

## PHASES OF THE LARGE- $N$ MATRIX MODEL AND NON-PERTURBATIVE EFFECTS IN 2D GRAVITY

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The large- $N$  solution of the one-matrix model of E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber is reconsidered for generic complex potential. A regular large- $N$  limit does not exist in some singular domain, which depends on the prescription chosen in order to make the matrix integral convergent at infinity. Near the  $m = 2$  critical point the singular domain (in the scaling variable  $x$  complex plane) is a sector of angle  $2\pi/5$  coinciding with the sector of poles of the “triply truncated” Painlevé I transcendent of Boutroux, which is therefore (although not real) the only solution of the string equation compatible with the matrix model and the loop equations for two-dimensional gravity. Our approach allows us to relate non-perturbative effects in the string equations to instantons in the matrix model and to discuss the flows between multicritical points.

### 1. Introduction

Two-dimensional quantum gravity may be formulated as a functional integral over two-dimensional riemannian manifolds. Discretizing this sum by reducing it to a sum over random triangulations allows us to formulate the theory in terms of random matrix models [1–3]. The topological expansion in terms of the genus of the manifolds is mapped onto the  $1/N$  expansion,  $N$  being the dimension of the matrix. The large- $N$  solutions of those matrix models (for a review see refs. [4, 5]) exhibit critical points. At those critical points continuum theories can be explicitly constructed at fixed order of the topological expansion. Explicit calculations of (for instance) scaling dimensions of operators and critical exponents allows us in many cases to identify those continuum theories with some specific conformal theories coupled to two-dimensional gravity.

Recently it was shown that by taking a double scaling limit a continuum limit for the whole topological expansion can also be constructed [6–8]. The remarkable

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underlying mathematical structure of these non-perturbative solutions (in particular the KdV hierarchy [9]), and its connection with other formulations of two-dimensional gravity (like topological gravity [10]) have been extensively studied in the last few months (for a review see e.g. ref. [11]). However this solution presents some ambiguities which are usually attributed to non-perturbative effects of the theory. For instance for the simplest case of pure gravity, the “string susceptibility”  $f(x)$ , where  $x$  is the scaling variable  $x = (g_c - g)N^{4/5}$ , satisfies a “string equation” which is the Painlevé I equation\*

$$x = f^2(x) - \frac{1}{6}f''(x). \quad (1)$$

This string equation determines uniquely all the coefficients of the topological expansion from the large- $x$  asymptotics of  $f$ ,

$$f(x) = x^{1/2} - \sum_{n \geq 1} a_n x^{(1-5n)/2}. \quad (2)$$

This series is asymptotic to any solution of eq. (1) which behaves as  $x^{1/2}$  as  $x \rightarrow +\infty$  but is not Borel summable since all the  $a_n$  are positive and grow like  $(2n)!$ . This reflects the fact that a general solution of eq. (1) depends on two integration constants. The large- $x$  behaviour fixes only one of them and there is an infinite family of solutions, which differ by terms exponentially small as  $x \rightarrow +\infty$ . It is of course very important (at least if one is interested in finding a consistent non-perturbative formulation of quantum two-dimensional gravity) to understand whether there are additional (if possible physical) criteria which allow us to fix those non-perturbative ambiguities, or if no real solution of (1) is acceptable.

Various authors [6–8, 13] suggested that those ambiguities are related to the fact that in the original matrix model the critical point associated to pure gravity corresponds to a potential unbounded from below. This seems related to the fact that it does not seem possible to flow from the  $m = 3$  tricritical point (which corresponds to a stable potential) to the  $m = 2$  critical point in the string equations [13, 14]. In a previous letter [15] the present author proposed to use the Schwinger–Dyson equations to deal with this problem. Starting from the matrix model the continuum version of the SD equations (which relate loops correlation functions) was obtained\*\*. It was shown that for pure gravity the existence of an infinite series of double poles on the real axis for any real solution of eq. (1) forbids us to use those solutions to construct loop amplitudes satisfying the SD equations. It was argued that only solutions analytic along the whole real axis were

\*The factor of  $1/6$  in eq. (1) differs from the usual factor of  $1/3$  because of the “doubling phenomenon” which occurs in matrix models with even potentials [12].

\*\*The continuum SD equations have recently been extended to gravity coupled to various matter fields, and have been derived from the string equations ( $[P, Q] = 1$ ) [16, 17] and from the topological gravity theory [17–19].

acceptable. Moreover it was conjectured that the *unique* complex solution (up to complex conjugation) of eq. (1) which satisfies this analyticity requirement could be obtained from the matrix model by starting from a potential bounded from below and by reaching the critical point by a proper analytic continuation.

In this paper we pursue this idea by reconsidering the large- $N$  solution of the one-matrix model of ref. [20] for generic complex potential. We first show, by studying the stability of this solution, that a regular large- $N$  limit, which corresponds to a limiting smooth distribution of the eigenvalues of the  $N \times N$  matrix along arcs in the complex plane, does not necessarily exist. In general the various phases of the matrix models are separated by singular domains (in the space of complex potentials), where no large- $N$  limit exists. The stable and singular domains are explicitly shown to depend on the boundary conditions at infinity in the matrix integral. In a second step we show on specific examples that the singular domains correspond exactly to the sectors where some solutions of the string equations have an infinite number of double poles, and that this allows us, for the example considered, to identify *uniquely* which solution of the string equation correspond to some specific critical point in the matrix model formulation.

This paper is organized as follows. In sect. 2 we formulate the existence and stability criteria of a large- $N$  solution of the matrix model, first on the example of the cubic potential and then for general (polynomial) potentials. In sect. 3 we consider in detail the case of the cubic potential and the critical point associated to “pure gravity” ( $m = 2$ ). We confirm our previous conjecture that only the so-called “triply truncated solution” of the Painlevé I equation (1) can be obtained from the one-matrix model. We also show explicitly how the non-perturbative imaginary part of this solution can be attributed to “instanton effects”, namely to the tunnelling of eigenvalues. In sect. 4 we consider the case of more general potentials of order 4 and 6, which allow us to discuss higher-order critical points, and general features of deformations between critical points. In sect. 5 we draw some conclusions.

