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# THE OPERATOR PRODUCT EXPANSION AND RENORMALONS: A COMMENT

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Recently, the ITEP group has analysed the structure of the operator product expansion in the $1/N$ expansion of some two-dimensional models. We show that, despite their claim, their analysis does not invalidate the conclusions of previous studies by the present author on the role of renormalon singularities in the operation product expansion. Some considerations on the applicability of the operator product expansion in QCD are also developed.

## 1. Introduction

Since its introduction by K. Wilson, the short-distance operator product expansion (OPE) [1] has been the object of vigorous theoretical investigations as well as of numerous applications in quantum field theory and in particle physics. Let us recall that, in its simplest form, the OPE states that the product of two local operators $A(x)$ and $B(0)$ may be expanded, as $x \to 0$, as a sum over all local operators $\mathcal{O}_n$:

$$A(x)B(0) \underset{x\to 0}{\simeq} \sum_n C_n^{AB}(x)\mathcal{O}_n(0). \tag{1.1}$$

In a recent series of papers, the present author has studied the structure of the OPE in asymptotically free field theories. The purpose of such a study was twofold. First, it is well known that in an asymptotically free field theory, the short-distance behaviour is related to the small coupling constant structure of the theory, and therefore to the summability of perturbation theory and to the role of non-perturbative effects. The understanding of such problems is of course of fundamental importance for gauge theories. Second, the OPE has been widely applied in the last years, under the impulsion of the ITEP group, to parametrize the non-perturbative aspects of QCD and to study various aspects of its spectrum, in the "QCD sum rules formalism". It was therefore important to understand the theoretical status of such non-perturbative applications of the OPE, which have met a good phenomenological success.

In [2,3], we studied the OPE in the $1/N$ expansion of the two-dimensional non-linear sigma models. In particular we pointed out that in order to give a sense to the expansion (1.1), one has to take into account the fact that the perturbative series which define the coefficients $C_n^{AB}(x)$ and the matrix elements of the composite operators $\mathcal{O}_n$ are in general not Borel-summable, because of renormalon singularities. A (partially arbitrary) resummation scheme has to be chosen in order to define each term of the r.h.s. of (1.1). In particular, there is no way to give an intrinsic sense (i.e. without reference to a particular subtraction scheme) to an object such as the gluon-condensate $\langle 0|(\alpha_s/\pi)G^a_{\mu\nu}G^{\mu\nu}_a|0\rangle$, which cannot be put on the same footing than observables with direct physical significance such as hadrons masses or the value of the chiral condensate. Let us note that this phenomenon was previously noticed in a different context, (the trace anomaly) by Adler in [4] (although its consequences were not fully appreciated). Our analysis of [3] was extended in [5] to models with instantons and in [6] to models with chiral condensates.

Of course, such an analysis raised, among others, the question of the theoretical foundation of the QCD sum rules formalism. Recently a series of papers by the ITEP group has appeared, where those questions are rediscussed [7,8]. In particular it is claimed that conclusions at variance with ours are obtained from similar studies of two-dimensional models. Since, in our opinion, a clear understanding of those issues is important for our comprehension of gauge theories, we want to discuss in this short paper this apparent discrepancy in a critical and pedagogical way.

Let us already state the main conclusions of the discussion which follows: We do not find any major contradiction between the analysis of the ITEP group and our own previous conclusions. In our opinion the disagreement claimed in [7,8] comes from two sources:

(i) From a problem of language: contrary to the assertion of [7,8], we have never said that it was impossible to define vacuum expectation values (v.e.v.) for local operators such as the gluon condensate. We have simply emphasized the fact that this was only possible once some particular renormalization scheme has been chosen and that the value for the condensates should depend on this scheme (in fact we even gave examples of such schemes in [3]). Indeed in [7,8] the ITEP group considers a form of the OPE (which is different from the one considered in [3]) where condensates do not suffer from renormalon singularities, but depend on an additional subtraction scale $\mu$.

(ii) From a partial misunderstanding of some of our results: the ITEP group attributes the appearance of imaginary parts in the v.e.v. of local operators as computed in [3] to an improper treatment of the anomalous dimensions of those operators. This is in fact not correct, and the above mentioned phenomenon is indeed related to renormalon singularities.

