The string worldsheet approach to AdS_3/CFT_2

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Abstract: These lecture notes provide the background material for the course given at the IPhT in November/December 2023. We attempt to give a self-contained overview of the state-of-the-art understanding of this topic, with an emphasis on the computation of worldsheet correlation functions, describing its almost 25-year-long rich history alongside a number of important recent developments.

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1 Introduction

How can we describe the universe moments after the big bang? What sets the scale of the cosmological constant derived from current astrophysical observations? How can we think of four-dimensional black holes at the microscopic level? What is quantum gravity? Can we unify it with the quantum field theory framework? How can we compute observables in generic quantum field theories at strong coupling? These are some of the burning open questions in high-energy physics. Most readers will probably be familiar with them, and they would also recognize that they are both conceptually and technically extremely hard to answer. Nevertheless, it is believed that, at least in principle, string theory could potentially provide answers for all of them!

One way to try to deal with these difficulties is to study some appropriate toy model. We could, for instance, consider larger symmetry structures such as conformal symmetry or supersymmetry, which may allow us to compute some relevant observables without having to resort to perturbative expansions. We could also lower the number of dimensions in order to reduce the number of propagating degrees of freedom, effectively simplifying the dynamics. Conversely, we could increase the number of dimensions, such that we have the possibility of landing on a critical string theory solution. Finally, we could also change the characteristics of the geometry under consideration, studying some of these problems in (asymptotically) Anti-de Sitter (AdS) spacetimes, as opposed to the asymptotically flat or asymptotically de Sitter backgrounds which are directly relevant for particle physics and cosmology. This would allow us to make use of the AdS/CFT correspondence, and more generically of the holographic principle, which states that certain theories of quantum gravity in $d + 1$ dimensions can be described equivalently in terms of their d-dimensional field theory duals. However, in doing so we must be careful when deciding what toy model to focus on. It should be simple enough so that it helps us in answering some of the above questions in a more controlled setting. On the other hand, it should not be oversimplified, otherwise it would fail to capture the relevant ingredients of the original system, and the whole exercise would become pointless.

In these notes we discuss one of the most interesting and most studied such toy models: that of strings in AdS_3. More precisely, this 2+1-dimensional background should be thought of as the extended part of a supergravity solution of the form $\text{AdS}_3 \times M_7$, where M_7 is some appropriate seven-dimensional compact manifold. We will mostly be agnostic about this internal sector, even though some of the results that we will present where the precise structure of M_7 comes into play correspond to the cases with $M_7 = S^3 \times M_4$, and where the four-dimensional sector is either T^4 or K3.

The propagation of strings in AdS_3 spacetimes combines highly non-trivial dynamics with a number of key ingredients which allow for an unprecedented level of computability, together with a considerable number of interesting applications. Let us briefly comment on some of these aspects that will be relevant for us:

- The AdS_3 geometry can arise as part of superstring solution in different ways. For instance, it describes the near-horizon region of the D1D5 system, which describes configurations sourced by a collection of n_1 D1-branes and n_5 D5-branes with a common spatial circle. One also takes the remaining four spatial directions of the D5 worldvolume as compactified on T^4 or K3. In the decoupling limit, this provides one of the original examples of the gauge-gravity duality envisioned in [1], which can be stated schematically as

  Strings theory on $\text{AdS}_3 \Leftrightarrow \text{CFT}_2$ on its asymptotic boundary.

- The dual conformal field theory (CFT) is known as the D1D5 CFT. As opposed to the well-known case of 4d $\mathcal{N} = 4$ U(N) super Yang-Mills (SYM) describing the low-energy dynamics of N overlapping D3-branes, the gauge theory associated to the D1D5 system is not conformal. It does, however, flow to a fixed point in the infrared (IR). Even though pinpointing the precise theory one lands on is perhaps surprisingly difficult, we can get a lot of mileage just from knowing that it must be a two-dimensional $\mathcal{N} = (4,4)$ superconformal field theory (SCFT). We will refer to it as the holographic CFT, boundary CFT, or spacetime theory interchangeably. The corresponding central charge must be of the form[1]

$$c_{\rm st} = 6 n_1 n_5. \tag{1.1}$$

This is an exact result, although it was originally derived in [2] from the gravity perspective by studying asymptotic symmetries, and only later related to the string theory/black hole perspective in [3, 4].

- Alternatively, the D1-branes can be seen as string-like instantons of the six-dimensional gauge theory on the D5-branes. The low-energy dynamics of the system should then describe the zero-modes of the corresponding topologically non-trivial solutions. For a single instanton, this can be thought of as an $\mathcal{N} = (4,4)$ CFT with target space given by the corresponding moduli space $\mathcal{M}_{\rm inst}$. The latter has (bosonic) dimension $4n_5$. Including the fermionic contributions, this gives a central charge $c^{(1)}_{\rm st} = 6n_5$, hence we denote this moduli space as $\mathcal{M}^{(1)}_{\rm inst} = \mathcal{M}_{6n_5}$.

- If one could ignore interactions, the fact that each D1-brane is indistinguishable from each other implies that the holographic CFT (HCFT) would be a symmetric product orbifold theory given by
$$\mathrm{Sym}^{n_1}\left(\mathcal{M}_{6n_5}\right). \tag{1.2}$$
By this we simply mean the product of n_1 copies of the SCFT with target space $\mathcal{M}_{6n_5}$, quotiented by the action of the permutation group S_{n_1}. The total central charge is then (1.1). Of course, this is an uncontrolled approximation. More generally, we can only expect the HCFT to be a deformation of a symmetric orbifold.

- One can further use U-duality [5] to relate the system under consideration with an analogous one with a single D5-brane and $N = n_1 n_5$ D1-branes. Here the single instanton moduli space has central charge 6, and we can heuristically think of it as parametrizing the location of the corresponding D1-brane on, say, T_4. At this point in moduli space, it is reasonable to expect that the dual theory is closely related to the orbifold model
$$\mathrm{Sym}^{N}\left(T^4\right). \tag{1.3}$$
This is usually referred to as the orbifold point (in the CFT moduli space). Here the holographic CFT is under exact control. One can use it, for instance, to calculate protected quantities such as the three-point functions of chiral primary operators in order to compare them with computations on the gravity side.

[1] More precisely, this is valid for the T^4 case, while for K3 one has an additional $+6$ contribution from a free, decoupled sector.

- The instanton moduli space $\mathcal{M}_{\text{inst}}$ should not be confused with the moduli space of the HCFT, parametrized by all its marginal deformations [6]. For the latter, the holographic correspondence dictates that it should be identified with the type II supergravity moduli space in the near-horizon regime of the D1D5 configuration compactified on M_4. Locally, this takes the form $\mathcal{M}^{\text{IR}}_{\text{sugra}}$ =SO(4,n)/SO(4)×SO(n), with $n = 5$ for $M_4 = T^4$ and $n = 21$ for $M_4 =$ K3 [7, 8]. Note that this is only a subgroup of the of the full $\mathcal{M}^{\text{UV}}_{\text{sugra}}$ =SO(5,n)/SO(5)×SO(n) for the asymptotically flat solution. As we flow from the UV region of the geometry to the IR one, some of the moduli get fixed. This is known as the *attractor mechanism* [9].

- By means of S-duality one can relate the above system with a solution sourced by a NS5-branes and (fundamental) F1-strings. Thus, in sharp contrast with higher-dimensional cases such as that of $\text{AdS}_5 \times S^5$, where the presence of D-branes implies that Ramond-Ramond (RR) fluxes must be turned on, one can obtain $\text{AdS}_3 \times S^3 \times M_4$ backgrounds which are sourced solely by fluxes of the Neveu-Shwarz-Neveu-Shwarz (NSNS) type. As a consequence, the worldsheet theory can be quantized, and the (perturbative) string dynamics can be studied exactly in α', i.e. well-beyond the supergravity regime. In particular, this allows us to study the mechanisms at work behind the holography duality with a far greater precision than in almost all other instances of AdS/CFT. The main ingredient of the worldsheet theory is the SL(2,$\mathbb{R}$) Wess-Zumino-Witten (WZW) model, since the corresponding group manifold (or rather its universal covering) is precisely global AdS_3 [10–15].

- Moreover, the D1D5 system has been a cornerstone for the string-theoretical description black holes. Black holes have a macroscopic entropy dictated by the Bekenstein-Hawking formula $S = A_{\text{h}}/4G$, where A_{h} is the area of the horizon and G is the gravitational coupling. It is in the context of the D1D5 system that the first string-theoretical microscopic computation of this entropy was obtained [3] by counting BPS excitations of the brane configuration in the weak coupling regime. This has also been the more fruitful arena for the Fuzzball program [16–18], which postulates that BHs can be understood as an averaged description of their microstates, some of which are coherent enough to be described in classical terms as geometries smooth and horizonless geometries which look like the corresponding BHs for a distant observer.

- All the tools discussed above can be used to describe string propagation in black hole (BH) backgrounds. For BHs in two dimensions, both Lorentzian and Euclidean, this is because they can be obtained by starting with the SL(2,$\mathbb{R}$) model and gauging appropriate currents [19]. For instance, the 2d Euclidean BH corresponds to the *cigar* geometry SL(2,$\mathbb{R}$)/U(1). On the other hand, three-dimensional asymptotically AdS_3 BHs are known as BTZ black holes [20]. They can be obtained by a set (discrete) orbifoldings of AdS_3. More recently, it was shown that, in the context of the Fuzzball program, one can also use related coset models to describe a (particularly symmetric) set of black hole microstates [21–28]. Hence, understanding string dynamics in AdS_3

can be seen as a necessary step for obtaining an exact description of string propagation in black hole and black hole microstate backgrounds.

- Finally, one also has an interesting relation with the decoupled theory on the NS5-branes, which is known as little string theory [29]. This is a non-gravitational theory with several stringy properties such as Hagedorn thermodynamics at high energies. It also constitutes one of the building blocks of an instance of holography beyond AdS, where the gravity background includes a dilaton that runs linearly with the radial direction. Here the dual theory is not a CFT, and not even a local quantum field theory. More recently, it was argued that this can also be understood in terms of certain particularly tractable irrelevant deformations of the original HCFT of the $T\bar{T}$-type [30–33].

The D1D5 system and strings in AdS_3 in particular therefore have an important number of very interesting aspects, which have been studied at length for over 25 years. In these notes we will work in the NS5-F1 frame and focus on the worldsheet theory, with a particular emphasis on the computation of its correlation functions. Other aspects of the model are discussed in several reviews, see [29, 34–40] for an incomplete list of references.

As stated above, in the RNS formalism employed here the relevant theory is the SL(2,$\mathbb{R}$) WZW model. Although the latter is believed to be exactly solvable, it presents a number of highly non-trivial features. This is mainly due to the fact that the target space under considerations is both Lorentzian and non-compact. As a result, one finds that the spectrum includes both a discrete sector, which encompasses the so-called short strings, and, above a certain threshold, a continuous one, representing the long strings configurations. Moreover, the spectral flow operation [41], which in simpler contexts such as the SU(2) WZW model [42] merely re-shuffles the different states, generates new, inequivalent representations in the AdS_3 setting. These must be included in the spectrum, otherwise the string theory description is simply inconsistent. The corresponding vertex operators have quite complicated operator product expansions (OPEs) with the symmetry currents of the model, which renders the computation of their correlation functions particularly involved. We will discuss recent results on how to overcome these difficulties[2], and also focus on the role of the dual CFT as seen from the worldsheet theory.

The structure of these notes is as follows. Section 2 is devoted to discussing the spectrum the SL(2,$\mathbb{R}$) WZW model [13, 14]. For this, we start by presenting the asymptotically flat NS5-F1 solution in type IIB supergravity compactified on $S^1 \times T^4$ and show how the decoupling limit leads to the near-horizon $\mathrm{AdS}_3 \times S^3 \times T^4$. In the string frame, the corresponding (squared) AdS_3 radius is set by n_5, in units of α'. We then focus on bosonic strings propagating in the AdS_3 for simplicity, characterized by the SL(2,$\mathbb{R}$) WZW model at level $k = n_5{+}2$. The classical limit, sometimes called the mini-superspace limit, then corresponds to large $k \sim n_5$. We discuss the isometries of AdS_3, which form an $\mathrm{SL}(2,\mathbb{R})_L{\times}\mathrm{SL}(2,\mathbb{R})_R$

[2]For simplicity we will restrict to studying the properties of the worldsheet CFT on the sphere, which gives the leading contribution at large N (which for us effectively means large n_1), although many of the results we described have been extended to higher genus, see for instance [43–46] and references therein.

algebra, and present the particle and string geodesics, highlighting the appearance of those associated to long strings states. We then move to the Euclidean geometry. We derive the semi-classical expression for the (unflowed) vertex operators of the model, and discuss their normalization conditions [47]. The rest of the section deals with the quantum theory, first using the Wakmioto free field representation, valid in the region near the boundary of AdS_3 [10, 11], and then move to the exact theory [12, 15, 48, 49]. Here we derive the spacetime symmetry generators from the worldsheet. We further highlight the role of the spectrally flowed representations, showing how they allow us to overcome an unphysical bound on the spacetime dimension of the states in the model, and sketching how they are taken into account in the torus partition function. Finally, we briefly review the main aspects of correlation functions and the OPE structure of the theory in the unflowed sector.

Section 3 is perhaps the more involved section of these notes at the technical level. It focuses on the computation of correlators involving spectrally flowed insertions. We discuss in detail the different representations of the corresponding vertex operators, and their interplay [15, 50–52]. The m-basis is that of eigenstates of the zero-modes of the Cartan currents. It allows for a pedagogical discussion of spectral flow and is best suited for applications to certain gauged models, such as the cigar theory. On the other hand, the conjugate x-basis resums the action of the rising zero-mode operators, and is best suited for holographic applications, as the complex coordinate x is identified with that of the boundary plane. Finally, an additional complex variable y is used to resum the action of *flowed* currents, allowing one to re-write a number of recursion relations satisfied by correlators involving operators with shifted spacetime weights [43] in terms of differential equations. We describe in detail how the appearance of (largely unknown) higher order poles in the OPEs between currents and flowed operators greatly complicates the computation of correlation functions. Then, we discuss how one can use a number of holomorphic covering maps to bypass these issues, leading to explicit, integral expressions for correlators with arbitrary spectral flow charges [43, 51–54]. As it turns our, the way these covering maps are used is reminiscent of their appearance in the computation of n-point functions in symmetric orbifold models, see [55–59]. In view of the above discussion on the nature of the dual CFT, this is of course not a coincidence.

In Section 4 we discuss the tensionless limit of the model, corresponding to $n_5 = 1$, or equivalently $k = 3$ in the bosonic language. We first show how massless higher-spin states arise in this context, where the AdS_3 space is string-size [60–62]. We then argue that it is at this point in the moduli space that the dual theory is exactly given by a symmetric orbifold, following [43, 63–65]. In particular, for strings in $AdS_3 \times S^3 \times T^4$ at $n_5 = 1$, the HCFT is precisely that of Eq. (1.2), with $\mathcal{M}_{6n_5} = \mathcal{M}_6 = T^4$. Strictly speaking, for this value of n_5 the RNS formalism breaks down, and one should use the hybrid formalism as in [63]. Nevertheless, many of the relevant features are visible in the bosonic SL(2,$\mathbb{R}$) model at level $k = 3$. We first motivate this by looking at the spectrum, and then describe how correlation functions – in particular those involving the worldsheet avatars of spacetime twist operators – localize on the loci defined by the existence of the corresponding covering maps [43, 66, 67]. Finally, even though the geometry is highly curved in this regime, we discuss the "geometric" interpretation of these features in terms of Wakimoto fields.

In Section 5 we continue our holographic exploration, now for $n_5 \geq 2$. We present the superstring worldsheet model on $\mathrm{AdS}_3 \times S^3 \times T^4$, and construct the vertex operators associated to the chiral primary operators of the boundary theory [68–72]. We then extend the techniques reviewed in the previous sections in order to compute their three-point functions for arbitrary spectral flow charges [73, 74]. We show that the worldsheet results precisely reproduce the structure constants and fusion rules of the exact chiral ring as computed from the D1D5 CFT at the orbifold point, see Eq. (1.3). This had to be the case, as these observables are protected by supersymmetry [6].

In Section 6 we move past protected observables, go back to the bosonic setting, and discuss the structure of the putative holographic CFT for $k > 3$. Here we mostly follow the presentation recently given in [75–77]. We motivate the holographic proposal by studying the theory on the long strings [78], and discuss the main differences between the cases with $k > 3$ and those with $2 < k < 3$. (As the target space is non-compact, the affine level of the SL(2,$\mathbb{R}$) WZW model is not necessarily integer). This leads to a deformation of a different symmetric orbifold model, where the structure is as in Eq. (1.2). The seed theory with target space $\mathcal{M}_{6n_5}$ now contains a Liouville-type factor, such that the total seed central charge is $c^{(1)} = 6n_5$. The non-compact scalar in question is related to the worldsheet field representing the radial direction in AdS_3. We motivate the precise marginal deformation involved, briefly discuss the differences between long and short strings from the holographic perspective, and sketch how the matching between certain residues of correlators at special points in the space of complex SL(2,$\mathbb{R}$) spins was obtained [76, 77].

The final two sections are devoted to a series applications of the worldsheet methods in more general contexts. First, in Section 7 we show how implementing a marginal deformation of the worldsheet CFT modifies the asymptotic structure of the geometry the strings propagate on [79]. In the UV region, the spacetime becomes flat, while the dilaton runs linearly along the radial direction. This is interpreted as a *reverse* RG flow from the boundary point of view, which is implemented by a specific, highly tractable, $T\bar{T}$-type irrelevant deformation [30–33, 80, 81]. This constitutes an interesting instance of non-AdS holography, where the dual theory, which leaves on the NS5-branes, is nothing but the little string theory mentioned above. We finish by discussing the Hagedorn behavior of the high-energy theormodynamics of these models.

Finally, we focus on black hole applications...

Some relevant background material is provided in the appendices. Appendix A dwells on WZW models in general, with a particular focus on the SU(2) example [42, 82–84]. In particular, we compare the role of spectral flow in this compact setting as compared to the SL(2,$\mathbb{R}$) case discussed in the main text. In Appendix B we review some important facts about BTZ black holes and asymptotically AdS_3 spacetimes. Appendix C provides some relevant aspects of Liouville theory and the DOZZ formula [85, 86], see also [87–90]. Finally, in Appendix D we briefly discuss symmetric product orbifold CFTs and the computation of some of their correlators by means of covering map techniques [55–59].

2 Geometry, spectrum and spacetime symmetries

Let us start by motivating the study of strings propagating in AdS_3. Consider type IIB superstring theory on

$$\mathrm{Mink}^{1,4} \times S^1_y \times T^4. \tag{2.1}$$

We further include n_5 NS5-branes wrapping the S^1_y (often called the y-circle) and the 4-torus, and n_1 F1-strings (i.e. fundamental strings) wrapped on S^1_y and smeared on the T^4. For this supersymmetric configuration which preserves 8 supercharges, the (string frame) geometry takes the form [91–93]

$$ds^2 = f_1^{-1}(r)(-dt^2 + dy^2) + f_5(r)(dr^2 + r^2 d\Omega_3^2) + dz_i dz^i, \tag{2.2}$$

where $y \sim y + 2\pi$, the z_i with $i = 1, \ldots, 4$ are coordinates on the T^4, r is the radial coordinate, and $f_{1,5}$ harmonic functions

$$f_{1,5}(r) = 1 + \frac{r_{1,5}^2}{r^2}\,, \quad r_5^2 = \alpha' n_5\,, \quad r_1 = \frac{g_s^2 \alpha' n_1}{v_4}. \tag{2.3}$$

Here g_s is the string coupling at infinity, l_s is the string length, and v_4 is the volume of the T^4 in string units, i.e. $v_4 \alpha'^2 = \mathrm{Vol}(T^4)$ with $\alpha' = l_s^2$. We also have a non-trivial dilaton Φ and NSNS 3-form field-strength H, namely

$$e^{2\Phi} = g_s^2 \frac{f_5}{f_1}\,, \quad H = 2\left(n_5 + \frac{g_s^2 n_1}{v_4}\frac{f_5}{f_1} *_6\right)\epsilon_{S^3}, \tag{2.4}$$

with ϵ_{S^3} the volume form on S^3 and $*_6$ the six-dimensional Hodge dual. Assuming the torus is roughly square, our smearing approximation is valid for $r \gg v_4^{1/4} l_s$.

