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ON THE AMBIGUITY OF COMPOSITE OPERATORS, IR RENORMALONS AND THE STATUS OF THE OPERATOR PRODUCT EXPANSION

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It is not possible in general to define in an unambiguous way the vacuum expectation value of local operators such that $\langle 0|\mathrm{Tr}\, G^{\mu\nu}G_{\mu\nu}|0\rangle$ used in the QCD sum rules formalism. This phenomenon is exhibited via the $1/N$ expansion of some two-dimensional models and explains the existence of IR singularities (renormalons) in the Borel plane. Such ambiguities are present in the general definition of composite operators in any UV free theory. The consequences for the status of the operator product expansion are discussed.

1. Introduction

One of the most recent applications of Wilson's short-distance operator product expansion (OPE) [1, 2] is the so-called "QCD sum rules formalism" proposed by Shifman, Vainshtein and Zakharof (SVZ) in [3]. The principle of this formalism is that the non-perturbative effects of QCD may be taken into account by assigning non-trivial vacuum expectation values to local operators such that $\mathrm{Tr}\,(G^{\mu\nu}G_{\mu\nu})$. Such a formalism has been widely applied to the study of the spectrum of QCD. In a recent paper we have studied the status of this expansion in two-dimensional models such that the non-linear sigma model and the Gross-Neveu model [4], and its relationship with the IR singularities present in the Borel transform of the perturbative series of such models, the so-called IR renormalons [5]. With the help of the $1/N$ expansion of those models, we recovered in a rigorous non-perturbative formulation the SVZ expansion. As a consequence we argued that the IR renormalons present in the perturbative part should be cancelled by those of the non-perturbative parts of the SVZ expansion.

In sect. 2 of this paper we show that the exact mechanism of cancellation of renormalons proposed in [4] is incorrect by exhibiting explicit counter-examples. This fact seems in discrepancy with our analysis of [4] and with the SVZ operator expansion and we shall see that it may be explained only if the notion of "condensates" as usually accepted is meaningless.

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Indeed, the analysis of sect. 3 reveals that, in a given UV subtraction scheme, the v.e.v. of composite operators *remains ambiguous* in the continuum limit and is not in general a real parameter with physical significance. This phenomenon leads to important modifications for some of the conclusions of [4], and is discussed in sect. 4.

First it is the existence of these ambiguities which explains the presence of IR renormalons and their location. In fact it becomes meaningful to speak about the v.e.v. of some local operators only once the lowest terms of the operator expansion have been defined by some (partially arbitrary) resummation prescription.

Second we shall argue that in QCD, among all local operators, only the fermion condensate $\bar{\psi}\psi$ and the topological charge density $F\tilde{F}$ are free of ambiguities.

Third we shall see that such ambiguities are present in the general definition of composite operators for any massive or massless UV free renormalizable field theory. The consequences for the status of the operator product expansion will be discussed. Finally we shall point out a possible connection of these problems with those of UV renormalons.

Let us end this section by recalling the content of [4], where the reader is referred for a general introduction to the problem of IR renormalons and for the notation and the technical arguments that we use throughout this paper. We are interested in the Green functions of the two-dimensional O(N) non-linear σ model, whose classical continuum action is:

$$S=\frac{N}{2g}\int \mathrm{d}^2x\,[\partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}+\alpha(\boldsymbol{S}^2-1)]\,, \tag{1.1}$$

where $\boldsymbol{S}=(S^i)_{i=1,N}$ is an N-component real field and α a Lagrange multiplier which fixes the constraint $\boldsymbol{S}^2=1$.

The Green functions of the quantum model may be constructed by usual perturbation theory in powers of the coupling constant g, where the model is renormalizable and asymptotically free [6], and independently by the $1/N$ expansion [7]. Let us consider a generic $O(N)$ invariant Green function*

$$G(x_1,\cdots,x_{2n})=\langle \boldsymbol{S}(x_1)\cdot \boldsymbol{S}(x_2)\cdots \boldsymbol{S}(x_{2n-1})\cdot \boldsymbol{S}(x_{2n})\rangle\,. \tag{1.2}$$

The main result of [4] is theorem A of subsect. 3.3 which states that within the $1/N$ expansion, G may be written as an operator expansion:

$$G(x_i;g)=\sum_i C_i(x_i;g)\langle O_i\rangle\,, \tag{1.3}$$

where the sum runs over all O(N) invariant composite operators O_i. The coefficients C_i are defined by a (partially arbitrary) Borel resummation prescription of a series in the renormalized coupling constant g obtained from perturbation theory. The

* Only O(N) invariant objects make sense because of the IR divergences of the perturbative expansion [8].

condensates $\langle O_i \rangle$ are non-perturbative terms of order

$$\langle O_i \rangle \sim \exp\left(-\frac{1}{g|\beta_2|}\dim O_i\right) : \beta_2 = -\frac{1}{2\pi} . \tag{1.4}$$

The first term C_0 (associated to the operator $\mathbb{1}$) corresponds to usual perturbation theory. The first non-perturbative term C_1 is associated to the "spin-wave condensate" $\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle$, and the next ones to operators with dimension $\geqslant 4$.