This paper is organized as follows. In sect. 2 we shall resume our point of view and the results of [3], then compare it with the analysis of the ITEP group, and point out where the disagreement is real and where it is only apparent. Although some details and the presentation differ, the material of sect. 2 is essentially the same as that of [3].

In sect. 3 we discuss the relation between these theoretical issues concerning the OPE and the practical aspects of the applicability of the OPE. We shall argue that the problems discussed above are not only mathematical subtleties but may be essential in order to understand the successes of the phenomenological uses of the OPE. We think that these considerations are new and answer to some of the questions raised by the ITEP group in [7].

Finally let us insist on the fact that we do not discuss in this paper the OPE in the Higgs model or in the Schwinger model, which are also discussed in [7, 8].

## 2. The two-dimensional O($N$) non-linear sigma model

### 2.1. THE OPE AND RENORMALONS

In the following we use the notations of [8], and work for simplicity in euclidean space. The bare action of the model is:

$$S = \tfrac{1}{2}\int d^2x \left[ \partial_\mu \sigma^a(x)\,\partial_\mu \sigma^a(x) + \frac{\alpha(x)}{\sqrt{N}}\left(\sigma^a(x)\sigma^a(x) - \frac{N}{f}\right)\right]. \tag{2.1}$$

$\sigma = (\sigma^a)_{a=1,N}$ is an $N$-component scalar field. The integration over the auxiliary field $\alpha$ enforces the constraint $\sigma^a(x)\sigma^a(x) = N/f$ where $f$ is the bare coupling constant. The $1/N$ expansion can be obtained by integrating out the $\sigma$ field and expanding the resulting effective action in $\alpha$. It involves the propagator of the $\sigma$ field at $N = \infty$:

$$\begin{aligned} D^{(\sigma)ab}(p) &= \int d^2x\, e^{ipx} \langle 0|\sigma^a(x)\sigma^b(0)|0\rangle \\ &= \frac{\delta^{ab}}{p^2 + m^2} \end{aligned} \tag{2.2}$$

and the propagator of the $\alpha$ field:

$$\begin{aligned} D^{(\alpha)}(p) &= \int d^2x\, e^{ipx} \left[\langle 0|\alpha(x)\alpha(0)|0\rangle - \langle 0|\alpha(0)|0\rangle^2\right] \\ &= -4\pi\sqrt{p^2(p^2+4m^2)} \Big/ \ln\left(\frac{\sqrt{p^2+4m^2} + \sqrt{p^2}}{\sqrt{p^2+4m^2} - \sqrt{p^2}}\right) \end{aligned} \tag{2.3}$$

in (2.2) and (2.3) $m$ is the dynamically generated mass:

$$m^2 = \Lambda^2 e^{-4\pi/f_R}, \tag{2.4}$$

where $f_R$ is the renormalized coupling constant and $\Lambda$ the renormalization scale. Finally at $N = \infty$ we have:

$$\langle 0|\alpha(0)|0\rangle = -\sqrt{N}\, m^2. \tag{2.5}$$

We do not describe here the details of the renormalization of the $1/N$ expansion.

Let us first recall the results of ref. [3]. We were interested in the structure of the perturbative expansion (with respect to $f_R$) of O($N$)-invariant Green functions such as:

$$G_2(x, m) = \langle 0|\sigma^a(x)\sigma_a(0)|0\rangle. \tag{2.6}$$

Since such an observable depends only on $|x|^2 m^2$, we may equivalently look at its short-distance expansion ($x \to 0$, $m^2$ fixed). In perturbation theory, it can be shown that the product of two operators $A$ and $B$ can be expanded in the OPE:

$$A(x)B(0) = \sum_n C_n^{AB}(x)\mathcal{O}_n(0), \tag{2.7}$$

where the coefficients $C_n^{AB}(x)$ contain, at each order of perturbation theory, only a given power of $x$ plus logarithmic corrections:

$$C_n^{AB}(x) \sim |x|^{-d_A - d_B + d_{\mathcal{O}_n}} \ln^P|x\Lambda|, \tag{2.8}$$

where $d_{\mathcal{O}}$ is the engineering dimension of the operator $\mathcal{O}$. Of course the $C_n^{AB}$ and the matrix elements of the operators $\mathcal{O}_n$ in (2.7) depend on the specific normal product algorithm chosen to define the operators $\mathcal{O}_n$, but it can be shown, by using for instance the minimal subtraction scheme, that indeed all given powers of $|x|$ can be recast in a given set of $C_n$.

