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A MODEL OF RANDOM SURFACES WITH NON-TRIVIAL CRITICAL BEHAVIOUR

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We study a model of planar random surfaces based on gaussian imbedding of simplicial lattices in a D-dimensional space. The model is shown to be equivalent to a planar ϕ^3 theory with exponential renormalized propagator for any dimension D. Scaling laws are derived and estimates for the exponents γ and ν by strong coupling expansions are obtained. The results differ from the predictions of mean field theory. γ depends on D and is probably zero for $D = 4$. The Hausdorff dimension is large (greater than four) but finite. The correlation length diverges at the critical point and the two-point function does not correspond to a free field theory. In general hyperscaling is not satisfied in such models.

1. Introduction

Models of random surfaces have raised great interest in the last few years, and should play an important role in many areas of theoretical physics: interface physics, lattice gauge theories, large-N limit of gauge theories, strings models, quantum gravity, etc. A lot of models have already been proposed and studied, but it is not clear, at least to the author, whether there is one, or many (possibly an infinite number) universal classes of random surfaces, what their critical properties are, and what their respective relevance to the problems mentioned above is.

Models of planar non-interacting surfaces made of plaquettes of an underlying hypercubic lattice in D-dimensional euclidean space [1–7] have an Hausdorff dimension d_H equal to four (for any $D \geqslant 2$) and correspond to a system of branched polymers [8] (see also the models of [9]). On the other hand, other surface models predict an infinite Hausdorff dimension: this is the case for the continuum model of [10] (at least for $D = +\infty$), for the Polyakov string model [11, 12], and for the models of [13, 14]. Most of these results have been obtained either by mean-field-like arguments, by computer simulations, or by combinations of rigorous analytic techniques and of numerical studies. The purpose of this paper is to study a new model of planar random surfaces, which has recently been proposed in [7, 15]. This

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model, which will be properly described in sect. 2, is obtained by considering all possible triangular lattices (with the topology of the sphere S_2) and by imbedding these lattices in D-dimensional euclidean space E_D with the simplest weighting factor: a gaussian one for each link of the lattice. This is a natural generalization in two dimensions of the gaussian chain model for polymers and, in some sense, a discrete version of the Polyakov string model. This model bears also a close similarity to the models for gauge theories introduced in [16] (see also [17]). As we shall see, this model has very interesting features and exhibits a non-trivial (i.e. different from mean field theory) critical behaviour.

In sect. 2, we introduce our model, define the partition function, the N-point functions, and the critical indices.

In sect. 3, we present some (rather elementary) analytic results. First it can be shown that our model is equivalent in any dimension D to a planar ϕ^3 theory with a (renormalized) exponential propagator, and therefore that it is also closely related to planar scalar field theories (this relationship was already shown for $D = 0$ in [15]). Second, we show that if the correlation length remains finite at the critical point, then the Hausdorff dimension d_H of the surface is infinite and its mean square extent grows like the logarithm of its area. On the contrary if the correlation length diverges, assuming that standard scaling holds, we derive scaling relations between the critical indices γ, ν, η (to be defined later) and the Hausdorff dimension d_H.

Sect. 4 is devoted to numerical studies of the model by techniques of strong coupling expansions. We first describe how the strong coupling series are constructed and what the observables considered are. Then we analyse the series up to order 8 for the mean area and the mean-square extent of the surface and give estimates for the corresponding exponents γ and ν for various values of the bulk dimension D. A ratio method and differential approximants have been used. Our results show, provided that usual scaling holds, that the correlation length diverges and are different from the predictions of mean field theory.

Finally in sect. 6 we discuss open problems and draw conclusions from our study.

2. The model

2.1. DEFINITION OF THE MODEL

Our model consists in imbedding, with a certain weight, in the D-dimensional euclidean space E_D all possible simplicial lattices (i.e. made of triangles glued along their edges) with the topology of the 2-sphere S_2 (i.e. an orientable surface with genus $g = 0$). Although one can consider different topologies, in this paper we shall consider only such planar closed surfaces. Problems associated with the inclusion of other topologies will be discussed later. More rigorously, we use the language of [7] and define a lattice S as an equivalent class of triangulations of S_2, i.e. the set of all triangulations of S_2 which are equivalent by isomorphism of S_2. In the following we shall denote

$$\mathrm{v(S)} = \{\text{set of vertices of S}\}$$

$$\ell(\mathrm{S}) = \{\text{set of lines of S}\}$$

$$\mathrm{t(S)} = \{\text{set of triangles of S}\}. \tag{2.1}$$

In general, such a lattice S may have a non-trivial group of symmetry g_S, the group of permutations of lines and vertices of S which leaves S invariant, i.e. which does not change the class of the corresponding triangulations, and preserves the planar structure of S. We will denote

$$C(\mathrm{S}) = \text{order of } g_S. \tag{2.2}$$

In order to exclude some singular configurations, we shall introduce some restrictions on the lattices S. We shall call an l-loop a set of l distinct lines $\{a_1, \ldots, a_l\}$ in $\ell(\mathrm{S})$ which form a closed loop in S. Then we consider

$$\mathbb{S}_0 = \{\text{lattices S (with the topology of } S_2)\}$$

$$\mathbb{S}_1 = \{\text{lattices S with no 1 loops}\}$$

$$\mathbb{S}_2 = \{\text{lattices S with no 1 and 2 loops}\}^{\star}. \tag{2.3}$$

We now define an intrinsic metric on each lattice S by assigning the same length ($=1$) to each line of S, that is by considering the triangles to be equilateral. Therefore to each vertex $i \in \mathrm{v(S)}$ is associated an element of area σ_i (the discrete analog of $\sqrt{g}$):

$$\sigma_i = \tfrac{1}{3} N_i, \tag{2.4}$$

where N_i is the number of triangles (or of lines) which meet at the vertex i. The total intrinsic area $|S|$ of S ($\int_S \sqrt{g}$) is

$$|S| = \sum_{i \in \mathrm{v(S)}} \sigma_i = \text{number of triangles in S}. \tag{2.5}$$

The intrinsic curvature is concentrated at the vertices, and is equal to the deficit angle. The curvature R_i at vertex i is therefore defined as

$$R_i = \pi \frac{(6 - N_i)}{N_i}. \tag{2.6}$$

⋆ $\mathbb{S}_2$ is analogous to the notion of skeleton surfaces of order 2 defined in [7].

The total curvature $|R|$ ($\int_S \sqrt{g}\, R$) is from Euler formula

$$|R| = \sum_{v(S)} \sigma_i R_i = 4\pi . \tag{2.7}$$

From (2.4) and (2.6) the total curvature squared ($\int \sqrt{g}\, R^2$) is

$$|R^2| = \sum_{v(S)} \sigma_i R_i^2 = 3\pi^2 \sum_i \frac{(6-N_i)^2}{N_i} . \tag{2.8}$$

In a second step we now imbed the lattice S in E_D. We choose the simplest imbedding by assigning a gaussian weight to each line of S. More precisely, we assign a position X_i to each vertex i of S and define an action $A(S, X)$ as

$$A(S, X) = \sum_{\substack{\text{lines } (i,j) \\ \in \ell(S)}} (X_i - X_j)^2 . \tag{2.9}$$

In order to integrate over all possible imbeddings we have to define a measure $d\mu(X)$, which may a priori depend on the surface S. In a discrete analogy with the continuum functional measure for a scalar field on a curved two-dimensional space, which is the one which allows one to recover the correct conformal anomaly [18], $D[\phi] = \prod_x d[|g(x)|^{1/4}\phi(x)]$, we choose

$$d\mu_i(X_i) = \prod_{\mu=1}^{D} d\left(\sigma_i^{1/2} X_i^{\mu}\right) = \sigma_i^{D/2}\, d^D(X) . \tag{2.10}$$

Physically this choice of measure means that we assign a large weight in the functional integral to vertices i with large volume element σ_i on S.

The action (2.9) has a zero mode, associated to translation invariance. It is eliminated by fixing the position of the center of gravity of the surface:

$$X_G = \frac{1}{|S|} \sum_{v(S)} \sigma_i X_i . \tag{2.11}$$

With all the previous definitions the partition function z of our model is

$$z(\beta, \alpha) = \sum_{S \in \mathbb{S}_2} \frac{1}{C(S)}\, e^{-(\beta|S| + \alpha|R^2|)} \int \prod_{v(S)} d\mu(X_i)\, \delta(X_G)\, e^{-A(S, X)} . \tag{2.12}$$

This action has two terms. A term depending on the intrinsic geometry of the surface $\beta|S| + \alpha|R^2|$ and a term depending on the imbedding $A(S, X)$. The coupling constants β and α play the role of a bare cosmological constant and of a higher-derivative coupling respectively. The symmetry factor $1/C(\mathrm{S})$ has to be introduced because in general there are many different ways of constructing a given simplex by gluing indiscernable triangles [15], or equivalently, because riemannian metrics with isometries are singular points in the space of all possible metrics on a given manifold.

In some sense, this model is a discrete version of the Polyakov string model, or of two-dimensional gravity coupled to a massless free scalar field, which is here the D-components field $\boldsymbol{X}$. The coupling constant α has the dimension $[\text{mass}]^{-2}$ and is expected to be irrelevant. Indeed as we shall see the critical properties of the model do not seem to depend on α, which has been introduced in the action for numerical reasons and to test the dependence of our results in the exact form of the action. The relevant variable is of course β, which has the dimension of a $[\text{mass}]^2$. As we shall see, there are analytic and numerical evidences that the partition function (2.12) exists for β large enough and becomes singular at a critical coupling β_c.

