ON THE CALCULABILITY OF NEWTON'S CONSTANT AND THE RENORMALIZABILITY OF SCALE INVARIANT QUANTUM GRAVITY

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We argue that Newton's constant is not unambiguously calculable in any non-finite pure matter theory. In finite, supersymmetric, conformally invariant theories it is calculable in terms of scalar expectation values. Implications for the renormalizability of scale invariant and conformally invariant quantum gravity are discussed.

Introduction. The idea that Newton's constant is not a fundamental parameter but is calculable in terms of fundamental dimensionless parameters has been considered in two different contexts. The first is the induced gravity program [1,2], where one attempts to compute the contribution to the induced Newton constant arising from the quantum fluctuations of the non-gravitational matter fields. The second context is fourth-order quantum gravitational theories (involving the squares of the scalar curvature and the Weyl tensor) where, for reasons to be elucidated, one asks under what conditions the Einstein action can be consistently excluded from the fundamental quantum action.

Recently one of us has argued [3] that in ordinary (non-supersymmetric) matter theories, despite contrary claims [1,2], Newton's constant and the cosmological constant were in general not calculable and were rather free parameters of the theory. The purpose of this letter is to extend this discussion to the case of supersymmetric matter theories and to scale invariant theories of gravity and their supersymmetric extensions. We will argue that Newton's constant is not unambiguously calculable in any non-finite pure matter theory, but that in certain scale invariant supergravity theories symmetries may protect it from ambiguities. On the contrary, in finite globally or locally supersymmetric theories, the inverse Newton constant is zero unless conformal invariance is spontaneously broken, in which case it is calculable in terms of scalar expectation values.

We begin by recalling the arguments of refs. [3,4]. Consider a matter field theory, conformally invariant at the classical level, quantized in a classical gravitational background field. The induced cosmological constant Λ_{ind} and the induced Newton constant G_{ind} are given by the formulae

$$G_{\text{ind}}^{-1} = -\mathrm{i}\frac{\pi}{6}\int \mathrm{d}^4x\, x^2[V(x)+U(x)], \tag{1}$$

$$\Lambda_{\text{ind}}/G_{\text{ind}} = -2\pi\langle T(0)\rangle_0 \tag{2}$$

$$= -\mathrm{i}\pi\int \mathrm{d}^4x[V(x)+U(x)]. \tag{3}$$

In (2) T is the trace of the stress energy tensor $T_{\mu\nu}$

$$T(x) = T_\mu^\mu(x) \tag{4}$$

V and U are given by

$$V(x) = \langle T\{\tilde{T}(x)\tilde{T}(0)\}\rangle_0, \tag{5}$$

with

$$\tilde{T}(x) = T(x) - \langle T(x)\rangle_0 \mathbf{1}, \tag{6}$$

and

$$U(x) = -2\mathrm{i}\langle g_{\mu\nu}(x)[\partial/\partial g_{\mu\nu}(x)]\sqrt{g(0)}\; T(0)\rangle_0. \tag{7}$$

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The subscript $\langle\ \rangle_0$ means that the vacuum expectation value is evaluated in the flat background metric $g^0_{\mu\nu} = \eta_{\mu\nu}$. Formulae (1) and (2) are due to Adler [1] and Zee [2]. Formula (3) for the cosmological constant corrects an incorrect formula (omitting the second term proportional to U) that has recurred in the literature.

$U(x)$ will in general be proportional to $\delta^4(x)$ and will not contribute to G_{ind} (because of the x^2 in (1)) unless there is a scalar field with a $-\zeta R\varphi^2/2$ interaction. In that case

$$U(x) = -6\mathrm{i}\zeta\partial^\mu\partial_\mu\delta^4(x)\langle\varphi^2(0)\rangle + \delta^4(x) \text{ terms}, \tag{8}$$

and it will contribute to G_{ind}.

These equations are in fact UV divergent and need to be renormalized. The correlation function $V(x)$ has a small x behavior given by the operator product expansion

$$V(x) = |x|^{-8}C_0(x) + \sum_{i=1}^{\infty} |x|^{-8+\dim O_i}C_i(x)\langle O_i\rangle, \tag{9}$$

where the $C_i(x)$ are dimensionless and contain only logarithms of $|x|$ (in perturbation theory). Therefore the formula (1) for G_{ind} is quadratically divergent.