## 2. The large- $N$ limit of the one-matrix model: existence and stability

Let us consider the “generic” one-matrix model defined by the integral over an  $N \times N$  hermitian matrix  $\Phi$ . After integrating out the  $SU(N)$  degrees of freedom one is left with the integral over the  $N$  eigenvalues  $\lambda_i$  of  $\Phi$ ,

$$Z = \int d\Phi \exp[-N \operatorname{tr} V(\Phi)] \propto \int \prod_{i=1}^N d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left[-N \sum_{i=1}^N V(\lambda_i)\right]. \quad (3)$$

Thus the problem reduces to the statistics of  $N$  “charges” on a line subject to a one-body potential  $V$  and to two-body logarithmic repulsive interactions. To discuss the  $m = 2$  critical point it suffices to restrict oneself to the cubic potential

which we can write as

$$V(\lambda) = g\lambda - \frac{1}{3}\lambda^3 \quad (4)$$

(the even quartic potential is known to lead to two different critical points [21]). The integral (3) diverges if one integrates over real  $\lambda$ 's. We *define* the model by taking for each  $\lambda_i$  a complex integration path which goes from  $-\infty$  to  $e^{i\pi/3}\infty$ . There are two alternate definitions of the model obtained by rotating the integration contour by  $2\pi/3$ , however they can be obtained from this particular one by changing the coupling constant  $g$  to  $ge^{\pm 2i\pi/3}$ .

The large- $N$  solution of ref. [20] is obtained by *assuming* that in the “thermodynamical limit”  $N \rightarrow \infty$  the integral (3) is dominated by a “mean-field” configuration such that the charges are evenly distributed on the real axis with an average density measure  $d\rho(\lambda)$  (normalized to unity) and such that relative fluctuations are suppressed by a factor of  $1/N^2$ . Then one has to extremize the action (we omit the overall  $N^2$  factor)

$$S = \int d\rho(\lambda) V(\lambda) - \int d\rho(\lambda) \int d\rho(\mu) \ln|\lambda - \mu|, \quad (5)$$

where  $\ln|\lambda|$  has to be understood as  $\frac{1}{2}(\ln(\lambda + i\epsilon) + \ln(\lambda - i\epsilon))$ . *Local variations* of the density lead to the saddle point equation

$$V'(\lambda) = 2 \int d\rho(\mu) \frac{1}{\lambda - \mu}, \quad (6)$$

satisfied iff  $\lambda$  belongs to the support of  $d\rho$ . In order to solve (6) one introduces the function

$$F(\lambda) = \int d\rho(\mu) \frac{1}{\lambda - \mu}, \quad (7)$$

analytic outside the support of  $d\rho$  and which goes as  $1/\lambda$  as  $\lambda \rightarrow \infty$  (from the normalization of  $d\rho$ ).  $F$  is nothing but the one-loop correlator  $\langle N^{-1} \text{Tr}(\lambda - \Phi)^{-1} \rangle$ . For the cubic potential (4) a generic solution is

$$F(\lambda) = \frac{1}{2} \left( (g - \lambda^2) + \sqrt{(g - \lambda^2)^2 + 4\lambda - x} \right), \quad (8)$$

where  $x$  is arbitrary. In general  $F$  will have two cuts, joining pairs of the four zeros of the quartic polynomial in the square root in (8) (let us label them  $a, b, c, d$ ). In ref. [20]  $x$  is adjusted so that  $F$  has only one cut  $[a, b]$ . This means that  $c = d$  are degenerate and this is possible for only three values of  $x$ . For the potential (4),  $x$  is chosen such that in the limit  $g \rightarrow +\infty$  the cut  $[a, b]$  reduces to the real local

minimum of  $V$  at  $\lambda \approx -\sqrt{g}$ . At the critical point  $g_c = 3.2^{-2/3}$  the double zero  $c$  coalesces with  $b$  and the extremum action  $S_c(g)$  is singular ( $S_c \sim (g - g_c)^{5/2}$ ).

This one-cut solution can be analytically continued to complex  $g$ . However, we must check whether it remains a physically acceptable minimum of  $S$ . There are in fact two requirements:

(i) Reality and positiveness of  $d\rho$ . The support of  $d\rho$  becomes now a curve  $C$  joining  $a$  to  $b$ . Indeed, for infinite  $N$  the eigenvalues  $\lambda_i$  can be considered as a function  $\lambda(x)$  of the continuous parameter  $x = i/N$  such that  $\lambda(0) = a$  and  $\lambda(1) = b$ . The function  $F$  has a discontinuity along  $C$ , and can be expressed as

$$F(\lambda) = \int_C d\mu \frac{u(\mu)}{\lambda - \mu}, \quad u(\mu) = \frac{dx}{d\mu}. \tag{9}$$

From eq. (6) along  $C$  we have

$$F(\lambda) = \frac{1}{2}V'(\lambda) \mp i\pi u(\lambda). \tag{10}$$

Hence if we consider the function

$$G(\lambda) = \int_a^\lambda d\mu (V'(\mu) - 2F(\mu)), \tag{11}$$

$a$  and  $b$  are branch points of  $G$ , and in particular along  $C$   $G$  is purely imaginary since

$$G(\lambda(x)) = \pm 2i\pi x. \tag{12}$$

Therefore we end up with the first requirement:  $G(\lambda)$  must be purely imaginary on a curve  $C$  which joins  $a$  to  $b$ .  $C$  is the support of the measure  $d\rho$ .