The expansion (1.3) was obtained by looking at the regularized theory at $d = 2 - \varepsilon$ dimensions and then by taking the limit $\varepsilon \to 0$. In the discussion of subsect. 3.4 of [4] we implicitly assumed that the $\langle O_i \rangle$ have a definite real limit as $\varepsilon \to 0$. From that assumption and (1.3) we concluded that the ordinary Borel transform $\hat{C}_0(s)$ of C_0 might have IR renormalons only at points

$$s = 8\pi, 12\pi, 16\pi, \dots , \tag{1.5}$$

in the Borel plane and that, for instance, the first renormalon of C_0 at $s = 8\pi$ should be exactly cancelled by the renormalon of $C_1 \langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle$. Such cancellations have to take place in order to suppress the ambiguities which are present in each separate terms of the r.h.s. of (1.3) and which are not present in its l.h.s. We shall now see that this last conclusion is in fact incorrect.

2. An example of non-cancellation of IR renormalons

Let us simply consider the propagator of the α field

$$\langle \alpha(x) \alpha(0) \rangle - \langle \alpha(x) \rangle \langle \alpha(0) \rangle , \tag{2.1}$$

in position space. At leading order in $1/N$ it is the Fourier transform of the G propagator studied in subsect. 2.3 of [4], namely

$$G(x) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \mathrm{e}^{ip\cdot x} G(p) , \tag{2.2}$$

where

$$G(p) = 4\pi [p^2(p^2 + 4m^2)]^{1/2} \left[\ln \left(\frac{\sqrt{p^2 + 4m^2} + \sqrt{p^2}}{\sqrt{p^2 + 4m^2} - \sqrt{p^2}} \right) \right]^{-1} ; \tag{2.3}$$

m^2 is the mass gap at $N = \infty$, related to the renormalized coupling constant g by

$$m^2 = \mu^2 \mathrm{e}^{-4\pi/g} , \tag{2.4}$$

where μ is the subtraction scale. For simplicity we set $\mu = 1$ and rescale

$$g = 4\pi f . \tag{2.5}$$

In [4] we wrote $G(p)$ as an inverse Borel transform

$$G(p)=\int_C \frac{\mathrm{d}s}{2i\pi}\,\mathrm{e}^{-s/f}\hat{G}(s)(p^2)^{1-s}. \tag{2.6}$$

The Borel transform $\hat{G}(s)(p^2)^{1-s}=\hat{G}(s,p)$ is analytic in s in the Borel plane with cuts on the positive real axis starting at integer values of s. C is an anti-clockwise contour around these cuts (see fig. 1). The discontinuities

$$\Delta_n(s)=\frac{1}{2i\pi}[\hat{G}(s\text{ above the cut})-\hat{G}(s\text{ under the cut})], \tag{2.7}$$

starting at $s=n\in\mathbb{N}$ may be computed from (2.3). The first ones are easily found to be

$$\Delta_0(s,p)=4\pi(p)^{1-s}\theta(s), \tag{2.8a}$$

$$\Delta_1(s,p)=4\pi(p^2)^{1-s}2(2-s)\theta(s-1), \tag{2.8b}$$

$$\Delta_2(s,p)=4\pi(p^2)^{1-s}(2s^2-9s+8)\theta(s-2). \tag{2.8c}$$

They are equal to the (ordinary) Borel transform of the terms of the expansion (1.3) for $G(p)$ associated to operators with dimensions 0, 2 and 4 respectively.

From (2.6) we get by Fourier transform the following representation for $G(x)$

$$G(x)=\int_C \frac{\mathrm{d}s}{2i\pi}\,\mathrm{e}^{-s/f}\hat{G}(s)\frac{1}{4\pi}\frac{\Gamma(2-s)}{\Gamma(s-1)}(\tfrac{1}{4}x^2)^{s-2}. \tag{2.9}$$

For Re $s<1$ $(x^2)^{s-2}$ has to be understood as the distribution "finite part of $(x^2)^{s-2}$". We note that $\Gamma(2-s)$ introduces poles at $s=2,3,\ldots$ which correspond precisely to IR renormalons. Indeed the Borel transform $\hat{G}(s,x)$ of $G(x)$ has cuts similar to those of $\hat{G}(s,p)$. From (2.8), the first discontinuity of $\hat{G}(s,x)$ is

$$\Delta_0(s,x)=(\tfrac{1}{4}x^2)^{s-2}\frac{\Gamma(2-s)}{\Gamma(s-1)}\theta(s), \tag{2.10}$$

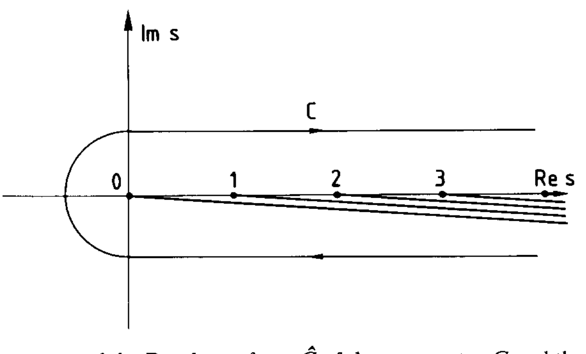

Fig. 1. The analytic structure of the Borel transform $\hat{G}$ of the propagator G and the integration contour C in (2.6) and (2.9).

and has a pole at $s=2$. But on the contrary the discontinuity at $s=1$ is

$$\Delta_1(s,x)=(\tfrac{1}{4}x^2)^{s-2}\frac{2\Gamma(3-s)}{\Gamma(s-1)}\,\theta(s-1)\,, \tag{2.11}$$

and is analytic at $s=2$. Clearly the singularity of Δ_0 at $s=2$ is not cancelled by Δ_1!

