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PLANAR DIAGRAMS, TWO-DIMENSIONAL LATTICE GRAVITY AND SURFACE MODELS

F. DAVID*

Service de Physique Théorique, CEN Saclay, 91191 Gif-sur-Yvette Cédex, France

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Some discrete lattice models for quantum two-dimensional euclidean gravity are shown to be equivalent to zero-dimensional planar field theories. Explicit expressions are given for partition functions. A universal continuum limit exists for open surfaces, but not for closed ones, and is argued to describe a space with negative average curvature. Extensions of those models to higher dimensions and to surface models are briefly discussed.

The quantization of the Einstein-Hilbert theory of gravity is known to present important conceptual as well as technical difficulties. Many suggestions for formulating regularized lattice versions of the theory have been made in order to face those problems [1-6]. In [5,6] Weingarten suggested reducing the functional integration over "all" d-dimensional riemannian manifolds to a discrete sum over manifolds made of d-dimensional hypercubes glued on their faces and belonging to some higher-dimensional flat hypercubic lattice. It was shown that if one sums over manifolds with an arbitrary topology, the path integral is divergent and the theory has no acceptable vacuum. The divergence disappears if the topology of the manifold is fixed. The two-dimensional case (random planar surfaces) has been extensively studied recently [7,8], but seems to describe in the continuum limit only a free field theory [8].

In this paper we want to consider related models of two-dimensional "abstract" surfaces, without any reference to some enveloping higher-dimensional lattice (this case was already suggested in [6]). We shall show that those models are equivalent to problems of diagram enumerations in scalar field theories. This fact makes it possible to write explicitly quantities like partition functions, and to ask questions about the existence and the meaning of a continuum limit in those models. With no restriction on the topology our models have no ground state, as in [5,6]. Restricting to the planar geometry, we shall show that there is no continuum limit for closed surfaces but that there is such a limit for open surfaces. This limit is universal

* Chercheur CNRS.

(different lattice models give the same results) and can be interpreted as a space with negative or null mean curvature.

We shall end this paper by considerations on possible extensions of these kind of models, their imbedding in an enveloping space and their possible relationship with other surface models.

Let us first present our models. We want to consider surfaces made by gluing together by their edges some flat elementary polygons. The simplest case (model I) corresponds to take as elementary pieces oriented equilateral triangles. We shall consider for simplicity the case of closed surfaces. A closed (connected) surface S with area $|S| = n$ is defined as a set of n triangles $(t_1 \ldots t_n)$ and of pairing of edges (which respects the orientation of the triangles)*. Since the n triangles are indiscernible, the contribution of each surface has to be divided by $(n!)$. The topology of the surface is characterized by its Euler number χ. The grand canonical partition function is therefore chosen as

$$Z = \sum_{S} \frac{1}{|S|!} e^{-E}, \qquad E = \beta|S| - \gamma\chi. \tag{1}$$

We have not added higher-dimensional terms, since they are expected to become irrelevant in the continuum limit if it exists.

The reader will have already recognized that our rules for defining a surface are very similar to the contraction rules which follow from Wick's theorem in the construction of diagrammatic expansions in terms of Feynman diagrams in field theory. Indeed, if, instead of considering the simplicial lattice made of the triangles of S, we consider the dual lattice (see fig. 1), each surface S is in one-to-one correspondence with a vacuum diagram G of a ϕ^3 theory, and the coefficient associated to each surface S is precisely the symmetry factor associated with the graph G**.

More precisely, our model is equivalent to the zero-dimensional matrix ϕ^3 model [9], defined by the action

$$S[\phi] = \tfrac{1}{2}\mathrm{Tr}\,\phi^2 + \frac{g}{\sqrt{N}}\mathrm{Tr}\,\phi^3, \tag{2}$$

where ϕ is an $N \times N$ hermitian matrix, with the relationship

$$g = e^{-\beta}, \qquad N = e^{\gamma}, \tag{3}$$

* One can also consider non-orientable surfaces by removing the orientability constraint on the triangles.