Since the model is asymptotically free, the powers of logarithm in (2.8) may be resummed in a series expansion in the effective coupling constant:

$$\bar{f}(x) \sim \frac{+1}{\beta_2 \ln|x\Lambda|}, \qquad \beta_2 < 0 \tag{2.9}$$

(for simplicity we neglect the corrections in $\ln\ln|x\Lambda|$, which come from the two loops contributions to the $\beta$ function and to the anomalous dimensions of the various operators $A$, $B$ and $\mathcal{O}_n$ in (2.7). They *do not play* any essential role in the discussion which follows).

In [3] we analysed the short-distance expansion of Green functions such as (2.6) within the $1/N$ expansion. Our results were the following:

One recovers on OPE of the form (2.7); for instance we may explicitly write:

$$G_2(x, m) = C_0(x, \Lambda) + C_1(x, \Lambda)\langle 0|\alpha(0)|0\rangle_\Lambda + \sum_{d_{\mathcal{O}}=4} C_{\mathcal{O}}(x, \Lambda)\langle 0|\mathcal{O}|0\rangle_\Lambda + \cdots \tag{2.10}$$

However, in order to give a definite sense to the sum (2.10), one has to take into account the fact, of course invisible in perturbation theory, that the series in $\bar{f}$ which defines $C_0, C_1, \dots$ are not Borel summable. For instance, the perturbative contribution $C_0(x, \Lambda)$ must be defined from its asymptotic series in $\bar{f}(x)$ by a Borel resummation procedure:

$$C_0(x, \Lambda) = \int_0^\infty \mathrm{d}b\, \mathrm{e}^{-b/\bar{f}} B_0(b). \tag{2.11}$$

However, the Borel transform $B_0(b)$ of $C_0$ can be shown to have branch points on the positive real axis at $b = -2/\beta_2,\ -4/\beta_2, \dots,$ denoted by the generic name of IR renormalons [9, 10], which come from divergences at small momenta in the definition of $B_0(b)$. The imaginary parts coming from such singularities, which are of order $m^2, m^4, \dots,$ are cancelled by similar singularities in the next terms of (2.10). For instance, the first condensate $\langle 0|\alpha(0)|0\rangle$ (proportional by the equations of motion to the spin-wave condensate $\langle 0|\partial_\mu\sigma^a\partial_\mu\sigma^a|0\rangle$), is non-perturbative and proportional to $m^2$, but suffers from non-perturbative divergences, coming here from integrations at large momenta and corresponding to UV renormalon singularities [9, 11, 12]. The corresponding imaginary part of order $m^2$ in $C_1$ $\langle 0|\alpha|0\rangle$ cancels exactly the imaginary part of $C_0$ so that the total sum is perfectly well defined and independent not only of the subtraction point (this is already true in perturbation theory), but of the resummation prescription chosen to define $C_0$.

The same phenomenon occurs of course for the next terms of the expansion. Therefore it was shown, at least within the $1/N$ expansion, that the OPE (2.7) makes perfect sense, once the renormalon phenomenon (which is peculiar to asymptotically free theories) has been identified and taken into account in the correct way.

Let us stress that the particular structure of the asymptotic expansion (2.7), described above, is by no means an exceptional phenomenon. It is familiar in quantum mechanics and in WKB theory, and more generally in the study of the asymptotic behaviour of solutions of linear differential equations. In particular the subtleties in the definition of composite operators have their analog (but for different reasons) in the definition of the contributions of instantons-antiinstantons configurations [13] in quantum mechanics.

