Nuclear Physics B209 (1982) 433–460

NON-PERTURBATIVE EFFECTS AND INFRARED RENORMALONS WITHIN THE 1/N EXPANSION OF THE O(N) NON-LINEAR SIGMA MODEL

F DAVID[1]

CEN-Saclay, 91191 Gif-sur-Yvette, Cedex France

Received 5 August 1982

We analyze the structure of the Borel transform of the two-dimensional O(N) non-linear σ model within its $1/N$ expansion We check the existence of IR singularities (IR renormalons) and the presence of non-perturbative terms which organize themselves in an operator expansion à la Shifman–Vainshtein–Zakharov We prove that renormalons cancel between the different terms of the operator expansion, so that there is a well-defined resummation procedure of the perturbative series. We suggest that this mechanism provides a general solution of the IR renormalons problem for massless UV free field theories

1. Introduction

In a well-known series of papers, Shifman, Vainshtein and Zakharov (SVZ) [1] suggested that it was possible to take into account the large distance non-perturbative effects of QCD with the help of the operator product expansion. The basic idea was the following; the product of two currents j^A, j^B (for instance) may be expanded into

$$T\{j^A(x), J^B(0)\} \underset{x\to 0}{=} \sum_n C_n^{AB}(x)\mathcal{O}_n , \tag{1.1}$$

where the sum runs over local operators $\mathcal{O}_n$ (properly renormalized) and the $C_n^{AB}(x)$ depend on x [2]. In [1], SVZ assumed that the expansion (1.1) holds beyond perturbation theory, so that one may write

$$\langle 0|T\{j^A(x), j^B(0)\}|0\rangle = \sum_n C_n^{AB}(x)\cdot\langle 0|\mathcal{O}_n|0\rangle , \tag{1.2}$$

where the vacuum expectation values $\langle 0|\mathcal{O}_n|0\rangle$ are allowed to be non-zero, and parametrize the non-perturbative effects of QCD (in perturbation theory, $\langle 0|\mathcal{O}_n|0\rangle = 0$ at all orders). The coefficients $C_n^{AB}(x)$ are given by perturbation theory.

Practically, SVZ and coworkers used the operator expansion (OE) (1.2) by matching it with dispersion relations (and by retaining the first operators of

[1] Physique Théorique CNRS

dimension ≤6 and the first order(s) in α_s for the C_n^{AB}) to look for instance at the meson spectrum of QCD. This "QCD sum rules formalism" has now a firm phenomenological status and the existence of "quark and gluon condensates" is recognized as an essential feature of gauge theories. Those condensates $\langle 0|\mathcal{O}_n|0\rangle$ have to be estimated from experimental data, Monte Carlo computations or semi-classical evaluations.

However, the validity of this expansion is not obvious, and was extensively discussed by SVZ in [1]. Indeed, it relies finally on a (somewhat phenomenological) extension of perturbative arguments. A related and important problem concerns the mathematical consistence of the expansion (1.2). The non-perturbative quantities $\langle 0|\mathcal{O}_n|0\rangle$ are exponentially small, typically

$$\langle 0|\mathcal{O}_n|0\rangle \sim \exp(d_n/\beta_2\alpha_s)\,, \tag{1.3}$$

where β_2 is the first term of the β function and d_n the canonical dimension of $\mathcal{O}_n$. It is not consistent to incorporate such terms as long as the perturbative series in α_s, $C_n^{AB}(x, \alpha_s)$, have not been properly summed. But those series are expected to be divergent so that even if there is a domain of physical parameters where the first perturbative corrections are smaller than the first non-perturbative ones (as argued in [1]) this situation disappears at large orders in α_s. Thus a resummation procedure for the perturbative series has to be defined. However, according to our present knowledge, difficulties are expected in the usual program of Borel summation of the perturbative series. Indeed, even disregarding the problems of instantons and of the behaviour of the Borel transform at infinity, one expects the presence of infrared (IR) singularities on the positive real axis of the Borel transform [3]. A simple argument to locate such singularities, usually called IR renormalons, has been given by Parisi [4]. In a massless UV free theory such as QCD ($\beta_2 < 0$), the Borel transformed effective coupling constant $\tilde{g}(p, b)$ should behave at small momenta as

$$\tilde{g}(p, \alpha_s) \sim |p|^{b\beta_2} \tag{1.4}$$

(b is the Borel variable). Inserting (1.4) in the (Borel-transformed) Dyson–Schwinger integral equations should give IR singularities at $b = -2n/\beta_2$, $n \in \mathbb{N}$.

Moreover, Parisi argued in [4] that (by analogy with IR divergences below 4 dimensions) these singularities could be classified in terms of the coefficients C_n of the OE (1.1), and so were related to the appearance of the non-perturbative expectation values (1.3). This relationship between the existence of an "IR tachyonic Landau singularity" (which corresponds to (1.4)) and some "vacuum instability" was previously noted by Gross and Neveu [5] and by Olesen [6]. However, it seems to us that this relationship has not received a more quantitative formulation and that the problem of the summation of the perturbative series of gauge theories has still to be understood.

In this paper we shall look at these points within the $1/N$ expansion of the O(N) non-linear sigma model at two dimensions. As gauge theories it is asymptotically free [9] and its perturbative expansion is made around a "wrong vacuum" since the classical theory describes $N-1$ interacting Goldstone bosons but the Mermin–Wagner–Coleman theorem [7] ensures the dynamical restoration of the O(N) symmetry and the non-perturbative generation of a mass gap for any positive coupling constant. In particular, the perturbative expansion has IR divergences which cancel only for the "physical" O(N) invariant observables [8]. The $1/N$ expansion takes into account these non-perturbative effects and is a powerful tool to study the theory, since it allows partial infinite resummation of the usual perturbative series [9]. Moreover, there are no instantons ($N>3$), thus the structure of non-perturbative effects is expected to be simpler.

Our purpose is to characterize the analytic structure of the Borel transform (with respect to the coupling constant) of any O(N) invariant observable at an arbitrary order of the $1/N$ expansion. The result is stated in theorem A (subsect. 3.3) and proves that, within the $1/N$ expansion:

there are non-perturbative terms which organize themselves formally in an operator expansion but have IR renormalons;

nevertheless, there is a Borel summation prescription for those terms which makes the SVZ operator expansion unambiguous and gives the right result; with such a prescription, IR renormalons are cancelled between the different non-perturbative terms.

Moreover, we shall argue this mechanism goes beyond the $1/N$ expansion and may provide a general solution for the problem of IR renormalons.

This paper is organized as follows:

In sect. 2 we introduce the O(N) model and its $1/N$ expansion (subsect. 2.1) and show that one recovers the operator expansion at leading order $N=\infty$ (subsect. 2.2) and more generally at the order of tree diagrams (subsect. 2.3), i.e. in cases where the perturbative series are convergent (no Borel transform is needed).

In sect. 3 we analyse the Borel transform of any order of the $1/N$ expansion. We adapt desingularization techniques of Bergère-Lam and the author [10, 11]. Basic definitions are given in subsect. 3.1. For technical reasons one first has to look at the "bare" theory below 2 dimensions (subsect. 3.2) and then to take the limit $d\to 2$ (subsect. 3.3), where the complete analytic structure of the Borel transform (in the first sheets) is obtained in theorem A. The result is discussed in subsect. 3.4.

In sect. 4 we discuss the possible validity of our result beyond the $1/N$ expansion and for other models. Various implications are examined.

Finally in appendix A the technicalities of the Borel transforms are recalled. In appendix B we discuss the obtaining of the coefficients of the operator expansion in perturbation theory for the O(N) model. Appendix C is devoted to another integral representation for the $1/N$ expansion needed in sect. 3 and used in appendix D for explicit computations of IR renormalons at first $1/N$ order.

2. The structure of the O(N) model for $N = \infty$

2.1 THE $1/N$ EXPANSION OF THE O(N) SIGMA MODEL

First we briefly recall how to obtain the $1/N$ expansion of the O(N) non-linear σ model. The generating functional reads

$$Z[J] = \int \mathscr{D}[S]\mathscr{D}[\alpha] \exp\left\{-\frac{N}{g_B}\int d^dx \{\tfrac{1}{2}(\partial_\mu S \cdot \partial_\mu S) + \tfrac{1}{2}\alpha(x)[S^2(x) - 1]\}\right\} \times \exp\left\{\int d^dx J(x) S(x)\right\}, \tag{2.1}$$

where $S(x)$ is a N-component real vector field defined in the d-dimensional euclidian space; the Lagrange multiplier $\alpha(x)$ fixes the constraint

$$S^2(x) = 1, \qquad \forall x. \tag{2.2}$$

g_B is the bare coupling constant and $J(x)$ the source term. Integrating over the S field we get

$$Z[J] = \int \mathscr{D}[\alpha] \exp\{-\tfrac{1}{2}NS_{\text{eff}}[\alpha]\} \exp\left\{\frac{g_B}{2N}\int \langle x|\frac{1}{-\Delta + \alpha(x)}|y\rangle J(x) J(y)\, dx\, dy\right\}, \tag{2.3}$$

with

$$S_{\text{eff}}[\alpha] = \text{Tr Ln}[-\Delta + \alpha(x)] - \frac{1}{g_B}\int d^dx\, \alpha(x). \tag{2.4}$$

The limit $N = \infty$ is obtained by taking the constant saddle point of S_{eff}, $\alpha(x) = \alpha_c$, given by

$$\langle x|\frac{1}{-\Delta + \alpha_c}|x\rangle = \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2 + \alpha_c} = \frac{1}{g_B}. \tag{2.5}$$

The integral (2.5) is UV divergent for $d \geqslant 2$. Using dimensional interpolation [12] and taking $d = 2 - \varepsilon$ (Re $\varepsilon > 0$), (2.5) makes sense and gives

$$\alpha_c = [g_B \Gamma(\tfrac{1}{2}\varepsilon)(4\pi)^{(\varepsilon-2)/2}]^{2/\varepsilon}. \tag{2.6}$$

At $N = \infty$ only the connected 2-point function survives and is

$$G_2(p) = \frac{g_B}{p^2 + \alpha_c(g_B)}, \tag{2.7}$$

so that $\alpha_c(g_B)$ is the square of the physical mass at $N = \infty$.