In ref. [1] it was argued that these divergences vanish if one uses an analytic regulator such as dimensional or ζ-function regularization, and that therefore, provided there are no dimension-2 operators (which can be ensured by gauge and chiral invariance), G_{ind} and Λ_{ind} are uniquely determined from the remaining finite parts. It was also argued that, if a massive regulator is used, the same unique answer will obtain by taking the finite part of G_{ind}.

In fact arguments have been given [3] strongly indicating that this is incorrect. In general, the finite part of a quantity with quadratic divergences cannot be uniquely defined, even if no logarithmic divergences are present. The way in which ambiguities appear, however, is extremely subtle.

To see that G_{ind} is ambiguous, consider expression (5) for $V(x)$. If the theory can be defined non-perturbatively $V(x)$, being a Green's function, will have an unambiguous value. C_0, however, cannot in general be uniquely defined. The reason is that the perturbation series for C_0 is not only not convergent; it is not even Borel summable. This can be inferred [‡1] in an asymptotically free theory [‡2] from inspection of a certain set of diagrams [5] or from the infrared behavior of the Green's functions [6]. If one attempts to resum the series using the Borel procedure, one expects to encounter the so-called "renormalon" singularities [3–6] on the positive Borel axis and a prescription must be adopted for going around these. C_0 will in general depend on the prescription.

If C_0 is ambiguous, how can V be unambiguous? The answer is that as C_0 is an infinite series in $\ln^{-1}|x|$, its non-perturbative ambiguity will be proportional to powers of x. It thus can and will be cancelled by ambiguities in the higher order C_i and $\langle O_i\rangle$, provided the same resummation prescription is adopted throughout.

In computing G_{ind} from $V(x)$ via eq. (1), one must separate the divergent contribution proportional to C_0. As we have just argued, this separation is not unique and G_{ind} will be ambiguous. Indeed, it was shown in ref. [3] that a previous model calculation of G_{ind} [1,7] in fact exhibited the expected ambiguity.

It should be added that there is no *proof* that G_{ind} will *necessarily* be ambiguous. What has been shown is that in the general case it is prescription dependent, but it is always possible that some unforseen mechanism could eliminate this dependence in particular models. In the next section we will show that supersymmetry alone does not provide such a mechanism, but that in *finite* supersymmetric models G_{ind} is unambiguous.

The supersymmetric case. The above arguments are very general, and ambiguities are expected to arise whenever certain operator mixings are allowed by dimensional and symmetry considerations. However, one does have to check if some symmetry forbids mixing and forces the ambigui-

‡1 In certain two-dimensional models such arguments may be made rigorous within the large-N expansion [4].

‡2 Of course in a non-asymptotically free theory non-perturbative quantities (such as G_{ind}) can not be defined at all.

ties to cancel for the particular matrix elements (1)–(3). For non supersymmetric theories there are none, but for supersymmetric theories many cancellations occur. Indeed, such cancellations occur for the cosmological constant. It is therefore important to investigate the formulae (1)–(3) for the case of supersymmetric matter field theories.

In order to preserve supersymmetry one must couple the theory to a classical supergravity background field [8]. The formulae (1)–(3) will appear as the reduction to the bosonic sector of the formulae for induced supergravity. Eq. (2) for the cosmological constant in terms of the trace anomaly is known to be a member of a supermultiplet along with the axial anomaly equation [9]. Supersymmetry prevents the trace anomaly from mixing with the operator **1** and the cosmological constant in finite and zero if supersymmetry is unbroken.

Unfortunately no such cancellations occur for the induced Newton constant. We first consider the case with no $-\zeta R\varphi^2/2$ interaction for which the $U(x)$ term can be ignored. From eq. (5) the function $V(x)$ may be written

$$V(x)=\sum_n \exp(ip_n\cdot x)|\langle 0|T(0)|n\rangle^2, \qquad (10)$$

from which it follows that G_{ind}^{-1} has a spectral representation

$$G_{\text{ind}}^{-1}=-\frac{4\pi}{3}\int_0^\infty \mathrm{d}\sigma^2\rho(\sigma^2)/\sigma^4, \qquad (11)$$

where the spectral function

$$\rho(q^2)=(2\pi)^3\sum_n \delta^4(p_n-q)|\langle 0|T(0)|n\rangle|^2 \qquad (12)$$

is positive semi-definite. This means that there cannot be cancellations between bosonic and fermionic states in (12). One can explicitly check, for instance for supersymmetric gauge theories, that as a consequence the divergent term $C_0(x)$ in eq. (9) is never zero in perturbation theory, as long as the trace anomaly itself does not vanish. Therefore in non-finite supersymmetric theories Newton's constant is never calculable. Let us note that in (3) the presence of the term $U(x)$ is essential in order to understand how $\Lambda_{\text{ind}}/G_{\text{ind}}$ (the vacuum energy) may be finite although G_{ind}^{-1} as given by (1) is infinite.