(ii) Global stability. If one moves a charge from  $\lambda_i$  on  $C$  to a position  $\lambda_f$  away from that support, the variation of the action is non-zero but

$$\Delta S \propto G(\lambda_f) - G(\lambda_i), \tag{13}$$

where  $G$  is the function defined by eq. (11). Such global variations are not dangerous if  $\Delta S$  has a positive real part. We therefore have the second requirement: *The contour of integration over the  $\lambda$ 's in eq. (3) can be deformed continuously into a contour which includes  $C$  and which do not cross any region where  $Re(G) < 0$ .* Note that although  $G$  is multivalued with branch points at  $a$  and  $b$ , its real part is unambiguously defined, provided that one does not cross  $C$ . Let us stress that the boundary conditions in eq. (3) enter explicitly here.

This analysis can easily be extended to study the stability of the general two-cut solution, and to a generic polynomial potential  $V(\lambda)$ . If  $V$  is a polynomial of

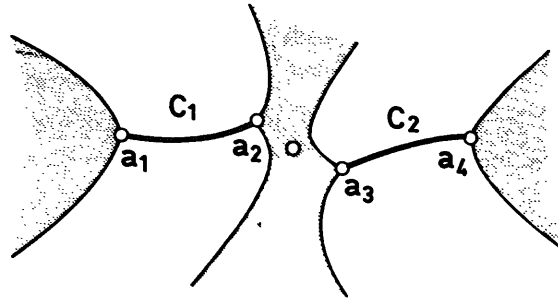


Fig. 1. General structure of the segments  $C_i$ , where the density of  $\lambda$ 's is non-zero, and of the domain  $D = \{\lambda; \text{Re}(G(\lambda)) > 0\}$  (shaded area) for a solution of the large- $N$  equations with  $n = 2$  cuts. The open circles  $\circ$  denote the zeros of  $G'(\lambda)$ .

degree  $m$ , the generic solution  $F$  which leads to (6) is

$$F(\lambda) = \frac{1}{2} \left( V'(\lambda) + \sqrt{V'(\lambda)^2 + N(\lambda)} \right), \quad (14)$$

where  $N(\lambda)$  is a polynomial of degree  $m - 1$ .  $F(\lambda)$  has  $2n$  branch points with  $n \leq m - 1$ .  $F$  will correspond to a (local) minimum of  $S$  if the integration contour over the  $\lambda$ 's in eq. (1) can be deformed continuously into a curve  $S$  which passes successively through the  $2n$  branch points  $(a_i, i = 1, 2n)$  of  $F$  in such a way that the function  $G$  defined by eq. (11) with initial point  $a_1$  has the following properties:

- (i) On the  $n$  segments  $C_j$  of  $S$  between successive pairs of branch points  $(a_{2j-1}, a_{2j})$   $G(\lambda)$  is purely imaginary. This implies that the measure  $d\rho$  is concentrated on the  $C_j$  and is real.
- (ii)  $G(a_{2j}) = G(a_{2j+1})$ . This implies stability of the action against move of charges from  $C_j$  to  $C_{j+1}$ .
- (iii)  $\text{Im}(G)$  is a monotonic function on the  $C_j$  which goes from 0 to  $4\pi$ . This implies positiveness and the correct normalization of  $d\rho$ .
- (iv)  $S$  must not cross any domain where  $\text{Re}(G) < 0$ . This is the global stability condition.

The general structure of the domain  $D = \{\lambda; \text{Re}(G(\lambda)) > 0\}$  and of the segments  $C_i$  is depicted in fig. 1. If for a given potential  $V(\lambda)$  more than one solution exist, one must compute explicitly the action  $S$  for each of those solutions and take the solution with smallest real part of  $S$ .

### 3. The cubic potential and the $m = 2$ critical point

Let us first discuss the case of the cubic potential (4). A priori there are three critical points at  $g_c, jg_c$  and  $j^2g_c$  ( $j = e^{2i\pi/3}$ ), corresponding to pure gravity, plus the trivial gaussian critical point at  $g = \infty$ . It is instructive to discuss first the

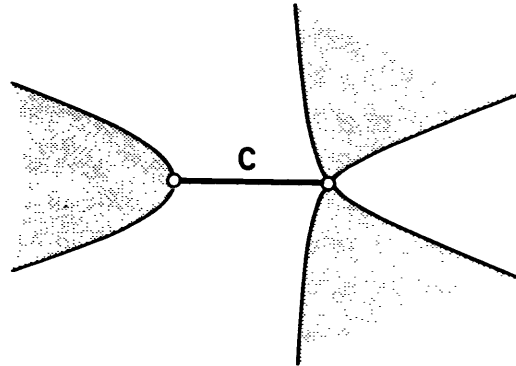


Fig. 2. C and D at the critical point.

structure of the phase diagram close to  $g = \infty$ . As discussed before as  $g \rightarrow +\infty$  the eigenvalues concentrate at the minimum  $\lambda = -\sqrt{g}$  of  $V$ . The function  $G(\lambda)$  defined by eq. (11) reduces to  $G(\lambda) = V(\lambda) - V(-\sqrt{g})$  and the issue of the stability of the saddle point  $-\sqrt{g}$  versus the alternate saddle point  $\sqrt{g}$  reduces to check whether the integration path that we have chosen, which goes from  $-\infty$  to  $e^{i\pi/3}\infty$ , does not cross a domain where  $\text{Re}(G(\lambda)) < 0$ . It is easy to check that this is satisfied if

$$-5\pi/3 < \text{Arg}(g) < \pi/3. \tag{15}$$

This means that the solution is discontinuous on the line  $\text{Arg}(g) = \pi/3$ , where one “switches” from one extremum of  $V$  to the other. The position of the discontinuity is dictated explicitly by the boundary conditions in the integral (1).

