MICROCANONICAL SIMULATIONS OF RANDOMLY TRIANGULATED PLANAR RANDOM SURFACES

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Received 6 December 1985

Results of Monte Carlo simulations of a model of random surfaces based on planar random triangulations with gaussian embedding in d-dimensional euclidean space are presented for various positive and negative values of d. The fractal dimension (Hausdorff dimension of the embedding) is large and decreases slightly with d. The spreading dimension (intrinsic Hausdorff dimension of the random lattice) is greater than two and increases with d. Flat regular lattices dominate at large negative d and very irregular ones for large positive d. There is a significant linear correlation between the effective action and the discrete Liouville action.

A possible and interesting approach to the construction of a continuum theory of random surfaces is to start from a discrete formulation of the problem which involves random triangulations [1–7] (for a general review of random surfaces theory see e.g. refs. [2,7]). The partition function is defined by a sum over all triangulations T of the sphere S_2 (i.e. two-dimensional simplicial complex with genus $g = 0$) [‡1] as

$$Z = \sum_{\mathrm{T}} \exp[-\beta(\mathrm{T})] \frac{1}{C(\mathrm{T})} \int \prod_{\mathrm{v}\in\mathrm{T}} \mathrm{d}^d(X_\mathrm{v}/\sqrt{\pi}) \times \exp\left(-\sum_{l=(\mathrm{v},\mathrm{w})} (X_\mathrm{v} - X_\mathrm{w})^2\right) \delta^d(X_{\mathrm{v}_0}) . \qquad (1)$$

$C(\mathrm{T})$ is the order of the symmetry group of T [5,6] and enters for combinatoric reasons. $|\mathrm{T}| = N_\mathrm{t}(\mathrm{T})$ is the number of triangles of T. The integral in (1) runs over the positions X_v in d-dimensional euclidean space of the $N_\mathrm{v}(\mathrm{T})$ vertices of T (except one vertex v_0 in order to eliminate the translation zero modes). The gaussian form of the action for the X's corresponds to "freeze" an intrinsic metric on the lattice T such that the intrinsic length of the links is constant. Let us note that, with the choice of measure for the X_v's in (1), the integration over the X's gives an effective action

$$S_{\mathrm{eff}}(\mathrm{T}) = \tfrac{1}{2} d \,\mathrm{Log}\{\mathrm{Det}'(-\Delta^{(2)})/|\mathrm{T}|\} , \qquad (2)$$

where $\Delta^{(2)}$ is the discrete laplacian acting on antisymmetric tensors (i.e. 2-forms) on T (and is an $N_\mathrm{t} \times N_\mathrm{t}$ matrix) [8] and where Det′ means the product of the $N_\mathrm{t} - 1$ non-zero eigenvalues of $\Delta^{(2)}$.

Another possible choice of measure in (1) for the X's is [5]

$$\prod_{\mathrm{v}} \mathrm{d}^d [X_\mathrm{v}\sqrt{C_\mathrm{v}(\mathrm{T})}] , \qquad (3)$$

where the coordination number $C_\mathrm{v}(\mathrm{T})$ is the number of triangles of T which meet at the vertex v. This choice corresponds to replace in the effective action (2) $\Delta^{(2)}$ by $\Delta^{(0)}$, the discrete laplacian acting on scalar functions (i.e. 0-forms) on the random lattice T. ($\Delta^{(0)}$ is an $N_\mathrm{v} \times N_\mathrm{v}$ matrix.) In the classical continuum limit these two actions are equivalent.

In this paper we shall present some results of Monte Carlo simulations performed on this model. Our restrictions will be the following. We consider planar triangulations (with genus 0) and with a *fixed* number of triangles. Moreover we exclude singular configurations where two vertices are joined by more than one link, or where the two extremities of a link are the same vertex.

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‡1 The extension to other topologies is straightforward.

