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RIGID RANDOM SURFACES AT LARGE d

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A model of two-dimensional random surfaces with extrinsic curvature energy is studied in the limit where the dimension of bulk space d is large. The large-d effective potential is constructed. For large surface tension the ground state is homogeneous and its properties are studied. For small enough surface tension, non-perturbative instabilities which break translation invariance in the plane of the membrane are shown to occur for large but finite wavelength. The relationships between this model, the bosonic string, the Liouville model and lattice random surface models are discussed.

1. Introduction

The influence of the rigidity (i.e. of the bending energy associated with the extrinsic curvature) on the statistical properties of two-dimensional surfaces with very small surface tension has been studied intensively during the last years. The importance of the rigidity was first pointed out by Helfrich [1], and by de Gennes and Taupin [2] for fluid membranes (such as vesicles or blood cells) and for microemulsions [3]. It was suggested that short-wavelength thermal undulations decrease the effective rigidity at large scales, generating a persistence length [2] beyond which normals to the surface are uncorrelated (crumpled phase). This effect was confirmed by the renormalization group calculation of Peliti and Leibler [4], and by further calculations by Förster [5] and Kleinert [6]★. Similar issues were discussed independently by Polyakov [9] in the context of string theories (see also ref. [10]). Both Förster [5] and Polyakov [9] suggested that string theories arise as effective theories for those models, the string tension being generated dynamically by dimensional transmutation as this is the case for the persistence length.

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★ Helfrich [7] and Förster [8] have recently emphasized that incompressibility of a fluid membrane may induce long-distance correlations between displacements of elements of the membrane. They argued that this effect may change the renormalization of the rigidity. However, no full renormalization-group treatment of such an effect has yet been given, and we shall not discuss this problem here.

Since then many publications have been devoted to this subject. Some authors have studied the classical solutions for a model of string with rigidity [11]. However, this should not be directly relevant for the understanding of the large-distance properties of the model, which are dominated by non-perturbative effects. Other studies [12, 13–15] are concerned with the properties of the quark-antiquark static potential in a model of string with rigidity (considered as an effective model for QCD flux tubes); the calculations in refs. [13, 14] are in fact incomplete, for a correct discussion of this problem we refer to ref. [15].

A useful tool to study non-perturbative phenomena in this model is the large-d limit, where d is the dimension of bulk space in which the two-dimensional surface is embedded. This large-d limit is quite similar to the large-n limit for n-component spin systems such as the $O(n)$ non-linear sigma model. It allows one to probe the structure of the surface at scales larger than the persistence length ξ_p in the critical domain where the surface tension is very small and the surface a very large object. This approach was initiated by one of the present authors in ref. [16], and by other authors [12–15, 17–19].

In this paper we shall present a detailed analysis of the model of random surfaces with bending rigidity and of its effective action in the large-d limit. A summary account of our results has already appeared in ref. [20]. Let us explain how this paper is organized and summarize our conclusions.

In sect. 2 we give the definition of the model and construct the large-d effective action. The model depends on two coupling constants: the bare bending rigidity κ_0 and the bare surface tension r_0, which corresponds physically to the chemical potential for the constitutive elements of the surface (whose typical size will define the ultraviolet cut-off). Then we find a homogeneous ground state (all points on the surface share the same properties) by extremizing the effective action. Such a ground state is found to have the properties expected from the renormalization group calculations. There is a critical line, defined by some value of the bare surface tension $r_0^{\text{crit}}(\kappa_0)$, where the surface becomes an infinite critical object. At this critical point, we still observe a persistence length and a non-zero surface tension. The β function agrees with the one-loop perturbative calculations of refs. [4–7], but we show that in the large-d limit, there is a 2-loop, contribution to β which was previously overlooked.

In sect. 3, we study the effect of small perturbations around this homogeneous ground state. We show that in the critical domain where the bare surface tension is close to (but larger than) the critical point, instabilities occur at finite wavelengths which destabilize the homogeneous ground state and lead to a spontaneous break-down of translation invariance in the plane of the surface.

In sect. 4 we discuss the physical significance of this large-d instability. According to the most plausible scenario, a first-order transition* should separate a homogeneous phase (which exists if the physical surface tension is large enough and where the large distance properties of the surface are described by the classical Nambu-

Goto model) from a non-homogeneous phase where the surface should become some fractal object at large distances. We also discuss rigid surfaces at finite d. We argue that if no instabilities exist, the effective action for the surface should be related to the Liouville model. The Liouville field ϕ appears as a bound state of the X field (describing the position of the membrane) which becomes massless at the critical point. The occurrence of tachyons in the Liouville model for $d>1$ leads us to conjecture that instabilities occur in the model of rigid surfaces for all values of the dimension for which the model makes sense (namely $d>2$). We also discuss the connection of this model with the bosonic string and with the random surface models defined on the lattice or with random triangulations.

2. The large d homogeneous solution

2.1. INTRODUCTION: THE LARGE d EFFECTIVE ACTION

Let us start with some definitions. The model describes a two-dimensional surface embedded in d-dimensional euclidean space. This surface is described by d bulk coordinates $X^i(\sigma)$ $(i=1,\dots,d)$, functions of two local coordinates σ^a $(a=1,2)$. The metric induced by the embedding is

$$g_{ab}(\sigma)=\partial_a X^i \partial_b X^j \delta_{ij}=\partial_a \boldsymbol{X}\cdot\partial_b \boldsymbol{X}. \tag{2.1}$$

The curvature properties of the surface are given by the second fundamental form:

$$\begin{aligned} K^i_{ab} &= D_a D_b X^i \\ &= \partial_a \partial_b X^i - \Gamma^c_{ab}\,\partial_c X^i , \end{aligned} \tag{2.2}$$

where Γ^c_{ab} is the Christoffel symbol. The intrinsic curvature is

$$R=\left(K^{ia}_a\right)^2-K^{ib}_a K^{ia}_b \tag{2.3}$$

and the extrinsic curvature is

$$\boldsymbol{K}^2=\left(K^{ia}_a\right)^2=(\Delta X)^2. \tag{2.4}$$

$\Delta=g^{ab}D_aD_b$ is the scalar laplacian. In units of length, X^i and g_{ab} have respectively dimension one and two; R and $(\boldsymbol{K})^2$ are the only scalar local operators with dimension -2 which, integrated over the surface, give a dimensionless action. It is well known that R depends only on the intrinsic metric g_{ab} while $\boldsymbol{K}^2$ depends explicitly on the embedding. Moreover, R is a total divergence and its integral over

* This nature of the transition at large d is still conjectural.

the surface gives the Euler characteristic (if the surface has no boundary) which is a topological invariant. We shall consider surfaces of fixed topology and thus we shall neglect this term in the action which reads finally:

$$S[X] = \frac{1}{2\alpha_0}\int d^2\sigma \sqrt{|g|}\,(K)^2 + r_0\int d^2\sigma \sqrt{|g|}\,. \tag{2.5}$$

$\kappa_0 = 1/\alpha_0$ is called the rigidity modulus; r_0 is the bare surface tension. κ_0 and r_0 have respectively dimension 0 and -2.

A one-loop calculation [4–6, 9] shows that the renormalized rigidity modulus $\kappa_R = 1/\alpha_R$ is given by

$$\kappa_R = \kappa_0 + \frac{d}{4\pi}\ln\frac{\mu}{\Lambda}\,. \tag{2.6}$$

μ is the renormalization scale and Λ some ultraviolet regulator.

We now derive the expression for the effective action in the large d limit. First we rescale the coupling constants into

$$\frac{1}{\alpha_0} = \frac{d}{\tilde{\alpha}_0}\,, \qquad r_0 = d\tilde{r}_0 \tag{2.7}$$

(we shall omit the $\tilde{}$ subscript in the following). As in ref. [9], we consider $g_{ab}(\sigma)$ as an auxiliary field and impose the constraint (2.1) by a Lagrange multiplier $\lambda^{ab}(\sigma)$. The generating functional $Z[J]$ is:

$$Z[J] = \int \mathscr{D}[X]\,\mathscr{D}[g]\,\mathscr{D}[\lambda]\exp\left(-d\left(S[X,g,\lambda] - \int d^2\sigma\, J\cdot X\right)\right), \tag{2.8}$$

$$S[X,g,\lambda] = \frac{1}{2\alpha_0}\int\sqrt{|g|}\,d^2\sigma\left[(\Delta X)^2 + \lambda^{ab}(\partial_a X\cdot\partial_b X - g_{ab})\right] + r_0\int\sqrt{|g|}\,d^2\sigma\,. \tag{2.9}$$

The functional integration measures $\mathscr{D}[X]$, $\mathscr{D}[g]$, $\mathscr{D}[\lambda]$ are defined in a covariant way with respect to the metric g_{ab} [21, 22]. The integration over X is gaussian and may be performed explicitly

$$Z[J] = \int \mathscr{D}[g]\,\mathscr{D}[\lambda]\exp(-dS_{\text{eff}}(g,\lambda,J))\,, \tag{2.10}$$

$$S_{\text{eff}}(g,\lambda,J) = \tfrac{1}{2}\operatorname{Tr}\ln\left(\Delta^2 - D_a\lambda^{ab}D_b\right) - \frac{1}{2\alpha_0}\int d^2\sigma\sqrt{|g|}\,\lambda^a_a + r_0\int d^2\sigma\sqrt{|g|}$$
$$- \frac{\alpha_0}{2}\int\!\!\int d^2\sigma\, d^2\sigma'\, J(\sigma)\left(\frac{1}{\Delta^2 - D_a\lambda^{ab}D_b}\right)_{\sigma\sigma'} J(\sigma')\,. \tag{2.11}$$

In the large-d limit the functional integral (2.10) is dominated by the saddle point of (2.11) with respect to g and λ. Defining as usual

$$X_{\rm cl}(\sigma) = -\frac{1}{d}\frac{\delta(\ln Z[J])}{\delta J(\sigma)} \tag{2.12}$$

and the effective action via the Legendre transform

$$\Gamma_{\rm eff}[X_{\rm cl}] = -\ln Z[J] - d\int \mathrm{d}^2\sigma\, J(\sigma)\cdot X_{\rm cl}(\sigma), \tag{2.13}$$

we get the expression for the effective action at large d:

$$\Gamma_{\rm eff}[X_{\rm cl}] = \Gamma_{\rm eff}[X_{\rm cl}, g[X_{\rm cl}], \lambda[X_{\rm cl}]] \tag{2.14}$$

with

$$\Gamma_{\rm eff}[X_{\rm cl}, g, \lambda] = dS[X_{\rm cl}, g, \lambda] + \tfrac{1}{2}d\,\mathrm{Tr}\ln\left(\Delta^2 - D_a\lambda^{ab}D_b\right). \tag{2.15}$$

$S[X_{\rm cl}, g, \lambda]$ is the classical action (2.9). $g[X_{\rm cl}]$ and $\lambda[X_{\rm cl}]$ are the saddle points of (2.15), $X_{\rm cl}$ being fixed, and are therefore functionals of $X_{\rm cl}$. $\Gamma_{\rm eff}[X_{\rm cl}]$ is explicitly reparametrization invariant. $\Gamma_{\rm eff}[X_{\rm cl}, g, \lambda]$ may be considered as the effective action for classical fields $X_{\rm cl}$, $g_{\rm cl}$, $\lambda_{\rm cl}$ and would have been obtained by the same process (introducing source terms for g and λ).