At $d = 2$ a wave function and a coupling constant renormalization are needed. We define the renormalized coupling constant g by

$$\frac{1}{g_B} = \frac{1}{g}Z(g), \qquad Z(g) = 1 + g\mu^{-\varepsilon}\Gamma(\tfrac{1}{2}\varepsilon)(4\pi)^{\varepsilon/2-1} \tag{2.8}$$

(where μ is the subtraction mass scale), so that at $d=2$

$$\alpha_c = \mu^2 e^{-4\pi/g} \tag{2.8}$$

The renormalized 2-point function is

$$G_2^R(p) = ZG_2(p) = \frac{g}{p^2+\alpha_c}. \tag{2.10}$$

Computing the fluctuations $\alpha = \alpha_c + \tilde{\alpha}$ around α_c in (2.3) we get the $1/N$ expansion. Its perturbative rules are illustrated in (fig. 1). The two propagators are the $\boldsymbol{S}$ propagator (fig. 1a) given by

$$D(p) = \frac{1}{p^2+\alpha_c}, \tag{2.11}$$

and the $\tilde{\alpha}$ propagator (fig. 1b) which is $-(1/N)G(p)$ with

$$G(p) = \left[\tfrac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+\alpha_c)((p+k)^2+\alpha_c)}\right]^{-1}. \tag{2.12}$$

The factors associated to $\boldsymbol{S}\cdot\boldsymbol{S}\cdot\tilde{\alpha}$ vertices and to internal $\boldsymbol{S}$ loops are illustrated in (figs. 1c, d). The internal $\boldsymbol{S}$ loops with only one or two $\tilde{\alpha}$ insertions are forbidden (figs. 1e, f).

The renormalizability of the $1/N$ expansion was shown in [13] within the BPHZ scheme (see also [14]). This point will be discussed in sect. 3.

2.2 THE ANALYTIC STRUCTURE AT $N=\infty$

As a first step let us discuss the analytic structure (in the coupling constant) of the leading order $N=\infty$. In a manner analogous to the one used in sect. 3, we first look at the bare theory at $d=2-\varepsilon$ (Re $\varepsilon>0$). Using the techniques of [11], the 2-point Green function given by (2.7) has the expansion

$$G_2(p) = \sum_{n=0}^{\infty}\left\{(-1)^n g_B \text{P.F.}\left[\frac{1}{(p^2)^{n+1}}\right]\alpha_c^n + a_n(d)\Delta^n\delta(p)g_B\int\frac{d^dk}{(2\pi)^d}\frac{(k^2)^n}{(k^2+\alpha_c)}\right\}. \tag{2.13}$$

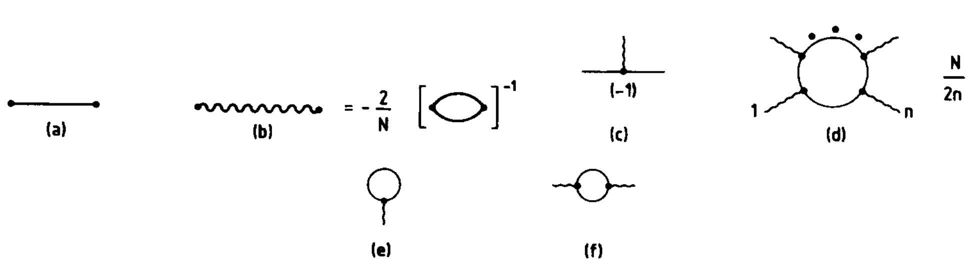

Fig 1 Elements of the $1/N$ expansion (a) The $\boldsymbol{S}$ propagator $D(p)$, (b) the $\tilde{\alpha}$ propagator $G(p)$, (c) the interaction vertex, (d) general internal $\boldsymbol{S}$ loop; (e, f) the internal loops forbidden in the expansion

P.F. stands for "Hadamard's finite part" [15] (P.F. $1/(p^2)^{n+1}$ is unambiguously defined for $-\frac{1}{2}d+n+1\notin\mathbb{N}$); $a_n(d)$ is a combinatorial factor ($a_n(d)=\Gamma(\frac{1}{2}d)/\Gamma(n+\frac{1}{2}d)\ 2^n(n!)^2$); Δ is the laplacian operator with respect to p. (2.13) is an expansion in powers of α_c since, using (2.5) and integration rules of dimensional interpolation [12, 16], we get

$$g_B\int\frac{d^dk}{(2\pi)^d}\frac{(k^2)^n}{(k^2+\alpha_c)}=(-\alpha_c)^n\,. \tag{2.14}$$

The point is that this expansion may be easily interpreted as an operator expansion: indeed α_c^n corresponds to $\langle\alpha^n\rangle$ at $N=\infty$ and the integral (2.14) is simply equal to the (bare) vacuum expectation value of $[\boldsymbol{S}(x)(-\Delta)^n\boldsymbol{S}(x)]$ for $N=\infty$. So we may rewrite (2.13) as

$$G_2(p)=\sum_{n=0}^{\infty}F_{1,n}(p,g_B)\langle\alpha^n\rangle+F_{2,n}(p)\langle\boldsymbol{S}(-\Delta)^n\boldsymbol{S}\rangle\,. \tag{2.15}$$

This is, in fact, an explicit realization of the operator expansion into non-analytic terms for massless superrenormalizable theories (here $d=2-\varepsilon$) [4, 11, 17, 18]. The operator α is related to the operator $\partial_\mu\boldsymbol{S}\cdot\partial_\mu\boldsymbol{S}$ via the equations of motion [9], namely

$$\alpha^n=(-\partial_\mu\boldsymbol{S}\cdot\partial_\mu\boldsymbol{S})^n+\text{contact terms}\,. \tag{2.16}$$

The limit $d=2$ of (2.13) is performed by taking into account the renormalization with (2.8) and (2.10). The composite operators α^n and $\boldsymbol{S}(-\Delta)^n\boldsymbol{S}$ need no additional renormalization at leading order in $1/N$. We get for the renormalized two-point function

$$G_2^R(p,g)=\sum_n(-1)^ngS[(p^2)^{-1-n};\mu]\cdot\langle\alpha^n\rangle+\sum_n a_m(d)\Delta^m\delta(p)\cdot\langle\boldsymbol{S}(-\Delta)^m\boldsymbol{S}\rangle\,, \tag{2.17}$$

where the distribution $S[(p^2)^{-1-n};\mu]$ is obtained by subtracting the pole of P.F. $(p^2)^{-1-n}$ at $\varepsilon=0$;

$$S[(p^2)^{-1-n};\mu]=\lim_{\varepsilon\to0}[\text{P.F.}\,(p^2)^{-1-n}+\mu^{-\varepsilon}\Gamma(\tfrac{1}{2}\varepsilon)a_n(d)\Delta^n\delta(p)]\,, \tag{2.18}$$

and is a finite distribution at $d=2$.

The expansion (2.17) is *exactly* the VSZ operator expansion (1.1). The ordinary perturbative part is given by the terms $n=0$, $m=0$ in (2.17), namely

$$G_2^R(p,g)_{\text{pert}}=\delta(p)+gS\left[\frac{1}{p^2};\mu\right], \tag{2.19}$$

but non-perturbative terms proportional to the v.e.v.,

$$\begin{gathered}\langle(-\partial_\mu\boldsymbol{S}\,\partial_\mu\boldsymbol{S})^n\rangle_{N=\infty}=\mu^{2n}\,e^{-4\pi n/g}\,,\\ \langle(\boldsymbol{S}\Delta^m\boldsymbol{S})\rangle_{N=\infty}=\mu^{2m}\,e^{-4\pi m/g}\,,\end{gathered} \tag{2.20}$$

have to be taken into account. One may check that the coefficients of the expansion coincide with those obtained from the perturbative OE (appendix B).

In the following we shall rescale the renormalized coupling constant

$$g \to g' = \frac{g}{4\pi} \tag{2.21}$$

in order not to deal with (4π) factors.

The (modified) Borel transform (A.1) is a Mellin transform with respects to the squared mass α_c/μ^2 (2.9). In this paper we often look first at the bare theory at $d = 2-\varepsilon$ ($\mathrm{Re}\,\varepsilon > 0$) in order not to deal with renormalization. Nevertheless we shall be interested into the analytic structure of the Mellin transform with respects to α_c:

$$\hat{f}(s) = \int_0^{\text{const}} \mathrm{d}\alpha_c\, \alpha_c^{s-1} f(\alpha_c)\,, \tag{2.22}$$

which is not the Borel transform with respect to g (or g_B) at $d = 2-\varepsilon$ anymore. For simplicity we shall use the term "Borel transform" for the Mellin transform (2.22), keeping in mind that it coincides with the Borel transform (A.1) only for $d = 2$, but that the inverse representation (A.2) always holds

$$f(\alpha_c) = \int_c \frac{\mathrm{d}s}{2i\pi} (\alpha_c)^s \hat{f}(s)\,. \tag{2.23}$$

The Borel transform of the propagator D (2.11) is then in momentum space for $d \leqslant 2$ [forgetting the distribution-like character of D and its singularities at $p = 0$, and integrating up to $\alpha_c = \infty$ in (2.22)]:

$$\hat{D}(p, s) = (p^2)^{-s-1}\Gamma(s+1)\Gamma(s) = (p^2)^{-s-1}\hat{D}(s)\,. \tag{2.24}$$

The poles at $s = -1, -2, \ldots$ are irrelevant and given by the behaviour at $\alpha_c = \infty$. The relevant poles of $\Gamma(s)$ at $s = 0, 1, \ldots$ give simply the Taylor expansion of D around $\alpha_c = 0$ (see fig. 2).

2.3 THE ANALYTIC STRUCTURE OF THE G PROPAGATOR

Let us now look at the $\tilde{\alpha}$ propagator $(-1/N)G(p)$ (2.12). Since $G = (D * D)^{-1}$ (where $*$ stands for the convolution product), using the expansions (2.13)–(2.17) of D we may expand G in terms of the operators α and $S(-\Delta^m)S$. The final result is

$$G(p) = G_0(p)\left[1 + \Sigma(p) G_0(p)\right]^{-1} \tag{2.25}$$

Fig 2 The operator expansion of the S propagator.

where $G_0(p)$ is the "perturbative" $\tilde{\alpha}$ propagator and is given at $d=2-\varepsilon$ by

$$G_0(p)=\left[\frac{1}{g_{\mathrm{B}}}\frac{1}{p^2}+\tfrac{1}{2}F_0(p)\right]^{-1}, \tag{2.26}$$

and at $d=2$ by

$$G_0(p)=\left[\frac{1}{g}\frac{1}{p^2}+\tfrac{1}{2}S_0(p;\mu)\right]^{-1}, \tag{2.27}$$

and where $\sum(p)$ contains the powers of α_c and is at $d=2-\varepsilon$ of the form

$$\sum(p)=\frac{1}{g_{\mathrm{B}}}\sum_{\substack{n,m\geq 0\\ n+m\neq 0}}\langle\alpha\rangle^n_{N=\infty}\langle \boldsymbol{S}(-\Delta)^m\boldsymbol{S}\rangle_{N=\infty}D_{n,m}(p)+\tfrac{1}{2}\sum_{p>0}\langle\alpha^p\rangle_{N=\infty}F_p(p), \tag{2.28}$$

and at $d=2$

$$\sum(p)=\frac{1}{g}\sum_{\substack{n,m\geq 0\\ n+m\neq 0}}\langle\alpha\rangle^n_{N=\infty}\langle \boldsymbol{S}(-\Delta)^m\boldsymbol{S}\rangle_{N=\infty}D_{n,m}(p)+\tfrac{1}{2}\sum_{p>0}\langle\alpha^p\rangle_{N=\infty}S_p(p;\mu). \tag{2.29}$$

In (2.26)–(2.29),

$$D_{n,m}(p)=(-1)^n a_m(d)\Delta^m(p^2)^{-n-1}, \tag{2.30}$$

and $F_p(p)$ ($S_p(p;\mu)$) is the "IR-finite part" (the "IR-subtracted part") of the 1-loop graph with p mass insertions [11] (see fig. 3).

