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A STUDY OF $(\boldsymbol{\phi}^2)^3_3$ AT $N=\infty$

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A study of various types of critical behavior and continuum limits of $(\boldsymbol{\phi}^2)^3_3$ at $N=\infty$ is presented with special emphasis on the BMB phenomenon. An estimate for the lowest value of N for which the BMB phenomenon could still occur is obtained. It is also conjectured that even for all finite N the BMB phenomenon does not survive.

1. Introduction

It is important to be able to tell how many free parameters a field theory of a given type might possess without losing its predictive power. The question whether dimensionless couplings in the bare lagrangian correspond to freely adjustable parameters in the continuum limit cannot, strictly speaking, be answered within perturbation theory if the couplings happen not to be asymptotically free. A variety of considerations indicate [1, 2] that ϕ^4_4, for example, becomes a free field theory if one insists on removing the ultraviolet cutoff completely. It was recently conjectured that $(\boldsymbol{\phi}^2)^3_3$ does have the general structure required for a 3-parameter continuum limit to exist [3, 4].

In order to clarify what the conjecture says it is useful to briefly review the general features that a globally O(N) symmetric field theory in 3 euclidean dimensions is commonly expected to possess [1]: it is assumed that a renormalization group (RG) transformation can be defined which effectively averages out the high-momentum components of the fields yielding a new local effective action. In the space of all possible actions one then finds a hierarchy of fixed points, each located on the boundary of the domain of attraction of its predecessor. In the symmetric phase, generically, the mass is finite and hence increases under the action of the RG transformation. Therefore, the most stable fixed point (TFP) corresponds to decoupled fluctuations. The boundary of the domain of attraction of the TFP is the one-dimension-less manifold of interactions on which the mass vanishes (the theory is critical). The RG transformation cannot take the interaction out of this manifold and, generically, drives it to an O(N) Wilson–Fisher fixed point. This fixed point (HFP) governs the critical behavior of the classical Heisenberg model. The HFP has one unstable direction which is tangent to a RG trajectory that connects the HFP to the TFP. A continuum field theory with only a mass as a free parameter can be built on this trajectory. The domain of attraction of the HFP on the critical surface also has a boundary. On this boundary the RG transformation drives one to the massless free field fixed point (GFP). The GFP controls the tricritical behavior of O(N) spin systems. The GFP has two unstable directions. They are tangent to a two-dimensional surface of RG trajectories. On the surface one can build a two-parameter family of interacting field theories. These are the $(\boldsymbol{\phi}^2)_3^2$ models which are perturbatively under control in the ultraviolet regime and can be rigorously constructed [5]. The GFP has one special property: one of the directions in its vicinity is only marginally stable. It corresponds to the $(\boldsymbol{\phi}^2)^3$ interaction. The marginality implies that, in perturbation theory, one can reduce the RG equations to a single differential equation in only one variable [6]. This is the Gell-Mann–Low "β-function" associated with the specific RG transformation. It is defined only perturbatively.

At $N=\infty$ the main change in the above picture is that, in the immediate vicinity of the GFP, the marginally stable direction becomes absolutely marginal giving rise to a line of fixed points. Each of these fixed points corresponds to a $(\boldsymbol{\phi}^2)_3^3$, $N=\infty$, scale-invariant field theory.

In order that a conventional, 3-parameter family of continuum $(\boldsymbol{\phi}^2)_3^3$ exist at finite N one additional fixed point, of one degree of stability less, is required. The RG trajectory emanating from the GFP tangent to the marginal direction should be attracted to this new (UV) fixed point.

The conjecture implying the existence of $(\boldsymbol{\phi}^2)_3^3$ was based on the $1/N$ expansion and underwent some evolution which we now briefly review. The model is defined by the following bare lagrangian:

$$A_{\mathrm{E}}[\boldsymbol{\phi}]=-\int\left[\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2+\tfrac{1}{2}\tilde{\mu}_0^2\boldsymbol{\phi}^2+\tfrac{1}{4}\tilde{\lambda}_0(\boldsymbol{\phi}^2)^2+\tfrac{1}{6}\tilde{\eta}_0(\boldsymbol{\phi}^2)^3\right], \tag{1.1}$$

where $\boldsymbol{\phi}(x)$ is an N-component scalar field. The large-N limit of (1.1) was investigated by Townsend [3], who noted that, to first nonvanishing order in $1/N$, the β-function of the model had an ultraviolet fixed point in the rescaled coupling $\tilde{\eta}_0$ N^2. This implied that at a finite, but possibly very large N, the UV fixed point would be a genuine feature of the theory. This view was also advocated by Pisarski [3]. An implicit assumption in these computations was that one could simply resum Feynman diagrams and organize the sum in orders of $1/N$. For the purpose of investigating the UV behavior of the theory one may set the 4-point coupling to zero. It is then true that only a finite number of diagrams contribute to a given order in $1/N$. To leading order the β-function (in $\eta=\tilde{\eta}N^2$) is zero and to subleading order it is

$$\beta=\frac{1}{N}\frac{3\eta^2}{2\pi^2}\left(1-\frac{\eta}{\eta^*}\right)+\mathrm{O}\left(\frac{1}{N^2}\right),\qquad \eta^*=192\,. \tag{1.2}$$

Bardeen, Moshe and Bander [4] used a variational method which does not rely on perturbation theory and is exact at infinite N. They found that the perturbation expansion in η breaks down at infinite N for $\eta > \eta_c = (4\pi)^2$. This invalidates the previous perturbative argument because $\eta^* > \eta_c$. However, a more detailed study of the properties of the $N=\infty$ theory led BMB to the conclusion that now η_c acts as an UV fixed point at which one has mass generation via dimensional transmutation and also spontaneous breaking of scale invariance accompanied by the appearance of a dilaton. BMB conjectured that at finite, but large N, the dilaton will get a small mass (due to the now anomalous scale invariance even for $\eta \leq \eta_c$) but that otherwise the physics would remain the same. As far as the existence of $(\phi^2)_3^3$ (for some finite but large N) the implications are the same as those of Townsend and Pisarski. The BMB calculation implies that the line of fixed points which emanates from the GFP ends at a point within the space of acceptable interactions. This point is the $N=\infty$ limit of the alleged finite-N UV fixed point.

The BMB analysis was carried out by approaching the point corresponding to η_c in a bare, cutoff version of (1.1) in a well-defined manner as the cutoff increased to infinity. The features of the continuum limit thus obtained were studied subsequently. The main purpose of this work is to identify the mechanism which gave rise to the BMB phenomena at $N=\infty$. This is done first by carrying out a complete analysis of the model at a finite cutoff which leads one to the construction of all possible continuum limits where $O(N)$-vector excitations are present and thus puts the BMB limit in the context of the more conventional possibilities. Second we establish the RG structure which made the existence of the BMB limit possible. With this understanding of the origin of the BMB phenomena we can speculate whether it is likely or not that it survives to finite N too.

Our analysis is basically a complete classification of the possible continuum limits of $O(N)$ symmetric N-component scalar field models in three dimensions at $N=\infty$. This is done both by the explicit solution of the model and via a RG transformation. In this sense the paper presents an explicit example where Wilson's [1] approach to the construction of a continuum field theory is seen in operation. For pedagogical purposes an attempt was made to make the paper as self-contained as possible. Therefore, we rederive herein some known results especially due to Ma [7].

The plan of the paper is as follows. In sect. 2 we compute the $N=\infty$ effective potential associated with the cutoff model in (1.1). With this input sect. 3 identifies the complete phase diagram in the three couplings space. Next the critical regions are identified and essentially two types of continuum limits are constructed. In sect. 4 the understanding of the relationship between the phase diagram and the BMB continuum limit is exploited to use the $\eta_0 \to \infty$ limit of (1.1) on the lattice to obtain an estimate for $\bar{N}$, where $\bar{N}$ is the largest N for which the BMB phenomenon is ruled out. An RG transformation is set up in sect. 5. Its fixed points and related scaling fields are found and it is shown that the continuum limits of sect. 3 are recovered. All this background is brought together in sect. 6 where the BMB limit is analyzed in detail from various points of view. A summary is given in sect. 7.

2. The effective potential at $N=\infty$

The effective potential is obtained from the effective action $\Gamma[\boldsymbol{\phi}_c]$ evaluated at constant fields. $\Gamma[\boldsymbol{\phi}_c]$ is the Legendre transform of the generating functional for connected vacuum correlations $W[\boldsymbol{J}]$.

$$e^{NW[\boldsymbol{J}]} \doteq \int [d\boldsymbol{\phi}]\, e^{A_E[\boldsymbol{\phi}]+\sqrt{N}\int \boldsymbol{J}\cdot\boldsymbol{\phi}\, d^3x}, \tag{2.1}$$

where $A_E[\boldsymbol{\phi}]$ has been defined in eq. (1.1). It consists of a kinetic energy and a potential energy term:

$$A_E[\boldsymbol{\phi}] = \int [\tfrac{1}{2}(\partial_\mu \boldsymbol{\phi})^2 + V_0(\boldsymbol{\phi}^2)]\, d^3x. \tag{2.2}$$

The field $\boldsymbol{\phi}$ is assumed to have no Fourier components with a 3-momentum that exceeds Λ, the ultraviolet cutoff, in magnitude.

The factor depending on V_0 in (2.1) is replaced by the inverse Laplace transform of its Laplace transform:

$$e^{-V_0(\boldsymbol{\phi}^2)} = \frac{1}{4\pi i}\int_{-i\infty}^{i\infty} d\tilde{M}^2 \int d\tilde{x}\, e^{-V_0(\tilde{x})+\frac{1}{2}\tilde{M}^2(\tilde{x}-\boldsymbol{\phi}^2)}. \tag{2.3}$$

A nontrivial large-N limit will be obtained if we rescale the couplings, fields and potential as follows:

$$\hat{\boldsymbol{\phi}} = \frac{1}{\sqrt{N}}\boldsymbol{\phi}, \qquad \hat{X} = \frac{1}{N}\tilde{x}, \qquad V_0(\boldsymbol{\phi}^2) = N\hat{V}_0(\hat{\boldsymbol{\phi}}^2). \tag{2.4}$$

After gaussian integration over $\hat{\boldsymbol{\phi}}$ an expression is obtained in which the N-dependence is explicit:

$$e^{NW[\boldsymbol{J}]} \propto \int [d\tilde{M}^2][d\hat{X}]\exp\Big\{-N\int \hat{V}_0(\hat{X}) + \tfrac{1}{2}N\int \tilde{M}^2\hat{X} - \tfrac{1}{2}N\,\mathrm{tr}[\log(-\partial^2+\tilde{M}^2)]$$
$$+\tfrac{1}{2}N\iint \boldsymbol{J}\frac{1}{-\partial^2+\tilde{M}^2}\boldsymbol{J}\Big\}. \tag{2.5}$$

At infinite N the path integral is dominated by either saddle points or the lower endpoint in the $\hat{X}$ integration. Throughout this paper only saddle points will matter. Assuming domination by a single saddle point (sp) the following expression for the effective action is obtained:

$$\frac{\delta W}{\delta J} = -\frac{1}{-\partial^2 + \tilde{M}^2_{sp}} J \equiv \hat{\phi}_c ,$$

$$\Gamma[\hat{\phi}_c] = \tfrac{1}{2}\int \hat{\phi}_c(-\partial^2 + \tilde{M}^2_{sp})\hat{\phi}_c - \tfrac{1}{2}\int \tilde{M}^2_{sp}\hat{X}_{sp} + \int \hat{V}_0(\hat{X}_{sp}) + \tfrac{1}{2}\,\mathrm{tr}\,[\log(-\partial^2 + \tilde{M}^2_{sp})I . \quad (2.6)$$

$\tilde{M}^2_{sp}$ and $\hat{X}_{sp}$ depend on $\hat{\phi}$ through J and the extremum requirement of the exponent of the integrand in eq. (2.5). J can be eliminated to get

$$2V_0'(\hat{X}_{sp}(x)) = \tilde{M}^2_{sp}(x)$$

$$\langle x|\frac{1}{-\partial^2 + \tilde{M}^2_{sp}}|x\rangle = \hat{X}_{sp}(x) - \hat{\phi}^2_c(x) . \quad (2.7)$$

For the computation of the effective potential we consider a constant source J. We shall henceforth implicitly assume that we deal only with classical fields $\hat{\phi}_c$ which can be obtained from a unique J. This holds for values of $\hat{\phi}^2_c$ that are larger or equal to the value that minimizes the effective potential [8]. We obtain

$$V_{eff}(\psi) = N\hat{V}_{eff}\left(\frac{1}{N}\psi^2\right)$$

$$\hat{V}_{eff}(\hat{\phi}^2_c) = \tfrac{1}{2}\tilde{M}^2_{sp}(\hat{\phi}^2_c - \hat{X}_{sp}) + \hat{V}_0(\hat{X}_{sp}) + \tfrac{1}{2}\int_{k^2 \leq \Lambda^2} \frac{d^3k}{(2\pi)^3}\log(k^2 + \tilde{M}^2_{sp}) , \quad (2.8)$$

with

$$2\hat{V}_0'(\hat{X}_{sp}) = \tilde{M}^2_{sp} ,$$

$$\int_{k^2<\Lambda^2} \frac{d^3k}{(2\pi)^3}\frac{1}{k^2 + \tilde{M}^2_{sp}} = \hat{X}_{sp} - \hat{\phi}^2_c . \quad (2.9)$$

We now scale out the cutoff Λ from the problem making all quantities dimensionless:

$$\phi_c = \frac{1}{\sqrt{\Lambda}}\hat{\phi}_c , \qquad X = \frac{1}{\Lambda}(\hat{X}_{sp} - \hat{\phi}^2_c) , \qquad M^2 = \frac{1}{\Lambda^2}\tilde{M}^2_{sp} ,$$

$$U_{eff}(\phi^2_c) = \Lambda^{-3}\,\hat{V}_{eff}(\hat{\phi}^2_c) , \qquad U_0(\phi^2_c) = \Lambda^{-3}\,\hat{V}_0(\Lambda\phi^2_c) . \quad (2.10)$$

Eq. (2.9) can be rewritten as

$$2U_0'(X+\phi_c^2)=M^2\,,$$

$$U_{\rm eff}(\phi_c^2)=U_0(\phi_c^2+X)-\tfrac{1}{2}\int_0^{M^2}{\rm d}m^2\,m^2\frac{{\rm d}f}{{\rm d}m^2}\,,$$

$$f(M^2)\equiv\int_{q^2\leq 1}\frac{{\rm d}^3q}{(2\pi)^3}\frac{1}{q^2+M^2}=\frac{1}{2\pi^2}\left(1-|M|\tan^{-1}\frac{1}{|M|}\right)=X\,. \tag{2.11}$$

f takes values between $1/2\pi^2$ (at $M^2=0$) and 0 (at $M^2=\infty$). Let $g(x)$ be the inverse of f with $0<x\leq 1/2\pi^2$. Eq. (2.11) simplifies to

$$g(X)=2U_0'(X+\phi_c^2)\,,$$

$$U_{\rm eff}(\phi_c^2)=U_0(X+\phi_c^2)\,,$$

$$U_{\rm eff}(\phi_c^2)=U_0(X+\phi_c^2)-\tfrac{1}{2}\int_{1/2\pi^2}^{X}{\rm d}t\,g(t)\,. \tag{2.12}$$

Since additive constants in $U_{\rm eff}$ and U_0 are devoid of any physical meaning it is better to work with derivatives:

$$U_{\rm eff}'(\phi_c^2)=U_0'(X+\phi_c^2)=\tfrac{1}{2}g(X)\,. \tag{2.13}$$

In the event that more than one solution $X(\phi_c^2)$ is found, the one corresponding to the dominating saddle can be seen to give the smaller value for $U_{\rm eff}(\phi_c^2)$.