** The reader may be worried that the factor associated to a surface S is not 1 but some rational number if the corresponding dual graph G has a non-trivial symmetry group (under the interchange of lines and vertex). This is not unexpected, since it is known that in general in the "space of all metric" over a given manifold, metrics with symmetries are singular points.

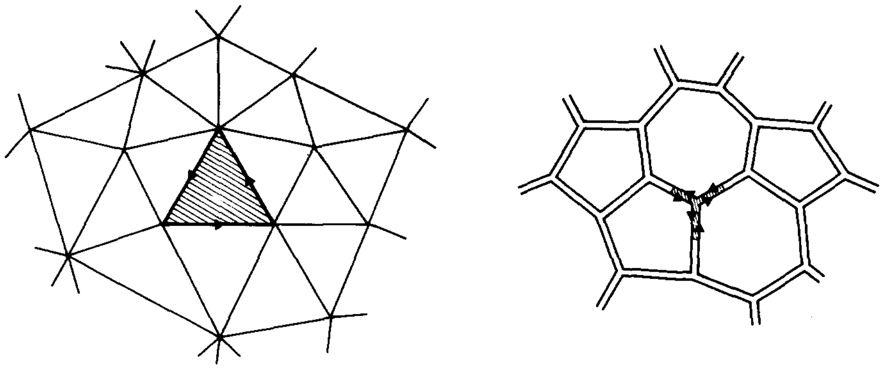

Fig. 1. A planar lattice made of oriented triangles (a) and the corresponding dual lattice (b). The length of the edges has been changed in order to map the lattices on flat space.

and the partition function (1) corresponds to (minus) the "vacuum energy" of the zero-dimensional model.

Since for finite N the action (2) is unbounded from below, it follows immediately that the partition function (1) is not convergent, as in [6]. The only possibility is to restrict ourself to the planar topology ($N = +\infty$ or $\gamma \to \infty$), in which case the integral

$$Z(\beta) = \lim_{N \to \infty} \frac{1}{N^2} \ln \int d^{2N}\phi\, e^{-S[\phi]} \tag{4}$$

makes sense for real g.

Let us now define two different versions of model I. Given some surface S, we call a ℓ-loop a set of ℓ *distinct* edges (of the triangles) in S which forms a loop on S. We shall define model I′ by the same formula (1) where the sum is restricted over surfaces S without 1-loops and model I″ by restricting (1) over surfaces S without 1- and 2-loops. It is easy to see that model I′ is equivalent to planar ϕ^3 without tadpoles (see fig. 2) and model I″ to planar ϕ^3 without tadpoles and self-energy insertions. This is simply achieved by adding to the action (2) a counterterm of the form

$$\Delta S[\phi] = \rho \operatorname{Tr} \phi + \tfrac{1}{2} x \operatorname{Tr} \phi^2, \tag{5}$$

and by adjusting ρ (for model I′) and ρ and x (for model I″) so that tadpoles (self-energy insertions) are cancelled.

Finally, we can define another model (model II) by considering surfaces made of oriented squares instead of triangles, and by taking the same action (1). This model

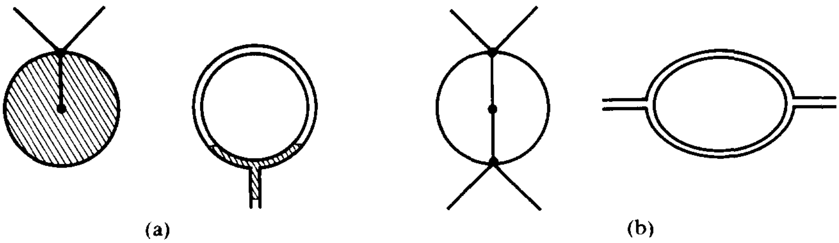

Fig. 2. (a) An example of a lattice with a 1-loop (the boundary of the dashed triangle) and the corresponding tadpole graph on the dual lattice. (b) An example of a lattice with a 2-loop and the corresponding self-energy graph on the dual lattice.

