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NON-PERTURBATIVE EFFECTS IN 2D GRAVITY AND MATRIX MODELS *

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1. Introduction

Two dimensional Euclidean quantum gravity may be formulated as a functional integral over 2–dimensional Riemannian manifolds. This infinite dimensional integral may be discretized in such a way that the topological expansion in terms of the genus of the manifold is mapped onto the $1/N$ expansion of some zero–dimensional matrix model [1]. The $N = \infty$ limit exhibits critical points which can be shown to describe the continuum limit of 2–dimensional gravity on a genus zero manifold, eventually coupled to some matter fields. Recently it was shown that a scaling limit can be constructed [2] . In this limit all the terms of the topological expansion survive and thus one obtains a fully non–perturbative solution for two dimensional gravity. However in the most interesting cases, in particular for pure gravity, the solution is defined as a solution of a non–linear differential equation of the Painlevé type and presents some non–perturbative ambiguities, related to the delicate issue of boundary conditions, which are usually attributed to some "non–perturbative effects" of the theory.

In this talk I shall review some attempts to get a better understanding of these effects. For simplicity and shortness I shall mainly deal with the case of pure gravity, which seems to embody the main problems. The approach that I have followed consists in trying to relate those non–perturbative issues to the non–perturbative effects which are present in the original matrix models.

2. The Scaling Limit

For completeness and in order to have consistent notations, let us recall explicitly how the scaling limit is obtained. We define the partition function for the Hermitian

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one matrix model as

$$
\begin{aligned}
Z \; &= \; \int d\Phi \; \mathrm{e}^{-N \; \mathrm{tr}(V(\Phi))} \\
&\propto \; \int \prod_{i=1}^{N} \; d\mu(\lambda_i) \; \Big(\prod_{i<j} (\lambda_i - \lambda_j) \Big)^2
\end{aligned}
\tag{2.1}
$$

where $d\mu(\lambda) = d\lambda \, \mathrm{e}^{-N\,V(\lambda)}$. Introducing orthonormal polynomials π_n with respects to $d\mu$

$$
\langle n | m \rangle \; = \; \int d\mu \; \pi_n \; \pi_m \; = \; \delta_{nm} \tag{2.2}
$$

one obtains by the standard manipulations the expression for the vacuum energy

$$
F \; = \; \ln \; Z \; \simeq \; \sum_{i=0}^{N-1} \; (N - i) \; \ln(b_i) \tag{2.3}
$$

where the coefficients b_i are related to the matrix elements of the operator Q of multiplication by λ over the π_n's

$$
\lambda \; \pi_n \; = \; Q_{nm} \; \pi_m \; = \; \sqrt{b_{n+1}} \; \pi_{n+1} \; + \; a_n \; \pi_n \; + \; \sqrt{b_n} \; \pi_{n-1} \tag{2.4}
$$

We shall consider the simplest case of the cubic potential, which can be written as

$$
V(\lambda) \; = \; g\lambda \; - \; \frac{\lambda^3}{3} \tag{2.5}
$$

From the relation $P + P^t \; = \; N \; V'(Q)$, where P is the operator $\frac{\partial}{\partial \lambda}$, one gets the recursion relations

$$
\begin{aligned}
0 \; &= \; g \; - \; (a_n^2 + b_n + b_{n+1}) \\
\frac{n}{N} \; &= \; - \; b_n \; (a_n + a_{n+1})
\end{aligned}
\tag{2.6}
$$

The large N limit is obtained by taking the continuum limit

$$
n/N \; \to \; x \quad ; \quad a_n \; \to \; a(x) \quad ; \quad b_n \; \to \; b(x) \tag{2.7}
$$

in the recursion relations (2.6). The critical point occurs when $a(x)$ and $b(x)$ becomes singular at $x = 1$. In our case this gives

$$
g_c \; = \; 3 \; 2^{-2/3} \quad , \quad a(1) \; = \; a_c \; = \; -2^{-1/3} \quad , \quad b(1) \; = \; b_c \; = \; 2^{-2/3} \tag{2.8}
$$

The scaling limit is obtained by rescaling by adequate powers of N

$$
\begin{aligned}
g \; &= \; g_c \; \big(1 + a^2 \, t\big) \qquad & n \; &= \; N \; \big(1 - a^2 \, x\big) \\
a_n \; &= \; a_c \, (1 - a \, v) \qquad & b_n \; &= \; b_c \, (1 - a \, u)
\end{aligned}
\tag{2.9}
$$

and by taking the limit

$$
N \; \to \; \infty \quad , \quad a^{5/2} N \; = \; \gamma^{-1}, \; x \text{ and } t \text{ fixed} \tag{2.10}
$$