As a consequence the discontinuity of $\hat{G}(s,x)$ at $s=2$ will be ambiguous. Indeed, using (2.8) we may write $\hat{G}(s,p)$ around $s=2$ as

$$\hat{G}(s,p)=4\pi\left[(2s^2-9s+8)\ln\left[\frac{1}{2-s}\right]+R(s)\pm i\pi(5-2s\right](p^2)^{1-s}, \tag{2.12}$$

where $R(s)$ is analytic in the disc $|s-2|<1$. The (+) and (−) signs refer to the values of $\hat{G}$ in the sheets reached from above or under the real axis respectively (see fig. 1). From (2.9) and (2.12) we get for the discontinuity of $\hat{G}(s,x)$ at $s=2$

$$\Delta_2(s,x)=\lim_{\varepsilon\to 0_+}\left[2\ln(\varepsilon)\delta(s-2)+(\tfrac{1}{4}x^2)^{s-2}\frac{\Gamma(2-s)}{\Gamma(s-1)}(2s^2-9s+8)\theta(s-2-\varepsilon)\right]$$
$$+(R(2)\pm i\pi)\delta(s-2)\,. \tag{2.13}$$

The first term $\lim_{\varepsilon\to 0}[\cdot]$ is some well-defined real distribution but there is a second term, localized at $s=2$ which is ambiguous and differs if we consider the discontinuity from above (+) or under (−) the real axis.

From the analysis of [4] $\Delta_2(s,x)$ is the Borel transform of the terms associated to the operators with dimension 4 in the operator expansion (1.3). A closer look reveals that the ambiguity is only proportional to the coefficient associated to the operator α^2, which is the only one present at order $1/N$ in the expression (1.3) for $G(x)$. The other operators, $\langle\partial_\mu\partial_\nu\boldsymbol{S}\,\partial_\mu\partial_\nu\boldsymbol{S}\rangle$ for instance, are only present at order $(1/N)^0$ and are unambiguous. Thus we are led to suspect that at order $1/N$, composite operators such as α^2 may be ambiguous. We now discuss in great detail this point and its consequences.

3. Why the v.e.v. of local operators are ambiguous

3.1. CONDENSATES IN THE $1/N$ EXPANSION

In order to understand this phenomenon, let us come back to the proof of theorem A in [4]. In the construction of the renormalized $1/N$ expansion, we first looked at the regularized theory at $d=2-\varepsilon$ dimensions ($\mathrm{Re}\,\varepsilon<0$). (The $1/N$ expansion allows a non-perturbative definition of dimensional interpolation). We first proved, via the 2nd proposition in subsect. 3.2 and eq. (3.30) of [4] that the Green function G at $d=2-\varepsilon$ may already be written as an operator expansion

$$G(x_i;g,\varepsilon)=\sum_i C_i(x_i;g,\varepsilon)\langle O_i\rangle_{g,\varepsilon}\,, \tag{3.1}$$

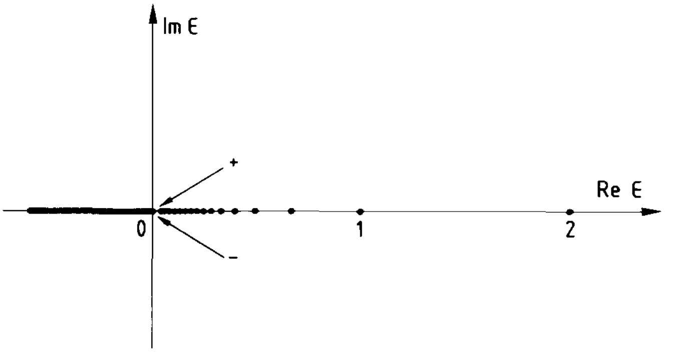

Fig. 2. The analytic structure in $\varepsilon = 2-d$ of some composite operator $\langle O_i \rangle$. $\varepsilon = 0$ is an accumulation of poles and the limit $\varepsilon \to 0$ must be taken from above (+) or under (−) the positive real axis.

where the coefficients C_i are perturbative series in g and the v.e.v. $\langle O_i \rangle_{g,\varepsilon}$ are of the form

$$\langle O_i \rangle_{g,\varepsilon} = g^{\dim O_i (1/\varepsilon)} \times [\text{perturbative series in } g] . \tag{3.2}$$

The C_i and the $\langle O_i \rangle$ are unambiguously defined as long as analytic and non-analytic powers of g do not coincide, namely as long as $\varepsilon \neq 2p/q$ (with $p < \dim O_i$). More precisely, each $\langle O_i \rangle$ appears to be a meromorphic function of ε in the right half plane Re $\varepsilon > 0$, with isolated poles at $\varepsilon = 2p/q$ ($0 \leq p \leq \dim O_i$, $q > 0$) which accumulate at $\varepsilon = 0$ (see fig. 2). Indeed we are dealing with composite operators with positive dimension which have not only logarithmic but quadratic, quartic ... ultraviolet divergences. The "normal product algorithm" used in the OPE to define an operator with dimension n must mix that operator only with operators with the same dimension, not with those with lower dimensions [2]. This is realized if one subtracts logarithmic divergences but takes only the "finite part" of quadratic, quartic divergences, or, to speak in terms of dimensional renormalization, if one subtracts the poles at $\varepsilon = 0$ but not those at $\varepsilon > 0$ which correspond to non-logarithmic divergences.

