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What is the intrinsic geometry of two-dimensional quantum gravity?

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The intrinsic geometry of 2D quantum gravity is discussed within the framework of the semi-classical Liouville Theory. We show how to define local reparametrization-invariant correlation functions in terms of the geodesic distance. Such observables exhibit strong non-logarithmic short-distance divergences. If one regularizes these divergences by a finite-part prescription, there are no corrections to KPZ scaling, the intrinsic fractal dimension of space-time is two, and no cascade of baby universes occurs. However we show that these divergences can be regularized in a covariant way and have a physical interpretation in terms of "pinning" of geodesics by regions where the metric is singular. This raises issues related to the physics of disordered systems (in particular of the 2D random field Ising model), such as the possible occurrence of replica symmetry breaking, which make the interpretation of numerical and analytical results a subtle and difficult problem.

1. Introduction

Despite the recent spectacular progress in the elaboration of quantum theories of gravity in two dimensions [1], in particular at the non-perturbative level (sum over all topologies) [2,3], some important issues remain to be understood even when no fluctuations of topology are allowed. The most important one is probably the $c > 1$ problem: in which phase does 2D quantum gravity live when coupled to unitary matter with central charge c larger than one? Another issue is to clarify the quantum geometry of space-time in the weak-coupling phase ($c < 1$). The striking agreement between the exact results obtained from the discretized version of the theory based on dynamical triangulations and random matrix models on one hand, and the results obtained from the continuum treatments of 2D gravity based on conformal field theories [4–6] or topological field theories [7] on the other hand, indicates that the theory has indeed some two dimensional character.

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However both the random matrix models and the topological gravity approach allow only to compute global observables such as expectation values of operators integrated over the whole 2D space. Recent numerical simulations of dynamical triangulations [8] have focused on the behavior of local observables which probe the "intrinsic" geometry of space-time at short distance. For technical reasons the most significant simulations deal only with pure gravity ($c = 0$), but are performed on very large lattices (up to $\sim 10^5$ triangles). Their results are at variance with the above picture and indicate that the intrinsic geometry of 2D space-time might be fractal. For instance the volume of the disk with radius r (the set of points at a geodesic distance $d < r$ from the origin) is found to grow much faster that r^2, and is argued to grow faster than any power of r. This indicates that space might have an "intrinsic fractal dimension" much larger than 2 (possibly infinite). Similarly the number of connected components of the circle of radius r (the boundary of the disk) was found to grow with r. The picture advocated in ref. [8] to explain those results is that space-time has a "spiky" structure much closer from the structure of a tree or a branched polymer than from that of ordinary flat 2D space. We think that it is important to have a better understanding of these problems, in particular to check to which extent the semi-classical concepts are valid in quantum gravity.

In this paper we try to discuss these problems within the continuum formulation of 2D gravity based on the Liouville model, which has proved to be very successful to compute global observables.

In sect. 2 we recall some basic facts about 2D gravity and Liouville theory. Then we show how reparametrization-invariant local observables can be defined in terms of the geodesic distance and how they can be calculated in the conformal gauge. However, while global observables often reduce to products of vertex operators integrated over the whole space, the local correlation functions that we discuss are non-local and much more complicated to deal with. In this paper we shall only present the results of one loop calculations, which are valid in the "semiclassical approximations" ($c \to -\infty$).

In sect. 3 we discuss the short-distance (UV) and long-distance (IR) divergences which are present in those observables. The IR divergences are shown to cancel because of reparametrization invariance. In addition to the usual logarithmic UV divergences which are present in global observables and signal the change of anomalous dimensions due to the coupling to gravity, we show that local observables have additional linear UV divergences which appear to be non-covariant (they depend on the background fiducial metric used in quantizing the Liouville theory).

In sect. 4 we treat those divergences in a naive way by a finite-part prescription. Then it appears that logarithmic UV divergences cancel completely at one loop, so that no anomalous scaling occurs for the internal geometry. The intrinsic dimension of space-time is found to be two, correlations of operators in terms of the geodesic distance between the points where they are located obey KPZ scaling, finally the Euler characteristic of the ball with radius r does not grow with r and therefore no "cascade of baby universes" occurs.

In sect. 5 we treat those divergences in a different way. We regularize them in a covariant way by defining the geodesic distance through the short-time behavior of the heat-kernel, i.e. through the "semi-classical" propagation of very massive "phantom" particles in the fluctuating metric. We show that those divergences survive this regularization, although in a slightly different form, and that they induce a singular non-analytic behavior of the local observables that we compute.

In sect. 6 we discuss the physical interpretation of those divergences. We show that they are associated to the "trapping" of the "phantom particles" by region where the Liouville field Φ is very large and negative, which corresponds to domains where the metric g_{ij} is singular in the conformal gauge. This phenomenon is very similar to what occurs in a well known class of problems of the statistical mechanics of disordered systems, namely the problem of random manifolds in random media. In fact we show that at the order where we work (first order in Liouville theory) our problem is closely related to the two-dimensional random-field Ising model (2D RFIM).

Finally in sect. 7 we discuss the possible consequences of this phenomenon. In fact we argue that this makes the whole approach of defining local observables in terms of geodesic distances in continuum 2D gravity quite problematic, especially if spontaneous replica symmetry breaking occurs. We also discuss briefly how "branching" might occur in 2D gravity: we suggest through a heuristic calculation that baby-universes are liberated only for $c > 1$ and we recall that the intrinsic dimension of branched polymers is not infinite, but two.

2. Geodesic distances and invariant observables in the Liouville theory

The basic hypothesis which underlines the formulation of 2D gravity as Liouville field theory is the following [5,9]. Starting from the action for 2D gravity coupled to some matter fields Ψ with central charge c and action S_{mat}

$$S(g, \Psi) = \int \mathrm{d}^2 x \sqrt{g}\, \Lambda + S_{\mathrm{mat}}(g, \Psi), \tag{2.1}$$

and fixing the conformal gauge

$$g_{ij}(x) = \hat{g}_{ij}\, \mathrm{e}^{\Phi(x)}, \tag{2.2}$$

where $\hat{g}$ is a fiducial metric, we assume that the effective theory after gauge fixing is of the form

$$S_{\mathrm{eff}}(\Phi, b, c, \Psi; \hat{g}) = S_{\mathrm{L}}(\Phi; \hat{g}) + S_{\mathrm{gh}}(b, c; \hat{g}) + S_{\mathrm{mat}}(\Psi; \hat{g}) \tag{2.3}$$

where S_L is the Liouville action

$$S_L(\Phi;\hat{g}) = \frac{1}{8\pi}\int d^2x\,\sqrt{\hat{g}}\left\{\frac{\beta}{2}\hat{g}^{ij}\,\partial_i\Phi\,\partial_j\Phi + (\beta+1)\hat{R}\Phi + \mu_R\,e^{\Phi}\right\}, \quad (2.4)$$

where we have normalized Φ such that the term proportional to the renormalized cosmological constant μ_R is e^{Φ}. S_{gh} is the action for the ghosts in the fiducial metric

$$S_{gh}(b,c;\hat{g}) = \int d^2x\,\sqrt{\hat{g}}\,b_{ij}\hat{D}^i c^j, \quad (2.5)$$

and S_{mat} is the action for the matter fields in the fiducial metric. Assuming that the fields Φ, b, c and Ψ are quantized with a regulator depending only on the fiducial metric $\hat{g}$ (this ensures that there are no diffeomorphism anomalies) and that the total central charge of the effective theory vanishes (this ensures that there are no conformal anomalies) fixes the value of the coupling constant β

$$\beta = \frac{13 - c + \sqrt{(25-c)(1-c)}}{12}. \quad (2.6)$$

β is nothing but $1-\gamma_{string}$. At that stage we deal carelessly with the ghosts and antighosts zero modes, which are associated respectively to conformal Killing vectors (and therefore to the Φ modes associated to global conformal diffeomorphisms, which should not be taken into account in the quantization of Φ), and to Teichmuller deformations (associated to the moduli dependence of the fiducial metric $\hat{g}$). Indeed they do not play any role in most of the discussion.

Non-global reparametrization-invariant observables can be constructed by considering correlation between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ at fixed geodesic distance $d(\boldsymbol{x},\boldsymbol{y};g)$. The geodesic distance is defined for a fixed metric g_{ij} in the standard way as the minimum of the length $\ell(\mathscr{C})$ over all paths $\mathscr{C}$ going from $\boldsymbol{x}$ to $\boldsymbol{y}$.

$$d(\boldsymbol{x},\boldsymbol{y};g) = \min_{\mathscr{C}_{x,y}}\left(\ell(\mathscr{C}_{x,y})\right) \quad (2.7)$$