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The principle of the updating of the configurations is the one first proposed in ref. [6]. We change the triangulation by taking a link at random and by proposing the flip of this link if it does not lead to any forbidden triangulation. The proposed flip is accepted or rejected according to the Metropolis algorithm. Such a process can be shown to be ergodic (all configurations are reached) and to respect balance [the counting factor $1/C(\mathrm{T})$ in (1) is automatically obtained].

In order to update the positions X_v of the vertices v we have used two methods.

Method I. The first one consists in performing separate Monte Carlo for the positions X_v. As in ref. [6] we have used the heat bath algorithm to update the positions of the vertices. Each flip is then performed at fixed vertex positions and the flip accepted or rejected according to the change in the total action by the standard Metropolis procedure.

Method II. The second one consists in computing directly the change in the effective action (2) during a flip and then to accept or reject the flip according to the Metropolis algorithm. For the size of the surfaces that we have considered, the most efficient procedure is the Scalapino–Sugar method which consists in storing the full propagator (i.e. the inverse of the connection matrix) and in computing exactly the change in the propagator and in the effective action at each flip. This procedure requires $\sim N_\mathrm{v}^2$ operations at each flip. We have also tried to compute the propagator by statistical methods (by using "pseudo-fermions" variables) and by iterative methods (conjugate gradient methods) but these methods are in fact much slower and less efficient in practice. One potential advantage of method II over method I is that the dimension d can be easily taken to be negative which is useful to compare this model to the Liouville string theory [9] in its weak coupling regime.

Let us now explain which observables have been considered. We have measured the square curvature density

$$R^2 = N_\mathrm{t}^{-1} \sum_\mathrm{v} (C_\mathrm{v} - 6)^2 / C_\mathrm{v} \,, \tag{4}$$

the effective action density (2)

$$\rho_\mathrm{eff} = -N_\mathrm{t}^{-1} (2/d)\, S_\mathrm{eff}$$
$$= -N_\mathrm{t}^{-1} \,\mathrm{Log}[\mathrm{Det}'(-\Delta^{(2)})/N_\mathrm{t}] \,, \tag{5}$$

and the discrete version of the Liouville action density [8]

$$\rho_\mathrm{Liouv} = N_\mathrm{t}^{-1} \sum_\mathrm{v} \sum_\mathrm{w} R_\mathrm{v} (-\Delta^{(0)} + P_0)^{-1}_{\mathrm{vw}} R_\mathrm{w} \,, \tag{6}$$

where $R_\mathrm{v} = (6 - C_\mathrm{v})/\sqrt{C_\mathrm{v}}$ is the curvature density, $-\Delta^{(0)}$ the scalar laplacian and P_0 the projector onto the zero mode of $\Delta^{(0)}$. These quantities have been chosen in order to study the connection of the model with the continuum Liouville theory, and the effect of the discretization on the conformal anomaly. In order to study the scaling properties of the surface we have considered two observables: The mean square extent of the surface,

$$\bar{X}^2 = \frac{1}{d} \left(\tfrac{1}{9} N_\mathrm{t}^{-2} \cdot \sum_\mathrm{v} \sum_\mathrm{w} C_\mathrm{v} C_\mathrm{w} \overline{(X_\mathrm{v} - X_\mathrm{w})^2} \right)$$
$$= N_\mathrm{t}^{-1} \,\mathrm{Tr}[(-\Delta^0 + P_0)^{-1} - P_0] \,, \tag{7}$$

whose scaling behaviour gives the fractal dimension d_F of the surface (i.e. the Hausdorff dimension of the embedding of the surface in bulk space) by

$$\langle X^2 \rangle \sim N_\mathrm{t}^{2/d_\mathrm{F}} \,, \quad N_\mathrm{t} \to \infty \,, \tag{8}$$

and the average intrinsic distance δ on the lattice T between two vertices δ_vw (defined as the minimal number of links joining v to w on T). Its scaling behaviour defines the so called spreading dimension (or topological dimension) d_s [10] (i.e. the intrinsic Hausdorff dimension of the lattice T) by

$$\langle \delta \rangle \sim N_\mathrm{t}^{1/d_\mathrm{s}} \,, \quad N_\mathrm{t} \to \infty \,. \tag{9}$$

The distance δ_vw has been extracted from the large mass limit of the propagator $(-\Delta + M^2)^{-1}$ by the formula

$$\delta_\mathrm{vw} = -\lim_{M^2 \to \infty} [M^2 (\mathrm{d}/\mathrm{d}M^2) \ln(-\Delta^{(0)} + M^2)^{-1}_{\mathrm{vw}} + 1] \,, \tag{10}$$

which follows from the random-walk expansion of the propagator. The measure of (6), (7) and (9) is therefore obtained by computing matrix elements of propagators on the random lattice T.