The $\mathrm{Tr}\ln(\ldots)$ in (2.15) is quadratically ultraviolet divergent. We have to introduce a regulator to define the trace in a way which does not break reparameterization invariance. We shall use a Pauli-Villars regulator by defining

$$\mathrm{Tr}_\Lambda \ln\left(\Delta^2 - D_a\lambda^{ab}D_b\right) = \mathrm{Tr}\left[\ln\left(\Delta^2 - D_a\lambda^{ab}D_b\right) - \ln\left(\Delta^2 - D_a\lambda^{ab}D_b + \Lambda^4\right)\right]. \tag{2.16}$$

Λ is a large mass and has dimension -1 in units of length. Zeta or heat-kernel regularizations could also have been used since they preserve reparameterization invariance.

2.2. THE SADDLE POINT EQUATIONS FOR PLANE CONFIGURATIONS

Eqs. (2.14) and (2.15) allow one, in principle, to compute the effective action for an arbitrary classical configuration, provided that the saddle point equations for g and λ have been solved! In practice, it is possible to solve those equations only in very simple situations. In ref. [16], the case of surfaces with the topology of the sphere (droplets) has been considered. In that case, a natural ansatz is to choose a metric g with constant curvature R and a classical field $X(\sigma) = cte$ (the center of mass of the surface). It was shown that such an ansatz was an extremum of the effective action provided that r_0 is larger than a critical value $r_0^{\rm crit}$ whose expression for the particular choice of regularization (2.16) is

$$r_0^{\text{crit}} = \frac{\Lambda^2}{8} - \frac{\Lambda^2}{8\pi} e^{-8\pi/\alpha_0}. \tag{2.17}$$

Moreover, the scalar curvature R of the surface is related to r_0 by an equation which reads, for r_0 close to r_0^{crit}:

$$R \simeq 24\pi\left(r_0 - r_0^{\text{crit}}\right). \tag{2.18}$$

Thus, as $r_0 \to r_0^{\text{crit}}$, $R \to 0$ and the surface becomes an object with intrinsic flat geometry and infinite area.

In this paper, we shall focus our attention to planar surfaces. An easy way to obtain such configurations for an arbitrary value of r_0 is to enforce periodic boundary conditions for the fields $\boldsymbol{X}$, g and λ

$$\begin{aligned} \boldsymbol{X}(\sigma_1 + L_1, \sigma_2) - L_1 \boldsymbol{u}_1 = \boldsymbol{X}(\sigma_1, \sigma_2 + L_2) - L_2 \boldsymbol{u}_2 = \boldsymbol{X}(\sigma_1, \sigma_2), \\ g_{ab}(\sigma_1 + L_1, \sigma_2) = g_{ab}(\sigma_1, \sigma_2 + L_2) = g_{ab}(\sigma_1, \sigma_2), \\ \lambda^{ab}(\sigma_1 + L_1, \sigma_2) = \lambda^{ab}(\sigma_1, \sigma_2 + L_2) = \lambda^{ab}(\sigma_1, \sigma_2). \end{aligned} \tag{2.19}$$

$\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two orthonormal vectors in bulk space. At the classical level, such conditions prevent a collapse of the classical extremum of (2.5) when $r_0 > 0$. Taking fluctuations into account, they allow us to obtain a flat saddle point even when $r_0 \neq r_0^{\text{crit}}$. The classical extremum of (2.5) corresponding to a plane is

$$\boldsymbol{X}_{\text{cl}} = \boldsymbol{X}_0 + \sigma_1 \boldsymbol{u}_1 + \sigma_2 \boldsymbol{u}_2. \tag{2.20}$$

We can compute the effective action of this classical configuration. Eq. (2.15) reads then:

$$\begin{aligned} \Gamma_{\text{eff}}[\boldsymbol{X}_{\text{cl}}, g, \lambda] = \tfrac{1}{2} d \operatorname{Tr} \ln\left(\Delta^2 - D_a \lambda^{ab} D_b\right) + d r_0 \int \sqrt{|g|}\, \mathrm{d}^2\sigma \\ + \frac{d}{2\alpha_0} \int \sqrt{|g|}\, \mathrm{d}^2\sigma\, \lambda^{ab}(\delta_{ab} - g_{ab}). \end{aligned} \tag{2.21}$$

Choosing $L_1 = L_2 = L$, we may look for a symmetric homogeneous saddle point

$$g_{ab}(\sigma) = \rho \delta_{ab}, \qquad \lambda^{ab}(\sigma) = \lambda g^{ab} = (\lambda/\rho) \delta_{ab}. \tag{2.22}$$

In the thermodynamic limit $L \to \infty$ and under these assumptions, eq. (2.21) can be computed exactly:

$$\frac{1}{d}\Gamma_{\text{eff}}[X_{\text{cl}}, g, \lambda] = \int d^2\sigma\rho\left[r_0 - \frac{\lambda}{\alpha_0} + \frac{\lambda}{\alpha_0\rho} - \frac{\Lambda^2}{8} - \frac{\lambda}{8\pi}(\ln\lambda - 1) + \frac{\lambda}{8\pi}\ln\Lambda^2\right]. \tag{2.23}$$

The saddle point is then given by the equations

$$\frac{\lambda}{\lambda_0}\left[\ln\frac{\lambda}{\lambda_0} - 1\right] + 1 = 8\pi\frac{\left(r_0 - r_0^{\text{crit}}\right)}{\lambda_0}, \tag{2.24a}$$

$$\rho = \frac{8\pi}{\alpha_0 \ln(\lambda/\lambda_0)}, \tag{2.24b}$$

where r_0^{crit} is given by (2.17) and λ_0 is defined as

$$\lambda_0 = \Lambda^2 e^{-8\pi/\alpha_0}. \tag{2.25}$$

Eq. (2.24a) has a unique solution $\lambda \geqslant \lambda_0$ ($\rho \geqslant 0$) for each $r_0 \geqslant r_0^{\text{crit}}$. As r_0 goes to r_0^{crit}, λ goes to λ_0 and the mean area $A = \rho L^2$ diverges as $(r_0 - r_0^{\text{crit}})^{-1/2}$. The surface becomes again an infinite flat object. Moreover, at the saddle point and for every $r_0 \geqslant r_0^{\text{crit}}$, eq. (2.23) reduces to

$$\frac{1}{d}\Gamma_{\text{eff}} = \int d^2\sigma \frac{\lambda}{\alpha_0} = L^2\frac{\lambda}{\alpha_0} \tag{2.26}$$

and leads to the expression for the physical surface tension τ_{phys}:

$$\tau_{\text{phys}} = \frac{\Gamma_{\text{eff}}}{dL^2} = \frac{\lambda}{\alpha_0}. \tag{2.27}$$

Our equations for the saddle point are in agreement with the calculations by Olesen and Yang [12] and by Braaten et al. [15] of the static potential $V(R)$ for a "smooth string" if one takes the large-R limit. However we disagree with the result by Alonso and Espriu [13], who used a regulator which is not reparametrization invariant.

2.3. RENORMALIZATION GROUP BEHAVIOR

We now discuss the renormalization group properties of the model at large d. Anticipating on the results of next section, we shall use the fact that, if the classical configuration X_{cl} is perturbed from the planar saddle point given by (2.20) by a small displacement $x_\perp$ orthogonal to the tangent plane, the correction to the effective action is found to be, at leading order in $x_\perp$

$$\delta\Gamma_{\text{eff}} = d\int \mathrm{d}^2\sigma\left(\frac{\lambda}{2\alpha_0}(\partial x_\perp)^2 + \frac{1}{2\alpha_0\rho}(\Delta x_\perp)^2\right) + \cdots . \tag{2.28}$$

The first term is simply the change in the area of the surface and the second is the first term in the expansion of the extrinsic curvature $(\boldsymbol{K})^2$. Hence, in addition to the physical surface tension $\tau_{\text{phys}} = \lambda/\alpha_0$, the physical rigidity κ_{phys} is given by

$$\kappa_{\text{phys}} = \frac{1}{\alpha_0\rho} = \frac{1}{8\pi}\ln\frac{\lambda}{\lambda_0} . \tag{2.29}$$

The renormalization group flows are defined in the standard way [23]. If one changes the short distance regulator $\Lambda \to \Lambda'$, which changes in the bare coupling constants $\alpha_0 \to \alpha_0'$ and $r_0 \to r_0'$ allow us to reabsorb the change in the regulator and give the same large-distance effective action? In our case the answer is simply given by

$$\tau_{\text{phys}} = \frac{\lambda}{\alpha_0} = \frac{\lambda'}{\alpha_0'}, \qquad \kappa_{\text{phys}} = \frac{1}{8\pi}\ln\frac{\lambda}{\lambda_0} = \frac{1}{8\pi}\ln\frac{\lambda'}{\lambda_0'} . \tag{2.30}$$

The critical point is ($\alpha_0^* = 0$, $r_0^* = \frac{1}{8}\Lambda^2$) and the renormalization group functions are defined as

$$\beta(\alpha_0') = \Lambda'\frac{\partial}{\partial\Lambda'}\alpha_0'\bigg|_{\alpha_0, r_0, \Lambda} , \tag{2.31}$$

$$\gamma(\alpha_0') = \Lambda'\frac{\partial}{\partial\Lambda'}\ln\left(r_0 - \tfrac{1}{8}\Lambda^2\right)\bigg|_{\alpha_0, r_0, \Lambda} . \tag{2.32}$$

From (2.25) and (2.30) we get

$$\Lambda^2\frac{\mathrm{e}^{-8\pi/\alpha_0}}{\alpha_0} = \Lambda'^2\frac{\mathrm{e}^{-8\pi/\alpha_0'}}{\alpha_0'} \tag{2.33}$$

and the β function is

$$\beta(\alpha) = -\frac{\alpha^2}{4\pi}\frac{1}{(1-\alpha/8\pi)} . \tag{2.34}$$

Similarly, from (2.24) we get the γ function

$$\gamma(\alpha) = -\frac{\alpha}{4\pi}\frac{1}{(1-\alpha/8\pi)} . \tag{2.35}$$

Those results agree in the large-d limit with the one-loop perturbative calculations which give [4–9]

$$\beta(\alpha) = -\frac{d\alpha^2}{4\pi}, \qquad \gamma(\alpha) = -\frac{(d-2)\alpha}{4\pi} \tag{2.36}$$

(the factor d comes from the rescaling (2.7))*.