is of course equivalent to a zero-dimensional ϕ^4 model, defined by the action

$$S[\phi] = \tfrac{1}{2}\mathrm{Tr}\,\phi^2 + \frac{g}{N}\mathrm{Tr}\,\phi^4, \tag{6}$$

with

$$g = -\mathrm{e}^{-\beta}. \tag{7}$$

A real β corresponds to a negative g, and the action (6) is unbounded from below for finite N. Therefore model II has no ground state if one sums over all possible topologies, and makes sense only in the planar case ($N = +\infty$), as for model I.

As already mentioned, the problem of counting the number of planar diagrams of scalar theories with a ϕ^3 or a ϕ^4 coupling has been already extensively studied and the generating functions for the number of diagrams obtained by a variety of methods [9–13]. In the following we shall use the notation and the results of [9]. Before coming to the explicit results, let us discuss the quantities we are interested in. In analogy with polymer problems as well as with other surface models, we are looking for some critical value of the coupling, β_c, where we can obtain surfaces S with an arbitrarily large area $|S|$. For that purpose, we shall look at the average area

$$\langle |S| \rangle = -\frac{\partial}{\partial\beta}\ln Z(\beta), \tag{8}$$

and see if it diverges, or becomes at least singular, at some β_c.

Let us first consider, as a trivial exercise, the 1-dimensional case where we consider lines L (assumed open with two ends), made of n segments with unit length. The partition function is trivially

$$Z(\beta) = \sum_{n=0}^{\infty} (\mathrm{e}^{-\beta})^n = \frac{1}{(1-\mathrm{e}^{-\beta})}, \tag{9}$$

and is defined only for $\beta > 0$. As $\beta \to 0_+$, the average length

$$\langle L \rangle = -\frac{\partial}{\partial\beta} \ln Z \simeq \frac{1}{\beta} \tag{10}$$

becomes infinite and we generate an arbitrarily long line. The results are similar for a closed line (ring).

Let us now consider model I. The partition function is (see eqs. (49) and (51) of [9])

$$Z(\beta) = \tfrac{1}{2} \sum_{k=1}^{\infty} (\mathrm{e}^{-\beta})^{2k} \frac{(72)^{2K} \Gamma(\frac{3}{2}K)}{\Gamma(K+1)\Gamma(\frac{1}{2}K+1)}, \tag{11}$$

and has a radius of convergence (in $g^2 = \mathrm{e}^{-2\beta}$) equal to

$$g_c^2 = \mathrm{e}^{-2\beta_c} = \tfrac{1}{108}\sqrt{\tfrac{1}{3}}\,. \tag{12}$$

But the general term of the series (11) behaves, for large K, as

$$\mathrm{e}^{(\beta_c - \beta)2K} K^{-7/2} \left(1 + \mathrm{O}\left(\frac{1}{K}\right)\right), \tag{13}$$

which means that the singularity of $Z(\beta)$ is in $(\beta - \beta_c)^{5/2}$ and therefore that only the third derivative of Z becomes infinite at β_c. As a consequence, the average surface $\langle |S| \rangle$ given by (8) (as well as the average surface squared $\langle |S|^2 \rangle$) remains finite as $\beta \to \beta_c$. The same phenomenon can be shown to occur for models I′ and I″, as well as for the quartic model II (of course the values of the critical coupling are different). An explicit calculation shows that at the transition point

$$\begin{aligned} \langle |S| \rangle_c &\simeq 3.06 \quad \text{model I}, \\ \langle |S| \rangle_c &\simeq 1.52 \quad \text{model II}, \end{aligned} \tag{14}$$

which is very small and has to be compared with the zero-temperature result ($\beta = +\infty$) ($\langle |S| \rangle_0 = 2$ for model I and $\langle |S| \rangle_0 = 1$ for model II). Therefore the transition at β_c is first-order-like, and the average area jumps from a finite value for $\beta = \beta_{c+}$ to infinity for $\beta = \beta_{c-}$. It does not seem possible to construct an interesting continuum limit from closed surfaces with genus 0.