So, $G(p)$ may also be written as an operator expansion of the form

$$G(p)=\sum_{\mathcal{O}}G_{\mathcal{O}}(p,g)\langle\mathcal{O}\rangle_{N=\infty}, \tag{2.31}$$

where the composite operators $\mathcal{O}$ are now of the form

$$\mathcal{O}=\alpha^n\prod_{j=1}^{J}(\boldsymbol{S}\Delta^{m_j}\boldsymbol{S}), \qquad n \text{ and } J\geq 0, \quad m_j>0, \tag{2.32}$$

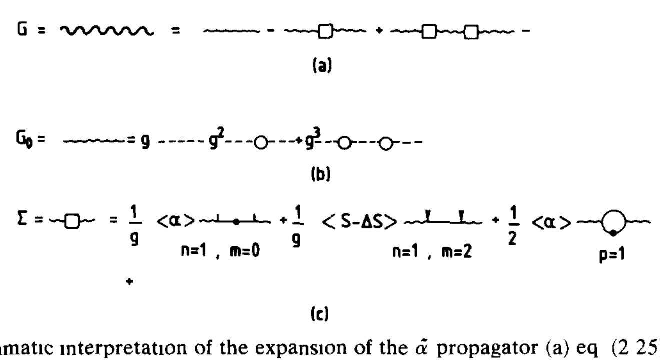

Fig 3 Diagrammatic interpretation of the expansion of the $\tilde{\alpha}$ propagator (a) eq (2 25), (b) expansion of the perturbative propagator G_0, (c) operator expansion of $\sum$ (2 28)–(2.30)

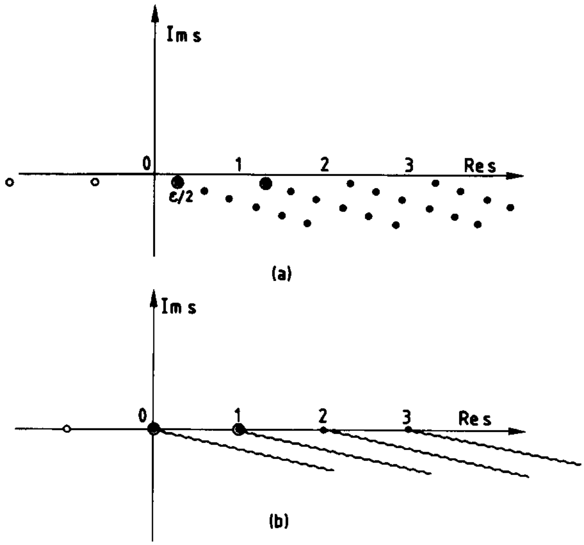

Fig 4. Analytic structure of the Borel transform of $G(p)$ at (a) $d=2-\varepsilon$ and (b) $d=2$.

and where $G_{\mathcal{O}}(p, g)$ is a series with a finite radius of convergence (depending on p) in g at $d=2$ (in g_B at $d=2-\varepsilon$). For $\mathcal{O}$ given by (2.32) we have $\langle\mathcal{O}\rangle_{N=\infty}=\alpha_c^{d_{\mathcal{O}}/2}$, where $d_{\mathcal{O}}=2(n+\sum_{j=1}^{J} m_j)$ is the dimension of $\mathcal{O}$.

From (2.31), we may write the Borel transform $\hat{G}$ of G as

$$\hat{G}(p, s)=(p^2)^{1+\varepsilon/2-s}\hat{G}(s), \tag{2.33}$$

where $\hat{G}(s)$ has the following analytic structure:

At $d=2-\varepsilon$, $\hat{G}(s)$ has single poles at s of the form

$$s_{n,k}=n+(1+k)\tfrac{1}{2}\varepsilon, \qquad n, k \in \mathbb{N}. \tag{2.34}$$

Each series of poles at fixed n corresponds to the expansion in g_B of the $G_{\mathcal{O}}$'s such that $d_{\mathcal{O}}=n$ (see fig. 4a)*.

At $d=2$, $\hat{G}(s)$ has now branch points at each $s=n\in\mathbb{N}^\star$.

According to appendix A, the discontinuity along the nth cut is given by the "ordinary" Borel transform $\tilde{G}_{\mathcal{O}}(p, s)$ of the $G_{\mathcal{O}}(p, g)$ for $d_{\mathcal{O}}=n$ [see (A.4)]. However, the fact that these series are convergent implies that each discontinuity is *analytic* in the whole complex s plane. An important consequence is that one meets such a cut at $s=n$ with *the same discontinuity* in the different Riemann sheets generated by the previous branch points at $p<n$. (See fig. 4b).

* In addition $\hat{G}(s)$ has irrelevant single UV poles at $s=(1-n+\frac{1}{2}\varepsilon)$ corresponding to the behaviour of G as $\alpha_c\to\infty$.

3. The general structure of the 1/*N* expansion

The result of the sect. 2 is that the operator expansion holds at the tree order of the $1/N$ expansion. Now we intend to understand whether this remains true, and how, within the next terms of this expansion. For that purpose we shall investigate the analytic structure of the Borel transform of an arbitrary amplitude of the $1/N$ expansion by using the desingularization techniques of [10, 11].

3.1 AN α PARAMETRIC REPRESENTATION FOR THE BOREL TRANSFORM

First we need a Schwinger–Symanzik representation for those amplitudes. We recall that the usual propagator $D(p)$ (2.11) may be written

$$D(p,\alpha_c)=\int_0^\infty d\alpha\ e^{-\alpha(p^2+\alpha_c)}. \tag{3.1}$$

Similarly, we write the propagator $G(p)$ (2.12) as

$$G(p,\alpha_c)=\int_0^\infty d\alpha\, M(\alpha,\alpha_c)\left(\frac{\partial}{\partial\alpha}\right)^2 e^{-\alpha p^2}, \tag{3.2}$$

with

$$M(\alpha,\alpha_c)=\int_{-i\infty}^{+i\infty}\frac{ds}{2i\pi}\alpha_c^s\frac{\hat{G}(s)}{\Gamma(1+s-\frac{1}{2}\varepsilon)}\alpha^{s-\varepsilon/2}, \tag{3.3}$$

where $\varepsilon=2-d$, $\hat{G}(s)$ is the Borel transform of G (2.33) and where the derivatives with respects to α in (3.2) are introduced in order to make the integral convergent at $\alpha=0$.

Let G be some graph of the $1/N$ expansion. We denote $\mathscr{D}(G)$ ($\mathscr{G}(G)$) the set of D (G) propagators of G. Using (3.1) and (3.3), the integrations over internal momenta may be performed in the standard way to get for the amplitude I_G of G the representation

$$I_G(p,\alpha_c)=\int_0^\infty \prod_a d\alpha_a\, M_a(\alpha_a,\alpha_c)\mathscr{D}_G[\exp\{-pd_G(\alpha)p\}P_G(\alpha)^{-d/2}], \tag{3.4}$$

where each α_a is associated to a line a of G, $M_a=\exp(-\alpha_a\alpha_c)$ if $a\in\mathscr{D}(G)$ and M_a is given by (3.3) if $a\in\mathscr{G}(G)$ and where

$$\mathscr{D}_G=\prod_{a\in\mathscr{G}(G)}\left(\frac{\partial}{\partial\alpha_a}\right)^2, \tag{3.5}$$

$pd_G(\alpha)p$ and $P_G(\alpha)$ are the usual Symanzik functions of G. Eq. (3.4) holds if the amplitude I_G is UV convergent. In order to make power-counting rules simple we associate to each line a,

$$\text{an IR degree } \underline{\delta}_a = -1 \qquad \text{if } a \in \mathcal{D}(\mathrm{G}) \\ = 1 + \tfrac{1}{2}\varepsilon \qquad \text{if } a \in \mathcal{G}(\mathrm{G})\,; \tag{3.6}$$

$$\text{and UV degrees } \bar{\delta}_a = -1-n \qquad n \in \mathbb{N}\,, \quad \text{if } a \in \mathcal{D}(\mathrm{G}) \\ = 1 - n - \tfrac{1}{2}\varepsilon k \quad n, k \in \mathbb{N}\,, \quad \text{if } a \in \mathcal{G}(\mathrm{G})\,. \tag{3.7}$$

The IR superficial degree of G is

$$\underline{\omega}(\mathrm{G}) = \tfrac{1}{2} dL(\mathrm{G}) + \sum_a \underline{\delta}_a + \frac{\Delta - N}{2}\,, \tag{3.8}$$

and the UV degrees of G are defined as

$$\bar{\omega}(G) = \tfrac{1}{2} dL(\mathrm{G}) + \sum_a \bar{\delta}_a + \frac{\Delta - N}{2}\,, \tag{3.9}$$

where $L(\mathrm{G})$ is the number of internal loops of G; Δ and N are respectively the number of derivative couplings in G and of derivatives with respect to some external momenta of G, if needed.

The main problem is that we cannot apply the standard techniques of [10, 11] to study the Borel transform of $I_{\mathrm{G}}(p, \alpha_{\mathrm{c}})$ at two dimensions for two basic reasons:

(a) First, those technics apply when the α integrand of (3.4) is FINE [19], that is has a "generalized Taylor expansion in every Hepp's sectors". This is not the case for the function $M_a(\alpha, \alpha_{\mathrm{c}})$ for the G-propagator, which contains infinite series of $1/\mathrm{Ln}\,\alpha$ as $\alpha \to 0$ coming from the cuts of $\hat{G}(s)$ in the representation (3.3).