2.2. CORRELATION FUNCTIONS

Similarly to strings models and other surface models, we define the N-point function $G_N(x_1, x_N)$ by restricting the sum (2.12) to surfaces whose some of the vertices coincide with the points $x_1 \dots x_N$: namely

$$G_N(\boldsymbol{X}_1 \dots \boldsymbol{X}_N) = \sum_{\mathrm{S}} \frac{1}{C(\mathrm{S})}\, \mathrm{e}^{(\beta|S| + \alpha|R^2|)} \times \int \prod_{\mathrm{v(S)}} \mathrm{d}\mu\,(\boldsymbol{X}_i)\, \mathrm{e}^{-A(\mathrm{S},\boldsymbol{X})} \prod_{a=1}^{N} \left[\sum_{j \in \mathrm{v(S)}} \sigma_j \delta(\boldsymbol{X}_j - \boldsymbol{X}_a) \right]. \tag{2.13}$$

Of particular interest is the 1-point function, which corresponds to the partition function of a surface attached to one point $\boldsymbol{X}$ in E_D, which is by translation invariance independent of $\boldsymbol{X}$, and is equal to

$$G_1(\boldsymbol{X}) = -\frac{\partial}{\partial\beta} z(\beta, \alpha), \tag{2.14}$$

and the two-point function $G_2(\boldsymbol{X}_1, \boldsymbol{X}_2) = G_2(\boldsymbol{X}_1 - \boldsymbol{X}_2)$.

Since the integration over $\boldsymbol{X}$ is gaussian in (2.12) and (2.13), it is of course easy to integrate explicitly over the $\boldsymbol{X}$'s. The action (2.6) may be written as

$$A(\mathrm{S}, \boldsymbol{X}) = \sum_{i,\, j \in \mathrm{v(S)}} \boldsymbol{X}_i D_{ij} \boldsymbol{X}_j, \tag{2.15}$$

with

$$D_{ij} = \begin{cases} -1 & \text{if } i \neq j,\ i, j \text{ belong to some line } \ell \text{ of S} \\ N_i = 3\sigma_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

Then we have

$$\int \prod_{\mathrm{v(S)}} \mathrm{d}\mu\,(X_i)\,\mathrm{e}^{-A(\mathrm{S},X)}\delta(X_{\mathrm{G}}) = \left[\pi^{1-L}\frac{\det_{L-1}D}{\prod_{\mathrm{v(S)}}\sigma_i}\right]^{-D/2}, \tag{2.17}$$

where $\det_{L-1}D$ means the determinant of any of the minors of D, i.e. of the submatrix $(L-1)\times(L-1)$ of D obtained by removing the line i and the column i of D for any $i \in \{1, L\}$, where L is the number of lines of S. The fact that $\det_{L-1}D$ is independent of the line i removed is a trivial consequence of the existence of the zero mode. The matrix D is related to the matrix elements of the scalar laplacian $-\Delta_{\mathrm{S}}$ on the lattice S by the relation

$$(-\Delta_{\mathrm{S}})_{ij} = \sigma_i^{-1/2}D_{ij}\sigma_j^{-1/2}, \tag{2.18}$$

and one can check the relation

$$\frac{\det_{n-1}D}{\prod_{\mathrm{v(S)}}\sigma_i} = \det{}'(-\Delta_{\mathrm{S}})\frac{1}{|S|}, \tag{2.19}$$

where $\det'(-\Delta)$ is the product of the $(L-1)$ non-zero eigenvalues of $-\Delta$. Let us note the additional factor $1/|S|$ where $|S|$ is given by (2.5), which is the contribution of the zero mode of the laplacian [14]. Therefore we have

$$z(\beta,\alpha) = \sum_{\mathrm{S}}\frac{1}{C(\mathrm{S})}\,\mathrm{e}^{-(\beta|S|+\alpha|R^2|)}\left[\pi^{S/2+1}\frac{\det'(-\Delta_{\mathrm{S}})}{|S|}\right]^{-D/2}. \tag{2.20}$$

For the correlation functions it is better to consider the Fourier transform

$$\tilde{G}_N(\boldsymbol{P}_1\ldots\boldsymbol{P}_N) = \int \mathrm{d}^D X_a \mathrm{e}^{iX_aP_a}G_N(X_1\ldots X_N). \tag{2.21}$$

One obtains for each lattice S a sum over the functions from $\{1,\ldots,N\}$ in the set of vertices of S, $\{a \to i(a);\, a=1, N\}$:

$$\tilde{G}_N(P_1\ldots P_N) = (2\pi)^D\delta^D\Big(\sum_a \boldsymbol{P}_a\Big)\sum_{\mathrm{S}}\frac{1}{C(\mathrm{S})}\,\mathrm{e}^{-(\beta|S|+\alpha|R^2|)}$$
$$\times \sum_{\substack{i\\ a\to i(a)}}\left[\pi^{|S|/2+1}\frac{\det'(-\Delta_{\mathrm{S}})}{|S|}\right]^{-D/2}\mathrm{e}^{-\boldsymbol{P}_i\Sigma_{ij}\boldsymbol{P}_j}\left[\prod_a \sigma_{i(a)}\right], \tag{2.22}$$

where the D-dimensional vector $\boldsymbol{P}_i$ is defined as

$$\boldsymbol{P}_i = \sum_a \delta_{i,i(a)} \cdot \boldsymbol{P}_a , \tag{2.23}$$

and where the matrix elements Σ_{ij} of the matrix Σ are given by

$$\Sigma_{ij} = \tfrac{1}{4} \left[\left(D_{lm} + \frac{\sigma_l \sigma_m}{\sum_k \sigma_k} \right)^{-1}_{ij} - \frac{1}{\sum_k \sigma_k} \right] . \tag{2.24}$$

Σ is the inverse of -4Δ in the subspace generated by the eigenvectors of $-\Delta$ with non-zero eigenvalue.

From (2.20) and (2.22) one sees that the bulk space dimension D may easily be extended from positive integer values to any real or complex value.

2.3. CRITICAL BEHAVIOUR AND CRITICAL INDICES

Since the total area $|S|$ of a lattice is always an even integer $|S| = 2n$, the partition function $z(\beta)$ and the correlation functions may be written as a power series in the variable $e^{-2\beta} = v$:

$$z(\beta) = \sum_n v^n a_n(\alpha, D) . \tag{2.25}$$

In our case the series starts at $n = 2$, which corresponds to the tetrahedron lattice. As already mentioned, the series (2.25) is expected to have a finite radius of convergence, and therefore to define an analytic function of β in the half-plane $\mathrm{Re}\,\beta > \beta_c$. At the critical coupling β_c the partition function and the correlation functions become singular, and one is interested in their critical behaviour as $\beta \to \beta_c$.

In analogy with other models, we shall consider the susceptibility χ defined as

$$\chi = \int d^D X G_2(X) = -\frac{\partial}{\partial\beta} G_1 = \frac{\partial^2}{\partial\beta^2} z(\beta) . \tag{2.26}$$

It is related to the mean area of a surface fixed to some point X_0 by one of its vertices, $\langle |S| \rangle$, by

$$|\langle S \rangle| = -\frac{\partial}{\partial\beta} \ln G_1 = \frac{\chi}{G_1} . \tag{2.27}$$

The exponent γ of χ is related to the large-order behaviour of the series (2.25), which is expected to be of the form

$$a_n \underset{n\to\infty}{\sim} B \cdot A^n n^{\gamma - 2} , \tag{2.28}$$

where $A = \mathrm{e}^{-2\beta_c}$. Then the singular part of χ at β_c behaves as

$$\chi(\beta) \sim (\beta - \beta_c)^{-\gamma}. \tag{2.29}$$

In particular we see that if γ is positive the mean area $\langle |S| \rangle$ diverges at the critical point and if γ is negative the mean area remains finite at β_c.

Other critical indices are associated to the two-point function which is expected to have an exponential decay at large distances:

$$G_2(X) \underset{X \to \infty}{\sim} \mathrm{e}^{-m(\beta)|X|}. \tag{2.30}$$

At the critical point, the mass gap $m(\beta)$ may or may not vanish. If it vanishes, the exponent ν is defined as usual by

$$m(\beta) \underset{\beta \to \beta_c}{\sim} (\beta - \beta_c)^{\nu}, \tag{2.31}$$

and for β close to β_c, so that $m \ll 1$, one expects that the two-point function has a scaling form for $1 \ll |X|$:

$$G_2(X) \sim |X|^{2-D-\eta} \mathrm{e}^{-|X| m(\beta)}, \tag{2.32}$$

which defines the "anomalous dimension" η.

Another observable is the mean-square extent (or gyration ratio) of a surface S, $\overline{X_S^2}$. Given a surface S it is defined as the average distance between two vertices of S:

$$\overline{X_S^2} = \frac{1}{|S|^2} \sum_{i,j \in \mathrm{v(S)}} \sigma_i \sigma_j \overline{(X_i - X_j)^2}$$
$$= \frac{1}{|S|^2} \frac{\int \mathrm{d}\mu(X) \mathrm{e}^{-A(\mathrm{S},X)} \delta(X_G) \left[\sum_{i,j} \sigma_i \sigma_j (X_i - X_j)^2 \right]}{\int \mathrm{d}\mu(X) \mathrm{e}^{-A(\mathrm{S},X)} \delta(X_G)}, \tag{2.33}$$

where the bar $(\overline{\ \cdot\ })$ means the average over the imbeddings of S with the weight (2.12). We consider the generating function $z_{\overline{X}^2}(\beta)$ defined by

$$z_{\overline{X}^2}(\beta) = \sum_{\mathrm{S}} \frac{1}{C(\mathrm{S})} \mathrm{e}^{-\beta|S| + \alpha|R^2|} \left[\frac{\det' - \Delta_{\mathrm{S}}}{|S|} \right]^{-D/2} (\overline{X_S^2}). \tag{2.34}$$

It is expected to have a series expansion in v of the form

$$z_{\overline{X}^2}(\beta) = \sum_n v^n b_n, \qquad b_n \sim B' A^n n^{\gamma' - 2}. \tag{2.35}$$

Let us consider the quantity b_n/a_n, where a_n is the term of order n of the series defining $z(\beta)$. This is nothing other than the average mean-square extent of surfaces S with *fixed area* $|S|$:

$$\bar{X}^2(|S|) = \frac{b_n}{a_n}, \qquad 2n = |S|. \tag{2.36}$$

The Hausdorff dimension d_{H} of the surface is defined by the large $|S|$ behaviour of this quantity, namely

$$\bar{X}^2(|S|) \underset{|S| \to \infty}{\sim} |S|^{2/d_{\mathrm{H}}}, \tag{2.37}$$

and is another critical quantity. We obviously have

$$d_{\mathrm{H}} = \frac{2}{\gamma' - \gamma}. \tag{2.38}$$

3. Some analytic results

3.1. RELATIONSHIP WITH SOME PLANAR FIELD THEORIES

There is a deep relationship between the models defined in sect. 2 and some planar field theories. This fact has already been noticed in zero dimensions ($D = 0$) in [15].

