CANCELLATIONS OF IR DIVERGENCIES IN TWO-DIMENSIONAL CHIRAL MODELS

F. DAVID
DPh-T, CEN-Saclay, 91190 Gif-sur-Yvette, France

Received 10 July 1980

For all two-dimensional chiral models which have a global symmetry, the invariant observables are proved to have an infra-red finite weak coupling perturbative expansion.

The two-dimensional σ-models (or chiral theories) have been extensively studied during the last years. Their geometrical structure leads, for a large class of models, to the existence of non-local [1] or local [2] classical conservation laws, which are proved to be preserved by quantization for the O(N) σ-model and lead to the factorization of the S matrix [3]. Another point of interest is the similarities of these models with four-dimensional gauge theories: asymptotic freedom, dynamical restoration of symmetry and the non-perturbative character of the particle spectrum.

In particular, the usual weak coupling expansion is performed in the spontaneously broken symmetry phase, and suffers from important IR divergencies. This is related to the fact that this phase cannot exist, from the Mermin–Wagner theorem [4]. It was conjectured by Elitzur [5], and proved by the present author [6], that, for the O(N) σ-model, that those IR divergencies cancel for any O(N) invariant observable (another analogy with what is expected in four-dimensional gauge theories).

In this paper we prove that this property is satisfied by all σ-models which possess a group of invariance, namely the models constructed on some homogeneous space. Such a space may be considered as the space G/H, where G is some Lie group and H some compact subgroup of G, or as a riemannian space E such that the metric is "the same" in the neighbourhood of every point of E (i.e., the group of transformations preserving the metric tensor sends any point of E in the whole space E). These two definitions are proven to be equivalent [7].

To deal with the most general models, we adopt the second, geometrical point of view. Given a riemannian space E, and considering some coordinate system $(\xi^i)_{i=1,N}$, a chiral field $\boldsymbol{\xi}$ with value into E may be constructed, whose euclidean action is

$$A_{HM} = \int d^Dx\, \tfrac{1}{2}\,[\partial_\mu\xi^i\partial_\mu\xi^j g_{ij}(\boldsymbol{\xi}) + HM(\boldsymbol{\xi})]\,, \tag{1}$$

g_{ij} is the metric tensor, $HM(\boldsymbol{\xi})$ is a "mass term" breaking geometrical invariance at the point $\boldsymbol{\xi} = 0$. We consider a general term of the form

$$M(\boldsymbol{\xi}) = M_{ij}\xi^i\xi_j + M_{ijk}\xi^i\xi^j\xi^k + \dots\,, \tag{2}$$

(M_{ij}) being symmetric positive definite. The vacuum expectation value of any observable $F(\boldsymbol{\xi})$ is given by the functional integral

$$\langle F\rangle_{HM} = \frac{1}{Z_0}\int \mathcal{D}[\boldsymbol{\xi}]\,F(\boldsymbol{\xi})\exp(-t^{-2}A_{HM})\,, \tag{3}$$

where $\mathcal{D}[\boldsymbol{\xi}] = \Pi_x\, d\boldsymbol{\xi}(x)(|g(\boldsymbol{\xi})|)^{1/2}$ is the invariant measure and t the coupling constant.

The perturbative expansion is obtained by expanding the metric tensor around $\boldsymbol{\xi} = 0$

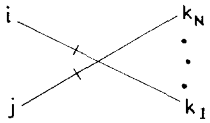
$$g_{ij}(\boldsymbol{\xi}) = g_{ij} + \xi^k g_{ij,k} + \xi^k\xi^l g_{ij,kl} + \dots\,, \tag{4}$$

so that we get the free propagator $D^{ij}(x-y)$ as the inverse of the operator [‡1]

$$D^{-1}_{ij} = -g_{ij}\Delta_x + HM_{ij}\,. \tag{5}$$

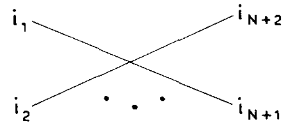
‡1 Diagonalizing M_{ij} in an orthonormal basis of g_{ij} we can express the propagator as a sum of usual scalar propagators: $D^{ij}(p) = \Sigma_{k=1}^N D^{ij}_{(k)}/(p^2 + Hm_k^2)$.