Next, we must renormalize the theory, introducing appropriate Λ-dependences of the parameters in $\hat{V}_0$, so as to insure that the physical $\hat{V}_{\rm eff}$ (see eqs. (2.8) and (2.10)) becomes a finite and nontrivial function of its argument in the limit $\Lambda\to\infty$. (It is well known that, to leading order in N, wave function renormalization is unnecessary.) This implies that we require $\lim_{\Lambda\to\infty}\Lambda^2U_{\rm eff}'(Z/\Lambda)$ (eq. (2.13)) to exist and be a nontrivial function of Z. Hence, for any Z, it must be true that $\lim_{\Lambda\to\infty}\Lambda^2 g(X_Z(\Lambda))$ exists with $X_Z(\Lambda)$ being the solution of

$$2U_0'\left(X_Z(\Lambda)+\frac{Z}{\Lambda},\Lambda\right)=g(X_Z(\Lambda))\,, \tag{2.14}$$

which corresponds to the dominant saddle. As we require that $g(X_Z(\Lambda))$ vanish with large Λ, $X_Z(\Lambda)$ must approach $1/2\pi^2$. Thus

$$X_Z(\Lambda)=\frac{1}{2\pi^2}+Y_Z(\Lambda)\,, \tag{2.15}$$

with $\lim_{\Lambda\to\infty}Y_Z(\Lambda)=0$. We now expand (2.14) for large Λ:

$$U_0'\left(\frac{1}{2\pi^2}+\frac{Z}{\Lambda}+Y_Z(\Lambda),\Lambda\right)=\tfrac{1}{2}(4\pi)^2Y_Z^2(\Lambda)[1-16Y_Z(\Lambda)+\cdots]\,. \tag{2.16}$$

At this point, we restrict our attention to an action of the form (1.1) and introduce the definitions

$$U_R(Z;\Lambda)\equiv U_0\left(\frac{1}{2\pi^2}+Z;\Lambda\right)-U_0\left(\frac{1}{2\pi^2};\Lambda\right),$$

$$\Lambda^{-3}\hat{V}_R(\Lambda Z;\Lambda)\equiv U_R(Z;\Lambda)\equiv \tfrac{1}{2}\mu_R^2 Z+\tfrac{1}{4}\lambda_R Z^2+\tfrac{1}{6}\eta_R Z^3\,. \tag{2.17}$$

Eq. (2.16), in the new language, now reads

$$\hat{V}'_R(Z+\Lambda Y_Z(\Lambda);\Lambda)=\tfrac{1}{2}(4\pi)^2(\Lambda Y_Z(\Lambda))^2(1-16Y_Z(\Lambda)+\cdots)\,. \tag{2.18}$$

Assuming that $y(Z)\equiv\lim_{\Lambda\to\infty}\Lambda Y_Z(\Lambda)$ exists and is nontrivial, the renormalized effective potential is now given by

$$\hat{V}'_{\rm eff}(\hat{\phi}^2)=\lim_{\Lambda\to\infty}\hat{V}'_R(\hat{\phi}^2+y(\hat{\phi}^2);\Lambda)=\tfrac{1}{2}(4\pi)^2y^2(\hat{\phi}^2)\,. \tag{2.19a}$$

Therefore, to achieve a renormalized theory, it is necessary to choose the Λ-dependences in $\hat{V}_0$ so that $y(z)$ exist. One clearly sufficient way to achieve this is to demand that $\lim_{\Lambda\to\infty}\hat{V}_R(T;\Lambda)$ exist for any T. This corresponds to the usual renormalization prescription, which can be seen as follows:

$$V_R(T;\Lambda)=\tfrac{1}{2}\mu_R^2\Lambda^2T+\tfrac{1}{4}\lambda_R\Lambda T^2+\tfrac{1}{6}\eta_R T^3\,. \tag{2.19b}$$

We must thus set

$$\mu_R^2=\hat{\mu}_R^2/\Lambda^2\,,$$
$$\lambda_R=\hat{\lambda}_R/\Lambda\,,$$
$$\eta_R=\hat{\eta}_R\,, \tag{2.19c}$$

where $\hat{\mu}_R^2$, $\hat{\lambda}_R$ and $\hat{\eta}_R$ are independent of Λ.

In terms of the original parameters in V_0, using (2.17), we get

$$\tilde{\mu}_0^2=\hat{\mu}_R^2-\frac{\Lambda}{2\pi^2}\hat{\lambda}_R+\frac{\Lambda^2}{(2\pi^2)^2}\hat{\eta}_R\,,$$

$$\tilde{\lambda}_0=\hat{\lambda}_R-\frac{\Lambda}{\pi^2}\hat{\eta}_R\,,$$

the usual counterterm structure.

We now proceed to calculate the renormalized $\hat{V}_{\rm eff}$ in terms of the physical couplings $\hat{\mu}_R^2$, $\hat{\lambda}_R$ and $\hat{\eta}_R$, by solving (2.19a) for y. This equation is much simpler than its unrenormalized counterpart (2.13) due to the disappearance of the dependence on the regularization method. The equation for y is of second order:

$$\hat{V}'_R(\hat{\phi}^2)+\hat{V}''_R(\hat{\phi}^2)y(\hat{\phi}^2)+\tfrac{1}{2}\hat{V}'''_R(\hat{\phi}^2)y^2(\hat{\phi}^2)=\tfrac{1}{2}(4\pi)^2y^2(\phi^2)\,,$$

$$\hat{V}'_R(\hat{\phi}^2)=\lim_{\Lambda\to\infty}\hat{V}'_R(\hat{\phi}^2;\Lambda)=\tfrac{1}{2}(\hat{\mu}_R^2+\hat{\lambda}_R\hat{\phi}^2+\hat{\eta}_R\hat{\phi}^4)\,. \tag{2.20}$$

For $\hat{\eta}_R \leq 16\pi^2 \equiv \eta_c$ the dominant saddle corresponds to

$$y(\hat{\phi}^2) = \frac{1}{\eta_c - \eta_R}\{\hat{V}''_R(\hat{\phi}^2) - [(\hat{V}''_R(\hat{\phi}^2))^2 + 2(\eta_c - \eta_R)\hat{V}'_R(\hat{\phi}^2)]^{1/2}\}. \tag{2.21}$$

Inserting (2.21) into (2.19) and performing the elementary integration leads to an explicit expression for the renormalized effective potential:

$$\chi \equiv 64\pi^2 \frac{1}{\hat{\lambda}_R}\hat{\phi}^2,$$

$$\begin{aligned}\hat{V}_{\text{eff}} = \frac{\hat{\lambda}_R^3}{3.2^{11}\pi^4(1-\hat{\eta}_R/\eta_c)^2}&\left\{\left[\tfrac{3}{2} + 48\pi^2\left(1-\frac{\hat{\eta}_R}{\eta_c}\right)\frac{\hat{\mu}_R^2}{\hat{\lambda}_R^2}\right]\chi\right.\\ &+\frac{3}{8}\left(1+\frac{\hat{\eta}_R}{\eta_c}\right)\chi^2 + \frac{1}{16}\frac{\hat{\eta}_R}{\eta_c}\left(1+\frac{\hat{\eta}_R}{\eta_c}\right)\chi^3\\ &\left.-\left[1+64\pi^2\left(1-\frac{\hat{\eta}_R}{\eta_c}\right)\frac{\hat{\mu}_R^2}{\hat{\lambda}_R^2} + \chi + \frac{1}{4}\frac{\hat{\eta}_R}{\eta_c}\chi^2\right]^{3/2}\right\} + \text{const.}\end{aligned} \tag{2.22a}$$

For $\hat{\mu}_R^2 = 0$ this is in agreement with the form guessed by Appelquist and Heinz [3] on the basis of perturbation theory (in the comparison one has to take into account that they work with complex fields). When $\eta_R = \eta_c$ the form of (2.22) is inconvenient and the effective potential becomes

$$\hat{V}_{\text{eff}} = \frac{1}{3.2^{11}\pi^4}\left\{\tfrac{1}{16}(\hat{\lambda}_R\chi)^3 - \frac{3}{8}\frac{[64\pi^2\hat{\mu}_R^2 - \frac{1}{4}(\hat{\lambda}_R\chi)^2]^2}{\hat{\lambda}_R + \frac{1}{2}\hat{\lambda}_R\chi}\right\} + \text{const.} \tag{2.22b}$$

When $\hat{\lambda}_R = 0$ this expression is unbounded from below if $\hat{\mu}_R^2 \neq 0$. The scale-invariant case $\hat{\mu}_R^2 = \hat{\lambda}_R = 0$ is bounded. In summary (2.22a) is acceptable when $\hat{\eta}_R \leq \eta_c$ with the exception of the region $\{\hat{\eta}_R = \eta_c, \hat{\lambda}_R = 0, \hat{\mu}_R^2 \neq 0\}$. The point $\hat{\eta}_R = \eta_c$ is a point where some nonanalyticity appears. This is most clearly seen by computing the physical mass m^2: for $\hat{\eta}_R \leq \eta_c$ and $\hat{\lambda}_R \neq 0$ m^2 vanishes when $\hat{\mu}_R^2 \to 0$ as $m^2 \sim \hat{\mu}_R^4/\hat{\lambda}_R^2$; for $\hat{\eta}_R < \eta_c$ and $\hat{\lambda}_R = 0$ $m^2 \propto \hat{\mu}_R^2/(1-\hat{\eta}_R/\eta_c)$.

Eq. (2.22) is not a necessary outcome of the renormalizability requirement, i.e. the requirement of finiteness of V_{eff} as $\Lambda \to \infty$. Indeed, instead of demanding the existence of $\lim_{\Lambda\to\infty} \hat{V}'_R(T;\Lambda)$ for all T we could demand

$$U'_R(0;\Lambda) = -U''_R(0;\Lambda)\frac{C_1}{\Lambda} + \frac{A}{\Lambda^2} + O\left(\frac{1}{\Lambda^3}\right). \tag{2.23}$$

This is equivalent to requiring $\lim_{\Lambda\to\infty} \hat{V}'_R(T;\Lambda)$ to exist for one value of T only; less adjustment is necessary in order to meet this requirement. With (2.23) one finds the following solution to eq. (2.16):

$$Y_Z(\Lambda) = \frac{C_1 - Z}{\Lambda} + \frac{C_2(Z)}{\Lambda^2} + O\left(\frac{1}{\Lambda^3}\right),$$

$$A + U''_R(0)C_2(Z) + \tfrac{1}{2}U'''_R(0;\Lambda)C_1^2 = 8\pi^2(C_1 - Z)^2. \tag{2.24}$$

The physical effective potential is

$$\hat{V}_{\text{eff}}(\hat{\phi}^2)=8\pi^2[C_1^2\hat{\phi}^2-C_1\hat{\phi}^4+\tfrac{1}{3}\hat{\phi}^6]+\text{const.}\,, \tag{2.25}$$

and has only one free parameter whose role is to set the scale. To obtain this continuum limit only a mass counterterm is necessary. Because of the identity

$$V'_{\text{R}}(T;\Lambda)=\Lambda^2\left[U'_{\text{R}}(0;\Lambda)+U''_{\text{R}}(0;\Lambda)\frac{T}{\Lambda}+\tfrac{1}{2}U'''_{\text{R}}(0;\Lambda)\frac{T^2}{\Lambda^2}\right], \tag{2.26}$$

we indeed see that for any $T\neq C_1$ $\lim_{\Lambda\to\infty}\hat{V}'_{\text{R}}(T;\Lambda)=\infty$. The continuum limit which leads to (2.25) cannot be obtained by perturbation theory renormalization. The existence of the phenomenon is not a large-N artefact [9]. We should remark that (2.26) is a limiting case of (2.22). One obtains (2.26) from (2.22) by taking the limit $\hat{\lambda}_{\text{R}}\to\infty$ with $\hat{\eta}_{\text{R}}$ and $\hat{\mu}^2_{\text{R}}/\hat{\lambda}_{\text{R}}$ kept fixed.

