A DUALITY PROPERTY OF PLANAR FEYNMAN DIAGRAMS

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Received 17 April 1985

It is found that the Fourier-transform of the amplitude of a planar Feynman diagram G can be written as the amplitude of the Feynman diagram $\tilde{\mathrm{G}}$, where $\tilde{\mathrm{G}}$ is the dual of G in the sense of graph theory, the propagators of $\tilde{\mathrm{G}}$ being the Fourier-transformed of the ordinary ones.

Surface models are currently becoming more and more popular. At the same time, the planar sector of field theories, viewed as a discretization of a surface theory, deserves more and more attention. It is our purpose in the present note to point out an interesting property of planar Feynman diagrams, which, although it may seem rather trivial, and has, in fact, already been used [1–3] [‡1], has never been explicitly formulated before to our knowledge.

Let us consider a planar (but otherwise arbitrary) Feynman diagram G. Its vertices will be indexed by v and its edges (or links or propagators) by ℓ. In a d-dimensional position space, such a diagram can be evaluated by

$$I_{\mathrm{G}} = K_{\mathrm{G}} \left(\prod_{\mathrm{v_i} \in \mathrm{G}} \int \mathrm{d}^d X_{\mathrm{v_i}} \right) \prod_{\ell \in \mathrm{G}} D(Y_\ell), \tag{1a}$$

$$Y_\ell = \sum_{\mathrm{v}} \mathcal{E}_{\ell \mathrm{v}} X_{\mathrm{v}}, \tag{1b}$$

where $\mathcal{E}$ is an incidence matrix having non-zero elements $\mathcal{E}_{\ell\mathrm{v}} = \pm 1$ only if an edge ℓ is incident with a vertex v. In order to have only differences of X_{v} for the Y_ℓ we require that, for any ℓ,

$$\sum_{\mathrm{v}} \mathcal{E}_{\ell \mathrm{v}} = 0, \tag{2}$$

In (1a) the index $\mathrm{v_i}$ refers to the internal vertices of G, i.e. those which are incident with at least two edges of G. K_{G} is a coefficient containing the coupling constant dependence and the symmetry factor associated with G. We can express $D(Y_\ell)$ as

$$D(Y_\ell) = (2\pi)^{-d} \int \mathrm{d}^d k_\ell \tilde{D}(k_\ell) \exp(\mathrm{i} k_\ell Y_\ell), \tag{3}$$

where $\tilde{D}(k_\ell)$ is the momentum-space representation of the propagator of the edge ℓ. If, on the other hand, we multiply (1a) by

$$\prod_{\mathrm{v_e} \in \mathrm{G}} \int \mathrm{d}^d X_{\mathrm{v_e}} \exp(-\mathrm{i} p_{\mathrm{v_e}} X_{\mathrm{v_e}}), \tag{4}$$

we obtain the Fourier-transform of I_{G}, $\tilde{I}_{\mathrm{G}}$ or the momentum-space representation of G. (Here, $\mathrm{v_e}$ refers to the external vertices of G).

The integration over all the position-space variables $X_{\mathrm{v_i}}, X_{\mathrm{v_e}}$ produces δ-functions on all vertices of G, giving

$$\tilde{I}_{\mathrm{G}} = K_{\mathrm{G}} \left(\prod_{\ell \in \mathrm{G}} (2\pi)^{-d} \int \mathrm{d}^d k_\ell \tilde{D}(k_\ell) \right) \times \prod_{\mathrm{v} \in \mathrm{G}} (2\pi)^d \delta^d(q_{\mathrm{v}}), \tag{5a}$$

$$q_{\mathrm{v_i}} = \sum_{\ell} \mathcal{E}_{\ell \mathrm{v_i}} k_\ell, \tag{5b}$$

$$q_{\mathrm{v_e}} = \sum_{\ell} \mathcal{E}_{\ell \mathrm{v_e}} k_\ell - p_{\mathrm{v_e}}. \tag{5c}$$

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[‡1] Ref. [3] has been communicated to us after the completion of the present note.