We now deal with the case of open planar surfaces. For model I we have to consider surfaces S made of triangles (squares) with the topology of a disc and with a boundary ∂S. The length of the boundary $|\partial \mathrm{S}|$ is simply the number of edges of triangles (squares) which belong to ∂S. If we now consider the dual lattice we see that S is in one-to-one correspondence with a planar diagram G of the ϕ^3 (or ϕ^4)

theory with $P = |\partial S|$ external legs, and that the counting factor of S is precisely the symmetry factor of G, as for closed surfaces. We can in fact define three different kinds of boundary conditions (b.c.), which correspond to considering various P-points functions:

(i) b.c. A = sum over surfaces corresponding to Green functions G_p;
(ii) b.c. B = sum over surfaces corresponding to connected functions C_p;
(iii) b.c. C = sum over surfaces corresponding to irreducible functions Γ_p.

(15)

The b.c. B and C can be shown to correspond to some "excluded volume effect" on the boundary.

If the length of the boundary is fixed, the general form of the action is

$$E = \beta|S| - \gamma\frac{1}{4\pi}|R|, \tag{16}$$

where $|R|$ is the total intrinsic curvature of S, which is no more a topological invariant. However, $|R|$ is related by the Euler formula to the total extrinsic curvature of the boundary ∂S in S and may be expressed as a boundary term. In a first step we shall neglect $|R|$ by setting $\gamma = 0$ and see how the average area of a surface with boundary of length P varies with β. For model II (square lattice) with b.c A, the partition function $Z_P(\beta)$, which has to be identified with the Green function $\langle \mathrm{Tr}(\phi^P)\rangle$ of the planar ϕ^4 model, admits a simple explicit algebraic expression (P has to be even)

$$Z_P(\beta) = \frac{P!}{(\frac{1}{2}P)!(\frac{1}{2}P+2)!}a^P\left(P+2-\tfrac{1}{2}Pa^2\right), \tag{17}$$

with

$$a^2 = \tfrac{1}{24}e^{\beta}\left(1-\sqrt{1-48\,e^{-\beta}}\right). \tag{18}$$

The partition function Z_P becomes singular at the critical coupling $\beta_c = \ln 48$, where a is singular, and its non-analyticity is of order $(\beta-\beta_c)^{3/2}$ (to be compared with the power $\frac{5}{2}$ for the closed case). This simply means that the general term of the expansion of Z_P in powers of $e^{-\beta}$ behaves like

$$Z_P = \sum_{k=0}^{\infty} a_k e^{-2k\beta}, \qquad a_k \underset{k\to\infty}{\sim} e^{2k\beta_c}k^{-5/2}. \tag{19}$$

Consequently, the average area

$$A(P,\beta) = -\frac{\mathrm{d}}{\mathrm{d}\beta}\ln Z_P = \frac{1}{2}\frac{P(P+2)(a^2-1)}{P(2-a^2)+4} \tag{20}$$

remains finite at β_c and has a singularity of order $(\beta-\beta_c)^{1/2}$.

However, it is interesting to see how this quantity behaves when the length of the boundary P becomes large. Above the critical coupling, the area grows linearly with P:

$$A(P,\beta) \underset{P\to\infty}{\sim} P\,C(\beta), \qquad \beta > \beta_c, \tag{21}$$

but the coefficient of proportionality diverges at β_c as

$$C(\beta) \sim \frac{1}{4\sqrt{\beta-\beta_c}}, \tag{22}$$

and at the critical coupling, the area grows like the square of the length of the boundary

$$A(P,\beta_c) \underset{P\to\infty}{\sim} \tfrac{1}{8}P^2, \tag{23}$$

as we could naively expect for a two-dimensional flat surface.

