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LOOP EQUATIONS AND NON-PERTURBATIVE EFFECTS IN TWO-DIMENSIONAL QUANTUM GRAVITY

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We present the loop equations of motion which define the correlation functions for loop operators in two-dimensional quantum gravity. We show that non-perturbative correlation functions constructed from real solutions of the Painlevé equation of the first kind violate these equations by non-perturbative terms.

1. Introduction

Two-dimensional quantum gravity may be formulated as a functional integral over the internal geometry of $2d$ manifolds. Some years ago, it was proposed to discretize this sum as a sum over random triangulations.[1] This allows us to map the discretized functional integral into an integral over random (Hermitian) matrices and to map the topological expansion (in terms of the genus of the $2d$ surface) into the large N expansion of the corresponding matrix model, where N is the dimension of the matrix. Various matrix models, which may correspond to pure gravity or to gravity coupled to some set of matter fields, can be solved by large N techniques and were shown to exhibit critical points where a continuum limit could be defined.[2] More recently those continuum limits were shown to agree with results obtained from continuum formulations of $2d$ gravity, based on conformal field theory techniques.[3] Most of those results were however restricted to *fixed* $2d$ topology.

Very recently it was shown that a continuum limit for the sum over all topologies of $2d$ manifolds may also be defined explicitly.[4,6] For pure gravity and gravity coupled to some matter fields with $c < 1$, it was shown that quantities such as the specific heat satisfy remarkable differential equations which define uniquely the perturbative topological expansion and which were suggested to lead to a non-perturbative definition of the theory.[4,8]

These results rely on a "constructive" approach. The discretized version of $2d$ gravity is defined explicitly by the random unitary matrix model. Integration over radial degrees of freedom reduces the problem to the statistics of the (real) eigenvalues of the matrix, which appears to be equivalent to the problem of non-

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interacting fermions in an external potential in 1 (space) dimension, thus to ordinary $1d$ quantum (statistical) mechanics.[9] However, the last step of this approach remains somewhat formal, since the ODE satisfied by the specific heat f defines only f up to non-perturbative terms which are invisible in the topological expansion. For the simplest case (pure gravity) suggestions have been made to fix these non-perturbative terms but it is not clear whether they really correspond to some physical requirement, or whether these ambiguities reflect the existence of new non-perturbative parameters of the theory.[4,8]

In this paper, we shall address this problem by using a different approach. In general a quantum field theory may be entirely defined by its Dyson-Schwinger equations, namely the equations of motion satisfied by the observables of the theory. Even if the equations of motion have been defined in perturbation theory, it is expected that they are satisfied by the full theory, irrespective of the phase in which the theory lives even if "non-perturbative effects" are present.

For $2d$ gravity the observables are probability amplitudes for loops. A natural question is how to define loop equations of motion in the continuum limit, and then to check whether non-perturbative $2d$ gravity satisfies these equations, and if these equations can be used to fix the non-perturbative parameters.

In this paper we deal with the pure gravity ($c = 0$) case. From the random matrix model, we define loop operators and write discretized loop equations. We then show how to take the continuum limit for these loop equations, and show that these equations allow us to compute recursively expectation values for loop operators at all order in the topological expansion. Finally we look at the consistency between the loop equations and the non-perturbative construction of $2d$ gravity. We show that for the non-perturbative constructions based on real solutions of the Painlevé equation of the first kind, loop equations cannot be satisfied, since they are necessarily violated by non-perturbative terms. In the conclusion we discuss the significance of this negative result and make some conjectures.

2. Loop Equations for Discretized Gravity

First we derive loop equations for discretized gravity, defined by the random matrix model, whose partition function is written as an integral over $N \times N$ Hermitian matrices,

$$Z = \int d\phi \, \exp\{-N \, \mathrm{Tr}[V(\phi)]\} \, . \tag{1}$$

These equations have already been derived in Refs. 10 and 11 and we shall discuss them mainly as an introduction to continuum loop equations. The operator corresponding to a loop with length K is[12,13] $1/N \, \mathrm{Tr}(\phi^K/K)$. A generating function for these operators is

$$W(L) = \frac{1}{N} \mathrm{Tr}(e^{L\phi}) \tag{2}$$

or its Laplace transform

$$\hat{W}(P) = \int_0^\infty dL e^{-LP} W(L) = \frac{1}{N} \mathrm{Tr}\left(\frac{1}{P - \phi}\right) . \tag{3}$$