In the conformal gauge (2.2), taking as fiducial metric the euclidean metric

$$\hat{g}_{ij} = \delta_{ij}, \quad (2.8)$$

and assuming that the conformal field Φ is small, it is in principle possible to compute order by order in Φ this distance, since for infinitesimal Φ there is only one geodesics which joins $\boldsymbol{x}$ to $\boldsymbol{y}$. Starting from the formula for the length of the path *

$$\ell=\int_0^1 \mathrm{d}s \; |\dot{\boldsymbol{x}}(s)| \; \mathrm{e}^{\Phi(x)/2}, \tag{2.9}$$

expanding to second order in Φ and solving the equation for the geodesics we get explicitly

$$d(\boldsymbol{x},\,\boldsymbol{y};\,\Phi)=|\boldsymbol{x}-\boldsymbol{y}|\int_0^1 \mathrm{d}s\left[1+\tfrac{1}{2}\Phi(\boldsymbol{x}(s))+\tfrac{1}{8}\Phi^2(\boldsymbol{x}(s))\right]$$

$$-\tfrac{1}{8}|\boldsymbol{x}-\boldsymbol{y}|^3\int_0^1 \mathrm{d}u\int_0^1 \mathrm{d}v\;\partial_\perp\Phi(\boldsymbol{x}(u))\;[u,v]\;\partial_\perp\Phi(\boldsymbol{x}(v)), \tag{2.10}$$

where $|\boldsymbol{x}-\boldsymbol{y}|$ is the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$ in the euclidean metric $\hat{g}$, $\boldsymbol{x}(s)=\boldsymbol{x}(1-s)+\boldsymbol{y}s$ interpolates linearly between $\boldsymbol{x}$ and $\boldsymbol{y}$, $\partial_\perp$ represents the partial derivative orthogonal to $\boldsymbol{y}-\boldsymbol{x}$,

$$\partial_\perp\Phi=\epsilon^i{}_j\frac{y^j-x^j}{|\boldsymbol{y}-\boldsymbol{x}|}\partial_i\Phi, \tag{2.11}$$

and finally the kernel $[u, v]$ is the 1D propagator with Dirichlet boundary conditions on the interval [0, 1], namely

$$[u,v]=v(1-u)\theta(u-v)+u(1-v)\theta(v-u). \tag{2.12}$$

From eq. (2.10) we can calculate to order $\mathrm{O}(\beta^{-1})$ various reparametrization invariant observables. For instance, the v.e.v. of the volume of the disk with radius r, $V(r)$, should be given by

$$\langle V(r)\rangle=\left\langle\int \mathrm{d}^2x\,\sqrt{g}\,F_r(d(\boldsymbol{o},\,\boldsymbol{x};\,g))\right\rangle, \tag{2.13}$$

where F_r is the step function

$$F_r(d)=\theta(r-d). \tag{2.14}$$

$\boldsymbol{o}$ is some arbitrary point taken to be the origin. In the conformal gauge (2.2) and

* It is not obvious how this formula gets renormalized by quantum fluctuations. However a renormalization of $\exp(\Phi/2)$ into $\exp(\alpha\Phi)$ should affect only the two-loops terms.

using eq. (2.10) we obtain, expanding to second order in Φ

$$\langle V(r)\rangle = V^{(0)}(r) + V^{(1)}(r) + V^{(2)}(r), \tag{2.15a}$$

$$V^{(0)}(r) = \int \mathrm{d}^2x\, F_r(|\boldsymbol{x}|), \tag{2.15b}$$

$$V^{(1)}(r) = \int \mathrm{d}^2x \Big\{ F_r(|\boldsymbol{x}|)\langle\Phi(\boldsymbol{x})\rangle + \tfrac{1}{2}|\boldsymbol{x}|\, F_r'(|\boldsymbol{x}|)\int_0^1 \mathrm{d}s \langle \Phi(\boldsymbol{x}(s))\rangle \Big\}, \tag{2.15c}$$

$$\begin{aligned} V^{(2)}(r) = {} & \int \mathrm{d}^2x\, F_r(|\boldsymbol{x}|)\tfrac{1}{2}\langle\Phi^2(\boldsymbol{x})\rangle \\ & + \int \mathrm{d}^2x\, F_r'(|\boldsymbol{x}|)\Big\{ |\boldsymbol{x}|\int_0^1 \mathrm{d}s \big\{\tfrac{1}{2}\langle\Phi(\boldsymbol{x})\Phi(\boldsymbol{x}(s))\rangle + \tfrac{1}{8}\langle\Phi^2(\boldsymbol{x}(s))\rangle\big\} \\ & - \tfrac{1}{8}|\boldsymbol{x}|^3\int_0^1 \mathrm{d}u\int_0^1 \mathrm{d}v\, [u,v]\langle\partial_\perp\Phi(\boldsymbol{x}(u))\,\partial_\perp\Phi(\boldsymbol{x}(v))\rangle\Big\} \\ & + \int \mathrm{d}^2x\, F_r''(|x|)\tfrac{1}{8}|\boldsymbol{x}|^2\int_0^1 \mathrm{d}u\int_0^1 \mathrm{d}v\langle\Phi(\boldsymbol{x}(u))\Phi(\boldsymbol{x}(v))\rangle. \end{aligned} \tag{2.15d}$$

Similarly one can calculate the v.e.v. for the number of connected components of the "circle" of radius r, $N(r)$. Indeed, it is related to the Euler characteristic of the disk, $\chi(r)$, by

$$\chi(r) = 2 - n(r). \tag{2.16}$$

We work at the level of the planar topology, and therefore the number of handles of the disk is zero. Since the disk is the set of points $\boldsymbol{x}$ such that $\mathrm{d}(\boldsymbol{x}) = \mathrm{d}(\boldsymbol{o}, \boldsymbol{x}; \hat{g}) \leqslant r$, we can use the formula

$$\chi(r) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_{d\leqslant r} \mathrm{d}^2x\, \partial_i\,\partial_j d\,\frac{\epsilon_{ik}\,\partial_k d\ \epsilon_{jl}\,\partial_l d}{|\partial_m d\ \partial_m d|}, \tag{2.17}$$

where ∂_i is the partial derivative with respect to x^i and everything is expressed in the fiducial euclidean metric $\hat{g}$. After somewhat lengthy calculations, involving integrations by part, and using translation invariance for the propagator

$\langle \Phi(\boldsymbol{x})\Phi(\boldsymbol{y})\rangle$, we obtain

$$\langle \chi(r)\rangle = -\frac{1}{2\pi}\left(X^{(0)}(r) + X^{(1)}(r) + X^{(2)}(r)\right), \tag{2.18a}$$

$$X^{(0)}(r) = \int \mathrm{d}^2x\, F_r'(|\boldsymbol{x}|)\frac{1}{|\boldsymbol{x}|}, \tag{2.18b}$$

$$\begin{aligned} X^{(1)}(r) = \int \mathrm{d}^2x\, F_r'(|\boldsymbol{x}|)\bigg\{ &\frac{1}{2|\boldsymbol{x}|}\int_0^1 \mathrm{d}s \langle \Phi(\boldsymbol{x}(s))\rangle \\ &+ \tfrac{1}{2}|x|\int_0^1 \mathrm{d}s\, s^2 \langle \partial_\perp^2 \Phi(\boldsymbol{x}(s))\rangle \bigg\} \\ &- \int \mathrm{d}^2x\, F_r''(|\boldsymbol{x}|)\tfrac{1}{2}\int_0^1 \mathrm{d}s \langle \Phi(\boldsymbol{x}(s))\rangle, \end{aligned} \tag{2.18c}$$

$$\begin{aligned} X^{(2)}(r) = \int \mathrm{d}^2x\, F_r'(|\boldsymbol{x}|)\bigg\{ &-\frac{1}{8|\boldsymbol{x}|}\int_0^1 \mathrm{d}s \langle \Phi^2(\boldsymbol{x}(s))\rangle \\ &+ \frac{1}{4|\boldsymbol{x}|}\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v \langle \Phi(\boldsymbol{x}(u))\Phi(\boldsymbol{x}(v))\rangle \\ &- \tfrac{1}{2}|\boldsymbol{x}|\int_0^1 \mathrm{d}s\, s \langle \partial_\perp \Phi(\boldsymbol{x}(s))\, \partial_\perp \Phi(\boldsymbol{x})\rangle \\ &+ \tfrac{1}{8}|\boldsymbol{x}|\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, \mathscr{D}[u,v] \langle \partial_\perp \Phi(\boldsymbol{x}(u))\, \partial_\perp \Phi(\boldsymbol{x}(v))\rangle \\ &+ \tfrac{1}{8}|\boldsymbol{x}|^3\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, \mathscr{E}[u,v] \langle \partial_\perp^2 \Phi(\boldsymbol{x}(u))\, \partial_\perp^2 \Phi(\boldsymbol{x}(v))\rangle \bigg\} \\ + \int \mathrm{d}^2x\, F_r''(|\boldsymbol{x}|)\bigg\{ &\tfrac{1}{4}\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v \langle \Phi(\boldsymbol{x}(u))\Phi(\boldsymbol{x}(v))\rangle \\ &+ \tfrac{1}{8}\int_0^1 \mathrm{d}s \langle \Phi^2(\boldsymbol{x}(s))\rangle \\ &+ \tfrac{1}{4}|\boldsymbol{x}|^2\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, u^2 \langle \partial_\perp^2 \Phi(\boldsymbol{x}(u))\Phi(\boldsymbol{x}(v))\rangle \\ &- \tfrac{1}{8}|\boldsymbol{x}|^2\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v [u,v] \langle \partial_\perp \Phi(\boldsymbol{x}(u))\Phi(\boldsymbol{x}(v))\rangle \bigg\} \\ + \int \mathrm{d}^2x\, F_r'''(|\boldsymbol{x}|)\tfrac{1}{8}|\boldsymbol{x}| &\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v \langle \Phi(\boldsymbol{x}(u))\Phi(\boldsymbol{x}(v))\rangle, \end{aligned} \tag{2.18d}$$

where in eq. (2.18d) $\mathscr{D}$ and $\mathscr{E}$ are given by

$$\mathscr{D}[u, v] = uv - 4[u, v] + (2v - 3u)\frac{\partial}{\partial u}[u, v] + (2u - 3v)\frac{\partial}{\partial v}[u, v],$$

$$\mathscr{E}[u, v] = [u, v](u - v)^2. \tag{2.19}$$

3. IR and UV divergences

If we want to compute (2.15) and (2.18), we must treat carefully the potential divergences which arise from the Liouville action (2.4). Indeed the v.e.v. for Φ must be computed in the flat fiducial metric (2.8) and for $\mu_R = 0$ (since we are interested in the critical properties of 2D gravity). Thus (2.4) reduces to the action of a free scalar field in two dimensions. The divergences are of two kind: Infrared divergences and ultraviolet divergences.