γ is here the "string coupling constant" and can be completely absorbed in the normalization. Therefore it will be set to unity. After expanding (2.9) in (2.6) one obtains that $v(x,t) = -u(x,t)$ and that $u(x,t)$ satisfies the "string equation"

$$ -\frac{1}{6}\frac{\partial^2 u}{\partial x^2} + u^2 = \frac{2}{3}x + t \tag{2.11} $$

which is nothing but the Painlevé I equation. From (2.3) the finite part of F in the scaling limit, $F(t)$, is equal to $-\int_0^\infty dx\; x\; u$, and therefore the "susceptibility" $f(t)$ is given by

$$ f(t) = F''(t) = -(3/2)^2\, u(0,t) \tag{2.12} $$

and obeys also a Painlevé I equation. Finally the operator Q given by (2.4) becomes the differential operator [3]

$$ Q = 2\, b_c^{1/2}\left(N^{2/5} + d^2 - 2u\right) \tag{2.13} $$

where $d = \partial/\partial x$. Using the free fermion formalism of [4] expectation values of operators in the original matrix model may be expressed as v.e.v. of one body operators for a system on N free fermions with Fock space generated by the one particle states $|n\rangle = \pi_n(\lambda)$. If one starts from the "loop operator" $W(\lambda)$, which is defined as

$$ W(\lambda) = N\;\mathrm{Tr}\left(\frac{1}{\lambda-\Phi}\right) \sim \Psi\,\frac{1}{\lambda - Q}\,\Psi^\dagger \tag{2.14} $$

(where $\Psi^\dagger$ and Ψ are the fermion field operators), the finite part of W in the scaling limit, $w(p)$, is defined by the rescaling

$$ w(p) \simeq 2\, b_c^{1/2}\, W(\lambda) \quad ; \quad \lambda = \lambda_c\left(1 + N^{-2/5}p\right) \quad ; \quad \lambda_c = 2\, b_c^{1/2} = 2^{-1/3} \tag{2.15} $$

The explicit expressions for the one- and two-loop v.e.v. are in the scaling limit (2.9)

$$ \langle w(p)\rangle = \int_0^\infty dx\; \langle x|\frac{1}{p - d^2 + 2u}|x\rangle \tag{2.16} $$

$$ \langle w(p)w(q)\rangle = \int_0^\infty dx \int_{-\infty}^0 dy\; \langle x|\frac{1}{p-d^2+2u}|y\rangle\langle y|\frac{1}{q-d^2+2u}|x\rangle \tag{2.17} $$

In the large t (or equivalently large x) limit u should fit with the large N solution and therefore should behave as $+t^{1/2}$. The problem is that equation (2.11) admits an infinite family of real solutions with this behavior as $x \to +\infty$. Moreover any such solution must have an infinite number of double poles on the negative real axis (see for instance [5]. The Laurent expansion around each pole x_0 is of the form

$$ u(x) = (x-x_0)^{-2} + o((x-x_0)^2) \tag{2.18} $$

and therefore from (2.12) each pole of f corresponds to a simple zero of the partition function Z. Two solutions of (2.11) have the same large t asymptotic expansion x

$$ f(t) = (3/2)^2\, t^{1/2} - \sum_{k=1}^{\infty} f_k\; t^{(1-5k)/2} \tag{2.19} $$

but differ by the position of (for instance) their largest pole. Linearizing (2.11) it is easy to see that the difference between two solutions behaves asymptotically as

$$\delta f \ \sim \ t^{-1/8} \ \exp\Big(- \frac{4 \ 3^{3/2}}{5} \ t^{5/4} \Big) \tag{2.20}$$

and is therefore exponentially small in the "string coupling constant" $t^{-5/2}$ [6]. This can be related to the fact that the coefficients f_k in (2.19) grow like $(2k)!$ and that the series (2.19) is not Borel summable [7].

Another (but related) problem occurs in the definition of the resolvent $\langle x|(p - d^2 + 2u)^{-1}|y\rangle$. The operator $-d^2 + 2u$ is not defined on the whole real axis because of the poles. A somewhat natural choice, proposed for instance in [4], consists in defining this operator between the largest pole x_0 and $+\infty$. Indeed viewing this operator as the Hamitonian of a particle in the potential u, the potential diverges enough at each pole to prevent tunnelling between the different "sectors". In other word one defines the resolvent by imposing that it vanishes at x_0 and $+\infty$ and plug it into the definition of the correlation functions (2.16),(2.17).