The pole at $\alpha = 8\pi$ has no physical meaning for the continuum theory. Indeed our calculation is equivalent to define the renormalized coupling constant α_R by the equation

$$\Lambda^2 \frac{e^{-8\pi/\alpha_0}}{\alpha_0} = \mu^2 \frac{e^{-8\pi/\alpha_R}}{\alpha_R} \tag{2.37}$$

(μ is the subtraction scale). One sees that the relationship between α_0 and α_R becomes singular for $\alpha_R \geqslant 8\pi$, while the physical quantities have no singularities. Other definitions of the renormalized coupling constant would remove the pole of the β function.

The result given by (2.34) for the β function differs from the previous calculations at large d [12, 13, 16–19] (including those by one of the present authors). The reason for such a discrepancy is the following. Most authors considered that the physical quantities (such as the surface tension and the correlation length) were only powers of λ, and that the factor $1/\alpha$ could be absorbed in a "wave function" renormalization of the X field (as suggested in ref. [9]). In fact there cannot be any wave function renormalization of X since it corresponds to the physical position of the membrane, and the physical length cannot get any anomalous dimension. However, the difference between (2.34) and previous results occurs only at two loops and does not change the physical content of the renormalization group equations.

2.4. ROTATIONAL INVARIANCE AND PERSISTENCE LENGTH

Finally for completeness we show in this subsection that at the critical point $r_0 = r_0^{\text{crit}}$ rotation invariance in bulk space, which is explicitly broken by the choice of boundary conditions, is restored, and that correlations between tangent planes are characterized by a persistence length which does not vanish at the critical point.

The orientation of the tangent plane at point with coordinates σ is characterized by the projection operator on the tangent plane, in d-dimensional bulk space, which reads

$$P^{ij}(\sigma) = g^{ab}(\sigma)\, \partial_a X^i(\sigma)\, \partial_b X^j(\sigma). \tag{2.38}$$

The correlation between the orientation of the surface at some point σ and the

* The one-loop result for γ in refs. [4, 24] is not correct [25].

reference plane (1,2) is measured by $\mathrm{Tr}[P_0 P(\sigma)]$ where $P_0^{ij} = \delta^{i1}\delta^{j1} + \delta^{i2}\delta^{j2}$ is the projector on the reference plane. To measure the expectation value of this observable to leading order in the $d \to \infty$ limit, it is enough to write $X = X_{\mathrm{cl}} + x_\perp$, where as in subsect. 2.3, $x_\perp$ is orthogonal to the reference plane, and to use the effective action (2.28) for $x_\perp$. Indeed each contraction of the fluctuations of the fields, ($x_\perp$, the metric g_{ab} and the Lagrange multiplier λ^{ab}) will contribute by a factor $1/d$, but since there are $d-2$ independent $x_\perp$ components, only the contributions of scalar products $x_\perp \cdot x_\perp$ will sum up to give a factor O(1). For instance

$$\begin{aligned}\left\langle \mathrm{Tr}[P_0 P(\sigma)]\right\rangle &= \left\langle g^{ab}(\sigma)\, \partial_a X_{\mathrm{cl}} \cdot \partial_b X_{\mathrm{cl}} \right\rangle \\ &= \left\langle \mathrm{Tr}\, g^{ab}(\sigma)\right\rangle \\ &= \left\langle \mathrm{Tr}(\delta_{ab} + \partial_a x_\perp \cdot \partial_b x_\perp)^{-1}\right\rangle . \end{aligned} \tag{2.39}$$

Using the previous remark, to leading order this equals

$$\mathrm{Tr}\left[[\delta_{ab} + d\langle \partial_a x_\perp\, \partial_b x_\perp\rangle]^{-1}\right] + \mathrm{O}(1/d), \tag{2.40}$$

where $\langle \partial_a x_\perp\, \partial_b x_\perp \rangle$ is computed with the effective action (2.28) and is therefore

$$\begin{aligned}\langle \partial_a x_\perp\, \partial_b x_\perp \rangle &= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{k_a k_b}{k^4/\rho + \lambda k^2} \frac{\alpha_0}{d} \\ &= \frac{\delta_{ab}}{2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{k^2 + \lambda\rho} \frac{\rho\alpha_0}{d} \\ &\simeq \frac{\delta_{ab}}{2} \frac{\rho\alpha_0}{d} \frac{1}{4\pi} \ln\frac{\Lambda^2}{\lambda}, \end{aligned} \tag{2.41}$$

where Λ is some UV regulator. Thus

$$\left\langle \mathrm{Tr}[P_0 P(\sigma)]\right\rangle = 2\left[1 + \rho\frac{\alpha_0}{8\pi}\ln\frac{\Lambda^2}{\lambda}\right]^{-1} + \mathrm{O}\left(\frac{1}{d}\right). \tag{2.42}$$

From eq. (2.24), we have seen that as $r_0 \to r_0^{\mathrm{crit}}$, $\rho \to \infty$. Thus $\langle \mathrm{Tr}[P_0 P]\rangle \to 0$ and the tangent plane becomes completely uncorrelated with the reference plane fixed by the boundary conditions. $\mathrm{O}(d)$ rotational invariance is restored at the critical point.

Similarly, one can consider the correlation between the tangent planes at two different points. Before doing this, it is useful to consider how the distance D in bulk space between those two points varies with the distance of their projections $|\sigma - \sigma'|$ on the reference plane. This distance is given by

$$D^2 = (\sigma - \sigma')^2 + [x_\perp(\sigma) - x_\perp(\sigma')]^2 \tag{2.43}$$

and in the large-d limit we have

$$\begin{aligned} D^2 &= (\sigma - \sigma')^2 + d\left\langle (x_\perp(\sigma) - x_\perp(\sigma'))^2 \right\rangle \\ &= (\sigma - \sigma')^2 + \frac{2\alpha_0}{\lambda} g\left((\sigma - \sigma')^2 \lambda\rho\right), \end{aligned} \tag{2.44}$$

where the function $g(\sigma^2)$ is given by

$$g(\sigma^2) = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} (1 - \mathrm{e}^{ik\sigma}) \left(\frac{1}{k^2} - \frac{1}{k^2 + 1} \right) \tag{2.45}$$

and behaves for large and small σ as

$$g(\sigma^2) \simeq \frac{1}{4\pi} \ln \sigma^2 \qquad \sigma^2 \gg 1 \tag{2.46a}$$

$$\simeq \sigma^2 \frac{1}{16\pi} \ln \frac{1}{\sigma^2} \qquad \sigma^2 \ll 1. \tag{2.46b}$$

Thus, close to the critical point ($r_0 \simeq r_0^{\text{crit}}$), ρ is large and there are three different regimes for D^2

$$|\sigma - \sigma'|^2 \ll \frac{1}{\lambda\rho}: \qquad D^2 \simeq \frac{2\alpha_0}{\lambda} \frac{1}{16\pi} (\sigma - \sigma')^2 \lambda\rho \log\left(\frac{1}{(\sigma - \sigma')^2 \lambda\rho} \right), \tag{2.47a}$$

$$\frac{1}{\lambda\rho} \ll |\sigma - \sigma'|^2 \ll \frac{\alpha_0}{2\pi\lambda}: \qquad D^2 \simeq \frac{2\alpha_0}{\lambda} \frac{1}{4\pi} \ln\left((\sigma - \sigma')^2 \lambda\rho\right), \tag{2.47b}$$

$$|\sigma - \sigma'|^2 \gg \frac{\alpha_0}{2\pi\lambda}: \qquad D^2 \simeq (\sigma - \sigma')^2. \tag{2.47c}$$

Thus at short distance, eq. (2.47a), the surface is a two-dimensional object. In the intermediate regime (2.47b), the logarithmic growth of D^2 with $(\sigma - \sigma')^2$ means that the surface has infinite Hausdorff dimension and is thus a crumpled object. The transition between those two regimes occurs for a distance

$$D^2 \sim \frac{\alpha_0}{\lambda}. \tag{2.48}$$

$\sqrt{\alpha_0/\lambda}$ appears as the persistence length, i.e. the scale length where the surface is crumpled and tangent planes becomes uncorrelated. Finally in the third regime (2.47c), the effect of the surface tension is observed and the surface becomes again a two-dimensional object, but oriented within the reference plane (1,2).

The estimate (2.48) for the persistence length, which therefore does not vanish at the critical point, is confirmed by a more rigorous estimate using the following rotationally invariant correlation function:

$$G(D) = \left\langle \frac{\int d^2\sigma \sqrt{g(\sigma)}\, \mathrm{Tr}[P(\sigma)P(0)]\, \delta[D^2 - D^2(\sigma,0)]}{\int d^2\sigma \sqrt{g(\sigma)}\, \delta[D^2 - D^2(\sigma,0)]} \right\rangle, \tag{2.49}$$

which measures correlation between tangent planes at two points as a function of their distance D in bulk space. A large-d calculation along the lines indicated above leads, at the critical point ($\rho \to \infty$) to

$$G(D) = \frac{1}{2}\left[\frac{1}{8\pi}\ln\frac{\Lambda^2}{\lambda}\right]^{-2} f^2(x^2), \tag{2.50}$$

where x is related to D via (2.44) and (2.45)

$$D^2 = \frac{2\alpha_0}{\lambda} g(x^2) \tag{2.51}$$

and where f is the two-dimensional massive propagator

$$f(x^2) = \frac{1}{(2\pi)^2}\int d^2k \frac{e^{ikx}}{k^2+1}. \tag{2.52}$$

Thus for $D \ll \sqrt{\alpha_0/\lambda}$, $G = O(1)$ and tangent planes are correlated. For $D \gg \sqrt{\alpha_0/\lambda}$, G decays very quickly as $\exp[-\exp(\pi D^2\lambda/\alpha_0)]$. (This very fast decay is a consequence of the infinite Hausdorff dimension of the surface at large scale.) However, $G(D)$ is a universal scaling function of $D\sqrt{\lambda/\alpha_0}$, which confirms our interpretation of $\sqrt{\alpha_0/\lambda}$ as the persistence length.

3. The stability of the homogeneous large d solution

3.1. THE QUADRATIC PART OF THE EFFECTIVE ACTION

In order to check if the homogeneous saddle point that we have obtained is the real extremum of the effective action which has to be taken into account in the large-d limit, we shall perform an analysis of the stability of this solution. This analysis is also important to understand which effective action governs the large distance properties of the surface. For that purpose we shall compute the second derivative of the effective action (2.15) with respect to variations of the position of the membrane X and also of the Lagrange multiplier λ^{ab} and the composite field g_{ab}.