Let S be some simplicial lattice as defined in subsect. 2.1 and $\tilde{\mathrm{S}}$ be the dual lattice of S, i.e. the lattice obtained by associating a vertex $\tilde{i}$ to each triangle t of S and connecting by a line $\tilde{\ell}$ the vertices corresponding to adjacent triangles. Each $\tilde{\mathrm{S}}$ is a graph of a planar ϕ^3 theory and it was shown in [15] that in zero dimensions the contribution of S in (2.12), which is simply $1/C(\mathrm{S})$, is precisely the contribution of the graph $\tilde{\mathrm{S}}$ in the expansion of the vacuum energy of the ϕ^3 planar theory. More precisely, a lattice S belonging to $\mathbb{S}_0$, $\mathbb{S}_1$ or $\mathbb{S}_2$ respectively is unequivocally associated to a general diagram of a ϕ^3 theory, to a diagram without tadpoles insertions, or to a diagram without tadpoles and self-energy insertions respectively (see fig. 1).

We show here that, up to some probably irrelevant terms (associated to the $\alpha|R^2|$ terms and the measure in (2.10)), this equivalence holds for any dimension D. The surface models defined in sect. 2 are equivalent to a planar ϕ^3 theory with an exponential propagator. The proof goes as follows. Let us consider the diagrams generated by the following action:

$$S = \int \mathrm{d}^D x \, \mathrm{Tr}(\phi \, \mathrm{e}^{-\Delta} \phi) + \frac{g}{\sqrt{N}} \mathrm{Tr}(\phi^3), \tag{3.1}$$

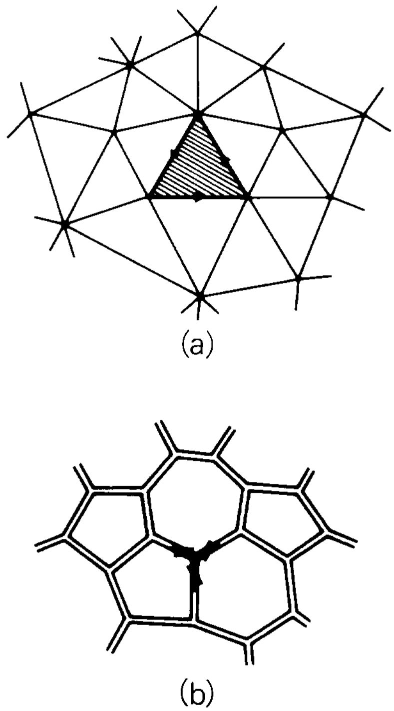

Fig. 1. A planar lattice S (a) and the corresponding dual lattice (b).

where $\phi(x)$ is an $N \times N$ hermitian matrix. In the limit $N \to \infty$, they correspond to the diagrams $\tilde{S}$ dual to the lattices S considered in sect. 2. Since the propagator is exponential

$$\tilde{D}(X-Y) = e^{-(X-Y)^2}, \tag{3.2}$$

the corresponding Feynman integral $I_{\tilde{S}}$ is convergent for any dimension D and is related to the determinant of any $(\tilde{n}-1)\times(\tilde{n}-1)$ minor of the matrix D defined by (2.16):

$$I_{\tilde{S}} = [\det_{\tilde{n}-1} D_{\tilde{S}}]^{-D/2}. \tag{3.3}$$

$\tilde{n} = |S|$ is the number of vertices of $\tilde{S}$. Now we use the well-known relation giving the determinant of minors of order 1 of the connection matrix D_G of any graph G, defined by (2.16)

$$\det_{n-1}(D_G) = \text{number of trees on G}, \tag{3.4}$$

where a tree T is a connected set of lines in G connecting all the vertices of the graph

G and without any internal loops [19]. Now, given some tree T on the lattice S, its complementary $\tilde{T}$, i.e. the set of lines which joins the centers of adjacent triangles of S and which does not cross a line of T, is a tree of the dual lattice $\tilde{S}$. Indeed, if $\tilde{T}$ is disconnected then T has internal loops and if $\tilde{T}$ has internal loops, T is disconnected*. Hence the number of trees on $\tilde{S}$ is equal to the number of trees on S and therefore the amplitude associated to the graph $\tilde{S}$ is equal to the amplitude of S.

Thus the only difference between the planar theory (3.1) and the surface model of sect. 2 lays in the additional term

$$\prod_{i \in v(S)} \sigma_i^{D/2} \times e^{-\alpha |R^2|}. \tag{3.5}$$

This term can be expanded in terms of the local curvature R_i given by (2.6)

$$\prod_{i \in v(S)} \exp\left\{ \sigma_i \left[\tfrac{1}{4} D \log 2 + \frac{D(\log 2 - 1)}{4\pi} R_i - \left(\frac{D}{8\pi^2} + \alpha \right) R_i^2 \right] + \dots \right\}. \tag{3.6}$$

The first term is a simple shift in β, the second one is a topological invariant and has no effect, the other ones are proportional to $|R^2|$, $|R^3|, \dots$ and are expected not to change the critical behaviour at the transition.

Hence we expect that in any dimension D, the surface model of sect. 2, defined from the set $\mathbb{S}_0$, is equivalent, at the transition point β_c, to a planar ϕ^3 theory regularized with an exponential bare propagator, and that the surface model defined from the set $\mathbb{S}_2$ is equivalent to a planar ϕ^3 theory with an exponential *renormalized* propagator.

3.2. SCALING RELATIONS

We now derive standard scaling relations in our model. Let us consider the two-point function $G_2(X)$. Its Fourier transform $\tilde{G}_2(p)$ is given by (2.22)–(2.24)

$$\tilde{G}_2(p) = \sum_{S \in \mathbb{S}_2} \frac{1}{C(S)} e^{-\beta |S|} a_S \sum_{(i,j) \in v(S)} \bar{\sigma}_i \sigma_j e^{-p^2 \bar{\Sigma}_{ij}} \tag{3.7}$$

where $\bar{\Sigma}_{ij}$ is defined by $-\Sigma_{ij}$ if $i \neq j$ and is set to be zero if $i = j$. a_S is the contribution of S to the partition function:

$$a_S = e^{-\alpha |R^2|} \left(\pi^{|S|/2+1} \frac{\det'(-\Delta_S)}{|S|} \right)^{-D/2}. \tag{3.8}$$

All the $\bar{\Sigma}_{ij}$ are $\geqslant 0$ and therefore, provided that $\operatorname{Re} p^2 > 0$,

$$G_1 = \tilde{G}_2(+\infty) \leqslant \tilde{G}_2(p^2) \leqslant \tilde{G}_2(0) = \chi. \tag{3.9}$$

* This holds only if S is planar.

For any euclidean momentum $\boldsymbol{p}$ the two-point function has the same radius of convergence $v_c = e^{-2\beta_c}$ as the partition function (incidentally this shows that there are no tachyons as long as $\beta > \beta_c$). The terms of the expansion of $\tilde{G}_2(p^2)$,

$$\tilde{G}_2(p^2) = \sum_{n=2}^{\infty} v^n a_n(p^2), \tag{3.10}$$

are expected to behave for large n as

$$a_n(p^2) \underset{n\to\infty}{\simeq} B(p^2) A^n n^{\gamma(p^2)}(1 + \ldots), \tag{3.11}$$

where $A = e^{2\beta_c}$ does not depend on p^2. Since $\tilde{G}_2(0) = \chi$,

$$\gamma(0) = \gamma, \qquad \gamma(+\infty) = \gamma - 1. \tag{3.12}$$

For $p^2 < 0$ the radius of convergence of (2.10), $v_c(p^2)$, may be smaller than v_c, and must decrease with p^2. Hence for v fixed $\tilde{G}_2(p^2)$ is analytic in p^2 as long as $p^2 > p_c^2$, being given by the equation $v_c(p_c^2) = v$. Therefore p_c^2 is nothing other than minus the square of the mass gap m of the model (see fig. 2):

$$v = v_c(p^2) \Leftrightarrow m^2(v) = -p^2. \tag{3.13}$$

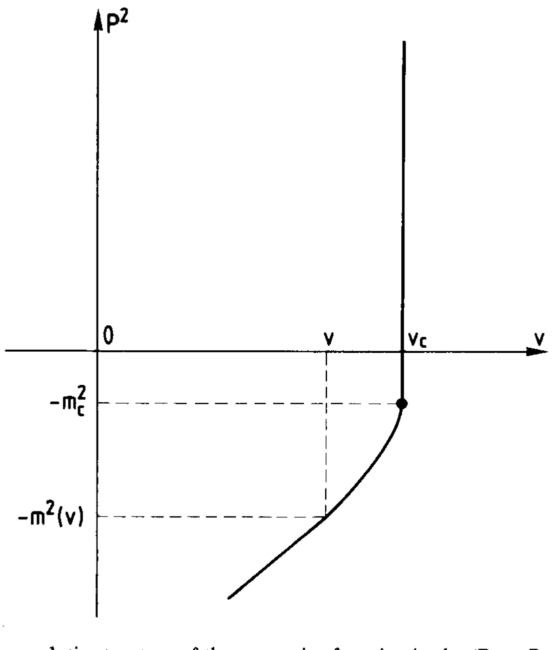

Fig. 2. The analytic structure of the two-point function in the (Re v, Re p^2) plane.

We now consider the behaviour of G_2 as $\beta \to \beta_c$. The mass gap $m(v)$ decreases as β increases and goes to a critical value m_c at β_c. There are two possibilities.