The loop equations are obtained simply by performing the change of variable $\phi \to \phi' = \phi + \varepsilon f(\phi)$ in (1). The measure changes as

$$d\phi \to d\phi' = d\phi\left(1 + \varepsilon\oint\frac{dz}{2i\pi}f(z)\left[\mathrm{Tr}\left(\frac{1}{z-\phi}\right)\right]^2\right), \tag{4}$$

while the action changes as

$$\mathrm{Tr}V(\phi) \to \mathrm{Tr}V(\phi) + \varepsilon\mathrm{Tr}[V'(\phi)f(\phi)]\,. \tag{5}$$

Taking for a particular function f the function

$$f(z) = \frac{1}{P-z}\,, \tag{6}$$

we get

$$d\phi = d\phi[1 + \varepsilon N^2[\hat{W}(P)]^2] \tag{7}$$

and

$$\mathrm{Tr}[V(\phi')] = \mathrm{Tr}V(\phi) + \varepsilon N\int_{-i\infty}^{+i\infty}\frac{dQ}{2i\pi}\frac{1}{Q-P}V'(Q)\hat{W}(Q)\,. \tag{8}$$

To be more specific we shall now restrict ourselves to the particular potential

$$V(\phi) = \frac{\mu}{2}\phi^2 - \frac{1}{3}\phi^3\,, \tag{9}$$

which is sufficient to get the critical point corresponding to pure gravity. We shall consider the connected correlation functions for M loop operators, $\langle W_1 \ldots W_M\rangle_c$, defined from the ordinary ones $\langle\ldots\rangle$ by

$$\langle W_1\ldots W_M\rangle = \sum_{\{X_I\}}\prod_{I=1}^{Q}\left\langle\prod_{i\in X_I}W_i\right\rangle_c (N^2)^{1-Q}\,, \tag{10}$$

where the sum runs over all partitions $\{X_I\}$ of the set of operators, Q being the number of elements of the partition. Using (7), (8) and (9) we get for $M > 0$

$$\begin{aligned}
&[\mu P - P^2 - 2\langle\hat{W}(P)\rangle_c]\langle\hat{W}(P)\hat{W}(P_1)\ldots\hat{W}(P_M)\rangle_c \\
&= \sum_{\substack{I\cup J=[1,M]\\ I,J\neq\varnothing}}\left\langle\hat{W}(P)\prod_{i\in I}\hat{W}(P_i)\right\rangle_c\left\langle\hat{W}(P)\prod_{j\in J}\hat{W}(P_j)\right\rangle_c \\
&\quad+\sum_{i=1}^{M}\left\langle\hat{W}(P_1)\ldots\frac{\partial}{\partial P_i}\left[\frac{\hat{W}(P_i)-\hat{W}(P)}{P_i-P}\right]\ldots\hat{W}(P_M)\right\rangle_c \\
&\quad+\frac{1}{N^2}\langle\hat{W}(P)\hat{W}(P)\hat{W}(P_1)\ldots\hat{W}(P_M)\rangle_c \\
&\quad+\langle[(\mu-P)W(0) - W'(0)]\hat{W}(P_1)\ldots\hat{W}(P_M)\rangle_c\,,
\end{aligned} \tag{11}$$

while for $M = 0$

$$(\mu P - P^2)\langle \hat{W}(P)\rangle_c = \langle \hat{W}(P)\rangle_c^2 + (\mu - P)\langle W(0)\rangle_c - \langle W'(0)\rangle_c + \frac{1}{N^2}\langle \hat{W}(P)\hat{W}(P)\rangle_c \,. \tag{12}$$

These equations, although lengthy, have a natural geometrical interpretation when formulated in terms of $W(L)$ (if one view L as a length variable) which is discussed in Ref. 12. They may be written in a compact functional form.[10,11] For instance (12) corresponds to

$$V'\left(\frac{\partial}{\partial L}\right)\langle W(L)\rangle_c = \int_0^L dL'\{\langle \hat{W}(L')\rangle_c \langle \hat{W}(L-L')\rangle_c + \frac{1}{N^2}\langle \hat{W}(L')\hat{W}(L-L')\rangle_c\}\,. \tag{13}$$

The loop equations contain two families of "constants of integration" which are the operators $W(0)$ and $W'(0)$ and which have to be fixed by some consistency requirement. The first one is easily fixed since

$$W(0) = \frac{1}{N}\mathrm{Tr}(\mathbb{1})\,, \tag{14}$$

so that $\langle W(0)\rangle_c = 1$ and $W(0)$ gives zero in higher connected correlation functions. The second one corresponds to

$$W'(0) = \frac{1}{N}\mathrm{Tr}(\Phi) \tag{15}$$

and is fixed as discussed below.