### 2.2. COMPARISON WITH THE CONCLUSIONS OF THE ITEP GROUP

Let us now compare in detail our conclusions with those of [7, 8]. In order to avoid the problems of renormalons, the authors considered a different form of the OPE, by introducing explicitly an additional momentum scale $\mu$ (in addition to the renormalization scale $\Lambda$). Roughly speaking, the coefficients of their expansion $C_n^{AB}(x, \mu)$ take into account the short-wavelength fluctuations of the field with momentum $|p| > \mu$ and the operators $\mathcal{O}_n(\mu)$ the long-wavelength fluctuations of the field with momentum $|p| < \mu$, so that the OPE takes the form:

$$A(x)B(0) = \sum_n C_n^{AB}(x,\mu)\mathcal{O}_n(\mu). \tag{2.12}$$

Their claim is that, contrary to (2.6), each term of (2.12) is perfectly well defined, but $\mu$-dependent, and that the OPE (2.12) is the fundamental object to consider. We agree perfectly with the first part of this assertion. Since the contributions of momenta $|p| < \mu$ have been subtracted in the definition of the $C_n^{AB}$, they cannot suffer from IR renormalons; similarly the operators $\mathcal{O}_n(\mu)$ cannot suffer from UV renormalons, which come only from large momenta. Of course there is a price to be paid for the disappearance of renormalons: each coefficient $C_n^{AB}(x)$ contains now non-perturbative as well as perturbative contributions and the condensates $\langle 0|\mathcal{O}_n(\mu)|0\rangle$ are a mixing of perturbative parts (proportional to powers of $\mu^2$) and of non-perturbative parts (proportional to powers of $m^2$), which depends on the specific algorithm chosen in order to define $\mathcal{O}_n$.

The two forms of the OPE (2.7) and (2.12) are two different and perfectly well-defined objects and the question of which one has to be considered depends on the problem studied. The first one really probes the short-distance behaviour of the theory (i.e. the vicinity of the fixed point of the RG transformations, $\bar{f} = 0$), while the second one probes the theory in the vicinity of a fixed finite scale, here $\mu$. The second one may be obtained from (2.7) by "integration" from $|x|^2 = 0$ to $|x|^2 = \mu^2$ and (2.7) may be obtained from (2.12) by looking at its behaviour as $\mu^2 \to +\infty$.

Let us develop this last point in more detail. For simplicity we shall consider the first non-trivial operator, the "spin-wave condensate", $\langle 0|\alpha|0\rangle$ at the first subleading order $1/N$. Its value at order $(1/N)^0$ is given by (2.5). At order $1/N$ it is given by the integral $I_1$ of the two-loops graph depicted in fig. 1. We therefore have,

$$\langle 0|\alpha|0\rangle = -\sqrt{N}m^2 + \frac{1}{\sqrt{N}}I_1 + O\left(\frac{1}{N^{3/2}}\right), \tag{2.13}$$

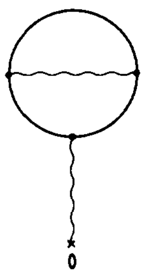

Fig. 1. The graph which contributes to $\langle 0|\alpha|0\rangle$ at order $1/N$. The straight lines are $D^{(\sigma)}$ propagators and the wavy lines $D^{(\alpha)}$ propagators.

with

$$I_1 = -\tfrac{1}{2}D^{(\alpha)}(0)\int\frac{d^2p}{(2\pi)^2}D^{(\alpha)}(p)\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+m^2)^2}\frac{1}{(p+k)^2+m^2}. \tag{2.14}$$

The $k$-integration is convergent and may be performed explicitly. The $p$-integration is quadratically divergent and will be regularized by a sharp cutoff $p^2 < \mu^2$. Performing as in [8] the substitution:

$$x = \left(\sqrt{1+\frac{p^2}{4m^2}} + \sqrt{\frac{p^2}{4m^2}}\right)^4 \tag{2.15}$$

we get finally an explicit expression in terms of the special functions $E_i$ and $E_1$*:

$$I_1(\mu) = m^2\Bigg[-E_i\left(\tfrac{1}{2}\ln(A(\mu))\right) + 2C + 2\ln\left(\tfrac{1}{2}\ln(A(\mu))\right)$$

$$+E_1\left(\tfrac{1}{2}\ln(A(\mu))\right) - 2\ln\left(1+\frac{4m^2}{\mu^2}\right)\Bigg], \tag{2.16}$$

with

$$A(\mu) = \left(\sqrt{1+\frac{\mu^2}{4m^2}} + \sqrt{\frac{\mu^2}{4m^2}}\right)^4. \tag{2.17}$$

C is the Euler constant. As done by the authors of [8] for the operator $\langle 0|\alpha(\mu)^2|0\rangle$, one can check here that the $\mu$-dependence of $C_1(x,\mu)$ $\langle 0|\alpha(\mu)|0\rangle$ cancels the $\mu$-dependence of $C_0(x,\mu)$ in the OPE (2.12) (at least up to order $\mu^{-2}$) at that order of the $1/N$ expansion.