3. Loop Equations

An alternative approach starts from the loop operator $W(\lambda)$ defined by (2.14) or its inverse Laplace transform

$$W(L) \ = \ N \ \mathrm{tr}\big(\mathrm{e}^{L\Phi}\big) \tag{3.1}$$

which corresponds (moreless) to the operator creating a hole (macroscopic loop) with length L in the two-dimensional worls sheet. The loop equations are the Schwinger–Dyson equations for the matrix model (2.1) and are derived simply by performing the change of variable $\Phi \to \Phi + \epsilon f(\Phi)$ in (2.1) (where $f(z)$ is an analytic function). The Jacobian for this change of variable is

$$J \ = \ 1 + \epsilon \oint \frac{dz}{2i\pi} \, f(z) \, \left(\mathrm{tr} \left(\frac{1}{z - \Phi} \right) \right)^2 + o\big(\epsilon^2\big) \tag{3.2}$$

From (3.2) one obtains easily the loop equation [8]

$$N^2 \, V'\Big(\frac{\partial}{\partial L}\Big) \, \langle W(L) \rangle \ = \ \int_0^L dL' \, \Big\{ \langle W(L') \rangle \langle W(L-L') \rangle + \langle W(L') W(L-L') \rangle \Big\} \tag{3.3}$$

or by Laplace transform

$$N^2 \, \big[V'(\lambda) \, \langle W(\lambda) \rangle \big]_< \ = \ \langle W(\lambda) \rangle^2 + \langle W(\lambda)^2 \rangle \tag{3.4}$$

where $[\ \ ]_<$ means the truncation to the powers λ^n with $n < 0$ in the Laurent expansion around $\lambda = \infty$. Including a source term for the loop operators W in the potential V one sees that the loop equation contains implicitely the infinite set of equations of motion for v.e.v. with an arbitrary number of loop operators. Those equations allows to compute recursively the correlation functions at all orders in the topological expansion.

The loop equation (3.4) takes a very simple form in the double scaling limit [9] . Indeed, defining the finite part of $W(\lambda)$ as

$$w(p) \;=\; \lambda_c \left(W(\lambda) \,-\, \frac{1}{2}\, V'(\lambda) \right) \tag{3.5}$$

and using (2.8), (2.9) and (2.15), (3.4) becomes

$$\langle w(p) \rangle^2 \,+\, \langle w(p) w(p) \rangle \;=\; \frac{1}{4}\, p^3 \;-\; \frac{3}{4}\, t\, p \,+\, \frac{1}{3}\, \langle P \rangle \tag{3.6}$$

where $\langle P \rangle$ is the v.e.v. of the "puncture operator" $P = -\partial/\partial t$ and depends only on the "renormalized cosmological constant" t. The scaling limit of loop equations involving more loop operators can be obtained in a similar way. For instance we have

$$2\langle w(p) \rangle \langle w(p) w(q) \rangle \,+\, \langle w(p)^2 w(q) \rangle \,+\, \frac{\partial}{\partial q} \left(\frac{\langle w(q) \rangle - \langle w(p) \rangle}{q \,-\, p} \right) \;=\; \frac{1}{3}\, \langle P\, w(q) \rangle \tag{3.7}$$

Those equations can be used to compute recursively (in the topological expansion) correlation functions in the scaling limit (see [10]

The interest of the loop equations is not merely calculational. In [11] and in [12] it was indeed shown that the loop equations can be written as recursion relations which follow from the string equation (2.11) , and from the fact that the partition function is the so-called τ-function of the corresponding KdV hierarchy. Moreover those recursion relations can also be obtained from the formulation of 2-d gravity as a topological fiels theory [13] . Therefore the three approaches (topological gravity, KdV hierarchy and loop equations) are equivalent, at least to all orders of the topological expansion. Let us show for instance explicitely the connection between (3.6) and the results of [11]. One can easily show that $w(p)$ has a large p expansion in powers $p^{-3/2-n}$, with $n \leq 0$, excepted for the one- and two-loops correlators. Defining the "finite part" $\tilde{w}$ of w as its $O(p{-}3/2))$ part, we get explicitely

$$\begin{aligned} \langle w(p) \rangle \;&=\; \frac{1}{2}\, p^{3/2} \;-\; \frac{3}{4}\, t\, p^{-1/2} \,+\, \langle \tilde{w}(p) \rangle \\ \langle w(p) w(p) \rangle \;&=\; \frac{1}{16}\, p^{-2} \,+\, \langle \tilde{w}(p) \tilde{w}(p) \rangle \end{aligned} \tag{3.8}$$

From (3.6) we get, if we perform the rescaling $t \to 2/3\, t$

$$\left[\left(p^{3/2} \,-\, t\, p^{-1/2} \right) \langle \tilde{w} \rangle \right]_{<} \,+\, \langle \tilde{w} \rangle^2 \,+\, \langle \tilde{w} \tilde{w} \rangle \,+\, \frac{1}{16\, p^2} \,+\, \frac{t^2}{4\, p} \;=\; 0 \tag{3.9}$$

This is[1] Eq. (2.14) of [11] if we identify $p^{3/2} \,-\, t\, p^{-1/2}$ with the derivative of the $m = 2$ singular potential $V'(p)$, and if we shift $\langle \quad \rangle \to \frac{1}{2} \langle \quad \rangle$ to take into account the "doubling phenomenon" which occurs in matrix models with even potential (see [13] and [14]), which fix the normalization used in [11] for the KdV hierarchy.