At that stage we must deal with the reparametrization invariance of the effective action. Indeed, in general, under a "gauge transformation"

$$\begin{aligned} X &\to X + (\varepsilon^a D_a) X, \\ g_{ab} &\to g_{ab} + D_a \varepsilon_b + D_b \varepsilon_a, \\ \lambda^{ab} &\to \lambda^{ab} + D^a \varepsilon^b - D^b \varepsilon^a. \end{aligned} \tag{3.1}$$

Γ_{eff} does not change (the D's denote covariant derivatives with respect to g). One needs to impose a gauge condition on the fluctuations around the classical configuration to avoid those flat directions. One possibility is to consider fluctuations δX of the position of the membrane normal to the classical configuration X_{cl}. We have found more convenient to impose a constraint on the metric g_{ab}, by using the well-known conformal gauge [9, 21, 22]

$$g_{ab}(\sigma) = \rho \delta_{ab} e^{\varphi(\sigma)} \tag{3.2}$$

(we neglect the transformations of the metric which correspond to Teichmüller deformations and which cannot be absorbed into a gauge transformation and a change in the conformal factor φ). Then the most general expansion of the classical fields around the plane saddle point is given for g_{ab} by (3.2), for X by

$$X(\sigma) = X_0 + \sigma^a u_a + x_\perp(\sigma) + v^a(\sigma) u_a \tag{3.3}$$

with

$$x_\perp(\sigma) \cdot u_a = 0, \qquad a = 1, 2 \tag{3.4}$$

(the v^a's denote tangential displacements in the plane of the surface) and for λ^a_b by

$$\lambda^a_b(\sigma) = \lambda \delta^a{}_b + \tau^a{}_b(\sigma). \tag{3.5}$$

ρ and λ are the solutions of the saddle point equations studied in sect. 2. Let us recall that the boundary conditions (2.19) were chosen in order to break explicitly rotational invariance and to obtain the plane given by (2.20) as a classical background. Therefore we have to take periodic boundary conditions for the fluctuations $x_\perp$, v^a, φ and τ^a_b, with period L, before taking the thermodynamic limit $L \to \infty$.

We now expand to second order $\Gamma_{\text{eff}}(\boldsymbol{X}, g_{ab}, \lambda_b^a)$ in $\boldsymbol{x}_\perp$, v^a, φ and τ_b^a:

$$\frac{1}{d}\Gamma_{\text{eff}}(\boldsymbol{X}, g_{ab}, \lambda_b^a) = L^2\tau_{\text{phys}} + \Gamma_{\text{eff}}^{(2)}(\boldsymbol{x}_\perp, v^a, \varphi, \tau_b^a) + \cdots, \tag{3.6a}$$

$$\Gamma_{\text{eff}}^{(2)} = \int\frac{d^2p}{4\pi^2}\begin{bmatrix} x_\perp^i(p) \\ v^a(p) \\ \varphi(p) \\ \tau_d^d(p)\end{bmatrix}^t \begin{bmatrix} \Gamma_{x^ix^j} & \Gamma_{x^iv^b} & \Gamma_{x^i\varphi} & \Gamma_{x^i\tau_f^e} \\ \Gamma_{v^ax^j} & \Gamma_{v^av^b} & \Gamma_{v^a\varphi} & \Gamma_{v^a\tau_f^e} \\ \Gamma_{\varphi x^j} & \Gamma_{\varphi v^b} & \Gamma_{\varphi\varphi} & \Gamma_{\varphi\tau_f^e} \\ \Gamma_{\tau_d^cx^j} & \Gamma_{\tau_d^cv^b} & \Gamma_{\tau_d^c\varphi} & \Gamma_{\tau_d^c\tau_f^e}\end{bmatrix} \begin{bmatrix} x_\perp^j(-p) \\ v^b(-p) \\ \varphi(-p) \\ \tau_f^e(-p)\end{bmatrix}. \tag{3.6b}$$

We have introduced the Fourier transforms of the fields $\boldsymbol{x}_\perp$, v^a, φ and τ_d^c. The Γ matrix (in (3.6b)) is hermitian. One sees from (2.15) that a lot of coefficients of Γ are obtained from the classical action (2.9). We get explicitly

$$\Gamma_{x^ix^j}(p) = \delta_{ij}\frac{1}{2\alpha_0\rho}\left[(p^2)^2 + \lambda\rho p^2\right] = \tfrac{1}{2}\delta_{ij}\left[\kappa_{\text{phys}}(p^2)^2 + \tau_{\text{phys}}p^2\right], \tag{3.7a}$$

$$\Gamma_{v^av^b}(p) = \tfrac{1}{2}\delta_{ab}\left[\kappa_{\text{phys}}(p^2)^2 + \tau_{\text{phys}}p^2\right], \tag{3.7b}$$

$$\Gamma_{v^a\tau_d^c}(p) = \frac{i}{4\alpha_0}(\delta_{ac}p_d + \delta_{ad}p_c), \tag{3.7c}$$

$$\Gamma_{v^a\varphi}(p) = 0; \qquad \Gamma_{x^iv^a}(p) = 0; \qquad \Gamma_{x^i\varphi}(p) = 0; \qquad \Gamma_{x^i\tau_d^c}(p) = 0. \tag{3.7d}$$

The calculation of the other coefficients needs the expansion of the $\operatorname{Tr}\ln(\cdot)$ in (2.15) in powers of φ and τ_d^c. We introduce the notation

$$\mathbb{O}(M) = \sqrt{|g(\sigma)|}\left[(\Delta^2 - D_a\lambda^{ab}D_b) + M^4\right], \tag{3.8}$$

where M is an arbitrary mass. The operator $\mathbb{O}(M)$ can be expanded to second order in φ and τ_b^a:

$$\mathbb{O}(M) = \mathbb{O}^{(0)}(M) + \mathbb{O}^{(1)}(M) + \mathbb{O}^{(2)}(M) + \cdots, \tag{3.9}$$

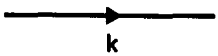
with

$$\begin{aligned} \rho\mathbb{O}^{(0)}(M) &= (\partial^2)^2 - \lambda\rho\,\partial^2 + (M^2\rho)^2, \\ \rho\mathbb{O}^{(1)}(M) &= -\partial^2(\varphi\,\partial^2) - \partial_a(\rho\tau_b^a\,\partial_b) + \varphi(M^2\rho)^2, \\ \rho\mathbb{O}^{(2)}(M) &= \tfrac{1}{2}\partial^2(\varphi^2\partial^2) + \tfrac{1}{2}\varphi^2(M^2\rho)^2. \end{aligned} \tag{3.10}$$

According to (2.16) we write

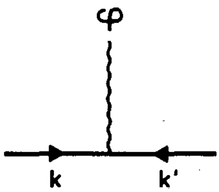
$$\mathrm{Tr}_\Lambda \ln\left(\Delta^2 - D_a \lambda^{ab} D_b\right) = \mathrm{Tr}\left[\ln[\rho \mathrm{O}(0)] - \ln[\rho \mathrm{O}(\Lambda)]\right]. \tag{3.11}$$

Let us now introduce diagrammatic notations. The propagator $[\rho \mathrm{O}^{(0)}(M)]^{-1}$ is given in momentum space by:

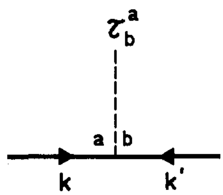


$$= \left(k^4 + \lambda \rho k^2 + \left(M^2\rho\right)^2\right)^{-1}. \tag{3.12a}$$

$\rho \mathrm{O}^{(1)}(M)$ leads to two vertices with one φ or τ external leg:

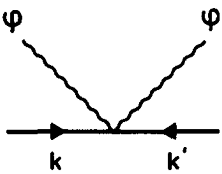


$$= -(k^2)(k')^2 + \left(M^2\rho\right)^2 \tag{3.12b}$$



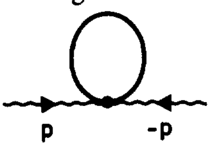
$$= -(k_a k'_b)\rho. \tag{3.12c}$$

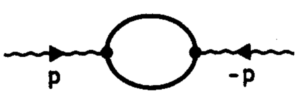
$\rho \mathrm{O}^{(2)}(M)$ leads to a vertex with two φ external legs:



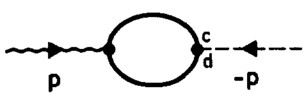
$$= \tfrac{1}{2}(k^2)(k')^2 + \tfrac{1}{2}(M^2\rho)^2. \tag{3.12d}$$

The expansion of (3.11) to second order may then be written diagrammatically, and we obtain the missing Γ coefficients:

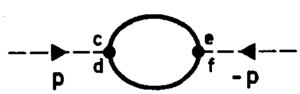
$\Gamma_{\varphi\varphi}(p) = \frac{1}{2}$


$- \frac{1}{4}$


$- \left(\frac{\lambda\rho}{2\alpha_0} - \frac{r_0\rho}{2}\right),$

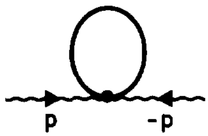
$\Gamma_{\varphi\tau^c_d}(p) = -\frac{1}{4}$


$- \frac{\rho}{4\alpha_0}\delta^c_d,$

$\Gamma_{\tau^c_d\tau^e_f}(p) = -\frac{1}{4}$


(3.13)

The diagrams in (3.13) must be understood in the regularization process (3.11). For instance, we have

$\frac{1}{2}$ 

$$= \tfrac{1}{4}\int \frac{d^2k}{4\pi^2}\left[\frac{(k^2)^2}{(k^2)^2+\lambda\rho k^2} - \frac{(k^2)^2+\Lambda^4\rho^2}{(k^2)^2+\lambda\rho k^2+\Lambda^4\rho^2}\right]$$

$$= \frac{1}{16\pi}(\rho\lambda)\ln\frac{\lambda}{\Lambda^2} + \mathrm{O}\left(\frac{1}{\Lambda}\right). \tag{3.14}$$

We thus obtain

$$\Gamma_{\varphi\varphi} = \frac{1}{2}\left[\frac{-(\lambda\rho)^2}{4\pi\sqrt{(p^2)^2+4\lambda\rho p^2}}\ln\frac{\sqrt{p^2+4\lambda\rho}+\sqrt{p^2}}{\sqrt{p^2+4\lambda\rho}-\sqrt{p^2}} + \frac{\lambda\rho}{8\pi} - \frac{p^2}{24\pi}\right]$$

$$+\left[-\frac{\rho\lambda}{16\pi}\ln\frac{\lambda}{\Lambda^2} - \frac{\rho\Lambda^2}{16} + \frac{\lambda\rho}{16\pi} - \frac{\lambda\rho}{2\alpha_0} + \frac{r_0\rho}{2}\right] + \mathrm{O}\left(\frac{1}{\Lambda}\right). \tag{3.15}$$