Our numerical simulations have been performed for values of the bulk dimension $d = 0, 12, -12$ and -96. For $d = 12$ we have used method I (Monte Carlo on the position) and for $d < 0$ we have used method II (Scalapino–Sugar). To initialize the system we started from a typical $d = 0$ configuration and have made 10^3

sweeps of the lattice at the final d for equilibrium before starting measurements [‡2]. During one sweep we tried to flip N_ℓ links (where N_ℓ is the total number of links of the lattice) and in method I the positions of N_v vertices (randomly chosen) are updated after each flipping sweep. The maximal number of vertices of the surfaces is 128 for $d = -96$, 256 for $d = -12$ and 512 for $d = 0$ and 12 ($N_t = 2N_v - 4$). During the measurement runs at $d = 0$ and $d = -12$ we have made 5×10^3 iterations and made a measurement each 10 iterations. For $d = +12$ those numbers are 5×10^4 and 10^2 and for $d = -96$ 2×10^3 and 10 respectively. The statistics has been checked systematically using the binning method. We found that the correlations between successive configurations increase strongly with d and N_v. This is why we have performed more sweeps between each measurement at $d = 12$ than at $d = 0$.

‡2 For $d = -96$ we made an "adiabatic" equilibrium by changing linearly d from 0 to -96 during the equilibrium period.

During measurement, the quantities (6), (7), (10) (including $\langle X^2 \rangle$ at $d = 12$) are obtained from an exact calculation of the propagators via an inversion algorithm. Finally let us stress that all simulations have been made with the effective action (2), which corresponds to the flat measure for the X's in (1).

In figs. 1 and 2 we present scatter plots of ρ_{eff} versus ρ_{Liouv} and of ρ_{eff} versus R^2, respectively, for samples of 200 surfaces with $N_v = 128$ vertices obtained for $d = -96$, $d = -12$, $d = 0$, $d = 12$ and for "tree-like" surfaces (T) obtained by a growing process by gluing a tetrahedron at random on a face of the surface (this is expected to be the most irregular surface). We observe a very strong linear correlation between ρ_{eff}, ρ_{Liouv} and R^2 (which persists for larger surfaces).

The observed correlation between ρ_{eff} and ρ_{Liouv} is a priori very encouraging, since in the continuum we expect such a linear relation from the conformal anomaly [9]. However, the slope obtained from fig. 1 is smaller by a factor of 0.7 to the exact factor $1/48\pi$

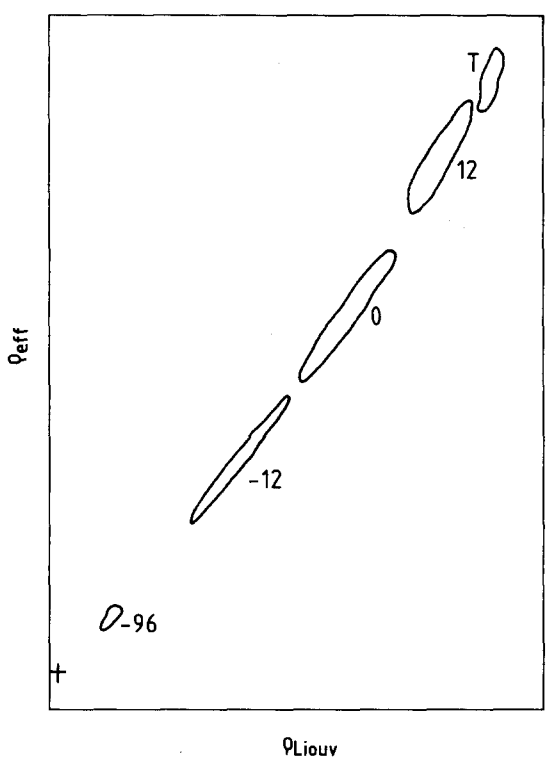

Fig. 1. Scatter plot of the effective action versus the Liouville action for $d = -96$, -12, 0 and 12 and for "tree-like" surfaces (T).