The computations presented so far were completely straightforward and the techniques are well known [10]. The analysis can be easily extended to any dimension d, $2<d\leqslant 4$, but this will not be done here. In principle, the computation of $1/N$ corrections is possible. The variational calculation of Bardeen, Moshe and Bander is in complete agreement with eq. (2.11). The agreement occurs because $V_{\text{eff}}(\boldsymbol{\phi}^2)$ is the vacuum energy density under the constraint that the spatial average of the quantum field be $\boldsymbol{\phi}$.

The perturbation theory renormalization does not affect the six-point coupling η_0 (see eq. (2.17)) as long as eq. (2.15) gives a dominating saddle with $Y_Z(\Lambda)\sim O(1/\Lambda)$. This is true for a finite range of η_0 and hence, to any order in perturbation theory the $\beta(\eta)$ function vanishes at leading order in $1/N$. BMB have noted that for $\eta_0>\eta_c$ another branch of the solutions to eq. (2.15) becomes the dominating saddle and perturbation theory breaks down. That something new might happen when $\hat{\eta}_{\text{R}}>\eta_c$ was already suggested by Townsend [3]. He suspected that $\hat{V}_{\text{eff}}$ might be singular at $\hat{\eta}_{\text{R}}=\eta_c$ (see eq. (2.22a)). Appelquist and Heinz [3] showed this not to be the case when $\hat{\mu}^2_{\text{R}}=0$ and $\hat{\lambda}_{\text{R}}\neq 0$. Our eq. (2.22b) extends this observation to the case $\hat{\mu}_{\text{R}}\neq 0$, $\hat{\lambda}_{\text{R}}\neq 0$. However, we have also seen that for $\hat{\lambda}_{\text{R}}=0$ and $\hat{\mu}^2_{\text{R}}\neq 0$ the limit $\hat{\eta}_{\text{R}}\to\eta_c$ leads to an effective potential which is unbounded from below and therefore, Townsend's concern seems to have been in place after all.

3. The phase diagram

As explicitly exhibited by the calculations in the previous section, the construction of continuum field theories is carried out in the vicinity of regions in the space of bare couplings where the system undergoes continuous phase transitions signaled by the vanishing of $g(X_Z(\Lambda))$ in eq. (2.16). For the value of Z corresponding to the minimum of the effective potential, the quantity g that vanishes is the mass squared. The infrared behavior of the system in the vicinity of the critical regions is controlled by the various fixed points of the RG transformation that were discussed in the introduction.

To study the critical regions, we need to obtain the phase diagram. For this, we have to find the solution of eq. (2.13) which is the dominating saddle for the $\boldsymbol{\phi}_c^2$ appropriate for the vacuum (i.e. the point where $U_{\rm eff}(\boldsymbol{\phi}_c^2)$ is at an absolute minimum). There are two possibilities:

$$\text{(a) broken symmetry:} \qquad \boldsymbol{\phi}_c^2 \neq 0 \quad U'_{\rm eff}(\boldsymbol{\phi}_c^2)=0\,, \tag{3.1a}$$

$$\text{(b) symmetric phase:} \qquad \boldsymbol{\phi}_c^2 = 0 \quad U'_{\rm eff}(\boldsymbol{\phi}_c^2)\neq 0\,. \tag{3.1b}$$

The two regions in the parameter space in which either (a) or (b) hold are separated by a transition surface. Part of it is critical.

In case (a) we have

$$U_{\rm eff}^{\rm vac} = \int_{1/2\pi^2}^{1/2\pi^2+\boldsymbol{\phi}_c^2} U_0'(t)\,{\rm d}t + U_0\left(\frac{1}{2\pi^2}\right), \qquad 2U_0'\left(\frac{1}{2\pi^2}+\boldsymbol{\phi}_c^2\right)=0\,, \qquad \phi_c^2>0\,, \tag{3.2}$$

whereas in (b) we get

$$U_{\rm eff}^{\rm vac} = \int_{1/2\pi^2}^{X} [U_0'(t)-\tfrac{1}{2}g(t)]\,{\rm d}t + U_0\left(\frac{1}{2\pi^2}\right), \qquad 2U_0'(X)=g(X)\,, \qquad X<\frac{1}{2\pi^2}\,. \tag{3.3}$$

Eqs. (3.2) and (3.3) can be brought into identical forms by defining in (3.2) $\bar{X}=1/2\pi^2+\boldsymbol{\phi}_c^2$ and in (3.3) $\bar{X}=X$ and by extending the domain of $g(t)$ to $t\geqslant 1/2\pi^2$ where we take $g\equiv 0$:

$$U_{\rm eff}^{\rm vac} = \int_{1/2\pi^2}^{\bar{X}} [U_0'(t)-\tfrac{1}{2}g(t)]\,{\rm d}t\,, \qquad 2U_0'(\bar{X})=g(\bar{X})\,. \tag{3.4}$$

Defining $g(t)=\hat{g}(t+1/2\pi^2)$ with $t\geqslant -1/2\pi^2$, eq. (3.4) can be rewritten with $Y=\bar{X}-1/2\pi^2$ as

$$U_{\rm eff}^{\rm vac} = \int_0^{Y} [U_{\rm R}'(t)-\tfrac{1}{2}\hat{g}(t)]\,{\rm d}t\,, \qquad U_{\rm R}'(Y)-\tfrac{1}{2}\hat{g}(Y)=0\,. \tag{3.5}$$

If $Y>0$ the symmetry is broken, and if $-1/2\pi^2<Y<0$ it is preserved.

The advantage of writing the equations as in eq. (3.5) is that a qualitative graphical analysis is made possible. The second equation in (3.5) identifies Y as the intersection of the graphs representing $U_{\rm R}'$ and $\frac{1}{2}\hat{g}$. The first equation in (3.5) allows us to find out which of the possible many intersections corresponds to the dominating saddle. One has to compare the sizes of the areas enclosed between the graphs of $U_{\rm R}'$ and $\frac{1}{2}\hat{g}$ from one intersection to the other. In our case $U_{\rm R}'$ is a parabola and not more than three intersections are possible. Fig. 1 schematically illustrates one such case. The three intersection points are labelled A, B, C. a_1, b_1, b_2 denote the magnitudes

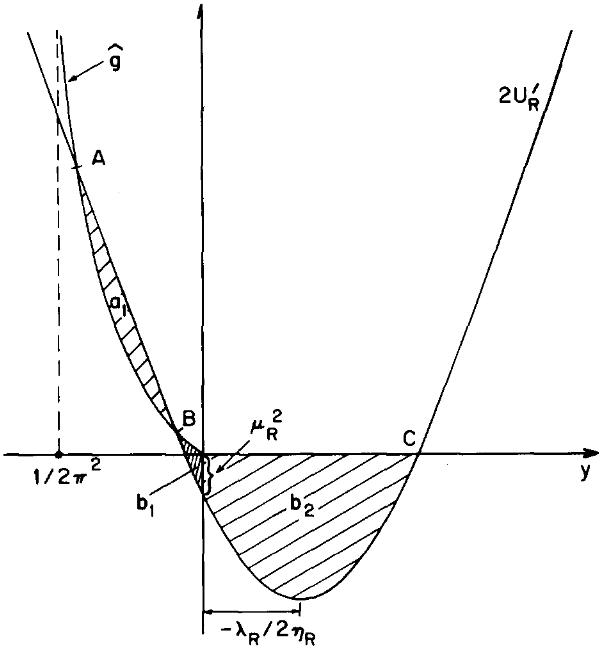

Fig. 1. Solving for the vacuum at $N=\infty$. Possible candidates are labelled A, B, C; the relative sizes of the areas a_1, b_1 and b_2 determine the dominating saddle.

of the various areas. From eq. (3.5) we get

$$U_{\mathrm{eff}}^{\mathrm{vac}} = \begin{cases} \mathrm{A}: & b_1 - a_1 \\ \mathrm{B}: & b_1 \\ \mathrm{C}: & -b_2 . \end{cases} \tag{3.6}$$

B is never a dominant saddle. A will dominate if $b_1 + b_2 < a_1$ and C otherwise. When C dominates the symmetry is broken.

The space of interactions is parametrized by

$$2U'_{\mathrm{R}}(X) = \mu_{\mathrm{R}}^2 + \lambda_{\mathrm{R}} X + \eta_{\mathrm{R}} X^2 , \qquad \eta_{\mathrm{R}} \geqslant 0, \qquad -\infty < \mu_{\mathrm{R}}^2, \lambda_{\mathrm{R}} < \infty . \tag{3.7}$$

In our qualitative analysis we shall first fix η_{R}, then λ_{R} and vary μ_{R}^2. It is necessary to distinguish between $0 < \eta_{\mathrm{R}} < \eta_{\mathrm{c}} \equiv (4\pi)^2$ and $\eta_{\mathrm{R}} > \eta_{\mathrm{c}}$. η_{c} is half the curvature of $\hat{g}$ at the origin (see eq. (2.16)).

Let $\eta_{\mathrm{R}} < \eta_{\mathrm{c}}$ be fixed. The minimum of V'_{R} is in the $y > 0$ half-plane iff $\lambda_{\mathrm{R}} < 0$. By inspection we see that if $\lambda_{\mathrm{R}} > 0$, we go through a second-order symmetry breaking transition when we vary μ_{R}^2 through zero. If $\lambda_{\mathrm{R}} < 0$, the symmetry breaking transition will be of first order, the dominating saddle moving from type C to type A (see fig. 1) when μ_{R}^2 is increased from negative values. However, because, unlike in fig. 1, the parabola representing U'_R is shallower than $\frac{1}{2}\hat{g}$, the transition will occur only at a positive value of μ_R^2. At $\lambda_R=0$ the second-order transition changes to first order and we have a tricritical point.

When $\eta_R>\eta_c$ and λ_R is large and positive we still have only a second-order transition at $\mu_R^2=0$. Similarly when λ_R is negative only a first-order transition occurs. If λ_R is positive but small we have a first-order transition which occurs already at a negative value of μ_R^2. For somewhat larger values of λ_R, but not too large, we can, while increasing μ_R^2 from very negative values, cross, first a second-order symmetry breaking transition at $\mu_R^2=0$ and later, at a positive value of μ_R^2, a first-order transition at which a saddle of type C (but located now in the $y<0$ half-plane) is replaced by one of type A. This latter transition is not a symmetry breaking one. For any $\eta_R>\eta_c$ there exists a λ_R for which the three solutions of the type A, B, C (all of them located now in the $y<0$ plane) become degenerate at some positive value of μ_R^2. At these points a continuous transition of the liquid–gas type occurs. These transitions happen inside the symmetric phase.

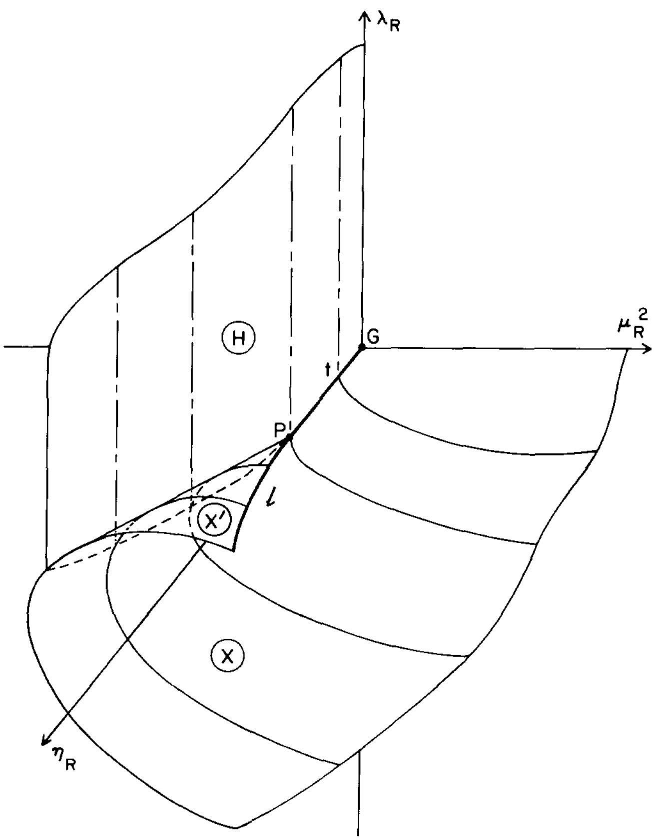

Fig. 2. Qualitative sketch of the $d=3$, $N=\infty$ phase diagram.

Fig. 2 summarizes the above qualitative features of the phase diagram. We have three regions of continuous transitions: the surface H which consists of part of the $\lambda_R > 0$, $\mu_R^2 = 0$ plane, the tricritical line t and the gas-liquid transition line ℓ. The lines t and ℓ meet at a point P where $\mu_R^2 = \lambda_R = 0$ and $\eta_R = \eta_c$. For $\eta_R > \eta_c$ the $\mu_R^2 = 0$, $\lambda_R > 0$ half-plane is cut by the first-order surface X which continues into the symmetrical phase with X′. The cut is one of the boundaries of H. X′ ends at ℓ. The main features of the diagram can be found also in ref. [11].

To give some feeling for the numbers involved we show in fig. 3 two fixed η_R slices of the phase diagram, one with $\eta_R < \eta_c$ and the other with $\eta_R > \eta_c$. The heavy line represents a first-order transition and the dashed one, one of second order. The graphs were computed numerically. The numerical study confirmed the qualitative features of fig. 2.