The saddle point equations (2.24) cancel the second term and lead to a finite coefficient $\Gamma_{\varphi\varphi}$:

$$\Gamma_{\varphi\varphi}(p) = \frac{1}{2}\frac{\tau_{\mathrm{phys}}}{\kappa_{\mathrm{phys}}}\left[-\frac{1}{4\pi}f(y) + \frac{1}{8\pi} - \frac{1}{24\pi}(y)^2\right]. \tag{3.16}$$

We have introduced the rescaled momentum

$$y = p\sqrt{\frac{\kappa_{\mathrm{phys}}}{\tau_{\mathrm{phys}}}} \tag{3.17}$$

and we define

$$f(y) = \frac{1}{\sqrt{(y^2)^2+4(y)^2}}\ln\left(\frac{\sqrt{y^2+4}+\sqrt{y^2}}{\sqrt{y^2+4}-\sqrt{y^2}}\right). \tag{3.18}$$

We obtain by the same process

$$\Gamma_{\varphi\tau^a_b}(p) = \frac{\rho}{32\pi}\left\{\delta_{ab}\left[1-(y^2+2)f(y) - \frac{8\pi}{\kappa_{\mathrm{phys}}}\right] + \left(\delta_{ab} - 2\frac{y^a y^b}{y^2}\right)[1-2f(y)]\right\}$$

(3.19)

and

$$\Gamma_{\tau_b^a \tau_d^c}(p) = -\frac{\rho^2}{16\pi}\frac{\kappa_{\text{phys}}}{\tau_{\text{phys}}}\left\{[\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}]F_1(y)\right.$$

$$+\left[\delta^{ac}\frac{y^b y^d}{y^2} + \delta^{bd}\frac{y^a y^c}{y^2}\right]F_2(y)$$

$$+\left[\delta^{ab}\frac{y^c y^d}{y^2} + \delta^{ad}\frac{y^b y^c}{y^2} + \delta^{bc}\frac{y^a y^d}{y^2} + \delta^{cd}\frac{y^a y^b}{y^2}\right]F_3(y)$$

$$\left.+\left[\frac{y^a y^b y^c y^d}{(y^2)^2}\right]F_4(y)\right\}, \tag{3.20a}$$

$$F_1(y) = \frac{1}{12y^2} + \frac{y^2}{24}\ln(y^2) - \frac{(y^2+1)^3}{12(y^2)^2}\ln(1+y^2) + \frac{(y^2+4)^2}{24}f(y), \tag{3.20b}$$

$$F_2(y) = \frac{-1}{3y^2} - \frac{y^2}{6}\ln(y^2) + \frac{2+3y^2+3(y^2)^2+2(y^2)^3}{6(y^2)^2}\ln(1+y^2)$$

$$-\frac{(y^2+1)(y^2+4)}{6}f(y), \tag{3.20c}$$

$$F_3(y) = -\frac{1}{3y^2} + \frac{y^2}{12}\ln(y^2) - \frac{(y^2)^3 - 3y^2 - 2}{6(y^2)^2}\ln(1+y^2)$$

$$+\frac{(y^2-2)(y^2+4)}{12}f(y), \tag{3.20d}$$

$$F_4(y) = 2\left(\frac{1}{y^2} - \frac{(1+y^2)}{(y^2)^2}\ln(1+y^2) + f(y)\right). \tag{3.20e}$$

The expression for $\Gamma_{\varphi\varphi}$ (3.16) has already been obtained in ref. [20]. This achieves the calculation of the quadratic part of the effective action.

3.2. THE STABILITY OF THE SADDLE POINT

We are now in a position to study the stability of the homogeneous saddle point. First, from (3.7d), one sees immediately that $x_\perp$ is not coupled to the other fields and that the quadratic part writes

$$\Gamma_{\text{eff}}^{(2)}(x_\perp, v, \varphi, \tau) = \int d^2\sigma \left(\kappa_{\text{phys}} \frac{(\partial^2 x_\perp)^2}{2} + \tau_{\text{phys}} \frac{(\partial_a x_\perp \cdot \partial_a x_\perp)}{2} \right) + \Gamma_{\text{eff}}^{(2)}(v, \varphi, \tau). \tag{3.21}$$

So, the saddle point is stable against small variations of $x_\perp$ and it is enough to check the stability with respects to variations in v, φ, τ. If this is the case, from (2.14), the calculation of the physical effective action $\Gamma_{\text{eff}}(x_\perp)$, which is obtained by extremizing $\Gamma_{\text{eff}}(x_\perp, v, \varphi, \tau)$ with respects to v, φ and τ, reduces to take $v = \varphi = \tau = 0$ in (3.21) (this is of course true only to leading order in $x_\perp$). This justifies eq. (2.28) used at the beginning of subsect. 2.3.

We now deal with the term $\Gamma_{\text{eff}}^{(2)}(v, \varphi, \tau)$. We first note that in the functional integral (2.8) which leads to the definition of the effective action, X and g_{ab} are real fields, while λ^a_b is a Lagrange multiplier which has to be integrated from $-i\infty$ to $+i\infty$. Thus, the variation τ^a_b of λ^a_b around the saddle point $\lambda\delta^a_b$ has to be considered as purely imaginary.

It is therefore not correct to check whether the matrix

$$\mathbf{C} = \begin{bmatrix} \Gamma_{v^1 v^1} & \Gamma_{v^1 v^2} & \Gamma_{v^1 \varphi} & \Gamma_{v^1 \tau_1^1} & \left(\Gamma_{v^1 \tau_1^1} + \Gamma_{v^1 \tau_1^2}\right) & \Gamma_{v^1 \tau_2^2} \\ \Gamma_{v^2 v^1} & \Gamma_{v^2 v^2} & \Gamma_{v^2 \varphi} & \Gamma_{v^2 \tau_1^1} & \left(\Gamma_{v^2 \tau_1^1} + \Gamma_{v^2 \tau_1^2}\right) & \Gamma_{v^2 \tau_2^2} \\ \Gamma_{\varphi v^1} & \Gamma_{\varphi v^2} & \Gamma_{\varphi\varphi} & \Gamma_{\varphi \tau_1^1} & \left(\Gamma_{\varphi \tau_2^1} + \Gamma_{\varphi \tau_1^2}\right) & \Gamma_{\varphi \tau_2^2} \\ \Gamma_{\tau_1^1 v^1} & \Gamma_{\tau_1^1 v^2} & \Gamma_{\tau_1^1 \varphi} & \Gamma_{\tau_1^1 \tau_1^1} & \left(\Gamma_{\tau_1^1 \tau_2^1} + \Gamma_{\tau_1^1 \tau_1^2}\right) & \Gamma_{\tau_1^1 \tau_2^2} \\ \begin{pmatrix} \Gamma_{\tau_2^1 v^1} \\ + \\ \Gamma_{\tau_1^2 v^1} \end{pmatrix} & \begin{pmatrix} \Gamma_{\tau_2^1 v^2} \\ + \\ \Gamma_{\tau_1^2 v^2} \end{pmatrix} & \begin{pmatrix} \Gamma_{\tau_2^1 \varphi} \\ + \\ \Gamma_{\tau_1^2 \varphi} \end{pmatrix} & \begin{pmatrix} \Gamma_{\tau_2^1 \tau_1^1} \\ + \\ \Gamma_{\tau_1^2 \tau_1^1} \end{pmatrix} & \begin{pmatrix} \Gamma_{\tau_2^1 \tau_2^1} & + & \Gamma_{\tau_2^1 \tau_1^2} \\ & + & \\ \Gamma_{\tau_1^2 \tau_2^1} & + & \Gamma_{\tau_1^2 \tau_1^2} \end{pmatrix} & \begin{pmatrix} \Gamma_{\tau_2^1 \tau_2^2} \\ + \\ \Gamma_{\tau_1^2 \tau_2^2} \end{pmatrix} \\ \Gamma_{\tau_2^2 v^1} & \Gamma_{\tau_2^2 v^2} & \Gamma_{\tau_2^2 \varphi} & \Gamma_{\tau_2^2 \tau_1^1} & \left(\Gamma_{\tau_2^2 \tau_2^1} + \Gamma_{\tau_2^2 \tau_1^2}\right) & \Gamma_{\tau_2^2 \tau_2^2} \end{bmatrix} \tag{3.22}$$

obtained from (3.6) has positive eigenvalues (there are only three independent τ coefficients since $\tau_2^1 = \tau_1^2$). The stability analysis is somewhat subtle, since we are dealing with the complex saddle point method. We shall proceed as follows. Let us write for convenience

$$\mathbf{C} = \begin{pmatrix} A(p) & B(p) \\ B^+(p) & C(p) \end{pmatrix}, \tag{3.23}$$

where A, B, C are 3×3 matrices (A and C are real symmetric, $A(p) = A(-p)$, $C(p) = C(-p)$ and $B^+(p) = B(-p)$).

(a) First we freeze v^a and φ and we look for a saddle point $\tau^a_b(v^c,\varphi)$ of $\Gamma^{(2)}_{\rm eff}(v^c,\varphi,\tau^a_b)$. It will be unique, since given by the solution of a linear equation (if $\det C\neq 0$)

$$\begin{pmatrix}\tau^1_1(v^c,\varphi)\\ \tau^1_2(v^c,\varphi)\\ \tau^2_2(v^c,\varphi)\end{pmatrix} = -C^{-1}(p)B^{+}(p)\begin{pmatrix}v^1(-p)\\ v^2(-p)\\ \varphi(-p)\end{pmatrix}. \tag{3.24}$$

(b) Then we check that this saddle point is stable with respect to *imaginary* variations $\tilde{\tau}^a_b$ around $\tau^a_b(v^c,\varphi)$.

Since the 3×3 submatrix $C(p)$ is real, this is equivalent to check that C is negative definite. This can be done by noting that its eigenvalues depend only on p^2 (by rotational invariance) and that if one of the components of p vanishes ($p_1=0$ or $p_2=0$), C becomes diagonal. We have checked numerically that in that case the diagonal elements are negative in the physical domain $p^2\geqslant 0$, $r_0\geqslant r_0^{\rm crit}$.

