AMBIGUITIES OF RENORMALIZED Φ_4^4 FIELD THEORY AND THE SINGULARITIES OF ITS BOREL TRANSFORM

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The analytic structure of the Borel transform of renormalized ϕ_4^4 theory can be deduced from the small regulator expansion of the regularized theory. The coefficients of Symanzik's local effective lagrangian describing this expansion are shown to be ambiguous, although well defined in perturbation. We deduce that the UV singularities of the Borel transform (renormalons) of ϕ_4^4 are proportional to insertions of local composite operators, as conjectured by Parisi. However, the renormalization group functions do not a priori contain renormalons. This can be proven at all orders of the $1/N$ expansion.

Renormalizable "infrared free" quantum field theories such as ϕ_4^4 or QED_4 are known to suffer from non-perturbative singularities at high energy [1]. The so-called "Landau ghosts" are related to the "triviality" of such theories, that is to the fact that renormalization group arguments [1,2] indicate that there is no consistent way to take the large cut-off limit of the regularized theory such that the renormalized coupling constant is not driven to zero. Those arguments have been consolidated by recent rigorous results from lattice ϕ_4^4 [3].

It is then important to understand what forbids in such theories to reconstruct consistent Green functions from their renormalized perturbative series, which are free of divergences. It was recognized by Gross and Neveu [4] that those renormalization effects are such that some individual Feynman amplitudes are positive and grow like the factorial of the number of vertex, and thus should produce a singularity on the positive real axis of the Borel transform of the perturbative series, making them non-Borel-summable. Such UV singularities were rediscovered by Lautrup [5] and 't Hooft [6] and are usually denoted by the generic name of "renormalons".

A next step was performed by Parisi [7] who conjectured that UV renormalons were proportional to the insertions of local irrelevant composite operators (such as ϕ^6, $\phi\Delta^2\phi$, ...). The argument is the following: the momentum dependence of the effective coupling constant $\bar{g}(p)$ of the ϕ_4^4 theory is (in the one loop approximation)

$$\bar{g}(p) = [1/g - \beta_2 \operatorname{Ln}(p/\mu)]^{-1}, \tag{1}$$

where β_2 is positive and is the first coefficient of the β function. We get for the Borel transform of (1)

$$\bar{b}(p) = (p/\mu)^{b\beta_2}. \tag{2}$$

Parisi argues that inserting $\bar{b}(p)$ in the Borel transformed Dyson–Schwinger equations defining the Green functions should generate UV singularities, when integrating at large momenta, as the Borel variable b increases. Moreover, (2) is very similar to the behaviour of $\bar{g}(p)$ for nonrenormalizable $\phi_{4+\epsilon}^4$ ($\epsilon > 0$):

$$\bar{g}(p) \simeq (p/\mu)^{\epsilon}, \tag{3}$$

where the BPHZ theorem tells us that UV divergences are proportional to the insertions of local operators with arbitrarily large dimensions. It is expected that the same structure occurs for renormalons.

In this letter we present a new approach to those problems which permit to describe precisely the analytic properties of the Borel transform of the renormalized ϕ_4^4 theory. In particular we shall give rigorous results at all orders of the $1/N$ expansion of the vectorial $(\boldsymbol{\phi}^2)^2$ model. The detailed analysis and complete proof will be presented in a subsequent paper.

The fundamental idea is to relate this question to the small regulator behaviour of the regularized ϕ_4^4 theory. Indeed let us consider some regularized form of the theory, characterized by some regulator a and by the bare coupling constant g_B. The renormalization group tells us that, g_B being fixed, the renormalized coupling constant g_R goes to zero with a as

$$a \simeq C\mu^{-1} \exp(-1/\beta_2 g_\mathrm{R}) . \tag{4}$$

From eq. (4) we see that the Borel transform with respect to g_R is related to the Mellin transform with respect to a, which describes the small a behaviour of the bare Green functions.

In order to make this idea more precise we shall consider the massless ϕ^4 theory and use for convenience a Pauli–Villar regulator, which corresponds to the bare action density

$$\mathcal{L} = \tfrac{1}{2}\boldsymbol{\phi}(-\Delta + a^2\Delta^2 + m_\mathrm{BO}^2)\boldsymbol{\phi} + (g_\mathrm{B}/8N)(\boldsymbol{\phi}^2)^2 . \tag{5}$$