This background is S-dual to the D1D5 system, considered in [1] as one of the original incarnations of the holographic duality, given by the AdS/CFT correspondence relating the near-horizon AdS_3 regime to the dual conformal field theory on the branes. In our NS frame, the decoupled regime is obtained by zooming to small r or taking n_1 and n_5 to be very large, so that one can drop the "1+" factors in the harmonic functions of Eq. (2.3). At the practical level, this can be achieved as a two-step procedure. We first take $r, g_s \to 0$ with r/g_s fixed (or equivalently, $n_1 \to \infty$ and $g_s \to 0$ with $g_s^2 n_1$ fixed). This intermediate step leads to the so-called five-brane decoupling limit, where the geometry is asymptotically flat but with a dilaton than runs linearly with r. In this setup, one can study a non-AdS version of holography. We will come back to this in section 7 below. The second step corresponds to taking $r, \alpha' \to 0$ with r/α' fixed, which is known as the string decoupling limit. This leads to

$$ds^2 = n_5\left[\frac{r^2}{\alpha'}(-dt^2 + dy^2) + \frac{\alpha'}{r^2}dr^2 + \alpha' d\Omega_3^2\right] + dz_i dz^i\,, \tag{2.5}$$

where we have further rescaled r with a factor $\sqrt{n_1 n_5 / v_4}$ for convenience, and

$$H = 2n_5(1 + *_6)\epsilon_{S^3}\,, \quad e^{2\Phi} = \frac{v_4 n_5}{n_1}. \tag{2.6}$$

The near-horizon geometry is then

$$\mathrm{AdS}_3 \times S^3 \times T^4, \tag{2.7}$$

the radii of both AdS_3 and S^3 in string units being set by n_5. On the other hand, n_1 only appears in the constant dilaton, such that the effective 6d string coupling is $g_6^2 = g_s^2/v_4 = n_5/n_1$. A weakly-coupled, low-energy supergravity description is accurate as long as $n_1 \gg n_5 \gg 1$. Finally, as the H-field becomes self-dual, the number of preserved supercharges is enhanced from 8 to 16.

This instance of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence is quite special. Indeed, all sources are of the NS type, and moreover the geometry (2.7) is that of a product of Lie groups, given by

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{U}(1)^4\,, \tag{2.8}$$

where, to be precise, we need to work with the universal cover of $\mathrm{SL}(2,\mathbb{R})$, for which the time direction is non-compact[3]. This means that the we can quantize the worldsheet theory. It takes the form of a (super) WZW model, and can be treated exactly in α', i.e. all the way down to small values of n_5. As we will discuss later on, the smallest possible value $n_5 = 1$ is actually rather exceptional, as the AdS_3 becomes string-size and the fundamental strings become tensionless [61, 62]. Of course, we still need n_1 to be large in order for the topological expansion in powers of e^{Φ} to be under control.

2.1 Geometric aspects and semiclassical considerations

Global AdS_3. Particle and string geodesics

Let us focus on the AdS_3 sector, and consider bosonic strings for simplicity, following [13]. In global coordinates, the line element and B-field read

$$ds^2 = k\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, dy^2\right)\,, \quad B = k\sinh^2\rho dt\wedge dy. \tag{2.9}$$

The geometry has constant negative curvature, whose isometries are generated by the six vector fields

$$\zeta^1_{L/R} = \frac{1}{2}\sin(x^\pm)\partial_\rho + \cos(x^\pm)\left[\tanh^{-1}(2\rho)\partial_\pm - \sinh^{-1}(2\rho)\partial_\mp\right]\,, \tag{2.10a}$$

$$\zeta^2_{L/R} = \frac{1}{2}\cos(x^\pm)\partial_\rho - \sin(x^\pm)\left[\tanh^{-1}(2\rho)\partial_\pm - \sinh^{-1}(2\rho)\partial_\mp\right]\,, \tag{2.10b}$$

$$\zeta^3_{L/R} = \partial_\pm\,, \tag{2.10c}$$

with $x^\pm = t \mp y$. These vector fields satisfy $[\zeta^a_L, \zeta^b_R] = 0$ for $a, b = 1, 2, 3$. By defining $J^a = i\zeta^a$ and $J^\pm = J^1 \pm iJ^2$ we obtain two copies of the SL(2,$\mathbb{R}$) algebra,

$$[J^3, J^\pm] = \pm J^\pm\,, \quad [J^+, J^-] = -2J^3\,. \tag{2.11}$$

The coordinates t and y paramtrize the (topological) cylinder that constitutes the conformal boundary of AdS_3. Hence, we identify $J^3_R + J^3_L$ and $J^3_R - J^3_L$ as generating translations along

[3] We will omit the distinction in the remainder of these notes.

the timelike and circular directions at fixed radial distance. The corresponding eigenvalues m_R and m_L (denoted as m and $\bar{m}$ in what follows) are then associated to the spacetime energy and the (angular) momentum along y, respectively, i.e.

$$m = \frac{E+n_y}{2} \qquad \bar{m} = \frac{E-n_y}{2}. \tag{2.12}$$

In the Lorentzian theory, which is what we are interested in, the spectrum is built out of states where $m, \bar{m} \in \mathbb{R}$.

Each point on AdS_3 defines an element g in SL(2,$\mathbb{R}$). Parametrising the latter via its isomoprhism[4] with $\mathrm{SU}(1,1)$, we can use the Pauli matrices to write

$$g = e^{iv\sigma_3} e^{\rho\sigma_1} e^{iu\sigma_3} \,. \tag{2.13}$$

with $u = \frac{1}{2}(t+y)$ and $v = \frac{1}{2}(t-y)$. The WZW action then takes the form

$$S_{\mathrm{WZW}} = \frac{k}{8\pi} \int d^2\sigma \, \mathrm{Tr}\left[g^{-1}\partial_\alpha g g^{-1}\partial^\alpha g\right] + k\Gamma \tag{2.14}$$

where Γ stands for the Wess-Zumino term discussed in Appendix A, see Eq. (A.8), and we have set $\alpha' = 1$. Defining the generators

$$t^1 = \frac{1}{2}\sigma_1 \,, \qquad t^1 = \frac{1}{2}\sigma_2 \,, \qquad t^3 = -\frac{i}{2}\sigma_3 \,, \tag{2.15}$$

we obtain the conserved currents

$$J^a_\alpha = k \, \mathrm{Tr}\left[t^a \partial_\alpha g g^{-1}\right] \,, \qquad \bar{J}^a_\alpha = k \, \mathrm{Tr}\left[t^a g^{-1}\partial_\alpha g\right] \,. \tag{2.16}$$

In terms of the right- and left-moving worldsheet coordinates $\tau^\pm = \tau \pm \sigma$, the equations of motion read

$$\partial_-(\partial_+ g g^{-1}) = 0 \,. \tag{2.17}$$

As reviewed in Appendix A, this implies that the conserved currents only depend on one of the light-cone coordinates. More precisely, we have $J^a = J^a(\tau^+)$ and $\bar{J}^a = \bar{J}^a(\tau^-)$. Combining $J^{1,2}$ into $J^\pm = J^1 \pm iJ^2$ as usual (and similarly for $\bar{J}^\pm$) we find that, in terms of the target space fields,

$$J^3 = k\left[\partial_+ u + \cosh(2\rho)\partial_+ v\right] \,, \qquad J^\pm = k \, e^{\mp 2iu}\left[\partial_+\rho \pm i\sinh(2\rho)\partial_+ v\right] \,, \tag{2.18}$$

while

$$\bar{J}^3 = k\left[\partial_- v + \cosh(2\rho)\partial_- u\right] \,, \qquad \bar{J}^\pm = k \, e^{\mp 2iv}\left[\partial_-\rho \pm i\sinh(2\rho)\partial_- u\right] \,. \tag{2.19}$$

[4] A generic group element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ can be parametrized in terms of an SU(1,1) element $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$, with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$. The explicit map is $2\alpha = a + d + i(b - c)$, $2\beta = b + c + i(a - d)$.

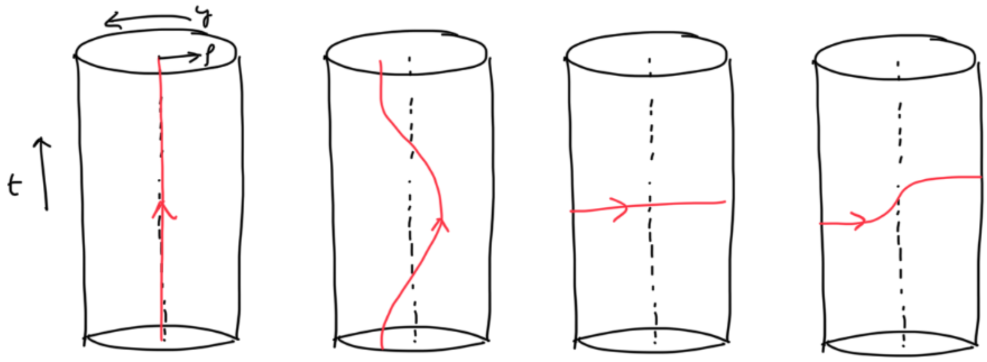

Figure 1. From left to right: Particle geodesics (a) timelike solution with $\rho_0 = 0$ (b) more general timelike configuration (c) basic spacelike solution (d) more general spacelike configuration.

The general solution to the equation of motion can be written as $g = g_+(\tau^+)g_-(\tau^-)$, where the closed-string boundary conditions impose $g_+(\tau^+ + 2\pi) = g_+(\tau^+)M$ and $g_-(\tau^- - 2\pi) = M^{-1}g_+(\tau^-)$ for some constant matrix monodromy $M \in \mathrm{SL}(2,\mathbb{R})$.

For strings propagating in $\mathrm{AdS}_3 \times M_7$ there will be an internal CFT, and one should also include the bc-system of reparametrization ghosts [94, 95]. Quantization is then implemented through the BRST procedure, such that, focusing on the right-moving sector, physical states must be in the cohomology of the charge

$$\mathcal{Q} = \oint \frac{d\tau^+}{2\pi} \left(c\, T_{\text{tot}} + \text{ghosts}\right), \tag{2.20}$$

where normal ordering is implied. Here $T_{\text{tot}}(\tau^+) = T(\tau^+) + T_{\text{int}}(\tau^+)$, where we reserve the symbol $T(\tau^+)$ for the contribution from the AdS_3 sector. In the classical limit the latter takes the form $T = k^{-1}\eta_{ab}J^aJ^b$, where the Killing metric has non-zero entries $\eta_{+-} = \eta_{-+} = \frac{1}{2}$ and $\eta_{33} = -1$. Hence, physical solutions must satisfy the Virasoro constraints, i.e. the corresponding T_{tot} and $\bar{T}_{\text{tot}}$ must vanish.

The simplest solutions are the time-like geodesics $\rho = 0$ and $t = \alpha\tau$, corresponding to $g = e^{i\alpha\tau\sigma_3}$, and the space-like geodesics with $t = y = 0$, and $\rho = \alpha\tau$, which give $g = e^{\alpha\tau\sigma_1}$. The parameter α sets either the energy or the radial momentum. respectively. Both solutions depend only on τ, hence they are effectively one-dimensional and correspond to particle geodesics. Only the time-like one, which has $J^\pm = 0$ and $J^3 = k\alpha/2$, giving $T = -\frac{k}{4}\alpha^2$, can solve the Virasoro constraint by setting $k\alpha^2 = 4h_{\text{int}}$, with $T_{\text{int}} = h_{\text{int}}$. For the space-like case, $J^3 = 0$ and $J^+ = J^- = k\alpha/2$, which leads to $T = +\frac{k}{4}\alpha^2$, hence for $\alpha \in \mathbb{R}$ we cannot solve the Virasoro condition while keeping $h_{\text{int}} \geq 0$. More general particle geodesics can be obtained by conjugation with constant matrices $h, \bar{h} \in \mathrm{SL}(2,\mathbb{R})$, i.e. by taking $g \to hg\bar{h}$. For instance, one can obtain for intance a time-like geodesics of the form

$$y = 0\,, \qquad \sinh\rho = \sinh\rho_0 \sin\alpha\tau\,, \qquad \tan t = \frac{\tan\alpha\tau}{\cosh\rho_0}\,, \tag{2.21}$$

(for $\tau > 0$). These geodesics are depicted in Fig. 1.

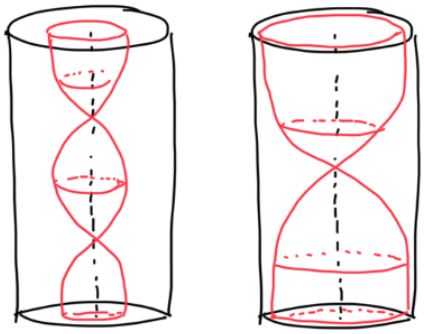

Figure 2. Short string (left) and long string (right) solutions with spectral flow $\omega = 1$.

Starting from a given classical solution $g = g_+ g_-$, one can generate new solutions by implementing $g_+ \to h_+(\tau^+) g_+$ and $g_- \to g_- h_-(\tau^-)$ for any $h_\pm \in$ SL(2,$\mathbb{R}$). The more interesting cases for us will be given by using

$$h_\pm = e^{\frac{i}{2}\omega_\pm \tau^\pm \sigma_3} \qquad \omega_\pm \in \mathbb{Z}\,. \tag{2.22}$$

In our parametrization, these shift the time and angular coordinates as

$$t \to t + \omega\tau\,, \quad y \to y + \omega\sigma\,. \tag{2.23}$$

Here we have chosen $\omega_- = \omega_+ \equiv \omega$ as required by the periodicity of the string worldsheet under $\sigma \to \sigma + 2\pi$ (because our time coordinate is non-compact). The resulting geodesics now depend explicitly on σ, hence we see them as describing the trajectory of a string. Roughly speaking, they correspond to rotating the original ones around the vertical $\rho = 0$ axis, while boosting them in the time direction. As a result, they wind around the y-circle ω times, leading to the configurations depicted in Fig. 2. However, in the interior of AdS$_3$ this circle becomes contractible, hence ω is not a conserved charge. Configurations with $\omega \neq 0$ will be denoted as spectrally flowed solutions. At the level of the currents evaluated on these new classical solutions, we find

$$J^3 = \tilde{J}^3 + \frac{k}{2}\omega\,, \qquad J^\pm = e^{\mp i\omega\tau^\pm}\tilde{J}^\pm\,, \qquad T = \tilde{T} - \omega\tilde{J}^3 - \frac{k}{4}\omega^2\,, \tag{2.24}$$

where $\tilde{J}^a$ and $\tilde{T}$ are those of the original solution.

When the original solution is a time-like geodesic, the Cartan current and physical state conditions read

$$J_0^3 = \tilde{J}_0^3 + \frac{k}{2}\omega\,, \qquad \tilde{T} - \omega\tilde{J}^3 - \frac{k}{4}\omega^2 + h_{\rm int} = -\frac{k}{4}\left(\omega+\alpha\right)^2 + h_{\rm int} = 0 \tag{2.25}$$

where $J_0^3 = \int_0^{2\pi}\frac{d\tau^+}{2\pi}J^3$, and similarly for the left-moving sector. Assuming $h_{\rm int} = \bar{h}_{\rm int}$, the energy reads

$$E = J_0^3 + \bar{J}_0^3 = \frac{k}{2}\omega + \frac{2}{\omega}\left(h_{\rm int} - \frac{k}{4}\alpha^2\right)\,, \tag{2.26}$$

and is bounded from above by $E_{\rm max} = \frac{k\omega}{2} + \frac{2h_{\rm int}}{\omega}$ and unbounded from below. An explicit example is given by the solution

$$y = \omega\sigma\,, \qquad \sinh\rho = \sinh\rho_0 \sin\alpha\tau\,, \qquad \tan t = \frac{\tan\alpha\tau - \cosh\rho_0 \tan\omega\tau}{\cosh\rho_0 + \tan\omega\tau\tan\alpha\tau}\,, \tag{2.27}$$

for some constant ρ_0, such that α sets the radial energy also here. In general, this represents a string bound to the interior of AdS_3 which oscillates around the center. These are the short string solutions. For $h_{\rm int} = \frac{k}{4}\omega^2$, we have to set $\alpha = 0$, and giving solutions with

$$y = \omega\sigma\,, \qquad \rho = \rho_0\,, \qquad t = \omega\tau\,, \tag{2.28}$$

such that the string stays at constant radial distance, while the energy comes only from the winding part, $E = E_{\rm max} = k\omega$.

So far, it appears that we cannot get string geodesics with $E > E_{\rm max}$. Here is where the the second type of solutions come into play, i.e. those obtained by spectrally flowing the space-like geodesics. Now we can solve the Virasoro condition event though $\tilde{T}$ changes sign. The resulting energy reads

$$E = \frac{k}{2}\omega + \frac{2}{\omega}\left(h_{\rm int} + \frac{k}{4}\alpha^2\right)\,, \tag{2.29}$$

which is now bounded from below by $E_{\rm min} = \frac{k\omega}{2} + \frac{2h_{\rm int}}{\omega}$. The threshold is thus at $E_{\rm min} = \frac{k\omega}{2}$, corresponding to $h_{\rm int} = 0$. For instance, we may have

$$y = \omega\sigma\,, \qquad \rho = \alpha|\tau|\,, \qquad t = \omega\tau\,, \tag{2.30}$$

These are the long strings, also known as scattering states, which come from radial infinity at $t \to -\infty$, then approach the center of AdS_3, and finally go back to the near-boundary region of the geometry as $t \to +\infty$. A slightly more general example corresponds to the spectrally flowed solution originating from that of Eq. (2.21), namely

$$y = \omega\sigma\,, \qquad \sinh\rho = \cosh\rho_0 \sinh\alpha\tau\,, \qquad \tan t = \frac{\tan\omega\tau + \tanh\alpha\tau\sinh\rho_0}{1 - \tan\omega\tau\tanh\alpha\tau\sinh\rho_0}\,. \tag{2.31}$$

Note that at $h_{\rm int} = \frac{k}{4}\omega^2$ this coincides with the short string solution. The latter can thus become unbounded and turn into long strings. These solutions are allowed to stay at constant ρ because the gravitational attraction is exactly compensated by the repulsion generated by the B-field. In this way, even very long strings located near the asymptotic boundary can remain at finite energy.

Euclidean AdS_3 and normalizability conditions

For holographic applications it will be useful to consider the correlation functions of the corresponding vertex operators in terms of their continuation to Euclidean signature. This is implemented by $t \to -it_E$, and the resulting manifold is known as H_3^+. The latter is

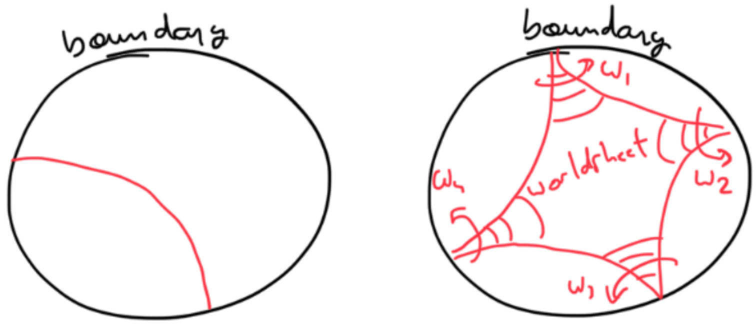

Figure 3. Geodesic for a an unflowed two-point function (left) and for a four-point function involving operators with non-trivial spectral flow charges (right).

not a group manifold, but can be understood as the coset space SL(2,$\mathbb{C}$)/SU(2). Changing coordinates to[5]

$$\gamma = e^{t_E - iy}\tanh\rho \qquad \phi = -t_E + \log(\cosh\rho), \tag{2.32}$$

the metric and B-field (up to an exact form) become

$$ds^2 = k\left(d\phi^2 + e^{2\phi}d\gamma d\bar{\gamma}\right)\,, \quad B = -\frac{k}{2}e^{2\phi}d\gamma\wedge d\bar{\gamma}\,. \tag{2.33}$$

Some of the classical solutions discussed above are depicted in the H_3^+ setting in Fig. 3.

The currents J^a take the form

$$k^{-1}J^+ = -\partial_\gamma \quad k^{-1}J^3 = -\gamma\partial_\gamma + \frac{1}{2}\partial_\phi\,, \quad k^{-1}J^- = -\gamma^2\partial_\gamma + \gamma\partial_\phi + e^{-2\phi}\partial_{\bar{\gamma}}. \tag{2.34}$$

Note that the final term in J^- vanishes at large ϕ. Finally, γ and $\bar{\gamma}$ are now complex coordinates on the boundary sphere.

We will be interested on the WZW models based on H_3^+ and SL(2,$\mathbb{R}$), where the dimensionless parameter k setting the size of the geometry in string units provides the affine algebra level. In order to build up some intuition, we start by considering the classical limit, $k \gg 1$, where only the zero-mode algebra is relevant. The cuadratic Casimir operator C, which reads

$$2C = -2J^3J^3 + J^+J^- + J^-J^+, \tag{2.35}$$

is then proportional to the corresponding Laplacian. In this limit scalar vertex operators of the model reduce to solutions of the Klein-Gordon equation. This is known in the literature as the mini-superspace limit [47].