[1] up to a factor 2 in the p^{-2} term, for which we have no explanation

One may however expect that the loop equations, which are the equations of motion for two dimensional gravity, and which have a simple and appealing geometrical interpretation in term of fusion and splitting of loops [8], are valid beyond perturbation theory. This is indeed what occurs in ordinary field theories: non-perturbative effects might change the v.e.v. of some operators but they do not affect the general form of the equations of motion. For pure gravity the equation (3.6) puts very strong constraints on the non-perturbative solutions, and in fact excludes all the real solutions discussed in the previous section. Indeed, if we start from a real solution of (2.11), and if we define the loop correlators by (2.16), (2.17), with the resolvent defined through the operator $Q = d^2 - 2u$ with support between the largest real pole x_0 of u and $+\infty$, the operator Q has a discrete spectrum ($e_0 > e_1 > e_2 > \ldots$), and therefore the resolvent $\langle x|(p-Q)^{-1}|y\rangle$ is a meromorphic function of p with simple poles located on the spectrum of Q. A straightforward calculation shows that the l.h.s. of (3.6) has then double poles with non-vanishing residues. For instance near the first pole we have

$$\langle w(p)\rangle^2 \,+\, \langle w(p)^2\rangle \;\simeq\; \frac{1}{(p\,-\,e_0)^2}\int_0^\infty dx\,|\psi_0(x)|^2 \tag{3.10}$$

where ψ_0 is the eigenfunction ($Q\psi_0 = e_0\psi_0$). This obviously contradicts (3.6), since the r.h.s. of (3.6) is a polynomial in p and cannot have double poles! In fact the residue of the double pole at $p = e_0$ in (3.10) behaves for large t as $\lambda \exp\left(-\mathrm{cst.}\, t^{5/4}\right)$. Thus the loop equations are violated by non-perturbative terms exactly of the same order as those presents in (2.20). This is of course not a coincidence.

The only way out of this problem is to find a potential u such that the operator Q has a continuous spectrum. A necessary condition is that $u(x)$ is analytic along the whole real axis. As we have seen, this is not possible for any real solution of (2.11). In fact only two *complex conjugate* solutions of (2.11)satisfy this requirement [5]. Those two solutions, which are denoted the “triply truncated solutions”, have the following properties. They have have an infinite set of double poles (with Laurent expansion given by (2.18)) in only one fifth of the complex x plane. In the remaining 4/5th, which for one of the solutions is the sector

$$-\,\frac{6\pi}{5} \,<\, \arg(x) \,<\, \frac{2\pi}{5} \tag{3.11}$$

the function u has at most a finite number of poles and behaves smoothly as $|x| \to \infty$ as $u(x) \sim x^{1/2}$. This analyticity domain contains the whole real axis and one might expect that the loop correlators defined via the resolvent by (2.16) and (2.17), which are of course no more real, satisfy the loop equations. As we shall see in the next section, there are strong evidences that those complex solutions are indeed obtained from the original matrix models, once the problem of the unboundness of the action is properly treated (at the mathematical level...).

4. Non-perturbative Effects in Matrix Models

The main feature of the original potential (2.5) used in the matrix model (2.1) is that it is unbounded from below. This is a general feature for any one matrix model which allows to reach the $m = 2$ critical point (corresponding to pure gravity). This is clear in the original large N solution of the model [15]. This solution relies on the $N = \infty$ eigenvalue density $d\rho(\lambda) = d\lambda\, u(\lambda)$, which must extremize the action

$$F \;=\; N^2 \int d\rho(\lambda)\, V(\lambda) \;-\; \int d\rho(\lambda) \int d\rho(\mu)\, \ln|\lambda - \mu| \tag{4.1}$$

From (4.1) the effective potential for *one* eigenvalue is

$$\Gamma(\lambda) \;=\; V(\lambda) - \int d\mu\, u(\mu)\, \ln|\lambda - \mu| \tag{4.2}$$

and the force exerced on one eigenvalue is

$$f(\lambda) \;=\; -\Gamma'(\lambda) \;=\; -V'(\lambda) + 2\,\mathrm{Re}\big(F(\lambda)\big) \tag{4.3}$$

where $F(\lambda)$ is nothing but the v.e.v. of the one loop operator

$$F(\lambda) \;=\; \int d\mu\, \frac{u(\mu)}{\lambda - \mu} \;=\; \lim_{N\to\infty} \frac{1}{N^2}\, \langle W(\lambda)\rangle \tag{4.4}$$