The reason for the line t ending at $\eta_R = \eta_c$ is that for $\lambda_R = \mu_R^2 = 0$ and $\eta_R > \eta_c$ a new saddle becomes dominant. It is because of this that the continuum effective potential computed in eq. (2.22) is right only for $\eta_R \leq \eta_c$. Bardeen, Moshe and

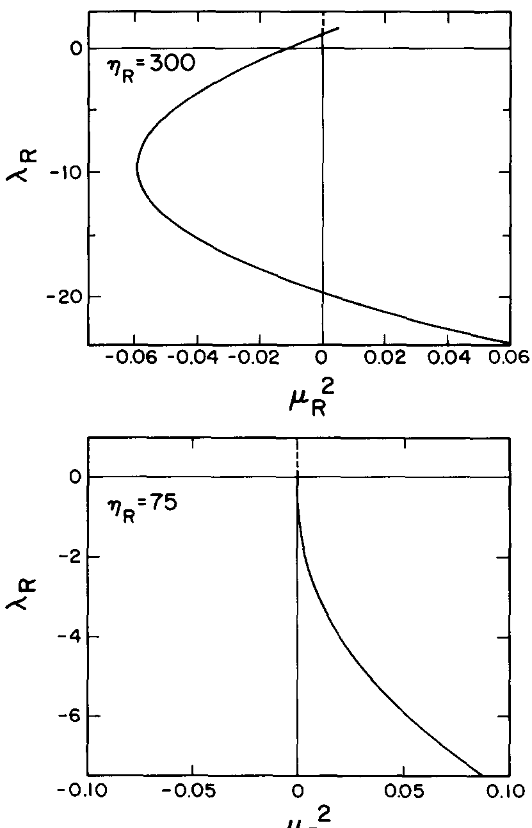

Fig. 3. Constant η-slices of the $d=3$, $N=\infty$ phase diagram.

Bander looked at $\eta_R > \eta_c$ and found an instability in the perturbative answer. As we can see this instability reflects the fact that the continuation of the perturbative answer to $\eta_R > \eta_c$ takes us into a metastable phase.

The point P where the lines ℓ and t meet is expected to have a special critical behavior. On t the O(N)-vector particles are massless and on ℓ an O(N)-scalar particle corresponding to $\boldsymbol{\phi}^2(x)$ is massless. At P we have both. The new type of continuum theory constructed by BMB around P inherits the massless excitation living on ℓ and gives a finite mass to the vectors. The nonvanishing of this mass reflects the spontaneous breaking of scale invariance and the massless scalar is the dilaton associated with this breaking.

The above phenomena occur because t and ℓ meet at P. However, the meeting of two one-dimensional manifolds embedded in a three-dimensional one is a nongeneric phenomenon. The occurrence of this accident may very well be restricted to $N=\infty$. If it is true that for any finite N, ℓ and t miss each other the BMB phenomenon will be strictly an $N=\infty$ curiosity.

When one inspects fig. 3 one might get the impression that the lines ℓ and t do have to meet. To dispel this impression we considered the $N=\infty$ phase diagram for $d=3+\varepsilon$, $\varepsilon>0$, with the same type of interaction. Since for $\varepsilon>0$ the ϕ^6 operator is irrelevant (nonrenormalizable) a BMB phenomenon should not occur. The analogue of fig. 3 is fig. 4 with $\varepsilon=0.5$. Lines of the ℓ and t type exist and the BMB phenomenon is indeed avoided by these lines never meeting. Both ℓ and t end on first-order transition surfaces without special endpoints. Therefore the BMB phenomenon can be avoided and there is no reason to expect it is not even at $d=3$ when $N<\infty$.

We should also remark that below three dimensions the tricritical line disappears at $N=\infty$. A tricritical line may very well exist for $d<3$ and for some finite N. This is another example of the delicate nature of the large-N limit, but, as it has no direct relevance to the present work it will not be discussed any further.

We now turn to the computation of some general characteristics of the various critical regions we have identified. More precisely we shall compute the critical exponents ν (related to the divergence of the correlation length) and α (related to the specific heat). In the generic case the values of ν and α depend only on the fixed points relevant for the critical regions and not on the details of the interactions. Different values for the exponents suggest therefore the existence of different types of fixed points.

The exponent ν governs the rate at which the physical mass of the vector particle vanishes when the bare mass approaches a critical value. Using eqs. (2.16), (2.17) and (3.3) we calculate the physical mass for small μ_R^2 ($\mu_R^2 \propto (\mu_0^2 - \mu_{0\,\mathrm{critical}}^2)$) in the vicinity of various critical regions:

$$M^2 = 2U_R'(\varepsilon) \equiv \mu_R^2 + \lambda_R \varepsilon + \eta_R \varepsilon^2 = g\left(\varepsilon + \frac{1}{2\pi^2}\right) \approx \eta_c \varepsilon^2 - 8\eta_c \varepsilon^3 \,. \qquad (3.8)$$

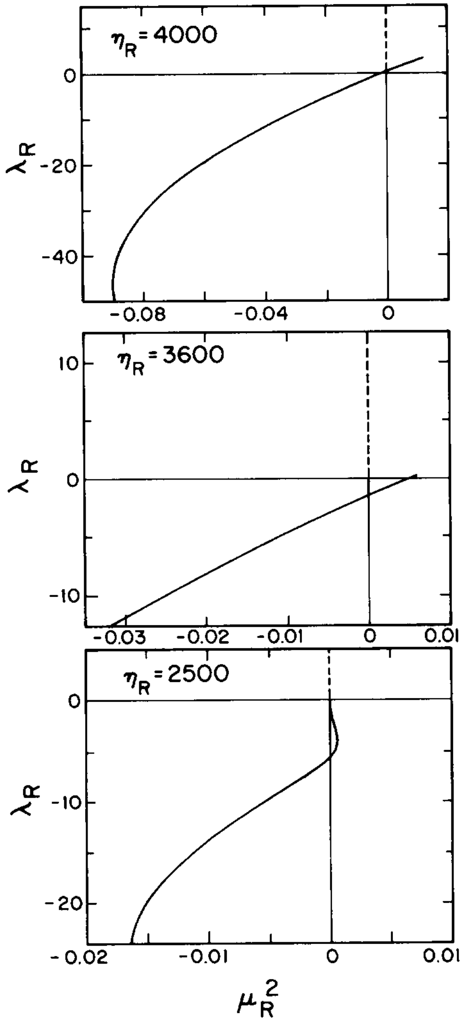

Fig. 4. Same as fig. 3 but for $d = 3.5$.

In eq. (3.8) $\varepsilon < 0$. When $\mu_R^2 = 0$ we are at a critical point CP. If the CP is on H, $\lambda_R > 0$ and hence $M \sim -\varepsilon \sim \mu_R^2$. If the CP is on t but not at P, then $\lambda_R = 0$ but $\eta_R < \eta_c$ implying $M \sim -\varepsilon \sim \sqrt{\mu_R^2}$. If the CP is at P then $\lambda_R = 0$, $\eta_R = \eta_c$ and hence $M \sim \varepsilon \sim \sqrt[3]{\mu_R^2}$. This means that the index ν has the values 1, $\frac{1}{2}$ and $\frac{1}{3}$ on H, t and P respectively. The fact that $\nu_P \neq \nu_t$ shows again that P is special. However, as we shall see in sect. 6, to state that a new type of fixed point governs the critical behavior at P would be misleading.

The exponent α, if positive, tells us how $d^2 U_{\text{eff}}(0)/d(\mu_R^2)^2$ diverges when we approach a critical point from the symmetric phase. In the notation of eq. (3.8) we have

$$U_{\rm eff}(0)=U_{\rm R}(\varepsilon)-\tfrac{1}{6}\eta_{\rm c}\varepsilon^3+\eta_{\rm c}\varepsilon^4+O(\varepsilon^5)\,. \tag{3.9}$$

Therefore

$$\frac{d^2U_{\rm eff}(0)}{d(\mu_{\rm R}^2)^2}=\frac{1}{2}\frac{d\varepsilon}{d\mu_{\rm R}^2}+[U_{\rm R}'(\varepsilon)-\tfrac{1}{2}\eta_{\rm c}\varepsilon^2]\frac{d^2\varepsilon}{d(\mu_{\rm R}^2)^2}$$
$$+[U_{\rm R}''(\varepsilon)-\eta_{\rm c}(\varepsilon)]\left(\frac{d\varepsilon}{d\mu_{\rm R}^2}\right)^2+\eta_{\rm c}\left(\frac{d}{d\mu_{\rm R}^2}\right)^2\varepsilon^4+\cdots.$$

If the CP is on H no divergence appears (for a discussion of this point see ref. [12]). If the CP is on t we get $\varepsilon\sim\sqrt{\mu_{\rm R}^2}$ and a divergence as $1/\sqrt{\mu_{\rm R}^2}$. If the CP is at P, $\varepsilon\sim(\mu_{\rm R}^2)^{1/3}$ and (3.9) becomes

$$\frac{d^2U_{\rm eff}(0)}{d(\mu_{\rm R}^2)^2}\sim\frac{1}{2}\frac{d}{d\mu_{\rm R}^2}\left[\mu_{\rm R}^2\frac{d\varepsilon}{d\mu_{\rm R}^2}\right]+\eta_{\rm c}\frac{d^2}{d(\mu_{\rm R}^2)^2}\varepsilon^4\sim(\mu_{\rm R}^2)^{-2/3}\,. \tag{3.10}$$

The exponent α is therefore <0, $\frac{1}{2}$, $\frac{2}{3}$ on H, t and P respectively. The hyperscaling relation $d\nu=2-\alpha$ is violated at P. This implies that at least one of the exponents at P does not follow only from the behavior of the RG transformation in the vicinity of the relevant fixed point but additional nonanalyticities are present. Again the point P is troublesome. In sect. 6 we shall argue that as far as the RG structure is concerned the point P is of the same type as the rest of the line t. All the special effects seen at P, including the BMB phenomenon, result from additional nonanalyticities, the existence of which is related to the meeting of the lines ℓ and t. There is little reason to assume that these additional nonanalyticities will somehow survive to finite N.

4. The strong coupling limit

The most distinctive feature of the $N=\infty$ phase diagrams analyzed in the previous section was the difference between the weak η-regime where a tricritical point exists and the strong η-regime where it disappears. The purpose of this section is to estimate to how small values of N the existence of the two regimes persists. In other words, we would like to find out from what value of N onwards the tricritical line has a finite extent. This value of N is a lower bound for the possible N's for which the BMB phenomenon has any chance of survival at all; given that the tricritical line indeed terminates for some N the more delicate question of direct relevance to BMB is whether the transition from strong to weak η takes place qualitatively as in fig. 3 or as in fig. 4. For the BMB mechanism to work the nongeneric case of fig. 3 has to occur. We shall not attempt here to study these more delicate features of the phase diagram.

Although we shall do only analytical work here we want our models to be directly amenable to computer simulations. This excludes the sharp momentum cutoff because it yields an action which is not strictly local. A lattice variant of (1.1) to which standard Monte Carlo simulation techniques can be applied is

$$A_{\mathrm{E}}^{\mathrm{L}}[\boldsymbol{\phi}]=-\sum_x\left\{\tfrac{1}{2}\sum_{\mu=1}^{3}[\boldsymbol{\phi}(x+\mu)-\boldsymbol{\phi}(x)]^2\right.$$
$$\left.+\tfrac{1}{2}\mu_{\mathrm{L}}^2\boldsymbol{\phi}^2(x)+\frac{1}{4N}\lambda_{\mathrm{L}}(\boldsymbol{\phi}^2(x))^2+\frac{1}{6N^2}\eta_{\mathrm{L}}(\boldsymbol{\phi}^2(x))^3\right\}. \qquad (4.1)$$

In (4.1) x is a lattice point and μ is a unit vector in one of the $\mu=1,2,3$ directions; $\boldsymbol{\phi}(x)$ is an N-component vector associated with the site x. Whenever we shall refer to the system in a finite volume, periodic boundary conditions are implicitly assumed.

The $\eta_{\mathrm{L}}=\lambda_{\mathrm{L}}=\mu_{\mathrm{L}}^2=0$ point is clearly tricritical. We shall make the plausible assumption that a tricritical line exists for at least a small range of positive η_{L}'s. We intend to establish whether the line extends out to $\eta_{\mathrm{L}}=\infty$. Hence we concentrate on the $\eta_{\mathrm{L}}=\infty$ limit of (4.1). This limit is particularly suited for Monte Carlo investigations because then the variable $\boldsymbol{\phi}$ becomes compact. It is useful to change variables:

$$\boldsymbol{\phi}(x)=\sqrt{N}\rho(x)\boldsymbol{\Omega}(x), \qquad \rho(x)\geqslant 0, \qquad \boldsymbol{\Omega}^2(x)=1,$$
$$\tilde{\mu}_{\mathrm{L}}^2=\frac{1}{\eta_{\mathrm{L}}}\mu_{\mathrm{L}}^2, \qquad \tilde{\lambda}_{\mathrm{L}}=\frac{1}{\eta_{\mathrm{L}}}\lambda_{\mathrm{L}}. \qquad (4.2)$$

The partition function of the model is

$$Z=\int\prod_x[\mathrm{d}\rho(x)\,\mathrm{e}^{-\eta_{\mathrm{L}}Nv(\rho^2(x))}]\prod_x\mathrm{d}\boldsymbol{\Omega}(x)$$
$$\times\exp\left\{-\tfrac{1}{2}\mathrm{N}\sum_{\mathrm{x}}\sum_{\mu=1}^{3}[\rho(\mathrm{x})\boldsymbol{\Omega}(x)-\rho(x+\mu)\boldsymbol{\Omega}(x+\mu)]^2\right\},$$
$$v(\rho^2)=\tfrac{1}{2}\tilde{\mu}_{\mathrm{L}}^2\rho^2+\tfrac{1}{4}\tilde{\lambda}_{\mathrm{L}}\rho^4+\tfrac{1}{6}\rho^6-\frac{N-1}{2N\eta_{\mathrm{L}}}\log(\rho^2). \qquad (4.3)$$