We now consider the $-\zeta R\varphi^2/2$ term. The allowed values of ζ are not arbitrary, but rather are fixed by consistent coupling to supergravity. Most of the couplings that appear in the literature have $\zeta=0$ [8] and do not lead to an unambiguous G_{ind}. However for couplings that preserve classical conformal invariance [10] (otherwise G_{ind} will be divergent at tree level), ζ is 1/6. In this case the $U(x)$ term in (1) merely serves to cancel contact terms arising $V(x)$, provided $\langle\varphi\rangle=0$. This can be seen by noting that, for a classically conformally invariant scalar action, the trace anomaly is given by

$$T(x)=\alpha\varphi(x)\partial S/\partial\varphi(x)+\beta S(x), \qquad (13)$$

where $S=\int \mathrm{d}^4x\, S(x)$ is the action, α is a constant depending on the conformal weight of the fields and β is a function of the coupling and renormalization constants. We have been careful not to omit terms proportional to the equations of motion. Using (13) and the Schwinger–Dyson equations, $V(x)$ can be written as

$$\begin{aligned}V(x)=&-(\alpha^2+2\alpha\beta)\delta^4(x)\langle\varphi(0)\partial S/\partial\varphi(0)\rangle\\&-\alpha^2\langle T\{\varphi(x)\varphi(0)\partial S/\partial\varphi(x)\partial\varphi(0)\}\rangle_c\\&+\beta^2\langle T\{S(x)S(0)\}\rangle_c \qquad (14)\end{aligned}$$

where subscript c denotes the connected part. The first term does not contribute to G_{ind} because of the x^2 in (1). Using the fact that

$$\partial S/\partial\varphi(x)\partial\varphi(0)=-\partial^\alpha\partial_\alpha\delta^4(x)+\delta^4(x)\text{ terms}, \qquad (15)$$

we see that the second term is of precisely the correct form to cancel the $U(x)$ contribution, which it indeed does (this can be checked in the free field calculations of ref. [11]). Thus G_{ind} is determined entirely from the last term in (14). Similar arguments can be given when spinors and vectors are included, except that the second term will also not contribute to G_{ind} (because spinor actions have only one derivative and vectors have $\alpha=0$). The last term in (14) has a positive definite spectral representation, so cancellations cannot occur. G_{ind}

will be divergent unless β is zero (i.e. the theory is finite), in which case it will vanish.

The case $\langle\varphi\rangle \neq 0$ is somewhat different. If the theory is not finite, G_{ind} is of course still not calculable. In the finite case, G_{ind} is calculable in terms of $\langle\varphi\rangle$ (if $\xi = 1/6$). After cancellations of contact terms, one finds

$$G_{\text{ind}}^{-1} = -4\pi\langle\varphi\rangle^2/3. \tag{16}$$

Scale and conformally invariant quantum gravity. The general scale invariant gravitational action may be written (up to topological terms) as

$$S = \int\left(\alpha^{-2}C_{abcd}C^{abcd} + \beta^{-2}R^2\right). \tag{17}$$

This action is invariant under (global) scale transformations as it contains no dimensionful parameters. For the special case $\beta^{-2} = 0$ there is also a (local) conformal invariance. It has been shown [12] that the general fourth-order action, which includes the Einstein and cosmological terms, is renormalizable. However, for a variety of reasons, some authors have considered excluding the Einstein and cosmological terms from the fundamental quantum action [13–15]. In this section we will discuss the consistency of such an exclusion. Before doing so we briefly mention why such an exclusion might be desirable.

One reason is the aesthetic notion that the fundamental laws of physics should contain no dimensionful parameters [1,2,13]. Since scale invariant gravity is asymptotically free [14], the Planck scale could then arise through dimensional transmutation. In the conformally invariant case, even the notion of length has no fundamental meaning and could arise only as a result of spontaneous symmetry breaking.

A further motivation to study the scale invariant theories is the zero energy theorem [15]. While the classical energy spectrum of the general fourth-order theories is believed to be unbounded from below, the spectrum of scale invariant theories includes only zero; even when coupled to arbitrary matter. This striking difference between the classical theories could persist to the quantum level and may be relevant to the unitarity problem.