Extremizing (4.1) leads to the equation

$$f(\lambda) \;=\; 0 \qquad \text{if } u(\lambda) \;\neq\; 0 \tag{4.5}$$

which means that the effective potential Γ is constant where eigenvalue density is non zero. The density of eigenvalues $u(\lambda)$ is given simply by the discontinuity of F

$$u(\lambda) \;=\; \frac{1}{\pi}\, \mathrm{Im}\big(F(\lambda - i\epsilon)\big) \tag{4.6}$$

The scaling limit is obtained here by performing the rescaling (2.9) for g and λ and letting $a \to 0$. We obtain for the force

$$f(p) \;=\; 2\,\mathrm{Re}\langle w(p)\rangle \quad ; \quad \langle w(p)\rangle^2 \;=\; \frac{1}{4}\left(p^3 - 3\,t\,p + \frac{4}{3}\langle P\rangle\right) \tag{4.7}$$

where $\langle P\rangle$ is some constant. This equation for $\langle w\rangle$ is nothing but the loop equation (3.6) at first order in the topological expansion, where the connected correlator $\langle ww\rangle$ vanishes. The puncture operator $\langle P\rangle$ is fixed by the requirement that $\langle w\rangle$ must have only one cut along $]-\infty, p_0]$. Indeed if this is not the case either u becomes complex, or it has support on two arcs (which is perfectly allowed) but is negative on one of them (which is impossible since u is a density) and moreover the effective potential is not the same on the two arcs. One obtains

$$\langle P\rangle \;=\; \frac{3}{2}\, t^{3/2} \;,\; \langle w(p)\rangle \;=\; \frac{1}{2}\,(\sqrt{t} - p)\, \sqrt{p + 2\sqrt{t}} \tag{4.8}$$

Hence the density $u(p)$ and the effective potential $\Gamma(p)$ for one eigenvalue

$$\begin{aligned} u(p) &= \frac{1}{2\pi}\,\mathrm{Re}\,\left[(\sqrt{t}-p)\sqrt{-p-2\sqrt{t}}\right] \\ \Gamma(p) &= \mathrm{Re}\,\left[\frac{2}{5}\,(3\sqrt{t}-p)(p+2\sqrt{(t)})^{3/2}\right] \end{aligned} \tag{4.9}$$

One sees that the effective potential goes to $-\infty$ as $p \to +\infty$ but that the eigenvalues, which are located on $(p < p_0 = -2\sqrt{t})$, are prevented to fall in this well by the "wall" $(-2\sqrt{t} < p < 3\sqrt{t})$ where $\Gamma > 0$, as long as t is positive. At the critical point $t = 0$, this wall disappears and the eigenvalues start to fall, hence the appearence of imaginary parts in the observables.

This classical picture is valid only for $N = \infty$. As long as N is finite, since N^{-1} plays the role of a "Planck constant", eigenvalues may cross the barrier and fall toward $+\infty$. As discussed in [6] this effect is exponentially suppressed at large N, and is therefore non–perturbative. Its amplitude can be estimates very easily by instanton technics[2]. The most probable phenomenon is that *one eigenvalues* crosses the wall while the $N-1$ others stay at equilibrium. The amplitude for such a process is given by $\exp(-Na^{5/4}\Gamma_{\mathrm{inst}})$, where "inst" corresponds to the configuration where the eigenvalue is at the top of the wall. From (4.9)

$$\Gamma_{\mathrm{inst}} \;=\; \Gamma(\sqrt{t}) \;=\; \frac{4}{5}\,3^{3/2}\,t^{5/4} \tag{4.10}$$

This is exactly the exponential factor in (2.20), which gives the amplitude of the leading non-perturbative effects contained in the string equations.

If one wants to work really at the nonperturbative level with the matrix model (2.1), that is at finite N, the partition function Z can be defined by the method of analytic continuation [16] . For the cubic potential (2.5) we take for the λ_i's in (2.1) a complex integration path going from $-\infty$ to (for instance) $\mathrm{e}^{i\pi/3}\infty$, which makes the matrix integral complex but perfectly well defined, for any complex value of g. The large N saddle point described above is not modified by this choice of contour, but now one can show that it is stabilized by this choice of boundary conditions. Indeed, in the scaling limit described above ($n \to \infty$, then $a \to 0$) the contour of integration for the eigenvalues goes now from $-\infty \leftarrow p$ to $p \to \mathrm{e}^{2i\pi/5}\infty$. Therefore this choice of boundary condition prevents the fall of the eigenvalues into the well $p \to +\infty$. Indeed, one can find a path which goes from the end point of the support of eigenvalues, $p_0 = -2\sqrt{t}$, to $\mathrm{e}^{2i\pi/5}\infty$, and which does not cross a region in the complex p plane where the effective potential Γ is negative.