Assuming that indeed at the critical coupling the surface S and the boundary ∂S do not develop some anomalous Hausdorff dimension, we can try now to construct a continuum limit by introducing a physical cutoff ε defined as the length of the edge of each elementary square, expressed in a physical unit, and by defining a physical area A_R and a physical boundary length L_R simply as

$$A_\mathrm{R} = \varepsilon^2|S|, \qquad L_\mathrm{R} = \varepsilon|\partial S|. \tag{24}$$

The continuum limit will be obtained by "renormalizing" $\beta(\varepsilon)$ and by taking the limit $\varepsilon \to 0$, L_R fixed. From (21) and (22) we must choose

$$\beta(\varepsilon) = \beta_c + \varepsilon^2\lambda_\mathrm{R} + \mathrm{O}(\varepsilon^3) \tag{25}$$

in order to obtain a finite limit for A_R. We find

$$A_\mathrm{R}(L_\mathrm{R},\lambda_\mathrm{R}) = \lim_{\varepsilon\to 0}\varepsilon^2 A\left(\frac{L_\mathrm{R}}{\varepsilon},\beta(\varepsilon)\right) = \frac{L_\mathrm{R}^2}{8+4L_\mathrm{R}\sqrt{\lambda_\mathrm{R}}}. \tag{26}$$

The "renormalized" coupling λ_R has the dimension $(\text{length})^{-2}$ and must be $\geqslant 0$.

The variance of the physical area $\langle A_R^2\rangle_{\text{conn.}} = \langle A_R^2\rangle - \langle A_R\rangle^2$ has also the continuum limit

$$\langle A_R^2\rangle_{\text{conn.}} = \frac{L_R^3}{8\sqrt{\lambda_R}\left(2 + L_R\sqrt{\lambda_R}\right)^2}, \tag{27}$$

and one can define a "specific heat" which in the "thermodynamic limit" $\langle A_R\rangle \to \infty$ goes to the constant

$$C = \frac{\langle A_R^2\rangle_{\text{conn.}}}{\langle A_R\rangle} \underset{L_R\to\infty}{\simeq} \frac{1}{2\lambda_R}. \tag{28}$$

A remarkable result is that this behaviour is universal. One can check that eqs. (21)–(23) still hold with other boundary conditions (b.c. B and C) and that they are still valid for triangular lattices (model I, I′ and I″). The continuum limit is always defined by eqs. (24), (25) and the physical area A_R has the universal form

$$\langle A_R\rangle = \frac{L_R^2}{a + bL_R\sqrt{\lambda_R}}. \tag{29}$$

The coefficients a and b depend on the specific model considered.

One can give a geometrical interpretation to eq. (29). For small surfaces ($L_R \ll \lambda_R^{-1/2}$), the mean area grows like the square of the perimeter; this is a good indication of the two-dimensional character of the surface. For large surfaces ($L_R \gg \lambda_R^{-1/2}$) the mean area grows only like the perimeter. This is reminiscent of a surface with constant negative curvature R (for instance the Poincaré disc), where the area of a circle A depends on its perimeter L as

$$A = \frac{4}{|R|}\left[\left(\pi^2 + \tfrac{1}{8}|R|L^2\right)^{1/2} - \pi\right], \tag{30}$$

and grows indeed like L^2 for small L but only like L for large L. Therefore the simplest interpretation of eq. (29) is that in the continuum limit we have some surface with a mean curvature proportional to $-\lambda_R$. At the critical point the curvature vanishes and we recover the scale-invariant result

$$\langle A_R\rangle = \frac{1}{a}L_R^2.$$

We now consider the role of the intrinsic curvature R. We shall give explicit results in the case of the triangular model I′ (ϕ^3 without tadpoles) with the boundary condition C (irreducible functions). The extension to other cases is probably possible. Let us consider a surface S (with a boundary ∂S) made of equilateral