We have two kinds of interaction vertices of order $t^N (N \geqslant 1)$ corresponding (a) to the expansion of g (g-vertices):



$$-\frac{t^N}{2} g_{ij,k_1 \ldots k_N} (p_i p_j) , \tag{6a}$$

(b) to the expansion of M (m-vertices):



$$-\frac{t^N}{2} HM_{i_1 \ldots i_{N+2}} . \tag{6b}$$

To deal not with UV divergencies we use dimensional regularization and calculate amplitudes at dimension $D = 2 - \epsilon$. Parametrization invariance is not broken and the measure terms are known to disappear.

We now analyse the IR divergencies of this general model. As the mass term is set to zero, the perturbative expansion diverges at dimension $D \leqslant 2$, since even the bare propagator becomes $g^{ij} p^{-2}$ and is not a well defined distribution [8]. In ref. [6], general methods of analysis of IR behaviour were developed and applied to the regularized O(N) model at dimension $D = 2 - \epsilon$. Those methods may be adapted to the general chiral case; indeed the IR structure of the graphs is the same, and only the algebra, related to the structure of the space E, is modified. We give only the final result of this study.

We consider a graph G which appears in the perturbative expansion of some function of the fields $O(\boldsymbol{\xi}(x))$. At dimension $D = 2 - \epsilon$, the amplitude $I_G(x, H)$ of the graph G has an IR asymptotic expansion in powers of $H^{-\epsilon/2}$ in terms of "dominant subgraphs" [6] E of G:

$$I_G(x, H) = \sum_{\substack{E \subsetneq G \\ \text{dominant}}} F_E(x) \cdot I_{[\tilde{G}/E]}(H) + O(H^{1-\epsilon L(G/2)}) , \tag{7}$$

$F_E(x)$ is the finite part of the amplitude of the dominant E and so is IR finite. The $I_{[\tilde{G}/E]}$'s are the IR divergent parts of the expansion (7) (They diverge like a power of $H^{-\epsilon/2}$) and are the amplitudes of the graphs $(\tilde{G}/E)$ obtained by shrinking E into one vertex into G.

The dominant subgraphs E of G are the subgraphs of G which may be considered as graphs of the operator O with "connected truncated insertions" of the field ξ^i at zero momenta, that is graphs of the operator

$$O^{\mathrm{c}}_{i_1 \ldots i_P}(x) = \frac{1}{P!} \, O \left\{ \prod_{\alpha=1}^{P} HM_{i_\alpha j_\alpha} \int \mathrm{d}x \, \xi^{j_\alpha}(x) \right\}_{\mathrm{C}} , \tag{8}$$

for some P. (The "C" means that graphs with disconnected part, where there are only insertions of ξ are not taken into account.) The reduced graphs $[\tilde{G}/E]$ may be considered as a graph of the operator

$$D^{i_1 \ldots i_P} = \xi^{i_1}(x) \ldots \xi^{i_P}(x) , \tag{9}$$

at some point x. From eq. (7) we construct the IR asymptotic expansion of the operator O at dimension $2 - \epsilon$.

$$\langle O(x) \rangle_{HM} = \sum_{P=0}^{\infty} \text{f.p.} \langle O^{\mathrm{c}}_{i_1 \ldots i_P}(x) \rangle \times \langle D^{i_1 \ldots i_P} \rangle_{HM} . \tag{10}$$

The "f.p." means the IR finite part of the operator $O^{\mathrm{c}}_{i_1 \ldots i_P}$ as $H \to 0$. The $\langle D^{i_1 \ldots i_P} \rangle_{HM}$ are the divergent parts of the expansion and diverge (perturbatively) as a power of $H^{-\epsilon/2}$. This expansion is valid at order N provided that $\epsilon < 2/N$, then only operators such that $P \leqslant N$ are present. (The summation over indices $i_1 \ldots i_P$ is understood.) Formula (10) is valid for any observable of any model. As pointed out in ref. [6], this result is quite similar to the Wilson operator product expansion [9].