[5]Note that this maps the boundary (Euclidean) cylinder to the complex plane parametrized by γ and $\bar{\gamma}$ at $\rho \to \infty$. In the Lorentzian setting, a similar change of variables gives the Poincaré metric on AdS$_3$. As discussed in Appendix B, this covers only part of the original manifold, that is, the diamond-shaped region known as the Poincaré patch. This is different from the Euclidean case, where $\gamma, \bar{\gamma}$ and ϕ cover all the hyperbolic manifold H_3^+.

In the H_3^+ case, a useful basis of solutions parametrized by a complex variable x is given by the functions

$$V_j(x) = \frac{2j-1}{\pi}\left(e^{\phi}|\gamma - x|^2 + e^{-\phi}\right)^{-2j}, \tag{2.36}$$

so that the mass can be read off from the Klein-Gordon equation associated to the metric (2.33),

$$\left[\Delta_{H_3^+} - \frac{4j(j-1)}{k}\right] V_j(x) = 0. \tag{2.37}$$

The operators J^a act on $V_j(x)$ as

$$J^+ \sim \partial_x \quad J^3 \sim x\partial_x + j\,, \quad J^- \sim x^2\partial_x + 2jx, \tag{2.38}$$

which shows that j is the spin of the corresponding representation. In other words, the functions $V_j(x)$ can be thought of as the components of a spacetime tensor of weights (j,j). It is interesting to look at how these behave at large ϕ,

$$V_j(x) \sim e^{2(j-1)\phi}\delta^{(2)}(\gamma - x) + \frac{(2j-1)}{\pi}e^{-2j\phi}|\gamma - x|^{-4j}\,, \tag{2.39}$$

where we have used a common representation of the complex Dirac delta function, namely

$$\delta^{(2)}(x) = \frac{2j-1}{\pi}\lim_{\varepsilon\to 0}\frac{\varepsilon^{2(2j-1)}}{(\varepsilon^2+|x|^2)^{2j}}\,, \qquad \mathrm{Re}\,j > \frac{1}{2}\,. \tag{2.40}$$

Note that in Eq. (2.39) we have omitted a series of corrections accompanying each of the two leading terms. We see from the first term that there is a leading divergence centered near $\gamma = x$, while the second term sets the decay rate at large ϕ for more general values of γ. We would like to identify these as the usual non-normalizable and normalizable terms that appear in holography.

For this, we employ an alternative description in terms of energy eigenstates, obtained by applying the Mellin-type transform

$$V_{jm\bar{m}} \equiv \int_{\mathbb{C}} d^2x x^{j-m-1}\bar{x}^{j-\bar{m}-1}V_j(x). \tag{2.41}$$

The integral is to be performed over the full complex plane, although when $m-j\in\mathbb{N}_0$ and $\bar{m}-j\in\mathbb{N}_0$ ($-m-j\in\mathbb{N}_0$ and $-\bar{m}-j\in\mathbb{N}_0$) it coincides with a double contour integral around $x=0$ ($x=\infty$) up to an overall factor. By means of the integral [15]

$$\frac{1}{\pi}\int_{\mathbb{C}} d^2x\, x^a\bar{x}^{\bar{a}}(1-x)^b(1-\bar{x})^{\bar{b}} = \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(-\bar{a}-\bar{b}-1)}{\Gamma(-\bar{a})\Gamma(-\bar{b})\Gamma(a+b+2)}\,, \tag{2.42}$$

this gives

$$V_{jm\bar{m}} \sim e^{2(j-1)\phi}\gamma^{j-m-1}\bar{\gamma}^{j-\bar{m}-1} + \frac{\Gamma(1-2j)\Gamma(j+m)\Gamma(j-\bar{m})}{\Gamma(2j)\Gamma(1+m-j)\Gamma(1-\bar{m}-j)}e^{-2j\phi}\gamma^{-j-m}\bar{\gamma}^{-j-\bar{m}} \tag{2.43}$$

at large ϕ. Note that one must have $m - \bar{m} \in \mathbb{Z}$ for the wave-function to be single-valued. Under this condition, the factor in front of the second term in (2.43) is actually symmetric under the exchange $m \leftrightarrow \bar{m}$ due to identity $\Gamma(z)\Gamma(1-z) = \pi \sin^{-1} \pi z$.

We now discuss the normalizability of these solutions. We work with the inner product associated to the L^2 norm. In terms of the coordinates $(\phi, \gamma, \bar{\gamma})$ this takes the form

$$\langle j, j' \rangle \equiv \frac{1}{\pi^3 k^{3/2}} \int d\phi d\gamma d\bar{\gamma}\, e^{2\phi} \overline{V}_j(x) V_{j'}(x)\,. \tag{2.44}$$

With this norm, which values of j correspond to states that should be included in the in the physical spectrum? As it turns out, the answer to this question is two-fold. For functions $V_{jm\bar{m}}$ with generic values of m and $\bar{m}$ and $\mathrm{Re}\, j \neq \frac{1}{2}$, either the first or the second term in Eq. (2.43) lead to a divergent norm upon integrating along ϕ. However, when $j = \frac{1}{2} + is$ with $s \in \mathbb{R}$ the solutions are delta-function normalizable. These states, which share many features with those appearing in Liouville theory [86], are those usually considered in the Euclidean model [49]. In the Lorentzian theory, they constitute the AdS_3 analogues of plane waves in flat space. They are associated to particles that come from the boundary, then approach interior of AdS_3 where they interact with the NS5 and F1 sources, and finally bounce back to infinity. The parameter s gives the radial momentum. Once we implementing spectral flow, they will correspond to the long string states discussed above. The relative coefficient appearing in (2.43) can be interpreted as the (classical limit of the) reflection coefficient

$$R_{jm\bar{m}} = \frac{\Gamma(1-2j)\Gamma(j+m)\Gamma(j-\bar{m})}{\Gamma(1-2j)\Gamma(1+m-j)\Gamma(1-\bar{m}-j)}, \tag{2.45}$$

which amounts to a pure phase shift for $m, \bar{m} \in \mathbb{R}$ (as long as $m - \bar{m} \in \mathbb{Z}$).

On the other hand, something special happens for real values of j, as there are certain values of m and $\bar{m}$ for which $R_{jm\bar{m}}$ develops simple poles. These occur at $m = \pm(j+n)$ with $n \in \mathbb{N}_0$, and similarly for $\bar{m}$. If we were interested in the H_3^+ per se, it would be natural to consider only complex values for m and $\bar{m}$ [47]. However, we are ultimately interested in the analytic continuation to Lorentzian (global) AdS_3, hence it makes more sense to take $m, \bar{m} \in \mathbb{R}$ as these quantum numbers parametrize the energy and momentum of the solution, see Eq. (2.12). Here lies one of the crucial differences between both models. Indeed, this allows to construct normalizable states by performing the following rescaling:

$$V_{jm\bar{m}} \to \tilde{V}_{jm\bar{m}} = \lim_{\bar{m} \to \pm(j+\bar{n})} \lim_{m \to \pm(j+n)} (j+n \mp m)(j+\bar{n} \mp \bar{m}) V_{jm\bar{m}} \qquad n, \bar{n} \in \mathbb{N}_0\,. \tag{2.46}$$

This has the effect of removing the first term in (2.43), while the second one remains finite. Hence, at least at the classical level, these configurations become normalizable for $j > \frac{1}{2}$, and should correspond to physical states. We will see below that in the quantum theory the reflection coefficient appears in the two-point functions. The poles described above then indicate the quantum numbers for which the external operators go on-shell (the m-basis employed here is the analogue of the usual Fourier basis, hence these singularities play the same role as the usual LSZ-type poles in QFT). This is consistent with our decision to include the corresponding states in the spectrum.

In order to go beyond the classical limit we need to study the WZW model based on H_3^+. As it follows from (2.33), the action reads

$$S = \frac{k}{\pi}\int d^2z \left(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma\right), \tag{2.47}$$

where the target space fields depend on the complex worldsheet coordinates $z, \bar{z}$. Of course, this can also be derived from Eq. (A.5) by an appropriate parametrization of the group elements, namely

$$g = \begin{pmatrix} e^{-\phi} + e^{\phi}\gamma\bar{\gamma} & e^{\phi}\gamma \\ e^{\phi}\bar{\gamma} & e^{\phi} \end{pmatrix}, \qquad g \in H_3^+ \simeq \frac{\mathrm{SL}(2,\mathbb{C})}{\mathrm{SU}(2)}. \tag{2.48}$$

Focusing on the holomorphic sector, the explicit expressions of the conserved currents read

$$k^{-1}J^+ = e^{2\phi}\partial\bar{\gamma}, \quad k^{-1}J^3 = e^{2\phi}\gamma\partial\bar{\gamma} - \partial\phi, \quad k^{-1}J^- = e^{2\phi}\gamma^2\partial\bar{\gamma} - 2\gamma\partial\phi - \partial\gamma. \tag{2.49}$$

Roughly speaking, these expressions can be understood which can be thought of as the one-forms dual to the killing vectors in Eq.(2.34). Similarly to the Lorentzian case, classical solutions are given by configurations that factorize as $g(z,\bar{z}) = g(z)\bar{g}(\bar{z})$. They can be parametrized as

$$\gamma = a(z) + \frac{\bar{b}(\bar{z})e^{-2c(z)}}{1 + b(z)\bar{b}(\bar{z})}, \quad \bar{\gamma} = \bar{a}(z) + \frac{b(z)e^{-2\bar{c}(\bar{z})}}{1 + b(z)\bar{b}(\bar{z})}, \tag{2.50}$$

$$\phi = c(z) + \bar{c}(\bar{z}) + \log[1 + b(z)\bar{b}(\bar{z})], \tag{2.51}$$

where the holomorphic functions $a(z)$, $b(z)$ and $c(z)$ are arbitrary. As will be justified below, the boundary conditions corresponding to the insertion of an operator characterized by a given value of j at a point x_0 on the asymptotic sphere and at z_0 on the worldsheet are of the form

$$\phi(z \sim z_0) \sim -\frac{j}{k}\log|z - z_0|^2, \quad \gamma(z \sim z_0) \sim x_0 + \mathcal{O}\left(|z - z_0|^{2j/k}\right). \tag{2.52}$$

The simplest classical configuration associated with a two-point function with operators $V_j(x_i, z_i)$ inserted at $z_1 = x_1 = 0$ and $z_2 = x_2 \to \infty$ thus reads

$$\phi(z,\bar{z}) = -\frac{j}{k}\log|z|^2, \quad \gamma(z) = 0, \tag{2.53}$$

which corresponds to $a(z) = b(z) = 0$ and $c(z) = -\frac{j}{k}\log z$ in Eqs. (2.50) and (2.51). This describes a geodesic joining the two boundary points through the interior of H_3^+. In the Lorentzian case, this corresponds to $\rho = 0$ and $t = 2j/k\,\tau$, i.e. it is the timelike geodesic (2.21) with $\alpha = 2j/k$.

We now consider small perturbations on top of the solution (2.53), following [15]. One can for instance perturb γ by turning on either one of the functions $a(z)$ and $\bar{b}(\bar{z})$ in (2.50). At first order, this gives either

$$\delta_{a,n}\gamma = \varepsilon z^n \quad \text{or} \quad \delta_{b,n}\gamma = \varepsilon\bar{z}^n|z|^{4j/k}, \quad n \in \mathbb{Z}. \tag{2.54}$$

They correspond to the action of the current modes J_n^+ and J_n^-, respectively. Indeed, in our parametrization the currents (2.49) are obtained through the usual formula $J^a = k\,\mathrm{Tr}\left[t^a \partial g g^{-1}\right]$ with generators

$$t^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad t^3 = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad t^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{2.55}$$

We are interested in perturbations giving non-trivial modifications at infinity, corresponding to solutions associated with the boundary insertion of operators of the form $(J_n^\pm V_j)$. We should thus only consider perturbations that are non-normalizable. Here the norm is defined using the target space metric, which gives

$$\int \frac{d^2z}{|z|^2}\left[(\delta\phi)^2 + e^{2\phi}\delta\gamma\delta\bar{\gamma}\right]. \tag{2.56}$$

For spacetime weights satisfying $\frac{1}{2} < j < \frac{k}{2}$, the non-normalizable modes are then given by $n = 0, -1, -2, \dots$ for J_n^+ and $n = -1, -2, \dots$ for J_n^-. Conversely, the normalizable modes can be thought of as annihilating the original operator V_j. For $J_n^\pm$ with $n > 0$ this is precisely what we expect for an affine primary state, see Appendix A. Additionally we expect $V_j(x_1 = 0, z_1 = 0)$ to be annihilated also by J_0^- since the Mellin transform of Eq. (2.41) tells us that we are dealing with the wave-function associated with an operator which has the lowest possible value of $m = j$, that is, a lowest-weight state in the corresponding representation of the zero-mode algebra.

However, the story is different for when $j \geq \frac{k}{2}$. Repeating the perturbative analysis for states with $j = \tilde{\jmath} + \frac{k}{2}\omega$, where $0 < \tilde{\jmath} < \frac{k}{2}$ and $\omega \in \mathbb{Z}$, we now find that the these states are annihilated by the modes $J_n^+ = 0$ with $n = \omega+1, \omega+2, \dots$ and $J_n^- = 0$ for $n = -\omega, -\omega+1, -\omega+2, \dots$. This means that we effectively shift the modes of $J^\pm$ in $\pm\omega$ units. Looking back at Eq. (2.24), which in the Euclidean model gives $J^\pm \to z^{\pm\omega} J^\pm$, we see this is what we expect from a state obtained by including ω units of spectral flow. As they are obtained by spectrally flowing time-like particle geodesics, these correspond to short string states. We will show below that this corresponds to a lowest-weight state of spin $\tilde{\jmath}$ with respect to a different SL(2,$\mathbb{R}$) subalgebra within the full current algebra of the WZW model, namely that generated by

$$\left\{\tilde{J}_0^+, \tilde{J}_0^-, \tilde{J}_0^3\right\} \equiv \left\{J_\omega^+, J_{-\omega}^-, J_0^3 - \frac{k}{2}\omega\right\}. \tag{2.57}$$

Finally, we consider the situation where $j = \frac{k}{2}\omega$. Here the two perturbations in (2.54) with $n = \pm\omega$ become exactly the same. On the other hand, a new solution appears:

$$\phi = -\frac{\omega}{2}\log|z|^2, \quad \gamma = \varepsilon z^\omega \log|z|^2. \tag{2.58}$$

This is precisely the (analytic continuation of the Lorentzian) solution for a long string with at small radial momentum, obtained by setting $\alpha \sim \varepsilon$ in Eq. (2.31). One finds that these states are annihilated by $J_n^\pm$ with $n > \pm\omega$. As will be discussed below, these spectrally flowed states are built from *unflowed* states which belong to the continuous representations of SL(2,$\mathbb{R}$), which have neither highest-weight states nor lowest-weight ones.

2.2 Wakimoto fields and spacetime symmetries

The current algebra is defined by the OPEs

$$J^a(z)J^b(w) \sim \frac{\eta^{ab}k/2}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{z-w}\,, \tag{2.59}$$

where k is the level of the affine algebra, while $-2\eta^{33} = \eta^{+-} = 2, f^{+-}{}_3 = -2$ and $f^{3+}{}_+ = -f^{3-}{}_- = 1$. The energy-momentum tensor and the central charge follow from the Sugawara construction, reviewed in Appendix A, and are given by

$$T_{\rm sl}(z) = \frac{1}{k-2} : -J^3(z)J^3(z) + \frac{1}{2}\left[J^+(z)J^-(z) + J^-(z)J^+(z)\right] :, \tag{2.60}$$

and

$$c_{\rm sl} = \frac{3k}{k-2}, \tag{2.61}$$

respectively. Note that the "-2" factors in the denominator are due to quantum effects, and where invisible in the classical limit, see for instance Eq. (2.37). Identical expressions hold for the anti-holomorphic sector.

At the classical level, we can introduce a pair of Lagrange multipliers $\beta, \bar\beta$ in order to rewrite the action (2.47) as

$$S = \frac{1}{2\pi}\int d^2z \left(\partial\phi\bar\partial\phi + \bar\beta\partial\bar\gamma + \beta\bar\partial\gamma - e^{-\sqrt{\frac{2}{k}}\phi}\beta\bar\beta\right), \tag{2.62}$$

Here we have rescaled $\phi \to \phi/\sqrt{2k}$ for normalization purposes. At large ϕ the interaction term can be neglected, hence the theory becomes that of a free scalar field and a so-called $\beta\gamma$-system. Moreover, in this regime the EOMs ensure that β, γ and $\partial\phi$ are holomorphic. At the quantum level, there is indeed a free field description valid in the above regime. This is known as the Wakimoto representation. However, one must take into account the non-trivial transformation of the path integral measure. This results in a small modification in the exponent of the interaction term, and further generates a dilaton linear in ϕ, leading to

$$S = \frac{1}{2\pi}\int d^2z \left(\frac{1}{2}\partial\phi\bar\partial\phi + \frac{Q_\phi}{4\sqrt{2}}R^{(2)}\phi + \bar\beta\partial\bar\gamma + \beta\bar\partial\gamma - e^{-\sqrt{\frac{2}{k-2}}\phi}\beta\bar\beta\right), \tag{2.63}$$

where $R^{(2)}$ is the worldsheet curvature scalar and $Q_\phi = -\frac{1}{\sqrt{k-2}}$. (The sign of the background charge Q_ϕ is purely conventional.) Near the asymptotic boundary, the interaction can be treated perturbatively, and the theory can be studied in terms of the free field OPEs, namely

$$\phi(z,\bar z)\phi(0) = -\log|z|^2\,, \quad \beta(z)\gamma(0) \sim \frac{-1}{z}\,. \tag{2.64}$$

The holomorphic currents are realized as

$$J^+ = \beta\,, \quad J^3 = (\beta\gamma) - \sqrt{\frac{k-2}{2}}\partial\phi\,, \quad J^- = (\beta\gamma^2) - \sqrt{2(k-2)}\gamma\partial\phi - k\partial\gamma. \tag{2.65}$$

We see from the OPE (2.64) that β acts as (minus) the derivative w.r.t. γ, hence for large k these expressions reproduce those given in Eq. (2.34) above, as expected. On the other hand, by solving the EOMs for $\beta, \bar{\beta}$ we go back to the currents of Eq. (2.49). We also have

$$T(z) = -\beta\partial\gamma - \frac{1}{2}\partial\phi\partial\phi - \frac{Q_\phi}{\sqrt{2}}\partial^2\phi\,, \tag{2.66}$$

with central charge

$$c = 2 + 1 + 6Q_\phi^2 = \frac{3k}{k-2}. \tag{2.67}$$

We see that the appearance of the charge Q_ϕ is necessary in order to generate the appropriate value of c.

The free field representation of the SL(2,$\mathbb{R}$) vertex operators reads

$$V_{jm\bar{m}} = \gamma^{j-m-1}\bar{\gamma}^{j-\bar{m}-1}e^{(j-1)\sqrt{\frac{2}{k-2}}\phi}, \tag{2.68}$$

and the corresponding worldsheet weights are given by

$$\Delta = -\frac{j(j-1)}{k-2}. \tag{2.69}$$

By using the OPEs in Eq. (2.64) we find that, in our conventions[6], the currents act as follows:

$$J^\pm(z)V_{jm}(0) \sim \frac{m \pm (1-j)}{z}V_{j,m\pm1}(0)\,, \quad J^3(z)V_{jm}(0) \sim \frac{m}{z}V_{jm}(0)\,. \tag{2.70}$$

for $m \neq \pm j$, while $J^\pm(z)V_{j,\pm j}(0) \sim 0$. Here we have suppressed the higher-order terms in the OPEs and omitted the anti-holomorphic indices for clarity.

The operators (2.68) form the so-called m-basis, where the Cartan current J^3 acts diagonally. We now construct the dual x-basis, i.e. the vertex operators whose wavefunctions are localized near $\gamma \sim x$ at the asymptotic boundary, see Eq. (2.36). For this we need to invert the transform defined in Eq. (2.41). Roughly speaking, this can be written as[7]

$$V_j(x,z) = \sum_{m,\bar{m}} x^{m-j}\bar{x}^{\bar{m}-j}\, V_{jm}(z)\,. \tag{2.72}$$

[6]The reader might prefer to work with a slightly more natural convention in which the coefficients generated by the action of $J_0^\pm$ vanish for $m = \pm j$, which would be analogous to what is usually done in the SU(2) case. However, for AdS$_3$ this actually introduces a number of inconvenient factors in different formulas. Our notation follows that of Refs. [15, 51, 53, 54].