The bare mass m_BO is chosen in such a way that the renormalized mass is equal to zero. According to Symanzik [8], at some order g_B^k, the small a expansion of some P-points irreducible bare function $\Gamma_\mathrm{B}^{(P)}$ is a sum of terms of the form $a^{2p}\,\mathrm{Ln}^q a$ $(p \geqslant 0, 0 \leqslant q \leqslant k)$ and may be described by a "local effective lagrangian" (LEL) containing terms of the form ϕ^6, ϕ^8, $\phi\Delta^2\phi$, etc.:

$$\mathcal{L}_\mathrm{eff} = \sum_{n=0}^{\infty} a^{2n} A_n , \tag{6a}$$

$$A_n = \sum_{\substack{\text{operators } O_i \\ \dim O_i = 2n+4}} Z_i(g_\mathrm{B}, a) O_i[\phi] . \tag{6b}$$

The functions $Z_i(g_\mathrm{B}, a)$ can be obtained in perturbation under the form

$$Z_i(g_\mathrm{B}, a) = \sum_{k,q} C_{k,q}\, g_\mathrm{B}^k (\mathrm{Ln}\, a\mu)^q , \tag{7}$$

where the coefficients $C_{k,q}$ are dependent upon the renormalization scheme chosen to define the operators O_i. As a consequence the dominant term ($n = 0$) corresponds to the usual coupling constant and wave function renormalizations

$$\Gamma_\mathrm{B}^{(P)}(g_\mathrm{B}, a) = Z^{-P}(g_\mathrm{B}, a\mu)\Gamma_\mathrm{Ren}^{(P)}(g_\mathrm{R}, \mu) + \mathrm{O}(a^2\,\mathrm{Ln}^{\cdot} a) , \tag{8}$$

where

$$\int_{g_\mathrm{B}}^{g_\mathrm{R}} \frac{\mathrm{d}t}{\beta(t)} = \mathrm{Ln}(a\mu) , \tag{9a}$$

and

$$Z(g_\mathrm{B}, a\mu) = \exp\left(\int_{g_\mathrm{B}}^{g_\mathrm{R}} \mathrm{d}t\ \gamma(t)/\beta(t)\right) . \tag{9b}$$

The Callan–Symanzik β and γ functions are subtraction scheme dependent (and here depend on the regularization). The subdominant terms of order a^2, a^4, ... in (8) correspond to the insertions of operators with dimension 6, 8, ... at zero momentum in $\Gamma^{(P)}$.

Now, if we sum the series in g_B, using renormalization group arguments to sum the Ln a (as done in ref. [8]), we expect for Γ_B a small a expansion of the form

$$\Gamma_\mathrm{B}(g_\mathrm{B}, a) \equiv \sum_{n\geqslant 0} a^{2n} \sum_{p\geqslant 0} \frac{1}{(\mathrm{Ln}\, a\mu)^p} \times \sum_{q=0}^{p} \mathrm{Ln}^q(\mathrm{Ln}\, a\mu) C_{n,p,q}(g_\mathrm{B}, \mu) . \tag{10}$$

From (10) we expect that the Mellin transform of Γ_B with respect to a

$$M(s) = \int_0^{\infty} \mathrm{d}a\, a^{-s-1}\Gamma_b(g_\mathrm{B}, a) , \tag{11}$$

is analytic for Re $s < 0$ and has branch points at positive even integer values of s. The discontinuity Δ_n at $s = 2n$ corresponds to the nth term in (10) and thus corresponds to the insertion of the corresponding operators of the LEL in Γ_B (see fig. 1).

We now restrict ourselves to the $1/N$ expansion of the ϕ_4^4 model, where explicit calculations can be done [9]. It is obtained by introducing a composite field σ and rewriting the action (5)

$$\mathcal{L} = \tfrac{1}{2}\boldsymbol{\phi}[-\Delta + a^2\Delta^2 + m_\mathrm{BO}^2 + (\mathrm{i}/\sqrt{N})\sigma]\boldsymbol{\phi} + (1/2g_\mathrm{B})\sigma^2 . \tag{12}$$

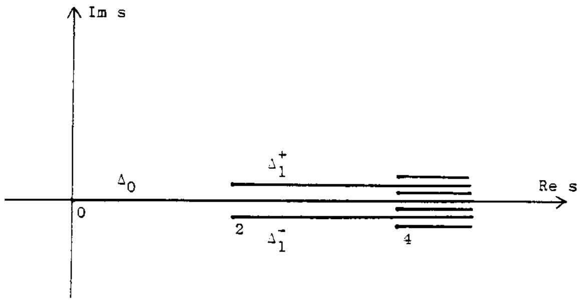

Fig. 1. The analytic structure of the Mellin transform $M(s)$ with respect to the UV regulator a.