The trouble, and our disagreement with the analysis of [7, 8], comes only when one tries in a naive way to study the limit $\mu^2 \gg m^2$ and to separate the perturbative contribution (proportional to $\mu^2$) from the non-perturbative one (proportional to $m^2$) in $\langle 0|\alpha(\mu)|0\rangle$. Indeed, in this limit, (2.16) simplifies into:

$$I_1(\mu) = -\tfrac{1}{4}\mu^2 e^{-\lambda}E_i(\lambda) + m^2\left[\frac{-1}{\lambda} + 2\ln\lambda + 2C\right] + O\left(\frac{m^4}{\mu^2}\right), \tag{2.18}$$

where

$$\lambda = \ln\left(\frac{\mu^2}{4m^2}\right), \qquad e^{-\lambda}E_i(\lambda) \simeq \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right). \tag{2.19}$$

* We use here the notations of [14].

The presence of $\lambda^{-1}$ and $\ln \lambda$ terms in (2.18) is associated to the anomalous dimension of the operator $\alpha$, which appears at order $1/N$.

If one follows the argument of the ITEP group [7,8] (which is done for the operator $\langle 0|\alpha^2|0\rangle$), the term proportional to $m^2$ in (2.18) defines the non-perturbative contribution to $\langle 0|\alpha|0\rangle$, and the presence of an imaginary part in the calculation of [3] of the non-perturbative contribution (done by using dimensional regularization) is associated to the term $\ln \lambda$ (which corresponds to a $\ln \varepsilon$ where $\varepsilon = 2 - d$ in the DR scheme).

This is not correct. The first term in (2.18), naively proportional to $\mu^2$, contains also a term proportional to $m^2$. Indeed, the full asymptotic expansion of $e^{-\lambda}E_i(\lambda)$ is:

$$e^{-\lambda}E_i(\lambda) \underset{\lambda \to \infty}{\equiv} \sum_{n=0}^{\infty} \frac{n!}{\lambda^{n+1}} \mp i\pi e^{-\lambda}. \tag{2.20}$$

The second, exponentially small term comes from the fact that the series in $1/\lambda$ in (2.20) is not only factorially divergent but is not Borel-summable, so that $e^{-\lambda}E_i(\lambda)$ may be represented by many equivalent integral representations:

$$e^{-\lambda}E_i(\lambda) = \fint_0^{\infty} \mathrm{d}b\, e^{-b\lambda} \frac{1}{1-b} \tag{2.21a}$$

$$= \int_0^{\infty} \mathrm{d}b\, e^{-b\lambda} \frac{1}{1-b} \mp i\pi e^{-\lambda} \tag{2.21b}$$

$$\arg(b) = \pm\varepsilon.$$

In (2.21) one sees explicitly the "renormalon" pole at $b = +1$. Hence, the correct definition of the continuum part of $\langle 0|\alpha|0\rangle$ is

$$\langle 0|\alpha|0\rangle \underset{\text{cont}}{=} -\sqrt{N}m^2 + \frac{1}{\sqrt{N}}m^2\left[-\frac{1}{\lambda} + 2\ln\lambda + 2C \mp i\pi\right] + O(1/N^{3/2}) \tag{2.22}$$

and the term of order $m^2$ in (2.18) is multivalued and has an imaginary part. Contrarily to the claim of [7,8], this has *nothing* to do with the presence of $1/\lambda$ and $\ln \lambda$ terms, which can be eliminated by taking into account the anomalous dimensions of $\alpha$. One can check explicitly here that the imaginary part obtained by working with a sharp cut-off is the same as the result obtained in [3] by using dimensional regularization (DR) in $d = 2 - \varepsilon$ dimensions and computing $\langle 0|\alpha|0\rangle$ in the limit $d = 2 \pm io - \varepsilon$, $\varepsilon \to 0$. There is therefore no way to get rid of the additional scale $\mu$ without meeting the renormalon singularity.