[7]To be precise, (2.72) holds for states in the discrete sector, where $j \in \mathbb{R}$. Hence, poles in the integrand of (2.41) coming from the expansion around $x = 0$ ($x = \infty$) are associated to states in the $\mathcal{D}_j^+$ ($\mathcal{D}_j^-$) representation. For states in the continuous sector, (2.72) is modified to account for the fact that, although $m - \bar{m}$ is an integer number, $m + \bar{m}$ can take arbitrary real values, which must be integrated over, leading to

$$V_j(x,z) = \frac{i}{(2\pi^2)}\sum_{m-\bar{m}}\int_{-\infty}^{\infty} d(m+\bar{m})\, x^{m-j}\bar{x}^{\bar{m}-j}\, V_{jm}(z)\,. \tag{2.71}$$

As in the classical limit, the zero-modes of the SL(2,$\mathbb{R}$) currents act as differential operators on such states, namely[8]

$$J^a(z)V_j(x,0) \sim \frac{1}{z}\,(D^a_x V_j)(x,0)\,, \tag{2.73}$$

with

$$D^+_x = \partial_x \quad D^3_x = x\partial_x + j\,, \quad D^-_x = x^2\partial_x + 2jx. \tag{2.74}$$

This shows that J^+_0 generates translations in x-space, hence, at least for discrete states, $V_j(x,z)$ can be understood as a lowest-weight operator translated from the origin, i.e.

$$V_j(x,z) = e^{xJ^+_0 + \bar{x}\bar{J}^+_0} V_{jj}(z) e^{-xJ^+_0 - \bar{x}\bar{J}^+_0}\,, \tag{2.75}$$

where we have identified $V_j(x=0,z) = V_{jj}(z)$. Conversely, $V_j(x \to \infty, z)$ can be related to $V_{j,-j}(z)$. In this sense, states in lowest-weight (highest-weight) representations, i.e. with $m = j+n$ ($m = -j-n$) with $n \in \mathbb{N}_0$ can be interpreted as in-going (out-going) modes.

It is instructive to compute the OPEs

$$\phi(z,\bar{z})V_j(x,0) \sim -(j-1)\sqrt{\frac{2}{k-2}}\log|z|^2 V_j(x,z)\,, \quad \gamma(z)V_j(x,0) \sim xV_j(x,0)\,. \tag{2.76}$$

The first expression justifies *a posteriori* the boundary conditions employed in Eq. (2.52), which should be valid in the limit of large k and large j, and was written in terms of the unrescaled variables. We would like to interpret the worldsheet operators $V_j(x,z)$ as local operators of a boundary conformal field theory. Eqs. (2.73) and (2.74) show that the global generators of the corresponding spacetime Virasoro algebra (not to be confused with the worldsheet Virasoro algebra) can be identified with

$$\mathcal{L}_{-1} = \oint dz J^+(z)\,, \quad \mathcal{L}_0 = \oint dz J^3(z)\,, \quad \mathcal{L}_1 = \oint dz J^-(z)\,. \tag{2.77}$$

The Wakimoto representation allows for a simple construction of the rest of the spacetime Virasoro modes, namely [10]

$$\mathcal{L}_n = \oint dz \left[(n+1)\gamma^n J^3 - n\gamma^{n+1}J^+\right]\,, \tag{2.78}$$

which is valid in the large ϕ region where the free field description is accurate. These modes are indeed BRST-invariant. The expression in Eq. (2.78) is unique up to BRST exact states, i.e. those with integrands proportional to $2\gamma^n J^3 - \gamma^{n+1}J^+ - \gamma^{n-1}J^-$, obtained by acting with $\mathcal{Q}$ on γ^n. The current OPEs combined with

$$J^3(z)\gamma^n(0) \sim -\frac{n}{z}\gamma^n(0)\,, \quad J^-(z)\gamma^n(0) \sim -\frac{n}{z}\gamma^{n+1}(0)\,, \tag{2.79}$$

imply that

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \mathcal{I}\frac{k}{2}(n^3-n)\delta_{n+m,0}\,, \tag{2.80}$$

[8] Here our convention differs in a sign from that used in [82]. As a result, the differential operators D^a_x must satisfy the zero-mode algebra only up to a global sign.

where

$$\mathcal{I} \equiv \oint \gamma^{-1}\partial\gamma\,. \tag{2.81}$$

It can be seen that this operator commutes with all $\mathcal{L}_n$. In a classical solution, $\mathcal{I}$ is an integer that counts how many times the string worldsheet wounds around the boundary circle. (In the Lorentzian framework we have $\gamma \sim e^{t-iy}$ at fixed ρ, hence $\mathcal{I}$ keeps track of how many times we go around the spatial circle upon shifting $\sigma \to \sigma + 2\pi$.) The expression for the spacetime B-field given in Eq. (2.33) then leads us to interpret this quantity as counting the number of fundamental strings sources present in our configuration. Let us stress that this is formally invisible from the perturbative worldsheet point of view when working at first order in the topological expansion. By replacing $\mathcal{I} \to n_1$ in (2.80) we read off the spacetime central charge

$$c_{\rm st} = 6kn_1\,. \tag{2.82}$$

However, it should be noted that in the quantum theory the role of the operator $\mathcal{I}$ is quite subtle; we will come back to it later on when discussing spectral flow. Finally, we can compute the action of the modes $\mathcal{L}_n$ on the vertex operators. By using Eq. (2.68), we get

$$[\mathcal{L}_n, V_{j,-m}(z)] = [n(j-1) - m]V_{j,-(m+n)}(z)\,. \tag{2.83}$$

This tells us that, up to the integration over the worldsheet coordinates z and $\bar{z}$, the m-basis operators $V_{jm\bar{m}}(z)$ can be interpreted as the (spacetime) Virasoro modes of local operator $V_j(x,z)$ of holomorphic weight j of the boundary theory[9].

We can also study what happens if there is an additional set of conserved Kac-Moody currents $K^a(z)$ in our string background, satisfying

$$[K^a_n, K^b_m] = i f^{ab}{}_c K^c_{n+m} + \frac{k'}{2} n \delta^{ab}\delta_{n+m,0} \tag{2.84}$$

for some integer k'. This is the case, for instance when the background is $\mathrm{AdS}_3 \times S^3 \times T_4$ (or K3), where there are currents $K^a(z)$ which generate the S^3 isometries. The former are identified with the R-symmetry generators of the holographic SCFT. We can use the Wakimoto fields to translate this into spacetime modes,

$$\mathcal{K}^a_n = \oint dz\, K^a(z)\gamma^n(z). \tag{2.85}$$

This leads to following extension of the spacetime symmetry algebra:

$$[\mathcal{K}^a_n, \mathcal{K}^b_m] = i f^{ab}{}_c \mathcal{K}^c_{n+m} + \mathcal{I}\,\frac{k'}{2} n \delta^{ab}\delta_{n+m,0}\,, \tag{2.86}$$

$$[\mathcal{L}_n, \mathcal{K}^a_m] = -m\mathcal{K}^a_{n+m}\,. \tag{2.87}$$

Since $\mathcal{I}$ is the same operator that appeared in the Virasoro case, we can interpret the spacetime central term as $k'_{\rm st} \equiv n_1 k'$.

[9] The odd minus sign in (2.83) comes from our conventions for the expansion in terms of the boundary coordinate x, see Eq. (2.72)

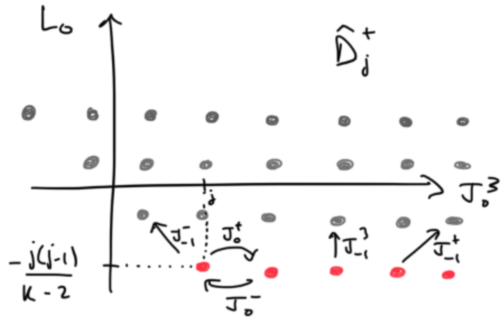

Figure 4. Weight diagram for the lowest-weight representation $\hat{\mathcal{D}}_j^+$

Let us discuss in more detail the allowed values for the spin quantum number j. Our semiclassical analysis indicated the presence of two types of modes normalizable ones: those with $j \in \frac{1}{2} + i\mathbb{R}$, which were only delta-function normalizable, and those with real $j > \frac{1}{2}$. However, we must be careful since this was done in the $k \to \infty$ limit, hence our conclusions could – and will – get modified in the quantum theory, i.e. at finite k. The broad picture is consistent with the well-known representation theory of the SL(2,$\mathbb{R}$) algebra. The relevant irreducible representations are either of the lowest/highest-weight type, or of the continuous type. A principal discrete series of lowest-weight is built out of the state $|j,j\rangle$, annihilated by j_0^-, by acting with j_0^+, thus spanning

$$\mathcal{D}_j^+ = \Big\langle\, |jm\rangle \ , \ m = j, j+1, j+2, \cdots \Big\rangle, \tag{2.88}$$

where $j_0^3|jm\rangle = m|jm\rangle$. Here $|jm\rangle$ represents the state created by inserting $V_{jm}(z)$ at the origin $z = 0$. The representations $\mathcal{D}_j^+$ are unitary for any $j > 0$. This also holds for their conjugate representations, denoted by $\mathcal{D}_j^-$, which are highest-weight representations defined analogously. On the other hand, the principal continuous series are given by

$$\mathcal{C}_j^\alpha = \Big\langle\, |jm\rangle \ , \ 0 \leq \alpha < 1 \ , \ m = \alpha, \alpha \pm 1, \alpha \pm 2, \cdots \Big\rangle. \tag{2.89}$$

These are unitary if $j = \frac{1}{2} + is$ with $s \in \mathbb{R}$. Strictly speaking, the representation with $s = \alpha - \frac{1}{2} = 0$ is reducible. Note that, as opposed to what happens for discrete states, in the continuous case the allowed values of m are not related to that of j.

With respect to the full current algebra, the states $|jm\rangle$ are affine primary states. They are annihilated by all positive modes $J^a_{n>0}$. The full affine modules $\hat{\mathcal{D}}_j^\pm$ and $\hat{\mathcal{C}}_j^\alpha$ are then generated by acting with $J^a_{n\leq 0}$. As an example, we show the weight diagram for the $\hat{\mathcal{D}}_j^+$ in the plane of eigenvalues of L_0 and J_0^3 in Fig. 4.

The above analysis was restricted to the zero-mode algebra. For WZW models the number of symmetry generators is enhanced to the full affine extension of the SL(2,$\mathbb{R}$) algebra at level k. The commutation relations of the generators J_n^a with $n \in \mathbb{Z}$ are contained in the OPEs (2.59). Each of the above states defines an affine primary state, satisfying

$$J_n^a|jm\rangle = 0\,, \qquad \forall\; n > 0, \tag{2.90}$$

from which the full affine module is generated by acting with J^a_n with $n<0$. However, as in the SU(2) case, there exist null descendants of the form

$$\left(J^-_{-1}\right)^{k-2j+1}|j\,j\rangle = 0\,. \tag{2.91}$$

For instance, the simplest null state is given by the $J^-_{-1}|\frac{k}{2}\frac{k}{2}\rangle$, since

$$\left|J^-_{-1}\left|\frac{k}{2}\frac{k}{2}\right\rangle\right|^2 = \left\langle\frac{k}{2}\frac{k}{2}\right|J^+_1 J^-_{-1}\left|\frac{k}{2}\frac{k}{2}\right\rangle = \left\langle\frac{k}{2}\frac{k}{2}\right|k-2J^3_0\left|\frac{k}{2}\frac{k}{2}\right\rangle = 0\,. \tag{2.92}$$

As discussed in Appendix A, this can be used to show that states with $j>\frac{k}{2}$ actually decouple [42]. Moreover, we have argued above that, at least semiclassically, states where j is an integer multiple of $\frac{k}{2}$ are rather special, as this is the value for which the long strings appear. This suggests that we should restrict to discrete representations with $\frac{1}{2}<j<\frac{k}{2}$. Indeed, it was shown in [15] that only such states satisfy the requisite no-ghost theorem in the bosonic string context. Moreover, and as we shall now see, this intuition turns out to be slightly off: the correct range will be given by

$$\frac{1}{2}<j<\frac{k-1}{2}\,. \tag{2.93}$$

Below we will provide an alternative argument for the upper bound on j based on spectral flow.

States belonging to the continuous representations are found to be problematic. Indeed, the mass-shell condition reads

$$-\frac{j(j-1)}{k-2}+h_{\text{int}}-1+N=0\,, \tag{2.94}$$

where N is the level of (worldsheet Virasoro) descendence. For $j=\frac{1}{2}+is$ we get $-j(j-1)=\frac{1}{4}+s^2$, hence this Virasoro constraint cannot be satisfied for $N>0$ assuming $h_{\text{int}}\geq 0$. Moreover, Eq. (2.37) shows that for any such state with $s>0$ the spacetime mass is violates the Breitenlohner-Freedman bound for AdS_3:

$$m^2 = 4j(j-1) = -1-4s^2 < m^2_{BF} = -1\,, \tag{2.95}$$

where we are working in units of the AdS_3 radius and in the large k regime. We conclude that these modes are tachyonic, and should be excluded. This is actually a relief: we argued above that j should be identified with the holomorphic weights in the boundary theory, and for this to be a unitary CFT we ought to avoid including operators with complex weights. In superstring theory on AdS_3, these tachyonic modes are excluded by the GSO projection.

However, the above discussion raises two very important problems. First, in our worldsheet description we have not found any physical operator corresponding to the short and long string solutions observed at the classical level. Moreover, we have not been able to construct any operator with a spacetime weight larger than $\frac{k}{2}$. We expect a large number of states above this value since the spacetime central charge is $c_{\text{st}}=6kn_1$, and we are working at large n_1. The solution to both issues was found in [15]: we are missing the spectrally flowed representations.

2.3 Spectrally flowed representations

Algebraically, spectral flow refers to a family of authomorphisms of the affine SL(2,$\mathbb{R}$) algebra at level k, namely the transformation $J^a_n \to \tilde{J}^a_n$ with

$$\tilde{J}^3_n = J^3_n - \frac{k}{2}\omega\delta_{n,0}\,, \quad \tilde{J}^\pm_n = J^\pm_{n\pm\omega}\,, \qquad \omega \in \mathbb{Z}\,. \tag{2.96}$$

The effect on the worldsheet Virasoro modes is

$$L_n \to \tilde{L}_n = L_n + \omega J^3_n - \frac{k}{4}\omega^2\delta_{n,0}\,. \tag{2.97}$$

All commutation relations are preserved by these transformations, and of course the affine level k and central charge $c = \frac{3k}{k-2}$ are also unchanged. Nevertheless, the effect on the different states is highly non-trivial. More precisely, we now show that whole new family of physical states is generated. These are distinct from the *unflowed* states that we have considered so far. This phenomenon is in contrast with what happens for WZW models based on compact groups such as SU(2), where spectral flow merely reshuffles the different primaries and descendants without producing any new states, see Appendix A.

Let us see how this works. Consider a state defined as an affine primary of spin j and projection m *with respect to the currents* $\tilde{J}^a_n$, for a given ω. We denote this state as $|jm\omega\rangle$. It satisfies

$$\tilde{J}^3_0|jm\omega\rangle = m|jm\omega\rangle\,, \quad \tilde{J}^\pm_{\pm w}|jm\omega\rangle = [m \pm (1-j)]|jm\omega\rangle\,. \tag{2.98}$$

We now restrict to $\omega > 0$ without loss of generality. Then, the following properties hold as well:

$$\tilde{J}^3_n|jm\omega\rangle = J^3_n|jm\omega\rangle = 0\,, \quad \tilde{J}^\pm_n|jm\omega\rangle = J^\pm_{n\pm\omega}|jm\omega\rangle = 0\,, \qquad \forall\; n > 0\,. \tag{2.99}$$

All other modes act non-trivially, except when dealing with a lowest/highest weight state with $m = \pm j$, for which we additionally have

$$\tilde{J}^\mp_0|j,\pm j,\omega\rangle = J^\mp_{\mp\omega}|j,\pm j,\omega\rangle = 0\,. \tag{2.100}$$

This shows that for $\omega > 0$ the state $|jm\omega\rangle$ is *not* an affine primary with respect to the action of the physical currents (by which we mean J^a_n) since generically it is not annihilated by the positive modes J^+_n with $n = 1, \dots \omega$. However, recall that physical states need not be affine primaries: we should only require that they are built upon (worldsheet) Virasoro primaries. Although the former implies the latter, the converse statement is not true [82]. Indeed, from Eq. (2.97) we see that $|jm\omega\rangle$ *is* a Virasoro primary, i.e. $L_{n>0}|jm\omega\rangle = 0$. Hence, such states (and their descendants) should be included in the spectrum, provided they satisfy the mass-shell condition. In order to identify the relevant quantum numbers we compute the action of the zero modes J^a_0 and L_0. By using Eq. (2.96) in combination with Eqs. (2.98) and (2.99), we derive

$$J^3_0|jm\omega\rangle = \left(\tilde{J}^3_0 + \frac{k}{2}\omega\right)|jm\omega\rangle = \left(m + \frac{k}{2}\omega\right)|jm\omega\rangle\,, \tag{2.101}$$

and

$$J_0^-|jm\omega\rangle = \tilde{J}_\omega^-|jm\omega\rangle = 0\,. \tag{2.102}$$

We thus identify $|jm\omega\rangle$ as the lowest-weight state in discrete representation of the physical zero-mode algebra with spin

$$h = m + \frac{k}{2}\omega\,. \tag{2.103}$$

Note that we have landed on a discrete lowest-weight representation *regardless of the unflowed representation we started with*. In other words, we need not impose any condition on whether j is real or not: we always end up with $h \in \mathbb{R}$. The highest-weight flowed primary from which the corresponding conjugate representation is generated is of the form $|j,-m,-\omega\rangle$. Finally, from Eq. (2.97) we get

$$L_0|j,\pm m,\pm\omega\rangle = \left(\tilde{L}_0 \mp \omega \tilde{J}_0^3 - \frac{k}{4}\omega^2\right)|j,\pm m,\pm\omega\rangle\,, \tag{2.104}$$

from where we read off the physical worldsheet weights of such spectrally flowed states,

$$\Delta = -\frac{j(j-1)}{k-2} - m\omega - \frac{k}{4}\omega^2 = -\frac{j(j-1)}{k-2} - h\omega + \frac{k}{4}\omega^2\,. \tag{2.105}$$

.

We now argue that states belonging to the spectrally flowed representations, which will be denote as $\mathcal{D}_j^{\pm,\omega}$ and $\mathcal{C}_j^{\alpha,\omega}$, are indeed new Virasoro primaries, as neither them nor their descendants can be obtained as a linear combination of states belonging to the unflowed representations. This can be seen for instance by looking at the allowed values for the corresponding worldsheet weights Δ. Recall that in each *unflowed* affine module, defined by the value of the corresponding spin j, the primary weight in Eq. (2.69) constitutes a lower bound for the eigenvalue of L_0. For (physical) discrete states thus have an overall lower bound of the form $\Delta > \Delta(j = \frac{k-1}{2})$, while for the continuous representations, which are tachyonic anyway, we get $\Delta > -\frac{1}{4(k-2)}$. In contrast, from Eq. (2.105) we immediately find that for in generic spectrally flowed sectors we can obtain arbitrarily negative values of Δ increasing the absolute value of m, see Fig. 5. It follows that these must be new physical states, which should be included in the spectrum.

The only exceptions correspond to the affine modules $\hat{\mathcal{D}}_j^{\pm,\omega=\mp 1}$. The reason for this can be understood in terms of a more general phenomenon known as the SL(2,$\mathbb{R}$) series identifications, which we now describe. From Eqs. (2.103) and (2.105) we see that that the states $|j,-j,\omega\rangle$ and $|\frac{k}{2}-j,\frac{k}{2}-j,\omega-1\rangle$ have exactly the same quantum numbers. Hence, they should be identified (up to a normalization constant, whose precise form will be obtained in due course). As these states can be used to generate their full affine modules, we get the following set of isomorphims:

$$\hat{\mathcal{D}}_j^{\pm,\omega} \simeq \hat{\mathcal{D}}_{\tilde{j}}^{\mp,\omega\pm 1}\,, \quad \tilde{j} = \frac{k}{2} - j\,. \tag{2.106}$$

In particular, $\hat{\mathcal{D}}_j^{\pm,\omega=\mp 1}$ are identified with unflowed representations. This is shown for a particular example in Fig. 6.

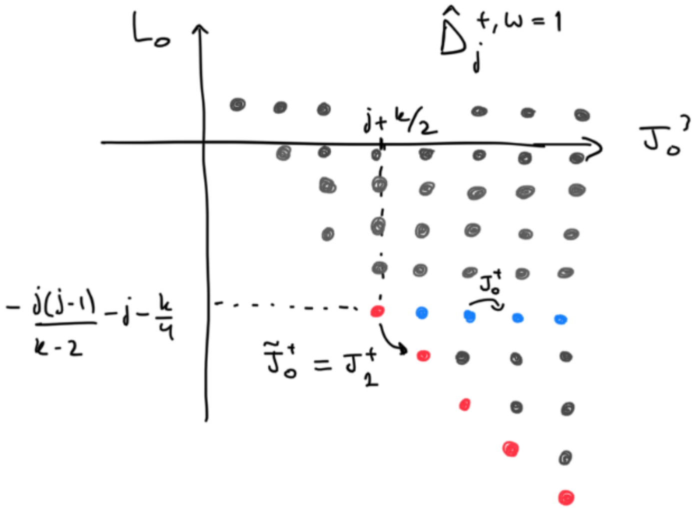

Figure 5. Weight diagram for the lowest-weight representation $\hat{\mathcal{D}}_j^{+,\omega}$ with $\omega = 1$.