We first discuss the loop equations in the planar limit ($1/N^2 = 0$), where the main features already appear. In the case for $M = 0$, Eq. (12) involves only $\langle \hat{W}(p)\rangle_c$ and is solved as

$$\langle \hat{W}(P)\rangle_c = \frac{1}{2}[(\mu P - P^2) - \sqrt{\Delta(P)}] \tag{16}$$

with

$$\Delta(P) = (\mu P - P^2)^2 + 4[P + \langle W'(0)\rangle_c - \mu]\,. \tag{17}$$

In Ref. 10, $\langle W'(0)\rangle_c$ is fixed by the requirement that $\langle \hat{W}(P)\rangle_c$ has only one cut $[a, b]$ in the complex P plane, which in the Gaussian limit $\mu \to \infty$ should be located at $[-2/\sqrt{\mu}, +2/\sqrt{\mu}]$ (this corresponds to the interval in which the eigenvalues of ϕ are located, according to Wigner's law). However, in general, $\Delta(P)$ has four simple zeros and $\hat{W}(P)$ two cuts, one located close to the origin and the other at large positive P. Thus $\langle W'(0)_c\rangle$ has to be fine-tuned so that the two zeros of Δ with the largest real part coalesce to give a double zero at P. $\Delta(P)$ is then of the form $\Delta(P) = (P - P_0)^2(P - a)(P - b)$, and $\langle \hat{W}(P)\rangle_c$ is analytic along the real axis for $P > b$.

We now turn to higher correlations functions ($M > 0$). In the planar limit the left-hand side of (11) involves the $(M + 1)$ loops correlation function, while the right-hand side involves only $M' \leq M$ loops correlation functions. Thus (11) may be used to compute recursively all the connected functions. However, the double zero P_0 of $\Delta(P)$ corresponds to a single zero of $[\mu P - P^2 - 2\langle \hat{W}(P)\rangle_c]$ and therefore the $(M+1)$ loops function $\langle \hat{W}(P)\hat{W}(P_1)\ldots\hat{W}(P_M)\rangle_c$ has a pole at P_0, unless the right-hand side of (11) vanishes at $P = P_0$. It is precisely this analyticity requirement which fixes uniquely the correlation function $\langle W'(0)\hat{W}(P_1)\ldots\hat{W}(P_M)\rangle_c$.

The same procedure can be used to go beyond the planar limit and to compute recursively the correlation functions at all orders in the topological expansion in powers of N^{-2}. Indeed, in order to extract the M-loops function at order N^{-2k} it is enough to know the M-loops functions for $M' \leq M + K$ at lower orders in N^{-2}. At each order of the recursion, the condition that the functions must be analytic at $P = P_0$ will fix $W'(0)$.

3. Loop Equations for Continuum Gravity

In the planar limit, the critical point $\mu = \mu_c$ is reached when the double zero $P_0(\mu)$ reaches the cut starting at $b(\mu)$. In the vicinity of the critical point, one may express the variables in terms of a regulator a (with dimension of length):

$$\begin{aligned} \mu &= \mu_c + a^2\Lambda\,, \\ P &= P_c + az\,, \end{aligned} \tag{18}$$

where the critical values μ_c and P_c are uniquely characterized by the requirement that in $\Delta(p,\mu)$ the terms of orders a and a^2 vanish identically. For our potential, this implies

$$P_c = [5 + 3\sqrt{3}]^{1/3}\,. \tag{19}$$

Then it appears that the M-loop function scales as $a^{5-(7/2)M}$, but for the 1-loop function which has a finite part equal to $1/2(\mu P - P^2)$. Defining renormalized correlation functions for continuum operators $\hat{w}$ as

$$\langle \hat{W}(P)\rangle_c = \frac{1}{2}(\mu P - P^2) + a^{3/2}\langle \hat{w}(z)\rangle_c\,, \tag{20}$$