The existence and the stability of a large N saddle point for *complex* t can easily be studied by complex saddle point methods. One can show that eigenvalues will still be located along the arc given by $\Gamma(p) = 0$, where Γ is given by (4.9) and corresponds now to the real part of the complex effective potential $\int_p 2\langle w\rangle$. There are two natural conditions of stability for this saddle point:

[2] as suggested by S. Shenker and J. Zinn-Justin

(*i*) The support of eigenvalues must connect $-\infty$ to the endpoint p_0. One can easily show that this happens only if

$$-\frac{6\pi}{5} \ < \ \mathrm{Arg}(t) \ < \ \frac{6\pi}{5} \tag{4.11}$$

(*ii*) one can still find a path which goes from p_0 to infinity such that $\Gamma(p) > 0$. With our choice of boundary condition this is possible if

$$-\frac{8\pi}{5} \ < \ \mathrm{Arg}(t) \ < \ \frac{2\pi}{5} \tag{4.12}$$

Thus the large N limit exists only in four-fifth of the complex t plane. One can show that there cannot exist a more complicated limit, such as a two arc phase, in the remaining sector. The instability in the singular sector $2\pi/5 < \mathrm{Arg}(t) < 4\pi/5$ corresponds precisely to instanton effects. Indeed it is on its boundary that the effective action of the instanton considered above vanishes.

The sector where the large N limit exists is *exactly* the same than the sector of analyticity of one of the "triply truncated solution" of (2.11). Since the planar limit is obtained from the scaling limit by letting x and $t \to \infty$, this allows to identify the triply truncated solution with the result of the scaling limit, if one start from the matrix model defined with the complex contour described above. The string susceptibility f and the loop amplitudes $\langle w \rangle$ will of course be complex, but with exponentially small imaginary parts (as $t \to \infty$) proportional to (2.20).

These arguments can be extended to other matrix models and to higher critical points. For instance in [16] the cases of the Painlevé II critical point and the $m = 3$ critical points are discussed in details. The non-perturbative effects in the corresponding string equations can also be attributed to instanton effects in the original matrix models. The same kind of arguments allows to study deformations between (multi)critical models [17] [18] [19] . In all know cases the conclusions of such a saddle point analysis are in perfect agreement with the analysis of non-perturbative effects in the string equations by Borel summation methods [7], and by WKB methods and the study of their monodromy properties [20].

5. Stochastic Quantization and the SUSY 1D String

Let us end by a few simple comments[3] about the proposal by Marinari and Parisi [21] to treat 2D gravity as the ground state of some supersymmetric 1d string model. The idea relies on the fact that in a model of the form (2.1), the average of a observable Q can be written as the ground state expectation value

$$\langle Q \rangle \ = \ \langle 0|Q|0 \rangle \tag{5.1}$$

[3] elaborated while I was writing these notes

of the observable Q in a quantum mechanical model with Hamiltonian

$$H_B \;=\; P^2 + V_B(\Phi) \quad ; \quad P = i\partial/\partial\Phi \quad ; \quad V_B \;=\; \frac{(V')^2}{4} \;-\; \frac{V''}{2} \tag{5.2}$$

This Hamiltonian H_B is the bosonic part of the supersymmetric quantum mechanical Hamiltonian $\mathbf{H}$ which can be obtained through the stochastic quantization of (2.1)and the associated Fokker-Planck Hamiltonian

$$\mathbf{H} \;=\; \begin{pmatrix} H_B & 0 \\ 0 & H_F \end{pmatrix} \;=\; \mathbf{Q}^2 \quad ; \quad \mathbf{Q} \;=\; \begin{pmatrix} 0 & iP + \frac{V'}{2} \\ -iP + \frac{V'}{2} & 0 \end{pmatrix} \tag{5.3}$$

where $\mathbf{Q}$ is the SUSY generator. The l.h.s. of (5.1) make sense only if V is bounded from below. In that case supersymmetry is unbroken, the ground state is bosonic and has zero energy, and (5.1) holds. In the case of interest here, V is unbounded from below but $\mathbf{H}$ is well defined and positive. Supersymmetry is broken and the two degenerate vacua $|0_B\rangle$ and $|0_F\rangle$ have a positive energy. The proposal of [21] (already made in [22]), is to define the v.e.v. of Q by (5.1) (taking of course the bosonic ground state). Some properties of this 1d supersymmetric theory in the scaling limit have been studied in [23], [24]. Of course the equations of motion of the original theory will be violated in the supersymmetric one by terms proportional to the supersymmetry breaking. Indeed the variation of the partition function under a field variation f can be written as