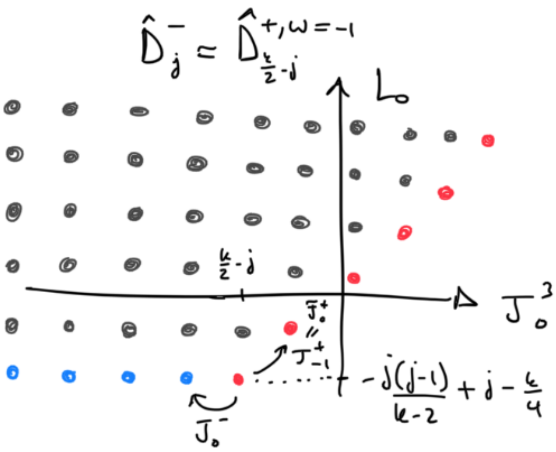

Figure 6. Equivalence between the representations $\hat{\mathcal{D}}_j^-$ and $\hat{\mathcal{D}}_{\frac{k}{2}-j}^{+,\omega}$ with $\omega = -1$.

We also note that this explains why we had to impose $j < \frac{k-1}{2}$ (as opposed to the classical upper bound $j < \frac{k}{2}$). Indeed, states in $\hat{\mathcal{D}}_j^{+,\omega=-1}$ with $j > \frac{k-1}{2}$ would be identified with unflowed states in a highest-weight representation of spin $\tilde{j} < \frac{1}{2}$, which are not normalizable.

Due to the series identifications (2.106), we should be careful and avoid overcounting when describing the spectrum of the model. This can be achieved for example by only including the discrete representations $\hat{\mathcal{D}}_j^{+,\omega\geq 0}$ and $\hat{\mathcal{D}}_j^{-,\omega\leq 0}$. Equivalently, we could work only with the $\hat{\mathcal{D}}_j^{+,\omega}$ representations and include all $\omega \in \mathbb{Z}$. Moreover, as in the unflowed sector, from the spacetime point of view discrete representations of the highest-weight type are interpreted as out-going modes while their lowest-weight counterparts are the in-going ones. In other words, the operators that create the states $|jm\omega\rangle$ and $|j,-m,-\omega\rangle$ contribute to the same vertex operator, denoted as $V_{jh}^{w}(x,z)$ with $h = m + \frac{k}{2}\omega$. This discussion holds for spectrally flowed representations of the zero-mode algebra regardless of whether they were built from unflowed states in the discrete or continuous sectors. Hence, we conclude that the physical spectrum of the SL(2,$\mathbb{R}$) WZW model is characterized by a Hilbert space of the form [13–15]

$$\mathcal{H} = \oplus_{\omega\in\mathbb{Z}} \left[\mathcal{H}_\omega^{\rm disc} \oplus \mathcal{H}_\omega^{\rm cont}\right] , \tag{2.107}$$

with

$$\mathcal{H}_\omega^{\rm disc} = \int_{\frac{1}{2}}^{\frac{k-1}{2}} dj\, \hat{\mathcal{D}}_j^{+,\omega} \otimes \hat{\mathcal{D}}_j^{+,\omega} , \tag{2.108}$$

and

$$\mathcal{H}_\omega^{\rm cont} = \int_{\frac{1}{2}+i\mathbb{R}} dj \int_0^1 d\alpha\, \hat{\mathcal{C}}_j^{\alpha,\omega} \otimes \hat{\mathcal{C}}_j^{\alpha,\omega}. \tag{2.109}$$

It was shown in [14] that this leads to a modular invariant partition function.

We now discuss in more detail the main properties of the spectrally flowed vertex operators. In the m-basis we denote them as $V^\omega_{jm}(z)$. For $\omega > 0$, they are formally defined by the following OPEs:

$$J^+(z)V^\omega_{jm}(w) = \frac{(m+1-j)V^\omega_{j,m+1}(w)}{(z-w)^{\omega+1}} + \sum_{n=1}^{\omega} \frac{(J^+_{n-1}V^\omega_{jm})(w)}{(z-w)^n} + \dots , \tag{2.110a}$$

$$J^3(z)V^\omega_{jm}(w) = \frac{\left(m+\frac{k}{2}\omega\right) V^\omega_{jm}(w)}{(z-w)} + \dots , \tag{2.110b}$$

$$J^-(z)V^\omega_{jm}(w) = (z-w)^{\omega-1}(m-1+j)V^\omega_{j,m-1}(w) + \dots , \tag{2.110c}$$

where the ellipsis indicate higher order terms. Note that the leading terms in the OPEs of $V^\omega_{jm}(z)$ with $J^\pm$ are known since they correspond to the action of the modes $J^\pm_{\pm\omega} = \tilde{J}^\pm_0$. In contrast, there is no simple expression available for the following orders. Similar equations hold for $\omega < 0$ with the roles of J^+ and J^- inverted. Even though we omit the antiholomorphic dependence, it should be clear that the operators $V^\omega_{jm}(z)$ have analogous OPEs with the currents $\bar{J}^a$.

We can also provide an x-basis construction for vertex operators with a non-trivial spectral flow charge. Indeed, for $\omega > 0$ all spectrally flowed m-basis primaries obtained above are annihilated by J_0^-. For each of them, the rest of the states in the corresponding zero-mode representation is obtained by acting repeatedly with J_0^+. Spectrally flowed x-basis operators are defined in analogy with those of the unflowed sector, see Eq. (2.72). More precisely, we have

$$V_{jh\bar{h}}(x,\bar{x},z,\bar{z}) = e^{xJ_0^+ + \bar{x}\bar{J}_0^+} V^\omega_{jm\bar{m}}(z) e^{-xJ_0^+ - \bar{x}\bar{J}_0^+} . \tag{2.111}$$

Note that here the holomorphic and anti-holomorphic spacetime weights $h = m + \frac{k}{2}\omega$ and $\bar{h} = \bar{m} + \frac{k}{2}\omega$ can be different. Nevertheless, in what follows we will mostly focus on spacetime scalars, that is, operators with $h = \bar{h}$. Hence, from now on we shall omit antiholomorphic quantities and simply use the shorthand $V^\omega_{jh}(x,z)$.

In Eq. (2.111) we have identified $V^\omega_{jh}(x=0,z) = V^\omega_{jm}(z)$ and $V^\omega_{jh}(x\to\infty,z) \sim V^{-\omega}_{j,-m}(z)$, so that when working in the x-basis it makes sense to restrict to $\omega \geq 0$. The defining OPEs (2.110) imply that for $V^\omega_{jh}(x,z)$ we have

$$J^+(w)V^\omega_{jh}(x,z) = \sum_{n=2}^{\omega+1} \frac{\left(J^+_{n-1}V^\omega_{jh}\right)(x,z)}{(w-z)^n} + \frac{\left(J^+_0 V^\omega_{jh}\right)(x,z)}{(w-z)} + \cdots\,, \tag{2.112a}$$

$$J^3(w)V^\omega_{jh}(x,z) = x\sum_{n=2}^{\omega+1} \frac{\left(J^+_{n-1}V^\omega_{jh}\right)(x,z)}{(w-z)^n} + \frac{\left(J^3_0 V^\omega_{jh}\right)(x,z)}{(w-z)} + \cdots\,, \tag{2.112b}$$

$$J^-(w)V^\omega_{jh}(x,z) = x^2\sum_{n=2}^{\omega+1} \frac{\left(J^+_{n-1}V^\omega_{jh}\right)(x,z)}{(w-z)^n} + \frac{\left(J^-_0 V^\omega_{jh}\right)(x,z)}{(w-z)} + \cdots\,, \tag{2.112c}$$

As in the unflowed sector, the zero modes act as differential operators in x,

$$\left(J^+_0 V^\omega_{jh}\right)(x,z) = \partial_x V^\omega_{jh}(x,z)\,, \tag{2.113a}$$

$$\left(J^3_0 V^\omega_{jh}\right)(x,z) = (x\partial_x + h)V^\omega_{jh}(x,z)\,, \tag{2.113b}$$

$$\left(J^-_0 V^\omega_{jh}\right)(x,z) = (x^2\partial_x + 2hx)V^\omega_{jh}(x,z)\,, \tag{2.113c}$$

while

$$\left(J^\pm_{\pm\omega}V^\omega_{jh}\right)(x,z) = \left[h - \frac{k}{2}\omega \pm (1-j)\right] V^\omega_{j,h\pm 1}(x,z)\,. \tag{2.114}$$

Importantly, in terms of the currents

$$J^+(x,z) = J^+(z)\,,\quad J^3(x,z) = J^3(z) - xJ^+(z)\,,\quad J^-(x,z) = J^-(z) - 2xJ^3(z) + x^2J^+(z) \tag{2.115}$$

we get[10]

$$J^3(x,w)V^\omega_{jh}(x,z) = \frac{h}{(w-z)}V^\omega_{jh}(x,z) + \cdots\,, \tag{2.116a}$$

$$J^-(x,w)V^\omega_{jh}(x,z) = (w-z)^{\omega-1}\left(J^-_{-w}V^\omega_{jh}\right)(x,z) + \cdots\,. \tag{2.116b}$$

This shows that when the currents are inserted at the same point in x as the vertex operators, the corresponding OPEs become analogous to the m-basis ones.

Let us now discuss the physical spectrum of the full string worldsheet theory. We restrict to the Virasoro primary sector for simplicity, and consider operators of the form

$$\mathcal{V}(x,\bar{x},z,\bar{z}) \equiv V^\omega_{jh}(x,\bar{x},z,\bar{z})V_{\text{int}}(z,\bar{z}). \tag{2.117}$$

Focusing on the holomorphic sector, we must impose the Virasoro constraint

$$-\frac{j(j-1)}{k-2} - h\omega + \frac{k}{4}\omega^2 + h_{\text{int}} - 1 = 0. \tag{2.118}$$

[10] Note that $J^-(x,z)$ is often written simply as $J(x,z)$ in the literature.

For states in lowest-weight representations we have $h = m + \frac{k}{2}$ with $m = j + n$ for some non-negative integer n. We can thus solve for j in Eq. (2.118), and also compute the spacetime weight, giving

$$h = n + \omega + \frac{1}{2} + \sqrt{(k-2)\left(h_0 - n\omega - \frac{\omega(\omega+1)}{2}\right)}\,, \tag{2.119}$$

with

$$h_0 = \frac{1}{4(k-2)} + h_{\text{int}} - 1. \tag{2.120}$$

Moreover, the allowed range for j given in Eq. (2.93) imposes the following bounds:

$$\frac{k}{4}\omega^2 + \frac{w}{2} \leq h_0 \leq \frac{k}{4}(\omega+1)^2 - \frac{w+1}{2}. \tag{2.121}$$

For a given h_0 in this range, setting $n = 0$ in (2.119) gives the lowest value of h. In particular, when the bound is saturated (for a fixed ω) we can write this as

$$h^{\min}(h_0^{\min}) = \frac{k}{4}\omega + \frac{h_0^{\min}}{\omega} = \frac{1}{2} + \frac{k}{2}\omega\,, \tag{2.122}$$

and

$$h^{\min}(h_0^{\max}) = \frac{k}{4}(\omega+1) + \frac{h_0^{\max}}{\omega+1} = -\frac{1}{2} + \frac{k}{2}(\omega+1). \tag{2.123}$$

We now move to continuous states. In this sector we interpret (2.118) as an equation for h instead of $j = \frac{1}{2} + is$, which gives

$$h = \frac{k}{4}\omega + \frac{1}{\omega}\left[\frac{s^2}{(k-2)} + h_0\right]. \tag{2.124}$$

The lowest possible value for h, say $h^{\min}$, clearly corresponds to $s = 0$.

For a given value of ω, the above discussion shows that, in the limit $j \to \frac{1}{2}$ or $j \to \frac{k-1}{2}$, i.e. when we saturate the bound (2.93), the minimal spacetime energy $E = h + \bar{h}$ for the corresponding short string states becomes exactly that of a continuous one with a winding charge given by either ω or $\omega + 1$. These are the only values of h_0 for which this happens. This means that, whenever they exist, that is, for values of h_0 satisfying (2.121), the state with lowest h always corresponds to a short string. On the other hand, when

$$\frac{k}{4}\omega^2 - \frac{w}{2} \leq h_0 \leq \frac{k}{4}\omega^2 + \frac{w}{2}\,, \tag{2.125}$$

the lowest energy eigenvalue is realized in the long string sector instead.

This is to be compared with our analysis of the classical string geodesics in global AdS_3. There we found that long strings can only give the lowest-energy states when we sit precisely at $h_{\text{int}} = \frac{k}{4}\omega^2$. In the classical regime, i.e. at large h and k, we have $h_0 \approx h_{\text{int}}$, see Eq. (2.120). We see from Eq. (2.125) that in the quantum theory this is extended to a strip of width ω.

Before finishing this section, we briefly go back to the free field description. Spectral flow corresponds to the transformation[11]

$$\gamma(z) \to z^{\omega}\gamma(z)\,, \quad \beta(z) \to z^{-\omega}\beta(z)\,, \quad \phi(z,\bar{z}) \to \phi(z,\bar{z}) - \omega\sqrt{\frac{k-2}{2}}\log|z|^2. \tag{2.126}$$

Vertex operators are then realized as

$$V^{\omega}_{jm\bar{m}} = z^{-m\omega}\bar{z}^{-\bar{m}\omega}\gamma^{j-m-1}\bar{\gamma}^{j-\bar{m}-1}e^{(j-1)\sqrt{\frac{2}{k-2}}\phi}. \tag{2.127}$$

By going to the x-basis we find

$$\gamma(z)V^{\omega}_{jh}(x,0) = xV^{\omega}_{jh}(x,0) + z^{\omega}V^{\omega}_{j,h-1}(x,0) + \cdots\,, \tag{2.128}$$

$$\phi(z,\bar{z})V^{\omega}_{jh}(x,0) \sim -\sqrt{\frac{2}{k-2}}\left(j-1+\frac{k-2}{2}\omega\right)\log|z|^2V^{\omega}_{jh}(x,0), \tag{2.129}$$

where we have used that since $J^+ = \beta$, γ is translated in x-space as $\gamma(x,z) = \gamma(z) - x$, while $\phi(x,\bar{x},z,\bar{z}) = \phi(z,\bar{z})$. The OPE with ϕ again justifies the boundary conditions used in Eq. (2.52) for spectrally flowed states. We also comment relation between the operator $\mathcal{I}$ defined in (2.81) when discussing the worldsheet realization of the spacetime Virasoro algebra. Eq. (2.126) gives

$$\mathcal{I} \to \oint z^{-\omega}\gamma^{-1}\partial\left(z^{\omega}\gamma\right) = \mathcal{I} + \omega. \tag{2.130}$$

Hence, strictly speaking, $\mathcal{I}$ cannot be interpreted as the identity operator of the spacetime theory, as it acts differently in each spectral flow sector.

2.4 Comments on the partition function

We now consider the partition function. Our treatment will follow the qualitative analysis [13], Appendix B. Thought the resulting expressions for the characters and partition function are useful form the intuitive point of view, they are somewhat ill-defined due to various divergencies. For a more precise treatment, which is outside of the scope of these notes, we refer the reader to [14, 65, 96, 97].

We study the characters of the discrete and continuous sectors separately. Let us first consider the discrete sector. We already know that, once spectral flow is taken into account, it is enough to work only with representations of the lowest-weight type, together with their spectrally flowed images. Naively, the computation should be somewhat similar to that of the SU(2) WZW model (see Appendix A) except for the range of allowed values for the spin j. However, it turns out that the computation is actually simplified. This is because in the SL(2,$\mathbb{R}$) case the expression analogous to Eq. (A.59) reads

$$||(j^-_{-1})^N|j,j\rangle||^2 = \prod_{n=1}^{N} n(k-2j-1+n)\,, \tag{2.131}$$

[11] This gives the correct transformation for J^3 thanks to the normal ordering constant ω appearing when flowing the product $(\beta\gamma)$, see Eq. (2.65).

implying that a null state can only be obtained by choosing $j = \frac{k-1+N}{2}$ for some positive integer N. As a consequence, no null states appear when imposing the physical restriction $\frac{1}{2} < j < \frac{k-1}{2}$! In the unflowed sector, this leads to global characters of the form

$$\chi_j^+(z) = \sum_{n=0}^{\infty} y^{j+n} = \frac{y^j}{1-y}, \tag{2.132}$$

where z is the chemical potential and $y = e^{2\pi i z}$. Including the affine descendants then gives

$$\chi_j^+(\tau, z) = \mathrm{Tr}_j^+ \left[q^{L_0 - \frac{c}{24}} y^{j_0^3} \right] = \frac{q^{-\frac{j(j-1)}{k-2} - \frac{k}{8(k-2)}} \chi_j^+(z)}{\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)}, \tag{2.133}$$

with τ the modular parameter, $q = e^{2\pi i \tau}$, and $c = \frac{3k}{k-2}$.

Let us now consider the spectrally flowed short string sector. Here the characters can be written as

$$\chi_j^{+,\omega}(\tau, z) = \mathrm{Tr}_j^+ \left[q^{L_0 - \omega j_0^3 - \frac{k}{4}\omega^2 - \frac{k}{8(k-2)}} y^{j_0^3 + \frac{k}{2}\omega} \right] = q^{-\frac{k}{4}\omega^2} y^{\frac{k}{2}\omega} \chi_j^+(\tau, z - \omega\tau), \tag{2.134}$$

which gives

$$\begin{aligned} \chi_j^{+,\omega}(\tau, z) &= \frac{q^{-\frac{j(j-1)}{k-2} - \omega j - \frac{k}{4}\omega^2 - \frac{k}{8(k-2)}} y^{j + \frac{k}{2}\omega}}{(1-yq^{-\omega}) \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n-\omega})(1-y^{-1}q^{n+\omega})} \\ &= \frac{q^{-\frac{j(j-1)}{k-2} - \omega j - \frac{k}{4}\omega^2 - \frac{k}{8(k-2)}} y^{j + \frac{k}{2}\omega}}{(1-yq^{-\omega}) \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n})(1-y^{-1}q^{n})} \prod_{n=1}^{\omega} \frac{1-y^{-1}q^n}{1-yq^{n-\omega}} \\ &= \frac{q^{-\frac{j(j-1)}{k-2} - \omega j - \frac{k}{4}\omega^2 - \frac{k}{8(k-2)}} y^{\frac{k}{2}\omega} \chi_j^+(z)}{\prod_{n=1}^{\infty}(1-q^n)(1-yq^{n})(1-y^{-1}q^{n})} \prod_{n=1}^{\omega} \frac{1-y^{-1}q^n}{1-yq^{-n}}. \end{aligned} \tag{2.135}$$

Here we recognize the overall shifts in the worldsheet weight and charge produced by spectral flow. We can also see that the last factor removes the modes associated to the action of $J^-_{-1}, \ldots, J^-_{-\omega}$ and adds those obtained by acting with $J^+_1, \ldots, J^+_\omega$, as expected. We can simplify this further using

$$\prod_{n=1}^{\omega} \frac{1-y^{-1}q^n}{1-yq^{-n}} = \prod_{n=1}^{\omega} \left(-y^{-1}q^n\right) = (-1)^\omega y^{-\omega} q^{\frac{1}{2}\omega(\omega+1)}, \tag{2.136}$$

so that

$$\chi_j^{+,\omega}(\tau, z) = \frac{q^{-\frac{1}{8} - \frac{1}{k-2}\left(j - \frac{1}{2} + \frac{k-2}{2}\omega\right)} y^{j - \frac{1}{2} + \frac{k-2}{2}\omega}}{(y^{-\frac{1}{2}} - y^{\frac{1}{2}}) \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n})(1-y^{-1}q^{n})}. \tag{2.137}$$

In order to obtain the contribution from discrete states to the torus partition function[12] we consider the diagonal combination of holomorphic and anti-holomorphic characters, integrate over the unflowed spins j and sum over the spectral flow charges ω. Remarkably, Eq. (2.137) shows that the characters depend only on the combination $t = j - \frac{1}{2} - \frac{k-2}{2}\omega$, so

[12]To be precise, here we also perform an analytic continuation both in the worldsheet and in spacetime, which allows us to take τ and z to be complex, while $\bar{\tau}$ and $\bar{z}$ are their complex conjugates.

that the integral in over $\frac{1}{2} < j < \frac{k-1}{2}$ and the sum over $\omega \in \mathbb{Z}$ combine to give an integral in t over the full real axis. This gives [13]

$$Z^{\rm disc}(\tau, z) = e^{\pi k \frac{(\mathrm{Im}\, z)^2}{\mathrm{Im}\,\tau}} |\Theta_1(\tau, z)|^{-2} \int_{-\infty}^{+\infty} dt\, e^{\frac{4\pi \mathrm{Im}\,\tau}{k-2} t^2 - 4\pi \mathrm{Im}\, z\, t} \sim \frac{e^{2\pi \frac{(\mathrm{Im}\, z)^2}{\mathrm{Im}\,\tau}}}{\sqrt{\mathrm{Im}\,\tau}\, |\Theta_1(\tau, z)|^2}, \tag{2.138}$$

where Θ_1 is the elliptic theta function. The final expression is manifestly modular invariant.