Since there is now an accumulation of poles on the positive real axis, in order to reach the limit $\varepsilon = 0$ one must give an imaginary part to ε and let ε go to zero from above or under the positive real axis (See fig. 2). The remarkable point, shown in [4], is that if $\varepsilon \to 0$ with Arg $\varepsilon > 0$ (respectively < 0), each $C_i(x_i; g, \varepsilon)$ tends toward the $C_i(x_i; g)$ obtained from its perturbative series by a Borel sum above (respectively under) the positive real axis. Similarly we shall see that the limit of $\langle O_i \rangle$ as $\varepsilon \to 0$ is different if Arg ε is positive or negative, contrary to what is supposed to be in [4].

3.2. THE AMBIGUITY OF CONDENSATES IN THE $1/N$ EXPANSION

Let us first consider the operator α^2 already discussed in sect. 2, and more precisely the quantity $\langle \alpha^2 \rangle - \langle \alpha \rangle^2$. At leading order $1/N$ the bare quantity is given by the

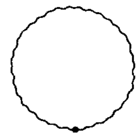

Fig. 3. The amplitude $\langle\alpha^2\rangle-\langle\alpha\rangle^2$ at order $1/N$. The wavy line is the α propagator G.

amplitude

$$I(\varepsilon)=\int \frac{\mathrm{d}^{2-\varepsilon}p}{(2\pi)^{2-\varepsilon}}\, G(p,\varepsilon) \tag{3.3}$$

corresponding to the tadpole graph of fig. 3. $G(p,\varepsilon)$ is the regularized α propagator given by (2.12) in [4]:

$$G(p,\varepsilon)=\left[\frac{1}{2}\int \frac{\mathrm{d}^{2-\varepsilon}p}{(2\pi)^{2-\varepsilon}}\,\frac{1}{(k^2+m^2)((p+k)^2+m^2)}\right]^{-1}, \tag{3.4}$$

m^2 is the mass gap. From the lemma of subsect. 3.3 in [4], the renormalized amplitude in the minimal subtraction scheme is

$$I^{\mathrm{MS}}(\varepsilon)=I(\varepsilon)+m^4K^{\mathrm{MS}}(\varepsilon)\,, \tag{3.5}$$

where $K^{\mathrm{MS}}(\varepsilon)$ is some counterterm analytic in the right half plane $\mathrm{Re}\,\varepsilon>0$. Using the integration rules of dimensional interpolation we get

$$\begin{aligned} I(\varepsilon)&=\frac{(4\pi)^{-1+\varepsilon/2}}{\Gamma(1-\frac{1}{2}\varepsilon)}\int_0^\infty \mathrm{d}(p^2)(p^2)^{-\varepsilon}G(p,\varepsilon)\\ &=\frac{(4\pi)^{-1+\varepsilon/2}}{\Gamma(1-\frac{1}{2}\varepsilon)}\, m^4\hat{G}(s=2,\varepsilon)\,, \end{aligned} \tag{3.6}$$

where $\hat{G}(s,\varepsilon)$ is the Mellin transform of G with respect to m^2, which coincides at $\varepsilon=0$ with the Borel transform $\hat{G}(s)$ of (2.6). $\hat{G}(s,\varepsilon)$ was studied in subsect. 2.3 of [4] and was shown to have single poles at

$$s_{n,k}=n+\tfrac{1}{2}\varepsilon(k+1)\,,\qquad n,k\in\mathbb{N}\,, \tag{3.7}$$

so that $\hat{G}(2,\varepsilon)$ is finite (as long as ε is different from the UV poles) and diverges as $\varepsilon\to0$, since the series of poles $s_{2,k}$ coalesces and becomes a cut at $s=2$. This gives a divergence which is precisely cancelled by the counterterm $K^{\mathrm{MS}}(\varepsilon)$ in (3.5). A finite part remains which is different if $\mathrm{Arg}\,\varepsilon$ is positive or negative as $\varepsilon\to0$. Indeed, the previous series of poles $s_{0,k}$ and $s_{1,k}$ become the cuts of $\hat{G}(s)$ at $s=0$ and $s=1$ given by (2.8a, b) and it is easy to see that the difference between the two limits is

$$\left[\lim_{\substack{\varepsilon\to0\\ \mathrm{Arg}\,\varepsilon<0}}-\lim_{\substack{\varepsilon\to0\\ \mathrm{Arg}\,\varepsilon>0}}\right]I^{\mathrm{MS}}(\varepsilon)=\frac{m^4}{4\pi}[\text{disc of }\hat{G}(s)\text{ at }s=2]=2i\pi m^4\,. \tag{3.8}$$

This corresponds exactly to the ambiguity of $\Delta_2(s, x)$ given by (2.13) and discussed in sect. 2.