(i) $m_c > 0$

The correlation length remains finite at the transition. Then the radius of convergence of $\tilde{G}_2(p^2)$ remains constant for $-m_c^2 < p^2 < 0$ and the large-order behaviour (3.11) still holds. The generic situation is that the index $\gamma(p^2)$ depends smoothly on p^2 (unless there is some cross-over between γ and a subdominant index). Then the general term of the series $\partial\tilde{G}_2(p)/\partial p^2$ behaves as

$$\frac{\partial}{\partial p^2} a_n = A^n B n^\gamma \frac{\partial}{\partial p^2} \gamma \ln n, \tag{3.14}$$

but $\partial\tilde{G}_2(p^2)/\partial p^2|_{p=0}$ is simply related to the generating function $z_{\overline{X}^2}(\beta)$ (2.34) by

$$-2D \frac{\partial}{\partial p^2} \tilde{G}_2(p^2)\Big|_{p=0} = \sum_{S} \frac{1}{C(S)} e^{-\beta|S|} a_S |S|^2 \overline{X}_S^2 = \frac{\partial}{\partial \beta^2} z_{\overline{X}^2}(\beta). \tag{3.15}$$

Hence the mean-square extent of a surface with fixed area (2.36) behaves for large $|S|$ as

$$\overline{X}^2(|S|) \underset{|S|\to\infty}{\simeq} -2D \frac{\partial}{\partial p^2} \gamma(p^2)\Big|_{p=0} \ln |S|. \tag{3.16}$$

Such a logarithmic behaviour indicates that the Hausdorff dimension of the surface is infinite and has already been obtained in other surface models [13, 14].

(ii) $m_c = 0$

The correlation length diverges at β_c and we have real critical behaviour. If we assume that the two-point function has the scaling form (2.32) close to the critical point, then the singular part of $\tilde{G}_2(0)$ is given by the large-x behaviour of $G_2(x)$:

$$\begin{aligned} \tilde{G}_2(0) &= \int d^D x\, G_2(x) \\ &\sim \int^{\infty} d\rho\, \rho^{1-\eta} e^{-\rho m(\beta)} \\ &\sim |m(\beta)|^{\eta-2} \sim (\beta - \beta_c)^{\nu(\eta-2)}. \end{aligned} \tag{3.17}$$

Since $\tilde{G}_2(0) = \chi$ we obtain the standard scaling relation

$$\gamma = \nu(2-\eta). \tag{3.18}$$

We may repeat the same argument for

$$\frac{\partial}{\partial p^2}\tilde{G}_2(0) = \int \mathrm{d}^D x\, |x^2| G_2(x)$$

$$\sim (\beta - \beta_c)^{\nu(\eta - 4)} \sim (\beta - \beta_c)^{-\gamma - 2\nu}. \tag{3.19}$$

But from (3.15) and (2.35) it should behave also as $(\beta - \beta_c)^{-\gamma'}$ and from (2.39) we get the relation

$$\nu = \frac{1}{d_H}, \tag{3.20}$$

which relates ν to the Hausdorff dimension d_H as defined from the mean square extent, and shows that for our model this definition coincides with the definition of [6, 7].

4. Strong coupling analysis

4.1. CONSTRUCTION OF THE SERIES

We now present an analysis of a strong coupling series of the model constructed above. In the following we shall *always* deal with the model defined with lattices of the set $\mathbb{S}_2$ (2.3), i.e. lattices without 1 or 2 loops. This restriction is only for practical and numerical reasons; results of other models will be presented elsewhere.

As we have seen, the observables of the model are defined as a series in the parameter $v = e^{-2\beta}$, the term of order v^n being a sum over surfaces S with area $|S| = 2n$. In order to construct the first terms of those series, we have first to enumerate all surfaces with fixed area and then to compute the corresponding observables, which are obtained (for the partition function and the N-point functions) from the computation of the determinant of the laplacian Δ_S and of the matrix elements of its inverse Σ.

Although the surfaces S of $\mathbb{S}_2$ may be constructed directly [16], we have found it more convenient and efficient to construct first their duals $\tilde{S}$, which are, as seen in subsect. 3.1, Feynman diagrams of a ϕ^3 theory (with no tadpole and self-energy insertions). Indeed these diagrams may be enumerated by simply using the Schwinger-Dyson equations of the planar ϕ^3 theory, which take a simple form [20].

Rather than considering connected vacuum graphs (which generate the partition function), we will generate the graphs of the three-point irreducible function $\Gamma_3(X_1, X_2, X_3)$ of the theory. Indeed, since tadpoles and self-energy insertions are forbidden, the derivative of the vacuum energy $E_0(g)$ of the planar ϕ^3 theory with respect to the coupling constant, which generates the diagram dual to $-\partial z(\beta)/\partial\beta$

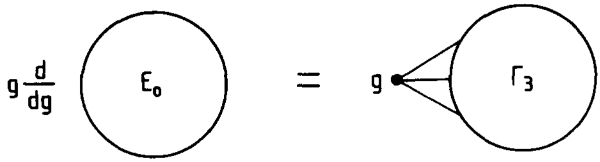

Fig. 3. Diagrammatic representation of eq. (4.1).

(i.e. the diagrams duals to $z(\beta)$ with a weight $|S|/C(\mathrm{S})$), is related to Γ_3 by

$$g\frac{\mathrm{d}}{\mathrm{d}g}E_0(g)=g\int \mathrm{d}^D X_1 \mathrm{d}^D X_2 \mathrm{d}^D X_3\, D(X_1)D(X_2)D(X_3)\Gamma_3(X_1,X_2,X_3)\,, \tag{4.1}$$

where D is the propagator (here $D(X)\sim \mathrm{e}^{-X^2}$) (see fig. 3). From the three-point irreducible function we generate the diagrams of $-\partial z(\beta)/\partial\beta$, with the correct symmetry factor.

The three-point irreducible function is constructed recursively by applying the Schwinger-Dyson (SD) equation, and can be constructed at order l from the p-point irreducible functions at order $l'<l$. The SD equations are easily written in a graphical way. Any planar p-point irreducible graph G with l loops may be decomposed in a unique way* into a chain of planar irreducible graphs $\mathrm{G}_1\dots\mathrm{G}_n$ with respectively $p_1\dots p_n$ external legs and $l_1\dots l_n$ internal loops with the constraints

$$\begin{gathered}p_1+\cdots+p_n=p+2n-1\,,\\ l_1+\cdots+l_n=l-1\,,\\ p_i\geqslant 3\,,\qquad l_i\geqslant 0\,,\qquad n\geqslant 1\end{gathered} \tag{4.2}$$

(see fig. 4). Iterating this decomposition up to $l=0$, where only the vertex with $p=3$ survives, we can construct in this way all planar p-point irreducible diagrams. Moreover, the symmetry factors are automatically obtained in that way, since a diagram with a factor $|S|/C(\mathrm{S})$ (which is always an integer) is constructed $|S|/C(\mathrm{S})$ times.

We have written a computer code which constructs all irreducible diagrams of the ϕ^3 theory, i.e. all possible "trees" of partitions of the form (4.2) (see fig. 5). In table 1 we have represented the number of such diagrams (with 3 external legs) as a function of the number of loops, and the area of the corresponding surfaces S, as obtained from the exact result [21] given by the 0-dimensional planar ϕ^3 theory.

* With a given labelling of external legs.

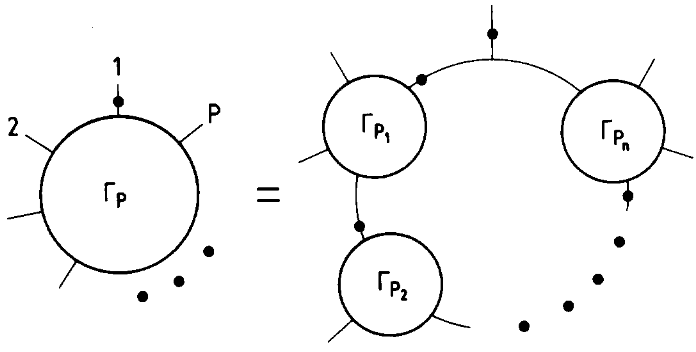

Fig. 4. Decomposition of a P-point irreducible graph of a ϕ^3 theory (without tadpoles and self-energy insertions) in terms of a chain of irreducible subgraphs, as given by (4.2).

Once a diagram $\tilde{\mathrm{S}}$ has been obtained, the dual lattice S is constructed, the matrix elements of the laplacian (2.18), its determinant (2.19) and its inverse (2.24) are numerically evaluated.

We obtain in this way the series associated with the total area

$$|S| = \sum_i \sigma_i, \tag{4.3}$$

and the mean-square extent times the squared volume

$$|S|^2 \overline{X}_S^2 = 2D \sum_{i,j} \overline{\Sigma}_{ij}. \tag{4.4}$$

From these series we shall estimate the critical indices γ and ν. We have also considered the total curvature squared $(\int \sqrt{g} R^2)$:

$$|R^2| = \sum_i \sigma_i R_i^2, \tag{4.5}$$

and the discrete analog of the Liouville action [14]:

$$S_{\text{Liouville}} = \sum_{i,j} \sigma_i R_i \Sigma_{ij} \sigma_j R_j, \tag{4.6}$$

in order to get some insight into the geometric properties of the surface and the relation of our model to the Polyakov string model. Unfortunately, we have not obtained any conclusive result for these two last quantities.

As can be seen from table 1, the number of graphs (and the computer time) grows exponentially by a factor ~ 8 at each order. The main part of our program constructs the diagrams and is not vectorizable by nature. However it is rather fast

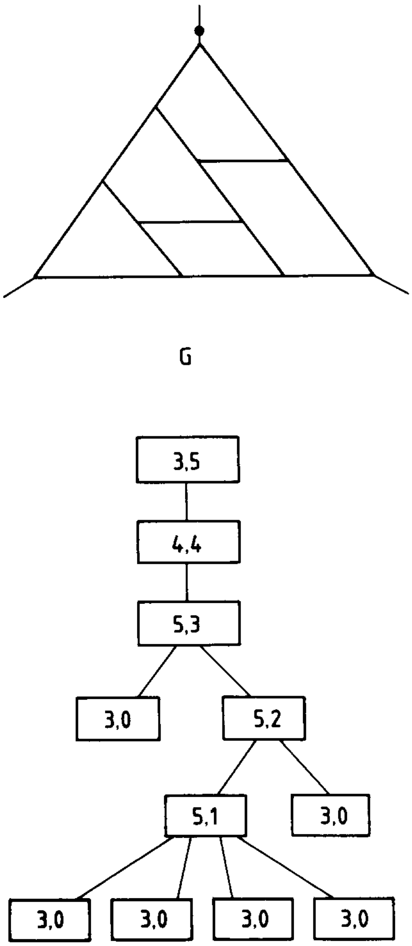

Fig. 5. Schematic decomposition of a planar graph G into a tree of partitions (4.2). Each box represents a couple (P, l). The end-points of the tree must be of the form $(3,0)$.

and represents, for graphs with $l = 7$ or 8, only 20% of CPU time. Most of the time is used to compute the determinant and the inverse of the laplacian for each graph. Moreover, once this has been computed, we obtain simultaneously the series for various observables, various dimensions D and various values of the $|R^2|$ parameter α. The total CPU time needed to compute the 8th first terms of the series is typically of the order of 1 min on a CRAY 1, which is rather modest.