On this point, and on this point only, we are in real disagreement with the authors of [7, 8], who seem to believe that the imaginary part comes from the anomalous dimension of the operator $\alpha$ and the presence of $\ln \varepsilon$ in the DR scheme. Indeed, as clearly explained in [3], when one uses the DR scheme, the existence of two different limits as $\varepsilon \to 0$ for $\langle 0|\alpha|0\rangle$ is related to the fact that $\varepsilon = 0$ is the limit of an infinite series of poles at $\varepsilon = 2/n$, $n = 1, 2, 3\ldots$ . This kind of singularities, related to the quadratic divergence of the operator $\alpha$, is much stronger than a $\ln \varepsilon$, which is indeed also present but does not cause any trouble.

To summarize this discussion, we hope that it is now clear for the reader that there is no *unique* way to define a purely non-perturbative value (i.e. proportional to $m^2$) for the condensate $\langle 0|\alpha|0\rangle$ and that one has to choose an (arbitrary) prescription to define its value and to sum the series defining $C_0(x)$ in the OPE (2.7). As explained above, the study of [7, 8] does not contradict this conclusion: it is possible to define an OPE of the form (2.12) by introducing an additional scale $\mu$. The arbitrariness in (2.7) (choice of the resummation prescription) and in (2.12) (choice of $\mu$) reflects the same phenomenon: the mixing of operators with different canonical dimensions, which cannot be avoided at a non-perturbative level.

## 3. On the practical uses of the OPE

In this last section we want to discuss briefly the relationship between the actual structure of the OPE (2.6) and the phenomenological uses of the OPE, in particular in the QCD sum rules formalism. As clearly stated in [7], in the practical applications of the OPE one accepts a simplified version:

(i) The coefficients of the OPE are calculated in perturbation theory, and in practice the first one or two terms of perturbative series are kept.

(ii) The non-perturbative effects are entirely contained in the vacuum expectation values of operators, which are assumed not to contain any perturbative contribution.

From our discussion as well as from the discussion of the ITEP group, it is clear that this is a very simplified version of the OPE. The crucial question is: Why does it work?

The answer of the ITEP group is the following: If one introduces a normalization point $\mu$, there seems to be a range of $\mu$, where $\mu$ is large enough with respect to $\Lambda$ so that the $\mu$-corrections in the first order of the coefficients are small and can be neglected, but small enough so that the value of the condensates $\mathcal{O}(\mu)$ does not seem to vary with $\mu$ (at least within a few percents). In other words, the case of QCD seems to be fortunate, for some reason, and close to the $O(N)$ non-linear sigma model at infinite $N$, where there are no problems with renormalons.

In our opinion there is perhaps a deeper reason for such a success, which is precisely related to the existence of renormalon singularities. Let us come back to

QCD and consider a correlation function such as

$$\pi(Q)=i\int e^{iqx}T\{J_A(x)J_B(0)\}\,d^4x. \tag{3.1}$$

In practice one uses to parametrize $\pi(Q)$ a representation of the form

$$\pi(Q)=\bar{C}_0(\alpha(Q))+\bar{C}_1(\alpha(Q))\left\langle 0\left|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^{\mu\nu}_a\right|0\right\rangle_{\text{phen.}}+\cdots, \tag{3.2}$$

where $\bar{C}_0(\alpha)$ and $\bar{C}_1(\alpha)$ are restricted to the first order in $\alpha$:

$$\bar{C}_0(\alpha)=a_0+a_1\alpha+\ldots,$$

$$\bar{C}_1(\alpha)=\frac{1}{Q^4}(b_0+b_1\alpha+\cdots) \tag{3.3}$$

and where $\alpha(Q)$ is the running coupling constant. Now let us compare (3.2) with the exact Borel representation of $\pi(Q)$:

$$\pi(Q)=\int_0^\infty db\,e^{-b/\alpha}B(b), \tag{3.4}$$

where the full Borel transform $B(b)$ is the discontinuity along the real axis of the function:

$$\hat{B}(b)=\int_0^\infty d\alpha\,e^{b\alpha}\pi(\alpha), \tag{3.5}$$

which has branch points at $b=0,\ -4/\beta_2,\ -6/\beta_2,\ldots$ corresponding to the terms of the OPE (2.7) associated to the operators of dimension $0,4,6$, etc. ....

