus

$$\omega^2(z) = (z+u)P_m^2(z,u),$$

where P_m is proportional to the polynomial part of the large z expansion of $z^m(1+u/z)^{-1/2}$ [7,8].

Analogous properties for $q > 2$ are maybe less well-known.

Operators of third order: $q = 3$. A short calculation shows that the trace $\omega(z)$ of the resolvent of the operator $Q = i(\mathrm{d}^3 - (3/4)\{\mathrm{d}, u\})$ is, in the spherical and scaling limit, proportional, up to a rescaling of x, to the expression (2.51) found in section2 2.4,2.5. The result relies, in particular, on the property that p which is even and relatively prime with $q = 3$, is necessarily of the form $p = 6m + 2$ or $p = 6m + 4$.

Generic situation. In the general case $q \geq 4$ the explicit functional form of the operator Q depends on the (p, q) models. In the semiclassical limit the coefficients w_i in equation (4.1) become proportional to u^i with in general p-dependent coefficients. However, it is easy to verify, using for example the string actions [5] in the semiclassical limit, that when $p = (2m + 1)q \pm 1$, the semiclassical form of the operator Q is m-independent. If for convenience we change the normalizations $u(x) \mapsto -2u(x)$, $z \mapsto 2i^q z$, and then set in (4.1):

$$y = 2u^{1/2} \cos\varphi,$$

we find for the denominator

$$y^q - qu(x)y^{q-2} + \cdots - 2z = 2\left(u^{q/2}\cos(q\varphi) - z\right).$$

Taking the residue in y we obtain for $\omega(z)$ the expression

$$\omega(z) \propto \int \mathrm{d}x \frac{u^{1/2}\sin\varphi}{u^{q/2}\sin(q\varphi)},$$

where φ is solution of the equation

$$u^{q/2}\cos(q\varphi) = z\,. \tag{4.2}$$

It follows

$$\begin{aligned} 2iu^{1/2}\sin\varphi &= \left(z + \sqrt{z^2 - u^q}\right)^{1/q} - \left(z - \sqrt{z^2 - u^q}\right)^{1/q}, \\ 2iu^{q/2}\sin(q\varphi) &= 2\sqrt{z^2 - u^q}. \end{aligned}$$

We now use the scaling relation $x = u^{(p+q-1)/2}$ and set $s = \sqrt{z^2 - u^q}$. We find

$$\omega(z) \propto \int^{\sqrt{z^2-a^2}} \mathrm{d}s \left(z^2 - s^2\right)^{(p-q-1)/2q} \left[(z+s)^{1/q} - (z-s)^{1/q}\right],$$

with $a = u^{q/2}$. In the two situations $p = (2m + 1)q \pm 1$ (thus $a = x^{1/(2m+2)}$ or $a = x^{q/(2mq+2q-2)}$) the integration yields an algebraic expression which, up to normalizations, agrees with the results (3.24),(3.27),(3.28) obtained for the corresponding (p, q) critical points of the $O(n)$ model. Note finally that these special (p, q) models include the unitary family $(q \pm 1, q)$.

5. Conclusion

We have exhibited all critical points of the $O(n)$ model on a random lattice generated by the Feynman diagram technique, in the case $n = -2\cos(\pi p/q)$. We have calculated, in the spherical limit using steepest descent, the resolvent and the singular free energy in the scaling limit. In particular we have found that if we parametrize p as $p = (2m+1)q \pm l$, $0 < l < q$, $\gamma_{\text{string}} = -2l/(p+q-l)$. For $l > 1$ this result is surprising since it differs from what is found for the (p, q) critical points of multimatrix models: $\gamma = -2/(p+q-1)$. One possible interpretation is that the operator which, in multimatrix models is coupled to the cosmological constant, is not present here*.

We have finally characterized the large order behaviour of the topological expansion confirming the expected $2k!$ behaviour already found in other matrix models.

We have then shown that in the special case $p = (2m+1)q \pm 1$ all the results we have obtained are identical to those found in the corresponding (p, q) string models by orthogonal polynomial techniques, and otherwise they are different.

Moreover the techniques developed here allow to obtain a number of additional results which will be presented in a separate article [11]. In particular we shall study the effect of other relevant operators, a question which has some subtle aspects since, as the example of the Ising model reveals, negative powers of the matrix S have sometimes to be considered. We shall characterize more precisely the large order behaviour. Finally we shall exhibit solutions for generic values of n between -2 and $+2$, and discuss the $n \to \pm 2$ limit.

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References

[1] I. Kostov, *Mod. Phys. Lett.* A4 (1989) 217, *Phys. Lett.* B266 (1991) 312;
M. Gaudin and I.Kostov, *Phys. Lett.* B220 (1989) 200.

[2] F. David, *Nucl. Phys.* B257[FS14] (1985) 45, 543;
J. Ambjørn, B. Durhuus and J. Fröhlich, *Nucl. Phys.* B257[FS14] (1985) 433; J. Fröhlich, in: *Lecture Notes in Physics,* Vol. 216, ed. L. Garrido (Springer, Berlin, 1985);
V. A. Kazakov, I. K. Kostov and A. A. Migdal, *Phys. Lett.* 157B (1985) 295; D. Boulatov, V. A. Kazakov, I. K. Kostov and A. A. Migdal, *Phys. Lett.* B174 (1986) 87; *Nucl. Phys.* B275[FS17] (1986) 641.

[3] M. Douglas and S. Shenker, *Nucl. Phys.* B335 (1990) 635;
E. Brézin and V. Kazakov, *Phys. Lett.* B236 (1990) 144;
D. Gross and A. Migdal, *Phys. Rev. Lett.* 64 (1990) 127; *Nucl. Phys.* B340 (1990) 333.

[4] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Nucl. Phys.* B241 (1984) 333.

[5] P. Ginsparg, M. Goulian, M. R. Plesser, and J. Zinn-Justin, *Nucl. Phys.* B342 (1990) 539.

[6] E. Brézin, M. Douglas, V. Kazakov, and S. Shenker, *Phys. Lett.* B237 (1990) 43;
D. Gross and A. Migdal, *Phys. Rev. Lett.* 64 (1990) 717;
C. Crnković, P. Ginsparg, and G. Moore, *Phys. Lett.* B237 (1990) 196.

[7] P. Ginsparg and J. Zinn-Justin, *Phys. Lett.* B255 (1991) 189.

[8] P. Ginsparg and J. Zinn-Justin, "Action principle and large order behaviour of non-perturbative gravity", LA-UR-90-3687/SPhT/90-140 (1990), to appear in *Random surfaces and quantum gravity*, proceedings of 1990 Cargèse workshop, edited by O. Alvarez, E. Marinari, and P. Windey.

[9] F. David, *Nucl. Phys.* B348 (1991) 507.

[10] M. R. Douglas, *Phys. Lett.* B238 (1990) 176.

[11] B. Eynard and J. Zinn-Justin, in preparation.

[12] I.K. Kostov and M. Staudacher, "Multicritical phases of the $O(n)$ model coupled to 2D gravity", SPhT/92-025, RU-92-6.