(b) Second, the amplitude has to be subtracted because of UV-divergent (sub)graphs. The point is that we need a subtraction scheme in the $1/N$ expansion which corresponds to a definite subtraction scheme in the usual weak coupling (perturbative) expansion. This is not the case for the BPH subtractions at zero momenta which were used in [13, 14]; indeed such subtractions are known to give IR divergences in the perturbative expansion which describes a massless theory. The modified soft mass renormalization scheme of [20] avoids that problem but introduces additional non-analyticity in the counterterms and is very difficult to handle explicitly*.

For those reasons we choose (as already done in sect. 2) to work in two steps:

First we look at the (bare) dimensionally regularized theory at $d = 2 - \varepsilon$ ($\mathrm{Re}\,\varepsilon > 0$), where the α-integrands are in fact FINE. This avoids point (a) and defines amplitudes meromorphic in the half-plane $\mathrm{Re}\,d < 2$.

Then we take into account renormalization and perform the limit $d \to 2$ by using dimensional renormalization [16] (the minimal subtraction scheme), which is known to respect the Ward identities of the $\mathrm{O}(N)$ invariance of the model [9].

* Moreover, the Ward identities of the model have to be restored by finite counterterms in the usual way

So we first study the bare amplitude I_G at $d = 2 - \varepsilon$ (Re $\varepsilon > 0$). Then the functions M_a are FINE and have the following expansion at $\alpha = 0$: *if* $a \in \mathscr{D}(G)$ we have obviously

$$M_a(\alpha, \alpha_c) = \sum_{n=0}^{\infty} \alpha^n \alpha_c^n d_n\,, \qquad d_n = \frac{(-1)^n}{n!}\,, \tag{3.10}$$

and if $a \in \mathscr{G}(G)$, from (3.3) and (2.34),

$$M_a(\alpha, \alpha_c) = \sum_{n,k} a^{n+k\varepsilon/2} \alpha_c^{n+(k+1)\varepsilon/2} g_{n,k}\,, \tag{3.11}$$

where

$$g_{n,k} = \frac{-1}{\Gamma(1+n+\frac{1}{2}k\varepsilon)} \operatorname{Res}\{\hat{G}(s); s_{n,k} = n + (k+1)\tfrac{1}{2}\varepsilon\}\,. \tag{3.12}$$

One can show that the amplitude I_G defined by (3.4) for d small enough is meromorphic in the half-plane $\{\text{Re}\, d < 2\}$ with poles at any d such that there is some connected one-particle irreducible (C1PI) subgraph S in G such that, for some set of $\{\bar{\delta}_a\}$ given by (3.7)

$$\bar{\omega}(s) = 0. \tag{3.13}$$

This leads to discrete series of poles at rational d with a point of accumulation at $d = 2$. So $d = 2$ will be in general an essential singularity, and there is a cut and other singularities on the real axis $d > 2$ (fig. 5). For Re $d < 2$ away from those discrete UV poles the convergent integral representation holds [21]:

$$I_G(p, \alpha_c) = \int_0^{\infty} d\alpha\, \mathscr{R}\left\{\prod_a M_a(\alpha_a, \alpha_c) \mathscr{D}_G[e^{-pd_Gp} P_G(\alpha)^{-d/2}]\right\}, \tag{3.14}$$

where the subtraction operator $\mathscr{R}$ is a sum of products of Taylor operators over all nests $\mathscr{N}$ of divergent subgraphs S of G [22]:

$$\mathscr{R} = \sum_{\mathscr{N}} \prod_{S \in \mathscr{N}} (-\tau_s^{-l(s)})\,. \tag{3.15}$$

Using (2.22) and the homogeneity properties of the integrand of (2.14), the Borel transform $\hat{I}_G(p, s)$ of I_G has the following integral representation:

$$\hat{I}_G(p, s) = \Gamma(s - \underline{\omega}(G)) \int_0^{\infty} d\alpha\, \mathscr{R}\left\{\prod_a M_a(\alpha_a, 1) \mathscr{D}_G[(pd_Gp)^{\omega(G)-s} \cdot P_G(\alpha)^{-d/2}]\right\}. \tag{3.16}$$

3.2. STRUCTURE OF THE BARE AMPLITUDES AT $d = 2 - \varepsilon$

The fact that the integrand of (3.16) is now FINE implies that the Borel transform $\hat{I}_G(p, s)$ is meromorphic in s and that the representation (3.16) is convergent for

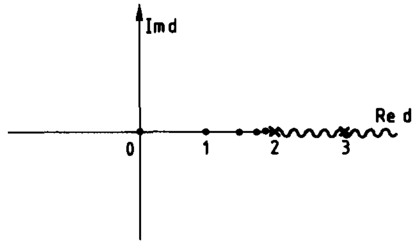

Fig. 5. Analytic structure of a regularized amplitude in $d: d=2$ is an essential singularity with a cut at $d>2$ and an accumulation of poles at $d<2$

Re s different from its poles. More precisely, one can extend the techniques used in [11] to classify those poles in terms of essential g-subgraphs E of G*. The result is the following:

Proposition 1: Let G be some graph and $\varepsilon \neq$ the UV poles of G given by (3.13). The Borel transform of G, $\hat{I}_{\mathrm{G}}(p, s)$, is meromorphic in the positive half-plane with single poles at values of s such that there is some g-essential $\mathrm{E} \subset \mathrm{G}$ and some choice of $\{\bar{\delta}_a, a \in \mathrm{E}\}$ such that

$$s = \underset{\sim}{\omega}(\mathrm{G}) - \bar{\omega}(\mathrm{E}) . \tag{3.17}$$

This structure corresponds to the following expansion of I_{G}:

$$I_{\mathrm{G}}(p, \alpha_{\mathrm{c}}) \simeq \sum_{\mathrm{E} \subseteq \mathrm{G}} F_{\mathrm{E}}(p, \alpha_{\mathrm{c}}) I_{(\mathrm{G/E})}(\alpha_{\mathrm{c}}) , \tag{3.18}$$

where the sum runs over the (infinite) set of g-essentials E in G.

$$I_{(\mathrm{G/E})}(\alpha_{\mathrm{c}}) = (\alpha_{\mathrm{c}})^{\omega(\mathrm{G/E})} I_{(\mathrm{G/E})}(1) \tag{3.19}$$

is the bare amplitude of the reduced graph (G/E)**.

$F_{\mathrm{E}}(p, \alpha_{\mathrm{c}})$ is the "IR-finite part" of $I_{\mathrm{E}}(p, \alpha_{\mathrm{c}})$ and is a (formal) series in $\{\alpha_{\mathrm{c}}^{n+(\varepsilon/2)k}; n, k \in \mathbb{N}\}$ of the form

$$F_{\mathrm{E}}(p, \alpha_{\mathrm{c}}) \simeq \sum_{\{\bar{\delta}_a\}} (\alpha_{\mathrm{c}})^{\sum_a (\delta_a - \bar{\delta}_a)} f_{\mathrm{E},\{\bar{\delta}_a\}}(p) , \tag{3.20}$$

obtained from I_{E} by:

(i) Insert the expansions (3.10), (3.11) of the functions M_a into the integral representation (3.14) of I_{E}.

(ii) Take the "finite part" of each term of this expansion, which gives the $f_{\mathrm{E},\{\bar{\delta}\}}$'s.

* According to [11], an essential g-subgraph of G is (at non-exceptional momenta) a (connected) subgraph E of G containing all its external vertices plus a family of derivatives versus external momenta of E internal to G

** (G/E) is obtained by reducing E to one vertex v in G and by putting on lines going to v the corresponding coupling derivatives The amplitude of (G/E) does not depend on the p's, so that we get the homogeneity relation (3.19)

Comment: The proof of this proposition is a straightforward extension of techniques of [11] and will not be given here. In our case the poles given by (3.17) have necessarily Re $(s)>0$. So any g-essential E gives series of sequences of poles at

$$s=\underline{\omega}(\mathrm{G/E})+n+\tfrac{1}{2}\varepsilon k\,, \qquad n,k\in\mathbb{N}\,. \tag{3.21}$$

We now have to check that this diagrammatic expansion corresponds to an operator expansion as for the $N=\infty$ order (2.15).

Proposition 2. Let $\mathcal{G}(p_i)$ be some bare Green function of the model at $d=2-\varepsilon$. Order by order within the $1/N$ expansion, the following expansion over all composite operators $\mathcal{O}_n$ (product of derivatives of α and $\boldsymbol{S}$ fields at the same point) holds

$$\mathcal{G}(p_i,\alpha_c)\simeq\sum_n \mathcal{G}_n(p_i,\alpha_c)\langle\mathcal{O}_n\rangle\,, \tag{3.22}$$

where each term of the $1/N$ expansion of $\mathcal{O}_n$ is a series in $\alpha_c^{\varepsilon/2}\propto g_B$ and so corresponds to a perturbative series in g_B and where each term of the $1/N$ expansion of the vacuum expectation value of $\mathcal{O}_n$, $\langle\mathcal{O}_n\rangle$, is proportional to $\alpha_c^{d_n}$ (where d_n is the canonical dimension of $\mathcal{O}_n$).

This expansion reflects the meromorphic structure of the Borel transform of $\mathcal{G}$, which has (order by order) series of single poles at

$$s=d_n+\tfrac{1}{2}\varepsilon k\,, \qquad k\in\mathbb{N}\,. \tag{3.23}$$

Proof. Diagrammatically, the reduced graphs (G/E) of the expansion (3.18) are obviously related to various composite operators $\mathcal{O}_n$ at different $1/N$ orders. Moreover, the sequences of poles (3.21) given by some E with $n\neq0$ (which give α_c^n contribution in F_{E}) come from the terms with $n\neq0$ of the expansion (3.10), (3.11) of the functions M_a. Those terms come from the operator expansions (2.15) and (2.31) of the propagators D and G, which involve vacuum expectation values (at order $N=\infty$) of operators of the form (2.32). When summing upon all graphs present in the $1/N$ expansion of $\mathcal{G}$, it is possible to reorganize all those contributions into an expansion of the form (3.22) (this needs a careful but not difficult analysis which will not be given here).

An explicit example is given in fig. 6. Let us note that many essentials contribute to the leading term ($n=0$) since the reduced graph may correspond to the observable $(\boldsymbol{S})^2=\mathbb{1}$.