3.1. INFRARED DIVERGENCES

The propagator of the free scalar field in 2 dimensions is known to be IR divergent. We can introduce an IR regulator by fixing the total volume of the 2D universe to be some constant A, once we have chosen a fiducial metric $\hat{g}$ corresponding to some compact manifold with given genus h and fixed moduli. Following ref. [10] we can introduce this constraint as a δ-function in the partition function for the Liouville field

$$Z_L(A) = \int \mathscr{D}_{\hat{g}}[\Phi]\ \delta\left(A - \int d^2x\ \sqrt{\hat{g}}\, e^{\Phi}\right)\ e^{-S_L(\Phi, \hat{g})}. \tag{3.1}$$

Splitting Φ into the laplacian zero mode Φ_0 and the fluctuating part $\tilde{\Phi}$

$$\Phi(x) = \Phi_0 + \tilde{\Phi}(x), \qquad \int d^2x\ \sqrt{\hat{g}}\, \tilde{\Phi} = 0, \tag{3.2}$$

we can integrate out explicitly the zero-mode Φ_0 and we get, using the fact that $\int d^2x\ \sqrt{\hat{g}}\, \hat{R} = 8\pi(1-h)$,

$$Z_L(A) = (\hat{A})^{-1/2}(A/\hat{A})^{-1-(\beta+1)(1-h)}$$
$$\times \int \mathscr{D}_{\hat{g}}[\tilde{\Phi}]\left(\hat{A}^{-1}\int d^2x\ \sqrt{\hat{g}}\, e^{\tilde{\Phi}}\right)^{(\beta+1)(1-h)} e^{-S_L(\tilde{\Phi})}, \tag{3.3}$$

where $\hat{A} = \int d^2x \sqrt{\hat{g}}$ is the background area. In the semiclassical limit $\beta \to \infty$ the fluctuations of $\tilde{\Phi}$ are of order $\beta^{-1/2}$. Therefore the non-local term in (3.3) reduces to a local interaction term

$$\left(\hat{A}^{-1} \int d^2x \sqrt{\hat{g}}\, e^{\tilde{\Phi}}\right)^{(\beta+1)(1-h)} \simeq \exp\left(\frac{\beta(1-h)}{2\hat{A}} \int d^2x \sqrt{\hat{g}}\, \tilde{\Phi}^2\right). \tag{3.4}$$

In the one-loop approximation, we can therefore compute correlation functions at fixed genus h, moduli and total volume A by choosing as fiducial metric $\hat{g}$ a representative of this class, for instance the metric with $\hat{R} = \text{constant}$, $\hat{A} = A$, and by using as effective action for $\tilde{\Phi}$ the free field action

$$S_{\text{eff}}(\tilde{\Phi}) = \frac{\beta}{8\pi} \int d^2x \sqrt{\hat{g}}\, \tfrac{1}{2} \left\{ \hat{g}^{ij}\, \partial_i \tilde{\Phi}\, \partial_j \tilde{\Phi} - \frac{8\pi(1-h)}{\hat{A}} \tilde{\Phi}^2 \right\}. \tag{3.5}$$

The effective mass term $m_{\text{eff}}^2 = -(1-\text{h})8\pi/\hat{A}$ is larger than or equal to 0 for $h > 0$. For $h = 0$ it is negative, but exactly equal to the first non-zero eigenvalue of the scalar laplacian Δ on the sphere with constant curvature. The three associated zero modes are the $l = 1$ spherical harmonics $\tilde{\Phi}_m = Y_m^1$, $m = -1, 0, 1$, which corresponds to global conformal diffeomorphisms on the sphere. Therefore they should not be taken into account when integrating over the $\tilde{\Phi}$ fluctuations and the quadratic form (3.5) is still strictly positive.

In fact the IR problem simplifies drastically if one considers reparametrization invariant observables like (2.13) or (2.17). Indeed it appears that (at least at one loop) they are IR finite. This is in fact quite natural and should follow from general results on IR divergences in Goldstone models in 2 dimensions [11]. Indeed the fact that Φ is massless follows from the invariance of the Liouville action (2.4) under a global shift

$$\Phi(x) \to \Phi(x) + \Phi_0 \tag{3.6}$$

(for $\mu_R = 0$ and asymptotically flat $\hat{g}$). Reparametrization invariant observables should be invariant under such a shift, since it may be absorbed into a change of coordinates

$$\boldsymbol{x} \to \boldsymbol{x}\, e^{-\Phi_0/2}. \tag{3.7}$$

This is confirmed by explicit one-loop calculations. For instance, assuming that Φ gets a non-zero v.e.v. $\langle\Phi\rangle$, (2.15c) and (2.18c) still vanish identically through partial integration

$$V^{(1)}(r) = \langle\Phi\rangle 2\pi \int_0^\infty dx\, x\left(F_r(x) + \tfrac{1}{2}xF_r'(x)\right) = 0,$$

$$\chi^{(1)}(r) = -\tfrac{1}{2}\langle\Phi\rangle 2\pi \int_0^\infty dx\left(F_r'(x) + xF_r''(x)\right) = 0. \tag{3.8}$$

This implies that we can perform all calculations in infinite flat space, without worrying about IR problems. The IR regularized propagator, which is of the form

$$\langle \Phi(x)\Phi(y)\rangle \simeq -(2/\beta)\ \ln\left(|x-y|^2/\hat{A}\right) \quad \text{if} \quad |x-y|^2 \ll \hat{A}, \tag{3.9}$$

leads for invariant observables to results which do not depend on the IR regulator $\hat{A}$. Thus we can shift the propagator by an arbitrary amount and replace eq. (3.9) by

$$\langle \Phi(x)\Phi(y)\rangle = -(4/\beta)\ \ln(|x-y|\Lambda), \tag{3.10}$$

where Λ is some finite mass scale.

3.2. ULTRAVIOLET DIVERGENCES

The problem of the short distance divergences is more delicate. In the original 2D gravity theory, discretized for instance by dynamical triangulations, the UV "proper cut-off" a expresses the fact that one should not consider fluctuations of the metric with intrinsic scale length $\ell^2 = \mathrm{d}x^i\ \mathrm{d}x^j g_{ij}(x) < a^2$. In our treatment of the effective Liouville theory, a is now considered as a "fiducial cut off". This means that fluctuations of all the fields, including the Liouville field Φ, must have scale length, expressed with respect to the fiducial metric $\hat{g}$, larger than a ($\ell^2 = \mathrm{d}x^i\ \mathrm{d}x^j \hat{g}_{ij}(x) < a^2$). The expectation values of the physical observables, when computed with this prescription, have in general UV divergences and therefore they depend on a. However they must not depend on the background metric $\hat{g}$. This requirement follows from a diffeomorphism anomaly consistency condition (absence of diffeomorphism anomalies) and fixes the conformal weight of the vertex operators which "dress" the local primary fields which are integrated over space in the calculations of global observables.

Let us concentrate on the observable $V(r)$ given by (2.15.) There are two kinds of divergences. The first one comes from the propagator at coinciding points. From eq. (3.9) it gives the logarithmic divergence

$$\langle \Phi^2(x)\rangle = -(4/\beta)\ \ln\left(a^2/\hat{A}\right) \tag{3.11}$$

(note that it depends on the fiducial metric through $\hat{A}$). It is the usual kind of divergence which is already present in the computation of global quantities involving vertex operators. The second kind of divergence comes from

$$\langle \partial_\perp \Phi(x)\ \partial_\perp \Phi(y)\rangle = \frac{4}{\beta}\frac{1}{|x-y|^2}. \tag{3.12}$$

Indeed this term gives in the double integral over u and v in eq. (2.15d) a linear divergence at $u = v$ of the form

$$\int_0^1 \mathrm{d}u\, \mathrm{d}v\, [u, v] \langle \partial_\perp \Phi(x)\, \partial_\perp \Phi(y) \rangle \sim \int \mathrm{d}u\, \mathrm{d}v\, [u, v] \frac{1}{|x|^2 (u-v)^2}. \quad (3.13)$$

If one follows the general recipe, one must regularize this linear divergence by cutting off the integral over u and v as soon as the two points $x(u)$ and $x(v)$ are at a distance less than the fiducial cut-off a, i.e. for $|u - v| < a/|x|\sqrt{\hat{g}}$. This gives a linear divergence in the fiducial cut-off, and a straightforward dimensional analysis shows that this gives a singular term in the one-loop contribution to the volume of the ball $V(r)$, (2.15d), of the form

$$V^{(2)}(r) \propto r^3/a, \quad (3.14)$$

with a positive coefficient. However one has to remember that r is a physical distance, expressed in term of the physical fluctuating metric, and V a physical area, while a is a fiducial cut-off. Therefore the coefficient in (3.14) should depend on the fiducial metric. An explicit calculation, using the IR cut-off discussed above, shows that in fact

$$V^{(2)}(r) \propto \frac{1}{6\beta} \sqrt{\frac{\hat{A}}{A}}\, \frac{r^3}{a}, \quad (3.15)$$

where A is the total area and $\hat{A}$ the fiducial area. Thus, because of this strange linear divergence, the dependence on the fiducial metric and the IR regulator does not disappear, contrary to what has been argued before. In the next sections we discuss two ways to get out of this contradiction.

4. The finite-part regularization

The first, and somewhat naive, point of view consists in arguing that since this divergence is $\hat{g}$-dependent, it signals that the observables we are interested in are not reparametrization invariant and that some counterterms have to be added to make them physical. An often used regularization procedure to deal with such power-like divergences caused by mixing of operators consists in using a finite part integration prescription to deal with all but the logarithmic singularities. At that stage we have no better justification for this procedure but it gives interesting results that we now describe.

In our case, when evaluating $V(r)$, this consists in treating the linear divergence at $u = v$ by the finite-part prescription

$$\int_{-\infty}^{+\infty} \mathrm{d}u \frac{1}{(u-v)^2} = 0. \quad (4.1)$$

We are then left with logarithmic divergences of the form $\int \mathrm{d}u\ |u-v|^{-1}$ that we regularize in the standard way with the fiducial cut-off a. We shall not give the details of the calculations. The various terms in eq. (2.15d) contribute to logarithmic divergences, which by dimensional analysis should amount to a term of the form $r^2 \ln(a)$. Similarly, the various propagators (3.9) and the integrations over u and v give logarithms of the distance, which amounts to terms of the form $r^2 \ln(r)$. The coefficient in front of this logarithmic term is universal, namely it does not depend on the exact form of the regulator.