As for the continuous representations $\mathcal{C}_j^\alpha$, the (unflowed) global character reads

$$\chi_j^\alpha(z) = y^\alpha \sum_{n \in \mathbb{Z}} e^{2\pi i z n} = 2\pi y^\alpha \sum_{m \in \mathbb{Z}} \delta(z + m). \tag{2.139}$$

Since this is non-zero only when z takes an integer value, we can set $y \to 1$ in various factors of the full affine character, leading to

$$\begin{aligned} \chi_j^\alpha(\tau, z) &= \frac{q^{\frac{1+4s^2}{4(k-2)} - \frac{k}{8(k-2)}} \chi_j^\alpha(z)}{\prod_{n=1}^\infty (1-q^n)(1-yq^n)(1-y^{-1}q^n)} \\ &= 2\pi \eta^{-3}(q) q^{\frac{s^2}{k-2}} \sum_{m \in \mathbb{Z}} e^{2\pi i m \alpha} \delta(z+m). \end{aligned} \tag{2.140}$$

Upon implementing spectral flow, we obtain the physical characters

$$\chi_j^{\alpha,\omega}(\tau, z) = 2\pi \eta^{-3}(q) q^{\frac{s^2}{k-2} + \frac{k}{4}\omega^2} \sum_{m \in \mathbb{Z}} e^{2\pi i m \left(\alpha - \frac{k}{2}\omega\right)} \delta(z - \tau\omega + m). \tag{2.141}$$

We now combine the holomorphic and anti-holomorphic characters, integrate over $s \in \mathbb{R}$ and also over $\alpha \in [0, 1)$. The latter integral forces m to be the same on both sides, while the former is a gaussian integral. When the dust settles, one finds

$$Z^{\rm cont}(\tau, z) = \frac{1}{\sqrt{\mathrm{Im}\,\tau} |\eta(\tau)|^6} \sum_{\omega, m = -\infty}^{+\infty} \delta^{(2)}(z - \omega\tau + m)\,, \tag{2.142}$$

which is formally modular invariant.

We finish by discussing the interpretation of the factor $\delta^{(2)}(z - \omega\tau + m)$. If we were to perform an orbifold identification and consider strings propagating in thermal AdS$_3$ (as opposed to global AdS$_3$), the relation between the Cartan generator J_0^3 on the worldsheet and the spacetime Virasoro mode $\mathcal{L}_0$ suggests that z should be interpreted as the spacetime modular parameter, i.e. that of the *boundary* torus. The worldsheet spectrum would also include twisted sectors, labelled by two integers c and d. In this context, the long-string partition function then ends up being proportional to [14, 65]

$$\sum_{a,b,c,d \in \mathbb{Z}} \delta^{(2)}(z(c\tau + d) - a\tau - d). \tag{2.143}$$

This localizes on configurations where there is a holomorphic map between the modular parameters of the worldsheet and boundary tori, i.e.

$$z = \frac{a\tau + d}{c\tau + d}. \tag{2.144}$$

This is consistent with our classical intuition: long strings can reach the near-boundary region, but we know from (2.47) that the corresponding contribution to the path integral are highly suppressed unless γ is holomorphic.

Given that γ can be thought of as parametrizing the AdS$_3$ boundary, there should then exist a holomorphic covering map from the worldsheet to the boundary. Roughly speaking, in Eq. (2.142) we have picked up the contributions of all such maps that are in the trivial (worldsheet) twist sector $(c, d) = (0, 1)$ and wind exactly ω times around the spacetime circle. These types of maps will play a prominent role in Sec. 3 below when computing correlators involving spectrally flowed insertions.

2.5 Exact description of the unflowed sector

So far we have discussed the spectrum of the theory, using both the classical limit of the model and its the free-field description for intuition. We now present the exact description [12, 15] of the unflowed sector of the SL(2,$\mathbb{R}$) WZW model. In this sector both vertex operators and their correlators can be understood in terms by analytic continuation from the Euclidean model on $H_3^+ \sim SL(2,\mathbb{C})/SU(2)$ [47–49, 98]. We leave the computation of spectrally flowed correlators to Section 3.

Until now we have neglected the interaction term in (2.63), which is only valid at the AdS$_3$ boundary. Near the boundary we can still use perturbation theory. However, the fact that we are dealing with a WZW model allows for a much greater computational power. Indeed, one can use both conformal and SL(2,$\mathbb{R}$) Ward Identities to derive various differential equations in z and x that must be satisfied by the correlators. Moreover, the Sugawara construction also implies the well-known Knizhnik-Zamolodchikov (KZ) equation. Even though they only appear in representations that are not part of the physical spectrum, further constraints can be derived from the existence of null vectors in SL(2,$\mathbb{R}$) affine modules. Several details related the derivation of these differential equations and their consequences are reviewed in Appendix A for the SU(2) case. One can obtain further constraints by imposing crossing symmetry. In the SU(2), H_3^+ and SL(2,$\mathbb{R}$) models it turns out that this is enough to derive two- and three-point functions (on the sphere) *exactly*! The same goes for several specific higher-point functions.

In the exact theory, vertex operators $V_j(x)$ are defined by their OPEs with the currents, see Eq. (2.73). The spectrum is constructed by starting with the *macroscopic* operators of [49], i.e. those belonging to the continuous representations. Operators in discrete representations are then defined by analytic continuation to real values of j. Since the conformal weight $(k-2)\Delta = -j(j-1)$ is symmetric under the exchange $j \to 1-j$, there is direct relation between operators with spins j and $1-j$. This is known as the reflection symmetry, and takes the following form:

$$V_j(x, z) = B(j) \int d^2x' \, |x - x'|^{-4j} V_{1-j}(x', z), \tag{2.145}$$

for some constant ν. The physical meaning of the reflection coefficient

$$B(j) = \frac{(k-2)\nu^{1-2j}}{\pi\gamma(\frac{2j-1}{k-2})}, \qquad \gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-\bar{x})} \tag{2.146}$$

will become clear shortly.

The global Ward identities imply that correlation functions

$$\langle V_{j_1}(x_1,z_1)\dots V_{j_n}(x_n,z_n)\rangle \tag{2.147}$$

must be annihilated by the differential operators

$$\sum_{i=1}^n \partial_{x_i}\,,\quad \sum_{i=1}^n x_i\partial_{x_i}+j_i\,,\quad \sum_{i=1}^n x_i^2\partial_{x_i}+2j_ix_i\,, \tag{2.148}$$

and

$$\sum_{i=1}^n \partial_{z_i}\,,\quad \sum_{i=1}^n z_i\partial_{z_i}+\Delta_i\,,\quad \sum_{i=1}^n z_i^2\partial_{z_i}+2\Delta_iz_i\,. \tag{2.149}$$

This fixes the dependence of two- and three-point functions on the insertion points z_i and x_i. For the two-point function, an important subtlety is that, as a consequence of the reflection symmetry (2.145), one must consider not only the usual solution $\langle V_{j_1}V_{j_2}\rangle \sim |x_{12}|^{-4j_1}$ for $j_1 = j_2$ (and with $x_{12} = x_1 - x_2$), but also a second, distributional solution of the form $\delta^{(2)}(x_{12})$, valid when $j_2 = 1 - j_1$. This comes from the distributional identity $x\delta'(x) = -\delta(x)$. More precisely, the two-point function reads

$$\langle V_{j_1}(x_1,z_1)V_{j_2}(x_2,z_2)\rangle = \frac{1}{|z_{12}|^{4\Delta_1}}\left[\delta^{(2)}(x_1-x_2)\delta(j_1+j_2-1)+B(j_1)\frac{\delta(j_1-j_2)}{|x_{12}|^{4j_1}}\right]\,. \tag{2.150}$$

These two terms match precisely what we would expect from the product of two of the asymptotic expressions given in Eq. (2.39). Indeed, for large k we have

$$B(j) = \nu^{1-2j}\frac{2j-1}{\pi}\frac{\Gamma(1-\frac{2j-1}{k-2})}{\Gamma(1+\frac{2j-1}{k-2})} \approx \frac{2j-1}{\pi} \tag{2.151}$$

as long as $\nu(k\to\infty)\to 1$, which reproduces the relative coefficient in (2.39). The exact value of ν is not fixed by internal consistency of the WZW model and can be chosen conveniently. In [49] this was taken to be

$$\nu = (k-2)\gamma\left(\frac{k-1}{k-2}\right)\,, \tag{2.152}$$

which effectively trivializes $B(j=1)$ for all k. On the other hand, three-point functions take the form

$$\langle V_{j_1}(x_1,z_1)V_{j_2}(x_2,z_2)V_{j_3}(x_3,z_3)\rangle = C(j_1,j_2,j_3)\left|\frac{x_{12}^{j_3-j_1-j_2}x_{23}^{j_1-j_2-j_3}x_{13}^{j_2-j_1-j_3}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3}z_{23}^{\Delta_2+\Delta_3-\Delta_1}z_{13}^{\Delta_1+\Delta_3-\Delta_2}}\right|^2\,. \tag{2.153}$$

The structure constants $C(j_1,j_2,j_3)$ were computed exactly in [48]. This was done by combining the constraints coming from null vectors belonging to representations with [53]

$$j=\frac{1}{2}\left[r+s(k-2)\right]\quad \text{where}\quad \begin{cases} r\le 0\,,\ s\le 0\,,\\ r>2\,,\ s>0\,. \end{cases} \tag{2.154}$$

with consistency conditions associated with crossing symmetry. This method closely follows the analogous computation performed in the context of Liouville theory in [85, 86]. These CFTs share a number of important features: Liouville theory is the simplest non-compact model with a Virasoro symmetry, while the H_3^+ model is the simplest one with a non-abelian Kac-Moody current algebra. As shown in [98], H_3^+ correlators can be derived from Liouville correlators. At the path integral level, this can be understood as follows: working with Wakimoto variables (2.63) and integrate out the $\beta\gamma$-system explicitly, thus leaving an effective action Liouville-type action for the non-compact scalar field ϕ, albeit with a modified background charge [99].

The computation of $C(j_1, j_2, j_3)$ is beyond the scope of these notes; see [89, 90] for a pedagogical derivation. The final result (known as the DOZZ formula for the Liouville case) takes the form [15]

$$C(j_1, j_2, j_3) = -\frac{G(1-J)}{2\pi^2 \nu^{J-1} \gamma\left(\frac{k-1}{k-2}\right) G(-1)} \prod_{i=1}^{3} \frac{G(2j_i - J)}{G(1-2j_i)}, \tag{2.155}$$

with $J = j_1 + j_2 + j_3$, where $G(j)$ is defined in Barnes double Gamma function. It is related to the function originally used in [86] by $G(j) = (k-2)^{-(k-2)^{-1} j(k-1+j)} \Upsilon^{-1}(-(k-2)^{-1/2} j)$. The properties of the function $G(j)$ that will be relevant for us are the following:

- For generic values of k, $G(j)$ has simple poles at

$$j = n + m(k-2) \quad \text{where} \quad \begin{cases} n \geq 0\,,\; m \geq 0\,, \\ n \leq -1\,,\; m \leq -1\,. \end{cases} \tag{2.156}$$

  When k is integer some of these become double poles.

- $G(j)$ satisfies the shift identities

$$G(j+1) = \gamma\left(-\frac{j+1}{k-2}\right) G(j)\,, \qquad G(j-(k-2)) = \frac{\gamma(j+1)}{(k-2)^{2j+1}} G(j)\,. \tag{2.157}$$

As a check, we can recover the two-function (2.150) by taking one of the insertions to be the identity. This is done by setting $j_3 = \varepsilon$ and carefully taking the limit $\varepsilon \to 0$. The x-dependence of the second term is easily obtained. In general, the structure constant vanishes for $j_3 \sim \varepsilon \to 0$ due to the factor $G(1-2j_3)$ in the denominator, although for $j_1 = j_2$ two extra terms in the numerator diverge as well. More precisely, we have

$$\begin{aligned} \frac{G(2j_1 - J)G(2j_2 - J)}{G(-1)G(1-2j_3)} &= \frac{G(j_{12} - \varepsilon)G(-j_{12} - \varepsilon)}{G(-1)G(1-2\varepsilon)} \\ = \frac{G(-1 + j_{12} - \varepsilon)G(-1 - j_{12} - \varepsilon)}{G(-1)G(-1-2\varepsilon)} & \frac{\gamma\left(\frac{\varepsilon + j_{12}}{k-2}\right)\gamma\left(\frac{\varepsilon - j_{12}}{k-2}\right)}{\gamma\left(\frac{2\varepsilon - 1}{k-2}\right)\gamma\left(\frac{2\varepsilon}{k-2}\right)} \\ \approx (k-2)\gamma\left(\frac{k-1}{k-2}\right)\frac{2\varepsilon}{\varepsilon^2 - j_{12}^2} &\to 2\pi(k-2)\gamma\left(\frac{k-1}{k-2}\right)\delta(j_{12})\,, \end{aligned} \tag{2.158}$$

with $j_{12} = j_1 - j_2$. The contact term is a bit more subtle. It can be picked up by noting that $x_{12}^{j_3-j_1-j_2}$ is meromorphic in j_3 *except* near $j_3 = j_1 + j_2 - 1$ when $x_1 = x_2$. This translates into the distributional identity [49, 100]

$$|x_{12}|^{2(\varepsilon-j_1-j_2)} \sim \frac{\pi\delta^{(2)}(x_{12})}{1-j_1-j_2+\varepsilon}\,, \tag{2.159}$$

which, in combination with the structure constant (which is regular), leads to the first term in (2.150).

From the three-point function one can derive the following (primary) OPE between vertex operators in continuous representations:

$$\begin{aligned} &V_{j_1}(x_1,z_1)V_{j_2}(x_2,z_2) \\ &= \int_{\frac{1}{2}+i\mathbb{R}_+} \frac{dj_3}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}} \int_{\mathbb{C}} d^2x_3 \frac{C(j_1,j_2,j_3)}{|x_{12}^{j_1+j_2-j_3}x_{23}^{j_2+j_3-j_1}x_{31}^{j_3+j_1-j_2}|^2} V_{1-j_3}(x_3,z_3)\,. \end{aligned} \tag{2.160}$$

The integrand is actually symmetric under $j_3 \to 1 - j_3$, as follows from Eqs. (2.145) and (2.42), which allows one to extend the integration region to the full line $j_3 = \frac{1}{2} + i\mathbb{R}$. Strictly speaking, Eq. (2.160) is valid only for spins satisfying [49]

$$|\text{Re}\;(j_1-j_2)\,| < \frac{1}{2} \quad \text{and} \quad |\text{Re}\;(j_1+j_2-1)\,| < \frac{1}{2}\,. \tag{2.161}$$

Nevertheless, one can use it compute the OPE for real values of j_1 and j_2 by analytic continuation. One must take into account, however, that the integrand has many poles in j_3, and as we move j_1 and j_2 some of these poles might cross the contour. When this happens, the OPE picks up a discrete set of additional contributions from the corresponding residues. Let us stress that these poles come not only from the structure constants but also from the powers of x_{ij}, in a similar fashion as what we saw above.

In special cases, more precisely when j_1 and/or j_2 belong to the set of degenerate representations of Eq. (2.154), these manipulations can lead to situations where the contribution from the integral vanishes exactly. In such cases the OPE becomes a sum over a finite set of contributions coming from the relevant residues. This allows one to derive the structure constants and operator product expansions for related models with finite, discrete spectra, such as minimal models and the SU(2) WZW model, see Eq. (A.57).

The full set of contributions to the $V_{j_1}V_{j_2}$ OPE is described in detail in [49]. For reasons that will become clear shortly, a particularly important case to consider that with $j_1 = 1$ and $j_2 = j$ in the physical range for discrete series in AdS$_3$, see (2.93). In the limit $z_{12} \to 0$, the leading term is regular and comes from the pole at $j_3 = j_1 + j_2 - 1$, i.e. exactly that of Eq. (2.159). This pole sits at $\text{Re}\, j_3 = 0$ when $\text{Re}\, j_1 = \text{Re}\, j_2 = \frac{1}{2}$, but for $j_1 = 1$ and $j_2 = j > \frac{1}{2}$ it moves to $\text{Re}\, j_3 > \frac{1}{2}$, thus crossing the integration line in the process. This leads to

$$V_1(x_1,z_1)V_j(x_2,z_2) \sim \int_{\mathbb{C}} d^2x_3\, \delta^{(2)}(x_{12})|x_{23}|^{2(2j+1)}|x_{31}|^{-2}V_{1-j}(x_3,z_2) = \delta^{(2)}(x_{12})V_j(x_2,z_2)\,, \tag{2.162}$$

where we have used (2.145). (The fact that we have a unit coefficient on the RHS follows from the choice in Eq. (2.152).) This OPE can be checked by considering the semi-classical expressions (2.39) near the boundary :

$$V_1(x,\bar{x})V_j(y,\bar{y}) \sim \delta^{(2)}(\gamma - x)e^{2(j-1)\phi}\delta^{(2)}(\gamma - y) \sim \delta^{(2)}(x-y)V_j(y,\bar{y}). \tag{2.163}$$

Finally, we can also consider the four-point functions of the H_3^+ WZW model. As usual, global conformal invariance implies

$$\langle V_{j_1}(x_1,z_1)V_{j_2}(x_2,z_2)V_{j_3}(x_3,z_3)V_{j_4}(x_4,z_4)\rangle = |Z[z_i,\Delta_i]X[x_i,j_i]|^2|\mathcal{F}(x,z)|^2\,, \tag{2.164}$$

where

$$Z[z_i,\Delta_i] = z_{21}^{-\Delta_1-\Delta_2+\Delta_3-\Delta_4}z_{31}^{-\Delta_1+\Delta_2-\Delta_3+\Delta_4}z_{32}^{\Delta_1-\Delta_2-\Delta_3+\Delta_4}z_{34}^{-2\Delta_4}\,, \tag{2.165}$$

$$X[x_i,j_i] = x_{21}^{-j_1-j_2+j_3-j_4}x_{31}^{-j_1+j_2-j_3+j_4}x_{32}^{j_1-j_2+j_3-j_4}x_{34}^{-2j_4}\,, \tag{2.166}$$

while the worldsheet and spacetime cross-ratios are defined as

$$z = \frac{z_{32}z_{14}}{z_{12}z_{34}}\,, \qquad x = \frac{x_{32}x_{14}}{x_{12}x_{34}}\,. \tag{2.167}$$

The function $\mathcal{F}(x,z)$ is the holomorphic conformal block. It is constrained by the KZ equation (A.37), which in the x-basis mixes differential operators in the variables z and x. This reads [49]

$$\partial_z\mathcal{F} = \frac{1}{k-2}\left(\frac{P_x}{z}+\frac{Q_x}{z-1}\right)\,, \tag{2.168}$$

with **NK:** *check signs*

$$P_x = x^2(x-1)\partial_x^2 - \left[(\kappa-1)x+2j_1-2j_4(x-1)\right]x\partial_x - 2\kappa j_4 x - 2j_1j_4 \tag{2.169}$$

and

$$Q_x = -x(x-1)^2\partial_x^2 + \left[(\kappa-1)(x-1)-2j_2-2j_4x\right](x-1)\partial_x + 2\kappa j_4(x-1) - 2j_2j_4, \tag{2.170}$$

where $\kappa = j_3 - j_1 - j_2 - j_4$. The conformal block is not known in closed form. Nevertheless, by inserting the OPE (2.160) one obtains a series expansion in powers of z related to the factorization of the four-point function in the chanel $14 \to 23$. More precisely, one has

$$|\mathcal{F}(x,z)| = \int_{\frac{1}{2}+i\mathbb{R}} dj\,\frac{C(j_1,j_4,j)C(j,j_2,j_3)}{B(j)}|\mathcal{F}_j(x,z)|^2 \tag{2.171}$$

where

$$\mathcal{F}_j(x,z) = z^{\Delta_j-\Delta_1-\Delta_4}x^{j-j_1-j_4}\sum_{n=0}^{\infty}f_{j,n}(x)z^n\,. \tag{2.172}$$

The KZ equation then implies that the zero-th order solution corresponds to the hypergeometric function

$$f_{j,0}(x) = {}_2F_1(j-j_1+j_2, j+j_3-j_4, 2j; x)\,, \tag{2.173}$$

suitably combined with a second solution corresponding to the exchange $j \to 1 - j$ in order to give a monodromy-invariant result [15]. The rest of the functions $f_{j,n}(x)$ can in principle be determined iteratively from the KZ equation as well. As it comes from the OPE (2.160), this expansion is strictly speaking only valid for external spins in the range (2.161) (replacing $j_2 \to j_4$), while a similar restriction holds for j_2 and j_3.