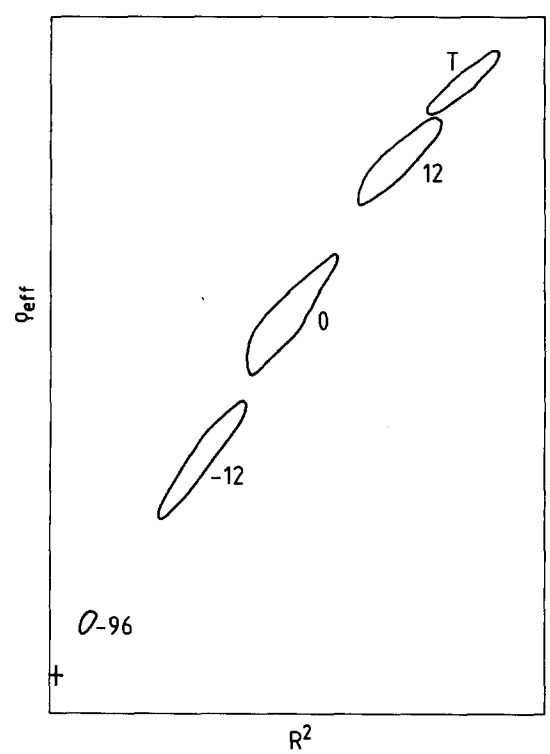

Fig. 2. Scatter plot of the effective action versus the curvature squared for the same configurations.

[9]. Moreover, the correlation between ρ_{eff} and R^2 indicates that ρ_{eff} and ρ_{Liouv} may depend strongly on the local structure of the surface, and therefore on the "high-energy" modes of the laplacian $\Delta^{(2)}$ (i.e. of the modes with large eigenvalues). On the contrary, the form of the conformal anomaly obtained in ref. [9] relies on an effective action calculation where the high-energy modes of the continuum laplacian have been integrated out and therefore rely on the contribution of the low-energy modes of Δ. This point of view seems to be corroborated by preliminary studies of the spectrum of the discrete laplacian on random surfaces [11].

In any way, from fig. 2 we see that the curvature-squared increases strongly with d and that it is very small for large negative d. The cross in figs. 1 and 2 corresponds to a flat torus with the same number of triangles. Arguments were given in ref. [6] that the flat regular lattice is a local minimum of the effective action (2). Our data clearly show that flat surfaces (at least at the scale that we are probing) are absolute minima for this action and dominate at $d = -\infty$.

Fig. 3 presents a Log–Log plot of $\langle X^2 \rangle$ versus N_{V} and fig. 4 a similar plot for $\langle \delta \rangle$, which give estimates of the fractal dimension d_{F} and of the spreading dimension d_{s} of the surface. There is reasonable evidence that we have reached the scaling region for $N_{\mathrm{V}} \geqslant 64$ in the cases $d = -12, 0, 12$, where the plot is linear, indicating the validity of the scaling laws (8)

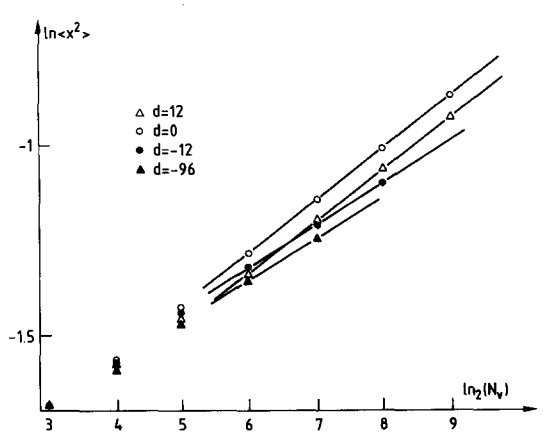

Fig. 3. Log–Log plot of the mean square extent $\langle X^2 \rangle$ versus the number of vertices N_{V} for $d = -96, -12, 0, 12$.