An observable O will be IR finite only if the operators $O^{\mathrm{c}}_{i_1 \ldots i_P}(x)$ have a zero limit as $H \to 0$ for any $P > 0$, so that $\langle O \rangle = \text{f.p.} \langle O \rangle$. We now prove that this is true only for invariant observables of models defined on homogeneous spaces. We consider some homogeneous riemannian space E and some coordinates ξ^i (with origin 0). To any point A of E we associate an isometric transformation τ_A on E which sends O into A and defines a new coordinate system $\bar{\xi}^i_A$ in a neighbourhood of A such that the metric tensor in the new system $(\bar{g}_A)$ is the same as in the first one (g). We may then consider the coordinates ξ^i of a point of E as a function of its coordinates $\bar{\xi}^j_A$ in the new system and of the coordinates α^k of A in the first system.

$$\xi^i = \xi^i(\bar{\xi}^j_A, \alpha^k) . \tag{11}$$

Moreover, the isometries τ_A are choosen such that, at the origin, this function is infinitely differentiable versus $\bar{\xi}^j_A$ and α^k. We now consider some invariant observable $O(g, \boldsymbol{\xi})$. (For instance the riemannian distance between the field at two points $\boldsymbol{\xi}(x)$ and $\boldsymbol{\xi}(y)$.)

The vacuum expectation value of O is expressed by the functional integral (3). If we made the change of variable $\xi \to \bar{\xi}_A$ into (3), the measure, the observable O and the free action are invariant, so that:

$$\langle O(\xi)\rangle_{HM} = \frac{1}{Z_0} \int \mathcal{D}[\bar{\xi}_A]\; O[\bar{\xi}_A]$$

$$\times \exp\left[-\frac{1}{t^2}\left(A_0[\bar{\xi}_A] + \int \mathrm{d}^D x\, HM(\xi)\right)\right], \qquad (12)$$

the only change occurs in the symmetry breaking term:

$$M(\xi) = M(\bar{\xi}_A, \alpha)$$

$$= M(0,\alpha) + 2\bar{\xi}^i_A D_i(\alpha) + \bar{M}(\bar{\xi}_A, \alpha)\,. \qquad (13)$$

In eq. (13) we have separated the two first terms of the expansion of M around $\bar{\xi}_A = 0$, so that $\bar{M}$ has the expansion:

$$\bar{M}(\bar{\xi}_A, \alpha) = \bar{M}_{ij}(\alpha)\bar{\xi}^i_A\bar{\xi}^j_A + \bar{M}_{ijk}(\alpha)\bar{\xi}^i_A\bar{\xi}^j_A\bar{\xi}^k_A + \dots\,. \qquad (14)$$

Defining a new parameter $\bar{\alpha}$ as

$$\bar{\alpha}^i = D_k(\alpha)\bar{M}^{ki}(\alpha) = \alpha^i + \mathrm{O}(\alpha^2)\,, \qquad (15)$$

we may invert the relation between the functions M and $\bar{M}$ and express the coefficients of the expansion of M, namely the $M_{i_1 \dots i_k}$, as functions of $\bar{\alpha}$ and of the coefficients $\bar{M}_{i_1\dots i_n}$. We have in fact a linear relation between M and $\bar{M}$:

$$M_{i_1\dots i_n}(\bar{\alpha}) = \sum_{2 \leqslant P \leqslant N} c^{j_1 \dots j_P}_{i_1 \dots i_N}(\bar{\alpha})\bar{M}_{j_1\dots j_P}\,, \qquad (16)$$

where the c's depend on $\bar{\alpha}$ (and are determined by the metric g and the isometry τ_A). We shall note the functional relation of M with $\bar{M}$ and $\bar{\alpha}$ by

$$M = M[\bar{M}, \bar{\alpha}]\,. \qquad (17)$$

Obviously, as $\bar{\alpha} = 0$, we have $M[\bar{M}, O] = \bar{M}$. With this notation, putting eq. (13) into eq. (12) and performing the same change of variables into Z_0, we get the fundamental identity valid for any invariant observable O:

$$\langle O(t\xi)\rangle_{H\bar{M}[M, t\alpha]}$$

$$= \frac{\langle O(t\xi)\exp[-H\alpha^i M_{ij}\int \xi^j\,\mathrm{d}x]\rangle_{HM}}{\langle \exp[-H\alpha^i M_{ij}\int \xi^j\,\mathrm{d}x]\rangle_{HM}}\,. \qquad (18)$$