4.2. SERIES ANALYSIS

In order to estimate the value of the critical coupling and of the corresponding exponents we have used two standard methods. The first one is a variant of ratio

TABLE 1
The number of irreducible planar diagrams with 3 external legs, no tadpoles and no self-energy, as a function of the number of loops l, and of the area $|S|$ of the corresponding dual lattice

l	$\|S\|$	Number of surfaces
1	4	1
2	6	3
3	8	13
4	10	68
5	12	399
6	14	2530
7	16	16 965
8	18	118 668
9	20	857 956

methods. If the general term of the series F,

$$F(v)=\sum_n f_n v^n, \tag{4.7}$$

behaves as

$$f_n = B\cdot A^n n^{\gamma-1}\left(1+\mathrm{O}\left(\frac{1}{n}\right)\right), \tag{4.8}$$

we first take the ratio

$$C_n^{(1)}=\frac{f_n}{f_{n-1}}\simeq A\left[1+\frac{(\gamma-1)}{n}+\mathrm{O}\left(\frac{1}{n^2}\right)\right]. \tag{4.9}$$

The value of A is then estimated by taking successive linear combinations of the C's:

$$C_n^{(p)}=\frac{nC_n^{(p-1)}-(n-p)C_{n-1}^{(p-1)}}{p}, \tag{4.10}$$

which should behave, if there is no confluent singularity, as

$$C_n^{(p)}\simeq A\left(1+\mathrm{O}\left(\frac{1}{n^p}\right)\right). \tag{4.11}$$

The value of the critical exponent γ is estimated by applying the same method to the series:

$$\begin{aligned} d_n^{(1)} &= n\left[C_n^{(1)}-C_n^{(2)}\right] \\ &\simeq A(\gamma-1)\left(1+\mathrm{O}\left(\frac{1}{n}\right)\right). \end{aligned} \tag{4.12}$$

In practice we shall limit ourselves to $p \leqslant 4$. The limitation of the method is the existence of confluent singularities, which are not eliminated, and the fact that for large p the coefficients of the subleading terms $O(1/n^{p+1})$ may become large.

The second method is a method of differential approximants [22]. We shall approximate F by the solution of the differential equation

$$P(v)v\frac{\partial}{\partial v}F(v)+Q(v)F(v)+R(v)=0. \tag{4.13}$$

$P(v)$, $Q(v)$ and $R(v)$ are polynomials of degree $p-1$, $q-1$, $r-1$, with $p+q+r-1=n$, n is the number of known terms of the series F and $Q(0)=1$. The singularities of F correspond to zeros of the polynomial P and γ is given by

$$\gamma=\frac{Q(v_0)}{v_0P'(v_0)}, \tag{4.14}$$

where v_0 is the closest zero of P from the origin.

This method is well adapted to our case where, as we shall see, the critical exponents are positive but small or negative and therefore out of the range of D log Padé approximants.

4.3. RESULTS AT $D=0$

Let us first present and discuss the results of our analysis at $D=0$. If $\alpha=0$ (no $|R^2|$ term) then the partition function corresponds to the generating function for the number of vacuum diagrams in the planar ϕ^3 theory [15] and is explicitly known, for the value of the critical coupling

$$v_c=\tfrac{27}{256}, \tag{4.15}$$

and the exponent γ

$$\gamma=-\tfrac{1}{2}. \tag{4.16}$$

So we have exact results to compare with the estimates coming from our series analysis.

Fig. 6 shows plots of the estimates $\gamma_n^{(p)}$ for the exponent γ for various values of α obtained by the ratio method,

$$\gamma_n^{(p)}=d_n^{(p)}/c_n^{(p)}, \tag{4.17}$$

for $p=3$ and 4 as a function of n^{-4}. We see that for all the values of α considered, the convergence is good and in agreement with the exact value (4.16). In particular the limit does not seem to depend on α, so that our universality hypothesis for γ seems reasonable.

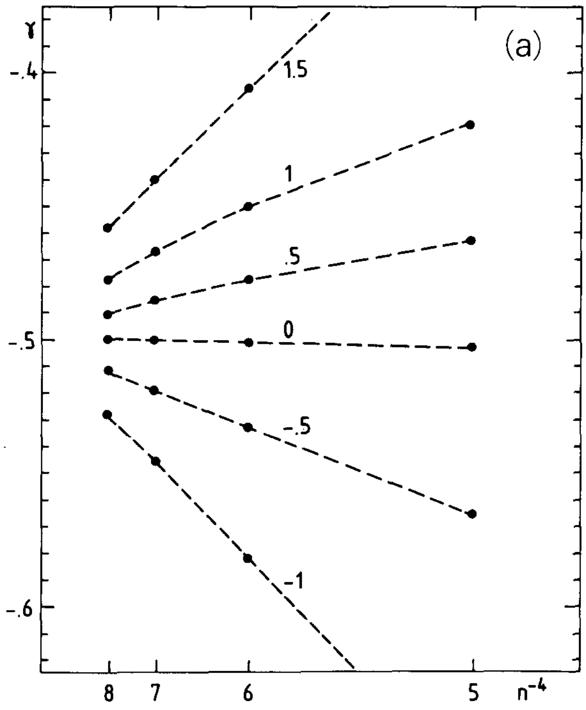

Fig. 6. Ratio approximants $\gamma_n^{(p)}$ at order $p = 3$ (a) and $p = 4$ (b) at $D = 0$ for various values of α.

Table 2 presents estimates for γ obtained by differential approximants. The results are rather stable and in agreement with previous results for small values of α ($0 < \alpha < 1$) but become less good for larger values of α, where the ratio method converges less quickly. From this example we could expect differential approximants to give good results when the ratio method works well, but to be more sensitive to a change in α.

We now turn to the exponent γ'. Fig. 7 presents estimates for γ' by the ratio method for $p = 3$ and 4. The convergence is less good than for γ and in particular it seems that there is a systematic dependence on α. We cannot tell whether this is due to the shortness of our series or if γ' depends effectively on α and is not universal. In any case we can reasonably say that (for the range of α considered)

$$\gamma' = -0.25 \pm 0.05 . \tag{4.18}$$

Table 3 presents the results of differential approximants for γ', which are compatible with (4.18).

Fig. 8 presents direct estimates of $2\nu = \gamma' - \gamma$ obtained by applying the ratio method to the ratio of the terms of the two series corresponding to γ (area) and γ'

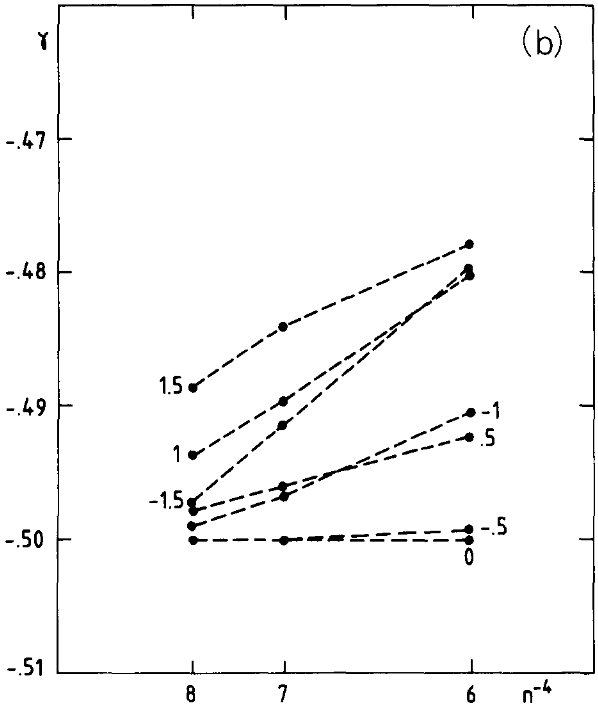

Fig. 6 (continued).

TABLE 2
Differential approximants of order $[P/Q/R]$ for γ at $D=0$ for various values of α

$[P,Q,R]$ \ α	−1	−0.5	0	0.5	1	1.5
[3/5/1]	−0.65	−0.55	−0.50	−0.50	−0.50	−0.45
[4/4/1]	−0.62	−0.54	−0.50	−0.50	−0.51	−0.34
[5/3/1]	−0.66	−0.55	−0.50	−0.50	−0.50	−0.44
[3/4/2]	−0.55	−0.51	−0.50	−0.50	−0.51	−0.55
[4/3/2]	−0.55	−0.51	−0.50	−0.50	−0.51	−0.55
[3/3/3]	−0.55	−0.51	−0.50	−0.50	−0.51	−0.55

(mean-square extent times the area). The results are compatible with previous results but here also we observe a systematic dependence on α.

For $\alpha=0$ we can get more precise estimates for γ' since we know the position of the singularity (4.15). Fig. 9 presents results of the ratio method once this information has been used for various values of p.