The last term in v cannot be neglected for very small ρ. As $\eta_{\mathrm{L}}\to\infty$ only the absolute minimum (or minima in the case of degeneracy) is important. v has two local minima:

$$\rho_1^2=\frac{N-1}{N}\frac{1}{\eta_{\mathrm{L}}\tilde{\mu}_{\mathrm{L}}^2}+\mathrm{O}\left(\frac{1}{\eta_{\mathrm{L}}^2}\right),$$
$$v_1\equiv\eta_{\mathrm{L}}v(\rho_1^2)=\frac{N-1}{N}[\tfrac{1}{2}+\tfrac{1}{2}\log\eta_{\mathrm{L}}+\tfrac{1}{2}\log\tilde{\mu}_{\mathrm{L}}^2]+\mathrm{O}\left(\frac{1}{\eta_{\mathrm{L}}}\right),$$
$$\rho_2^2=\rho_0^2+\frac{N-1}{N\rho_0^2}\frac{1}{\eta_{\mathrm{L}}(\tilde{\lambda}_{\mathrm{L}}+2\rho_0^2)}+\mathrm{O}\left(\frac{1}{n_{\mathrm{L}}^2}\right),$$
$$\rho_0^2\equiv-\tfrac{1}{2}\tilde{\lambda}_{\mathrm{L}}+\tfrac{1}{2}\sqrt{\tilde{\lambda}_{\mathrm{L}}^2-4\tilde{\mu}_{\mathrm{L}}^2},$$

$$v_2 \equiv \eta_L v(\rho_2^2)$$
$$= \frac{N-1}{N}\{\tfrac{1}{2}\eta_L \rho_0^2[\tilde{\mu}_L^2 + \tfrac{1}{2}\tilde{\lambda}_L \rho_0^2 + \tfrac{1}{3}\rho_0^4] - \tfrac{1}{2}\log \rho_0^2\} + O\left(\frac{1}{\eta_L}\right). \quad (4.4)$$

To obtain a nontrivial limit we adjust the couplings such that ρ_0^2 and $v_2 - v_1$ stay finite as $\eta_L \to \infty$. Then the integral over $\rho(x)$ in (4.3), at each x, becomes a sum over $\rho(x) = \rho_0$ and $\rho(x) = 0$. We introduce some more conventional notation:

$$\beta = \rho_0^2 > 0, \qquad \mu = v_1 - v_2 - 3\beta. \quad (4.5)$$

The two-parameter model describing the $\eta_L \to \infty$ limit has a partition function given by

$$Z = \prod_x \left(\sum_{\rho(x)=0} e^{N\mu\rho(x)} \right) \prod_x \left(\int d\boldsymbol{\Omega}(x) \right)$$
$$\times \exp\left\{ N\left[\beta \sum_x \sum_{\mu=1}^{3} \rho(x)\rho(x+\mu)\boldsymbol{\Omega}(x)\cdot\boldsymbol{\Omega}(x+\mu) \right] \right\}. \quad (4.6)$$

The variables $\rho(x)$ are vacancy variables with chemical potential $N\mu$. In the absence of vacancies the model is equivalent to an O(N) classical Heisenberg ferromagnet with inverse temperature $N\beta$.

At infinite N the phase diagram can be obtained directly for (4.6). However, the limits $\eta_L \to \infty$ and $N \to \infty$ are interchangeable and therefore we can first apply the analysis of the previous section to (4.1). In that analysis we have to replace the sharp momentum cutoff function $f(M^2)$ in eq. (2.4) by its lattice analogue:

$$f_L(M^2) = \int_{|q_\mu|<\pi} \frac{d^3q}{(2\pi)^3} \frac{1}{4\sum_{\mu=1}^{3} \sin^2(\tfrac{1}{2}q_\mu) + M^2},$$
$$g_L(f_L(M^2)) = M^2, \qquad g_L(x) = 0 \quad \text{for } x > f_L(0) \equiv \beta_c. \quad (4.7)$$

One obtains a phase diagram separated in two regions by a first-order transition line $\mu(\beta)$:

$$\mu(\beta) = -3\beta + \tfrac{1}{2}\log\beta + \tfrac{1}{2} + \tfrac{1}{2}\int_\beta^{\beta_c} g_L(\beta')\, d\beta' + \frac{1}{2}\int_{|q|\leq\pi} \frac{d^3q}{(2\pi)^3} \log\left[4\sum_{\mu=1}^{3} \sin^2 \tfrac{1}{2}q_\mu\right]. \quad (4.8)$$

For $\mu > \mu(\beta)$ there are absolutely no vacancies ($N = \infty$) and we have the expected second-order transition of symmetry breaking at $\beta_c = f_L(0) \approx 0.2527$. For $\mu < \mu(\beta)$ the vacancies occupy all the sites and no dynamics are left. The point $\mu = \mu(\beta_c)$, $\beta = \beta_c$ is the endpoint of the second-order line on the first-order line; this point is not a tricritical point.

The features of the phase diagram at $N<\infty$ can be understood approximately. On the boundaries of the half-plane $\{-\infty<\mu<\infty; \beta\geq 0\}$ we have the following situation:

(i) At $\mu=+\infty$ there are no vacancies and a second-order symmetry breaking transition is expected to occur at β_N^c. Increasing N has a disordering effect and hence we expect β_N^c to monotonically increase from the Ising value (≈ 0.221) to the $N=\infty$ value (≈ 0.253). As a phenomenological* relationship we take

$$\beta_N^c \sim 0.253-\frac{0.032}{N}. \tag{4.9}$$

(ii) At $\mu\to\infty$ and $\beta\to+\infty$ with $\mu\sim -3\beta$ the $\mathbf{\Omega}$'s will be completely aligned if there are no vacancies, and completely random if there are. Therefore we have a first-order transition line which asymptotically behaves like $\mu\sim-3\beta$ and across which both the vacancy density and the $\mathbf{\Omega}$-magnetization change discontinuously.

(iii) At $\beta=0$ the free energy is analytic in μ (except at $N=\infty$). Therefore, for any $N<\infty$ the first-order line will not reach the $\mu=\beta=0$ point. The main issue is now to decide how the line in (iii) and the line in (i) join. That they have to join is clear, because, for any fixed μ, increasing β from 0 to ∞ has to induce an $O(N)$ symmetry breaking transition at some $\beta^c(\mu; N)$. This transition can be of first or second order. Since the vacancies have a disordering effect $\beta^c(\mu; N)$ is monotonically increasing as μ decreases from infinity where $\beta^c(\infty; N)=\beta_N^c$. The two lines can either join smoothly, in which case a tricritical point exists or the first-order line can continue into the $O(N)$-disordered phase. In this phase the first-order line represents a liquid–gas transition signaled by a discontinuous change in the vacancy density; however the line has no symmetry breaking role any more and the average of $\mathbf{\Omega}(x)$, the magnetization, vanishes on both its sides.

To establish which type of line meeting we expect we would like to integrate out the $\mathbf{\Omega}$-fluctuations to obtain an effective vacancy system. This can be done approximately both for β small and β large. It is convenient to replace $\rho(x)$ by an Ising spin variable $s(x)$:

$$s(x)=1-2\rho(x). \tag{4.10}$$

The effective action for $s(x)$ represents to leading order both in strong and weak β-coupling, an Ising system in an external magnetic field:

$$S_{\text{eff}}[s]=H_{\text{eff}}\sum_x s(x)+\beta_{\text{eff}}\sum_x\sum_{\mu=1}^{3} s(x)s(x+\mu), \tag{4.11}$$

where

$$\frac{H_{\text{eff}}}{N}=\begin{cases}\beta\gg\beta_c(\mu;B) & -\frac{1}{2}\mu-3\beta/2\\ \beta\ll\beta_c(\mu;N) & -\frac{1}{2}\mu-\frac{3}{4}\beta^2\end{cases},\qquad \frac{1}{N}\beta_{\text{eff}}=\begin{cases}\beta\gg\beta_c(\mu;N) & \frac{1}{4}\beta\\ \beta\ll\beta_c(\mu;N) & \frac{1}{8}\beta^2.\end{cases} \tag{4.12}$$

* The $1/N$ dependence of β_N^c is only a guess.

The vacancy density would change discontinuously when $H_{\rm eff}=0$ and $\beta_{\rm eff}>\beta_{\rm Ising}\approx 0.221$. Let us consider the case where N is rather large, $N>5$ say. Since for $\beta \gg \beta_c(\mu; N)$ we would have $\beta_{\rm eff}\approx \frac{1}{4}N\beta$ we see that by the time β is large $\beta_{\rm eff}$ is certainly such. Therefore we expect both the vacancy density and the magnetization to change discontinuously as $H_{\rm eff}$ changes sign. Hence the two transitions could separate, most likely, when the first-order vacancy transition occurs at zero magnetization. This is what happened at $N=\infty$. We are therefore only to consider the case $\beta<\beta_c(\mu; N)$. To get there a first-order transition without a symmetry breaking one at the same time we need to meet two requirements:

$$H_{\rm eff}\approx 0\,,$$
$$\beta_{\rm eff}>0.221\,,\qquad \beta<\beta_c(\mu; N)\,. \tag{4.13}$$

$\beta_c(\mu;\infty)$ is μ-independent for $\mu>\mu(\beta_c)$. Therefore, for large N the μ-dependence of $\beta_c(\mu; N)$ is weak and we can approximate $\beta_c(\mu; N)\approx\beta^c_N$ (see (4.9)). In reality β^c_N is a lower bound on $\beta_c(\mu, N)$. With (4.12) and (4.13) we get the following conditions:

$$\frac{1.33}{\sqrt{N}}=\sqrt{(8/N)0.221}<\beta<0.253-\frac{0.032}{N}\,,$$
$$\mu\approx -1.5\beta^2\,. \tag{4.14}$$

The first line in (4.14) implies $N\geq 28$. Therefore we conclude that when N is decreased from infinity the tricritical line will reach an infinite extent at $N\approx 28$. The conditions (4.14) are superposed on the infinite-N phase diagram in fig. 5. The approximate agreement between the line $\mu\approx -1.5\beta^2$ and the exact $N=\infty$ result strengthens our faith in the rather crude approximations we employed.

We conclude this section by stating that the BMB cannot take place for values of N smaller than a number around 30.

5. The renormalization group analysis

The construction of the continuum limits in sect. 2 proceeded by a direct approach. A bare interaction which depends on a few parameters is picked, regions where a mass vanishes are identified, and using this mass as a scale, one tries to adjust the other parameters in such a way that, up to wave function renormalization, all correlations have a finite and hopefully nontrivial continuum limit. The direct approach basically requires one to solve the theory along with its construction. The continuum limit is found by a trial and error method. The origin of the special features discovered by BMB is somewhat obscured in this type of approach.

In the RG approach the problem of constructing a continuum limit is conceptually separated from the problem of solving it [1]. In principle, the search for the

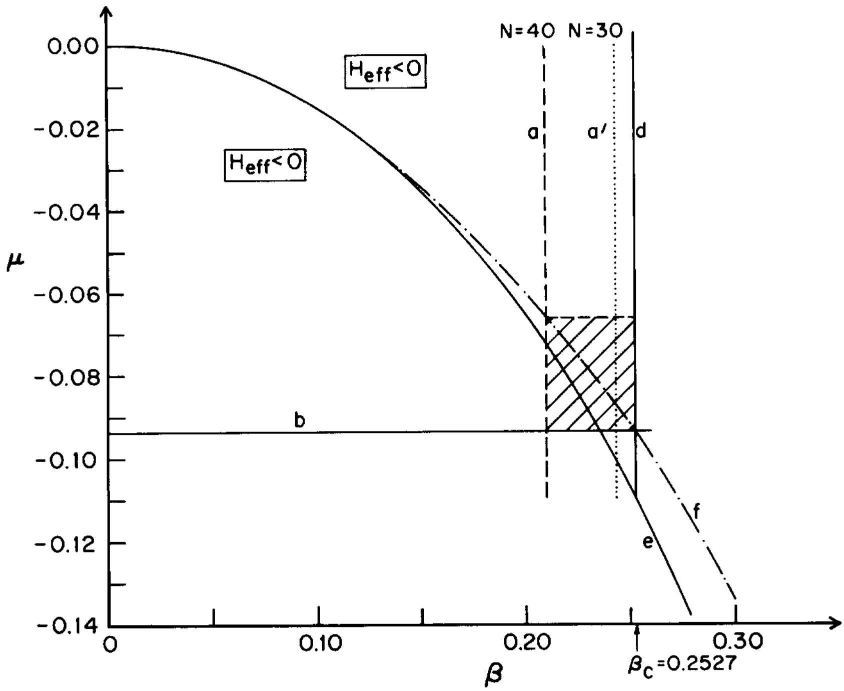

Fig. 5. The lattice vacancy model. e and d are exact at $N=\infty$. e and f are first-order transition lines and d is a second-order one. f is the approximation used in the text; it is N-independent. Lines a and a' are the values of β for which $\beta_{eff}=\beta_c$ (Ising) for $N=40$ and 30 respectively. Line b is at the value of μ at which f and d meet; for μ below this value the second-order transition disappears in the approximation employed. The hatched area is where two transitions will occur at constant μ for $N=40$.

continuum limits is systematic. The price paid for these advantages is the necessity of working in a not really well-defined space of all possible short-range interactions obeying some symmetry invariances. In this space of cutoff (bare) lagrangians a transformation R_s is defined. In our case we assume a sharp momentum cutoff. R_s transforms a lagrangian with cutoff Λ into one with cutoff Λ/s $(s\geq 1)$. The transformed lagrangian, after wave function renormalization, is required to generate an effective action for classical fields with momentum less than Λ/s, of exactly the same form as the original one would. By definition $R_{s_1s_2}=R_{s_1}R_{s_2}$. Let $\Gamma[\boldsymbol{\phi};\Lambda;\mathscr{L}]$ denote the effective action obtained from $\mathscr{L}$ with a cutoff Λ. Then

$$\Gamma[\boldsymbol{\phi};\Lambda;\mathscr{L}]=\Gamma\left[f_s(\boldsymbol{\phi});\frac{\Lambda}{s};R_s(\mathscr{L})\right]. \tag{5.1}$$

We shall frequently denote (when there is no danger of confusion) $f_s(\boldsymbol{\phi})\equiv\boldsymbol{\phi}_s$ and $R_s(\mathscr{L})\equiv\mathscr{L}_s$. Since all our analysis is at $N=\infty$ we shall take $\boldsymbol{\phi}_s=\boldsymbol{\phi}$.