In fact the final result for $V(r)$ is that there are no logarithmic corrections. The various logarithms which come from the various terms in eq. (2.15d) cancel each other at the end of the calculation. Thus the volume of the ball with radius r is given at one loop by

$$\langle V(r)\rangle = \pi r^2 + (C/\beta) r^2 + \dots, \tag{4.2}$$

with C some constant.

The same conclusion can be drawn, within this finite-part prescription, for the Euler characteristics of the disk (2.18) which gives the number of connected components $N(r)$ of the circle with radius r (this corresponds in ref. [8] to the "number of baby-universes" generated at "time" r). When looking at eq. (2.18d) it seems that the existence of operators with four $\partial_\perp$ implies that there are divergences in a^{-3} and a^{-2}. However the explicit calculation shows that these divergences cancel and that we are left with the familiar a^{-1} and $\ln(a)$ divergences. Using the f.p. prescription to get rid of the a^{-1} divergences, we found after a somewhat lengthy calculation that

$$\langle \chi(r)\rangle = 1 + (D/\beta) + \dots. \tag{4.3}$$

Finally we have checked for consistency that if we compute the correlation function of two primary operators Ψ with conformal weight $\Delta^{(0)}$ at fixed geodesic distance r, that we define for instance as

$$\langle \Psi(0)\Psi(r)\rangle = \frac{\left\langle \int \mathrm{d}x \int \mathrm{d}y\ \Psi(x)\Psi(y)\theta(r-d(x,y))\right\rangle}{\left\langle \int \mathrm{d}x \int \mathrm{d}y\ \theta(r-d(x,y))\right\rangle}, \tag{4.4}$$

it still obeys KPZ scaling, namely that

$$\langle \Psi(0)\Psi(r)\rangle \sim r^{-4\Delta} \quad r\to\infty, \tag{4.5}$$

with $\Delta = \Delta^{(0)} + (1/\beta)\Delta^{(0)}(1-\Delta^{(0)})$ which correspond to leading order in $1/\beta$ with the KPZ formula for the scaling dimension Δ of Ψ dressed by gravitation.

Unfortunately in sect. 5 we shall see that this cancellation of logarithmic divergences, although quite interesting, is not the end of the story.

5. The covariant regularization

We now propose a covariant definition of the geodesic distance between points which should regularize to all orders the singularities that we have discussed before. Let us consider a massive scalar field $\Psi^a(a = 1, n)$, coupled to the metric by the standard action

$$S(g, \Psi) = \tfrac{1}{2}\int d^2x \sqrt{g}\left(g^{ij}\, \partial_i\Psi\, \partial_j\Psi + m^2\Psi\Psi\right) \tag{5.1}$$

In a classical metric the geodesic distance can be obtained through the large-mass limit of the propagator

$$d(x, y; g) = \lim_{m\to\infty}\left(-(1/m)\ \ln(\langle\Psi(x)\Psi(y)\rangle)\right) \tag{5.2}$$

where here $\langle\dots\rangle$ means the v.e.v. over Ψ. This follows easily from the random-walk representation of the propagator for a scalar field

$$\langle\Psi(x)\Psi(y)\rangle = \int_0^\infty d\tau \int_{r(0)=x}^{r(\tau)=y} \mathscr{D}[r(\sigma)]\ \exp\left(-m^2\tau - \int_0^\tau d\sigma\ |\dot{r}(\sigma)|/4\right), \tag{5.3}$$

where τ is the proper time.

The $\langle\ \rangle$ may be taken out of the logarithm by the well-known replica trick. We introduce n replica Ψ^a, $a = 1, n$ of Ψ and we write

$$\ln^p(\langle\Psi(x)\Psi(y)\rangle) = \lim_{n\to 0}\left(\frac{\partial}{\partial n}\right)^p \left\langle \prod_{a=1,n} \Psi^a(x)\Psi^a(y)\right\rangle. \tag{5.4}$$

One advantage of this formulation is that if the metric g fluctuates, this definition of the geodesic distance stays valid, since we only have to take the v.e.v. over both Ψ^a and over the gravitational field. For instance the observable $V(r)$ becomes

$$\langle V(r)\rangle = \lim_{m\to\infty}\lim_{n\to 0}\int d^2x \sqrt{g}\, F_r\left(-\frac{1}{m}\frac{\partial}{\partial n}\right)\langle\overline{\Psi}(o)\overline{\Psi}(x)\rangle, \tag{5.5}$$

where $\overline{\Psi} = \prod_{a=1,n}\Psi^a$. The fact that we take the limit $n = 0$ means that the metric g is quenched when averaging over the massive field Ψ, instead of being annealed as in ordinary global observables.

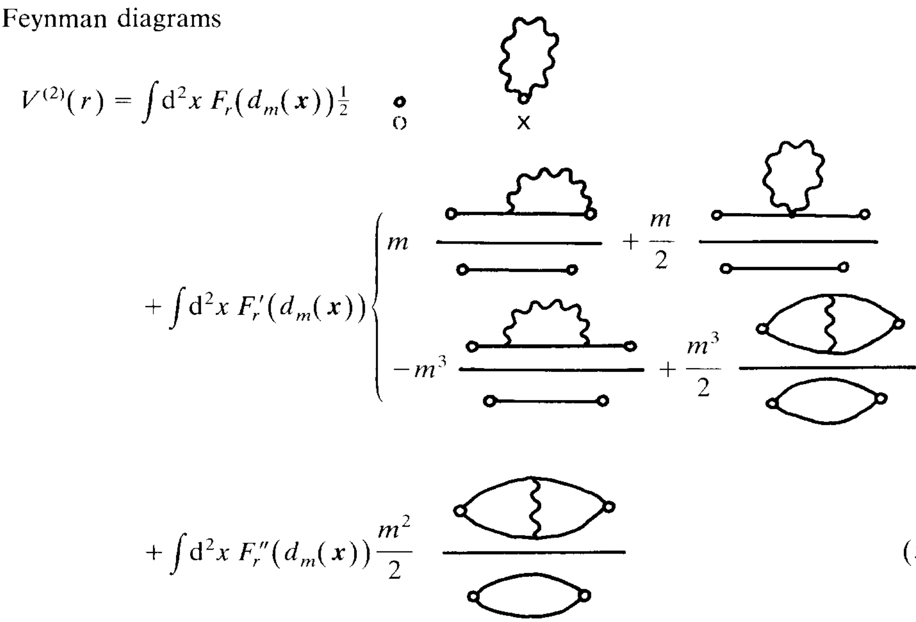
Another advantage of this procedure is that the geodesic distance is now expressed in term of correlators of matter fields coupled to gravity, which can be treated as the other matter fields. In the conformal gauge (2.2) we obtain the effective action for the Ψ^a,

$$S(\Psi, \Phi; \hat{g}) = \tfrac{1}{2}\int \mathrm{d}^2x \sqrt{\hat{g}}\left\{\hat{g}^{ij}\, \partial_i\Psi^a\, \partial_j\Psi^a + m^2\, \mathrm{e}^{\Phi}\Psi^a\Psi^a\right\}. \tag{5.6}$$

Finally the most important point is that the large mass m plays the role of an UV regulator which smoothes the linear divergences that we discussed before. Indeed, in the functional sum over paths (5.3) for large but finite mass the dominant contributions are peaked around the saddle point (a particle moving along the geodesic with constant proper seed $m/2$) but fluctuate by an amount of order $1/m$. Since m is a physical mass it corresponds to a covariant short distance regulator. We expect the metric to be probed only at length-scales larger than the cut-off scale $1/m$.

A fully covariant definition of the observables is therefore obtained by first quantizing the whole Liouville theory (Φ, ghosts, matter and replica Ψ^a), taking the continuum limit $a \to 0$, and then looking at the limit $m \to \infty$. Note that in doing so the matter central charge is shifted by $c \to c + n$. The corresponding change in the Liouville coupling constant $1/\beta$ is of order $1/\beta^2$. Possible contributions in (5.5) from this renormalization of β come from $(\partial\beta^{-1}/\partial n)|_{n=0}$ and are at least of order $1/\beta^2$.

In practice this procedure leads already at one loop to somewhat lengthy calculations. We give below the formula for $\langle V(r)\rangle$ at one loop in terms of Feynman diagrams



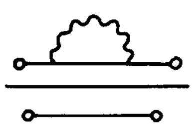
(5.7)

The thick lines represent the massive Ψ propagator, the wavy line the Φ propagator for the Liouville field (3.9). Finally $d_m(x) = -(1/m)\ln(\langle\Psi(o)\Psi(x)\rangle)$ is the "regularized distance" in flat space given by eq. (5.2). One should notice that in the perturbation expansion ratios of Feynman diagrams appear. Each thick line represents in fact one replica propagating in the fluctuating metric and the quantum fluctuations of Φ induce interactions between different replica, as is clear from the last two diagrams.

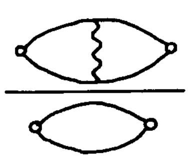
For finite m, as discussed before, (5.7) is IR finite. It is also UV finite. Indeed, the only UV divergences come from the first and third diagrams, which contain a Liouville "tadpole". It is easy to see that they cancel exactly (through an integration by part). Therefore (5.7) does not depend on the fiducial cutoff, nor on the fiducial metric, as expected.