We get the $1/N$ expansion in terms of Feynman diagrams involving the usual ϕ propagator

$$D(p) = (p^2 + a^2 p^4)^{-1} , \tag{13}$$

and the σ propagator which corresponds to the sum of the "bubbles diagrams"

$$G(p) = \left(\frac{1}{g} + \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} D(k) D(p-k) \right)^{-1} . \tag{14}$$

The Mellin transforms of D and G have already the analytic structure described above. Using tools of perturbation and renormalization theory (α-parametric representation, desingularization operators) it is possible to show that this structure extends to any amplitude of the $1/N$ expansion [10]. Let us discuss only the final result.

Theorem. At any order of the $1/N$ expansion, the Mellin transform $M(s)$ of any bare irreducible function $\Gamma_B(x; g_B, a)$ has the following analytic structure:

(a) $M(s)$ is analytic away from the positive real axis with branch points at $s = 2n$ (n non-negative integer).

(b) The discontinuity $\Delta_n(s)$ of $M(s)$ at $s = 2n$ exhibits the so-called "Stokes phenomenon": there are 2^n possible determinations of Δ_n, depending on the sheet developed by the branch points at $s = 2p < 2n$ that we consider.

(c) Each $\Delta_n(s)$ may be decomposed into terms $F(s)$ which correspond to nonambiguous (single valued) insertions of composite operators in Γ, times multivalued coefficients $R(s)$ conjugate to those operators and which carry the ambiguity. More precisely, we can write

$$\Delta_n(x;s) = \sum_{\substack{\{O_i\} \\ \Sigma_i (\dim O_i - 4) = 2n}} F_{\{O_i\}}(x;s) \otimes_i R_i(s + \dim O_i - 4) . \tag{15}$$

In (14) the sum runs over all families of composite operators O_i with dimension greater or equal to 4 and $\otimes$ means the usual Borel convolution product

$$f \otimes g(s) = \int_0^s dt\, f(t)\, g(s-t) \tag{16}$$

(the Borel transform of the usual product is the Borel convolution product of the Borel transforms).

The functions F and the coefficients R are such that

(d) $F_{\{O_i\}}(x;s)$ corresponds diagrammatically to the Mellin transform of the function Γ with insertions of the operators O_i at zero momenta (in some renormalization scheme which makes those insertions UV finite) and is analytic for $-2 < \mathrm{Re}\, s < +2$, with branch points at $s = 2m$ (m integer > 0).

(e) Each $R_i(s)$ is defined only for $\mathrm{Re}\, s > 0$ and is of the form:

$$R_i(s) = \theta(s) \times \text{an } \textit{analytic} \text{ function along the positive real axis} . \tag{17}$$

This analyticity property is true only for appropriate renormalization procedures.

(f) The ambiguity of Δ_n is contained in the R_i's, that is, there are 2^{n_i} determinations of the R_i's ($n_i = \frac{1}{2} \dim O_i - 4$), which depend on the sheet that we consider.

The consequences for the Borel transform of ϕ_4^4. Let us now discuss the consequences of the above result for the analytic structure of the Borel transform of renormalized ϕ_4^4. Of course the following considerations will be rigorous only within the $1/N$ expansion but are expected to be valid for finite N, since no other non-perturbative effects (such as instantons) can modify this structure.

(i) Using (4), (9a) and (9b) the functions $F_{\{O_i\}}(s)$ can be shown to define by formal inverse Borel transform the perturbative renormalized Green functions of the ϕ_4^4 theory (with insertions of the operators O_i)

$$\Gamma_{\mathrm{Ren}\{O_i\}}(x;g_{\mathrm{R}}) = \text{“}\int_0^\infty\text{”}\, \mathrm{d}s\, \mathrm{e}^{-s/\beta_2 g}\, F_{\{O_i\}}(x;s)\,. \quad (18)$$

(ii) The coefficients R_i are nothing else than the Mellin transform of the coefficients Z_i of the LEL (6a,b) which describes the limit $a \to 0$. Therefore the terms involving irrelevant operators with dimension greater or equal to six are ambiguous (multivalued with $2^{(\dim Q-4)/2}$ determinations), although well defined in perturbation theory [see (7)].

(iii) Since the second discontinuity at $s = 2$ has two determinations $\Delta_1^+ \neq \Delta_1^-$ (see fig. 1), the first discontinuity $\Delta_0(s)$ has a cut at $s = 2$, which is the first UV renormalon of the corresponding renormalized Green function Γ_{Ren}. The corresponding discontinuity at $s = 2$ of $\Delta_0(s)$ is equal to $\Delta_1^+(s) - \Delta_1^-(s)$, and is therefore proportional to the insertion of dimension-6 operators [from (ii)]. This can be easily generalized to prove that all UV renormalons (the singularities of the functions $F(s)$ on the positive real axis) come from the ambiguities of the LEL (6) and are therefore proportional to insertions of composite operators. This explains and proves the conjecture by Parisi [7].