The problem of constructing a continuum limit is to find a bare lagrangian $\mathscr{L}_{0,s}$ with a cutoff Λ and such an explicit dependence on s $(s\geq 1)$ that the following limit exists:

$$\lim_{s\to\infty} \Gamma[\boldsymbol{\phi}; \Lambda_{\mathrm{R}} s; \mathcal{L}_{0,s}] = \lim_{s\to\infty} \Gamma[\boldsymbol{\phi}; \Lambda_{\mathrm{R}}; R_s(\mathcal{L}_{0,s})]\,. \tag{5.2}$$

In (5.2) Λ_{R} is finite but arbitrary. The existence of the limit on the right-hand side depends on whether $\lim_{s\to\infty} R_s(\mathcal{L}_{0,s})$ exist within the space of acceptable interactions. If it does, then $\bar{\mathcal{L}} = \lim_{s\to\infty} R_s(\mathcal{L}_{0,s})$ is a UV cutoff lagrangian which has no dependence on the bare cutoff. To get the correlation functions of the continuum limit one has to solve the model with $\bar{\mathcal{L}}$. While the solution may be very hard to obtain it is explicitly obvious that the UV divergences have been eliminated.

The existence of a continuum limit depends now only on the properties of R_s. For the limit to have some predictive power it is necessary to get an $\bar{\mathcal{L}}$ which depends on only a finite number of free parameters. A simple way to obtain this is for R_s to have a fixed point with a finite number of unstable directions. These unstable directions, with the fixed point at their origin, generate a curvilinear (in terms of the bare couplings in $\mathcal{L}_{0,s}$) coordinate system spanning a finite-dimensional submanifold of the space of all interactions. This manifold of RG trajectories represents the complete set of continuum field theories one can build around the given fixed point. Since, by definition, the transformation R_s does not take us out of the space of interactions, the trajectories end at fixed points of higher degrees of stability.

It is impossible to apply the above ideas in their full generality even in the rather simple $N=\infty$ situation. We therefore truncate the space of interactions to lagrangians of the form

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \boldsymbol{\phi})^2 + V(\boldsymbol{\phi}^2)\,. \tag{5.3}$$

It is convenient to replace the requirement of (5.1) which gives identical Green functions in x-space by the requirement that the Green functions be the same in momentum space (after imposition of momentum conservation). The appropriate truncation then becomes

$$V_{\mathrm{eff}}[\boldsymbol{\phi}^2; \Lambda; V] = V_{\mathrm{eff}}\left[\boldsymbol{\phi}^2; \frac{\Lambda}{s}; V_s\right]. \tag{5.4}$$

We now rescale by N as in eq. (2.8) and also go to unitless couplings and fields (see eq. (2.10)):

$$U_{\mathrm{eff}}[Z; U] = s^{-3} U_{\mathrm{eff}}[sZ; U_s]\,. \tag{5.5}$$

It makes more physical sense to view R_s as acting on the derivatives:

$$U'_{\mathrm{eff}}[Z; U'] = s^{-2} U'_{\mathrm{eff}}[sZ; U'_s]\,. \tag{5.6}$$

The functional dependence of U'_{eff} on U' is known from eq. (2.13):

$$U'_{\mathrm{eff}}[Z; U'] = U'(Z+X) = \tfrac{1}{2}g(X)\,. \tag{5.7}$$

The last equality defines X as a function of Z. We shall consider the equations only locally and therefore we can neglect for the time being the problem of competing saddles. A more explicit form for U'_s can be obtained:

$$U'_s(y) = U'_{\rm eff}[y-x_1\,;\,U'_s] = s^2 U'_{\rm eff}\left[\frac{y-x_1}{s}\,;\,U'\right] = s^2 U'\left(\frac{y-x_1}{s}+x_2\right), \qquad (5.8)$$

with

$$g(x_1) = 2U'_s(y)\,, \qquad g(x_2) = \frac{1}{s^2}2U'_s(y)\,. \qquad (5.9)$$

Eqs. (5.8) and (5.9) can be combined with the help of the function f defined in eq. (2.11):

$$U'_s(y) = s^2 U'\left(\frac{1}{s}y - \frac{1}{s}f(2U'_s(y)) + f\left(\frac{1}{s^2}2U'_s(y)\right)\right). \qquad (5.10)$$

Eq. (5.10) was obtained by Ma [7] by explicitly integrating out the high-momentum components of the fields. Our derivation is simpler technically but not conceptually because it makes use of the exact solubility (in an implicit form) of the models in eq. (5.3) at $N=\infty$.

We now introduce F, F_s, the inverse function to $2U'$, $2U'_s$ (which is defined at least locally). Eq. (5.10) then simplifies to

$$F_s(u) - f(u) = s\left[F\left(\frac{u}{s^2}\right) - f\left(\frac{u}{s^2}\right)\right]. \qquad (5.11)$$

$F_s - f$ is just the inverse of $2U'_{\rm eff}\,[x, U_s]$, as can be easily shown, and thus also transforms homogeneously under the renormalization group (see eq. (5.6)). It is easy to find the fixed points

$$F^*(u) = f(u) + c\sqrt{u}\,. \qquad (5.12)$$

To know which values of c are acceptable we have to extend the locally defined F^* to a globally defined inverse. The inverse is the derivative of the potential U^* and has to be defined for any nonnegative value of its argument. $f(u)$ can be written as

$$f(u) = -\frac{1}{4\pi}\sqrt{u} + f_{\rm A}(u)\,, \qquad f_{\rm A}(u) = \frac{1}{2\pi^2}(1+\sqrt{u}\tan^{-1}\sqrt{u})\,. \qquad (5.13)$$

$f_{\rm A}(u)$ is analytic in u. As long as $c \neq (1/4\pi)$ u cannot change sign and $U^{*\prime}$ has to be always positive. It is easy to see that $f(u)/\sqrt{u}$ decreases monotonically to zero when $u \to \infty$ implying that

$$\min_{0\le u\le\infty} (f(u) + c\sqrt{u}) > 0\,, \qquad \forall c > 0\,, \qquad (5.14)$$

and therefore that $U^{*\prime}(\phi^2)$ will be undefined for ϕ^2 small enough. Hence c cannot be positive unless it is equal to $1/4\pi$. The two possibilities left are $-\infty<c<0$ and $c=1/4\pi$.

$c=-\infty$ implies $u\equiv 0$ or $U^{*\prime}(\phi^2)\equiv 0$ and therefore corresponds to the gaussian fixed point. For any $-\infty<c<0$ one can reconstruct a $U^{*\prime}(\phi^2)$ which is analytic at $\phi^2=0$. For $\phi^2<1/2\pi^2$ one inverts $f_A(u)+(c-1/4\pi)\sqrt{u}$ and for $\phi^2>1/2\pi^2$ one inverts $f_A(u)-(c-1/4\pi)\sqrt{u}$. There is no nonanalyticity at $\phi^2=1/2\pi^2$. The parametrization in terms of c in eq. (5.12) should be understood to be relevant for $\phi^2\leqslant 1/2\pi^2$. At $c=0$ one still can invert F^* to obtain $U^{*\prime}(\phi^2)$ but, because of eq. (5.14), $U^{*\prime}(\phi^2)$ now diverges as $1/\phi^2$ when ϕ^2 approaches zero. At $c=1/4\pi$ one inverts $f_A(u)$ by analytically continuing to negative u-values. $U^{*\prime}(\phi^2)$ is monotonic because its inverse, $f_A(u)$, has no branch cut signularities. We shall see later that $c=1/4\pi$ corresponds to the HFP to which the surface H is attracted, $-\infty\leqslant c<0$ corresponds to a line of fixed points to which the tricritical line t is pointwise attracted and $c=0$ is the BMB FP to which the critical point P flows.

In order to be able to follow the RG flows even far away from the fixed points it is convenient to parametrize the space of interactions by a set of parameters which undergo only a rescaling under R_s [13]. Such parameters are sometimes referred to as scaling fields. We shall use two "coordinate patches", one appropriate for the vicinity of the $c=1/4\pi$ fixed point and the other for the vicinity of the $-\infty\leqslant c\leqslant 0$ line of fixed points.

The interactions in the vicinity of the HFP are parametrized by the Taylor coefficients of an analytic function $F(u)-f_A(u)$ (see eq. (5.13)) where F is, as usual, the inverse of $2U'$. From eq. (5.11) we see that

$$F(u)-f_A(u)=\sum_{n=1}^{\infty} b_n u^{n-1}\,,$$

$$F_s(u)-f_A(u)=\sum_{n=1}^{\infty} b_n s^{y_n} u^{n-1}\,,\qquad y_n=3-2n\,. \tag{5.15}$$

The exponents y_n are those of the spherical model. The HFP corresponds to $b_n=0$, $\forall n\geqslant 1$.

Around the HFP we can construct a continuum limit as explained at the beginning of this section. We need a $U_{0,s}(x)$ which has a corresponding $F^0(u;s)$ given by

$$F^0(u;s)=f_A(u)+\frac{a(s)}{s}+\sum_{n=2}^{\infty} b_n(s)u^{n-1}\,,$$

$$\lim_{s\to\infty} a(s)=a<\infty,\qquad \lim_{s\to\infty} b_n(s)=b_n<\infty\,,\qquad n=2,\dots\infty\,. \tag{5.16}$$

The requirements of $b_n(s)$ can be relaxed but this is not necessary. Assuming for the moment that such a $U_{0,s}(x)$ can be found we now identify the continuum limit. It corresponds to $\lim_{s\to\infty} R_s[F^0(u;s)]=f_A(u)+a$ and has only one free parameter. In the symmetric phase the RG trajectory is $f_A(u)+K$ with $K\leqslant 0$. When K goes from 0 to $-\infty$ the theory goes from the HFP to the TFP. To every point on the trajectory there corresponds an effective potential given by eq. (2.13):

$$U'_{\rm eff}(\phi_c^2) = 8\pi^2(\phi_c^2 - K)^2 . \tag{5.17}$$

This form is equivalent to eq. (2.25) and the identification of the trajectory with one of the continuum limits of sect. 2 is complete.

We now have to show how in our polynomial bare interaction an s-dependence can be introduced in $U_{0,s}(x)$ such that (5.16) holds. Any bare interaction for which $2U'(x)$ has a simple zero somewhere can be inverted to an analytic function:

$$F(u) = f_A(u) + c_1 + \sum_{n=2}^{\infty} c_n u^{n-1} . \tag{5.18}$$

The coefficients c_n are smooth functions of the coefficients in $2U'(x)$. c_1 is defined by

$$U'\left(\frac{1}{2\pi^2} + c_1\right) = U'_R(c_1) = 0 . \tag{5.19}$$

Demanding c_1 to be very small enforces a constraint on U'_R:

$$U'_R(0) \equiv \mu_R^2 = \text{very small} . \tag{5.20}$$

As long as $U''_R(0) \neq 0$ the other c_n's will stay finite as $c_1 \to 0$ and the requirements of eq. (5.16) can be met. One can view c_1 as parametrizing the distance of the bare interaction from the region of the critical surface ($c_1 = 0$) which is attracted to the HFP. When $U''_R(0) = 0$ and c_1 is small the interaction is close to the boundary of the domain of attraction of the HFP. There the parametrization of (5.18) becomes singular. To summarize, a continuum theory around the HFP can be built if the manifold of bare interactions intersects its domain of attraction. In the generic case this simply means that the dimensionality of this manifold is at least as large as the number of unstable directions at the given FP.

The parametrization in eq. (5.18) is not applicable to functions U' which never vanish (and, to be acceptable, have to be always positive). Such a function has a minimum at some $\bar{X}$, $U'(\bar{X}) = \frac{1}{2}\bar{u} > 0$, $U''(\bar{X}) = 0$ but, in the generic case, $U'''(\bar{X}) \neq 0$. The inverse function to $2U'(x)$, F, is double valued and can be parametrized as

$$F(u) = \bar{X} \pm \sqrt{u - \bar{u}}\, F_1(u - \bar{u}) + F_3(u - \bar{u}) , \tag{5.21}$$

where $F_{1,3}$ are analytic at zero and $F_3(0) = 0$. The function $f_A(u)$ is analytic everywhere on the real axis and therefore

$$F(u) - f_A(u) = F_2(u - \bar{u}) \pm \sqrt{u - \bar{u}}\, F_1(u - \bar{u}) = \sum_{n=1}^{\infty} b_n (u - \bar{u})^{(n-1)/2} . \tag{5.21a}$$

Under the RG transformation we get

$$F_s(u)-f_A(u)=\sum_{n=1}^{\infty} b_n s^{y_n}(u-s^{y_0}\bar{u})^{(n-1)/2}\,, \qquad y_n=2-n\,. \tag{5.21b}$$

These are the gaussian exponents. There are two relevant parameters ($\bar{u}$ and b_1) and a marginal one (b_2). The manifold of RG trajectories which represent continuum limits is parametrized by

$$F(u)=f_A(u)+A(u-B)^{1/2}+D\,. \tag{5.22}$$

The inverse of $2U'_{\rm eff}(x)$ is $F(u)-f(u)=A\sqrt{u-B}+(1/4\pi)\sqrt{u}+D$ and therefore

$$\phi_c^2=A\sqrt{2U'_{\rm eff}(\phi_c^2)-B}+\frac{1}{4\pi}\sqrt{2U'_{\rm eff}(\phi_c^2)}+D\,. \tag{5.23}$$

Defining $y(\phi_c^2)=-(1/4\pi)\sqrt{2U'_{\rm eff}(\phi_c^2)}$ and comparing with eq. (2.20) we see that we obtained the expected continuum limit and that the relation between the renormalized parameters of eq. (2.20) and A, B, D is given by

$$\mu_R^2+\lambda_R X+\eta_R X^2=\frac{\Lambda_R^2}{A^2}\left(\frac{X}{\Lambda_R}-D\right)^2+B\Lambda_R^2\,. \tag{5.24}$$

For $B=D=0$, $F(u)$ in eq. (5.22) becomes a fixed point on the gaussian line of fixed points; a comparison with eq. (5.12) gives

$$B=D=0\to A^2=\left(\frac{1}{4\pi}-c\right)^2. \tag{5.25}$$

For any A^2 (the sign of A has no meaning) which is acceptable, we have a 2-dimensional manifold parametrized by B and D. When $D\to-\infty$ with A, B fixed we go to the TFP. The parametrization of eq. (5.21a) can be used even if $\bar{u}<0$, but in this case also eq. (5.18) is applicable. This region of overlap of our two coordinate patches does not intersect the critical surface. For the purpose of constructing continuum limits $\bar{u}$, and hence B, need not be constrained to positive values. Taking $D\to-\infty$ with $B=1/A^2(p-D)D$ (p and A fixed) eq. (5.22) goes into $F(u)=f_A(u)+\frac{1}{2}p$, which is a point on the trajectory connecting the HFP to the TFP. This relationship reflects the one noted in sect. 2 (see the discussion after eq. (2.26)). One cannot have $A^2<1/(4\pi)^2$ in eq. (5.22) if $B=D=0$ because of eq. (5.25) and the discussion after eq. (5.14). The field theory corresponding to $B=D=0$ is scale invariant.