The reader may object that such a phenomenon is perhaps not present for the first non-trivial operator, $\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle$, which is the counterpart of the gluon condensate in gauge theories. In fact, using the same kind of argument, one can show that the "spin-wave condensate", which is equal, from the equations of motion of the model, to

$$\partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} = -\alpha\,, \tag{3.9}$$

has the same kind of ambiguity as the operator α^2 discussed above. From (3.9), $\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle$ is given by

$$\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle = -m^2 + \frac{1}{N} J^{\mathrm{MS}} + \mathrm{O}\left(\frac{1}{N^2}\right), \tag{3.10}$$

where J^{MS} is the amplitude of the graph of fig. 4. However the analysis involves an amplitude with 2 loops and overlapping divergences and is rather tedious. In appendix A we calculate this ambiguity by showing that the first non-perturbative term of the OPE of the irreducible two-point function $\Gamma_2(p)$ in the $1/N$ expansion is ambiguous at order $1/N$. Since this term is proportional to $\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle$ we get the ambiguity, which is:

$$\Bigg[\lim_{\substack{\varepsilon \to 0 \\ \mathrm{Arg}\, \varepsilon < 0}} - \lim_{\substack{\varepsilon \to 0 \\ \mathrm{Arg}\, \varepsilon > 0}} \Bigg] \langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle = \frac{1}{N} 2 i \pi m^2 + \mathrm{O}\left(\frac{1}{N^2}\right). \tag{3.11}$$

3.3. DISCUSSION

Before going to the consequences of such ambiguities for the status of the OPE and the problem of IR renormalons we discuss in more details some points of our result.

Such ambiguities come from the fact that there are quadratic divergences (poles at Re $\varepsilon > 0$) in the $\langle O_i \rangle$ which are not subtracted in the minimal subtraction scheme. The reader may object that such divergences are already present in the graphs of

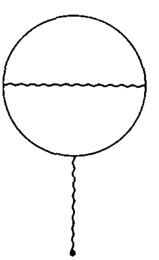

Fig. 4. The amplitude $\langle \alpha \rangle$ at order $1/N$. The straight line is the $\boldsymbol{S}$ propagator $1/(p^2 + m^2)$.

the Green function (1.2) (see for instance the 2-point function in appendix A) and ask whether the Green functions are well defined. In fact those quadratic divergences can be shown to cancel themselves between the different graphs of the Green functions, because they correspond to trivial insertions of the operator $\mathbb{1}$. Thus although such ambiguities are present in some individual amplitudes of the $1/N$ expansion, they disappear from the Green functions (1.2) involving only operators with dimension zero.

One may also question the way we have computed the v.e.v. of local operators, namely the fact that we have used in a non-perturbative way dimensional interpolation and then taken the limit $\varepsilon \to 0$. Instead one may compute those v.e.v. first with an IR cut-off, in order to make perturbation theory valid, and then set that cut-off to zero. We have looked at the v.e.v. of some operators of the non-linear σ model when a constant external magnetic field H is added to the action. For $H \neq 0$ IR renormalons disappear from Green functions which are given by the Borel sum of their perturbative series. However, if one tries to reconstruct the v.e.v. of local operators from their perturbative series

$$\langle O \rangle_H \equiv \sum_{n=0}^{\infty} g^n O_n(H) , \tag{3.12}$$

the series (3.12) appears not to be Borel summable. More precisely, there are renormalons on the positive real axis in the Borel plane, which give the same ambiguities as those obtained previously for $H=0$! Therefore, the fact that the v.e.v. of local operators are ambiguous is a universal renormalization effect related to the subtractions needed in order to define perturbatively such operators. In particular it should appear in massive as well as massless theories!

Finally, one can try to extract the $\langle O \rangle$'s from the lattice theory. Let us for instance use the method proposed in [9] for the evaluation of the gluon condensate to compute $\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle$. The lattice version of that operator is

$$\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle_{\text{lattice}} = \frac{1}{a^2} \left\langle \sum_\mu (\boldsymbol{S}_{i+\mu} - \boldsymbol{S}_i)^2 \right\rangle , \tag{3.13}$$

where a is the lattice spacing. The continuum operator should be obtained by subtracting the divergent part of order $1/a^2$

$$\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle_{\text{lattice}} = \frac{1}{a^2} \left[\sum_{k=0}^{\infty} (g_{\text{R}}^k) O_k(\ln a) \right] + \langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle_{\text{continuum}} + \mathrm{O}(a^2) , \tag{3.14}$$

where O_k is a polynomial of order k in $\ln a$. But the problem remains the same! The series which defines the divergent part has in fact a renormalon at $s=1$. According to the summation prescription that we choose to define this divergent part (the so-called perturbative tail), we get a different value for $\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S} \rangle_{\text{continuum}}$, with the same ambiguity as above.

4. The consequences for the problem of IR renormalons and the status of the operator product expansion

4.1. IR RENORMALONS AND THE SVZ OPERATOR EXPANSION

We are now in a position to correct the conclusions of ref. [4] by taking into account the ambiguities of the condensates. The following conclusions have to replace points (1) to (5) in sect. 4 of [4].

