A COMMENT ON INDUCED GRAVITY

François DAVID [1]
The Institute for Advanced Study, Princeton, NJ 08540, USA

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Because of the non-perturbative ambiguity of the stress-energy tensor in theories with dynamical breaking of conformal invariance, there is no unique self-consistent way to compute the Newton's and cosmological constants of the corresponding induced gravitation theories, when leading quadratic or quartic divergences are present.

Following an approach initiated by Zel'dovich [1] and Sakharov [2], Adler has proposed that gravity may be considered as an effective long distance effect which arises from dynamical symmetry breaking in an underlying conformally invariant quantum field theory [3]. Neglecting the fluctuations of the space–time metric $g_{\mu\nu}(x)$ and quantizing the matter fields in the corresponding curved background metric, one gets an effective action in terms of the metric which, in the long distance and low energy limit, reduces to the Einstein–Hilbert gravitational action

$$S_{\text{ind}} = \frac{1}{16\pi G_{\text{ind}}} \int d^4x \sqrt{-g}\,(R - 2\Lambda_{\text{ind}})\,. \tag{1}$$

Adler [4,5] and Zee [6] have shown that the induced Newton constant G_{ind} and the induced cosmological constant Λ_{ind} can be written in terms of correlation functions of the stress-energy tensor $T^{\mu\nu}(x)$ in the flat metric ($g^0_{\mu\nu} = \delta_{\mu\nu}$):

$$\Lambda_{\text{ind}}/G_{\text{ind}} = -2\pi \langle T^\mu_\mu(0)\rangle_0\,, \tag{2}$$

$$(16\pi G_{\text{ind}})^{-1} = -\frac{i}{96}\int d^4x\, x^2 \langle T\{\tilde{T}^\mu_\mu(x), \tilde{T}^\nu_\nu(0)\}\rangle_0\,, \tag{3}$$

where

$$\tilde{T}^{\mu\nu}(x) = T^{\mu\nu}(x) - \langle T^{\mu\nu}(x)\rangle\,. \tag{4}$$

Restricting oneself to a pure Yang–Mills theory for simplicity, the stress-energy tensor is related to the action density via the trace-anomaly relation

$$T^\mu_\mu(x) = [\beta(g)/2g]\ \mathrm{Tr}[F^{\mu\nu}(x)\,F_{\mu\nu}(x)]\,. \tag{5}$$

One of the problems raised by such an approach is that the formula (2) for $\Lambda_{\text{ind}}/G_{\text{ind}}$ presents quartic ultraviolet divergences (of order a^{-4} where a is some UV regulator). Similarly, from the operator product expansion, the integrand of the rhs of (3) behaves for small x as

$$\langle T\{\tilde{T}^\mu_\mu(x), \tilde{T}^\nu_\nu(x)\}\rangle \underset{x\to 0}{=} (x^2)^{-4}\{\log(x^2)\} + (x^2)^{-2}\{\log(x^2)\}\langle T^\mu_\mu(0)\rangle_0 + O((x^2)^{-1})\,. \tag{6}$$

Therefore, the leading term of (6) gives quadratic UV divergences for G_{ind}. Adler [4,5] and Zee [6] have in fact argued that such divergences are formal ones and disappear when one uses an analytic regulator such as dimensional regularization or a ξ-function regularization. This corresponds to the well-known integration rule in a 2ω-dimensional space [7].

$$\int d^{2\omega}p\,(p^2)^{-\alpha} = 0 \qquad \forall \omega, \alpha \tag{7}$$

or equivalently, to a Hadamard's finite part description for the integration at $x^2 = 0$. With such a prescription one can get rid of such divergences *at all orders* of perturbation theory and get a finite, scheme-independent result for G_{ind} and Λ_{ind}. For gauge theories Λ_{ind} appears related to the "gluon condensate" which has

[1] On leave from SPhT Saclay, France.

been extensively discussed since the works of Shifman et al. [8]. Khuri has discussed in detail under what conditions one can get a positive G_{ind} in such schemes [9].