$$\langle \hat{W}(P_1)\ldots\hat{W}(P_M)\rangle_c = a^{5-(7/2)M}\langle \hat{w}(z_1),\ldots,\hat{w}(z_M)\rangle_c\,, \tag{21}$$

and the "string coupling constant" G as

$$G = N^{-2}a^5\,, \tag{22}$$

in the continuum limit $a \to 0$, Λ, z and G being fixed, the equations of motion reduce to

$$\langle \hat{w}(z)\rangle_c^2 + G\langle \hat{w}(z)\hat{w}(z)\rangle_c = Az^3 - Bz\Lambda + \langle P\rangle_c \tag{23}$$

and for $M > 0$

$$2\langle \hat{w}(z)\rangle_c\langle \hat{w}(z)\hat{w}(z_1)\ldots\hat{w}(z_M)\rangle_c + G\langle \hat{w}(z)\hat{w}(z)\hat{w}(z_1)\ldots\hat{w}(z_M)\rangle_c$$
$$+ \sum_{I\cup J=[1,M]}\langle \hat{w}(z)\prod_{i\in I}\hat{w}(z_i)\rangle_c\langle \hat{w}(z)\prod_{j\in J}\hat{w}(z_j)\rangle_c$$

$$+ \sum_{i=1}^{M} \langle \hat{w}(z_1) \ldots \frac{\partial}{\partial z_i} \frac{\hat{w}(z_i) - \hat{w}(z)}{z_i - z} \ldots \hat{w}(z_M) \rangle$$

$$= \langle P\hat{w}(z_1) \ldots \hat{w}(z_M) \rangle , \tag{24}$$

where A and B are some strictly positive constants and P is the singular part (of order $a^{3/2}$) of the operator $W'(0)$.

Equations (23)and (24) are simpler than the discrete loop equations (11), (12) and are universal, since the non-universal coefficients A and B may be absorbed into a rescaling of $\hat{w}$, z and Λ. Λ is the renormalized cosmological constant and G the renormalized "string" coupling constant. In fact, up to a rescaling, all observables depends only on $z/\sqrt{\Lambda}$ and the scaling variable $[\Lambda G^{2/5}]$ but our convention allows a clearer discussion of the "semi-classical limit" $G \to 0$.

The renormalized loop operator $w(l)$ defined from $\hat{w}(z)$ by inverse Laplace transform,

$$w(l) = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2i\pi} e^{zl} \hat{w}(z) , \tag{25}$$

is proportional to the original operator creating a loop of length $l = Ka$:

$$\frac{1}{N} \mathrm{Tr}(\phi^K) = a^{-5/2} (P_c)^{(1+l/a)} w(l) . \tag{26}$$

Thus, in the continuum limit l corresponds really to the length of the loop. The a dependent factor in (26) corresponds to some kind of wave-function renormalization.

The continuum loop equations (23) and (24) contain the unknownlocal operator P, which is determined by a consistency condition similar to the one fixing $W'(0)$. As was done before we can solve iteratively the loop equations. At genus zero order ($G = 0$), (23) reads (with proper normalization) $\langle \hat{w}(z) \rangle_c^2 = z^3 - 3\Lambda z + \langle P \rangle_c$ so that for generic $\langle P \rangle_c$,$\langle \hat{w}(z) \rangle_c$ has a cut in the right half plane (Re $z > 0$) and another cut on the negative real axis. The path of integration in (25) when defining $\langle w(l) \rangle$ has to be taken to the right of both cuts, otherwise $\langle w(l) \rangle$ would be non-zero for negative length! Then $\langle w(l) \rangle$ grows exponentially for large l, which is quite unphysical. The condition that $\langle \hat{w}(z) \rangle$ has only one cut, on the negative real axis, is equivalent to the requirement that $\langle w(l) \rangle$ must decrease exponentially for large l and fixes $\langle P \rangle$ to be equal to $2\Lambda^{3/2}$, so that

$$\langle \hat{w}(z) \rangle_{G=0} = (z - \Lambda^{1/2}) \sqrt{z + 2\Lambda^{1/2}} \tag{27}$$

has a single zero on the positive real axis instead of a cut. Starting from (27), the loop equations can be solved iteratively to compute loops correlation functions at arbitrary order in the topological expansion in powers of G. At each step the requirement that there are no singularities on the positive real axis in the z variables (equivalent to the requirement that the amplitudes for large loops must be exponentially small) fixes the matrix elements of the unknown operator P.