(c) Once we have extremized $\Gamma^{(2)}_{\rm eff}(v,\varphi,\tau)$ with respect to τ, we have to study the stability of the resulting effective action for v and φ,

$$\begin{aligned}\Gamma^{(2)}_{\rm eff}(v^a,\varphi) &= \Gamma^{(2)}_{\rm eff}\big(v^a,\varphi,\tau^c_d(v^b,\varphi)\big)\\ &= \int\frac{{\rm d}^2p}{(2\pi)^2}\begin{pmatrix}v^a(p)\\ \varphi(p)\end{pmatrix}^{\rm t}\big(A(p)-B(p)C^{-1}(p)B^{+}(p)\big)\begin{pmatrix}v^b(-p)\\ \varphi(-p)\end{pmatrix}\end{aligned} \tag{3.25}$$

with respect to real variations of v^a and φ. For this purpose we repeat the previous analysis by first freezing φ and looking at the saddle point $v^a(\varphi)$ of $\Gamma^{(2)}_{\rm eff}(v^a,\varphi)$. If the eigenvalues of the 2×2 submatrix $D(p)$ obtained by taking the first two rows and columns of the hermitian matrix $(A-BC^{-1}B^{+})$ are positive, $v^a(\varphi)$ minimizes $\Gamma^{(2)}(v^a,\varphi)$. In that case also one can use rotational invariance to simplify the calculation and check numerically that these eigenvalues are indeed positive in the physical domain $p^2\geqslant 0$, $r_0\geqslant r_0^{\rm crit}$.

(d) Thus the problem of the stability of the homogeneous saddle point described in subsect. 2.2 is reduced to the study of the sign of the effective action for the conformal field φ. Indeed, the previous analysis shows that for a non-zero fixed φ, the corresponding saddle point obtained by perturbing the homogeneous saddle point is stable with respect to variations of all other fields $x_\perp$, v^a and τ^a_b. The effective action for φ, which is obtained by extremizing (3.25) with respect to v^a,

$$\begin{aligned}\Gamma^{(2)}_{\rm eff}(\varphi) &= \min_{\text{real } v^c}\ \max_{\text{real } \tau^{ab}} \Gamma^{(2)}_{\rm eff}(v^c,\varphi,\tau^a_b)\\ &= \min_{\substack{\text{real } v^c\\ \text{real } x_\perp}}\ \max_{\text{real } \tau^a_b} \Gamma^{(2)}_{\rm eff}(x_\perp,v^c,\varphi,\tau^a_b)\end{aligned} \tag{3.26}$$

is in fact simply expressed in term of the φ-φ element of the inverse of the matrix $\mathbf{C}$ in (3.22) or of the inverse of the matrix Γ in (3.6b) (Γ is the second derivative of the effective action and thus Γ^{-1} is nothing but the propagator which will appear in the large-d expansion.) We have

$$\Gamma_{\text{eff}}^{(2)}(\varphi) = \int \frac{d^2p}{(2\pi)^2}\varphi(p)\frac{1}{(\mathbb{C}^{-1})_{\varphi\varphi}(p)}\varphi(-p)$$

$$= \int \frac{d^2p}{(2\pi)^2}\varphi(p)\frac{1}{(\Gamma^{-1})_{\varphi\varphi}(p)}\varphi(-p). \tag{3.27}$$

The sign of this effective action has been studied numerically.

In fig. 1, we plot the dependence in the rescaled two-dimensional momentum y

$$|y|^2 = \frac{\kappa_{\text{phys}}}{\tau_{\text{phys}}}|p|^2 \tag{3.28}$$

of the second derivative of the effective action $\Gamma_{\text{eff}}^{(2)}(\varphi)$ given by (3.27)

$$G(y^2) = \frac{\kappa_{\text{phys}}}{\tau_{\text{phys}}}\frac{1}{2}\frac{\partial^2\Gamma_{\text{eff}}^{(2)}(\varphi)}{\partial\varphi(p)\partial\varphi(-p)}\bigg|_{\varphi=0} = \frac{\kappa_{\text{phys}}}{\tau_{\text{phys}}}\frac{1}{(\mathbb{C}^{-1})_{\varphi\varphi}(p)} \tag{3.29}$$

for various values of the surface tension r_0. The dependence in r_0 is expressed in

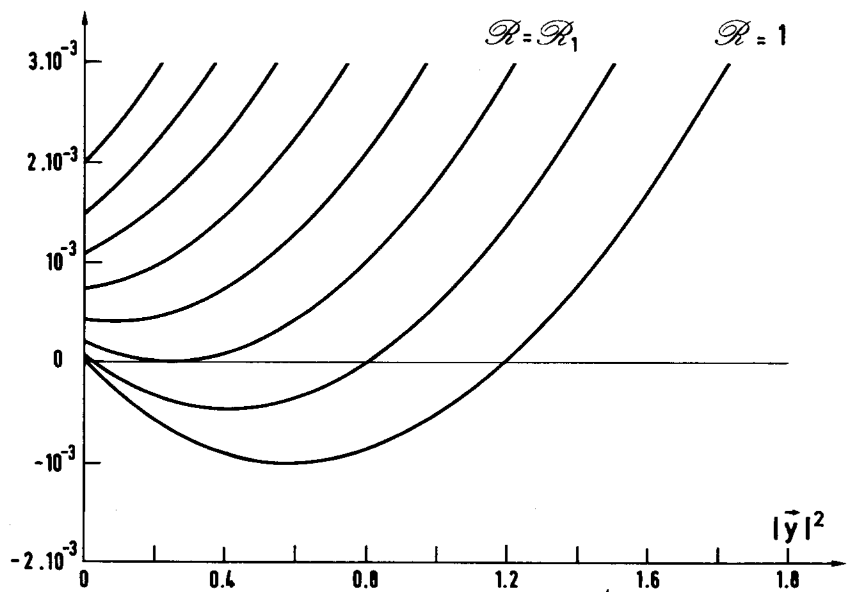


Fig. 1. The second derivative of the effective action for φ, $1/(\Gamma_{\varphi\varphi}^{-1})\cdot(\kappa_{\text{phys}}/\tau_{\text{phys}})$, as a function of the two-dimensional squared momentum $|y|^2 = |p|^2\kappa_{\text{phys}}/\tau_{\text{phys}}$, for various values of the surface tension (labelled by $\mathscr{R}$).

term of the dimensionless ratio

$$\mathscr{R} = \frac{\tau_{\text{phys}}(r_0)}{\tau_{\text{phys}}(r_0^{\text{crit}})}, \tag{3.30}$$

which increases continuously from 1 to $+\infty$ as r_0 increases from the critical point r_0^{crit} given by (2.17) to $+\infty$. The main properties of G are the following:

(a) For any $\mathscr{R} \geqslant 1$ ($r_0 \geqslant r_0^{\text{crit}}$) and for large momenta $|y|^2 \gg 1$, G is positive and is found to behave like

$$G(y^2) \sim y^2 \ln(y^2). \tag{3.31}$$

This is the bevavior expected by perturbative power counting, since the model is asymptotically free.

(b) For $\mathscr{R}$ larger than a critical value $\mathscr{R}_1 > 1$

$$\mathscr{R}_1 = 1.103 \pm 0.001 \tag{3.32}$$

(i.e. for r_0 larger than a critical value $r_1^{\text{crit}} > r_0^{\text{crit}}$) G is positive for every euclidean momenta $|y|^2 \geqslant 0$.

(c) For $\mathscr{R} = \mathscr{R}_1$ ($r_0 = r_1^{\text{crit}}$), G vanishes at some finite momentum

$$|y|^2 = |y|^2_{\text{crit}} \simeq 0.25. \tag{3.33}$$

(d) For $1 < \mathscr{R} < \mathscr{R}_1$ ($r_0^{\text{crit}} < r_0 < r_1^{\text{crit}}$), G becomes negative for some interval of momenta $|y|^2_{\text{min}} < |y|^2 < |y|^2_{\text{max}}$, and is positive away from this domain.

(e) At the critical point $\mathscr{R} = 1$, ($r_0 = r_0^{\text{crit}}$) (although the rescaling (3.28) becomes singular since κ_{phys} vanishes), the domain extends from zero momenta to some finite value. Thus

$$|y|_{\text{min}} = 0 \qquad \text{at } \mathscr{R} = 1. \tag{3.34}$$

Moreover, for small momenta, G is found to behave linearly with $|y|^2$ as

$$G(|y|^2) = -\frac{1}{96\pi}|y|^2. \tag{3.35}$$

From this study, if $\mathscr{R} < \mathscr{R}_1$, the second derivative of the effective action for conformal deformations of the metric of the surface has negative eigenvalues. Therefore, the homogeneous saddle point for the large-d effective action studied in sect. 2 becomes unstable and does not describe anymore the physical ground state of the surface. In the next section we discuss the physical significance of this result.

In ref. [19], Pisarski presented an analysis of the stability of the large-d homogeneous ground state; his conclusions strongly differ from ours and we shall discuss now the reasons for the discrepancies. Pisarski concludes that the ground state is unstable at large momenta $|y|^2 \gg 1$ and for all values of the bare surface tension ($r_0 > r_0^{\text{crit}}$ or $\mathscr{R} > 1$) while we found an instability only for $\mathscr{R} > \mathscr{R}_1$ and in a finite range of $|y|^2$. The main reason for this difference is the following. Pisarski takes as a definition for the second derivative potential for φ the φ-φ element of the matrix of second derivatives of the effective potential, $\Gamma_{\varphi\varphi}$, which is given by (3.16), and which is indeed negative. However, as seen in (3.27) the correct definition for the effective potential is $1/(\Gamma^{-1})_{\varphi\varphi}$. Since the hermitian matrix Γ is not positive definite, the fact that $(\Gamma)_{\varphi\varphi}$ is negative does not imply that $(\Gamma^{-1})_{\varphi\varphi}$ is negative (at variance with the claim in ref. [16]). This fact can be seen by looking at the sign of the eigenvalues of the matrix $\mathbf{C}$ given by (3.22). At large momenta one would expect three negative eigenvalues (corresponding to the three imaginary τ's) and three positive ones (corresponding to the three real fields v^1, v^2 and φ). In ref. [19], Pisarski performs such an analysis but makes a rotation $\tau \to i\tau$ in order to deal with a real field τ. However this is not a unitary transformation and the eigenvalues of the matrix $\mathbf{C}'$ obtained by this rotation are not directly related to those of $\mathbf{C}$. For these reasons we think that the conclusions of ref. [19] are incorrect although the starting point, namely the form for the effective potential $\Gamma(X, g, \lambda)$, is the same as ours.

A less important difference lies in the fact that longitudinal displacements v^i are not taken into account in ref. [19]. We have checked that this changes only by a few percents the form of the effective potential for φ. Finally, Pisarski claims that large momentum instabilities exists for $d > 13$. As we shall see in next section, the small momentum instabilities that we display are expected to exist for $d > 1$.