Table 4 presents the results of differential approximants where we have forced the polynomial P in (4.13) to have a zero at the exact singularity. From these results we

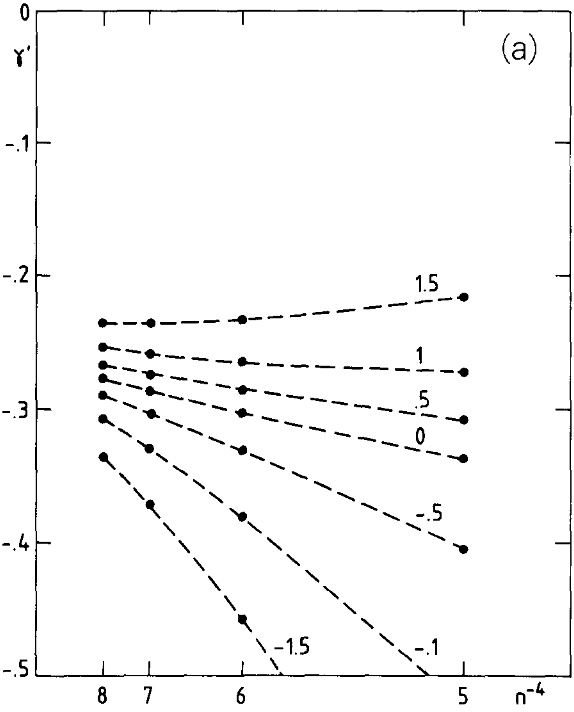

Fig. 7. Ratio approximants $\gamma_n'^{(p)}$ at order $p=3$ (a) and $p=4$ (b) at $D=0$ for various values of α.

estimate that at $\alpha = 0$

$$\gamma' = -0.275 \pm 0.015, \qquad \nu = 0.112 \pm 0.008. \tag{4.19}$$

Finally we have tried to estimate the critical behaviour of the quantities $|R^2|$ and $S_{\text{Liouville}}$ but the results are not at all conclusive, and we do not even recover the correct position of the critical coupling. This means that the corresponding series are much less well behaved and more irregular at the first orders. This is not completely unexpected, in particular for $S_{\text{Liouville}}$: indeed the Liouville action is expected to be an effective action valid at large scales, i.e. for large surfaces. At the order considered here we are in fact still dealing with rather small surfaces. We could expect the situation to improve by considering much larger series.

We now discuss the consequences of our estimations. From subsect. (3.1), if the mass gap does not vanish at the critical point, then $\gamma' = \gamma$. Since from our estimates, γ' is greater than γ, it follows that the mass gap *must vanish* and the two-point function* is expected to have a scaling behaviour. Our final estimates for η and the

* Correlations functions can be defined in zero dimension by simple analytic continuation in D without any problems.

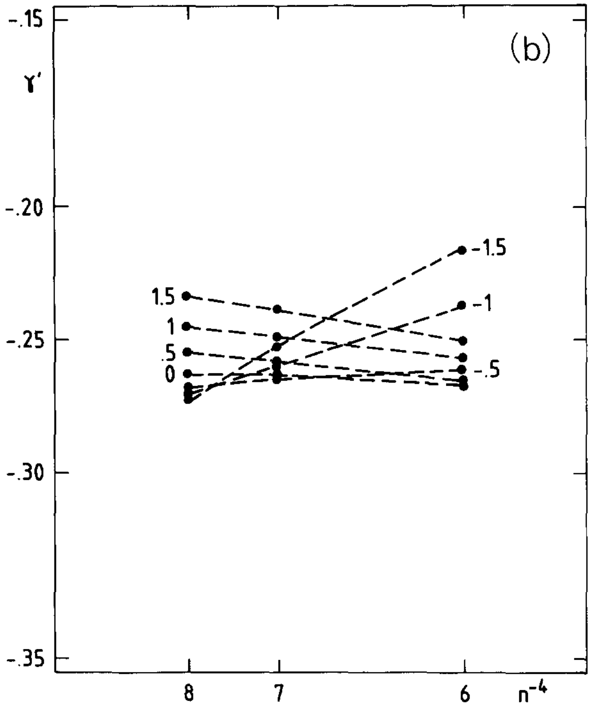

Fig. 7 (continued).

TABLE 3
Differential approximants of order $[P/Q/R]$ for γ' at $D=0$ for various values of α

$[P,Q,R]$ \ α	0	0.5	1
[3/5/1]	−0.31	−0.27	−0.26
[4/4/1]	−0.30	−0.26	−0.25
[5/3/1]	−0.31	−0.27	−0.26
[3/4/2]	−0.27	x	−0.24
[4/3/2]	−0.27	x	−0.24
[3/3/3]	−0.27	x	−0.24

The cross x at $\alpha = 0.5$ indicates that a spurious singularity closer to the origin invalidates the method.

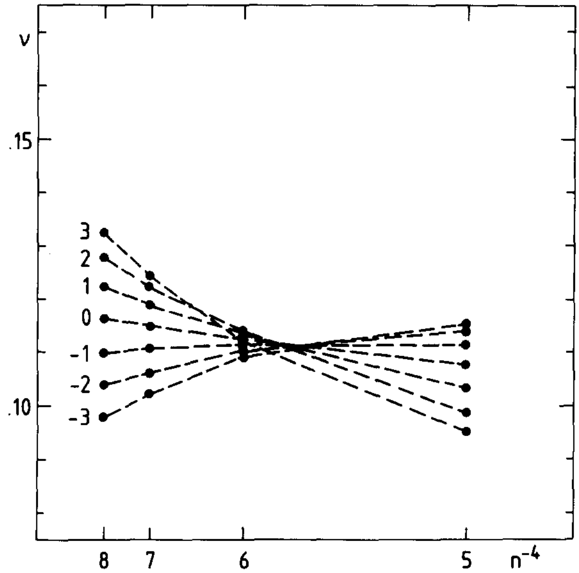

Fig. 8. Ratio approximants $\nu_n^{(p)}$ at order $p = 3$ for various values of α, as obtained from the ratio term by term of the series giving γ and γ'.

Hausdorff dimension are for $D = 0$ and $\alpha = 0$

$$\eta = 6.35 \pm 0.35,$$

$$d_{\mathrm{H}} = 8.75 \pm 0.75, \tag{4.20}$$

which are clearly not those of mean field theory. In particular the value of η indicates that the continuum limit (for at least the two-point function) is not a free field theory. Let us note that the fact that $\eta > 2$ is, to our knowledge, not forbidden by any fundamental principle. One simply expects the inequality

$$\eta > 2 - D \quad \text{or} \quad \gamma < \nu D \tag{4.21}$$

to hold, in order to have a two-point function decreasing at large distances. Another interesting point is that γ is negative and therefore the mean area $\langle |S| \rangle$ of a surface (attached to a point) remains finite at the transition. The fact that ν is non-zero means that however, the fluctuations of the surface are sufficient to induce correlations between points of space at arbitrarily large distances, so that the correlation length diverges at the critical point.

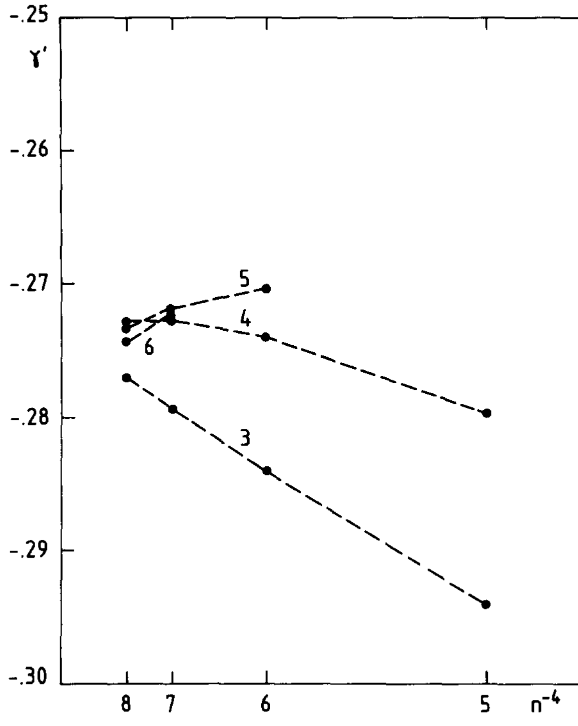

Fig. 9. Ratio approximants for γ' of order $p = 3,4,5,6$ at $D = 0$, $\alpha = 0$, using the exact value for the singularity.

TABLE 4
Differential approximants for γ and γ' at $D = 0$, $\alpha = 0$, using the exact value of the singularity

$[P,Q,R]$	γ	γ'
[3/6/1]	−0.5008	−0.286
[4/5/1]	−0.5003	−0.281
[5/4/1]	−0.5004	−0.281
[6/3/1]	−0.5010	−0.287
[3/5/2]	−0.5001	−0.273
[4/4/2]	−0.5000	−0.273
[5/3/2]	−0.5001	−0.273
[3/4/3]	−0.5000	−0.273
[4/3/3]	−0.5000	−0.273

4.4. RESULTS AT $D \neq 0$

We now present the results of our analysis for non-zero dimensions D. As we have seen, D appears only as a parameter in the series (as the number of components N in vector models) and may be taken positive or negative. As for $D = 0$, we have studied the series for various values of α, expecting the convergence of our methods of extrapolation to be better for some optimal value of α (assuming that the exponents do not depend on α), and in order to test the stability of our results. However, the imprecision increases as D goes further from the "solvable case" $D = 0$, and in addition we have no exact results for γ; this increases the error bars on ν. For these reasons we have not been able to get clear conclusions for $|D| \gtrsim 10$. We have also used, in addition to the action (2.8) for surfaces S, a slightly different action which gives better results by replacing $\alpha|R|^2$ in (2.8) by

$$\alpha \sum_{i \in \mathrm{v(S)}} \sigma_i R_i^2 \to \alpha \sum_{i \in \mathrm{v(S)}} \ln \sigma_i . \tag{4.22}$$

From (3.6) this change affects, up to a finite renormalization of β and α, only terms of order $R^3, \dots$. It is also equivalent to changing the measure $d\mu(X)$ over the

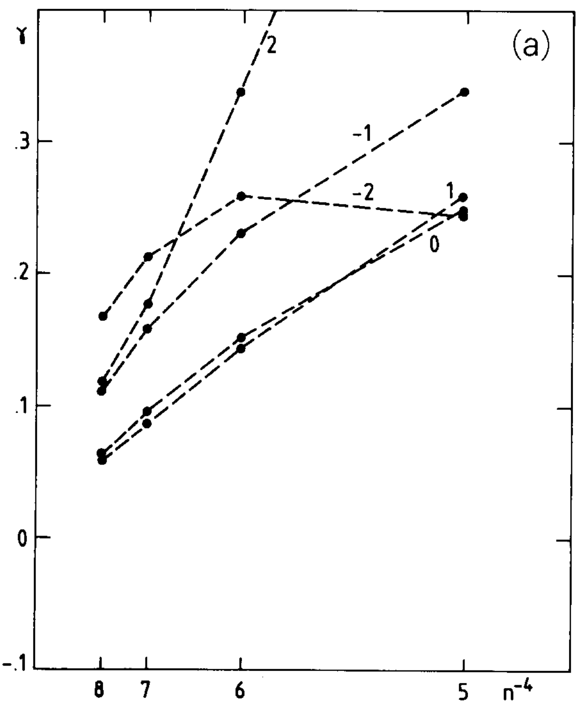

Fig. 10. Ratio approximants for γ at order $p = 3$ (a) and $p = 4$ (b) at $D = 4$.