For other values of j_i one must proceed by analytic continuation. This is necessary in order to compute, for instance, the correlation funciton of four unflowed states in the discrete representations of AdS$_3$, which have $j_i \in \mathbb{R}$. As discussed for the OPE expansion, the procedure is much more interesting than it sounds. It was famously shown in [15] by studying the factorization limit of such four-point functions that the s-channel includes contributions associated with the propagation long string states with $\omega = 1$. The precise expressions are consistent with the three-point functions which violate spectral flow conservation, to be discussed in Sec. 3 below. This shows, once again, that the AdS$_3$ model would be inconsistent without the inclusion of spectrally flowed states.

Exact derivation of spacetime symmetry generators

Let us now go back to string theory on AdS$_3 \times M_{\rm int}$. A generic physical vertex operator takes the form

$$\Phi = P(J,\dots)\bar{P}(\bar{J},\dots)V_j(x,\bar{x},z,\bar{z})V_{hint}(z,\bar{z})\,, \tag{2.174}$$

where P and $\bar{P}$ are polynomials with overall scaling degree N and $\bar{N}$, such that the physical state conditions read

$$-\frac{j(j-1)}{k-2} + h_{\rm int} - 1 + N = -\frac{j(j-1)}{k-2} + \bar{h}_{\rm int} - 1 + \bar{N} = 0. \tag{2.175}$$

We should be able to build the spacetime chiral and Virasoro algebras without relying on the free-field approximation. Moreover, the construction should be independent of the details of the internal geometry.

The key ingredient is given by the SL(2,$\mathbb{R}$) vertex operator with $j = 1$. This is a $(h,\bar{h}) = (1,1)$ tensor in spacetime, and it further has vanishing worldsheet weight Δ. Although V_1 is not the worldsheet identity, i.e. the operator with $j = 0$, it is related to it through the analytic continuation of the H_3^+ reflection formula (2.145) away from $j \in \frac{1}{2}+i\mathbb{R}$, namely $V_0(x,z) = V_0(0,z) = B(0)\int d^2x'\, V_1(x',z)$. Using the operator V_1 allows us to solve the Virasoro conditions simply by setting $h_{\rm int} = \bar{h}_{\rm int} = 0$ and $N = \bar{N} = 1$. The proposal for the worldsheet avatars of the local operators generating the (holomorphic) spacetime symmetry algebra are then as follows [12]. For the chiral currents and the operator $\mathcal{I}$ we have

$$\mathcal{I} = \frac{1}{(k-2)^2}\int d^2z\, \left(J\bar{J}V_1\right)(x,\bar{x},z,\bar{z})\,, \tag{2.176}$$

with $\left(J\bar{J}V_1\right)$ stands for the normal-ordered product of $J^-(x,z)\bar{J}^-(\bar{x},\bar{z})V_1(x,\bar{x},z,\bar{z})$, and

$$\mathcal{K}^a(x) = -\frac{1}{k-2}\int d^2z\, K^a(z)\left(\bar{J}V_1\right)(x,\bar{x},z,\bar{z})\,, \tag{2.177}$$

while the spacetime energy momentum tensor is

$$\mathcal{T}(x) = \frac{1}{2(k-2)} \int d^2 z \, \left(\left[(\partial_x J) \partial_x + 2(\partial_x^2 J) \right] \bar{J} V_1 \right) (x, \bar{x}, z, \bar{z}) \, . \tag{2.178}$$

The SL(2,$\mathbb{R}$) currents are inserted so that $(J\bar{J}V_1)$ and $(\bar{J}V_1)$ transform as operators of spin (0,0) and (1,0) under the action of the zero-mode algebra. We now show that these operators satisfy the expected OPEs in spacetime, namely

$$\mathcal{K}^a(x)\mathcal{K}^b(y) \sim \frac{\delta^{ab} k' \mathcal{I}}{2(x-y)^2} + \frac{i f^{ab}{}_c K^c(y)}{x-y} \, , \tag{2.179a}$$

$$\mathcal{T}(x)\mathcal{T}(y) \sim \frac{c_{\text{st}}^{(1)} \mathcal{I}}{2(x-y)^4} + \frac{2T(y)}{(x-y)^2} + \frac{\partial_y T(y)}{x-y} , \tag{2.179b}$$

with $c_{\text{st}}^{(1)} = 6k$ and $\mathcal{I}$ acting as the spacetime identity up to a constant, at least when working in the unflowed sector of the theory, and

$$\mathcal{K}^a(x)\mathcal{V}_h(y,\bar{y}) \sim \frac{t^a \mathcal{V}_h(y,\bar{y})}{(x-y)} \, , \tag{2.180a}$$

$$\mathcal{T}(x)\mathcal{V}_h(y,\bar{y}) \sim \frac{h\mathcal{V}_h(y,\bar{y})}{(x-y)^2} + \frac{\partial_y \mathcal{V}_h(y,\bar{y})}{(x-y)} \, . \tag{2.180b}$$

Here the vertex operators are $\mathcal{V}_j(x,\bar{x}) = \int d^2 z \, V_j(x,\bar{x},z,\bar{z}) V_{\text{int}}(z,\bar{z})$.

The main obstacle technical obstacle is that the definitions in Eqs. (2.177) and (2.178) are given in terms of integrals in z over the full complex plane, as opposed to contour integrals. In order to bypass this issue, we use the following trick. Since we are only interested in the singular terms, the relevant information is still available upon applying $\partial_{\bar{x}}$ on both sides of all OPEs in Eqs. (2.179) and (2.180), and using the distributional identity

$$\partial_{\bar{x}} \left[(x-y)^{-1} \right] = \pi \delta^{(2)}(x-y) \, . \tag{2.181}$$

Since $\mathcal{K}^a(x)$ and $\mathcal{T}(x)$ are supposed to be holomorphic only the contact terms should survive. On the worldsheet side, we use the the following identity for V_1:

$$\partial_{\bar{x}} \left(\bar{J}^- V_1 \right) (x,\bar{x},z,\bar{z}) = (k-2) \, \partial_{\bar{z}} V_1(x,\bar{x},z,\bar{z}) \, , \tag{2.182}$$

which is a direct consequence of the Sugawara construction. This leads to

$$\partial_{\bar{x}} \mathcal{K}^a(x) = \oint_{\mathcal{C}} dz \, K^a(z) V_1(x,\bar{x},z,\bar{z}) \, , \tag{2.183}$$

and

$$\partial_{\bar{x}} \mathcal{T}(x) = \frac{1}{2} \oint_{\mathcal{C}} dz \, \left[(\partial_x J^-) \partial_x + 2(\partial_x^2 J^-) \right] V_1(x,\bar{x},z,\bar{z}) \, , \tag{2.184}$$

which hold inside generic correlation functions. Indeed, the integrands in the original definitions are total derivatives *up to* contact terms from the (derivatives of the) OPEs with other vertex operators in the correlator, hence the integration contour $\mathcal{C}$ should be understood as a series of small curves surrounding those additional insertions. For instance, for a given correlator we get for example

$$\langle \partial_{\bar{x}} \mathcal{K}^a(x)\mathcal{V}(y,\bar{y})\dots\rangle \sim \int d^2w \oint_w dz \langle V_1(x,\bar{x},z,\bar{z})V_j(y,\bar{y},w,\bar{w})K^a(z)W_{\text{int}}(w,\bar{w})\dots\rangle\,, \tag{2.185}$$

where we have focus on the contribution from a small circle around the first additional insertion. A key property of the operator V_1 is the OPE obtained in Eq. (2.162). As a direct consequence of this, (2.185) becomes

$$\langle \partial_{\bar{x}} \mathcal{K}^a(x)\mathcal{V}(y,\bar{y})\dots\rangle \sim \pi\delta^{(2)}(x-y)t^a\langle \mathcal{V}(y,\bar{y})\dots\rangle\,, \tag{2.186}$$

which confirms (2.180a). Similarly, we can start with

$$\langle \partial_{\bar{x}} \mathcal{K}^a(x)\mathcal{K}^b(y)\dots\rangle \sim \int d^2w \oint_w dz \langle K^a(w)V_1(x,\bar{x},z,\bar{z})K^b(z)\left(\bar{J}V_1\right)(y,\bar{y},w,\bar{w})\dots\rangle\,. \tag{2.187}$$

This contains single-pole and double-pole contributions from the OPE of the worldsheet currents. Proceeding as before and using

$$\begin{aligned}(k-2)\lim_{z\to w}\partial_z V_1(x,\bar{x},z,\bar{z})\left(\bar{J}V_1\right)(y,\bar{y},w,\bar{w}) &= \lim_{z\to w}\partial_x\left(JV_1\right)(x,\bar{x},z,\bar{z})\left(\bar{J}V_1\right)(y,\bar{y},w,\bar{w})\\ &= \partial_x\delta^{(2)}(x-y)(J\bar{J}V_1)(y,\bar{y},w,\bar{w}) \end{aligned}\tag{2.188}$$

for the double pole residue, we get

$$\langle \partial_{\bar{x}} K^a(x)K^b(y)\dots\rangle \sim \pi\langle\left[\delta^{(2)}(x-y)if^{ab}{}_cK^c(y)+\partial_x\delta^{(2)}(x-y)(k'/2)\delta^{ab}\mathcal{I}\right]\dots\rangle\,, \tag{2.189}$$

as expected. A similar set of computations with $\mathcal{T}(x)$ confirms the rest of the spacetime symmetry algebra. For a more detailed study of the properties of $\mathcal{I}$ see [12].

As a sanity check, we now make contact with the free field description. At large ϕ, we combine the currents of Eq. (2.49) into

$$\bar{J}(\bar{x},\bar{z}) \approx k\left[(\bar{\gamma}-\bar{x})^2e^{2\phi}\bar{\partial}\gamma - 2(\bar{\gamma}-\bar{x})\bar{\partial}\phi - \bar{\partial}\bar{\gamma}\right]\,. \tag{2.190}$$

Starting from (2.177) and using $\Phi_1 \approx \delta^{(2)}(x-\gamma)$ and that γ is holomorphic near the boundary, we obtain

$$\mathcal{K}^a(x) = \int d^2z\, K^a(z)\bar{\partial}\bar{\gamma}\,\delta^{(2)}(x-\gamma) = \int d^2z\, K^a(z)\bar{\partial}\left(\frac{1}{x-\gamma}\right) = \oint dz\,\frac{K^a(z)}{x-\gamma}\,, \tag{2.191}$$

so that, upon expanding in powers of x/γ we find

$$\mathcal{K}^a(x) = \sum_n x^{-n-1}\oint dz\, K^a(z)\gamma^n\,, \tag{2.192}$$

thus reproducing (2.85). Similarly, for the operator $\mathcal{I}$ we have

$$\mathcal{I} = \int d^2z\,\partial\gamma\bar{\partial}\bar{\gamma}\,\delta^{(2)}(x-\gamma) = \oint dz\,\frac{\partial\gamma}{x-\gamma} = \sum_n x^{-n}\oint dz\,\gamma^{n-1}\partial\gamma = \oint dz\,\gamma^{-1}\partial\gamma\,. \tag{2.193}$$

Finally, for the case of $\mathcal{T}(x)$ we have

$$\mathcal{T}(x) = \frac{1}{2}\int d^2z \, \left[(\partial_x J^-)\partial_x + 2(\partial_x^2 J^-)\right] \bar{\partial}\left(\frac{1}{x-\gamma}\right) = \oint dz \, \left[\frac{J^3(x,z)}{(x-\gamma)^2} + \frac{2J^+(z)}{x-\gamma}\right] , \tag{2.194}$$

where in the last step we have used that $x\delta'(x) = -\delta(x)$. This leads to

$$\mathcal{T}(x) = \sum_n x^{-n-2} \oint dz \, \left[(n+1)\gamma^n J^3(z) - n\gamma^{n+1} J^+(z)\right] , \tag{2.195}$$

in accordance with Eq. (2.78).

3 Correlation functions and spectral flow

In this section we study correlators with spectrally flowed primary insertions. We start by considering a small number of particular cases by using m-basis methods and discuss the corresponding the spectral flow violation rules [15, 40, 50, 101]. We then describe the constraints imposed by the so-called local Ward identities, which ultimately imply a set of recursion relations between primary correlators. We then introduce the y-transform, the analogue of the x-basis for spectrally flowed operators [51]. This allows us to recast the recursion relations in terms of differential equations. The latter can be solved in full generality for three-point functions [54]. Finally, we state the conjectured solution for four-point functions put forward in [53].

3.1 m-basis flowed correlators and parafermions

The SL(2,ℝ) WZW model is believed to be exactly solvable. By this, we mean that it should be possible to compute all the relevant CFT data, namely the spectrum, the OPE coefficients (or equivalently the three-point functions), and the dynamics as captured by the four-point functions, which ought to be crossing symmetric. One of the main interests of these notes is to describe the state-of-the-art in what has, for a long time, delayed the possibility of saying it *is* actually solved: the computation of correlation functions involving spectrally flowed insertions.

For pedagogical reasons, we choose to follow the historical timeline, and first discuss the original computations of [15, 50], see also [101–105]. We thus focus on the first non-trivial cases for two- and three-point functions. The main tool comes from the parafermionic so-called decomposition, where one factorizes the SL(2,ℝ) fields into an operator belonging to the SL(2,ℝ)/U(1) coset model, and an exponential operator associated to the remaining U(1) factor. The former is known as the (semi-infinite) *cigar* theory due to shape of the resulting geometry . It actually corresponds to the euclidean version of the two-dimensional black-hole, see [19]. At the level unflowed vertex operators, this decomposition reads

$$V_{jm\bar{m}} = \Psi_{jm} e^{m\sqrt{\frac{2}{k}}\varphi} \,, \tag{3.1}$$

where, as usual, we have omitted the anti-holomorphic components and labels. Here φ is a canonically normalized (holomorphic) scalar which bosonizes the Cartan current, i.e.

$$J^3(z) = -\sqrt{\frac{k}{2}}\partial\varphi(z), \qquad \varphi(z)\varphi(w) \sim -\log(z-w), \tag{3.2}$$

while $\Psi_{jm\bar{m}}$ is a coset operator satisfying

$$\Delta[\Psi_{jm\bar{m}}] = -\frac{j(j-1)}{k-2} + \frac{m^2}{k} \,. \tag{3.3}$$

This is known as a parafermion: its OPEs with the currents contain fractional powers in $\mathbb{Z} + \frac{2}{k}$. In geometric terms, one can interpret the quantum numbers of $\Psi_{jm\bar{m}}$ as follows. The momentum along the asymptotic circle (which has radius $\sqrt{k}$) is given by $m - \bar{m} \in \mathbb{Z}$, while the winding corresponds to $m + \bar{m} \in k\mathbb{Z}$. While the former is conserved, there is no reason to expect that winding is conserved as well since the circle is contractible [50]. Similarly, the currents factorize as $J^{\pm} = \sqrt{k}\,\psi^{\pm} e^{\pm\sqrt{\frac{k}{2}}\varphi}$ since they have charge ± 1, where $\Delta[\psi^{\pm}] = 1 + \frac{1}{k}$.

The coset decomposition is quite useful for describing spectrally flowed (m-basis) primary states. Indeed, one can easily check that the expression

$$V^{\omega}_{jm}(z) = \Psi_{jm}(z) e^{\left(m+\frac{k}{2}\omega\right)\sqrt{\frac{2}{k}}\,\varphi(z)} \tag{3.4}$$

satisfies all the defining properties of the operator on the LHS, namely its worldsheet weight and OPEs with the SL(2,$\mathbb{R}$) currents. Importantly, the parafermionic sector remains untouched.

Let us use this to compute the x-basis two-point functions with spectral flow. As usual, the global Ward identities imply that[13]

$$\langle V^{\omega_1}_{j_1 h_1}(x_1, z_1) V^{\omega_2}_{j_2 h_2}(x_2, z_2)\rangle = \frac{\delta^{(2)}(h_1 - h_2) D(h_i, j_i, \omega_i)}{|x_{12}|^{4h_1} |z_{12}|^{4\Delta_1}}\,, \tag{3.5}$$

and can only be non-vanishing for either $j_2 = j_1$ or $j_2 = 1 - j_1$. Our goal is to compute the structure constant $D(h_i, j_i, \omega_i)$. For this, we can take $x_1 = z_1 = 0$ and $x_2 = z_2 = \infty$, in which case the x-basis flowed operators become m-basis flowed primaries. More explicitly, we have

$$\begin{aligned}
\delta^{(2)}(h_1 - h_2) D(h_1, j_i, \omega_i) &= \langle V^{\omega_1}_{j_1 h_1}(0,0) V^{\omega_2}_{j_2 h_2}(\infty,\infty)\rangle = \langle V^{\omega_1}_{j_1 m_1}(0) V^{-\omega_2}_{j_2, -m_2}(\infty)\rangle \\
&= \langle \Psi_{j_1 m_1}(0) \Psi_{j_2,-m_2}(\infty)\rangle \langle e^{\left(m_1+\frac{k}{2}\omega_1\right)\sqrt{\frac{2}{k}}\,\varphi(0)} e^{-\left(m_2+\frac{k}{2}\omega_2\right)\sqrt{\frac{2}{k}}\,\varphi(\infty)}\rangle\,,
\end{aligned} \tag{3.6}$$

with $m_i = h_i - \frac{k}{2}\omega_i$, and where we have made use of (3.4). The point is that the parafermionic correlator is *the same* as the one appearing in the unflowed two-point function, which in particular imposes $m_1 = m_2$. Since the z-dependence has been stripped off, the only role of the free-field correlator is to impose the additional charge conservation condition $\omega_1 = \omega_2$, consistent with the requirement $h_1 = h_2$. (Had we not set the z-dependence beforehand, the free-field factors would have provided the additional terms in the weights as compared to the unflowed operators, namely $\Delta_i - \Delta_i(\omega_i = 0) = -m_i\omega_i - \frac{k}{4}\omega_i^2$.) Consequently, by means of (2.41) we can transform the unflowed two-point function (2.150) to the m-basis, and read off the desired result, which gives [15]

$$\begin{aligned}
&\langle V^{\omega_1}_{j_1 h_1}(x_1, z_1) V^{\omega_2}_{j_2 h_2}(x_2, z_2)\rangle = \\
&\qquad = \frac{\delta_{\omega_1,\omega_2}\delta^{(2)}(h_1 - h_2)}{|z_{12}|^{4\Delta_1}|x_{12}|^{4h_1}}\left[\delta(j_1 + j_2 - 1) + \frac{\pi\delta(j_1 - j_2) B(j_1)\gamma(j_1 + m_1)}{\gamma(2j_1)\gamma(1 - j_1 + m_1)}\right].
\end{aligned} \tag{3.7}$$

The computation for three-point functions which conserve spectral flow, i.e. those for which $\omega_1 + \omega_2 = \omega_3$, is completely analogous[14]. The result is not very illuminating, so we will

[13] In principle, the spacetime Ward identities allow for a contact term with $h_1 \neq h_2$, but this clashes with the worldsheet Ward identities [51].

[14] When all three ω_i are non-zero charge conservation enforces an additional constraint on the j_i.

not write it here explicitly. It can be constructed from the unflowed m-basis three-point function, which were computed in [106] in terms of hypergeometric functions ${}_3F_2$.

Are we done? If the spectral flow charge was a conserved quantity, the answer would be yes. However, this need not be the case. Consider the simplest possible example, namely the three-point function with $\boldsymbol{\omega} = (1,0,0)$:

$$\langle V^1_{j_1h_1}(0,0)V_{j_2}(x_2,1)V_{j_3}(\infty,\infty)\rangle = \langle V^1_{j_1m_1}(0)V_{j_2}(1,1)V_{j_3,-j_3}(\infty)\rangle \,. \tag{3.8}$$

Here the vertex operator in the middle is still in the x-basis. Indeed, when inserted at a generic point, say $x_2 = 1$, i.e. away from $x_2 = 0$ or $x_2 = \infty$, this is a linear combination of an infinite number of m-basis operators, see Eq. (2.71). Charge conservation implies that only a mode with $m_2 = m_3 - m_1 - \frac{k}{2}$ can contribute, giving

$$\langle \Psi_{j_1m_1}(0)\Psi_{j_2,m_2}(1)\Psi_{j_3,-m_3}(\infty)\rangle\langle e^{\left(m_1+\frac{k}{2}\right)\sqrt{\frac{2}{k}}\,\varphi(0)}e^{m_2\sqrt{\frac{2}{k}}\,\varphi(1)}e^{-m_3\sqrt{\frac{2}{k}}\,\varphi(\infty)}\rangle \,.$$

An analogous charge conservation condition holds in the anti-holomorphic sector, which we have omitted. Hence, the parafermion correlator is unlike any appearing in the unflowed sector: momentum is conserved, as it should, but the total winding is not (by one unit). As argued above, we do not expect this to vanish.