4. Discussion

4.1. THE SPONTANEOUS BREAKDOWN OF TRANSLATION INVARIANCE AT LARGE d

Let us first briefly recall the properties of the homogeneous large-d solution obtained in sect. 2. We enforced boundary conditions on the surface such that it was forced to stay in a preferred orientation, characterized by a plane (generated by the two vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in eq. (2.19). When taking the thermodynamic limit ($L \to \infty$), for a microscopic surface tension r_0 large enough ($r_0 > r_0^{\text{crit}}$), the average position of the surface is still parallel to this plane, which means that the rotational invariance $O(d)$ of the original action (2.5) is spontaneously broken to its subgroup $O(d-2) \times O(2)$. The surface fluctuates around its average position, so that the mean area projected onto a unit element of area on the reference plane, $\langle A \rangle / L^2$, is finite but larger than 1.

The correlations between normals (or tangent planes) to the surface decay exponentially at large distance with a finite correlation length ξ_{p}. Beyond this scale, the effect of the rigidity is negligible and the effective action for the normal displacement of the membrane is found to be gaussian, and characterized by a positive effective surface tension τ_{phys} given by (2.27)

$$S_{\text{eff}} = \int d^2\sigma\, \tau_{\text{phys}} \tfrac{1}{2}(\partial x_\perp \cdot \partial x_\perp). \tag{4.1}$$

A critical point was found ($r_0 = r_0^{\text{crit}}$) where the mean area of the surface becomes infinite ($\rho \to \infty$). At this point the rotational symmetry O(d) is restored. However, it was found that the effective surface tension and the correlation length ξ_{p} are still finite, as predicted by the renormalization group arguments. The large distance properties of this homogeneous saddle point correspond to the predictions of the large-d limit of the interface model of Wallace and Zia [26–28] for $\varepsilon = 2$ (where ε is the dimensionality of the interface). In particular, the gaussian form of the action (4.1) implies that the fractal dimension of the surface is infinite.

We now discuss the physical significance of the results of sect. 3. Let us first recall the strategy of the large-d limit. The effective action for the physical field X_{cl} is obtained by extremizing the complete effective action for X_{cl}, g and λ with respect to all the fields but X_{cl}. For $r_0 > r_1^{\text{crit}}$ (i.e. $\mathscr{R} > \mathscr{R}_1$) this program can be achieved explicitly for any almost plane configuration $X_{\text{cl}} = X_0 + \sigma^a u_a + x_\perp$ ($x_\perp$ small) by taking $\varphi = 0$ (and $v = \tau = 0$). We thus reobtain the homogeneous saddle point of sect. 2 and the effective action for X_{cl} reads, according to (3.21)

$$\Gamma_{\text{eff}}(X_{\text{cl}}) = \tau_{\text{phys}} \int dA_{\text{cl}} + \tfrac{1}{2}\kappa_{\text{phys}} \int dA_{\text{cl}}\, K_{\text{cl}}^2 + \text{higher order terms}, \tag{4.2}$$

where dA_{cl} and K_{cl}^2 are respectively the element of area and the extrinsic curvature of the average position X_{cl} of the surface. All the properties described above are then valid and we thus justify the renormalization scheme of subsect. 2.3.

For $r_0^{\text{crit}} \leqslant r_0 \leqslant r_1^{\text{crit}}$ (i.e. $1 \leqslant \mathscr{R} \leqslant \mathscr{R}_1$), this process does not hold. In this case, the effective action for both X_{cl} and φ becomes unstable at the homogeneous saddle point ($X_{\text{cl}} = \sigma^a u_a$, $\varphi = 0$) with respect to small fluctuations of φ with some finite momenta. The homogeneous solution of sect. 2 is thus no more stable and, as the main consequence, the initial symmetry of translation in the plane of the membrane will be spontaneously broken.

The nature of the real vacuum is unknown to us. We did not succeed in finding explicit non-homogeneous saddle points of the effective action (2.15) and this should be a very difficult task. We expect that the membrane will develop structures (bumps, fingers...) with typical size given by the correlation length ξ_{p}.

One may notice that this instability is a non-perturbative phenomenon, since it appears for τ_{phys} of the order of $\tau_{\text{phys}}(r_0^{\text{crit}}) = \Lambda^2 e^{-8\pi/\alpha_0}/\alpha_0$ ($\mathscr{R}_1 = \tau_{\text{phys}}(r_1^{\text{crit}})/\tau_{\text{phys}}(r_0^{\text{crit}}) \simeq 1.103$). We are not in a position to check the order of the transition between these two phases. This involves indeed the cubic term $\Gamma_{\text{eff}}^{(3)}$ in the expansion of Γ_{eff} in φ for $|y|^2 = |y_{\text{crit}}|^2$. However, a simple look at the diagrams involved in $\Gamma_{\text{eff}}^{(3)}$ shows that there is no reason why $\Gamma_{\text{eff}}^{(3)}(y_{\text{crit}})$ should vanish. It is thus reasonable to predict the existence a first order transition which occurs for some value of r_0 larger than r_1^{crit}. The critical point r_1^{crit} that we have exhibited by a local stability analysis is simply the boundary of the domain where the homogeneous solution is metastable.

In the new phase where translation invariance is spontaneously broken, the area of the surface should be larger than in the homogeneous phase. As r_0 is decreased the area should diverge at a new physical critical point $r_0'^{\text{crit}}$. At this point the surface is expected to be some fractal object, which has completely lost its intrinsic dimensionality (two). This critical point $r_0'^{\text{crit}}$ is described by another universality class than the naive critical point r_0^{crit}.

This phenomenon is very probably related to the "branching" which occurs in models of random surfaces made of plaquettes on a d-dimensional hypercubic lattice [29]. It was proven for large d [30] that at the critical point where the area becomes infinite, branched polymer configurations dominate the partition function (this remains presumably true for any $d > 2$). Models of random surfaces on a lattice without curvature energy [31], as well as models based on random triangulations [31, 32], may be considered as discretized effective models for the rigid surfaces studied here, valid only at scales much larger than the persistence length ξ_{p}. From this point of view, ξ_{p} should be identified with the cut-off a (the lattice spacing) of the discrete models.

4.2. THE EFFECTIVE THEORY AT FINITE d

Up to now, our conclusions are valid only in the limit $d \to \infty$. One would like to know whether the transition that we have exhibited disappears for small enough dimension of euclidean space in which the surface is embedded. In such a case, the effective theory for the surface at the critical point r_0^{crit} where the area diverges and the $O(d)$ symmetry is restored should be related to the string models, as first suggested in refs. [5, 9].

In this subsection we shall study these questions and try to make the connection between strings and random surfaces with rigidity more precise. For that purpose we shall try to construct an effective action for the surface at the critical point. As it will turn out, under the assumption that no instability occurs, this effective action can only be the Liouville action, which was first introduced by Polyakov [33] in the study of the string models. It turns out that the Liouville model is not physically acceptable if the dimension of space, d, is larger than 1. Most of our results will consist in fact in a rederivation, in a different context, of classical results in string theory. However, we think that their physical significance is somewhat different in our case (and perhaps easier to understand), and that they will lead to further insight on the relationship between the bosonic string model, the Liouville field theory and models of random surfaces.

Let us start from the model defined by the action (2.9). We shall quantize it by using the conformal gauge. For that purpose we choose a reference metric $\dot{g}_{ab}$ and impose the gauge constraint

$$g_{ab}-\tfrac{1}{2}\dot{g}_{ab}\left(\dot{g}^{cd}g_{cd}\right)=0\,, \tag{4.3}$$

which enforces the condition $g_{ab}(\sigma)=\dot{g}_{ab}(\sigma)e^{\psi(\sigma)}$. We shall not consider the degrees of freedom associated to Teichmüller deformations of the metric, which are not taken into account by this gauge fixing. There is only a finite number of them in general and they will not play any role in our discussion. With the gauge fixing (4.1) we have to add to the action (2.9) a Faddeev-Popov term involving anticommuting ghost fields $b_{ab}(\sigma)$ and $c^a(\sigma)$ (where $b_{ab}(\sigma)$ is a traceless symmetric tensor with respects to $\dot{g}$, i.e. $\dot{g}^{ab}b_{ab}=0$). Such a term has been first written by Polyakov in [9] and reads [9,21,22]

$$S_{\mathrm{FP}}(b_{ab},c^a,g)=\int\sqrt{g}\,b_{ab}\left(D^a c^b+D^b c^a\right). \tag{4.4}$$

We shall now make the following assumptions:

(i) There is a critical point r_0^{crit} where the surface becomes an infinitely extended object and where the O(d) rotation invariance is restored.

(ii) A mass scale m is generated dynamically by dimensional transmutation

(iii) At the critical point, there is no instability which breaks the homogeneity of the surface. The intrinsic geometry of the surface is therefore still two-dimensional at scales $l\gg 1/m$.

Hypotheses (1) and (2) are simply the properties expected from renormalization group calculations. Hypothesis (3) means that at the critical point, the ground state has the properties of the naive homogeneous ground state constructed in sect. 2. The intrinsic geometry of the surface depends only on the induced metric g_{ab} (and not on the embedding in bulk space) and may be characterized for instance by its spreading dimension d_{s} [34]. We therefore assume that $d_{\mathrm{s}}=2$ and that the Hausdorff dimension (i.e. the dimension of the embedding of the surface in bulk space) is infinite. In addition, we shall make a fourth assumption, also based on the large d results of sect. 3. We have seen that the conformal field φ (which is a bound state of the X's) becomes massless at the critical point ($r_0=r_0^{\mathrm{crit}}$). Indeed the second derivative of the effective action for φ, $G(p^2)$, vanishes at zero momentum. This appearance of a massless state is in fact not expected. It means that the effective action becomes invariant under global shifts $\varphi\to\varphi+\varphi_0$, which correspond to global rescalings of the metric $g_{ab}\to g_{ab}e^{\varphi_0}$, or to global rescalings of the coordinates on the surface $\sigma^a\to e^{-\varphi_0/2}\sigma^a$. Such dilations are in fact allowed at the critical point, since the surface is an infinite object with the intrinsic geometry of the infinite flat plane. The instability discussed above occurs only because the kinetic term for φ is negative. Our fourth assumption, which is in fact a consequence of assumptions (i) and (iii), is the following:

(iv) At the critical point a massless bound state ϕ of the elementary fields X exists, which is associated to global rescalings of the internal metric on the surface.

With those assumptions, we now construct an effective action for the massless modes of the surface, which should describe the physics of the surface at scales much larger than the dynamically generated length scale $\xi_p = 1/m$. Those massless modes must be:

(i) a d-component X field describing the position of the membrane (which is nothing but the Golstone boson associated to the spontaneous breakdown of translation invariance in d-dimensional space);
(ii) the large wavelength components of the ghost fields b_{ab} and c^a, which have to stay massless because of BRS invariance;
(iii) the bound state ϕ associated to global rescaling.