Finally, the general fourth-order action is unbounded in euclidean space (unless Λ is order one in Planck units) while the scale invariant action is positive semi-definite (for α^{-2}, $\beta^{-2} > 0$). An unbounded action usually signals the non-existence of a ground state, and makes it difficult or impossible to use non-perturbative methods such as lattices or instantons.

However renormalizability, both perturbative and non-perturbative, is a requirement that one should impose on the fundamental gravitational action. If we couple matter to the action (17) without putting it in a supermultiplet with the graviton, the results of the previous section immediately imply that G_{ind} will not be calculable. It is thus a free parameter of the theory and should be included in the fundamental action.

For the purely gravitational case the arguments are not so readily applied. A more complicated expression replaces (1) for the gravitational contributions to G_{ind} [1]. One cannot use a spectral decomposition to argue that there are no cancelations rendering G_{ind} finite. Nevertheless, there are no symmetries protecting it and it seems very likelv that G_{ind} is not calculable.

The situation is quite different for (extended) scale invariant supergravity. In this case everything is tied together in a supermultiplet and there could be symmetries protecting G_{ind} from divergences. For example, the addition of the Einstein action and its Rarita–Schwinger superpartner entails a massive pole in the Rarita–Schwinger propagator. Thus there could be a chiral symmetry which forbids the Einstein action and protects G_{ind}. In the case of $N = 3,4$ extended supersymmetry, higher derivative actions can be written in terms of superfields, but the Einstein action and its superpartner cannot. Thus one might again expect G_{ind} to be protected. In sum the renormalizability of scale invariant supergravity (without the Einstein term) remains an open question.

Perhaps the most interesting case is extended conformal supergravity. It has been argued [16] that there exists a class of finite locally superconformal theories similar to the class of finite Yang–Mills theories [16,17]. In this case there are of course no problems with renormalons, and G_{ind}^{-1} will be zero unless conformal invariance is spontaneously broken. We will return to this question in a future publication [18].

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References

[1] S. Adler, Rev. Mod. Phys. 54 (1982) 729, and references therein.
[2] A. Zee, Phys. Rev. D23 (1981) 858.
[3] F. David, Phys. Lett. 138B (1984) 383.
[4] F. David, Nucl. Phys. B234 (1984) 237.
[5] G. 't Hooft, Erice lecture notes (1977), ed. A Zichichi (Plenum, New York, 1978).
[6] G. Parisi, Nucl. Phys. B150 (1979) 163.
[7] A. Zee, Phys. Rev. Lett. 48 (1982) 295.
[8] See e.g. J. Wess and J. Bagger, Supersymmetry and supergravity (Princeton U.P., Princeton, 1983).
[9] S. Ferrara and B. Zumino, Nucl. Phys. B87 (1975) 207.
[10] See e.g. E. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. B217 (1983) 489;
B. de Wit and J. van Holten, Nucl. Phys. B184 (1981) 77.
[11] H. Muratani and S. Wada, Phys. Rev. D29 (1984) 637
[12] K.S. Stelle, Phys. Rev. D16 (1977) 953.
[13] B. Hasslacher and E. Mottola, Phys. Lett. 95B (1980) 237.
A. Zee, Phys. Lett 109B (1983) 183, UW preprint 40048-32P2, to be published in Ann. Phys.
M. Kaku, Nucl. Phys. B203 (1982) 285.
[14] J. Julve and M. Tonin, Nuovo Cimento 46B (1978) 137,
E. Fradkin and A.A. Tseytlin, Nucl. Phys. B201 (1982) 469; B203 (1982) 157.
[15] D. Boulware, G. Horowitz and A. Strominger, Phys. Rev. Lett. 50 (1983) 1726.
[16] P.S. Howe, K.S. Stelle and P.K. Townsend, Nucl. Phys. 236B (1984) 125;
P. West, Proc. Shelter Island Conf., to be published, and references therein.
[17] S. Mandelstam, in: Proc. 21st Intern Conf. on High energy physics eds. P. Petiau and M. Porneuf (Paris, 1982);
L. Brink, O. Lindgren and B. Nilsson, University of Texas preprint UTTG-1-82;
S. Hamidi, J. Patera and J.H. Schwartz, Caltech preprint 68-1115 (1984).
[18] V.P. Nair and A. Strominger, in preparation.