(i) Within the $1/N$ expansion of the non-linear σ model, the Green functions involving only operators with dimension 0 may be expressed as an operator expansion

$$G(x_i;g)=\sum_i C_i(x_i,g)\langle O_i\rangle\,, \tag{4.1}$$

provided that the following resummation prescriptions are adopted in order to deal with the ambiguities of the C_i and the $\langle O_i\rangle$.

(a) The coefficients C_i of the operator expansion are defined from their perturbative series by a Borel sum above (respectively under) the positive real axis in the Borel plane.

(b) Simultaneously the condensates $\langle O_i\rangle$ are defined by their limit as $\varepsilon\to 0$ with $\operatorname{Arg}\varepsilon>0$ (respectively $\operatorname{Arg}\varepsilon<0$).

Thus the v.e.v. of the operators with dimension n make sense only once a resummation prescription has been chosen for the terms with dimension $<n$ in the operator expansion (4.1).

(ii) Since the condensates, and in particular the first one $\langle\partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}\rangle$, are ambiguous, the Borel transforms of the C_i have IR renormalons at integer values of $-\frac{1}{2}s\beta_2$ in the Borel plane*. In particular the first renormalon of the perturbative part C_0 is at $s=-2/\beta_2$ (contrary to the claim of [4]) and gives an ambiguity which cancells that of $C_1\langle\partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}\rangle$ given by the first condensate. Therefore, it is not because the v.e.v. of composite operators are non-zero that there are IR renormalons, but rather because these v.e.v. are not unambiguously defined. As a consequence one can prove the conjecture made by Parisi in [5], namely that all IR renormalons are proportional to the coefficients of the operator expansion.

(iii) As expected, the v.e.v. of the operators O_i with dimension n satisfy a renormalization group equation of the form

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\gamma_n(g)\right]\langle O_i\rangle=0\,, \tag{4.2}$$

which determines those v.e.v. up to numerical factors C_i

$$\langle O_i\rangle = C_i\mu^{\dim O_i}\exp\left[\frac{\dim O_i}{\beta_2 g}+\mathrm{O}(\ln g)\right], \tag{4.3}$$

* We recall that β_2 is the first coefficient of the β function and is negative.

which contains the ambiguity, since the RG functions β and γ do not have renormalons.

(iv) We naturally conjecture that such features are valid beyond the $1/N$ expansion of the non-linear σ model and are in particular also valid for four-dimensional gauge theories. Thus we see that the formalism at the basis of the QCD sum rules has a mathematical status which is much more subtle than usually assumed. In particular we see no reason why the gluon condensate

$$\langle 0|\mathrm{Tr}\, G^{\mu\nu}G_{\mu\nu}|0\rangle\,, \tag{4.4}$$

should not have the ambiguities of the spin-wave condensate in the non-linear σ model. Such problems arise from the fact that we are dealing with dynamical phenomena in a renormalizable asymptotically free theory. Therefore they cannot be seen when looking at the sum rules formalism in two-dimensional gauge theories [10], non-relativistic models [11], as well as in the case of spontaneous symmetry breaking [12]. An important question remains to determine whether such ambiguities have important consequence for the phenomenological applications of QCD sum rules or not.

(v) We have not discussed in this paper the effect of instantons and how they can be taken into account in a mathematically consistent way in the operator product expansion. This will be the subject of a separate paper [13].

4.2. WHICH OPERATORS ARE NOT AMBIGUOUS?

One may ask whether there are composite operators which do not have such ambiguities. Since this problem comes from the presence of quadratic or higher UV divergences which have to be subtracted in order to define these operators, only operators with only logarithmic UV divergences make sense. In QCD two kinds of operators survive; first the quark condensate

$$\langle 0|\bar{\psi}\psi|0\rangle \tag{4.5}$$

which is protected by chiral symmetry and second the topological charge density

$$q=\langle 0|\frac{g^2}{32\pi^2}\mathrm{Tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu})|0\rangle\,, \tag{4.6}$$

which is protected by CP symmetry. This is pleasing since those operators have a direct physical meaning through the PCAC relations and chiral symmetry breaking for $\bar{\psi}\psi$ and through the ABJ axial anomaly and the U(1) problem for q. One may explicitly check the existence of those two operators in the $1/N$ expansion of the Gross–Neveu model [14] for (4.5) and the CP^{N-1} model [15] for (4.6).

4.3. THE DEFINITION OF COMPOSITE OPERATORS AND THE STATUS OF THE SHORT-DISTANCE OPE

Now we want briefly to discuss the fact that the ambiguities that we have displayed in the definition of the v.e.v. of composite operators should be present in any Green function involving these operators. Indeed the arguments and the explicit calculations that we have performed for vacuum to vacuum matrix elements may be repeated with the same conclusions in the general case.