We now discuss the vicinity of the real critical point  $g_c = 3/2^{2/3}$ . At the critical point the support of the eigenvalues C and the domain D are depicted in fig. 2. As shown for instance in ref. [15], the scaling limit near the critical point is obtained by rescaling,

$$\left. \begin{aligned} \lambda &= \lambda_c(1 + az), & \lambda_c &= 2^{-1/3} \\ g &= g_c(1 + a^2x), & g_c &= 3 \cdot 2^{-2/3} \end{aligned} \right\} a \rightarrow 0. \tag{16}$$

In this limit the function  $G' = V' - 2F$  scales as

$$G'(\lambda) = a^{3/2} 2^{1/3} (\sqrt{x} - z) \sqrt{z + 2\sqrt{x}}. \tag{17}$$

Thus in the rescaled variable  $z$  the support C goes from  $-\infty$  to  $z_c = -2\sqrt{x}$ . The analysis of the consistency of this one-cut solution reduces to the study of the primitive

$$G(z) = \int_{\lambda_c}^{\lambda} d\mu G'(\mu) = a^{5/2} \frac{2}{5} (3\sqrt{x} - z)(z + 2\sqrt{x})^{3/2}. \tag{18}$$

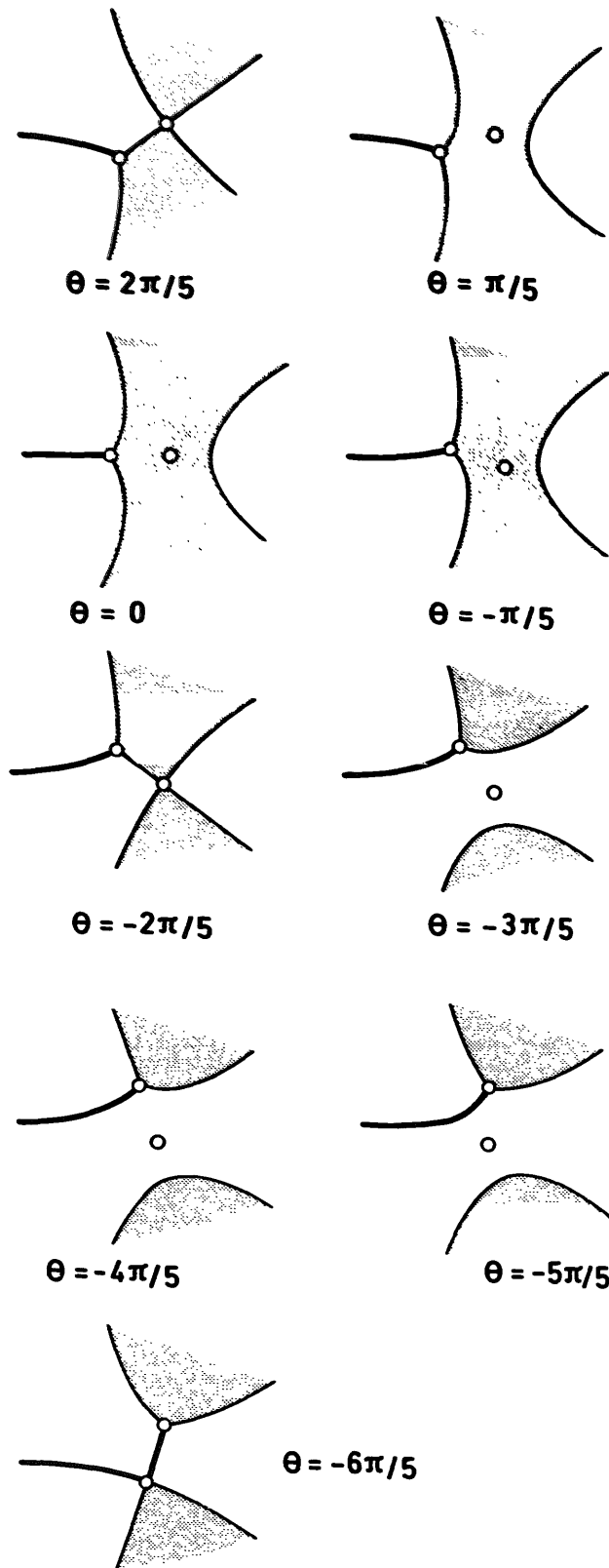


Fig. 3. Variations of C and D (in the rescaled variable  $z$ ) as  $\theta = \text{Arg}(x)$  varies from  $2\pi/5$  to  $-6\pi/5$ .



In fig. 3 we have depicted how the domain D where  $\text{Re}(G)$  is positive and the support C, which is the curve joining  $-\infty$  to  $z_c$  where  $G$  is purely imaginary, are deformed when one turns around the critical point by changing  $\text{Arg}(x)$ .

C is continuous as long as  $-6\pi/5 < \text{Arg}(x) < 6\pi/5$  but get a cusp at  $\pm 6\pi/5$ . At that cusp the density of eigenvalues *vanishes linearly* along C. Thus as  $\text{Arg}(x) \rightarrow \pm 6\pi/5$  the support C tends to break up into two segments.

On the other hand, global stability is ensured only if  $z_c$  is connected to  $e^{2i\pi/5}\infty$  through D. This is possible only if  $-8\pi/5 < \text{Arg}(x) < 2\pi/5$ . As  $\text{Arg}(x) \rightarrow 2\pi/5$  the domain D is pinched at the point  $z = \sqrt{x}$ . For  $\text{Arg}(x) = 2\pi/5$ ,  $G(\sqrt{x}) = 0$  and eigenvalues will start to be attracted by the pinching point. Therefore the domain where the one-cut solution exists is the sector

$$-6\pi/5 < \text{Arg}(x) < 2\pi/5. \tag{19}$$

One might wonder if in the remaining sector a two-cut configuration minimizes the action. However one can convince oneself that, from topological considerations on the curves where  $\text{Arg}(G)$  is constant, a two-cut configuration can exist only if one starts from a potential  $V(\lambda)$  with degree  $n \geq 4$ . Indeed, from fig. 1 in that case the domain D must have at least four extensions to  $\infty$ . Since  $G(\lambda) \sim V(\lambda)$  at  $\infty$  this means that  $V$  is at least quartic. Thus in the sector  $2\pi/5 < \text{Arg}(x) < 4\pi/5$  there is *no smooth thermodynamical limit when the number of "charges" goes to  $\infty$* .