If the explicit calculations should quickly become very cumbersome, one can obtain from (23) and (24) the large z behavior of the operator $\hat{w}(z)$, that is, the small length limit of the loop operator $w(l)$. Indeed, the dominant term of $\langle \hat{w}(z)\rangle_c$ at large z comes only from the genus zero contribution (27) which behaves as $z^{3/2}$. Hence one can prove that

$$\hat{w}(z) \sim \frac{1}{2z^{3/2}} P \quad \text{as } z \to \infty \tag{28}$$

except when $\hat{w}(z)$ appears in one- and two-loop functions at genus zero, which have additional more singular powers of z. Thus P is nothing but the so-called "puncture operator" which inserts an infinitesimal loop on the $2d$ surface.

4. Loop Equations and Non-Perturbative Gravity

We now discuss the relation between loop equation and the recent non-perturbative construction of two-dimensional gravity. This construction has already been discussed by many authors and we shall only start from the basic results for pure gravity.[4,8] In the continuum limit, the problem reduces to the study of quantum mechanics in 1 dimension with Hamiltonian

$$H = u(x) - G\frac{\partial^2}{\partial x^2}\,, \tag{29}$$

where the potential $u(x)$ obeys the Painlevé equation,

$$u^2(x) - \frac{G}{3}u''(x) = x\,. \tag{30}$$

Connected functions for the loop operator $\hat{w}(z)$ may be expressed simply by considering H as a one-body operator for a system of free fermions, with one-fermion states $|x\rangle$ labelled by their energy $E = -x$, with the Fermi level localized at the renormalized cosmological constant $E_F = -\Lambda$. Indeed, in the continuum limit, connected functions of the operators $\hat{w}(z)$ reduce to the vacuum expectation values of the corresponding products of the resolvents $1/(z + H)$. For instance,[7,8]

$$\langle \hat{w}(z)\rangle_c = \int_\Lambda^{+\infty} dx \langle x|\frac{1}{z+H}|x\rangle\,, \tag{31}$$

$$\langle \hat{w}(z_1)\hat{w}(z_2)\rangle_c = \int_\Lambda^{+\infty} dx \int_{-\infty}^{\Lambda} dy \langle x|\frac{1}{z_1+H}|y\rangle\langle y|\frac{1}{z_2+H}|x\rangle\,, \tag{32}$$

etc. Strictly speaking, these equations are valid only as formal power series in G and the integral in (31) has to be defined with a finite part prescription to deal with the divergences at ∞ (which are related to the non-scaling finite part in the free energy and the 1-loop connected function). $u(\Lambda)$ corresponds to the string susceptibility and should be positive in the planar limit ($\Lambda \to \infty$ or $G = 0$). Thus for $G = 0$, $u(x) = +x^{1/2}$ and it is known with that this initial condition, from Eq. (30), all terms of the perturbation expansion of u in terms of G are known:

$$u(x) = x^{1/2} - \frac{G}{24}x^{-2} - \frac{49}{1152}G^2 x^{-9/2} + \dots\,. \tag{33}$$

Similarily as $G \to 0$, the resolvent behaves as

$$\langle x| \frac{1}{z+H} |x\rangle \underset{G\to 0}{\simeq} \frac{1}{2} \frac{G^{1/2}}{[z + x^{1/2}]^{1/2}} , \tag{34}$$

hence

$$\langle \hat{w}(z) \rangle \underset{G\to 0}{=} \frac{4}{3} \sqrt{g} \left(z - \frac{1}{2}\sqrt{\Lambda} \right) (z + \sqrt{\Lambda})^{1/2} , \tag{35}$$

which coincides, up to finite rescalings, with (27).