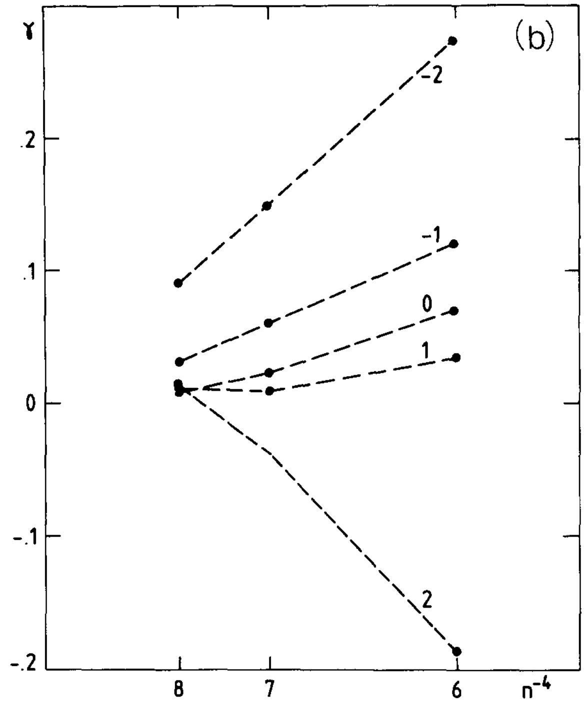

Fig. 10 (continued).

position of the vertices into

$$d\mu'(X_i) = d^D(X_i)\,\sigma_i^{D/2+\alpha}. \tag{4.23}$$

We shall present detailed results only for the "physical case" $D = +4$. Fig. 10 presents estimates for γ by the ratio method for the action (2.8), for $p = 3$ and 4 and different values of the parameter α. Fig. 11 presents similar estimates for γ'. Fig. 12 presents direct estimates for ν by the ratio method applied to the series formed by the ratio term by term of the two previous series. Differential approximants have also been used and give compatible results, but are, as for the $D = 0$ case, much more sensitive to changes of α around the optimum value. As for $D = 0$ there seems to be a systematic dependence on α of ν, which is not present for γ. Within the range of parameters considered for the action, our final estimates are

$$\gamma = 0.0 \pm 0.05\,,$$

$$\nu = 0.13 \pm 0.06\,. \tag{4.24}$$

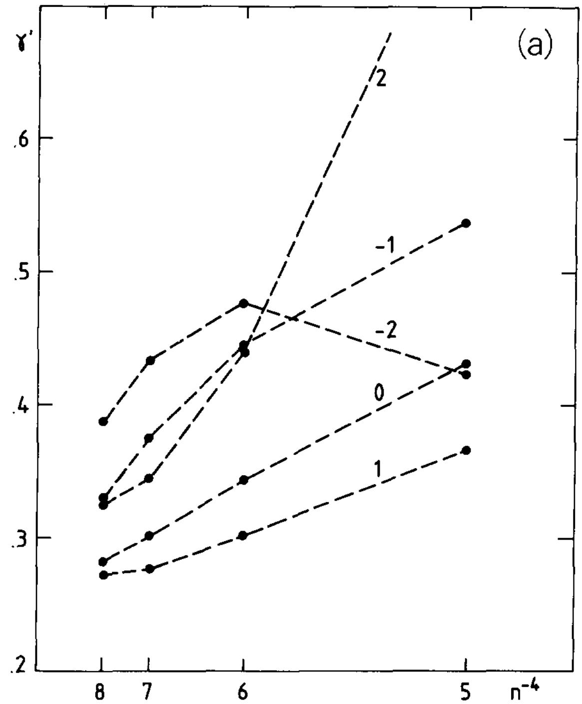

Fig. 11. Ratio approximants for γ' at order $p = 3$ (a) and $p = 4$ (b) at $D = 4$.

Estimates have been obtained in the same way for $-2 < \alpha < 2$ in dimensions $D = -8$, -4, 8 and 12. They are collected in figs. 13 and 14, where they are also compared with the predictions of other models of surfaces.

Let us first discuss the results for the susceptibility exponent γ. Within the range of α considered, and within our estimated error bars, γ does not seem to vary with α. γ increases clearly with the dimension D, starting from ~ -2 for $D = -8$ and seems to saturate around 0.5 for large positive D. It crosses the critical value $\gamma = 0$ at a critical dimension D_c which is remarkably close to four*. For $D > D_c$, the mean area of a surface $|\langle S \rangle|$, as defined by (2.29), must diverge at the critical point, but remains finite for $D < D_c$.

We now discuss our results for ν. ν does not seem to depend strongly on D, but for any D we have observed a systematic dependence on α, which we may attribute to the imprecision of our method or to a real non-universality of ν. Within our range of D, we can only say that ν is clearly greater than zero (and hence the correlation length diverges) and less than the mean field value 0.25 (at least for $D \leqslant 8$). In

* This term of the critical dimension has nothing to do with the usual notion of the upper or lower critical dimension of critical phenomena.

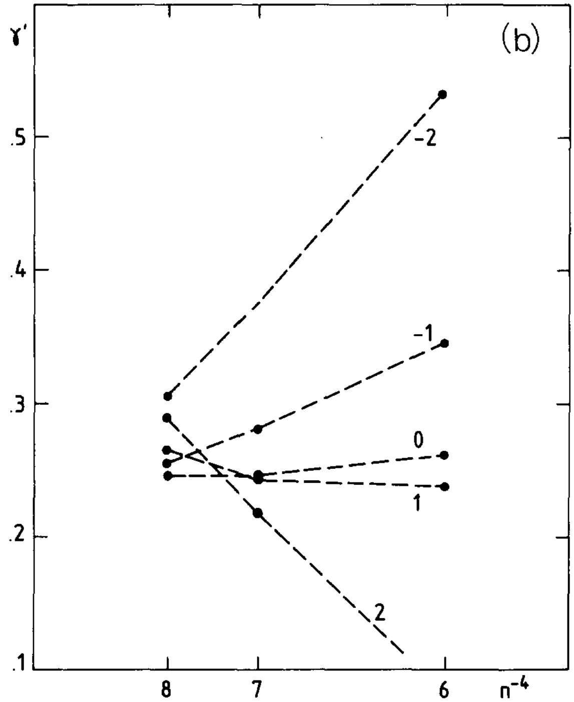

Fig. 11 (continued).

particular there does not seem to be any drastic change at the critical dimension $D_c \simeq 4$.

It is interesting to compare our results with the predictions of other models of random surface. In the models of [1–7] the mean field exponents $\gamma = \frac{1}{2}$, $\nu = \frac{1}{4}$ are exact for any dimension D ($D \geqslant 2$). Our results differ strongly from these mean field values, for at least $D \leqslant 8$. In particular the exponent η must be non-zero, indicating that the two-point function cannot correspond to a free field. It is possible that γ coincides with $\frac{1}{2}$ above some upper critical dimension but from our results we cannot extract any information on this question. In any case our model seems to belong to a different universality class than the model of [1–7].

In the string model of Polyakov [11] the values for γ and ν are at one loop* (at first order in the coupling constant $g^2 = 48\pi/(26 - D)$, i.e. for large negative D) [12]

$$\gamma = \tfrac{1}{6}(D-7), \qquad \nu = 0. \tag{4.25}$$

* It has been argued that this result for γ is exact.

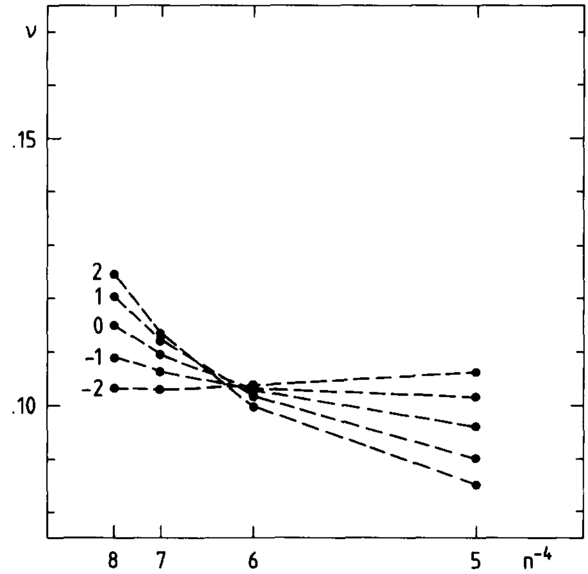

Fig. 12. Direct ratio approximants for ν at order $p = 3$ at $D = 4$.

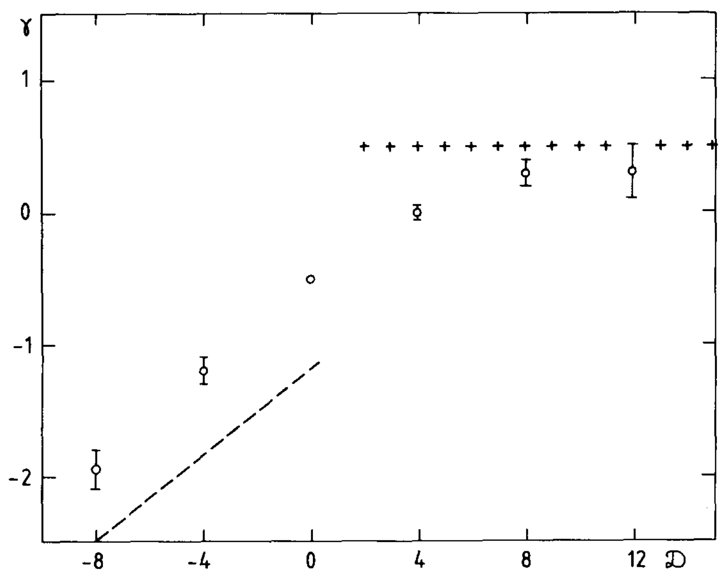

Fig. 13. Final estimates for γ in dimensions -8, -4, 0, 4, 8 and 12. The crosses indicate the mean field prediction of the model of [1–7], the dashed line the prediction of the Polyakov string model [11, 12].