3.3 STRUCTURE OF THE RENORMALIZED AMPLITUDES AT $d=2$

As already discussed in subsect. 3.1, the subtraction schemes at zero momenta of [13, 14, 20] are not suited to our study of the renormalized theory at $d=2$. For that reason we shall use dimensional renormalization (the minimal subtraction scheme or MS) [16] which is known to lead to an IR-finite perturbative expansion

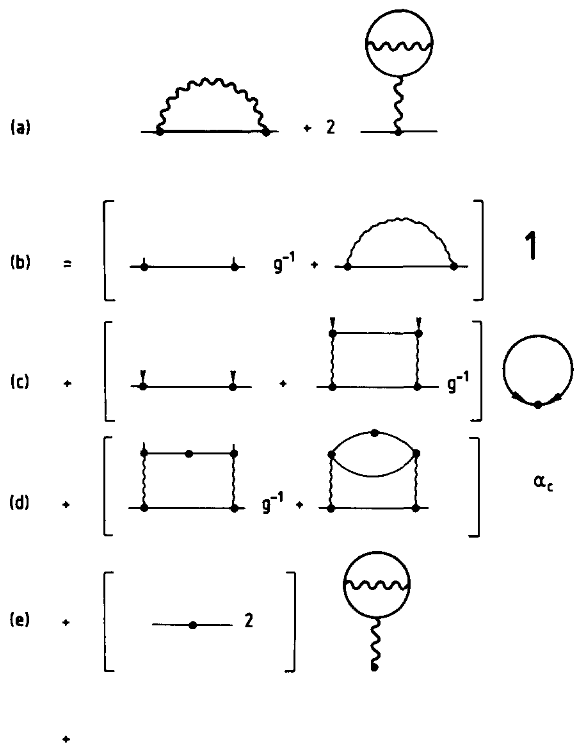

Fig. 6. Diagrammatic interpretation of the first terms of the expansion (3.22) for the irreducible two points function at order $(1/N)^1$. (a) the two graphs of $1/N$ expansion, the terms (b), (c), (d) correspond to operators $\langle \mathbb{1} \rangle$, $\langle (\partial_\mu \boldsymbol{S}) \rangle^2$ and $\langle \alpha \rangle$ at order $N = \infty$, the term (e) to $\langle \alpha \rangle$ at order $(1/N)^1$.

[8] and which respects the Ward identities [9] (the O(N) invariance) and the "quantum chirality identities" [13, 20] ($S^2 = 1$) in the perturbative phase. Unfortunately, there is no corresponding explicit subtraction scheme in the $1/N$ expansion; indeed, we have seen in subsect. 3.1 that $d = 2$ may be a branch point and/or an essential singularity, which cannot be subtracted as a pole (via some Cauchy integration for instance). For that reason we must define the MS scheme implicitly in the following way.

Lemma: (a) Let G be some graph of the $1/N$ expansion. The renormalized amplitude I_G^{MS} of G is defined at $d = 2 - \varepsilon$ (Re $\varepsilon > 0$) by

$$I_G^{MS}(p, \alpha_c) = \sum_{\{S\}} I_{G/US}(p_i, \alpha_c) \prod_S (\alpha_c)^{\omega^*(S)} K_S(g; \mu), \tag{3.24}$$

where:

(i) The sum is performed over all families (eventually empty) of disjoint C1PI divergent subgraphs S of G (at $d = 2$).

(ii) Each counterterm $K_S(g;\mu)$ is defined as a series in the renormalized coupling constant g [given by (2.6) and (2.8) as a function of α_c] of the form

$$K_S(g;\mu)=\sum_n g^n \mu^{-\varepsilon n} k_{S,n}(\varepsilon)\,, \tag{3.25}$$

obtained by summing the counterterms (of the MS scheme) of the perturbative expansion corresponding to S (consequently each $k_{S,n}(\varepsilon)$ is a polynomial in $1/\varepsilon$). The series (3.25) has a finite radius of convergence as long as ($\mathrm{Re}\,\varepsilon>0$) and so defines an analytic function of g (or α_c).

(iii) μ is the usual subtraction mass scale [the same as in (2.8)]

(iv) In (3.24) and in the following $\omega^*(s)$ stands for $\omega(s)$ at $\varepsilon=0$. S is divergent at $d=2$ iff $\omega^*(s)\in\mathbb{N}$.

(b) The amplitude I_G^{MS} has a limit as $\varepsilon\to0$ provided that

$$\mathrm{Arg}\,\varepsilon\in\,]-\tfrac{1}{2}\pi,0[\cup]0,\tfrac{1}{2}\pi[\,.$$

Comments: The fact that (3.24) defines a finite amplitude at $d=2$ is not obvious: indeed the counterterms are defined perturbatively and the bare amplitudes contain also non-perturbative terms (propositions 1 and 2). One may introduce an IR cut off (finite volume or external symmetry breaking term) which eliminates the non-perturbative terms in (3.22). The counterterms of the minimal subtraction scheme do not depend on the IR cut off so that one may sum up the perturbative expansion order by order to get an $1/N$ expansion which is now UV finite, then, set the IR cut off to zero and recover (3.24). A complete and rigorous proof is rather delicate and will not be given here.

With this subtraction scheme we can now go to $\varepsilon=0$. The main result is:

Theorem A. (a) At each order of the $1/N$ expansion, the Borel transform of any renormalized Green function $\mathscr{G}^{MS}(p_i,s)$ at two dimensions is analytic in s away from the positive real axis and has branch points at each entire point $s\in\mathbb{N}$.

(b) Each discontinuity $\Delta_p\hat{\mathscr{G}}^{MS}$ at $s=p$ *in the first sheet* (fig. 7a) may be written as a sum over all operators $\mathcal{O}_n$ of dimension $d_n=p$:

$$\Delta_p\hat{\mathscr{G}}^{MS}(s)=\sum_{\substack{\mathcal{O}_n\\ d_n=p}}\hat{\mathscr{G}}_n^{MS}(s)*\hat{\mathcal{O}}_n^{MS}(s)\,, \tag{3.26}$$

where $\hat{\mathscr{G}}_n^{MS}(s)$ is the ordinary Borel transform (see appendix A) of the term dual to $\mathcal{O}_n$, $\mathscr{G}_n^{MS}$, in the formal operator expansion obtained by the techniques exposed in appendix B. More precisely, each $\mathscr{G}_n^{MS}$ is at each order of the $1/N$ expansion a

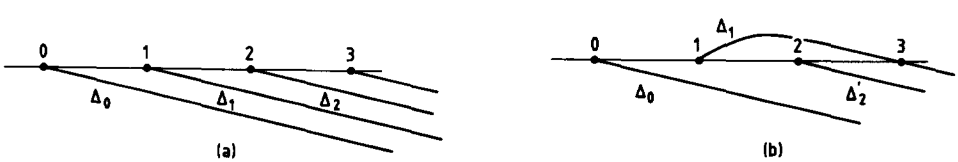

Fig. 7. The two different cuts at $s=2$ which give the first IR renormalon.

(divergent) series of the form

$$\mathcal{G}_n^{\mathrm{MS}}(g) = \sum_k g^k G_{n,k}\,, \tag{3.27}$$

and $\tilde{\mathcal{G}}_n^{\mathrm{MS}}(s)$ is given by the convergent series*

$$\tilde{\mathcal{G}}_n^{\mathrm{MS}}(g) = \sum_k \frac{s^{k-1}}{\Gamma(k-1)} G_{n,k}\,. \tag{3.28}$$

Similarly $\tilde{\mathcal{O}}_n^{\mathrm{MS}}$ is the ordinary Borel transform of $\langle \mathcal{O}_n^{\mathrm{MS}} \rangle$, and is given by the discontinuity at $s = p$ of the Borel transform $\hat{\mathcal{O}}_n^{\mathrm{MS}}(s)$**, $*$ is the Borel convolution product which reads

$$\tilde{g} * \tilde{\mathcal{O}}(s) = \int_p^s \mathrm{d}u\, \tilde{\mathcal{O}}(u) \tilde{\mathcal{G}}(s-u)\,. \tag{3.29}$$

(c) The discontinuities at $s = p$ ($p \geqslant 2$) in the different sheets corresponding to the branch points at $q < p$ are in general different from the first one given by (3.26).

Proof: The principle of the proof is the following. We first look at the analytic structure of the Borel transform of $\mathcal{G}^{\mathrm{MS}}$ at $d = 2 - \varepsilon$. Using prop. 2 and lemma 1, one can show that, order by order in the $1/N$ expansion, $\hat{\mathcal{G}}^{\mathrm{MS}}(p_i, s)$, is meromorphic like $\hat{G}$ with infinite series of single poles at $s = n + \frac{1}{2}\varepsilon k$ ($n, k \in \mathbb{N}$) corresponding to the new operator expansion

$$\mathcal{G}^{\mathrm{MS}}(p_i, \alpha_c) = \sum_n \mathcal{G}_n^{\mathrm{MS}}(p_i, \alpha_c) \langle \mathcal{O}_n^{\mathrm{MS}} \rangle\,, \tag{3.30}$$

where $\langle \mathcal{O}_n^{\mathrm{MS}} \rangle$ is now the v.e.v. of the renormalized operator $\mathcal{O}_n$ and is (in the $1/N$ expansion) of the form

$$\langle \mathcal{O}_n^{\mathrm{MS}} \rangle = \alpha_c^{d_n} \{\text{series in } g_{\mathrm{B}}\}\,,$$

and where each $\mathcal{G}_n^{\mathrm{MS}}(p_i, \alpha_c)$ is a perturbative series in g_{B} (and so in g) convergent for $\mathrm{Re}\,\varepsilon > 0$; each term of the corresponding series in g, $G_{n,k}(\varepsilon)$ corresponds to IR and UV-subtracted amplitudes of the perturbation theory and so has a limit as $\varepsilon \to 0$ which is the $G_{n,k}$ of (3.27).

We now have to take the limit $\varepsilon \to 0$ with $\mathrm{Arg}\,\varepsilon = \theta$ fixed. The crucial point is that each series of poles at $s = n + \frac{1}{2}\varepsilon k$, n fixed, coalesces to give a cut along $s = n + \lambda$, $\arg \lambda = \theta$, and that no other singularity appears as $\varepsilon \to 0$. This may be shown by the following argument, developed in appendix C.

The amplitudes subtracted at zero momenta according to the usual Zimmermann scheme may be represented by a "complete Mellin representation" [23]. In this representation, (given in appendix C) general arguments show that there are no other singularities than the above cuts at $\varepsilon = 0$. Then one can argue that the finite

* The term $k = 0$ has to be understood as $\delta(s)$.

** $\hat{\mathcal{G}}_n^{\mathrm{MS}}(s)$ has in fact a single branch point at $s = p$ so that $\tilde{\mathcal{O}}_n^{\mathrm{MS}}(s)$ is analytic on the real axis $s > d_n$.

counterterms needed to recover the MS subtracted amplitude do not destroy the structure and simply modify the discontinuities. So we get part (a).