It is not very difficult to extend this analysis to the case of a generic complex coupling constant  $g$ . The whole phase diagram is depicted in fig. 4. The one-cut solution extremizes the action in the white domain. No regular large-N limit exists in the dashed domain, which connects the three critical points  $\infty$ ,  $g_c$  and  $jg_c$ . In this domain we expect that zeroes of the partition function  $Z$  becomes dense as

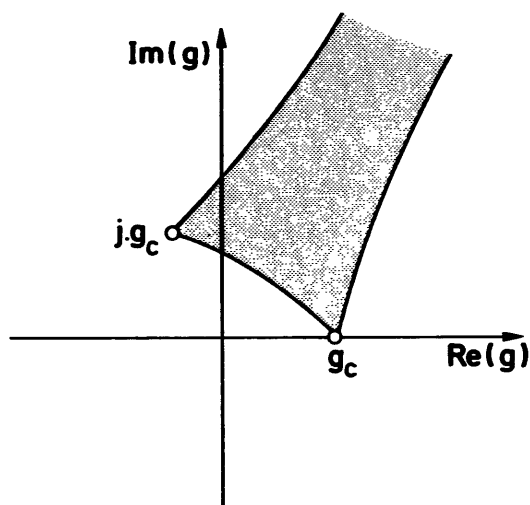


Fig. 4. The phase diagram of the cubic model in the complex  $g$ -plane. In the shaded domain no large-N solution exists.

$N \rightarrow \infty$ . The critical point at  $j^2 g_c$  cannot be reached with our choice of boundaries in the integral (3).

The structure of the phase diagram allows us in fact to determine *entirely* which solution of the Painlevé equation (1) is compatible with the original matrix model. Indeed, let us recall that the string susceptibility  $f(x)$  is the second derivative of the free energy  $-\ln Z$  with respect to the scaling variable  $x = (g_c - g)N^{4/5}$ . We expect that taking the scaling limit (16) at the critical point on the large- $N$  solution ( $N \rightarrow \infty$  while  $g$  fixed, then  $a \rightarrow 0$ ) should be consistent with taking the large- $x$  limit in the non-perturbative solution ( $N \rightarrow \infty$  while  $x$  fixed, then  $x \rightarrow \infty$ ). The first limit exists only in the sector (19) and gives  $f(x) = \sqrt{x}$ . From the study of the general solutions of eq. (1), it is known (see e.g. ref. [22]) that they have an infinite set of double poles in the whole complex  $x$ -plane, and that some (infinite) family of solutions have a finite number of poles in sectors of angle  $4\pi/5$ . There is only one solution which has a finite number of poles in the whole sector (19) with angle  $8\pi/5$ . In this sector this solution (which is denoted in ref. [22] the “triplely truncated solution”) has regular large- $x$  asymptotics given by eq. (2). In the remaining singular sector  $2\pi/5 < \text{Arg}(x) < 4\pi/5$ ,  $f$  has still an infinite set of poles, since it behaves asymptotically as  $\sqrt{x} \wp(x^{5/4})$ , where  $\wp$  is a Weierstrass function. Thus no regular large- $x$  limit exists in that singular sector. The conclusion is that if one starts from the one-matrix integral (3) with the integration contour specified above, the scaling limit for  $f$ , if it exists, must be the triplely truncated solution with no poles in the lower half-plane.

This argument is in fact universal and does not depend on the specific choice of the potential  $V$ . Indeed to reach a critical point associated to pure gravity it is sufficient that locally at some end point of the support of eigenvalues the structure depicted in figs. 2 and 3 occurs. Since our analysis depends only on the local properties of the function  $G'$ , our conclusions are valid for general potentials.

Finally one can explicitly associate the exponentially small non-perturbative ambiguities of general solutions of the string equation to tunneling effects of eigenvalues in the potential  $V^*$ . Indeed, with the cubic potential (4) and the normalization (16), in the double scaling limit  $a \rightarrow \infty$ ,  $a^5 N^2 = 1$ , the susceptibility  $f = -(\partial^2/\partial x^2)\ln Z$  satisfies the equation

$$f^2(x) - \frac{1}{6}f''(x) = \left(\frac{3}{2}\right)^4 x. \quad (20)$$

Solutions of eq. (20) analytic for large positive  $x$  are defined up to non-perturbative terms of order

$$\delta f \sim x^{-1/8} \exp\left(-\frac{4}{5}(3)^{3/2} x^{5/4}\right). \quad (21)$$

\* This was suggested to us by S. Shenker and J. Zinn-Justin.

In particular, the triply truncated solution of eq. (20) has an imaginary part which behaves for large  $x$  as (21). If we come back to the large- $N$  solution,  $N \operatorname{Re}(G(z))$  is nothing but the effective potential for one eigenvalue. It is constant and vanishes on  $C$ , is positive for  $-2\sqrt{x} < z < 3\sqrt{x}$  and negative (and unbounded from below) for  $z > 3\sqrt{x}$ . The instanton configuration with lowest action corresponds to *one* eigenvalue sitting at the top of the wall at  $z = \sqrt{x}$ . The action of this instanton is in the double scaling limit

$$S_{\text{inst}} = NG(\sqrt{x}) = \frac{4}{5}(3)^{3/2} N(a^2x)^{5/4} = \frac{4}{5}(3)^{3/2} x^{5/4}. \tag{22}$$

This is *exactly* the coefficient in the exponential in (21). Estimating the contribution of the fluctuations of the eigenvalue at the top of the wall should give the (imaginary) coefficient in front of the exponentially small term in the free energy with the correct  $x^{-1/8}$  power<sup>\*</sup>.

#### 4. $\Phi^4$ and $\Phi^6$ potentials and multicritical points

This analysis can be applied to more general potentials and to the multicritical points. Let us first consider the quartic potential

$$V(\lambda) = \frac{1}{2}g\lambda^2 + \frac{1}{4}\lambda^4. \tag{23}$$

We take the real axis as contour of integration in the matrix integral (3). Already at the classical level  $g = \infty$  there are two phases: phase I for  $-3\pi/4 < \operatorname{Arg}(g) < 3\pi/4$  where eigenvalues concentrate at one minimum  $\lambda_c = 0$ , and phase II for  $3\pi/4 < \operatorname{Arg}(g) < 5\pi/4$  where the eigenvalues are equally concentrated at the two minima  $\lambda_c = \pm \sqrt{g}$ .