The purpose of this note is to explain why such a prescription is in fact not consistent and why one cannot get rid of quadratic or quartic UV divergences in such a simple way. This follows from the observation we have made in some two-dimensional models, where composite operators such as T^{μ}_{μ} may be computed in a non-perturbative way via the $1/n$ expansion [10]. It appears that when computed with an analytic regulator, such operators are finite but present *non-perturbative ambiguities* which make them *complex* and *multivalued.* For instance, for Yang–Mills theories the stress-energy tensor is expected to have an imaginary part and to be of the form

$$\langle T^{\mu}_{\mu}(0)\rangle_0 = (A \pm \mathrm{i}B)\, M^4\,, \tag{8}$$

where M is a renormalization group invariant scale mass.

Such an ambiguity comes from the fact that the quartic divergences in T^{μ}_{μ} are contained in a series in the bare coupling constant which is not Borel summable. According to the summation prescription chosen to define that divergent part (above or below the positive real axis in the Borel plane), we get two possible determinations for the finite part, corresponding to the two values of $\langle T^{\mu}_{\mu}(0)\rangle_0$ in (8).

The existence of these ambiguities is a general problem which occurs in the definition of the terms of the Wilson's operator product expansion. Therefore, one can show that the leading term in (6), which corresponds to the quadratic divergences of G^{-1}_{ind}, suffers from the same problem and that the finite part of (3) has the same kind of ambiguity, namely:

$$G^{-1}_{\text{ind}} = (C \pm \mathrm{i}D)\, M^2\,. \tag{9}$$

Let us point out that this ambiguity for G_{ind} was already noticed by Adler (see the appendix B of ref. [5]). He indeed noticed that the finite part of the integral (3) may be defined by dimensional regularization as the limit as ω goes to 2 (2ω being the dimension of space–time) of a regularized form of the integral (3). This regularized integral has in fact a cut starting from $\omega = 1$ to $\omega = +\infty$, so that the limits as $\omega \to 2 \pm \mathrm{i}0_+$ are different (and complex conjugate). This is the ambiguity that we have displayed in (9). In order to avoid that problem, Adler argued that hermiticity requires that the result of (3) must be real, so that one must take the real part of (9) for G^{-1}_{ind}. This recipe is in fact incorrect: hermiticity requires indeed that the result must be real; however, any linear combination of the form

$$G^{-1}_{\text{ind}} = \Big[(\tfrac{1}{2} - \mathrm{i}y) \lim_{\omega\to 2+\mathrm{i}0_+} + (\tfrac{1}{2} + \mathrm{i}y) \lim_{\omega\to 2-\mathrm{i}0_+}\Big]\, G^{-1}_{\text{ind}}(\omega) \tag{10}$$

satisfies that requirement and gives an arbitrary result for G^{-1}_{ind}

$$G^{-1}_{\text{ind}} = (C + yD)\, M^2\,. \tag{11}$$

The same argument holds obviously for Λ_{ind}.

Some authors have computed Λ_{ind} and G^{-1}_{ind} in SU(N) gauge theories using the dilute instanton gas approximation, and obtained an UV finite result [11]. This does not contradict our argument. The above ambiguities come from the mixing of the operators T^{μ}_{μ} and $(\tilde{T}^{\mu}_{\mu})^2$ with the operator $\mathbf{1}$, and can be shown to occur only in the sector with zero topological charge (provided that $N > 12/11$ and $6/11$, respectively). A general discussion on the role of instantons in the renormalization of composite operators is made in ref. [12].

Therefore, it appears that the arbitrariness contained in the subtraction of the quadratic or quartic UV divergences of (2) and (3) is not suppressed in general by using an analytic regulator, but simply transferred at a non-perturbative level. There may perhaps be some cases where Im Λ and Im G^{-1} vanish (the simplest example is provided by the two-dimensional O(N) non-linear sigma model in the limit $N = \infty$ [10]), but this seems to us an exceptional and non-generic situation. In particular, this arbitrariness should also be present when one takes into account the fluctuation of the metric itself, for instance, by using a higher derivatives theory of gravitation [13].

The only general way to avoid that problem, which comes from the mixing of the operator T^{μ}_{μ} with the operator $\mathbf{1}$, is to use a supersymmetric theory of matter. It is well-known that in that case, quadratic divergences disappear and for instance, in supersymmetric Yang–Mills, $(\beta/g)\,\mathrm{Tr}\,F^2$ is a member of a super multiplet and cannot mix with **1**. However, in such a case, one must couple the matter field to the background metric in a supersymmetric way, and one is inevitably led to the study of induced supergravity. It should be interesting to see what are the analogs of the formulas (2) and (3) in that case.

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