Starting now from the fact that those cuts are the limit of the series of poles, one uses (3.30) to get (3.26). Indeed, for $\varepsilon \neq 0$, $\operatorname{Arg} \varepsilon = \theta$, the discontinuity along the line $\{s = p + e^{i\theta}x\}$ may be written in the form (3.26), but now $\tilde{\mathscr{G}}_n^{\mathrm{MS}}$ is defined on the line $\operatorname{Arg} s = \theta$ as a sum of Dirac distributions at $s = \frac{1}{2}\varepsilon k$, $k \in \mathbb{N}$ and $\tilde{\mathscr{O}}_n^{\mathrm{MS}}$ on the line $\operatorname{Arg}(s-p) = \theta$ as a sum of Dirac distributions at $s = p + \frac{1}{2}\varepsilon k$, $k \in \mathbb{N}$.

Now the distribution $\tilde{\mathscr{G}}_n^{\mathrm{MS}}$ with discrete support at $\varepsilon > 0$ tends towards (3.28) as $\varepsilon \to 0$; indeed, using (2.6) and (2.8), to the term g^k in (3.27) corresponds the distribution

$$\Delta_\varepsilon^{(k)} = \sum_{m=0}^{\infty} \Gamma(\tfrac{1}{2}\varepsilon)^{-k} \frac{(k+m-1)!}{m!(k-1)!} \delta(s - (k+m)\tfrac{1}{2}\varepsilon) , \tag{3.31}$$

which tends towards $s^{k-1}/\Gamma(k)$ as $\varepsilon \to 0$. $\tilde{\mathscr{O}}_n^{\mathrm{MS}}$ being equal to the discontinuity of $\hat{\mathscr{O}}_n^{\mathrm{MS}}$ for any $\varepsilon \geqslant 0$, we finally get part (b) of the theorem.

The arguments developed here do not permit us to look at the singularities in other sheets than the first one. The CM representation is a more adequate tool for that problem. In appendix D we use it to look explicitly at point (c) at order $(1/N)^1$.

3.4 THE STATUS OF THE OPERATOR EXPANSION AND IR RENORMALONS

Now we can see how the operator expansion makes sense. Using (3.26) and the inverse Borel transform (2.23) we get that the operator expansion

$$\mathscr{G}^{\mathrm{MS}}(p_i, g) = \sum_n \mathscr{G}_n^{\mathrm{MS}}(p_i, g) \langle \mathscr{O}_n^{\mathrm{MS}}(g) \rangle , \tag{3.32}$$

is *exact* at each order of the $1/N$ expansion with the following resummation prescriptions which make it unambiguous.

(i) All $\mathscr{G}_n(p_i, g)$ are defined as the Borel sum (A.6) of (3.27) with the *same* prescription of integration on a complex line above (or under) the positive real axis.

(ii) The non-perturbative quantities $\langle \mathscr{O}_n^{\mathrm{MS}}(g) \rangle$ are defined without ambiguities since each ordinary Borel transform $\tilde{\mathscr{O}}_n^{\mathrm{MS}}$ is analytic for $\operatorname{Re} s > d_n$.

The remarkable point is that we have obtained that result quite independently of the presence and of the nature of the IR renormalons on the positive real s axis of the ordinary Borel transform $\tilde{\mathscr{G}}_n$. Those singularities come obviously from point (c) of theorem A. However, since the discontinuities $\Delta_p \hat{G}$ given by (3.26) are real for $p < s < p+1$, they are equal in the first sheet above or under the positive real axis; one concludes that the first IR renormalon of the perturbative series $\tilde{\mathscr{G}}_0^{\mathrm{MS}}$ is at $s = 2$ (and not at $s = 1$) and corresponds to the operators of dimension 4 (see fig. 7).

Similarly, the first IR renormalon of $\tilde{\mathscr{G}}_1^{\mathrm{MS}}$ (corresponding to the operator $(\partial \boldsymbol{S})^2$) is at $s = 1$. From (3.32) when adding the two contributions associated to the operators $\mathbb{1}$ and $(\partial S)^2$, the corresponding singularities at $s=2$ must cancel, leaving the next renormalon at $s=3$. This mechanism holds at all orders; namely, when adding the terms corresponding to the operators of dimensions $\leqslant P$ in the operator expansion, all renormalons at $s \leqslant P+1$ cancel.

Of course one would like to have more details on the nature of those singularities in the $1/N$ expansion. Unfortunately, the techniques developed here cannot cope with that problem in general. We have checked in appendix C the presence of such renormalons at the order $1/N$. At that order those singularities of the Borel transform $\tilde{\mathscr{G}}_0(s)$ are single poles at $s = 2, 3, 4, \ldots$ and are checked to be proportional to the terms $\tilde{\mathscr{G}}_n(s)$ dual to some of the operators of the corresponding dimension. The renormalon at $s=2$ is for instance dual to the operator $\alpha \cdot \alpha$, the renormalons at $s=3$ dual to $\alpha \cdot \alpha \cdot \alpha$ and $\partial_\mu \alpha \cdot \partial_\mu \alpha$, etc.... One expects that such a feature remains at next orders.

4. General discussion

(1) Let us first outline the results of this paper: At any arbitrary order of the $1/N$ expansion, we have shown that in the O(N) σ model:

(i) There are IR renormalons at $s = 4\pi n$, $n \geqslant 2$ on the positive real axis of the Borel transform of the perturbative expansion.

(ii) Non-perturbative terms proportional to vacuum expectation values of all invariant operators are present.

(iii) Those terms are organized in an operator expansion à la SVZ at all orders.

(iv) There is a well-defined Borel summation prescription of the perturbative series which deals with IR renormalons and makes the operator expansion unambiguous. Basically, the operator expansion manages to cancel the IR renormalons of its different perturbative parts, as explained in subsect. 3.4.

(2) The technics of this paper may be applied without difficulties to the two-dimensional U(N) Gross–Neveu Model [5]. One can show similarly that the vacuum expectation values of the fermion condensates associated to the spontaneous breakdown of the (discrete) $\mathbb{Z}_2$ chiral symmetry organize in a SVZ operator expansion which cancels the corresponding IR renormalons.

(3) It seems reasonable to think that those results are not modified when summing the $1/N$ expansion to get the σ model for finite N (the $1/N$ expansion is likely Borel summable [24, 25]). One expects that the branch points of the Borel transform are only shifted to $s = 2n/\beta_2$ with $\beta_2 = (N-2)/2\pi N$ but that their structure remains the same. The cancellation of IR renormalons which takes place is, in fact, the only possible way to make the SVZ operator expansion consistent with the Borel resummation procedure. So we conjecture that the same phenomenon occurs in four-dimensional gauge theories (with massive or massless fermions). It is interesting to note that SVZ found heuristically that an optimal summation procedure for the

operator expansion was their "Borel improvement" [1], that is precisely the use of the Borel transform.

However, it seems to us that any direct attempt to get a rigorous proof of that fact (even for the σ model for finite N) needs a non-perturbative use of the Dyson–Schwinger equations and is a formidable program [26].

(4) The mechanism described above which deals with IR renormalons is somewhat different to what happens for instantons in quantum mechanics or in massive fields theories [27].

In the case of instantons, one can write any quantity E as

$$E(g) = \sum_{n=0}^{\infty} E^{(n)}(g), \tag{4.1}$$

where $E^{(n)}(g) \sim O([E^{(1)}(g)]^n)$ is the sum of n instanton contributions. However, each $E^{(n)}$ is ambiguous even at leading order $O([E^{(1)}(g)]^n)$ (this is related physically to the instability of instantons–anti-instantons configurations, and mathematically to Stockes phenomenon, i.e. to the different ways to catch those configurations in the functional integral). This ambiguity in the definition of $E^{(n)}$ is raised only when making precise the integration prescription around the corresponding singularity at $s = nS_0$ of the Borel transform of $\sum_{p=0}^{n-1} E^{(p)}(g)$ (S_0 is the action of the instanton).

In the operator expansion (3.32), the terms of order $O([e^{2/\beta_2 g}]^n)$ given by operators $\mathcal{O}_k$ of dimension $d_k = 2n$, are ambiguous only at next order $O([e^{2/\beta_2 g}]^{n+1})$, but the v.e.v. $\mathcal{O}_k(g) = \langle 0|\mathcal{O}_k|0\rangle$ are given by perturbation theory only up to a numerical factor. Indeed, the $\mathcal{O}_k$'s satisfy the RG equation*

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_n(g)\right] \mathcal{O}(g) = 0\,, \tag{4.2}$$

which determines the $\mathcal{O}_k$'s up to dimensionless factors C_k**:

$$\mathcal{O}_k(g) = C_k \mu^{2n} \exp \int^{g} du \frac{-2n + \gamma_n(g)}{\beta(g)}\,. \tag{4.3}$$

(5) However, the cancellation of renormalons at $s = -(2/\beta_2)(n+1)$ between the term of order n and those of order $<n$ fixes strong constraints on the C_k's. For instance, the existence of a renormalon at $s = -4/\beta_2$ fixes the value of the first perturbative term $\langle 0|(\partial_\mu S)^2|0\rangle$. An interesting question is: can all non-perturbative quantities be fixed in that way? In such a case one could say that (formally) perturbation theory contains enough information to recover the full theory. One could also imagine numerical estimations of the first non-perturbative terms based on that scheme.

* γ_n is an operator mixing all $\mathcal{O}_k$ with dimension $2n$.

** The RG functions $\beta(g)$ and $\gamma_n(g)$ are Borel summable and so computable from perturbative theory This may be checked in the $1/N$ expansion; the reason why there are no IR renormalons, at least in the MS scheme, is that β and the γ's are not modified when there is an IR cut off

(6) In sect. 3 we never made precise the integrability at infinity of the Borel transform. In fact we expect no problem order by order in the $1/N$ expansion but it is possible that in the full theory this condition breaks down, so that the Borel sum itself is only asymptotic [3]*. With this restriction this does not change our conclusions.

(7) In the case of the non-linear σ model, the existence of a "spin wave condensate", $\langle 0|(\partial_\mu \boldsymbol{S})^2|0\rangle \neq 0$, is obviously related to the restoration of the O(N) symmetry. This is self-evident in the limit $N=\infty$ since we have then**

$$\langle 0|\partial_\mu \boldsymbol{S}\, \partial_\mu \boldsymbol{S}|0\rangle = -(\text{physical mass})^2\,. \tag{4.4}$$

(8) The presence of condensates is often justified by the existence of non-perturbative effects such as instantons [1]. It is interesting to note that the existence of instantons is not necessary to get such condensates.