Now one has to study the large m limit of (5.7). This amounts to study for fixed m the large $|x|$ behavior of the diagrams, or equivalently in momentum space the structure of the Landau singularity at the first cut at $p^2 = -m^2$ (for one-replica diagrams) or at $p^2 = -4m^2$ (for the two-replica diagram). This can be done explicitly, for instance by using a multiple Mellin representation for the diagrams [12]. Details on the calculations are given in appendix A. The final result is the following:

The second and sixth diagrams contribute to (5.7) by terms at most of order $r^2\ln(rm)$ and are therefore at most logarithmically divergent. The important terms are the fourth and fifth diagrams, which appear to be linearly divergent. More precisely

m^4  $\sim (A\ln(mx) + B)(mx)^2 + \mathrm{O}(mx),$ (5.8)

while

$\frac{m^4}{2}$  $\sim (A\ln(mx) + B)(mx)^2 + C(mx)^{3/2} + \mathrm{O}(mx),$ (5.9)

with A, B, C some constants ($C = -\sqrt{\pi}/64 < 0$). Thus the linear divergence cancel and we are left with a square-root divergence in the UV cut-off m. The final result is

$$\langle V(r)\rangle \simeq \pi r^2 + \frac{1}{\beta}\left(\frac{\pi^{3/2}}{64} r^{5/2} m^{1/2} + \ldots\right) + \ldots. \tag{5.10}$$

This non-analytic divergence has in fact the same origin as the linear divergence in the fiducial cut-off discussed in sect. 4. As discussed in appendix A, it occurs at coinciding proper-time for the two internal vertices of the diagram in eq. (5.9), that is when the two internal vertices comes close to each other. In other words, it comes from the short-distance singular behavior of the interaction between replica induced by the fluctuations of the metric. The fact that one obtains a square-root divergence $\sim (rm)^{1/2}$ instead of a linear divergence $\sim r/a$ (as expected from sect. 3) is not really surprising. Indeed in flat space ($\Phi = 0$) the random walk has lateral fluctuations of order $(mr)^{1/2}$. Hence the "effective length-cut-off" beyond which the metric is probed is $(r/m)^{1/2}$ instead of $1/m$, as naively expected. Since similar diagrams should appear also in the expression for other local covariant quantities such as $\langle \chi(r) \rangle$, we expect (although we have not done the explicit calculations), that such square-root singularies are generic. We discuss the significance of these singularities in sect. 6.

6. Geodesics in random metrics and directed polymers in random media

At first sight eq. (5.10) means that the metric is so singular that the average volume of a ball with radius r grows much faster than r^2. Since one expects that much stronger divergences will appear at higher orders in the perturbative expansion in $1/\beta$, this seems to corroborate the idea put forward in ref. [8] that space-time in 2D gravity has a fractal structure. However in our opinion one should get a better understanding of the precise origin of these divergences before drawing definite conclusions. For that purpose let us come back to the random walk representation (5.3) for the propagator. When analyzing the origin of the divergence in eq. (5.9) one can see that it is contained neither into the fluctuations of the proper-time of the two replica, nor in the longitudinal fluctuations of $\boldsymbol{r}$ along the directions of the geodesic. Therefore we keep only the transverse fluctuations in the definition of the distance (5.2) and we replace in the propagator (5.3) the action quadratic in $\boldsymbol{r}$ simply by the length L of the walk, that is

$$\langle \Psi(o) \Psi(x) \rangle \to \int \mathscr{D}[\epsilon(z)]\ \mathrm{e}^{-mL[\epsilon]}, \tag{6.1}$$

where

$$L[\epsilon] = \int_0^x \mathrm{d}z \left(1 + \dot{\epsilon}^2\right)^{1/2} \mathrm{e}^{\Phi(z,\epsilon)/2}, \tag{6.2}$$

z is the coordinate in the $\boldsymbol{x}$ direction and ϵ the coordinate orthogonal to z (in flat space). Moreover at leading order in $1/\beta$, and for the purpose of studying the leading divergence, one may expand (6.2) and keep the terms quadratic in ϵ and linear in Φ, thus getting

$$L_{\text{eff}}[\epsilon] = x + \tfrac{1}{2} \int_0^x \mathrm{d}z \left(\dot{\epsilon}(z)^2 + \Phi(z, \epsilon(z)) \right). \tag{6.3}$$

In doing so we have broken explicitly general covariance but this is not important for the analysis of the leading divergence. To find the path with minimal length between o and x is reduced to find the configuration for ϵ which minimizes the action functional (6.3). This form of action is exactly the one which appears in the continuous formulation of the problem of directed polymers in random media [13]. This problem, and its extension to random manifolds in random media, has been recently the subject of numerous investigations. It is related to interfaces in the presence of quenched random inpurities [14], to surface growth through the KPZ equation [15] (not to be confused with the KPZ of ref. [4]), to randomly stirred fluids through the Burgers equation [16], and many questions and physical issues in these problems are similar to those encountered in spin glasses [17]. In (6.3) ϵ represents the transverse position of a one-dimensional interface with tension unity, subject to the random potential Ψ, and m is the inverse temperature. However there are important differences. Most studies of directed polymers deal with correlations for the disorder which are gaussian and local, and characterized by the 2-points correlator of the form

$$\overline{\Phi(z,\epsilon)\Phi(z',\epsilon')} = \lambda\delta(z-z')f(\epsilon-\epsilon'), \tag{6.4}$$

with f some function which is generally of the form

$$f(\epsilon-\epsilon') \sim \frac{1}{1-\gamma}|\epsilon-\epsilon'|^{2(\gamma-1)} \quad |\epsilon-\epsilon'| \to \infty \tag{6.5}$$

($\overline{\ldots}$ means the average over the disorder). In our case the correlations for Φ are non-local, since given by eq. (3.10). Moreover (6.3) is only an approximation valid to leading order in the "disorder strength" $\lambda \equiv 1/\beta$ and to higher order there will be more complicated n-point correlations for the disorder.

The important phenomenon for random directed-polymers is that at low temperature there is competition between the tension, which tends to make the polymer straight ($\epsilon \simeq$ const.), and the random potential, which attracts the polymer in the regions where Φ is large and negative. Simple dimensional analysis shows that for $\gamma < 2$, that is for long-range correlations, the disorder is a relevant perturbation, since λ has engineering dimension $z^{\gamma-2}$. Therefore it is expected that the disorder will always (no matter how small λ is) roughen the polymer, since the gain in energy obtained by visiting regions with large negative Φ, even if they are located at large ϵ, will always overcome the cost in tension energy. The difficult question is to characterize the state(s) of the polymer, in particular the value of the "wandering exponent" ζ which relates the transverse fluctuations of the polymer $(\Delta\epsilon)^2$ to its longitudinal extend x by

$$(\Delta\epsilon)^2 \sim x^{2\zeta}. \tag{6.6}$$

In our case, that is for geodesics in random metrics, the concept of wandering exponent is not covariant but one can look at other observables which are reparametrization invariant and which probe the fluctuations and the correlations between random walks. First the average effective-length of the walk between $\boldsymbol{o}$ and $\boldsymbol{x}$, $\overline{\langle L_{\mathrm{eff}}\rangle}$ (where $\langle \dots \rangle$ and $\overline{\dots}$ denote respectively the average over the path position ϵ and over the metric Φ), corresponds to the average free energy and is found to be, by the very calculation which gives eq. (4.2),

$$\overline{\langle L_{\mathrm{eff}}\rangle} = x + \frac{1}{\beta}\left(-\frac{\sqrt{\pi}}{64} m^{1/2} x^{3/2} + \dots \right) + \dots . \tag{6.7}$$

This means that the path indeed goes through domains where the potential energy $\Phi(z, \epsilon)$ is negative and overcomes the kinetic energy $\dot{\epsilon}^2$. Another interesting quantity is the number of intersections (in the spin-glass terminology the overlap) between two walks. If one considers two independent random walks a and b, the overlap is defined as

$$Q_{ab} = \int_0^x \mathrm{d}z\, \delta(\epsilon_a(z) - \epsilon_b(z)). \tag{6.8}$$

This definition is not covariant but may easily be made covariant without affecting the most divergent terms we are interested in. The calculation is detailed in appendix B. We obtain at one loop

$$\overline{\langle Q_{ab}\rangle} \propto (mx)^{1/2} + (C/\beta)(mx)^2 + \dots, \tag{6.9}$$

with C a positive constant. The classical $(mx)^{1/2}$ term corresponds to the probability for two random walks to have met after x steps. The fluctuation term is much larger. This means that the two random walks are much more correlated. This is consistent with the idea that they are pinned by the regions with large negative Φ, in which they intersect more often. The same conclusion may be drawn from computing the average minimal (geodesic) distance between two paths (which in some sense measures the transverse fluctuations).

In fact, as far as the leading divergence in mr is concerned and if we stay at the semi-classical level in Liouville, we shall see that the exact correlator (3.10) for the Liouville field is equivalent to a local correlator (in z) of the form (6.4), (6.5) with $\gamma = 1/2$! A hint to this fact is provided by the explicit results (6.7), (6.9). Indeed the first-order correction in the disorder strength scales as $mx^{3/2}/\beta$ times the leading term (the x-term in (6.7) is a zero-point energy in (6.3) which has to be subtracted). This means that the disorder strength $1/\beta \sim \lambda$ had dimension $z^{-3/2}$. This is corroborated by the following general argument. We have argued that the divergence comes from the short-distance behavior of the Liouville correlator between two different replica a and b. Such a Liouville correlator can be written as

$$-\ln\left[(x-x')^2+(\epsilon_a(x)-\epsilon_b(x'))^2\right]$$

$$=\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{d}t\frac{t^{s-1}}{\Gamma(s)}\mathrm{e}^{-t((x-x')^2+(\epsilon_a(x)-\epsilon_b(x')^2)}\Big|_{s=0}$$

$$=\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{d}t\frac{t^{s-1}}{\Gamma(s)}\mathrm{e}^{-t(x-x')^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\sqrt{4\pi t}}\mathrm{e}^{iq(\epsilon_a(x)-\epsilon_b(x'))-q^2/4t}\Big|_{s=0}, \quad (6.10)$$

and taking the average over ϵ of the vertex operators $\mathrm{e}^{iq\epsilon}$ leads to some integrand which depends, among other variables, on x and x'. The singular term arises because as m goes to ∞, the region where $x'\simeq x$, q and t are large, so that $t\sim m^2$ and $t(x-x')^2\sim q^2/t\sim \mathrm{O}(1)$ dominates some part of the integrand (see appendix B). This means that we can assume that the term $\mathrm{e}^{iq\epsilon}$ is slowly varying and that the gaussian integration over x' can be performed. We are left with the distribution

$$\simeq\delta(x-x')\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{d}t\frac{1}{2}\frac{t^{s-2}}{\Gamma(s)}\int_{-\infty}^{\infty}\mathrm{d}q\ \mathrm{e}^{-q^2/4t+iq(\epsilon_a(x)-\epsilon_b(x))}\Big|_{s=0}$$

$$=-\delta(x-x')\left|\epsilon_a(x)-\epsilon_b(x')\right|, \quad (6.11)$$

which correspond to the form (6.5) for $\gamma=1/2$.