In this section we have shown explicitly that all the continuum limits obtained in sect. 2 by the direct method can be gotten from our RG analysis. We are now prepared to discuss the BMB limit from both points of view and to try to identify the various sources of the phenomena exhibited by the BMB limit. This will be done in the next section.

6. The Bardeen–Moshe–Bander continuum limit

BMB exploited the special properties of the critical point P to construct a new type of continuum limit. The basic equation is (2.18) and is reproduced here for convenience:

$$\hat{V}'_{\mathrm{R}}(Z+\Lambda Y_Z(\Lambda);\Lambda)=\tfrac{1}{2}(4\pi)^2(\Lambda Y_Z(\Lambda))^2(1-16Y_Z(\Lambda)+\cdots)\,. \tag{6.1}$$

A finite renormalized V_{eff} is obtained when the dominating saddle solution of eq. (6.1), $Y_Z(\Lambda)$, has the property that both sides of the equation approach finite and nontrivial limits as $\Lambda\to\infty$ for any Z.

When $\eta_{\mathrm{R}}>\eta_{\mathrm{c}}\equiv(4\pi)^2$, and $\hat{V}'_{\mathrm{R}}$ is renormalized in the perturbation theory way, the solution with $\Lambda Y_Z(\Lambda)\to y(Z)$ which was correct for $\eta_{\mathrm{R}}<\eta_{\mathrm{c}}$, is no longer the dominating $\Lambda\to\infty$ saddle and the perturbation-theory continuum limit is lost. This is true for some range of values for λ_{R} and μ^2_{R} and in particular for $\lambda_{\mathrm{R}}=\mu^2_{\mathrm{R}}=0$. The BMB limit is obtained by taking $\eta_{\mathrm{R}}\to\eta_{\mathrm{c}}^+$ with $\lambda_{\mathrm{R}}=\mu^2_{\mathrm{R}}=0$ fixed. In the symmetric phase the value $Z=0$ in (6.1) is special since $2\hat{V}'_{\mathrm{eff}}(0)$ is the mass of the vector particles. For $\eta_{\mathrm{R}}=\eta_{\mathrm{c}}+\delta$ with $\delta>0$ small, the dominating solution to (6.1) with $Z=0$ is $Y_Z(\Lambda)=-\frac{1}{16}\delta+\mathrm{O}(\delta^2)$ giving a mass

$$M=\tfrac{1}{4}\pi\Lambda\delta(1+\mathrm{O}(\delta))\,. \tag{6.2}$$

Hence the mass can be kept finite as $\Lambda\to\infty$ by taking $\delta\sim 1/\Lambda$. When δ is finite the branch of solutions to (6.1) which gives the correct mass of eq. (6.2) will dominate also for small and positive values of Z. When Z is increased further however, the lower solution takes over and the derivative of the effective potential goes through a discontinuous jump. This can be understood with the help of the graphical analysis in sect. 3. By eq. (2.13) the evaluation of $U'_{\mathrm{eff}}(\boldsymbol{\phi}^2_{\mathrm{c}})$ is seen to be similar to the analysis of the vacuum in only that the parabola representing U'_{R} in fig. 1 has to be shifted towards the left. For any value of $\boldsymbol{\phi}^2_{\mathrm{c}}>0$ the points of types A, B and C are all in the left-hand half-plane. When $\boldsymbol{\phi}^2_{\mathrm{c}}$ increases the point A is eventually replaced by C, causing the jump. The smaller δ is, the smaller is the value of Z where the jump occurs. In the limit $\delta\to 0$ the discontinuity in the derivative occurs at the origin and is vanishing small. Therefore, in the BMB limit, the derivative of the effective potential is given, for any nonvanishing value of the classical field, by the $\eta_{\mathrm{R}}\to\eta_{\mathrm{c}}$ limit of the perturbation theory result in eq. (2.21) with $\mu^2_{\mathrm{R}}=\lambda_{\mathrm{R}}=0$:

$$\left.\frac{\mathrm{d}\hat{V}_{\mathrm{eff}}}{\mathrm{d}\boldsymbol{\phi}^2}\right|_{\boldsymbol{\phi}^2\neq 0}=\frac{(\boldsymbol{\phi}^2)^2}{2(1/\sqrt{\eta_{\mathrm{R}}}+1/\sqrt{\eta_{\mathrm{c}}})^2}\xrightarrow[\eta_{\mathrm{R}}\to\eta_{\mathrm{c}}]{}2\pi^2(\boldsymbol{\phi}^2)^2\,. \tag{6.3}$$

However, at the origin we have

$$\left.\frac{\mathrm{d}\hat{V}_{\mathrm{eff}}}{\mathrm{d}\boldsymbol{\phi}^2}\right|_{\boldsymbol{\phi}^2=0}=\tfrac{1}{2}m^2\,, \tag{6.4}$$

where m is arbitrary. The point $\boldsymbol{\phi}^2=0$ represents the vacuum in the symmetric case we are investigating. The discontinuity of the derivative of $\hat{V}_{\text{eff}}$ at the origin implies therefore that all 1PI Green functions of 4 fields or more diverge at zero momenta. This happens due to a massless O(N)-scalar excitation which leaves only the two-point function finite because of the exact O(N) symmetry.

BMB argued that the massless excitation plays the role of a dilaton. Had we picked $m^2=0$ in (6.4) the theory would be scale invariant. When $m^2 \neq 0$ this invariance is broken. BMB view this breaking both as spontaneous (and hence the dilaton) and as a reflection of dimensional transmutation (the mass has been induced by a specific dependence of a dimensionless coupling (η_R) on the cutoff Λ). Regardless of the interpretation the origin of the massless excitation is clear: it appeared when a discontinuity in the derivative of the bare effective potential was made vanishingly small and, at the same time, was pushed to the origin where the quantity in question is identifiable with the mass of the O(N)-vector particles. The points where this mass has a discontinuity that just vanished are all on the line ℓ. At the point P we are also on t and therefore both the discontinuity and the individual values can be made to vanish. If the line ℓ did not meet the region where the mass of the vector particles vanishes, a continuum limit on ℓ would represent a scalar field theories without any O(N)-vector particles. In this theory one could make the scalars massless and scale invariance would be preserved. At P we can keep the O(N)-vector particles at a finite mass even when the scalar mass vanishes and the BMB phenomenon is made possible.

Our discussion until this point was from the point of view of the direct approach to the construction of a continuum limit. It was difficult to assess to what degree the two interpretations BMB give to their phenomenon are meaningful. The RG analysis we shall present below will show that unlike in the cases discussed in the previous section there is no one-parameter RG trajectory representing the various masses of the BMB continuum limit, but rather this limit is represented by a single point, the endpoint of the gaussian line of fixed points ($c=0$ in (5.12)):

$$2U^{*\prime}(\phi^2)=g(\phi^2)\,, \tag{6.5}$$

g was defined below eq. (2.11). The various values for the mass do not come in as parametrizing possible RG flows but simply as a free parameter in the solution of the field theory, a parameter which represents the arbitrary expectation value of an order parameter related to scale invariance.

We proceed to show this explicitly by solving the theory defined by U^* in (6.5). Eq. (6.5), strictly speaking defines U^* only for $\boldsymbol{\phi}^2 \leq 1/2\pi^2$ but the definition can be extended by analyticity. $g(\phi^2)$ is the inverse of f (eq. (2.11)), which by eq. (5.13) can be written as $f=f_A-(1/4\pi)\sqrt{u}$ where $f_A(u)$ is an analytic function. We define $\tilde{g}(\phi^2)$ to coincide with $g(\phi^2)$ for $\phi^2 \leq 1/2\pi^2$ and with the inverse of $\tilde{f}=f_A+(1/4\pi)\sqrt{u}$ for $\phi^2>1/2\pi^2$. Setting $2U^{*\prime}(\phi^2)=\tilde{g}(\phi^2)$ a function is obtained which is analytic on the positive real axis but not at the origin. $U^*(\phi^2)$ is sketched in fig. 6.

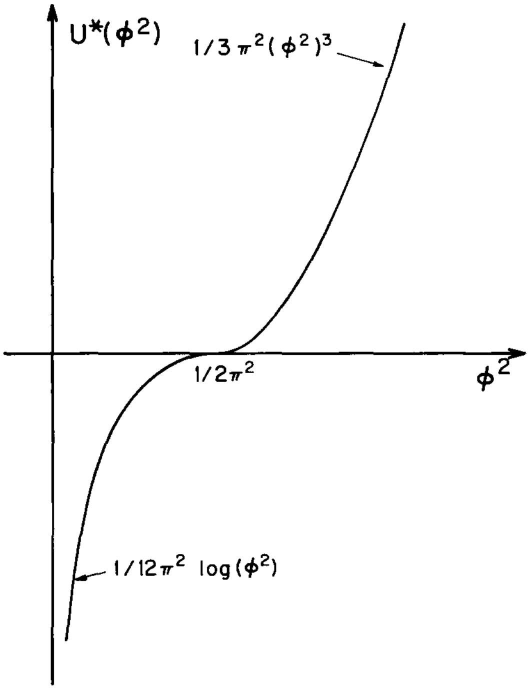

Fig. 6. Qualitative sketch of the potential corresponding to the BMB fixed point.

The system represented by U^* has, as usual, all UV divergences eliminated (i.e. momenta are bounded by unity), but is nevertheless a continuum theory. All one has to do to go to physical variables is to introduce Λ_R^{-1}, an arbitrary definition of the unit of length. We might worry that due to the divergence of $U^*(\phi^2)$, $(\underset{\phi^2\to 0}{\sim}(1/12\pi^2)\log(\phi^2))$ the theory does not exist. This divergence is however cancelled by the centrifugal repulsion residing in the $\boldsymbol{\phi}$-functional integration measure. To see this enclose the system in a volume L^3:

$$\boldsymbol{\phi}(x)=\frac{1}{L^{3/2}}\sum_k \boldsymbol{\phi}_k\cdot e^{ikx}. \tag{6.6}$$

The logarithmic piece of U^* contributes to the action the following term:

$$\frac{N}{12\pi^2}\int dx\ln\left[\frac{1}{L^3}\sum_k \boldsymbol{\phi}_k^2+\sum_{k\neq q}\boldsymbol{\phi}_k\cdot\boldsymbol{\phi}_q\,e^{i(k-q)x}\right]. \tag{6.7}$$

The integration measure is $\prod_{k^2<1} d\boldsymbol{\phi}_k$. The possible divergence of the functional integral can be studied by checking what happens when we scale $\boldsymbol{\phi}\to\lambda\boldsymbol{\phi}$ with a very small λ. From the action (see eq. (6.7)) we get a term $-(NL^3/12\pi^2)\ln\lambda^2$ and from the measure we get

$$N \sum_k \ln \lambda \approx N \frac{L^3}{2\pi^3} \int_{k^2<1} \mathrm{d}^3 k \ln \lambda = \frac{NL^3}{12\pi^2} \ln \lambda^2 , \tag{6.8}$$

and hence the cancellation.

We now go on to actually solve the model defined by U^*. From eq. (2.13) we get

$$U'_{\mathrm{eff}}(\phi_c^2) = U^{*\prime}(x+\phi_c^2) \equiv \tfrac{1}{2}\tilde{g}(x+\phi_c^2) = \tfrac{1}{2} g(x) . \tag{6.9}$$

For $\phi_c^2 = 0$ the last equation is an identity satisfied by any $0 < x \leq 1/2\pi^2$. $g(x)$, the mass, is therefore arbitrary, varying between 0 and ∞. For $\boldsymbol{\phi}_c^2 > 0$ only one solution exists in the range $0 < x \leq 1/2\pi^2$ and it satisfies $\bar{x} + \boldsymbol{\phi}_c^2 > 1/2\pi^2$. The value $\bar{g} \equiv g(\bar{x})$,

$$\tilde{f}(\bar{g}) = f(\bar{g}) + \boldsymbol{\phi}_c^2 , \tag{6.10}$$

implying $(1/2\pi)\sqrt{\bar{g}} = \boldsymbol{\phi}_c^2$ or $U'_{\mathrm{eff}}(\boldsymbol{\phi}_c^2) = 2\pi^2(\boldsymbol{\phi}_c^2)^2$ in agreement with eq. (6.3). Note that rescaling by Λ_R has no effect and therefore our model is scale invariant. Our analysis has established explicitly the equivalence between the BMB continuum limit and the theory governed by U^*.