However, the problem of taking into account instantons in the operator expansion in a mathematically consistent way has still to be understood [29] and is obviously related to a correct understanding of the dense instanton gas problem [30].

I am grateful to E. Brézin and J. Zinn-Justin for their interest and for a careful reading of the manuscript, and to C. de Calan for useful discussions. I thank also Pr. K. Symanzik and H.J. de Vega for stimulating discussions.

Appendix A

BOREL TRANSFORMS

A convenient object to study non-Borel summable functions is the (modified) Borel transform [31]

$$\hat{f}(s) = \int_0^{\text{const}} \frac{\mathrm{d}g}{g^2}\, \mathrm{e}^{s/g} f(g)\,. \tag{A.1}$$

If $f(s)$ is analytic in some disc $\mathrm{Re}\,(g^{-1}) > \rho^{-1}$ tangent to the origin $g=0$ (and has an *ad hoc* behaviour as $g\to 0$, which we do not make precise), $\hat{f}(s)$ is analytic away from the positive real axis $s\in \mathbb{R}^+$ and the following inverse representation holds***:

$$f(g) = \int_{\mathrm{c}} \frac{\mathrm{d}s}{2i\pi}\, \mathrm{e}^{-s/g} \hat{f}(s)\,, \tag{A.2}$$

where the (anticlockwise) contour c encircles $\mathbb{R}^+$ (fig. 8) (A.1) is nothing other than

* 't Hooft's argument for such a singularity is based on the existence of an infinite number of resonances at arbitrary large energy. This is not the case for the non-linear σ model [28].

** $\partial_\mu \boldsymbol{S}\, \partial_\mu \boldsymbol{S}$ is subtracted according to some normal product algorithm and may perfectly well have a negative vacuum expectation value

*** $\hat{f}(s)$ depends on the upper bound of integration in (A.1) but its discontinuity along $\mathbb{R}^+$ does not.

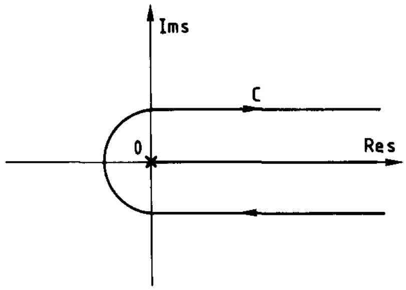

Fig. 8 Integration contour for the inverse Borel transform (A.2)

a Laplace transform with respect to $1/g$, or a Mellin transform with respect to $\exp(-1/g)$.

If $f(g)$ has an asymptotic expansion

$$f(g)=\sum_{k=0}^{\infty} g^k C_k\,, \tag{A.3}$$

and is Borel summable (i.e. satisfies the Nevanlinna–Sokal theorem [32]), the discontinuity at $s=0$, $\Delta_0\hat{f}$, given by

$$\Delta_0\hat{f}(s)=\frac{1}{2i\pi}\left[\hat{f}(s+i\varepsilon)-\hat{f}(s-i\varepsilon)\right], \tag{A.4}$$

is equal to the ordinary Borel transform $\tilde{f}(s)$ of f,

$$\tilde{f}(s)=\sum_k s^{k-1}\frac{C_k}{\Gamma(k)}, \tag{A.5}$$

where $s^{-1}/\Gamma(0)$ has to be understood as $\delta(s)$. (A.2) is nothing other than the usual inverse Borel transform

$$f(g)=\int_0^{\infty} \mathrm{d}s\ \mathrm{e}^{-s/g}\tilde{f}(s)\,. \tag{A.6}$$

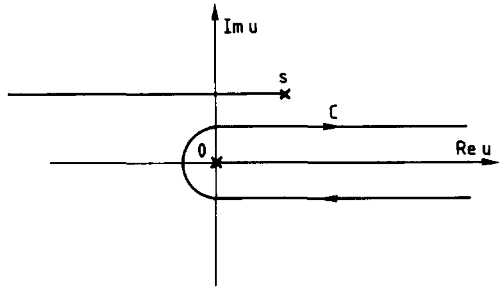

Fig. 9 Integration contour from the Borel convolution (A 7).

When f is not Borel summable, as in instanton problems, $\hat{f}(s)$ has other branch points on the positive real axis which have to be taken into account when one writes (A.2) as an integral over the discontinuities.

In this paper the term Borel transform denotes in general the transform (A.1) and we are more precise when dealing with the ordinary Borel transform (A.5), (A.6).

We finally recall that the Borel transform of a product $f_1 \cdot f_2$ is the Borel convolution product $\hat{f}_1 * \hat{f}_2$:

$$\hat{f}_1 * \hat{f}_2(s) = \int_c \frac{du}{2i\pi} \hat{f}_1(u) \hat{f}_2(s-u) , \tag{A.7}$$

where the contour c encircles $\mathbb{R}^+$ (or $s - \mathbb{R}^+$) as in fig. 9.

Appendix B

THE OPERATOR EXPANSION IN PERTURBATION THEORY

A simple way to get the terms of the (formal) operator expansion (1.2) in perturbation theory is to see it as an IR expansion when some IR cut off goes to zero. In the case of the 2-dimensional non-linear σ model one may consider the theory in a finite volume V or put an external constant magnetic field H.

General techniques of studying asymptotic estimates in perturbation theory [2, 10, 11] can be used to get the following result: within perturbation theory, the following expansion holds as V $(1/H) \to \infty$ for any observable $\mathcal{G}$:

$$\mathcal{G}(V) \cong \sum_n \mathcal{G}_n(V) \mathcal{O}_n(V) , \tag{B.1}$$

where the sum runs over all composite operators $\mathcal{O}_n$.

$\mathcal{O}_n(V)$ is equal to the vacuum expectation value of the renormalized operator $\mathcal{O}_n$ and is of the form

$$\mathcal{O}_n(V) = V^{-d_n/2} \sum_{k=0}^{\infty} g^k P_{(n),k}(\mathrm{Ln}\, V) , \tag{B.2}$$

where d_n is the dimension of the operator $\mathcal{O}_n$ and each $P_{(n),k}$ a polynomial (of order $\leqslant k$) in Ln V.

The term dual to $\mathcal{O}_n$, $\mathcal{G}_n(V)$, is analytic in V^{-1}, and more precisely

$$\mathcal{G}_n(V) \simeq \sum_{k=0}^{\infty} g^k G_{n,k}(V^{-1}) , \tag{B.3}$$

where each $G_{n,k}(V^{-1})$ is a (formal) series in V^{-1} and corresponds to a sum of UV and IR-*subtracted*[*] Feynman amplitudes.

[*] In the sense of [4, 12].

The key of the proof of the IR finiteness of the non-linear σ model [8] is the fact that, for invariant observables, among all operators of dimension 0, only the invariant one $\mathbb{1}$ survives in (B.1). Similarly one expects that in general, only O(N) invariant operators $\mathcal{O}_n$ are present in (B.1). In fact, to the "naively" invariant operators one must add new ones involving the operator $\Delta\sigma/\sigma$, which is the perturbative analog to the operator α (in the parametrization $\boldsymbol{S} = (\boldsymbol{\pi}, \sigma = \sqrt{1-\pi^2})$); it may namely be written as the derivative of the action versus the constraint and so is related to the action $(\partial_\mu \boldsymbol{S}\ \partial_\mu \boldsymbol{S})$ via the equations of motion [9]. [9].

Taking $V=\infty$ the $\mathscr{G}_n$ are finite perturbative series (B.3) and the $\mathcal{O}_n$ are zero order by order in g (B.2). It is those $\mathcal{O}_n$ which are assumed to be non-zero in the SVZ operator expansion.

Appendix C

The complete Mellin (CM) representation of Feynman amplitudes [23] provides a systematic tool to study its analytic properties. Indeed, an amplitude is written as an integral of the Mellin type, the CM integrand is a product of Γ functions and of linear powers of external invariants and of internal masses, its analytic structure allows the determination of any asymptotic expansion. Another advantage is that the renormalization according to the Zimmermann scheme takes a very simple form: it results in a modification of the integration path without any change in the integrand [23]. It is not difficult to extend the CM representation to our problem. Starting from the representation (3.4) we get for some convergent graph G:

$$I_{\mathrm{G}} = \int_{\mathrm{C_0}\cap\Delta_{\mathrm{G}}} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_k S_k^{y_k}\Gamma(-y_k) \prod_a \alpha_c^{-\varphi_a}\Gamma_a(\varphi_a)\,, \tag{C.1}$$

where the variables x_j and y_k are attached respectively to each one-tree j or two-trees k of G. S_k is the cut invariant corresponding to the two-trees k (and is quadratic in external momenta). The linear function $\varphi_a(x,y)$ associated to the line a is

$$\varphi_a = \sum_j u_{aj}x_j + \sum_k u_{ak}y_k - \underline{\delta}_a\,, \tag{C.2}$$

where u_{aj} $(u_{ak})=0$ or 1 following the line a belongs or not to the one-tree j (two-trees k). The terms $\Gamma_a(\varphi)$ are given by

$$\Gamma_a(\varphi) = \begin{cases} \hat{D}(-\varphi)/\Gamma(-\varphi-\underline{\delta}_a) = \Gamma(\varphi)\,, & \text{if } a \in \mathscr{D}(\mathrm{G})\,, \quad \text{(C.3a)} \\ \hat{G}(-\varphi)/\Gamma(-\varphi-\underline{\delta}_a)\,, & \text{if } a \in \mathscr{G}(\mathrm{G})\,. \quad \text{(C.3b)} \end{cases}$$

The integration symbol means

$$\int_{-i\infty}^{+i\infty} \frac{\mathrm{d}x_j}{2i\pi}\frac{\mathrm{d}y_j}{2i\pi}\,\delta\Big(\sum_j x_j + \sum_k y_k + \tfrac{1}{2}d\Big)\,, \tag{C.4}$$

and that (x, y) belongs to the intersection of the cell C_0:

$$C_0 = \{x, y \mid \mathrm{Re}\, x > 0, \mathrm{Re}\, y > 0\}, \tag{C.5}$$

and of the UV convergence domain Δ_G:

$$\Delta_G = \{x, y \mid \mathrm{Re}\, \varphi_a > 0, \forall a \in G\}. \tag{C.6}$$

The only change with [23] is the φ_a's and the functions Γ_a which have now a more complex analytic structure (with cuts at negative $\varphi = -n$). Similarly, a renormalized amplitude according to the Zimmermann scheme (subtractions at zero momenta) I_G^R is

$$I_G^R = \sum_c \mu_c \int_{c \cap \Delta_G} J_G(x_i, y_j, S_k) \alpha_c^{-\Sigma_a \varphi_a}, \tag{C.7}$$

where the sum runs over cells c delimited by the singularities of the function $\prod_i \Gamma(-x_i) \prod_k \Gamma(-y_k)$ in (C.1). The multiplicity factor μ_c is an integer $\neq 0$ only for a finite number of cells c such that $c \cap \Delta_G \neq \varnothing$ and does not depend on the functions Γ_a. J_G is the same integrand as in (C.1). The Borel transform $\hat{I}_G^R$ is for $\mathrm{Re}\, s < 0$:

$$\hat{I}_G^R(s) = \sum_c \mu_c \int_{c \cap \Delta_G} \delta\Big(s + \sum_a \varphi_a\Big) J_G(x_i, y_j, S_k), \tag{C.8}$$

and its analytic structure is obtained by translating s into the domain $\mathrm{Re}\, s > 0$ ($\mathrm{Re}\, \varphi < 0$). Without getting into a general study, one may deduce that the singularities are branch points at the same positions as in the case of usual propagators, and are the right limit of the series of poles occurring at $d = 2 - \varepsilon$, as claimed in subsect. 3.3.