In eq. (18) we have scaled $\xi \to t\xi$ and $\alpha \to t\alpha$ in order to obtain the usual weak coupling perturbative expansion, and inverted the notation M and $\bar{M}$. If we expand the r.h.s. of eq. (18) in powers of α, we note that the term of order P is precisely the vacuum expectation value of the operator $O^{\mathrm{c}}_{i_1\dots i_p}$ defined in eq. (8).

$$\text{r.h.s.}(18) = \sum_{P=0}^{\infty} (-1)^P \alpha^{i_1} \dots \alpha^{i_P} \langle O^{\mathrm{c}}_{i_1\dots i_P}\rangle_{HM}\,. \qquad (19)$$

Indeed, expanding the exponential at the numerator we get $1/p!$ times P insertions of $HM_{ij}\xi^j$ at zero momenta, and the disconnected insertions are eliminated by the denominator.

The l.h.s. of eq. (18) corresponds to a renormalization of the symmetry breaking term which depends on $t\alpha$. Then the term of order P in α ($P \geqslant 1$) and of order N in t is related to derivatives versus the symmetry breaking term (that is versus the $M_{ij\dots}$) of terms of order $N' < N$ of the perturbative expansion of $\langle O\rangle$.

So, if the terms of order $N' < N$ (in t) of $\langle O\rangle$ have been proved to be IR finite, and so independent of M as $H \to 0$, the terms of order $N' \leqslant N$ of $O^{\mathrm{c}}_{i_1\dots i_p}$ are proved to have a zero IR limit as $H \to 0$ for $P \geqslant 1$. Their finite parts are then equal to zero and from eq. (10) (which gives the asymptotic IR behaviour of $\langle O\rangle$) we deduce that the term of order N of $\langle O\rangle$ is IR finite. So a recursive proof shows that any invariant observable of the regularized theory at ($D = 2 - \epsilon$) is IR finite at any order of perturbative expansion.

This result holds independently of the regularization (for instance for the lattice theory). We do not deal with the problems of renormalization at $D = 2$ (continuous limit) nor with the physics of such general models. The models defined on symmetric spaces are proved to be renormalizable and the counterterms are IR finite [10,11] so that renormalization preserves the IR finitness of invariant observables. These models include the CP^N models, the grassmannian models and the principal chiral fields. It seems very plausible that those IR cancellations are also present in the supersymmetric extensions of these σ-models [12].

I would like to thank E. Brézin for a critical reading of this manuscript.

References

[1] V.E. Zakharov and A.V. Mikhailov, Sov. Phys. JETP 47 (1978) 1017;
H. Eichenherr and M. Forger, Nucl. Phys. B155 (1979) 381;
E. Brézin, C. Itzykson, J. Zinn-Justin and J.B. Zuber, Phys. Lett. 82B (1979) 442.
[2] K. Pohlmeyer, Commun. Math. Phys. 46 (1976) 207.
[3] A.M. Polyakov, Phys. Lett. 72B (1977) 224;
M. Lüscher, Nucl. Phys. B135 (1978) 1.
[4] N.D. Mermin and H. Wagner, Phys. Rev. Lett. 17 (1966) 1132.
[5] S. Elitzur, Institute of Advanced Study preprint (1979).
[6] F. David, Saclay preprint, DPh-T/64/80 (May 1980).
[7] S. Kobayashi and K. Nomitzu, Foundations of differential geometry, Vol. 2 (Interscience, New York, 1969).
[8] L. Schwartz, Théorie des distributions (Hermann, Paris, 1957–1959).
[9] K. Wilson, Phys. Rev. 179 (1969) 1499.
[10] E. Brézin, J. Zinn-Justin and J.C. Le Guillou, Phys. Rev. D14 (1976) 985.
[11] E. Brézin, S. Hikami and J. Zinn-Justin, Nucl. Phys. B165 (1980) 528.
[12] E. Witten, Phys. Rev. D16 (1977) 2991;
P. Di Vecchia and S. Ferrara, Nucl. Phys. B130 (1977) 93.