The real solutions of (30) such that $u(x) \sim x^{1/2}$ at infinity are known to have an infinite series of double poles on the real axis, which accumulate at $x = -\infty$.[14] Moreover, there is an infinite family of such solutions, which may be labelled for instance by the position of the first pole on the real axis. Those solutions have the same asymptotic expansion as a formal power series in G. Indeed they differ only by exponentially small terms of order

$$\Delta u \simeq x^{-1/8} \exp\left(-\frac{4}{5}\sqrt{\frac{6}{G}} x^{5/4} \right) . \tag{36}$$

At any double pole x_i, the potential u diverges as

$$u(x) \cong \frac{2G}{(x-x_i)^2} + O[(x-x_i)^2] . \tag{37}$$

This is enough for all eigenfunctions of the Hamiltonian H (29) and for the resolvent $\langle x | 1/(z + H) | y\rangle$ to vanish at x_i. In other words, there is no tunneling through the poles and the eigenstates stay localized between two successive poles $[x_i, x_{i-1}]$ or between the first pole x_1 and $+\infty$. This leads various authors[5,7,8] to suggest that a non-perturbative definition of two-dimensional gravity could be obtained by taking a real solution of (29) characterized by its first pole x_1, and by defining the correlation functions by Eqs. (31) and (32) when taking the resolvent $\langle x | 1/(z + H) | y\rangle$, which has support $[x_1, +\infty]$ and vanishes on $[-\infty, x_1]$ ("perturbative phase"), or even by taking the resolvent with support between two successive poles $[x_{i+1}, x_i]$ ("non-perturbative phase"). The singularity at $\Lambda = x_1$ might correspond to a "condensation of handles". With such a proposal, the main issue is obviously to understand the meaning of the non-perturbative parameter x_1, which label the "non-perturbative solutions", and to understand whether it can be fixed by some physical requirement or whether it corresponds to a new physical parameter of the theory, like the θ-angle in $4d$ gauge theories.[5]

In fact, it is easy to see that none of these solutions satisfies the continuum equations of motion (23), (24). Let us consider the "perturbative phase" where we define the 1- and 2-loop correlation functions by (31) and (32) by taking the resolvent with support $[x_1, +\infty]$. As stressed in Ref. 7, a non-perturbative property of the Hamiltonian H (29), when quantized in the interval $[x_1, \infty]$, is that it has a discrete spectrum with eigenvalues λ_i. Each eigenfunction should behave as

$$\psi_i(x) \simeq (x - x_1)^2, \qquad x \to x_1$$
$$\psi_i(x) \simeq \exp -\frac{4}{5}\sqrt{G}\, x^{5/4}, \quad x \to \infty. \tag{38}$$

Therefore $\langle \hat{w}(z)\rangle_c$ has a single pole for each eigenvalue $z = -\lambda_i$, while $\langle \hat{w}(z)\hat{w}(z)\rangle_c$ has a double pole. If we write the left-hand side of the equation of motion (23) for $M = 0$, we see that

$$\langle \hat{w}(z)\rangle_c \langle \hat{w}(z)\rangle_c + \langle \hat{w}(z)\hat{w}(z)\rangle_c \underset{z\to-\lambda_i}{=} \frac{1}{(z+\lambda_i)^2}\int_\Lambda^\infty dx|\psi_i(x)|^2. \tag{39}$$

Thus the coefficient of the double pole, although exponentially small in Λ from (38), is non-zero. However the right-hand side of (23), although a complicated function of Λ, is a polynomial of degree 3 in z and cannot have any double poles. Thus if H has a discrete spectrum loop equation, (23) cannot be satisfied!

The situation is worse if we take as support for H and for the resolvent the interval between two poles of u, since then the coefficient of the double pole is not even exponentially small in Λ. In fact, one can even take in (38) and (39) the resolvent as defined in any interval $[x_i, +\infty]$, or even for any $[a, +\infty]$, a not being necessarily a pole. Indeed, the resolvent $\langle x|1/(z + H)|y\rangle = R(x, y; z)$, as defined by the equation

$$(z + H_x)R(x, y; z) = (z + H_y)R(x, y; z) = \delta(x - y) \tag{40}$$

with the boundary condition

$$R = 0 \text{ as } x \text{ or } y = a \text{ or } +\infty, \tag{41}$$

will be a meromorphic function of x and y with single poles at the double poles x_i of u, but one can check that the correlation functions defined by (31), (32) have poles at each $\Lambda = x_i$ but no cuts and are therefore also acceptable in the interval $\Lambda \in [x_1, +\infty]$. However, in any case the coefficients of the double poles of (33) will be given by (39) and cannot vanish identically.

5. Discussion

We have still far from a complete understanding of the relation between loop equations and the non-perturbative formulation of $2d$ gravity. In this section we shall discuss some open problems.