In order to recover the MS subtracted amplitude, one has to introduce finite counterterms. One has in fact

$$I_G^{MS}(p) = \sum_{\{S\}} I_{G/US}^R(p) \prod_S (I_S^{MS}), \tag{C.9}$$

where the sum runs over family of disjoint ClPI divergent subgraphs S. The vertex corresponding to S in (G/US) has to be oversubtracted according to the superficial degree of S ($\bar{\omega}(S) \geq 0$). The counterterms I_S^{MS} are renormalized amplitudes taken at zero external momenta. They can depend on g only as

$$I_S^{MS}(g) = \alpha_c^{\omega(S)} \times (\text{series in } g),$$

and so correspond only to a discontinuity at $s = \underline{\omega}(S)$ in the Borel plane.

Appendix D

Let us apply the CM representation to study the Borel transform of the $1/N$ order of the two-point function Γ_2. It is expressible in terms of the two graphs of fig. 6. The second one does not depend on the external momenta and by homogeneity it gives a single cut at $s=1$. So we are interested in the one-loop graph S_1.

Using the CM representation for its amplitude subtracted at zero momenta, one gets for its Borel transform the representation

$$\hat{I}^{R}(s)=\int_{-i\infty}^{+i\infty}\frac{du}{2i\pi}\frac{\Gamma(u-s)\Gamma(2-u)\Gamma(s-1)}{\Gamma(2-s)}(p^2)^{1-s}\Gamma_G(-u)\Gamma_D(u-s)\,, \quad (D.1)$$

where the functions Γ_D and Γ_G are given respectively by (C.3a) and (C.3b). (D.1) holds for $-1<\operatorname{Re}s<\operatorname{Re}u<0$. The integrand has then double poles at $u=s-k$ ($k\in\mathbb{N}$) and branch points at $u=k\in\mathbb{N}$. Going into the domain $\operatorname{Re}s<0$, $\hat{I}^{R}$ has a singularity if the integration contour is pinched between two of those singularities (fig. 10). So one gets immediately the branch points at $s=n\in\mathbb{N}$.

To get the MS subtracted amplitude, according to (C.3), one needs two counter-terms which modify the discontinuities of $\hat{I}$ at $s=0$ and $s=1$, but not the next ones. So we start from (D.1) to study the singularities at $s=n\geqslant 2$ in the different sheets.

Moving the integration contour around the poles ($u=s-n$) we get for $\hat{I}^{R}(s)$ the (formal) sum

$$\hat{I}^{R}(s)=\sum_{n=0}^{\infty}(p^2)^{1-s}\frac{\Gamma(s-1)}{\Gamma(2-s)}\left(\frac{(-1)^n}{n!}d_n\frac{d}{ds}+b_n\right)\Gamma_G(n-s)\Gamma(2-s+n)\,, \quad (D.2)$$

where $d_n=(-1)^n/n!$ is the residue of Γ_D at n and b_n is irrelevant in the following.

The discontinuities of $\hat{I}(s)$ in the first sheet are given by theorem A. The existence of an IR renormalon at $s=2$ comes from the difference between that discontinuity at $s=2$ and the discontinuity in the sheet reached by passing under $s=1$ (fig. 7).

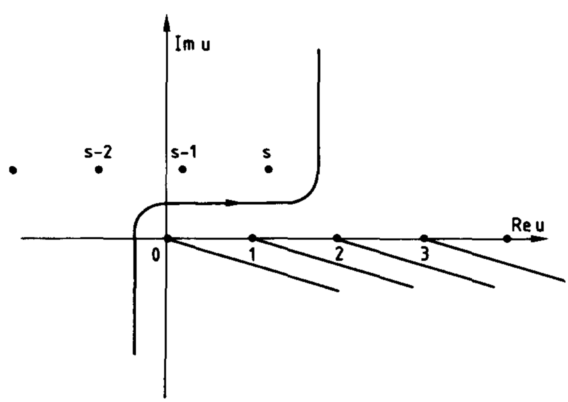

Fig 10. Analytic structure and integration contour for the integral (D 1)

The difference of $\hat{I}^{R}(s)$ between those two sheets comes from the discontinuity of Γ_G at $s=1$; one gets for that difference

$$\hat{I}^{R}(s)-\hat{I}'^{R}(s)=(p^2)^{1-s}\frac{\Gamma(s-1)}{\Gamma(2-s)}\left(d_0\frac{d}{ds}+b_0\right)2i\pi\Delta_1\Gamma_G(s)\Gamma(2-s)$$

$$=-2i\pi\frac{d_0}{p^2}\Delta_1\Gamma_G(2)\Gamma(1)\frac{1}{s-2}+\text{regular function at } s=2\,, \quad \text{(D.3)}$$

where $\Delta_1\Gamma_G(s)=\Delta_1\hat{G}(s)/\Gamma(s-1)$ is the discontinuity of Γ_G at $s=1$ and is analytic for $\text{Re}\, s>1$.

From (D.3), the first renormalon at $s=2$ (that is the first singularity of the discontinuity of $\hat{I}^{R}(s)$ at $s=0$) is a *single pole* and is proportional to d_0/p^2, which corresponds graphically to the term dual to the operator $\alpha\cdot\alpha$.

The same arguments hold for the next discontinuities at $n>2$. At that order of $1/N$ expansion, one may check that the various renormalons are always single poles and are proportional to terms dual to composite operators involving only the α field. For instance, we get renormalons at $s=3$ in terms of the duals of the operators $\alpha\cdot\alpha\cdot\alpha$ and $\partial_\mu\alpha\cdot\partial_\mu\alpha$. A closer look at (D.3) suggests that those singularities may be classified in terms of nests of essentials but we need to study diagrams with many loops in order to make this idea more precise and such a study becomes very complicated.

References

[1] M.A. Shifman, A.I Vainshtein and V.I. Zakharov, Nucl. Phys B147 (1979) 385, 447, 519, M.A. Shifman, A I Vainshtein, M.B. Voloshin and V I Zakharov, Phys Lett. 77B (1978) 80

[2] K. Wilson, Phys. Rev 179 (1969) 1499; W. Zimmermann, *in* Lectures on elementary particles and quantum field theory, ed. S. Deser, M Gisaru and M Pendleton (M.I.T. Press, Cambridge, Mass 1971) vol. 1

[3] G. 't Hooft, *in* Erice Lectures Notes 1977, ed. A Zichichi (Plenum Press, New York, 1978)

[4] G Parisi, Nucl. Phys. B150 (1979) 163

[5] D J Gross and A. Neveu, Phys Rev. D10 (1974) 3235

[6] P. Olesen, Phys Lett. 73B (1978) 327

[7] N D Mermin and H. Wagner, Phys Rev. Lett. 17 (1966) 1133, S. Coleman, Comm. Math. Phys. 31 (1973) 259

[8] F. David, Comm Math. Phys. 81 (1981) 149

[9] E Brézin, J Zinn-Justin and J C Le Guillou, Phys. Rev. D14 (1976) 2615

[10] M.C. Bergère and Y M.P. Lam, Comm Math. Phys 39 (1974) 1, Asymptotic expansion of Feynman amplitude. Part II – the divergent case, F.U.B. preprint, H.E.P May 74/9, unpublished

[11] M C. Bergère and F. David, Non-analyticity of the perturbative expansion for super-renormalizable massless field theories, Preprint DPh T.82/16 Ann. of Phys. to be published

[12] E.R Speer and M.J. Westwater, Ann Inst. H Poincaré A14 (1971) 1

[13] I Ya. Aref'eva, Theor. Math Phys. 36 (1978) 24

[14] J H. Lowenstein and E R. Speer, Nucl. Phys. B158 (1979) 397

[15] L. Schwartz, Théorie des distributions (Paris, Hermann, 1966)

[16] G 't Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189; E.R Speer, Dimensional and analytic renormalization, *in* Renormalization theory, Erice Summer School, ed G Velo and A.S Wightman (1975)

[17] K. Symanzık, Nuovo Cım Lett 8 (1973) 771
[18] R Jackiw and S Templeton, Phys Rev. D23 (1981) 2291,
T Appelquıst and R. Pisarskı, Phys. Rev D23 (1981) 2305
[19] M C Bergère, C de Calan and A P C Malbouısson, Comm. Math Phys. 62 (1978) 137
[20] I Ya. Aref'eva, E R. Nissimov and S.J. Pacheva, Comm. Math Phys. 71 (1980) 213
[21] M.C Bergère and F. Davıd, J Math Phys. 20 (1976) 1244
[22] M.C. Bergère and Y.M P Lam, J Math. Phys 17 (1976) 1546
[23] C de Calan and A.P.C Malbouısson, Ann. Inst. H Poıncaré 32 (1980) 91,
C. de Calan, F David and V Rıvasseau, Comm Math. Phys 78 (1981) 531
[24] H J de Vega, Phys Lett 98B (1981) 280
[25] J. Frohlıch, A. Mardın and V. Rıvasseau, Comm Math. Phys 86 (1982) 87
[26] J. Bros and D Iagolnıtzer, Ann of Phys 137 (1981) 104
[27] J. Zınn-Justın, Nucl. Phys B192 (1981) 125
[28] A B. Zamolodchikov and A.I. Zamolodchıkov, Ann of Phys 80 (1979) 253
[29] V.A. Novıkov, M A Shıfman, A I. Vaınshtein and V.I. Zakharov, Nucl. Phys B174 (1980) 378
[30] G. Munster, Z. Phys. C12 (1982) 43
[31] W.Y. Crutchfield, Phys Rev D19 (1979) 2370
[32] A.D Sokal, J Math Phys 21 (1980) 261