The whole phase diagram in the complex  $g$ -plane is depicted in fig. 5. In phase I the eigenvalues are distributed along one arc. In phase II the eigenvalues are distributed along two arcs (symmetric with respect to the origin). These two phases are separated by the shaded singular domain where no large- $N$  solution exists. The two critical points for imaginary  $g$  correspond to the standard  $m = 2$  critical points of pure gravity. The critical point at  $g_c = -2$  is the critical point described in ref. [21], where the segment  $C$  breaks into two segments. It has recently been shown that in the double scaling limit this critical point is associated to the Painlevé II equation [14] which was previously encountered in unitary matrix models [23]. This equation writes

$$x = \frac{1}{4}o^2 - 2e, \quad 0 = eo - o'', \tag{24}$$

<sup>\*</sup> A priori this coefficient cannot be zero, since it is proportional to the inverse of the square root of the curvature of the effective potential  $G$  at the top of the wall, and since when applying the steepest descend method one necessarily pass through the saddle point.

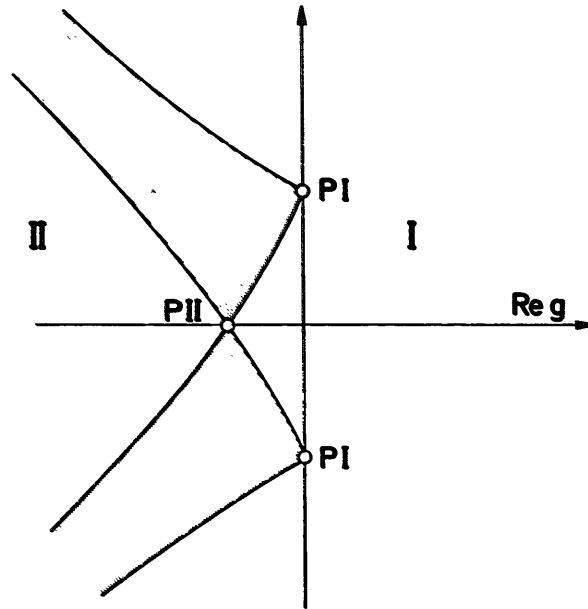


Fig. 5. The phase diagram of the quartic model in the complex  $g$ -plane. In the shaded domain no large- $N$  solution exists. In domain I (resp. domain II) a one-cut (resp. two-cut) solution exists. The two PI critical points correspond to the two solutions of  $m = 2$  pure gravity. The PII critical point is associated to the Painlevé II equation.

where  $x$  is the scaling variable,  $x \propto (g - g_c)$ .  $e$  and  $o$  are the scaling functions. It is argued in ref. [14] that the physical solution is the unique solution with fixed boundary conditions,

$$\begin{aligned} o &\sim x^{1/2}, & e &\sim x^{-2}, & x &\rightarrow +\infty, \\ o &\sim 0, & e &\sim x, & x &\rightarrow -\infty. \end{aligned} \quad (25)$$

In both regimes the exponentially small terms are of order  $\exp(-\text{const.}|x|^{3/2})$ . They become dominant if  $|\text{Arg}(|x|)| > \pi/3$ , and therefore in those two sectors we expect that this solution of eq. (24) has an infinite number of poles. As for the Painlevé I case it is precisely in those two sectors that there is no large- $N$  solution for the matrix model.

Let us now discuss the tricritical ( $m = 3$ ) point, which corresponds to gravity coupled to Lee–Yang matter [24]. It can be obtained by starting from a matrix model with potential

$$V(\lambda) = \frac{1}{2}g_1\lambda^2 + \frac{1}{4}g_2\lambda^4 + \frac{1}{6}\lambda^6, \quad (26)$$

and by fine tuning  $g_1$  and  $g_2$  so that the support  $C$  of the eigenvalues is one arc with end points  $\pm\lambda_c$  and that near those end points the effective potential

$G' = V' - 2F$  behaves as

$$G'(\lambda) \sim (\lambda - \lambda_e)^{5/2}. \tag{27}$$

Before going to explicit calculations one can already draw general conclusions by considering the allowed configurations for the support of eigenvalues  $C$  and the stability domain  $D$ . At the tricritical point this configuration in the vicinity of the end point  $\lambda_e$  is depicted in fig. 6a. It has three distinct sectors starting from  $\lambda_e$  and going to  $\infty$ . Therefore there must be three distinct tricritical solutions, which depends on the sector chosen as boundary condition at  $\infty$  in the integral (3). Only the solution corresponding to the sector which contains the real axis can be real. It should correspond to the real solution of the tricritical string equation found numerically in ref. [13]. When moving away from the tricritical point by changing  $g_1$  or  $g_2$  but keeping them real,  $C$  and  $D$  can only be deformed as depicted in fig. 6. Fig. 6b corresponds to the deformation of the tricritical point into the ordinary critical point associated to the Painlevé I equation. However it is clear that the real tricritical solution cannot be deformed into a critical solution since the path of integration (the real axis) is no more contained in  $D$ . On the contrary the two complex tricritical solutions can be deformed into the corresponding complex critical solutions. Fig. 6c corresponds to the deformation of the tricritical point into the critical point described in ref. [21] where the support  $C$  breaks up into two segments, which is associated to the Painlevé II equation [14, 23]. One sees that the real tricritical solution can a priori be deformed into this critical solution. Figs. 6d–g describe deformations into the noncritical case ( $m = 1$ ).