We have thus obtained a better physical understanding of the non-logarithmic short-distance divergences that appear in the semi-classical calculations. However, as we shall discuss in sect. 7, these results raise new questions about the intrinsic geometry of 2D gravity.

7. Discussion

We end with a list of questions and some remarks.

7.1. RENORMALIZATION OF β

We have seen that at first order in the "Liouville coupling constant" $1/\beta$, the randomness induced by the quantum fluctuations of the metric acts exactly as the randomness of a quenched potential on a directed polymer. For this last system this means that its long-distance properties are not described by the weak-disorder expansion, that the disorder strength is a relevant variable which is renormalized and that is should flow to some non-trivial IR fixed point (see ref. [14] and references therein). In our case the disorder strength is $1/\beta$ which is related to the central charge c of the matter sector by eq. (2.6). Our first-order calculation suggests that $1/\beta$ is also a relevant coupling which has to be renormalized in the IR. In particular it must be stressed that we have not found any local counterterm which could be added to the action (5.6), (2.4) to cancel the divergence (this would have provided a justification for the finite-part regularization of sect. 4). However it is known that β cannot be (and is not) renormalized when computing global observables. It is therefore quite mysterious that β seems to be renormalized, and flows toward the strong-coupling region, when one is interested into local observables.

7.2. RELATION WITH THE 2D RFIM

The fact that the geodesics behave like directed polymers in $1+1$ dimensions with $\gamma = 1/2$ is also very puzzling. Indeed this model should describe (in the SOS approximation, that is neglecting overhangs and handles) the behavior of the interface of the random field Ising model in 2 dimensions at low temperatures [14]. However it is now rigorously proven that the 2D RFIM is disordered at any positive temperature [18]. This corroborates the early argument by Imry and Ma [19] that $D=2$ is the lower critical-dimension for the RFIM, and the scaling arguments which lead to the Flory estimate for the wandering exponent $\zeta^{\mathrm{F}} = (5-D)/3$ [20]. Indeed, for $D \leqslant 2$, $\zeta \geqslant 1$. This means that the SOS approximation is not valid, that rotation invariance is restored and that the notion of interface does not make sense anymore. Of course we cannot draw any conclusion in our case, since the equivalence with the 2D RFIM is valid only to first order in $1/\beta$. However it is very suggestive that already at first order the quantum fluctuations of the metric have such a strong effect.

7.3. REPLICA SYMMETRY BREAKING

This raises a fundamental issue. When defining the geodesic distance through the large-mass limit of the propagator by eq. (5.2), we have made an implicit but very important assumption: The number of distinct geodesics between the two points with length close to the minimal one must not grow too fast with the distance between the points. Otherwise, in addition to the length, the entropy of the geodesics should also contribute to (5.2). This entropy contribution is in fact already present when defining the distance between points through (5.2) on a flat regular lattice (before coupling to gravity). Let us for instance consider the two points $(0, 0)$ and (N, N) on the square lattice (see fig. 1). Their distance in lattice unit calculated with the definition of ref. [8] is $2N$, while the distance calculated through (5.2) is of course $\sqrt{2}N$, as long as we are in the continuum limit $1 \ll m^{-1} \ll N$. The difference comes from the large entropy (of order N) of the

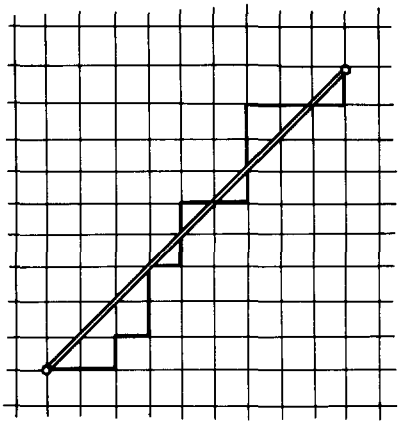


Fig. 1. Typical "microscopic geodesic" (black line) and the "macroscopic" geodesic (double line) between two points on the regular square lattice.

paths with minimal length on the lattice (let us call them "microscopic geodesics"). Our definition is the correct one since the most probable geodesics are located along the diagonal, and there is only one "macroscopic geodesic".

In a random metric, there is the possibility that many different "macroscopic geodesics" exist between two points. This phenomenon should correspond to the occurrence of replica symmetry breaking (RSB) in the replica theory given by (5.6) [21]. RSB might play an essential role for directed polymers. Indeed, Mézard and Parisi have recently rederived the Flory value for the wandering exponent ζ^{F} with a variational Hartree–Fock method [22]. In this approximation they showed that hierarchical RSB does occur in the whole region where disorder is relevant at arbitrarily small temperature, and was essential to recover the Flory exponents.

In our case we cannot conclude anything from the first-order calculation that we have presented here. However we see no reason to exclude the possibility that RSB occurs, since the quantum fluctuations of the metric are already very strong in the semi-classical regime. This can be seen for instance by computing the fluctuations of the overlap between two geodesics. The calculation goes along the same lines as the one of the overlap. We obtain

$$\overline{\langle Q_{ab}Q_{ab}\rangle - \langle Q_{ab}\rangle\langle Q_{ab}\rangle} \propto mx + (D/B)(mx)^{5/2} + \dots, \tag{7.1}$$

with D some positive constant. Thus although quantum fluctuations of the metric increase the mean overlap between two geodesics, they also increase the fluctuations of the overlap. This means that two different paths, although they are close to each other a long part of their journey, may separate themselves for a long time. This picture is of course very crude. It is corroborated by the computation of the fluctuations of the distance between two different paths; quantum fluctuations decrease this mean distance but at the same time increase its fluctuations.

If RSB occurs, we expect (5.2) not to be valid anymore, since it measures the total mean free energy of the geodesics (that is their length minus their entropy). Thus using (5.2) should lead to an underestimate of the physical macroscopic distance between two points, and therefore to a possible overestimate of the intrinsic fractal dimension. However it is not clear if one can define in an unambiguous way the geodesic distance, owing to the possible very complicated structure of the space of macroscopic geodesics. Perhaps this problem can be solved by introducing explicitly terms which break replica symmetry in the action (5.6) [23]. It is also not clear which definition can be used in the numerical simulation. It would be of course very interesting to test those ideas in numerical simulations (although the experience for the 3D SK spin glass shows that this should be quite difficult).

7.4. WORMHOLES IN LIOUVILLE THEORY

An important question is to understand if "wormhole" configurations are important in the functional integral over metric. Indeed one expects that such configurations would be important if there is a proliferation of baby-universes, as advocated in ref. [8]. Let us present a very crude, but simple, estimate of the importance of such configuration in the Liouville theory.

Let us consider the following configuration, depicted on fig. 2. A "baby universe" with constant positive curvature R_{baby} is sewed to a flat "parent universe" through a small bottleneck with radius ϵ (see fig. 2). The corresponding

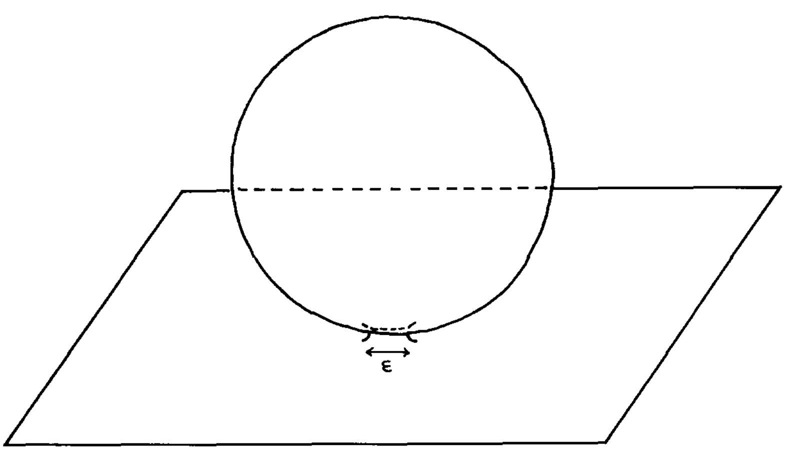


Fig. 2. A baby-universe with constant positive curvature R connected to euclidean flat space through a wormhole with diameter ϵ.

metric $g = \exp(\Phi)$ is

$$\Phi(x) = \begin{cases} 0 & \text{if} \quad |x| \geqslant \epsilon \\ -2\ \ln\left(\dfrac{1+\rho|x|^2}{1+\rho\epsilon^2}\right) & \text{if} \quad |x| < \epsilon, \end{cases} \tag{7.2a}$$

with

$$R_{\text{baby}} = \frac{8\rho}{\left(1+\rho\epsilon^2\right)^2} \simeq \frac{8}{\rho\epsilon^4}. \tag{7.2b}$$

The Liouville action (2.4) for this configuration is for small ϵ

$$S_{\text{baby}} = \beta\left(\ln\left(1+\rho\epsilon^2\right) - \frac{\rho\epsilon^2}{1+\rho\epsilon^2}\right) \simeq -\beta\ \ln\left(R_{\text{baby}}\epsilon^2\right) \tag{7.3}$$

and is divergent for $\epsilon = 0$. We want to use a Kosterlitz–Thouless-like argument to estimate the probability to have such a baby-universe *with fixed internal volume* $A_{\text{baby}} \simeq 8\pi/R_{\text{baby}}$. We estimate the free energy as (7.3) minus the entropy of the baby-universe, which is of order $\ln(A_{\text{parent}}/\epsilon^2)$, where A_{parent} is the volume of the parent universe. Therefore the free energy diverges with ϵ as

$$F_{\text{baby}} \simeq (\beta - 1)\ \ln\left(1/\epsilon^2\right) \tag{7.4}$$

As long as $\beta > 1$, that is $c < 1$, it is large and positive. This crude calculation suggests that in the whole weak-coupling phase such configurations (baby-universes connected to the parent-universe by a microscopic small "wormhole") are suppressed by a power-like factor in the functional integral over metric. One can repeat the same argument when comparing the probability to have a large universe with volume A and curvature $R \sim 8\pi/A$ with that of having two universes with volumes $\sim A/2$ and curvatures $\sim 2R$ connected by a small wormhole. As long as $c < 1$ the probability to split vanishes in this approximation.