$$\langle -f' + f\,V' \rangle \;=\; \langle 0_B|\{\mathbf{Q},\mathbf{F}\}|0_B\rangle \quad ; \quad \mathbf{F} \;=\; \begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix} \tag{5.4}$$

and (5.4) vanishes only if $\mathbf{Q}|0_B\rangle = 0$. However we are dealing with a model of 2d gravity coupled to supersymmetric matter which is perfectly self consistent and which may define a physically interesting theory containing 2d gravity.

Following [21] and [23] we have to consider the ground state of a system of N fermions in the potential $V_B = N(\lambda + (g - \lambda^2)^2/4)$. The $N = \infty$ limit can therefore be studied by the WKB approximation [15]. In the planar scaling limit ($N \to \infty$, then $a \to 0$) the potential V_B becomes

$$v(p) \;=\; p^3 \;-\; 3\,t\,p \tag{5.5}$$

The particle density $\rho(e,p)$ and the integrated particle density $\rho(p) = \int de\,\rho(e,p)$ are respectively

$$\rho(e,p) \;=\; \frac{1}{2\pi\,\sqrt{e - v(p)}}\,\theta(e - v(p)) \quad ; \quad \rho(p) \;=\; \frac{1}{\pi}\,\sqrt{e_F - v(p)}\,\theta(e_F - v(p)) \tag{5.6}$$

The Fermi energy e_F is fixed by the normalization condition

$$\nu(e_F) \;=\; \frac{1}{\pi}\int_{v<e_F} dp\,\sqrt{e_F - v(p)} \;=\; 0 \tag{5.7}$$

(where the divergence at $-\infty$ is treated by a finite part prescription).

In the weak coupling region $t > 0$, where SUSY is unbroken, the solution of (5.7) is given by $e_F = -2\,t^{3/2}$, which corresponds to the value of v at the local minimum $p = \sqrt{t}$. Then we recover exactly the large N solution, as expected, since we have $\rho(p) = u(p)$, where $u(p)$ is the eigenvalue density given by (4.9). e_F is identified with $\frac{4}{3}\,\langle P\rangle$. For $t < 0$ SUSY is spontaneously broken and $v(p)$ has no real local minimum. However (5.7) still has a unique real solution, since $\nu(e)$ is defined on $]-\infty,\infty[$, with $\nu'(e) > 0$, and since $\nu(e) \sim \pm|e|^{5/6}$ as $e \to \pm\infty$. Since $\nu(0) > 0$, e_F is negative, and scales as $e_F = \mathbf{c}\,(-t)^{3/2}$ with $\mathbf{c}$ some transcendental number.

This has some nasty effects on the physical observables of the theory. Indeed, according to the rule (5.1), the v.e.v. of the loop operator which creates a loop with length ℓ is given (playing with Laplace transform) by

$$\langle w(\ell)\rangle \;=\; \int_{-i\infty}^{i\infty} \frac{dp}{2i\pi}\, \mathrm{e}^{p\ell}\, \sqrt{v(p) - e_F} \tag{5.8}$$

For $\ell > 0$ we wrap the contour around the cut $]-\infty, p_0]$ and we obtain the expected result $\langle w(\ell)\rangle = \int dp\,\rho(p)\,\mathrm{e}^{p\ell}$. For $\ell < 0$, if $t > 0$ we get $\langle w\rangle = 0$, but if $t < 0$ the integrand $\sqrt{v(p) - e_F}$ has a second cut right to the contour of integration and therefore $\langle w(\ell)\rangle$ does not vanish ! Moreover for large negative ℓ it behaves as

$$\langle w(\ell)\rangle \;\sim\; \ell^{-3/2}\,\mathrm{Re}\left(\mathrm{e}^{p_1 \ell}\right) \tag{5.9}$$

where p_1 is one of the two complex conjugate zeros of $(v - e_F)$. The amplitude for a loop with negative length oscillates wildly and can even be negative. The existence of such “unphysical states” is a serious problem if one wants to interpret the 1d SUSY string as a pure 2d gravity theory. In the planar limit they appear only for $t < 0$ but in the scaling limit loops with negative length should have a non-zero, but exponentially small, amplitude for positive t.