We would now like to explicitly identify the dilaton. For that we write the path integral at the BMB FP in polar coordinates: $\boldsymbol{\phi}(x) = \rho(x)\boldsymbol{\Omega}(x)$ with $\boldsymbol{\Omega}^2(x) = 1$ and $\rho(x) > 0$.

$$\int [\mathrm{d}\boldsymbol{\Omega}] \left[\rho^N \frac{\mathrm{d}\rho}{\rho} \right] \mathrm{e}^{-N \int [\frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}\rho^2 (\partial_\mu \boldsymbol{\Omega})^2 + U^*(\rho^2)]} . \tag{6.11}$$

The variables in (6.11) are analogous to those used in sect. 4. We shall identify ρ with the dilaton field by showing that ρ_0, the vacuum expectation value of ρ, is arbitrary. The $\boldsymbol{\Omega}$-field is a Heisenberg field with an inverse temperature given by $\rho_0^2 N$. One can integrate over $\boldsymbol{\Omega}$ at infinite N using the saddle-point approximation

$$\int [\mathrm{d}\boldsymbol{\Omega}] \, \mathrm{e}^{-(1/2)\rho_0^2 N \int (\partial_\mu \boldsymbol{\Omega})^2} = \mathrm{e}^{NL^3 w(\rho_0^2)} ,$$

$$w(\rho_0^2) = \tfrac{1}{2} \int_{1/2\pi^2}^{\rho_0^2} \mathrm{d}t \, g(t) - \tfrac{1}{2} \log(\rho_0^2) . \tag{6.12}$$

In eq. (6.12) we used the fact that for $\rho(x) = \rho_0$ the sharp momentum cutoff on the $\boldsymbol{\phi}(x)$ variables applies to $\boldsymbol{\Omega}(x)$. Comparing with eq. (6.9) we see that $w(\rho_0^2)$ cancels against $U_{\mathrm{eff}}(\rho_0^2)$ and the measure term as long as $0 < \rho_0^2 \leq 1/2\pi^2$. Hence ρ_0 is undetermined in this interval. $\rho_0^2 = 1/2\pi^2$ corresponds to a massless $\boldsymbol{\Omega}(x)$ because it is the value for the large-N critical temperature of the Heisenberg model with sharp momentum cutoff. $\rho_0^2 \to 0$ obviously makes the O(N)-vector particles infinitely massive. The allowed range of ρ_0 spans all possible values for the vector mass.

Our analysis has established the appearance of a finite mass for the vector particles in the BMB continuum limit as a reflection of spontaneous scale-invariance breaking.

The dilaton associated with the breaking is the massless excitation of the field $\rho(x)$. $\rho(x)$ couples directly to $(\partial_\mu \mathbf{\Omega})^2$ as it should. Our RG analysis however implies that the mass which appears in the continuum model is not to be associated with a location along a renormalized trajectory and hence unlike BMB we do not view the phenomenon as an example of dimensional transmutation.

This then implies that the endpoint of the gaussian line of fixed points is not special from the RG viewpoint. The special phenomena occurring there result from the ability of the infinite-N theory to have a nonanalytic relationship between the interaction and the Green functions even with a finite UV cutoff present. Hence, there is in our opinion, no reason to expect these phenomena to survive to finite N this implying in turn that the question of the existence of a UV nontrivial $(\boldsymbol{\phi}^2)_3^3$ interaction at any finite N is as open as it was before the BMB discovery.

Our analysis of the BMB FP would be incomplete if we did not explain the source for the different values for the critical exponents ν obtained in sect. 3. We turn now to this issue. We feel that a discussion of the exponent α also would be superfluous. The problem is to explain why at P the relation $\nu = 1/y_0$, where y_0 is defined in eq. (5.21b), is violated. It does hold everywhere else in the critical regime.

We first review the general considerations which lead to the relationship between ν and y_0.

Any potential parametrized by $\bar{u}, b_1, b_2, \ldots$ as in eq. (5.21a) will lead to a mass m^2 (we discuss only the symmetric phase):

$$F(u) = f_A(u) + \sum_{n=1}^{\infty} b_n (u - \bar{u})^{(n-1)/2} ,$$

$$m^2 = h(\bar{u}, b_1, b_2, \ldots) . \tag{6.13}$$

Under $R_S m^2 \to s^2 m^2$ by definition and on the basis of eq. (5.21b) we get

$$m^2 = \bar{u} h\left(1, \frac{b_1}{\sqrt{\bar{u}}}, b_2, b_3\sqrt{\bar{u}}, b_4 \bar{u}, \ldots\right) . \tag{6.14}$$

Suppose now that we start with

$$2U'(\phi^2) = \mu_R^2 + \eta_R (\boldsymbol{\phi}^2)^2 ,$$

$$\Rightarrow F(u; \mu_R^2, \eta_R) = f_A(u) + \sum_{n=1}^{\infty} \bar{b}_n(\mu_R^2, \eta_R)[u - \bar{u}(\mu_R^2, \eta_R)]^{(n-1)/2} ,$$

$$\bar{u}(\mu_R^2, \eta_R) = \mu_R^2 ,$$

$$\bar{b}_n(\mu_R^2, \eta_R) = \begin{cases} 1/\sqrt{\eta_R} \text{ for } n=2 \\ 0 \text{ for } n=2k, \quad k \geq 2 \\ B_{2k+1}(\mu_R^2) \text{ for } n = 2k+1, \qquad k \geq 0 , \end{cases}$$

$$\sum_{k=0}^{\infty} B_{2k+1}(\mu_R^2) Z^k = \frac{1}{2\pi^2} - f_A(\mu_R^2 + Z) \text{ for any } Z . \tag{6.15}$$

From (6.14) we get

$$m^2=\mu_R^2 h\left(1,-\frac{1}{2\pi^2}\tan^{-1}|\mu_R|,\frac{1}{\sqrt{\eta_R}},B_3(\mu_R^2)\mu_R,\ldots\right). \tag{6.16}$$

If the function h in (6.16) has a finite limit as $\mu_R\to 0$ we obtain

$$M^2\sim\mu_R^2 h\left(1,0,\frac{1}{\sqrt{\eta_R}},0,0,\ldots\right), \tag{6.17}$$

and the expected value of ν. It is easy to compute h in (6.17): for $\bar{u}=1$, $b_2=1/\sqrt{\eta_R}$ and all other b_n's vanishing the mass is equal to the value of the function inverse to $F(u)-f(u)$ at the origin:

$$F(m^2)-f(m^2)=\frac{1}{\sqrt{\eta_c}}\sqrt{m^2}-\frac{1}{\sqrt{\eta_R}}\sqrt{m^2-1}=0\,. \tag{6.18}$$

(The minus sign had to be picked because the value 0 is attained on the lower branch.) With (6.18) we get in (6.17)

$$M^2=\mu_R^2\frac{1}{1-\eta_R/\eta_c}, \tag{6.19}$$

and we see that the formula breaks down when $\eta_R\to\eta_c$. For $\eta_R=\eta_c$ the arguments of h in (6.17) define, via (6.13), a function F whose inverse diverges as $1/(\boldsymbol{\phi}^2)^2$ at the origin. This means that when $\boldsymbol{\phi}^2\to 0$ very large values of u become important and the neglect of the small terms $b_3\sqrt{\bar{u}}$, $b_4\bar{u},\ldots$ in (6.14) is, a priori, unwarranted. A straightforward analysis shows that only the $b_3\sqrt{\bar{u}}$ term has to be kept when computing the value of h in eq. (6.14). One finds that h diverges and satisfies the following equation (at $\eta_R=\eta_c$):

$$\frac{1}{8\pi\sqrt{h}}-\frac{\mu_R}{2\pi^2}h=\mathrm{O}(\mu_R)\,,\qquad \mu_R\to 0\,.$$

$$h\sim\left(\frac{4\pi}{\mu_R}\right)^{2/3}. \tag{6.20}$$

Together with eq. (6.17) this gives $M^2\sim(\mu_R^2)^{2/3}$ in accordance with sect. 3.

Had we started with a U' with $\eta_R>\eta_c$, U'_S would have developed, at some finite S, a discontinuity at some $\phi^2>0$. The result of eq. (6.20) comes about by pushing the discontinuity to $\boldsymbol{\phi}^2=0$. It is clear that the mechanism operating here is of the same origin as the mechanism used to obtain the BMB continuum limit. This is strictly an $N=\infty$ mechanism because only at infinite N is it impossible to claim that the integration over the fields in a finite momentum shell, that is the action of R_S with a finite S, is an analytical mapping in the space of hamiltonians. The $N=\infty$ limit might, and in our case does, induce some nonanalyticity.

We expect therefore that the phenomenon would disappear at any finite N. We expect that the situation at infinite N would reflect itself in the appearance of a crossover-type of behavior, with $m^2 \sim (\mu_R^2)^{\nu_1}$ for small μ_R^2 (but not too small) going over to $m^2 \sim (\mu_R^2)^{\nu_2}$ ($\nu_2 \neq \nu_1$) when μ_R^2 becomes even smaller.

7. Summary

This paper presented a study of the various continuum limits of $(\phi^2)_3^3$ at $N=\infty$ with special emphasis on the BMB phenomenon. A complete analysis of the phase diagram in the three coupling model showed that the BMB phenomenon occurred because the tricritical line ended at the same point where a liquid gas Ising type transition did. Thus the critical regime on the $\mu_R^2 = \lambda_R = 0$ line becomes a closed segment one of its ends being the BMB point. The RG analysis showed that this segment flows pointwise into a one-dimensional set of fixed points. This set was also closed. A by-product of this analysis is an extension of the BMB phenomenon to arbitrary potentials. The fixed point corresponding to the BMB critical point is the endpoint of the above mentioned line of fixed points. This is a peculiar point because in any vicinity of it there exists an interaction on which the action of the RG transformation becomes nonanalytic after a finite amount of evolution. It was argued that both the meeting of the critical lines in the three coupling space and the above-mentioned peculiarity of the fixed point are phenomena which occur only because $N=\infty$. Because of that we conjecture that the BMB phenomenon, in the sense that it predicts a new type of fixed point, will disappear at any finite N. The expected relic of the $N=\infty$ situation is that for very large N, a regime where strong crossover phenomena between Heisenberg-like and Ising-like critical behaviors occur.

In the course of obtaining the above results we elucidated (at $N=\infty$) the difference between continuum theories which are renormalized perturbatively and those which are not.

We also initiated an investigation on the lattice. In this paper we analyzed only the strong coupling limit of the three coupling model. The resulting vacancy-spin system was used to develop an understanding of the mechanisms which cause the gross features of the phase diagram and also provided an estimate for a lower bound on N, $\bar{N}$. Only for $N \geq \bar{N}$ can one hope that the BMB phenomenon survives and even this seems to us very unlikely.

One natural question to ask is what the $1/N$ corrections are. We have not investigated this question in detail but we suspect that, whenever we are in the vicinity of the liquid-gas second-order point the $1/N$ expansion, which is essentially a mean field expansion, would break down because of the infrared divergences one expects in 3 dimensions. At the liquid-gas second-order transition point the parameter N plays a rather artificial, loop counting, role*. As the BMB phenomenon is related to a meeting of the two critical regions the question whether the BMB

* This is very different from the $O(N)$-spin second-order transition point.

phenomenon survives to finite N cannot, presumably, be answered by computing $1/N$ corrections.

A summary account of part of our findings has appeared in ref. [14].

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References

[1] K.G. Wilson and J.B. Kogut, Phys. Reports 12C (1974) 75
[2] M. Aizenman, Phys. Rev. Lett. 47 (1981) 1; Comm. Math. Phys. 86 (1982) 1;
J. Fröhlich, Nucl. Phys. B200 (1982) 281;
B. Freedman, P. Smolensky and D. Weingarten, Phys. Lett. 113B (1982) 481
[3] P.K. Townsend, Nucl. Phys. B118 (1977) 199 (see especially appendix B, p. 216); Phys. Rev. D14 (1976) 1715; D15 (1975) 2269;
T. Appelquist and U. Heinz, Phys. Rev. D25 (1982) 2620;
R.D. Pisarski, Phys. Rev. Lett. 48 (1982) 574
[4] W.A. Bardeen, M. Moshe and M. Bander, Phys. Rev. Lett. 52 (1983) 1188
[5] J. Glimm and A. Jaffe, Quantum physics, a functional integral point of view (Springer, New York, 1981)
[6] K.G. Wilson, Rev. Mod. Phys. 47 (1975) 773
[7] S. Ma, Rev. Mod. Phys. 45 (1973) 589; Phys. Lett. 43A (1973) 475
[8] S. Coleman, R. Jackiw and H.D. Politzer, Phys. Rev. D10 (1974) 2491
[9] K.G. Wilson, Phys. Rev. D6 (1972) 419
[10] S. Coleman, in Pointlike structures inside and outside hadrons, ed. A. Zichichi (Plenum, New York, 1981);
V.J. Emery, Phys. Rev. B11 (1974) 239;
M. Halpern, Nucl. Phys. B173 (1980) 504
[11] S. Sarbach and T. Schneider, Phys. Rev. B16 (1977) 347
[12] S. Ma, in Phase transitions and critical phenomena, vol. 6, ed. Domb and Green (Academic Press, London, 1976)
[13] F.J. Wegner, in Phase transitions and critical phenomena, vol. 6, ed. Domb and Green (Academic Press, London, 1976)
[14] F. David, D.A. Kessler and H. Neuberger, Phys. Rev. Lett. 53 (1984) 2071