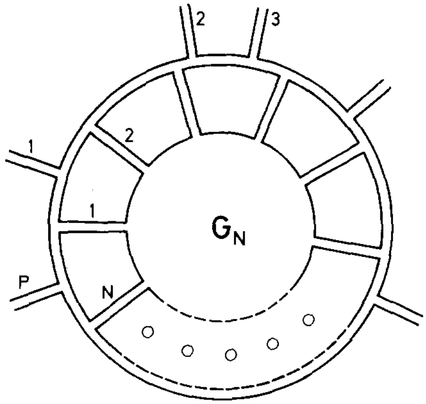

Fig. 3. The decomposition of a planar irreducible ϕ^3 graph without tadpoles into an external loop and an internal part.

triangles. The integral of the intrinsic curvature may be expressed as a sum over the internal vertices v of S:

$$\frac{1}{4\pi}\int_S \sqrt{g}\, R = \tfrac{1}{6}\sum_{v\in S-\partial S}(6-n_v), \tag{31}$$

where n_v is the number of triangles of S which meet at the vertex v. Using the Gauss-Bonnet formula we can reexpress this as a sum over vertex of the boundary

$$\frac{1}{4\pi}\int_S \sqrt{g}\, R = 1-\tfrac{1}{6}\sum_{v\in\partial S}(3-n_v). \tag{32}$$

This may be expressed in terms of the dual lattice, that is in terms of diagrams of a ϕ^3 theory. Indeed, let us consider a planar one-particle irreducible graph G with P external legs of the ϕ^3 theory, with no tadpoles ($P > 3$). Due to the planar geometry, we can always decompose G in a unique way into an external loop L with P external legs and N internal legs which are connected to the remaining part of the graph G-L, which appears therefore as a (not necessarily connected and irreducible) graph with N external legs (see fig. 3). Reciprocally one can obtain all irreducible graphs in that way. Taking into account the symmetry factors this is expressed by the relation between Green functions G_N and irreducible functions Γ_p for $p > 3$ in the planar $g\phi^3$ model without tadpoles:

$$\Gamma_p(g) = \sum_{N=0}^{\infty}(-3g)^{N+P}\frac{(N+P-1)!}{N!(P-1)!}G_N(g). \tag{33}$$

The counting factor is simply the number of different loops with P external legs and N internal legs.

It is now easy to see that the number of vertices on the boundary of a surface S is equal to the number of external legs of the dual graph G and that the quantity $\Sigma_{v \in \partial S} n_v$ is simply equal to $N + 2P$ where N is the number of internal legs of the external loop of G. Therefore, the total intrinsic curvature (32) is given by

$$\frac{1}{4\pi} \int_S \sqrt{g} R = 1 - \tfrac{1}{6}(P - N), \tag{34}$$

and the partition function corresponding to the action (16) for model I′, b.c. C, for a surface with boundary of fixed length P is

$$Z_P(\beta, \gamma) = \sum_{N=0}^{\infty} (-3g)^{N+P} z^{6+N-P} \frac{(N+P-1)!}{N!(P-1)!} G_N(g), \tag{35}$$

where $z = e^{\gamma/6}$ and $g = e^{-\beta}$. (36)

The generating function $G_N(g)$ was shown in [9] to admit the integral representation

$$G_N(g) = \int d\lambda\, \lambda^N v(\lambda, g), \tag{37}$$

where the function $v(\lambda, g)$ has a compact support [2a, 2b] and is the density of eigenvalues of the matrix ϕ in the limit $N \to \infty$. Therefore we have for Z_P the integral representation

$$Z_P(\beta, \gamma) = (-1)^P z^{6-P} \int d\lambda \left(\lambda z + \frac{1}{3g}\right)^{-P} v(\lambda), \tag{38}$$

with the explicit form for $v(\lambda)$

$$v(\lambda) = (3g\lambda + 1 - \tau) \sqrt{-\lambda^2 - \frac{2\sqrt{\tau}}{1 - 2\tau}\lambda - \frac{9\tau - 4}{(1 - 2\tau)^2}}, \tag{39}$$