This heuristic argument suggests that the $c = 1$ barrier can be interpreted as the onset of the liberation of baby-universes in 2D gravity. This is consistent with the conjecture that for $c > 1$ branched configurations dominate the functional integral over metrics (with fixed topology). Our argument might be related to the one of Cates [24] who interprets the $c = 1$ barrier as the onset of liberation of singular spikes with deficit angle 2π. However he does not take into account the renormalization of the puncture operator $\exp(\Phi)$ (this amounts to take $\beta = (25 - c)/6$ instead of (2.6) in (2.4)). With the correct normalization and following his argument one finds that "spikes with deficit angle 4π" are liberated at $c = 1$. But a small wormhole connecting two flat universes can be viewed as such a pathological spike.... Our argument may also be viewed as a poor man's version of the one of ref. [25] which states (for a lattice random surface model) that branching occurs as soon as $\gamma_{\text{string}} > 0$. Of course much work is needed to see if this argument can be made more rigorous. In particular it would be interesting to understand if it applies for the theories with $c < 1$ but with "dangerous non-local operators" discussed by Seiberg [26] (we have assumed that there is no particular coupling between the matter sector and the metric at the wormhole; this is probably correct for simple unitary matter such as $n = c$ massless bosonic fields) or for the theories with $c > 1$ discussed by Kutasov and Seiberg [27].

7.5. INTRINSIC DIMENSION OF REAL BRANCHED POLYMERS

Finally let us end with a simple remark, but based on an exact result. In ref. [8] it is argued that an infinite intrinsic dimension is characteristic of a branched polymer phase. In fact the intrinsic fractal dimension of a branched polymer is 2! The simplest, but indirect, argument consists in considering a gaussian branched polymer, and neglecting self-interaction (this is certainly valid if $D > 6$). Then it is known that the Hausdorff dimension of the polymer, defined by the average distance in physical space between two points, is 4 [28], which means that

$$\langle (r_i - r_j)^2 \rangle \propto N^{1/2}, \tag{7.5}$$

where N is the number of elements of the polymer and i and j two random points. Let d be the average number of links between those two points. The average distance in physical space depends only on d, since all the contributions of the branches factorize. What remains is a linear gaussian polymer with length d between i and j. Therefore

$$\langle (r_i - r_j)^2 \rangle \propto d. \tag{7.6}$$

Thus the average distance on the polymer between two points scales with its volume as $\langle d \rangle \propto N^{1/d_{\text{intr}}}$ and we get $d_{\text{intr}} = 2$. Of course some other quantities, such as the spectral dimension d_s, differ for true branched polymers ($d_s = 4/3$) and for 2D quantum gravity in the weak coupling phase ($d_s = 2$).

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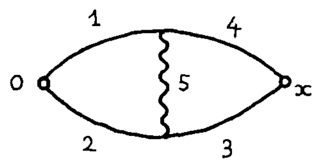
Appendix A

In this Appendix we give the details of the derivation of eq. (5.9). In the Schwinger representation the amplitude for the graph



is given by

$$I(x,m)=\frac{1}{(4\pi)^3}\int_0^\infty \mathrm{d}\alpha \frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{(4\alpha_5)^{-\epsilon}}{\Gamma(\epsilon)}N^{-1}\,\mathrm{e}^{-m\Sigma-x^2P/4N}\big|_{\epsilon=0} \tag{A.1}$$

where Σ, P and N are the usual Symanzik polynomials

$$\Sigma=\alpha_1+\alpha_2+\alpha_3+\alpha_4,$$

$$P=\alpha_5(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+(\alpha_1+\alpha_2)(\alpha_3+\alpha_4),$$

$$N=\alpha_5(\alpha_1+\alpha_4)(\alpha_2+\alpha_3)+(\alpha_1\alpha_2\alpha_3+\alpha_2\alpha_3\alpha_4+\alpha_3\alpha_4\alpha_1+\alpha_4\alpha_1\alpha_2). \tag{A.2}$$

With the change of variable

$$\alpha_1=\sigma_1 u,\qquad \alpha_4=\sigma_1(1-u),$$

$$\alpha_2=\sigma_2 v,\qquad \alpha_3=\sigma_3(1-v),$$

$$N'=\alpha_5+\sigma_1 u(1-u)+\sigma_2 v(1-v), \tag{A.3}$$

we get

$$\frac{1}{(4\pi)^3}\int_0^\infty \mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\int_0^1 \mathrm{d}u\,\mathrm{d}v\int_0^\infty \mathrm{d}\alpha_5\frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{(4\alpha_5)^{-\epsilon}}{\Gamma(\epsilon)}N'^{-1}$$

$$\times\,\mathrm{e}^{-(m^2\sigma_1+x^2/4\sigma_1)}\,\mathrm{e}^{-(m^2\sigma_2+x^2/4\sigma_2)}\,\mathrm{e}^{-x^2(u-v)^2/4N'}\big|_{\epsilon=0}. \tag{A.4}$$

We represent the last exponential as

$$\mathrm{e}^{-x^2(u-v)^2/4N'}=\int_{-i\infty}^{i\infty}\frac{\mathrm{d}s}{2i\pi}\left(\frac{x^2(u-v)^2}{4N'}\right)^{-s}\Gamma(s), \tag{A.5}$$

and we perform the integration over α_5 to get

$$\frac{1}{(4\pi)^3}\int_0^\infty d\sigma_1\, d\sigma_2 \int \frac{ds}{2i\pi}\int_0^1 du\, dv\; e^{-(m^2\sigma_1 + x^2/4\sigma_1)}\, e^{-(m^2\sigma_2 + x^2/4\sigma_2)}$$

$$\times \frac{d}{d\epsilon} 4^{s-\epsilon}(x\,|u-v|)^{-2s} \frac{\Gamma(1-\epsilon)\Gamma(\epsilon - s)\Gamma(s)}{\Gamma(1-s)\Gamma(\epsilon)}$$

$$\times \left[\sigma_1 u(1-u) + \sigma_2 v(1-v)\right]^{s-\epsilon}\Big|_{\epsilon=0} \tag{A.6}$$

(the integration over the imaginary part of s has to be done for $0 < \mathrm{Re}\ s < \epsilon$)

We can now study the large-m limit. As $m \to \infty$ the σ integral becomes dominated by the saddle point $\sigma_1 = \sigma_2 = x/2m$. If we perform the σ integration by the saddle point method, we obtain

$$\frac{1}{(4\pi)^3}\frac{\pi x}{2m^3} e^{-2mx} \int \frac{ds}{2i\pi}\int du\, dv$$

$$\times \frac{d}{d\epsilon}\left(\frac{2}{m}\right)^{s-\epsilon} x^{-(s+\epsilon)} \frac{\Gamma(1-\epsilon)\Gamma(\epsilon-s)\Gamma(s)}{\Gamma(1-s)\Gamma(\epsilon)}$$

$$\times |u-v|^{-2s}(u(1-u) + v(1-v))^{s-\epsilon}. \tag{A.7}$$

Integrating over u and v we obtain a meromorphic function of s with a series of poles along the positive real axis. Some of them come from the Γ functions in (A.7), the others from the divergence of the u and v integrations if s is large enough. Now if one integrates over s from $-i\infty$ to $+i\infty$, shifting the contour of integration to the right and picking a pole at $s = s_0$ gives a residue of order $(xm)^{-s_0}$. Starting from $0 < \mathrm{Re}\ s < \epsilon$ the first pole is at $s = \epsilon$ and comes from the function $\Gamma(\epsilon - s)$. Its residue gives the leading term of the large-m limit of (A.1). Taking the derivative with respect to ϵ and setting ϵ to zero gives finally

$$I_0(x, m) = \frac{1}{128\pi^2} x m^{-3}\, e^{-2xm}(-2\ln(x) + 3). \tag{A.8}$$

Note that it is less than 0 at large x because the Liouville propagator is proportional to $-\ln(x)$ and hence less than 0 at large x.

The second pole is at $s = 1/2$ and comes from the u and v integration. Indeed, the integral diverges along $u = v$ because of the $|u-v|^{-2s}$ term as soon as $s \geqslant 1/2$. The residue gives for the sub-leading term

$$I_{1/2}(x, m) = -\frac{1}{256\sqrt{\pi}} x^{1/2} m^{-7/2}\, e^{-2xm}. \tag{A.9}$$

The other poles, as well as the terms neglected in the σ integration, contribute only at order $m^{-4}\exp(-2mx)$. Combining with the large-m behavior of the scalar propagator

$$\frac{1}{4\pi}\left(\frac{2\pi}{xm}\right)^{1/2} e^{-xm} \tag{A.10}$$

gives eq. (5.9). The same technique allows us to study the graph

and leads to eq. (5.8). One sees in this calculation that the fluctuations of the proper-times σ_1 and σ_2 of the two "replica" particles 1 and 2 are not important and that the singular term occurs for $u = v$, that is when the two particles have the same intermediate proper-time $\alpha_1 = \alpha_2$.