6. Conclusion

The various approaches to the scaling limit for two dimensional quantum gravity give different points of view on the non-perturbative effects in the theory. Remarkably those effects can be understood (and to some extend calculated) within the matrix model formulation, and they are deeply connected to the unboundness of the potential. At the present stage my feeling is that pure 2d quantum gravity has a somewhat similar status than QED_4 for negative e^2 [25]. It is a well defined theory in perturbation theory. It is renormalizable and asymptotically free. However the vacuum is unstable under the formation of handles (a process somewhat analogous to $\mathrm{e}_+\mathrm{e}_-$ pairs creation for QED) and it seems that no physically acceptable stable vacuum can be reached. The fact that similar issues appear also in critical strings [26] and that $3+1$ ordinary gravity is also unstable under conformal modes means that the understanding of this kind of problems is crucial for the elaboration of a quantum theory of gravity.

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References

[1] J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B257 (1985) 433.
F. Favid, Nucl. Phys. B257 (1985) 45.
V. A. Kazakov, Phys. Lett. 150B (1985) 282; V. A. Kazakov, I. K. Kostov and A. A. Migdal, Phys. Lett. 157B (1985) 295.

[2] E. Brézin and V. A. Kazakov, Phys. Lett. 236B (1990) 2125.
M. R. Douglas and S. H. Shenker, Nucl. Phys. B335 (1990) 635.
D. J. Gross and A. A. Migdal, Phys. rev. Lett. 64 (1990) 27.

[3] M. R. Douglas, Phys. Lett. 238B (1990) 2125.

[4] T. Banks, M. R. Douglas, N. Seiberg and S. H. Shenker, Phys. Lett. B 238 (1990) 279.

[5] E. Hille, "Ordinary Differential Equations in the Complex Domain", Pure and Applied Mathematics, J. Wiley & Sons, 1976.

[6] S. H. Shenker, "The Strength of Nonperturbative Effects in String Theory", these proceedings.

[7] P. Ginsparg and J. Zinn-Justin, these proceedings.

[8] S. R. Wadia, Phys. Rev. D 24 (1981) 970.
A. A. Migdal, Phys. Rep. 102 (1983) 199.

[9] F. David, Mod. Phys. Lett. A5 (1990) 1019.

[10] J. Ambjørn and Y. M. Makeenko, preprint NBI-HE-90-22, May 1990.
J. Ambjørn, J. Jurkiewicz and Y. M. Makeenko, preprint NBI-HE-90-41, August 1990.

[11] R. Dijkgraaf, H. Verlinde and E. Verlinde, "Loop Equations and Virasoro Constraints in Non-Perturbative 2-D Gravity", preprint PUPT-1184 IASSNS-HEP-90/48, May 1990

[12] M. Fukuma, H. Kaway and R. Nakamaya, "Continuum Schwinger-Dyson Equations and Universal Structures in Two dimensional Quantum Gravity", preprint UT-562 KEK-TH-251, May 1990.

[13] E. Witten, Nucl. Phys. B340 (1990) 281.
R. Dijkgraak and E. Witten, Nucl. Phys. B342 (1990) 486.
E. Verlinde and H. Verlinde, "A Solution of Two Dimensional Topological Gravity", preprint PUPT-1176, 1990

[14] C. Bachas and P. M. S. Petropoulos, Phys. Lett. 247B (1990) 363.
C. Bachas, "On Triangles and Squares", these proceedings.

[15] E. Brezin, C. Itzykson, G. Parisi and J.-B. Zuber, Commun. Math. Phys. 59 (1978) 35.

[16] F. David, "Phases of the Large N Matrix Model and non-perturbative Effects in 2d Gravity", preprint SPhT/90/090, July 1990.

[17] M.Douglas, N. Seiberg and S. Shenker, Phys. Lett. B244 (1990) 381.

[18] J. Jurkiewicz, Phys. Lett. B 245 (1990) 178.

[19] G. Bhanot, G. Mandal and O. Narayan, "Phase Transitions in 1-Matrix Models", preprint IASSNS-HEP-90-52, May 1990.
G. Mandal, these proceedings.

[20] G. Moore, Commun. Math. Phys. 133 (1990) 261.

[21] E. Marinari and G. Parisi, Phys. Lett. 240B (1990) 375.

[22] J. Greensite and M. Halpern, Nucl. Phys. B242 (1984) 167.

[23] M. Karliner and A. Migdal, “Nonperturbative 2D Quantum Gravity via Supersymmetric String”, preprint PUPT-1191, July 1990.

[24] J. Ambjørn, J. Greensite and S. Varsted, “A Non-perturbative Definition of 2D Quantum Gravity by the Fifth Time Action”, preprint NBI-HE-90-39, July 1990.

[25] F. J. Dyson, Phys. Rev. 85 (1952) 32.

[26] D. J. Gross and V. Periwal, Phys. Rev. Lett. 60 (1988) 2105.