with

$$3g = \sqrt{\tau}(1 - 2\tau). \tag{40}$$

For $\lambda \neq 0$, Z_P is always singular at β_c (corresponding to $\tau_c = \frac{1}{6}$). β_c is independent of γ and corresponds to the point where the coefficient of the negative square root singularity of $v(\lambda)$ (corresponding to $\sqrt{\lambda - 2a}$) vanishes. The continuum limit result (29) remains valid, but with coefficients a and b depending on γ. The mean total curvature

$$\left\langle \frac{1}{4\pi} R_{\text{bulk}} \right\rangle = \frac{\partial}{\partial \gamma} Z_p(\beta, \gamma), \tag{41}$$

may be computed in the same way. After lengthy calculations, it appears that for γ close to zero, it diverges in the continuum limit like $1/\varepsilon$. This is not unexpected and means that the metric becomes singular near the boundary. In order to obtain a finite result, we have in fact to renormalize γ (which is a marginal variable). There is a critical value $\gamma_c = \frac{1}{6}\ln\frac{3}{5}$ where the total curvature has a continuum limit. γ has to be renormalized as a function of the cutoff ε:

$$\gamma(\varepsilon) = \gamma_c + \varepsilon\mu_R + O(\varepsilon^2), \tag{42}$$

and we define the continuum limit by the limit $\varepsilon \to 0$, L_R, λ_R, μ_R fixed. One gets a finite value for the mean total curvature (41), which behaves, in the large volume limit, as

$$\frac{1}{4\pi}\langle R_{\text{bulk}}\rangle \underset{L_R \to \infty}{\simeq} L_R\left(c\mu_R - d\sqrt{\lambda_R}\right), \tag{43}$$

where c and d are *positive* numerical constants. We see from (43) that, as expected, μ_R acts as a chemical potential for the curvature, which is proportional to μ_R and to the length of the boundary. At the critical point ($\gamma = \gamma_c$ or $\mu_R = 0$), we expect to have "subtracted" the effect of the boundary. Then, for $\lambda_R > 0$, the total curvature is *negative* and proportional to L_R, hence to the volume A_R, in agreement with our interpretation of the continuum limit as a space with negative curvature proportional to λ_R. For $\lambda_R = 0$ the total curvature is zero, as expected for a flat surface.

Before considering extensions of these simple models, let us discuss in a more critical way our conclusions. In the derivation of the continuum limit and its interpretation, we have assumed that the surface and its boundary do not develop an anomalous Hausdorff dimension. As we have seen, this assumption does not lead to any contradiction (as far as we have gone) but one cannot exclude completely more complicated behaviours. A real check would need the computation of "local" instead of global quantities (for instance the mean area of a sphere as a function of its radius in the surface [14]). Unfortunately, the diagrammatic interpretation is not useful in computing such quantities and some numerical work will probably be needed.

Another objection to this approach to construct "quantum gravity" is that we start from a completely discrete model where the length of the links is fixed and where the curvature takes integer values. One may think that the Regge-calculus approach [3,4], where one considers a lattice with fixed topology and where the lengths of the links are the dynamical variables is more appropriate. However, it is well known that in the continuum limit, models where the fields take discrete values may be equivalent to model with continuous fields (for instance the Ising model and the ϕ^4 theory). The Regge approach suffers also from problems (a continuum limit is far less easy to construct than here and it is difficult to recover the usual conformal anomaly) [15]. It is of course easy to couple the two models by summing over all possible planar lattices *and* then integrating over all possible link lengths for each lattice, with an appropriate action.

In the same spirit than in [6], we can extend those models to higher dimensions by considering manifolds made by gluing d-dimensional regular simplices (or hypercubes). It is possible to construct field theories where Feynman diagrams are dual of such lattices but it is not possible to write the integral of the curvature, which is now a relevant quantity, in terms of these theories. Let us point out that for $d > 2$, if a continuum limit exists for such models, it is quite possible that the space will develop an anomalous dimension $d_{\mathrm{H}} \neq d$ at short distances, as suggested by the β-function of pure Einstein gravity in $2+\varepsilon$ dimensions, which has a non-trivial UV fixed point [16].