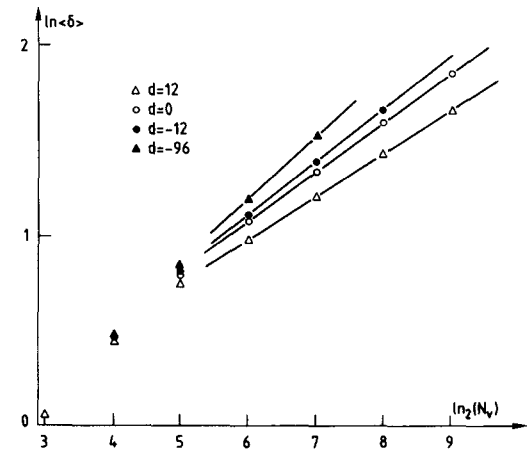

Fig. 4. Log–Log plot of the mean intrinsic distance $\langle \delta \rangle$ versus the number of vertices N_{V} for $d = -96, -12, 0, 12$.

and (10). The corresponding estimates for d_{F} and d_{s} are reported in table 1.

We observe that d_{F} and d_{s} depend rather smoothly on d. The fractal dimension decreases slightly as d increases, and is large. The estimate for $d = 0$ is consistent with the estimate of ref. [5] coming from strong coupling series, and close to the estimate of ref. [6] at $d = 3$. The spreading dimension is clearly greater than 2, and increases with d.

For $d = -96$ the scaling region has not yet been reached [‡3] and we cannot give precise estimates for d_{F} and d_{S}. However, it seems plausible that $d_{\mathrm{F}} > 13$ and that $2.0 < d_{\mathrm{s}} < 2.5$. This is consistent with the conjecture that flat regular lattices dominate at $d = -\infty$, which gives $d_{\mathrm{F}} = \infty$, $d_{\mathrm{s}} = 2$.

Runs at $d = 48$ have been made to see which configurations dominate for very large d. However, one

[‡3] Since typical surfaces are very flat, long-range fluctuations are essential.

Table 1
Estimates for d_{F} and d_{s} for $d = -12, 0, 12$.

d	d_{F}	d_{s}
−12	12.5(3)	2.55(5)
0	9.9(4)	2.8(1)
12	9.7(5)	3.2(2)

seems to get trapped very easily into metastable states and no definite conclusion has yet been reached. Let us only mention that the configurations which are generated seem close to "tree-like" surfaces made by gluing at random tetrahedra on their faces. For such surfaces one expects $d_F = \infty$ and $d_s = \infty$. In any way there is no evidence at that stage that for large positive d branched polymer configurations (for which $d_F = 4$ and $d_s = 2$) dominate.

Finally, let us mention that a recent numerical simulation with the alternative measure (3) (i.e. with the discrete scalar laplacian) gives for positive bulk dimensions d very different estimates for the fractal dimension d_F [12]. We have performed some preliminary runs which confirm that indeed d_F and d_s seem to depend strongly on the form of the action [11]. It is very important to understand if universality is really violated in those models and which phenomenon is responsible for this violation. Another point would be to perform grand-canonical simulations, where the number of triangles of the random lattice is allowed to vary, in order for instance to measure the exponent γ of the susceptibility of the model. Algorithms may be constructed for such simulations [13] and we hope to present some results in the future.

The simulations were performed using the CRAY X MP at Saclay. We are very grateful to J. Jurkiewicz, A. Krzywicki and B. Petersson for interesting discussions and for communicating to us the results of their simulations prior to publication. One of us (F.D.) thanks also H. Hermann and J.M. Luck for useful discussions.

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