Although we have shown that for the non-perturbative definitions of $2d$ gravity proposed insofar the loop equations are violated by non-perturbative terms, we have not been able to show directly that, to all orders in the perturbative expansion, the two constructions coincide. We have only checked by explicit calculations to the lowest orders and for correlation functions with a small number of loops that the two approaches give the same result.

The loop equations (23), (24) have been derived from the random matrix model. Their left-hand side which seems somewhat complicated has in fact a simple geometrical interpretation in terms of loop operators $w(l)$. Indeed, then it corresponds to insertion and deletion of one loop while keeping the total length of the loops constant.[10,12] We have taken these equations as a definition of $2d$ gravity but it is not excluded that some additional terms (for instance some "non-perturbative condensate") might appear. A "loop field theory" derivation of the loop equations (perhaps in the spirit of string field theory) would be very helpful. One may notice that the loop equations for "ordinary" $2d$ gravity bear some similarities with the recursion relations written by Witten for topological $2d$ gravity,[15] at this moment we are however unable to elaborate further in this direction.

We have shown that it is quite implausible that real solutions of the Painlevé equation (30) might lead to a non-perturbative definition of $2d$ gravity. However, one might speculate that there is a relation between *complex* solutions of (30) and the original matrix model. This original model suffers from the defect that, in order to get a continuum limit corresponding to pure gravity, the action $\mathrm{Tr}[V(\phi)]$ is unbounded from below. However, the partition function may be defined by analytic continuation. For instance, starting from the potential

$$V_\lambda(\phi) = \frac{\phi^2}{2} - \frac{\lambda}{4}\phi^4 , \tag{42}$$

the partition function (1) is defined for $\mathrm{Re}\,\lambda < 0$. Rotating simultaneously λ and the integration path for the matrix elements of ϕ in the complex plane, one can easily show that $Z(\lambda)$ has only a square root singularity at $\lambda = 0$ and may be analytically continued into the whole doubly covered punctured plane $\mathbb{C} - \{0\}$. Thus for finite but large N and in the vicinity of the critical point λ_c for the $N = \infty$ theory, which is on the positive real axis, the matrix model admits two (complex conjugate) definitions, which are for instance obtained by iterating the recurrence relations for the coefficients R_n of the orthogonal polynomials, starting from the two possible analytic continuations for the initial term $R_0(\lambda) = \int_{-\infty}^{+\infty} dx \exp[-N V_\lambda(x)]$. In the planar limit $N = \infty$, $a \to 0$, these two definitions should give the two possible determinations on the whole real axis of the susceptibility $f(\Lambda)$, namely

$$f(\Lambda) = \begin{array}{ll} \sqrt{\Lambda} & \text{if} \Lambda > 0, \\ \pm i\sqrt{|\Lambda|} & \text{if} \Lambda < 0 . \end{array} \tag{43}$$

Similarily, in the scaling limit for $G \neq 0$, it is plausible that the two determinations of the matrix model gives for the susceptibility f the two solutions of the Painlevé equation (30) which behave for both $\Lambda \to +\infty$ and $\Lambda \to -\infty$ as $\sqrt{\Lambda}$. Indeed, according to the analysis of Boutroux,[14] the Painlevé equation has a unique "triply truncated solution" (up to complex conjugation) with the asymptotics

$$u(x) \underset{x \to \pm\infty}{\simeq} \sqrt{x} \tag{44}$$

and no infinite set of double poles in a sector around the whole real axis. It would be worthwhile to prove (or disprove) this conjecture and to check whether the loop equations (23)–(24) are satisfied by the correlation functions obtained from the resolvent for this particular solution. This solution is nevertheless unphysical, since for large positive Λ, the susceptibility $f(\Lambda)$ should has an exponentially small but non-vanishing imaginary part

$$\mathrm{Im}\, f(\Lambda) \simeq \Lambda^{-1/8} \exp -\frac{4}{5}\sqrt{\frac{6}{G}}\Lambda^{5/4}, \tag{45}$$

which reflects the "instanton-like" imaginary part present in the partition function $Z(\lambda)$ of the original matrix model.

Finally it should be interesting to write loop equations for the $c \neq 0$ models like the "multicritical gravity" models or gravity coupled to various conformal field theories.

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