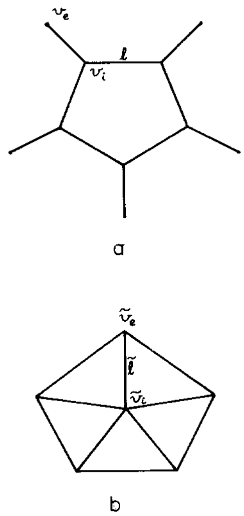

Fig. 1. A planar diagram G (a) with internal and external vertices and its dual diagram $\tilde{G}$ (b).

Let us now define the dual diagram of G, which we call $\tilde{G}$. $\tilde{G}$ is constructed in the following way (see fig. 1):

(i) Inside each (internal) face of G draw a vertex $\tilde{v}_i$ of $\tilde{G}$.

(ii) Draw a vertex $\tilde{v}_e$ of $\tilde{G}$ between any two consecutive external lines of G (which define an external face of G).

(iii) Draw new edges $\tilde{\ell}$ of $\tilde{G}$, bonding the vertices $\tilde{v}$, across the edges ℓ of G and in one-to-one correspondence with them. (To each ℓ corresponds an $\tilde{\ell}$.)

We know that, for planar diagrams, any propagator ℓ is the border of two adjacent faces. Therefore, any momentum k_ℓ (or p_{v_e}) can be considered as the difference of two momenta $K_{\tilde{v}}$ associated with two adjacent faces of G sharing the edge ℓ. Expressing all momenta k_ℓ as functions of the momenta $K_{\tilde{v}}$ associated with the faces of G, we trivially satisfy all the constraints

$$q_v = 0, \tag{6}$$

implied by the δ-functions of (5a). But each $K_{\tilde{v}}$ is, of course, through (i) and (ii), associated with a vertex $\tilde{v}$ of $\tilde{G}$. Therefore, we write (5a) as

$$\tilde{I}_G = K_G (2\pi)^{d(n_v - n_\ell)} \times \left(\prod_{\tilde{v}_i \in \tilde{G}} \int \mathrm{d}^d K_{\tilde{v}_i} \right) \prod_{\tilde{\ell} \in \tilde{G}} \tilde{D}(P_{\tilde{\ell}}), \tag{7a}$$

$$P_{\tilde{\ell}} = \sum_{\tilde{v}} \tilde{\mathcal{E}}_{\tilde{\ell}\tilde{v}} K_{\tilde{v}}, \quad \sum_{\tilde{v}} \tilde{\mathcal{E}}_{\tilde{\ell}\tilde{v}} = 0, \tag{7b, c}$$

where n_v (n_ℓ) is the number of vertices (edges) of G and $\mathcal{E}$ is the incidence matrix of $\tilde{G}$. Non-zero elements $\tilde{\mathcal{E}}_{\tilde{\ell}\tilde{v}} = \pm 1$ only exist if an edge $\tilde{\ell}$ is incident with a vertex $\tilde{v}$.

Comparing (1) and (7), and identifying K_G with $K_{\tilde{G}}$, we get the identity

$$\tilde{I}_G(p_{v_e}) = (2\pi)^{d(1-n_{\tilde{v}_i})} I_{\tilde{G}}(K_{\tilde{v}_e}), \tag{8}$$

which can be phrased as:

> The Fourier transform of the Feynman integral of a diagram G can be obtained by writing the Feynman integral for the diagram $\tilde{G}$ dual to G and using Fourier-transformed propagators instead.

We can mention at least one possible use of this property [2,3]. Suppose the diagram G corresponds to a cubic interaction and that it is drawn on a surface S. Then, the Fourier-transform of I_G is naturally expressed as a function over a triangulation of S, the variables $K_{\tilde{v}}$ being considered as d-component scalar fields defined on that surface. Moreover, $\tilde{D}$ can be written (in the case of a massive scalar field) as

$$\tilde{D}(P_{\tilde{\ell}}) = \int_0^\infty \mathrm{d}\alpha \exp(-\alpha m^2 - \alpha P_{\tilde{\ell}}^2). \tag{9}$$

If P^2 is viewed as $\Delta^\mu K \, \Delta_\mu K$ we have a discrete version of a lagrangian density $\alpha(\partial_\mu K)^2$ on the surface S, α being viewed as describing some intrinsic metric on S. This may help to draw a connection between planar field theory and surface theories [4].

We would like to thank M. Bergère for discussions and warm encouragements.

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