Those global features are confirmed by explicit calculations. Recent analytical and numerical investigations of the phase diagram of the large- $N$   $\Phi^6$  matrix model by Jurkiewicz [25] and by Bhanot et al. [26] show that the  $m = 3$  tricritical point can be deformed into the  $m = 1$  noncritical point and into the PII critical point but that it is impossible to reach the  $m = 2$  critical point. The stability criteria used by these authors are similar to those discussed in sect. 2. However they discuss neither the case of complex potentials nor the dependence on the integration path in the matrix model (implicitly the integration over the  $\lambda$ 's is made on the real axis). Our approach allows explicit calculations for generic complex potentials. Starting from the potential (26) the tricritical point is given by

$$\lambda_e = \left(\frac{32}{5}\right)^{1/6}, \quad g_1^{\text{tricr}} = \frac{15}{8}\lambda_e^4, \quad g_2^{\text{tricr}} = -\frac{5}{2}\lambda_e^2. \tag{28}$$

The scaling limit is obtained by rescaling,

$$\left. \begin{aligned} \lambda &= \lambda_e(1 + az) \\ g_1 &= g_1^{\text{tricr}}(1 + 2a^3T_0 + a^2T_1) \\ g_2 &= g_2^{\text{tricr}}(1 + a^3T_0 + a^2T_1) \end{aligned} \right\} a \rightarrow \infty. \tag{29}$$

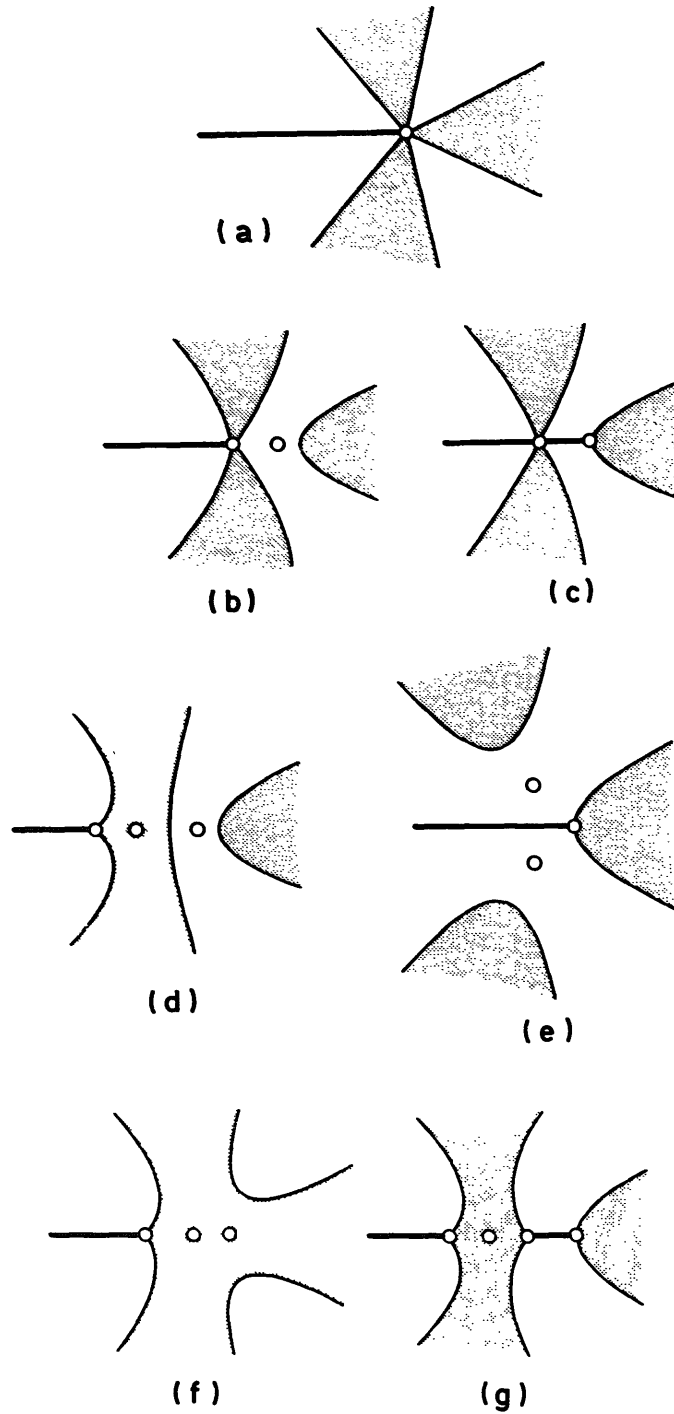


Fig. 6. Structure of C and D near the tricritical point for real potentials: (a) the tricritical point; (b) deformation into the ordinary PI critical point; (c) deformation into the PII critical point; (d)–(g) deformations into noncritical points.

The one-cut solution becomes after some algebra

$$G'(\lambda) = a^{5/2} 2^{5/2} \lambda_e^5 \sqrt{z - z_c} (z - z_+) (z - z_-), \tag{30}$$

with

$$\begin{aligned} \frac{8}{3} z_c^3 - 2T_1 z_c + T_0 &= 0, \\ z_{\pm} &= -\frac{1}{4} \left( z_c \pm \sqrt{-5z_c^2 + \frac{15}{2} T_1} \right). \end{aligned} \tag{31}$$

We denote the rescaled coupling constant by  $T_0$  and  $T_1$  since they are similar to the scaling fields  $T_i$  which allow us to couple relevant operators  $O_i$  to the  $m = 3$  critical theory (see for instance ref. [14]).  $T_0$  is nothing but the cosmological constant  $x$  and  $T_1$  couples to the  $m = 1$  theory. A shift in  $z$  ( $z = \bar{z} + \sqrt{T_1}/2$ ) maps the equation for  $z_c$  into

$$\begin{aligned} \frac{8}{3} \bar{z}_c^3 + 2\tilde{T}_2 \bar{z}_c^2 + \tilde{T}_0 &= 0, \\ \tilde{T}_2 = 2\sqrt{T_1}, \quad \tilde{T}_0 = T_0 - \frac{2}{3} T_1^{3/2}, \end{aligned} \tag{32}$$

where, as we shall see later,  $\tilde{T}_2$  couples to the  $m = 2$  ordinary critical theory.