(iv) The renormalization group functions β and γ and the anomalous dimensions of the composite operators O_i are obtained from derivatives with respect to g_{R} (g_{B} being fixed) of the coefficients of the LEL, that is of the R_i's. From point (e) of the theorem, it follows that it is possible to define renormalization schemes such that the renormalization group functions do not have UV renormalons (this was also a long-standing conjecture [11]). An important and still unsolved question is to know whether dimensional renormalization satisfies this condition (this is essential for the consistency of Wilson's ϵ expansion in critical phenomena [12]).

Let us end with some more general remarks. Our analysis may be applied without difficulties to massive ϕ_4^4 and to other infrared free theories such as QED_4, with the same conclusions. Recently some authors have proposed various “non-orthodox” quantization procedures for such theories in order to try to escape triviality [13–15]. However, in order to construct a theory whose Green functions have the same perturbative expansion as ϕ_4^4 (for instance), one has to “add” to the lagrangian an infinite number of exponentially small counterterms associated to irrelevant operators with arbitrarily large dimensions, in order to cancel the ambiguities due to UV renormalons. According to the notion of LEL [8], this should be equivalent to introduce again some fundamental UV cutoff Λ of order

$$\Lambda \simeq \exp(1/\beta_2 g_{\mathrm{R}})\,, \quad (19)$$

in the theory. The question is to know whether it is possible to do this without violating some of the axioms of continuum quantum field theory or modifying the field content of the theory.

Finally, one may ask why the coefficients of Symanzik's LEL are ambiguous, although they are well defined in perturbation theory. This is in fact because those coefficients suffer from infrared ambiguities which make them non-Borel summable and are related to the problem of IR renormalons [16,17]. (Those coefficients correspond to some irreducible functions of the massless theory at zero momenta.) However, that problem is not present in UV free theories such as two-dimensional nonlinear sigma models or four-dimensional gauge theories, where the LEL is well defined and the action improvement program [16] perfectly consistent. In such theories UV renormalons problems occur only in the definition of composite operators with strictly positive dimension [18].

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References

[1] L.D. Landau et al., Collected papers of L.D. Landau (Gorden and Breach, New York, 1965).
[2] I. Pomeranchuck, V. Sudakov and K. Ter Martirosyan, Phys. Rev. 103 (1956) 784.
[3] D. Brydges, J. Fröhlich and T. Spencer, Commun. Math. Phys. 83 (1982) 123;
M. Aizenman, Commun. Math. Phys. 86 (1982) 1.
[4] D.J. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235.
[5] B. Lautrup, Phys. Lett. 69B (1977) 109.
[6] G. 't Hooft, Erice lecture notes (1977).
[7] G. Parisi, Phys. Lett. 76B (1978) 65; Phys. Rep. 49 (1979) 215.
[8] K. Symanzik, in: New developments in quantum field theory and statistical mechanics, eds. M. Lévy and P.K. Mitter (Plenum, New York, 1977) p. 265; in: Recent developments in gauge theories, eds. G. 't Hooft et al. (Plenum, New York, 1980) p. 313.

[9] R. Abe, Prog. Theor. Phys. 48 (1972) 1414; 49 (1973) 49;
S.K. Ma, Phys. Rev. A7 (1973) 2172.
[10] M.C. Bergère and F. David, in preparation.
[11] E. Brezin, unpublished.
[12] K.G. Wilson and M.E. Fisher, Phys. Rev. Lett. 28 (1972) 240;
K.G. Wilson, Phys. Rev. D7 (1973) 2911.
[13] J.R. Klauder, Phys. Rev. D24 (1981) 2599;
R.J. Rivers, J. Phys. A16 (1983) 2521.
[14] G. Gallavoti and V. Rivasseau, Phys. Lett. 122B (1983) 269.
[15] N.N. Khuri, preprint RU 82/B/62 (1983).
[16] G. Parisi, Nucl. Phys. B250 (1979) 163.
[17] F. David, Nucl. Phys. B209 (1982) 433; Saclay preprint 83/59 (1983); Nucl. Phys. B, to be published.
[18] K. Symanzik, in: Mathematical problems in theoretical physics, Lecture notes in physics, Vol. 153, eds. R. Schrader et al. (Springer, Berlin, 1982); in: Les Houches Lecture notes (August 1982), to be published;
B. Berg, S. Meyer, I. Montvay and K. Symanzik, Desy preprint 83-015 (March 1983), to be published.