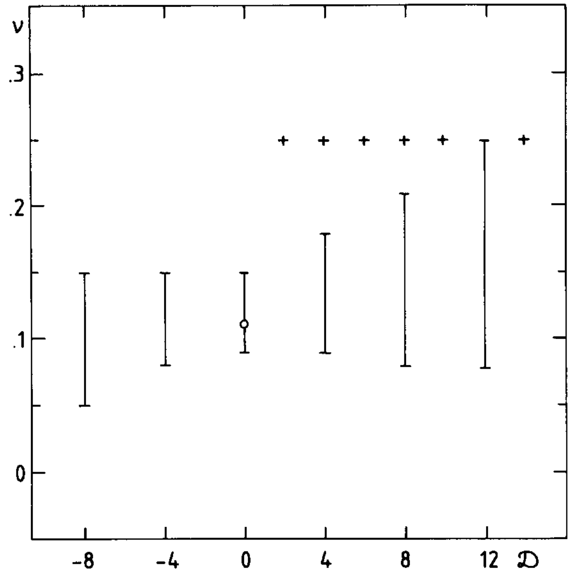

Fig. 14. Our estimates for ν in dimensions -8, -4, 0, 4, 8, 12 for values of α between -2 and $+2$. The circle at $D = 0$ indicates the more precise estimate obtained for $\alpha = 0$.

We observe that for large negative D our model gives for γ an estimate which is not so far from this prediction. However ν, although small, is not zero. It is interesting to note that our estimates for γ seem to interpolate between the prediction of Polyakov model for negative D and the prediction of mean field theory for positive D.

5. Discussion

In this paper we have presented and studied by strong coupling series a model of planar closed random surfaces. Estimates for the critical exponents γ and ν have been obtained. From these results and assuming that standard scaling holds in that model, we have shown that, at least for dimensions of the bulk space $-4 \leqslant D \leqslant +8$, the model presents a transition at some critical value of the coupling constant β, where the mean area of the surface becomes singular and where the mass gap vanishes. At that point the surface becomes a critical object with a Hausdorff dimension which is large but finite. In this range of D, the critical exponents do not coincide with the mean field exponents. γ depends strongly on D and the two-point function does not correspond to a free field theory. There is evidence for a critical dimension $D_c \simeq 4$, above which the mean area of the surface diverges, and below which the mean area remains finite at the transition.

This last fact is of course interesting if one thinks about the possible links between surface models and gauge theories. However, as we have seen, the behaviour of the mass gap does not seem to change at $D=4$. Perhaps the behaviour of the string tension is singular at $D=4$? The study of the string tension requires estimates of the loop correlation functions, which is numerically more difficult. Let us note anyway that if at $D=4$, $\gamma=0$, from the scaling relation (3.18) $\eta=2$ and the two-point function behaves at short distance, in the continuum limit, as

$$G_2(|\boldsymbol{X}|) \sim |\boldsymbol{X}|^{-4}, \tag{5.1}$$

in contrast with the plaquette-plaquette correlation function in gauge theories, whose behaviour at short distance is given by perturbation theory, and is

$$\langle 0|\operatorname{Tr} F^2(\boldsymbol{X}) \operatorname{Tr} F^2(0)|0\rangle_{\text{conn.}} \sim |\boldsymbol{X}|^{-8}. \tag{5.2}$$

Of course many problems remain to be studied in these surface models. In order to construct a continuum theory one has first to define a continuum limit for all N-point functions in euclidean space. Let us note that extensions of scaling arguments suggest that if at the critical point the mass gap vanishes, then the N-point function should scale as

$$G_N(\lambda \boldsymbol{X}_1, \ldots, \lambda \boldsymbol{X}_N) \sim \lambda^{(D+(\gamma-2)/\nu)+N(1/\nu-D)} G_N(\boldsymbol{X}_1 \ldots \boldsymbol{X}_N). \tag{5.3}$$

This means that the observable which appears in the N-point functions, $\int \mathrm{d}^2\xi \sqrt{g}(\xi)\delta^D(\boldsymbol{X}-\boldsymbol{X}(\xi))$, gets the scaling dimension $d_{\mathrm{H}}-D$, a fact which is not unexpected, and confirms the relation $1/\nu = d_{\mathrm{H}}$. In addition we get a "vacuum anomalous dimension" $D+(\gamma-2)/\nu$, which is not present in local field theories. The scaling relation (5.3) holds for polymer models, as well as for the surface model of [1–7], where the "vacuum anomalous dimension" is present, (except for $D=6$). In the language of critical phenomena such a phenomenon is associated with a violation of hyperscaling, which is not surprising in our case, since as we have seen, the correlation length and the "mean size" of the surface do not scale with the same exponents (for $D<D_{\mathrm{c}}$ the latter even does not diverge but has a weaker singularity). In the language of field theory this means that a wave function renormalization and a renormalization of the "string tension" will not be sufficient to make the observables finite, but that in addition an overall multiplicative renormalization will have to be performed.

Another problem lies in the fact that, at least order by order in $\mathrm{e}^{-\beta}$, the N-point functions do not satisfy reflection positivity. One may hope, but we have no proof, that this property is recovered at the critical point. This is crucial in order to construct in Minkowski space a continuum theory which respects the usual axioms of field theory (except of course locality).

Let us emphasize that all these considerations are restricted to the case of planar, non-interacting surfaces. Interactions are described in surface models by the breaking and joining of surfaces, that is by considering surfaces with arbitrary Euler number χ. In our case (as in the case of [1–7]) if one sums over surfaces with arbitrary topologies, the partition function $z(\beta)$ diverges for any finite β, since the model is related to a ϕ^3 theory where the field ϕ has now a finite number of components, which has no ground state. This seems a disease common to all known discrete surface models. For all the reasons mentioned above, it is not clear, even if a surface model exhibits a non-trivial critical behaviour, whether it corresponds to the quantum theory of some extended objects in a local limit.

Finally let us stress that the results that we have presented here are based on the analysis of rather short series. It will be necessary to check these results with longer series, with other numerical methods such as Monte Carlo simulations, and with exact results.

Note added

After the completion of this work we received a preprint by Ambjørn, Durhuus and Fröhlich [23] which discusses surface models related to those studied in this paper. In particular, preliminary results of a Monte Carlo simulation are presented, which seem compatible with a large Hausdorff dimension.

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References

[1] D. Weingarten, Nucl. Phys. B210 (1980) 285
[2] T. Eguchi and H. Kawai, Phys. Lett. 110B (1982) 143; 114B (1982) 247
[3] H. Kawai and Y. Okamoto, Phys. Lett. 130B (1983) 415
[4] B. Berg and A. Billoire, Phys. Lett. 139B (1984) 297;
B. Berg, A. Billoire and D. Foerster, Nucl. Phys. B251 [FS13] (1985) 665
[5] B. Durhuus, J. Fröhlich and T. Jónsson, Nucl. Phys. B225 (1983) 185; Phys. Lett. 137B (1984) 93
[6] B. Durhuus, J. Fröhlich and T. Jónsson, Nucl. Phys. B240 (1984) 453
[7] J. Fröhlich, Proc. Sitges Conf. on Statistical mechanics, Sitges, Barcelona, June 1984 (Springer, Berlin, 1985)
[8] G. Parisi, Phys. Lett. 81B (1979) 357;
J.M. Drouffe, G. Parisi and N. Sourlas, Nucl. Phys. B161 (1981) 397
[9] T. Eguchi and T. Filk, Phys. Lett. 149B (1984) 38
[10] D.J. Wallace and R.K.P. Zia, Phys. Rev. Lett. 43 (1979) 308;
M.J. Lowe and D.J. Wallace, Phys. Lett. 93B (1980) 433; J. Phys. A13 (1980) L381;
F. David, Phys. Lett. 102B (1981) 193
[11] A.M. Polyakov, Phys. Lett. 103B (1981) 207

[12] A.B. Zamolodchikov, Phys. Lett. 117B (1982) 87;
J. Jurkiewicz and A. Krzywicki, Phys. Lett. 148B (1984) 148
[13] A. Billoire, D. Gross and E. Marinari, Phys. Lett. 139B (1984) 75;
D. Gross, Phys. Lett. 138B (1984) 185;
B. Duplantier, Phys. Lett. 141B (1984) 239
[14] M. Bander and C. Itzykson, Saclay preprint PhT 84-127 (1984), Proc. seminar series on Non-linear equations in field theory, Paris VI and Meudon, Sept. 83–May 84 (Springer, Berlin) to appear; Irvine preprint no. 84-74; Nucl. Phys. B, in print
A. Jevicki and M. Ninomiya, Phys. Lett. 150B (1985) 115
[15] F. David, Nucl. Phys. B257 [FS14] (1985) 45
[16] J.M. Drouffe, Nucl. Phys. B218 (1983) 89;
J.M. Drouffe and H. Kluberg-Stern, Hadron spectrum in a euclidean invariant regularization of QCD at strong coupling, Saclay Preprint T85/071, Nucl. Phys. B, to be published
[17] V.A. Kazakov, Phys. Lett. 150B (1985) 282
[18] S. Hawking, Comm. Math. Phys. 55 (1977) 133;
K. Fujikawa, Phys. Rev. D23 (1981) 2262
[19] K. Symanzik, Prog. Theor. Phys. 20 (1958) 690
[20] J. Koplik, A. Neveu and S. Nussinov, Nucl. Phys. B123 (1977) 109;
P. Cvitanovic, P.G. Lauwers and P.N. Scharbach, Nucl. Phys. B203 (1982) 385
[21] E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, Comm. Math. Phys. 59 (1978) 35
[22] G.S. Joyce and A.J. Guttmann, J. Phys. A5 (1972) L81
[23] J. Ambjørn, B. Durhuus and J. Fröhlich, Diseases of triangulated random surface models and possible cures, Nordita preprint 85/2, Nucl. Phys. B257 [FS14] (1985) 433