The “purely” tricritical theory is obtained by setting  $T_1$ , or equivalently  $\tilde{T}_2$ , to zero.  $T_0 = \tilde{T}_0$  is the cosmological constant  $x$ . We can study how the effective potential  $G$  changes as  $x$  is rotated in the complex plane, as done previously for the  $m = 2$  case. If we start from real positive  $x$ ,  $z_c \propto -x^{1/3}$  is negative and we are in the situation depicted in fig. 6f (but the two zeros  $z_{\pm}$  are not on the positive real axis). This situation is allowed for the three possible integration contours ( $\text{Arg}(z) = 0, \pm 4\pi/7$ ) at infinity.

The dependence on the integration contour appears when  $x$  becomes complex. Let us first take a contour such that  $\text{Arg}(z) = 4\pi/7$  as  $|z| \rightarrow \infty$ , the solution can be continuously deformed in the sector

$$-\frac{3}{7}(3\pi - \theta) < \text{Arg}(x) < \frac{3}{7}(\pi - \theta), \quad \theta = \arctan(1/\sqrt{5}), \tag{33}$$

which contains the whole real axis and the lower half-plane. In this sector the susceptibility  $f(x)$  is analytic and is proportional to  $x^{1/3}$ . Therefore with this boundary condition in the double scaling limit  $a \rightarrow 0$ ,  $a^7 N^2 = 1$ , we must obtain the (presumably) unique but complex solution of the string equation  $R_3(f) = x$  which behaves as  $x^{1/3}$  as  $x \rightarrow +\infty$  and  $j^2|x|^{1/3}$  as  $x \rightarrow -\infty$ . This solution can be deformed explicitly into the  $m = 2$  critical solution, as argued above. Indeed setting now  $\tilde{T}_0 = 0$ ,  $\tilde{T}_2 < 0$  the solution stays real but  $z_- = z_c$ , which shows that we are now in the situation depicted in fig. 6b.

Let us now consider the real contour  $\text{Arg}(z) = 0$ . The solution for real positive  $x$  stays stable only in the sector

$$-\frac{3}{7}(\pi + \theta) < \text{Arg}(x) < \frac{3}{7}(\pi + \theta). \quad (34)$$

However for real negative  $x$  the solution with positive  $z_c \propto (-x)^{1/3}$  is allowed since it corresponds to the situation of fig. 6e. This solution is stable in the sector

$$-\frac{3}{7}\theta < \text{Arg}(-x) < \frac{3}{7}\theta, \quad (35)$$

and in this sector the susceptibility is proportional to  $(-x)^{1/3}$ . Therefore with this contour we must obtain in the double scaling limit the real solution of ref. [13] of the string equation. This solution can also be deformed into an  $m = 2$  solution. However this can be achieved for  $\tilde{T}_0 = 0$  and  $\pi/7 < \text{Arg}(\tilde{T}_2) < 5\pi/7$ , which corresponds to a complex potential, and is therefore unphysical.

This line of argument is quite general. A  $m$ -critical point corresponds to an effective potential  $G$  of the form

$$G(\lambda) \sim (\lambda - \lambda_c)^{(2m+1)/2}. \quad (36)$$

It can be reached from a matrix model with real boundary conditions only for  $m$  odd, and it is almost obvious that a real odd-critical solution cannot be deformed into an even-critical solution, as argued in ref. [13], as long as complex deformations are forbidden.

## 5. Conclusion

In conclusion we have shown that the large- $N$  solution of the hermitian matrix model of ref. [20] can be generalized to complex potentials and that one can discuss in this framework the dependence on the boundary conditions in the matrix integral. A regular large- $N$  solution does not always exist, or may be unstable, but we have found on the example of pure gravity that the region in phase space where there is no solution coincides with the domain where poles exist for some solutions of the string equation, and that this fixes uniquely which solution is compatible with the matrix model formulation. For pure gravity no real solution of the Painlevé I equation is allowed. The only acceptable solutions are the two (complex conjugate) triply truncated solutions of Boutroux, and corresponds to the two possible ways to reach an  $m = 2$  critical point by analytic continuation in the space of potentials for the one-matrix model. They have exponentially small imaginary parts which can be interpreted as instanton effects of the original matrix model.

Those features seem general, since they persist on the example of the higher critical points that we have considered. This suggests that many of the global



features of the solutions of the string equations, in particular the crucial issue of the boundary conditions, can be understood already at the semiclassical level in the matrix model formulation.

Let us come back to the problem of the nature of the flows between the various multicritical points. We have seen on the example of the  $m = 3$  theory that by looking at the structure of the lines of constant argument of the effective potential  $G(\lambda)$  in the complex plane and at their deformations as the potential  $V(\lambda)$  is changed, one can see which deformations are possible between the various critical points. In a recent paper [27] Moore, among many interesting things, studies by WKB methods the linear differential equations, whose compatibility conditions give the KdV string equations, and their monodromy properties. The structure of the Stokes lines in the complex plane for the spectral parameter  $\lambda$  that he finds in the case of low  $m$  is strikingly similar to the structure of the equiphase lines for the function  $G$  that we have found. This similarity suggests that there is probably a deep connection between those two treatments\*.

Finally let us briefly quote some open problems. It would be interesting to find a general treatment of multicritical points, in particular for the more general  $(p, q)$  systems which can be obtained from multi-matrix models. Our treatment of complex potentials should in particular allow us to treat non-perturbatively the singular potentials of ref. [8]. But obviously one of the most important issues is to understand if there is a way to define a consistent *real* solution of pure two-dimensional gravity or of gravity coupled to unitary matter.

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\* Note however that in ref. [27] it is claimed that no isomonodromy deformation allows a flow from  $m = 3$  to  $m = 2$ , contrary to what we have found in sect. 4. It is not clear to us whether Moore's analysis puts extra constraints on the flows or if his argument has to be modified when dealing with complex potentials. Indeed in our case the integration path over  $\lambda$  is not real, while Moore's argument relies on the value of the Stokes data on the real line.

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