Finally, we can construct various surface models by embedding the models that we have considered in a bulk d-dimensional space. Beside the lattice-imbedding of [5, 7, 8] which describes a free theory of branched polymers with Hausdorff dimension 4, one can construct continuous imbedding by assigning to each vertex i of a lattice S a position given by a vector $\boldsymbol{X}_i$ in $\mathbb{R}^d$. Possible actions are the sum of the areas of the corresponding triangles in $\mathbb{R}^d$ (as in [17, 18]) or more generally any translationally and rotationally invariant symmetric positive function of the positions of the vertices of each triangle [19]. One can also assign a position $\boldsymbol{X}_{\tilde{i}}$ to each vertex $\tilde{i}$ of the dual lattice and choose an action of the form

$$S = \sum_{\substack{\text{links of the dual}\\ \text{lattice } (\tilde{i},\tilde{j})}} f\left(|\boldsymbol{X}_{\tilde{i}} - \boldsymbol{X}_{\tilde{j}}|^2\right), \tag{44}$$

where f is some positive increasing function. In particular one can choose $f(|X_i - X_j|^2) = |X_i - X_j|^2$. In this case the triangular (or square) lattice model is equivalent to a planar d-dimensional ϕ^3 (or ϕ^4) theory with gaussian propagators*. We do not know if those models belong to the same universality class than the model of [5, 8] or to some other class (or classes). A critical question is to determine the "most probable" intrinsic geometry of the underlying lattice (before considering its imbedding). If the lattice is really a two-dimensional object (as suggested by our study for $d = 0$) then the imbedding will be a surface-like object with infinite Hausdorff dimension [17–19]. On the contrary if it is a tree-like object (typical examples are provided by the so-called parquet graphs), then its imbedding will be a branched-polymer-like object, with Hausdorff dimension 4 [23] as in [5, 8].

A possible signal to discriminate such behaviours is the large-order behaviour of the partition function for open surfaces with a fixed boundary, which is expected to

* Such models have been recently proposed by Fröhlich in [20]. The possibility that planar scalar theories might describe some surface models at the critical coupling where their perturbative series diverge has already been suggested by Greensite in [21], and studied in the case of "fishnet diagrams". For earlier considerations see [22].

be of the form

$$Z(\beta)=\sum_K \mathrm{e}^{(\beta_c-\beta)K} A_K, \qquad A_K \underset{K\to\infty}{\sim} K^{-\varepsilon}. \tag{45}$$

If $\varepsilon > 2$ (this is the case for $d=0$ where $\varepsilon = 2.5$) the average surface remains finite at β_c. If $\varepsilon < 2$, the average surface diverges (this is the case for the model of [5, 8] where $\varepsilon = \frac{3}{2}$) at β_c. A study of the expression of the vacuum energy $E^1(g)$ in the planar ϕ^4 model in $d=1$* (obtained in [9] by using the equivalence of this quantum mechanical system with a free Fermi gas) suggests that the singularity of $Z(\beta)$ is in that case of the form $(\beta-\beta_c)\ln|\beta-\beta_c|$, which corresponds to the marginal case $\varepsilon = 2$. One may speculate that ε depends on the dimension d and that perhaps $\varepsilon > 2$ for $d<1$ and $\varepsilon < 2$ for $d>1$. For comparison, let us note that the Liouville string theory [24] describes for $d=-\infty$ a surface with constant negative curvature imbedded in $\mathbb{R}^d$ with a gaussian weight (and therefore with "infinite" Hausdorff dimension [18, 19, 25, 26]) and has been shown to be tachyon-free for $d \leqslant 1$, where there seems to be a qualitative change in the ground state and the spectrum of the theory [27]! New analytical techniques and numerical work will be needed in order to study such surface models and to clarify these questions.

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* This corresponds to the imbedding of the dual of a square lattice with the action of the form (44) with $f(X^2)=|X|$.

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