$$\pi(Q)=C_0(Q)+C_1(Q)\left\langle 0\left|\frac{\alpha_s}{\pi}G^2\right|0\right\rangle+\ldots. \tag{3.6}$$

When one approximates the full $C_0(Q)$ by its first terms $\bar{C}_0$, one neglects all the high-order terms of the factorially divergent series in $\alpha$ corresponding to $C_0$. But the large-order behaviour of this series is governed by the IR renormalon at $b=-4/\beta_2$, which is proportional to $C_1(Q)$. Therefore it is a priori not a bad approximation to replace the remaining part of $C_0-\bar{C}_0$, plus the non-perturbative part proportional to the gluon condensate (the sum of the two terms does not have any ambiguity) by a term proportional to $C_1$, the coefficient of proportionality being precisely the phenomenological value attributed to the gluon condensate in (3.2), and to adjust this coefficient in order to reproduce the term that we have neglected. This is clearly

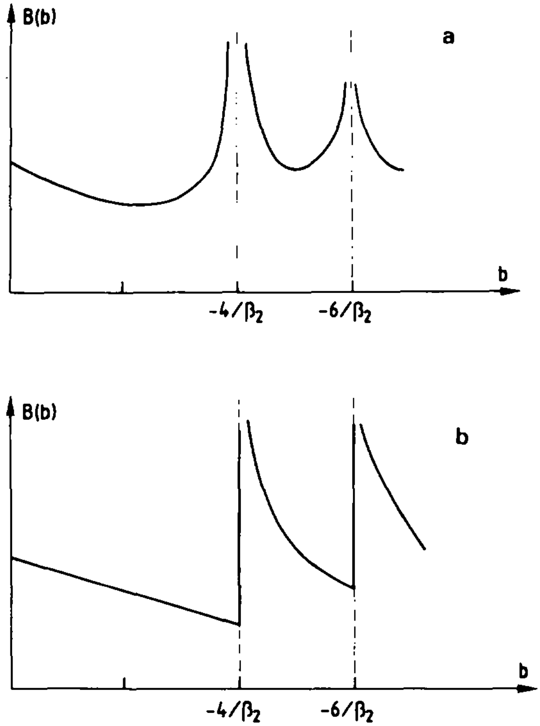


Fig. 2. The structure of the Borel transform $B(b)$. On fig. 2a is depicted the exact structure of $B(b)$, with IR renormalons at $b = -4/\beta_2$ and $-6/\beta_2$. On fig. 2b is depicted the ansatz obtained from the simplified version (3.2) of the OPE.

depicted in fig. 2 where we have represented the exact structure of the Borel transform $B(b)$ of $\pi(Q)$, with in particular the renormalon at $b = -4/\beta_2$, and the ansatz for the Borel transform obtained by representing $\pi(Q)$ by an approximate OPE of the form (3.2). The crucial point is that, if the Borel transform of $C_0$ near $-4/\beta_2$ is dominated by the renormalon singularity, then such an ansatz is expected to be successful, with the *same* value for the phenomenological gluon condensate, for *different* correlations functions, with a reasonable precision.

Of course one may imagine modelling the Borel transform $B(b)$ for larger values of $b$ in the same way; this corresponds to introduce condensates with higher dimensionality (6, ...) in (3.2).

We think that this kind of argument may explain why the practical uses of the OPE described above, where renormalons singularities are neglected (or equivalently the need for a normalization point $\mu$) are successful in QCD, although there is no theoretical reason to think that renormalon singularities are unimportant (unlike the non-linear sigma models at large $N$). Of course, if one could have very long perturbative series in QCD, such an ansatz would not be sufficient, and the real structure of the Borel transform should be taken into account; this has been done with success in other applications of field theory, such as the calculation of critical exponents.

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