Because of euclidean invariance, the effective action for the X's must be of the form

$$S_{\text{eff}}(X) = \frac{1}{4\pi\alpha_s'} \int \sqrt{\dot{g}}\, \dot{g}^{ab}\, \partial_a X \cdot \partial_b X, \tag{4.5}$$

where the "intercept" α_s' is proportional to $1/m^2$. The effective action for the ghosts will be taken to be the Faddeev-Popov action (4.4). The situation for ϕ is slightly more subtle. We may of course write a dimensionless kinetic term for ϕ,

$$\int \sqrt{\dot{g}}\, \dot{g}^{ab}\, \tfrac{1}{2} \partial_a \phi\, \partial_b \phi, \tag{4.6}$$

but, since we have assumed that the surface has the intrinsic geometry of an infinite plane, there is another term which is invariant under the transformation $\phi \to \phi +$ const, namely

$$\int \sqrt{\dot{g}}\, \dot{R} \phi, \tag{4.7}$$

where $\dot{R}$ is the two-dimensional intrinsic curvature associated to $\dot{g}$. Indeed, on the plane, $\int \sqrt{\dot{g}}\, \dot{R} = 0$. With an adequate normalization of ϕ, we can write the most general effective action for ϕ as

$$S_{\text{eff}}(\phi) = \frac{\eta}{48\pi} \int \sqrt{\dot{g}} \left(\tfrac{1}{2} \dot{g}^{ab}\, \partial_a \phi\, \partial_b \phi + \dot{R} \phi \right), \tag{4.8}$$

where η is at that stage an unknown coupling constant. Eq. (4.8) is simply the massless Liouville action which plays a central role in the Polyakov string model [9] as well as in the bosonization of ghosts in string theories [35].

The total effective action is therefore of the form

$$S_{\mathrm{eff}} = S_{\mathrm{eff}}(X) + S_{\mathrm{FP}}(b, c, \dot{g}) + S_{\mathrm{eff}}(\phi) \tag{4.9}$$

and has been formulated with the help of the classical reference metric $\dot{g}$, which allowed us to introduce a fiducial metric on the surface. However, this metric is a gauge fixing parameter which has no physical significance, and the physics described by the effective action (4.9) must not depend on it. In particular, conformal changes in the metric $\dot{g}_{ab} \to \dot{g}_{ab}\mathrm{e}^{\varphi}$ must not change the free energy F of the system. But the φ dependence of F is nothing but the conformal anomaly and is known to be of the form

$$\frac{\delta F}{\delta\varphi(\sigma)} = -c\frac{1}{48\pi}\sqrt{\dot{g}(\sigma)}\,\dot{R}(\sigma) + \mathrm{O}\left(\frac{1}{m^2}\right). \tag{4.10}$$

Here the mass scale m plays the role of an ultraviolet regulator, since the effective action (4.9) is valid only at momenta $|p| < m$. c is the central charge of the theory. In our case an explicit calculation gives

$$c = d - 26 + \eta + 1. \tag{4.11}$$

The first two terms in the r.h.s. of eq. (4.11) come respectively from the X's and the ghosts contributions. They have been first obtained by Polyakov [9]. The last two terms come from ϕ. The third one, η, arises because the action (4.8) is not classically conformally invariant. Indeed a classical solution ϕ_{cl} obeys the equation of motion

$$-\Delta\phi_{\mathrm{cl}} + \dot{R} = 0 \tag{4.12}$$

and a change in the metric $\dot{g} \to \dot{g}\,\mathrm{e}^{\varphi}$ induces a change $\phi_{\mathrm{cl}} \to \phi_{\mathrm{cl}} - \varphi$ which induces a corresponding change in the classical action $S_{\mathrm{eff}}(\phi_{\mathrm{cl}}) \to S_{\mathrm{eff}}(\phi_{\mathrm{cl}}) - (\eta/48\pi)\int\sqrt{\dot{g}}\,\dot{R}\varphi$. The last term in (4.11) comes from the fluctuations of ϕ around the classical solution.

Thus the physical requirement that the free energy of the system does not depend on the metric $\dot{g}_{ab}$ imposes that $c = 0$. This fixes the value of the coupling constant η to be

$$\eta = 25 - d. \tag{4.13}$$

This is in fact in agreement with our large-d calculation. From eq. (3.35), the kinetic part for the effective action of φ at small momenta is found to be

$$\Gamma^{(2)}_{\mathrm{eff}}(\varphi) = -\frac{d}{96\pi}\int \mathrm{d}^2\sigma(\partial\varphi)^2 \tag{4.14}$$

which coincides at large d with (4.8) and (4.13) and explains why at large d and small momenta the effective action for φ is nothing but given by the conformal anomaly, as first pointed out in ref. [16]. Eq. (4.13) was known from the study of the quantization of the massive (or interacting) Liouville model [36].

Thus we see that the kinetic term for ϕ is negative as long as $d > 25$, and therefore we expect that the large-distance instabilities discussed in this paper will exist at least for $d > 25$. However, as we shall see, the Liouville model suffers from some additional problems in the strong coupling region $1 < d \leqslant 25$, and we expect that such instabilities will still appear in this domain, when considering other observables than those associated to conformal deformations of the surface.

In order to compute correlation functions between points of the surface, we need to introduce the following observable, which is nothing but the density at point $\boldsymbol{X}$

$$\rho(\boldsymbol{X}) = \int \mathrm{d}^2\sigma \sqrt{g(\sigma)}\, \delta^d(\boldsymbol{X} - \boldsymbol{X}(\sigma)). \tag{4.15}$$

Its Fourier transform in d-dimensional space is the vertex operator

$$V(\boldsymbol{P}) = \int \mathrm{d}^2\sigma \sqrt{g(\sigma)}\, \mathrm{e}^{i\boldsymbol{P}\cdot\boldsymbol{X}(\sigma)}. \tag{4.16}$$

A natural form for the vertex operator in the effective theory (4.9) is

$$V_{\mathrm{eff}}(\boldsymbol{P}) = \int \mathrm{d}^2\sigma \sqrt{\dot{g}(\sigma)}\, \mathrm{e}^{i\boldsymbol{P}\cdot\boldsymbol{X}(\sigma) + A\phi(\sigma)}. \tag{4.17}$$

It is necessary to introduce a term $A\phi$ because we want that vacuum expectation values of products of vertex operators do not depend on the metric $\dot{g}$.

This will be true if the conformal weight of V_{eff} is zero. This conformal weight is given by

$$h = 1 - \frac{\alpha_s'}{4} P^2 - A + \frac{6}{\eta} A^2 = 0, \tag{4.18}$$

where the first term comes from the factor $\sqrt{\dot{g}}$, the second one from the fluctuations of $\boldsymbol{X}$ and the third and fourth terms from the classical and fluctuating parts of ϕ respectively. In the classical limit $\alpha_s' \to 0$, $\eta \to \infty$, where fluctuations of $\boldsymbol{X}$ and ϕ may be neglected, we must have $A = 1$. Thus $h = 0$ fixes A to depend on $\boldsymbol{P}$ as

$$A = \frac{\eta}{12}\left(1 - \sqrt{1 - \frac{24}{\eta}\left[1 - \frac{\alpha_s'}{4} P^2\right]}\right). \tag{4.19}$$

We immediately see that A is real for large momenta $\boldsymbol{P}$ but get an imaginary part if

$$P^2 < P^2_{\mathrm{crit}} = \frac{24-\eta}{6\alpha'_{\mathrm{s}}} = \frac{d-1}{6\alpha'_{\mathrm{s}}}. \tag{4.20}$$

Thus, if $d > 1$, the vertex operator (4.17), which allows to compute correlations between the position of points of the surface (and to obtain physical quantities such as the mean density or the fractal dimension) gets an imaginary part for small euclidean momenta. This should reflect a large distance instability and means that the large distance properties of the vacuum of the theory are not those we started from in order to construct our effective theory. It is tempting to conjecture that this instability is the instability that we have displayed in sect. 3, which breaks homogeneity of the surface and destroys its two-dimensional character at large distance.

Finally, let us discuss the relationship between the effective theory (4.9), the bosonic string and the Liouville model. Taking the formal limit $\eta \to 0_+$ $(d \to 25_-)$, A becomes very small but purely imaginary. Let us rescale

$$\phi = \left(\frac{24}{\alpha'_{\mathrm{s}}\eta}\right)^{1/2} X_{26}, \qquad A = i\left(\frac{\alpha'_{\mathrm{s}}\eta}{24}\right)^{1/2} P^{26}. \tag{4.21}$$

Then the second term in the effective action for ϕ disappears and $S_{\mathrm{eff}}(\phi)$ reads

$$\lim_{d\to 25_-} S_{\mathrm{eff}}(\phi) = \frac{1}{4\pi\alpha'_{\mathrm{s}}}\int \sqrt{\hat{g}}\,\hat{g}^{ab}\,\partial_a X_{26}\,\partial_b X_{26}, \tag{4.22}$$

while the vertex operator (4.17) reads

$$\lim_{d\to 25_-} V_{\mathrm{eff}}(\boldsymbol{P}) = \int \mathrm{d}^2\sigma\sqrt{\hat{g}}\,\mathrm{e}^{i(\boldsymbol{P}\cdot\boldsymbol{X} + P_{26}\cdot X_{26})}. \tag{4.23}$$

The constraint (4.16) becomes

$$\boldsymbol{P}^2 + P_{26}^2 = \frac{4}{\alpha'_{\mathrm{s}}}. \tag{4.24}$$

We recover the 26-dimensional bosonic string model provided that we view the field ϕ as the missing 26th dimension; the vertex operator becomes the vertex operator describing tachyon emission since (4.24) is the mass shell condition for the tachyon. Thus at $d = 25$, the original euclidean symmetry $\mathrm{E}(d) \times \mathrm{E}(1)$ (the last E(1) corresponds to shifts in ϕ) is enlarged to the larger group E(26). Of course this argument is unfortunately quite formal because of the "tachyonic" instabilities discussed before.

In the literature most attention has been devoted to the quantization of the interacting Liouville model. In the context of surface models with rigidity this corresponds to construct an effective action for the membrane away from the critical point ($r_0 > r_0^{\text{crit}}$). Then we must add to the effective action (4.8) a mass term of the form

$$\mu^2 \int \sqrt{g}\, e^{\Phi}, \tag{4.25}$$

where the mass μ^2 is of the order of ($r_0 - r_0^{\text{crit}}$). The free model (4.8) becomes an interacting model which is much more difficult to study. However the main features discussed in this subsection (eqs. (4.13) and (4.18), existence of tachyons for $d > 1$) where previously shown to hold also in the interacting case [36], although they are much more difficult to derive. One may also notice that if $\mu^2 > 0$ and $\eta > 0$ ($d < 25$), the natural ground state for the surface with no boundaries is a space with negative curvature (hyperbolic plane), while one would expect the surface to be an object with a finite size and therefore a positive curvature. This is a further indication that something is wrong with this effective action in the domain $d < 25$ (at least for describing random surfaces).

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