Let us consider the O(N) non-linear σ model with an IR cut-off (for instance in a finite volume) in order to simplify the discussion. The perturbative series of Green functions involving composite operators do have renormalons and are not Borel summable. This is in fact not surprising since such objects appear in the short-distance OPE which reads:

$$\langle S(\lambda x_1)\cdots S(\lambda x_p)S(y_1)\cdots S(y_q)\rangle \underset{\lambda\to 0}{=} \sum_i C_i(\lambda x_1,\ldots,\lambda x_p)\langle O_i(0)S(y_1)\cdots S(y_q)\rangle\,. \tag{4.7}$$

But we have seen that the coefficients $C_i(\lambda x_1\cdots\lambda x_p)$, which are the same as those in the SVZ expansion, are ambiguous because of the presence of IR renormalons. Similarly the terms $\langle O_i(0)S(y_1)\cdots S(y_q)\rangle$ have to be ambiguous, and are defined in (4.7) only once a summation prescription has been chosen for the previous terms of the operator expansion. Let us emphasize that this phenomenon takes place because the operators O_i have been subtracted with the counterterms of the massless theory (for instance in the minimal subtraction scheme). There exist subtraction schemes (like the PBHZ scheme at zero momenta) where the ambiguities may be recast in the counterterms and where composite operators are well-defined. For instance we have seen that $\partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}_{\rm MS}$ is not defined, however it follows from (4.7) that

$$\partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}_{\rm MS} - \mathbb{1}\langle\partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}\rangle_{\rm MS} = \partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}_{\rm BPHZ} \tag{4.8}$$

is unambiguous.

4.4. IR AND UV RENORMALONS

Finally we want to suggest a general explanation for the existence of such ambiguities in the definition of composite operators in UV free theories. Indeed, their Borel transform is expected to have UV singularities on the *negative* real axis at $s=2n/\beta_2$; $n=1,2,\ldots$ in the Borel plane [14, 16]. It has been conjectured by Parisi [17] that such UV renormalons are proportional to local insertions of composite operators with dimension $2n+2$. If this argument is correct and if we consider some composite operator O with dimension p, nothing prevents O to mix with the operators with dimension $q<p$ to give UV renormalons on the *positive* real axis at $s=(q-p)/\beta_2(0\leq q<p)$ in the Borel transform of Green functions involving one insertion of O. This is precisely the locations of the ambiguities that we have found

previously so that we conjecture that it is this mechanism of UV renormalons which makes the notion of composite operators ambiguous and therefore is responsible for the existence of IR renormalons in massless UV free theories.

I thank J. Zinn-Justin for his interest and for helpful discussions.

Appendix

Let us consider the irreducible two-point function $\Gamma_2(p)$ in the $1/N$ expansion. Its first terms are given in fig. 5. Let us note by $J(p)$ the term of order $1/N$ given by the two graphs of fig. 5.

From the analysis of [4] (see fig. 6) Γ_2 may be written as an operator expansion

$$\Gamma_2^{\mathrm{MS}}(p) = \Gamma_{2(0)}^{\mathrm{MS}}(p) + \Gamma_{2(1)}^{\mathrm{MS}}(p)\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S}\rangle + \cdots , \tag{A.1}$$

with

$$\Gamma_{2(0)}^{\mathrm{MS}} = p^2 + \mathrm{O}\left(\frac{1}{N}\right), \tag{A.2}$$

$$\Gamma_{2(1)}^{\mathrm{MS}} = -1 + \mathrm{O}\left(\frac{1}{N}\right). \tag{A.3}$$

We shall show that the term $\Gamma_{2(1)}^{\mathrm{MS}}\langle \partial_\mu \boldsymbol{S}\, \partial^\mu \boldsymbol{S}\rangle$ is ambiguous at order $1/N$. For that purpose, let us consider the term of order $1/N$, $J^{\mathrm{MS}}(p)$, of Γ_2. First we relate $J^{\mathrm{MS}}(p)$ computed in the minimal subtraction scheme to its value computed in the Zimmermann scheme of subtraction at zero momenta $J^{\mathrm{R}}(p)$ via eq. (C.9) of ref. [4]

$$J^{\mathrm{MS}}(p) = J^{\mathrm{R}}(p) + J^{\mathrm{MS}}(p=0) + p^2 \frac{\partial}{\partial p^2} J^{\mathrm{MS}}(p)\big|_{p=0} \tag{A.4}$$

In Zimmermann's subtraction scheme, the amplitude of the second graph is zero, since it does not depend on p. On the other side the amplitude of the first graph has been written in appendix D of ref. [4]. From eq. (D.1) the (modified) Borel transform of $J^{\mathrm{R}}(p)$ is

$$\hat{J}^{\mathrm{R}}(p) = \frac{1}{4\pi} \int_{\substack{\sigma - i\infty \\ \sigma<0}}^{\sigma+i\infty} \frac{\mathrm{d}u}{2i\pi}\, \frac{\Gamma(u-s)\Gamma(2-u)\Gamma(s-1)\Gamma(u-s)}{\Gamma(2-s)\Gamma(u-1)}\, \hat{G}(u)(p^2)^{1-s}, \tag{A.5}$$

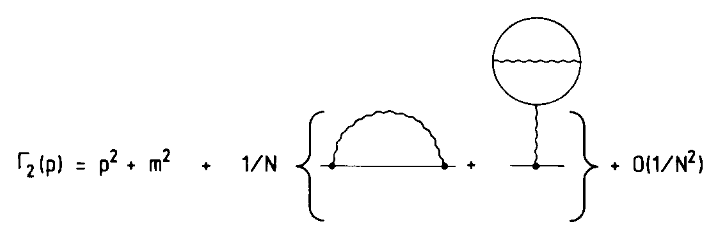

Fig. 5. The truncated 2-point function up to order $1/N$.