Appendix B

In this appendix we detail the calculation of the overlap Q_{12} between two walks (labelled here by 1 and 2) to first order in $1/\beta$. We have to introduce two sets of replica ϵ_a^1 and ϵ_a^2 ($a = 1, n$). The v.e.v. for the overlap can now be written as

$$\langle Q_{12}\rangle = \lim_{n\to 0}\left\langle \frac{1}{n}\sum_{c=1}^{n}\int_0^x \mathrm{d}z\, \delta\big(\epsilon_c^1(z) - \epsilon_c^2(z)\big)\right\rangle. \tag{B.1}$$

Integrating over the Liouville field Φ we obtain the replica action which is at leading order

$$S_{\mathrm{rep}} = \frac{m}{2}\int_0^x \mathrm{d}z \sum_{a=1}^{n}\sum_{\alpha=1}^{2}(\dot\epsilon_a^\alpha)^2$$
$$-\frac{m^2}{8}\int_0^x \mathrm{d}z_1\,\mathrm{d}z_2 \sum_{a,b=1}^{n}\sum_{\alpha,\beta=1}^{2}\left(-\frac{2}{\beta}\right)\ln\Big[(z_1-z_2)^2 + \big(\epsilon_a^\alpha(z_1) - \epsilon_b^\beta(z_2)\big)^2\Big]. \tag{B.2}$$

Representing the δ function in eq. (A.1) by a Fourier transform

$$\delta\big(\epsilon_c^1 - \epsilon_c^2\big) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}\lambda}{2\pi}\mathrm{e}^{i\lambda(\epsilon_c^1-\epsilon_c^2)} \tag{B.3}$$

and using the representation (6.10) for the interaction term we obtain finally for the term of order $1/\beta$ of eq. (B.1) a representation as a sum of exponentials of linear combinations of the ϵ's, with coefficients which are linear in λ and q. Applying Wick's theorem to average over the ϵ's we obtain exponentials of quadratic forms in λ and q. Performing the gaussian integration we are left with

$$\frac{1}{n}\sum_{c=1}^{n}\sum_{a,b=1}^{n}\sum_{\alpha,\beta=1}^{2}\int_0^x \mathrm{d}z\,\mathrm{d}z_1\,\mathrm{d}z_2$$

$$\times\frac{\mathrm{d}}{\mathrm{d}s}\int \mathrm{d}t\frac{t^{s-3/2}}{\Gamma(s)}\frac{\sqrt{\pi}}{2}\mathrm{e}^{-t(z_1-z_2)^2}\left(\det\left[M^{\alpha\beta}_{abc}(z_1, z_2, z)\right]\right)^{-1/2}, \quad \text{(B.4)}$$

where $M^{\alpha\beta}_{abc}(z_1, z_2, z)$ is a 2×2 matrix of the form

$$\begin{pmatrix} (1/4t)+\frac{1}{2}([z_1,z_1]+[z_2,z_2]-2\delta_{ab}\delta^{\alpha\beta}[z_1,z_2]) & \frac{1}{2}(\delta_{ac}\sigma^{\alpha}[z_1,z]-\delta_{bc}\sigma^{\beta}[z_2,z]) \\ \frac{1}{2}(\delta_{ac}\sigma^{\alpha}[z_1,z]-\delta_{bc}\sigma^{\beta}[z_2,z]) & [z,z] \end{pmatrix}, \quad \text{(B.5)}$$

where $\sigma^1 = 1$, $\sigma^2 = -1$ and where $[z_1, z_2]$ is the massless 1D propagator with Dirichlet b.c. on the interval $[0, x]$, namely

$$[z_1, z_2] = (1/m)\left[z_1(1-z_2/x)\theta(z_2-z_1) + z_2(1-z_1/x)\theta(z_1-z_2)\right]. \quad \text{(B.6)}$$

There are four different contributions in (B.4), depending on the values of a, b and c. Their respective weights are $a=b=c\to n$, $a=b\neq c\to n(n-1)$, $a=c\neq b$ or $b=c\neq a\to 2n(n-1)$, $a\neq b\neq c\neq a\to n(n-1)(n-2)$.

We now look at the large-m limit. Since the 1D propagator is of order m^{-1}, from (B.5) t has to be rescaled as $t\to mt$. Then the integrand in (B.4) scales as a power of m, but for the gaussian term $\exp(-mt(z_1-z_2)^2)$. It becomes peaked around the region $z_1 = z_2$, which is precisely the singular region where the interaction between the replica is singular. Therefore we keep only the contribution of the region $z_1 \simeq z_2$ and we perform explicitly the gaussian integration $\int \mathrm{d}z_2 \exp(-mt(z_1-z_2)^2)$ in (B.5). Writing explicitly the sum over α and β we are left with an integral of the form

$$\int_0^x \mathrm{d}z_1\,\mathrm{d}z\int_0^\infty \mathrm{d}t\frac{\pi}{t^2}\left(\begin{vmatrix}(1/4t)+[z_1,z_1] & [z_1,z] \\ [z_1,z] & [z,z]\end{vmatrix}^{-1/2}\right.$$

$$-4\begin{vmatrix}(1/4t)+[z_1,z_1] & \frac{1}{2}[z_1,z] \\ \frac{1}{2}[z_1,z] & [z,z]\end{vmatrix}^{-1/2}$$

$$\left.+3\begin{vmatrix}(1/4t)+[z_1,z_1] & 0 \\ 0 & [z,z]\end{vmatrix}^{-1/2}\right). \quad \text{(B.7)}$$

We have already taken the derivative with respect to s and set $s=0$. Finally we can perform the t integration explicitly. We obtain

$$\int_0^x \mathrm{d}z_1 \, \mathrm{d}z \; 8\pi \left(\frac{[z_1, z_1]}{[\overline{z, z}]} \right)^{1/2} \left(4(1-F/4)^{1/2} - (1-F)^{1/2} - 3\right), \qquad \text{(B.8)}$$

with

$$F = \frac{[z_1, z]^2}{[z_1, z_1][z, z]}, \qquad \text{(B.9)}$$

and one can check that the integrand is always larger than 0, so that (B.8) gives x^2 times some positive constant. Reminding that the interaction term is proportional to $m^3/4\beta$ we have thus obtained eq. (6.9).

References

[1] J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B257 (1985) 433
F. David, Nucl. Phys. B257 (1985) 45
V.A. Kazakov, Phys. Lett. 150B (1985) 282; V.A. Kazakov, I.K. Kostov and A.A. Migdal, Phys. Lett. B157 (1985) 295

[2] E. Brézin and V.A. Kazakov, Phys. Lett. B236 (1990) 2125;
M.R. Douglas and S.H. Shenker, Nucl. Phys. B335 (1990) 635;
D.J. Gross and A.A. Migdal, Phys. Rev. Lett. 64 (1990) 27

[3] Random surfaces, quantum gravity and strings, Proc. Cargèse Workshop, 1990, eds. O. Alvarez, E. Marinari and P. Windey, to appear

[4] V. Knizhnik, A. Polyakov and A. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819

[5] F. David, Mod. Phys. Lett. A3 (1988) 1651;
J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509

[6] M. Goulian and M. Li, Phys. Rev. Lett. 66 (1991) 2051;
P. Di Francesco and D. Kutasov, Correlation functions in 2D string theories Princeton preprint PUPT-1237 (1991);
V.S. Dotsenko, Three-points correlation functions of the minimal conformal theories coupled to 2D gravity, preprint LPTHE-91-18 (1991)

[7] E. Witten, Nucl. Phys. B340 (1990) 281;
J. Distler, Nucl. Phys. B342 (1990) 523;
R. Dijkgraaf and E. Witten, Nucl. Phys. B342 (1990) 486;
R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 435;
E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 457

[8] M. Agishtein and A.A. Migdal, Int. Jour. Mod. Phys. C1 (1990) 165;
M. Agishtein, J. Richardson, L. Jacobs and A.A. Migdal, Mod. Phys. Lett. A5 (1990) 12;
M. Agishtein and A.A. Migdal, Nucl. Phys. B350 (1991) 690;

[9] T.L. Curthright and C.B. Thorn, Phys. Rev. Lett. 48 (1982) 1309; Ann. of Phys. (N.Y.) 147 (1983) 365; 153 (1983) 147;
J.L. Gervais and A. Neveu, Nucl. Phys. B238 (1984) 125; 396

[10] A. Gupta, S. Trivedi and M. Wise, Nucl. Phys. B340 (1990) 475

[11] S. Elitzur, Nucl. Phys. B212 (1883) 501;
F. David, Commun. Math. Phys. 81 (1981) 149; Phys. Lett. B96 (1980) 371

[12] C. de Calan and A.P.C. Malbouisson, Ann. Inst. Poincaré 32A (1980) 91;
C. de Calan, F. David and V. Rivasseau, Commun. Math. Phys. 78 (1981) 531
[13] M. Kardar and Y.C. Zhang, Phys. Rev. Lett. 58 (1987) 2087;
[14] T. Natterman and J. Villain, Phase Transitions 11 (1988) 5
[15] M. Kardar, G. Parisi and Y.C. Zhang, Phys. Rev. Lett. 56 (1986) 889
[16] D. Forster, D.R. Nelson and M.J. Stephen, Phys. Rev. A16 (1977) 732
[17] D. Derrida and H. Spohn, J. Stat. Phys. 51 (1988) 817
[18] M. Aizenmann and J. Wehr, Commun. Math. Phys. 130 (1990) 489
[19] Y. Imry and S.-K. Ma, Phys. Rev. Lett. 35 (1975) 1399
[20] J. Villain, J. Physique Lett. 43 (1982) L551;
G. Grinstein and S.K. Ma, Phys. Rev. B28 (1983) 2588
[21] M. Mézard, G. Parisi and M. Virasoro, Spin glass theory and beyond (World Scientific, Singapore, 1987)
[22] M. Mézard and G. Parisi, J. Phys. A23 (1990) L1229; Replica field theory for random manifolds, preprint LPTENS (1990)
[23] G. Parisi and M.A. Virasoro, J. Physique 50 (1989) 3317
[24] M.E. Cates, Phys. Lett. B161 (1988) 719;
A. Krzywicki, Phys. Rev. D41 (1990) 3086
[25] B. Durhuus, J. Fröhlich and T. Jonsson, Nucl. Phys. B240 (1984) 453
[26] N. Seiberg, Notes on quantum Liouville theory and quantum gravity, *in* Random surfaces, quantum gravity and string, Proc. Cargèse Workshop, 1990
[27] D. Kutasov and N. Seiberg, Phys. Lett. B251 (1990) 67
[28] P.H. Zimm and W.H. Stockmayer, J. Chem. Phys. 17 (1949) 1301;
P.G. de Gennes, Biopolymers 6 (1968) 175