which may be written as the formal sum of the residues at $u-s=n\in\mathrm{N}$

$$\hat{J}^{\mathrm{R}}(p)=\frac{(p^2)^{1-s}}{4\pi}\frac{\Gamma(s-1)}{\Gamma(2-s)}\sum_{n=0}^{\infty}\left[\frac{1}{(n!)^2}\left(\frac{\mathrm{d}}{\mathrm{d}s}+2\gamma_n\right)\left(\frac{\Gamma(2-s+n)}{\Gamma(s-n-1)}\hat{G}(s-n)\right)\right].$$

In (A.6) $\hat{G}(s)$ is the Borel transform of the α propagator and has discontinuities at $s=0, 1, 2,\ldots$ given by (2.8). The γ_n are numerical factors

$$\gamma_0=-\gamma, \qquad \gamma_n=-\gamma+\sum_{k=1}^{n}\frac{1}{k}, \tag{A.7}$$

where γ is Euler's constant.

Since $J^{\mathrm{MS}}(p)$ is unambiguously defined for any p, so are $J^{\mathrm{MS}}(0)$ and $(\partial/\partial p^2)\mathrm{J}^{\mathrm{MS}}(0)$. From dimensional consideration, their Borel transform $\hat{J}^{\mathrm{MS}}(0)$ and $(\partial/\partial p^2)\hat{J}^{\mathrm{MS}}(0)$ have simply one unambiguous discontinuity at $s=1$ and $s=0$ respectively. Thus, if the discontinuities of $\hat{J}^{\mathrm{MS}}(p)$ are ambiguous, those ambiguities are identical to those of $\hat{J}^{\mathrm{R}}(p)$, which we study now.

The discontinuity at $s=0$ comes only from the term $n=0$ in (A.6) and is equal to:

$$\operatorname*{disc}_{s=0}\hat{J}^{\mathrm{R}}=\left[\frac{\mathrm{d}}{\mathrm{d}s}+2\gamma_0\right]\times\operatorname*{disc}_{s=0}\hat{G}(s)\frac{(p^2)^{1-s}}{4\pi}-\operatorname*{disc}_{s=0}\left[\hat{G}(s)(\psi(2-s)+\psi(s-1))\frac{(p^2)^{1-s}}{4\pi}\right], \tag{A.8}$$

where ψ is the logarithmic derivative of Γ. Now the discontinuity at $s=1$ comes from the terms $n=0$ and $n=1$ and is

$$\begin{aligned}\operatorname*{disc}_{s=1}\hat{J}^{\mathrm{R}}=-(s-2)^2\Bigg[&\left(\frac{\mathrm{d}}{\mathrm{d}s}+2\gamma_1\right)\operatorname*{disc}_{s=1}\hat{G}(s-1)\frac{(p^2)^{1-s}}{4\pi}\\&-\operatorname*{disc}_{s=1}\hat{G}(s-1)(\psi(3-s)+\psi(s-2))\frac{(p^2)^{1-s}}{4\pi}\Bigg]\\&+\left(\frac{\mathrm{d}}{\mathrm{d}s}+2\gamma_0\right)\operatorname*{disc}_{s=1}\hat{G}(s)\frac{(p^2)^{1-s}}{4\pi}\\&-\operatorname*{disc}_{s=1}\hat{G}(s)(\psi(2-s)+\psi(s-1))\frac{(p^2)^{1-s}}{4\pi}.\end{aligned} \tag{A.9}$$

The ambiguity comes from the last term of (A.9), namely $\hat{G}(s)\psi(s-1)$. Indeed, similarly to what was done in (2.12), we may write $\hat{G}(s)$ at $s=1$ as

$$\hat{G}(s)=4\pi\left[2(2-s)\ln\frac{1}{1-s}+S(s)\pm i\pi\right], \tag{A.10}$$

where $S(s)$ is analytic in the disc $\{|s-1|<1\}$ and $\pm$ refers to the sheet above or under the real axis.

Since $\psi(s-1)$ has a pole at $s=1$, (A.9) is finally of the form

$$\operatorname*{disc}_{s=1} \hat{J}^{\mathrm{R}} = \text{a real distribution} \mp i\pi\delta(s-1)\,. \qquad \text{(A.11)}$$

Now we have to recall from [4] that $\mathrm{disc}_{s=1}\,\hat{J}^{\mathrm{MS}}$ is the ordinary Borel transform of $\Gamma^{\mathrm{MS}}_{2(1)}\langle\partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}\rangle$, namely that:

$$\Gamma^{\mathrm{MS}}_{2(1)}\langle\partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}\rangle = \int_0^\infty \mathrm{d}s (m^2)^s \operatorname*{disc}_{s=1} \hat{J}^{\mathrm{MS}}\ . \qquad \text{(A.12)}$$

From (A.3) and (A.11), (A.12) we see that the ambiguity comes from $\langle\partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}\rangle$ and that

$$\left[\lim_{\substack{\varepsilon\to 0\\ \mathrm{Arg}\,\varepsilon<0}} - \lim_{\substack{\varepsilon\to 0\\ \mathrm{Arg}\,\varepsilon>0}}\right]\langle\partial_\mu \boldsymbol{S}\,\partial^\mu \boldsymbol{S}\rangle = 2i\pi m^2\frac{1}{N} + \mathrm{O}\left(\frac{1}{N^2}\right). \qquad \text{(A.13)}$$

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