In order to compute it, we take advantage of the fact that, as can be infered from the parafermionic decomposition, the state with $\omega = 1$ can be constructed explicitly from the fusion of unflowed operators. More explicitly, we have

$$\lim_{z'\to z}(z-z')^m V_{jm}(z')V_{\frac{k}{2}\frac{k}{2}}(z) = V^{\omega=1}_{jm}(z)\,. \tag{3.9}$$

For this reason,

$$V_{\frac{k}{2}\frac{k}{2}}(z) = e^{\sqrt{\frac{k}{2}}\varphi} \tag{3.10}$$

is known as the *spectral flow operator*. Even though it is outside of the physical range (2.93), this is a very interesting operator for the following reasons:

- its parafermionic component $\Psi_{\frac{k}{2}\frac{k}{2}}$ is simply the identity, as can be seen from Eq. (3.3),
- it has a null descendent at level 1, namely

$$J^-_{-1}\Big|\frac{k}{2},\frac{k}{2}\Big\rangle = 0\,, \tag{3.11}$$

  as discussed around Eq. (2.92), and
- the Virasoro mode L_{-1} acts very simply as

$$L_{-1}\Big|\frac{k}{2},\frac{k}{2}\Big\rangle = -J^3_{-1}\Big|\frac{k}{2},\frac{k}{2}\Big\rangle. \tag{3.12}$$

This can be extended to arbitrary ω [52]. In the x-basis, the generalization of Eq. (3.9) to arbitrary $\omega > 0$ gives

$$V^\omega_{jh}(x,z) = \lim_{\varepsilon,\bar{\varepsilon}\to 0}\varepsilon^{m\omega}\bar{\varepsilon}^{\bar{m}\omega}\int d^2y\, y^{j-m-1}\bar{y}^{j-\bar{m}-1}V_j(x+y,z+\varepsilon)V^{\omega-1}_{\frac{k}{2}\frac{k}{2}\omega}(x,z). \tag{3.13}$$

To understand how this works, we can for instance evaluate it at $x=0$, giving

$$\begin{aligned} V^{\omega}_{jh}(0,z) &= \lim_{\varepsilon,\bar{\varepsilon}\to 0} \varepsilon^{m\omega}\bar{\varepsilon}^{\bar{m}\omega} \int d^2y\, y^{j-m-1}\bar{y}^{j-\bar{m}-1} V_j(y,z+\varepsilon) V^{\omega-1}_{\frac{k}{2},\frac{k}{2}\omega}(0,z) \\ &= \lim_{\varepsilon,\bar{\varepsilon}\to 0} \varepsilon^{m\omega}\bar{\varepsilon}^{\bar{m}\omega} V_{jm}(z+\varepsilon) V^{\omega-1}_{\frac{k}{2}\frac{k}{2}}(z) = V^{\omega}_{jm}(z)\,, \end{aligned} \tag{3.14}$$

as expected. Another simple check comes from studying the action of the zero-mode currents on (3.13). For J_0^+, this gives ∂_y when acting on the spectrally unflowed vertex $V_j(x+y,z+\varepsilon)$ in the integrand, while on the spectral flow operator it acts as $\partial_x - \partial_y$, such that the total action is characterized by ∂_x. The action of $J_0^3(x) \equiv J_0^3 - xJ_0^+$, is slightly more interesting to derive. Using $J_0^3(x) = J_0^3(x+y) + yJ_0^+$, we get

$$\begin{aligned} \left(J_0^3 V^{\omega}_{jh}\right)(x,z) &= \lim_{\varepsilon,\bar{\varepsilon}\to 0} \varepsilon^{m\omega}\bar{\varepsilon}^{\bar{m}\omega} \int d^2y\, y^{j-m-1}\bar{y}^{j-\bar{m}-1} \times \\ &\qquad \left(j + y\partial_y + \frac{k}{2}\omega\right) V_j(x+y,z+\varepsilon) V^{\omega-1}_{\frac{k}{2},\frac{k}{2}\omega}(x,z) \\ &= \left(m + \frac{k}{2}\omega\right) V^{\omega}_{jh}(x,z) = hV^{\omega}_{jh}(x,z), \end{aligned} \tag{3.15}$$

consistent with (2.110). A similar computation can be performed for $J_0^-(x)$.

For the particular case of $\omega=1$, this tells us that we can deduce spectral flow violating three-point function (3.8) from an unflowed four-point function! Moreover, this is a special four-point function which can be derived exactly due to the fact that one of the insertions, namely the spectral flow operator $V_{\frac{k}{2}}(x)$, has a null descendant. This provides a differential constraint which, combined with the usual KZ equation, allows us to fix the dependence on the spacetime and worldsheet cross-ratios. More explicitly, we consider

$$\begin{aligned} &\langle V_{j_1}(x_1,z_1)V_{j_2}(x_2,z_2)V_{j_3}(x_3,z_3)V_{\frac{k}{2}}(x_4,z_4)\rangle = \\ &\tilde{C}(j_1,j_2,j_3)|\mathcal{F}(x,z)|^2 \left| \frac{x_{32}^{(j_1+j_4-j_2-j_3)} x_{34}^{-2j_4} x_{31}^{(j_4+j_2-j_1-j_3)} x_{21}^{(j_3-j_1-j_4-j_2)}}{z_{32}^{(\Delta_2+\Delta_3-\Delta_1-\Delta_4)} z_{34}^{2\Delta_4} z_{31}^{(\Delta_1+\Delta_3-\Delta_4-\Delta_2)} z_{21}^{(\Delta_1+\Delta_4+\Delta_2-\Delta_4)}} \right|^2, \end{aligned} \tag{3.16}$$

where

$$z = \frac{z_{32}z_{14}}{z_{12}z_{34}}\,, \quad x = \frac{x_{32}x_{14}}{x_{12}x_{34}}\,, \quad \Delta_4 = -\frac{k}{4}\,, \quad j_4 = \frac{k}{2}. \tag{3.17}$$

Eqs.(3.11) and (3.12) imply that an extra insertion of the operators

$$O^{\rm NS} \equiv \oint_{z_4} dz' \frac{J^-(x_4,z')}{(z'-z_4)}\,, \tag{3.18}$$

$$O^{\rm KZ} \equiv \partial_{z_4} + \oint_{z_4} dz' \frac{J^3(x_4,z')}{(z'-z_4)}, \tag{3.19}$$

annihilates the correlator. By turning the contours around, one derives the following two conditions for the conformal block:

$$\left[\frac{x}{z} - \frac{x-1}{(z-1)}\right] x(x-1)\partial_x \mathcal{F} = \left[\kappa\left(\frac{x^2}{z} - \frac{(x-1)^2}{(z-1)}\right) + \frac{2j_1 x}{z} + \frac{2j_2(x-1)}{(z-1)}\right]\mathcal{F}\,, \tag{3.20}$$

and

$$-\partial_z\mathcal{F}=\frac{x(x-1)}{z(z-1)}\partial_x\mathcal{F}+\left[\frac{j_1}{z}+\frac{j_2}{z-1}+\kappa\left(\frac{x}{z}-\frac{x-1}{z-1}\right)\right]\mathcal{F}\,, \tag{3.21}$$

where $\kappa=j_3-j_1-j_2-j_4$. Up to a multiplicative constant, these are solved exactly by

$$\mathcal{F}(x,z)=z^{j_1}(z-1)^{j_2}x^{2j_2+\kappa}(x-1)^{2j_1+\kappa}(z-x)^{j_4-j_1-j_2-j_3}. \tag{3.22}$$

We can also fix the constant $\tilde{C}$ in (3.16) by considering the factorization limit, see the discussion around Eq.(2.171). Indeed, by considering the null-state equation, this time for a three-point function of the form $\langle V_{j_1}V_{j_2}V_{\frac{k}{2}}\rangle$, one finds that it vanishes unless $j_2=\frac{k}{2}-j_1$. The precise statement can be derived directly from the structure constants (2.155), giving

$$C\left(j_1,\frac{k}{2},j\right)\sim\delta\left(j_1+j-\frac{k}{2}\right)\,, \tag{3.23}$$

where we have omitted a j-independent albeit k-dependent factor of the form $\sqrt{B(j_1)B(\frac{k}{2}-j_1)}$, see (2.151). Hence, there is a single intermediate channel, leading to

$$\tilde{C}\left(j_1,j_2,j_3\right)\sim B\left(\frac{k}{2}-j_1\right)^{-1}C\left(\frac{k}{2}-j_1,j_2,j_3\right)\sim B(j_1)C\left(\frac{k}{2}-j_1,j_2,j_3\right). \tag{3.24}$$

We have thus computed (3.16) exactly.

We can now use this to compute the three-point function (3.8) by means of Eq. (3.13). Employing purely holomorphic notation (and ignoring a couple of overall signs) we have

$$\begin{aligned}&\langle V^1_{j_1h_1}(0,0)V_{j_2h_2}(1,1)V_{j_2h_2}(\infty,\infty)\rangle &(3.25)\\ &=\lim_{\varepsilon\to0}\varepsilon^{m_1}\int dy\,y^{j_1-m_1-1}\langle V_{j_1}(y,\varepsilon)V_{j_2}(1,1)V_{j_3}(\infty,\infty)V_{\frac{k}{2}}(0,0)\rangle\,.\\ &\approx\tilde{C}(j_1,j_2,j_3)\lim_{\varepsilon\to0}\varepsilon^{j_1+m_1}\int dy\,y^{j_2+j_3-m_1-\frac{k}{2}-1}(y-\varepsilon)^{\frac{k}{2}-j_1-j_2-j_3} &(3.26)\end{aligned}$$

where we have used that for these insertion points $x_{12}=y-1$, $z_{21}=\varepsilon-1$, $x=\frac{y}{y-1}$ and $z=\frac{\varepsilon}{\varepsilon-1}$, and that we are interested in the $\varepsilon\to0$ limit. The latter trivializes once we change variables $y\to y\varepsilon$, so the correlator does not vanish in general. By using (2.42) we finally obtain

$$\langle V^1_{j_1h_1}(0,0)V_{j_2h_2}(1,1)V_{j_2h_2}(\infty,\infty)\rangle=\frac{C\left(\frac{k}{2}-j_1,j_2,j_3\right)B(j_1)\gamma\left(j_1+m_1\right)\gamma(j_2+j_3-h_1)}{\gamma\left(j_1+j_2+j_3-\frac{k}{2}\right)}\,, \tag{3.27}$$

where $h_1=m_1+\frac{k}{2}$.

This is an important result. We now pause in order to emphasize its physical interpretation, which had a strong impact in the study of strings in AdS$_3$. We started by considering a curved background sourced by n_5 NS5-branes ($n_5\sim k$ in the supergravity regime) and n_1 of fundamental strings. In the near-horizon region the geometry becomes AdS$_3\times M_{\text{int}}$. Naively, one would expect the charges n_1 and n_5 to be fixed quantities. Nonetheless, upon setting $j_2,j_3\in\mathbb{R}$ and $j_1\in\frac{1}{2}+i\mathbb{R}$, we find that the correlator in Eq.(3.27) describes a non-zero probability for a process where two short strings join to form a long one. As discussed in earlier sections, this long string can then travel all the way to the boundary in finite proper time, and escape from the AdS_3 region. In other words, the system can dynamically emit an F1-string, so that n_1 can actually change! In this case it changes by one unit because the long string in question is singly wound.

This interpretation raises an interesting question regarding the holographic CFT. Our discussion of the spacetime Virasoro algebra led us to conclude that the corresponding central charge was given by $c_{\text{st}} = 6kn_1$. What do we mean then, when we say that n_1 should be considered dynamical? In passing, we note that there have been some clues along the way pointing to the fact that taking this as a fixed central charge for the spacetime theory might be a bit too quick. This is related to the appearance of the operator $\mathcal{I}$ (see Eqs. (2.81) and (2.176)) in the spacetime OPEs involving $\mathcal{T}(x)$ and $\mathcal{K}^a(x)$. As we briefly discussed early on, $\mathcal{I}$ is not exactly the identity operator of the boundary theory: it only behaves as such when working in a fixed spectral flow sector. Roughly speaking, its expectation value counts the number of strings in the background.

The resolution to this puzzle was given in [107] where the authors argued that we are actually working in the grand canonical ensemble. By this we mean that on the boundary are combining an infinite number of individual holographic CFTs with many different central charges, and fixing an associated chemical potential. In order to fix n_1 one should perform a Legendre transformation by adding a source for $\mathcal{I}$. We will come back to this interpretation in section 6 below.

For now, we go back to our study of spectrally flowed three-point functions. Note that, from Eq. (3.27), we can deduce the parafermion correlator with one unit of winding violation. We can then use it to compute all SL(2,$\mathbb{R}$) correlation functions which, so to speak, violate total spectral flow conservation by one unit, i.e. those with $\omega_1+\omega_2-\omega_3 = \pm 1$. Thus, we ask again (at least for three-point functions): are we done?

3.2 Spectral flow selection rules

This section is not called *Conclusions and outlook*, so the reader might suspect the answer is no. We are actually quite far from being done. In order to see this, we now study the spectral flow selection rules more systematically [15, 40]. We focus on correlators of Virasoro primary operators, the descendant case can be treated similarly [40]. Moreover, we star from the knowledge that, for two point functions, spectral flow *is* conserved, as it was argued above the parafermionic decomposition[15]. This extends to the full affine modules, such that, up to the series identifications discussed around Eq. (2.106), the different spectral flow sectors are orthogonal to each other.

The distinction between correlators of m-basis vertex operators and those involving the x-basis ones will be important in what follows. Let us first work in the m-basis, where ω can be non-negative. Consider the state

$$\prod_{i=1}^{n-1} V^{\omega_i}_{j_i m_i}(z_i)|0\rangle \,. \tag{3.28}$$

[15]Independent arguments can be found in [40] and [51].

The defining OPEs (2.110) imply that it is annihilated by

$$\mathcal{J}^+ = \oint dz\, J^+(z) \prod_{i=1}^{n-1} (z-z_i)^{\omega_i+1}. \tag{3.29}$$

Since the different spectral flow sectors are orthogonal to each other, this means that, upon using the VV OPEs (which is not known precisely) to write the generic state (3.28) as a sum of individual operators with definite spectral flow charges, denoted as ω_n, each of them inserted at the origin, they must all be annihilated separately. Now, in terms of modes we have $\mathcal{J}^+ = J^+_p + b_+ J^+_{p-1} + \dots$, where the highest mode has $p = \sum_{i=1}^{n-1}(\omega_i+1)$ and b is some constant. Since there are no null states in these representations, this implies an upper bound on ω_n, namely $\omega_n \leq p-1$. One can play a similar game with the operator

$$\mathcal{J}^- = \oint dz\, J^-(z) \prod_{i=1}^{n-1} (z-z_i)^{-\omega_i+1}\,, \tag{3.30}$$

such that $\mathcal{J}^- = J^-_q + b_- J^-_{q-1} + \dots$, with $q = \sum_{i=1}^{n-1}(1-\omega_i)$, which leads to the lower bound $\omega_n \geq 1-q$. These combine into

$$2-n \leq \omega_n - \sum_{i=1}^{n-1} \omega_i \leq n-2. \tag{3.31}$$

In other words, if we consider the inner product with a conjugated operator by shifting $\omega_n \to -\omega_n$, we obtain the m-basis selection rules for flowed correlators[16]:

$$\left| \sum_{i=1}^{n} \omega_i \right| \leq n-2. \tag{3.32}$$

Hence, for flowed three-point functions in the m-basis we were actually already done, as spectral flow can only be violated by one unit.

However, somewhat surprisingly at first, the story is different for the correlation functions in the x-basis. We now consider the state

$$\prod_{i=1}^{n-1} V^{\omega_i}_{j_i h_i}(x_i, z_i)|0\rangle\,, \tag{3.33}$$

where we now take $\omega_i \geq 0\ \forall\, i$. The current $J^+(z) = J^+(x,z)$ is the same at any point in x-space, hence the first part of our previous derivation involving $\mathcal{J}^+$ goes through. However, this is not so for J^- since $J^-(x_i,z) \neq J^-(x_j,z)$ when $x_i \neq x_j$. As a consequence, the operator $\mathcal{J}^-$ does *not* annihilate the state (3.33). Hence, we end up with a single bound

[16] More stringent selection rules can be obtained by specifying how many of the involved operators are in the continuous and discrete representations, respectively, or by working with highest/lowest weight states, which can be always achieved by acting with the currents. operators. Since this will not be necessary for us, we refer the interested reader to [15, 40].

instead of two. Consequently, we obtain a much weaker restriction[17]: non-vanishing x-basis spectrally flowed n-point functions must satisfy

$$\omega_i - \sum_{j\neq i} \omega_j \leq n-2\,. \tag{3.34}$$

One way to understand this is as follows. The translation from $x=0$ to an arbitrary point in x-space can be seen as an automorphism of the current algebra. In other words, the modes $J^a_n(x)$ (all evaluated at the *same* value of x) satisfy exactly the same algebra as the original modes $J^a_n = J^a_n(0)$ for any x. Hence, one could think about performing the spectral flow operation along any of the resulting Cartan generators. Now for an m-basis correlator we always sit at the origin and consider spectral flow along a fixed direction in *isospin* space, namely the one defined by the Cartan current J^3 at $x=0$. (Equivalently, for $x\to\infty$ we simply flip the sign of ω). This is different in the x-basis, as for each individual operator $V^{\omega_i}_{j_i h_i}(x_i,z_i)$ one considers spectral flow along the direction of $J^3(x_i)$, defined by the corresponding boundary insertion point. Here lies the fundamental difference between both types of correlation functions [63].

3.3 Spectrally flowed correlators and covering maps

The analysis of the previous section shows that there are many more non-trivial correlators that we need to compute. The problem is that the tools we have used so far are not really useful in this context. On the one hand, m-basis techniques cannot be used as we do not expect generic correlators to have a well-defined limit where we send all x_i to 0 or ∞. On the other hand, while the formula (3.13) works for arbitrary spectral flows, its application quickly becomes cumbersome. For instance, for a three-point function with all $\omega_i>0$ we would need to compute a complicated six-point function with three unflowed insertions and three *generalized* spectral flow operators.

3.3.1 The recursion relations

These issues where first addressed in [43, 51]. The authors noticed that, similar to the usual global Ward identities, consistency of the correlators with the OPEs between vertex operators and currents implied a series of constraints, which are in principle powerful enough to compute all spectrally flowed correlation functions in terms of the unflowed ones. These were dubbed *local* Ward identities, and as we now describe, they take the form of recursion relations in the h_i quantum numbers.

Let us denote

$$F = \left\langle \prod_{j=1}^{n} V^{\omega_j}_{j_j h_j}(x_j,z_j) \right\rangle\,, \tag{3.35}$$

and define

$$F^i_m = \left\langle \left(J^+_m V^{\omega_i}_{j_i h_i}\right)(x_i,z_i) \prod_{j\neq i} V^{\omega_j}_{j_j h_j}(x_j,z_j) \right\rangle. \tag{3.36}$$

[17]Although x-basis correlators can in principle be expanded in terms of m-basis ones (with descendant insertions), there is in principle no contradiction since this sum contains an infinite number of terms. Nevertheless, it would be useful to see this explicitly.

Although we know a couple of these quantities a bit more explicitly, namely $F_0^i = \partial_{x_i} F$ and

$$F^i_{\omega_i} = \left(h_i - \frac{k}{2}\omega_i + 1 - j_i\right) \left\langle V^{\omega_i}_{j_i,h_i+1}(x_i,z_i) \prod_{j\neq i} V^{\omega_j}_{j_j h_j}(x_j,z_j) \right\rangle , \tag{3.37}$$

which follows from (2.114), the rest of the F^i_m with $m = 1,\ldots,\omega_i - 1$ are unknown, at least so far. Counting only the F^i_m with $m > 0$, we thus have a total of $\sum_{i=1}^n (\omega_i - 1)$ unknowns. For later convenience, we also define

$$\begin{aligned} F^{i,-}_{-\omega_i} &= \left\langle \left(J^-_{-\omega_i} V^{\omega_i}_{j_i h_i}\right)(x_i,z_i) \prod_{j\neq i} V^{\omega_j}_{j_j h_j}(x_j,z_j) \right\rangle \\ &= \left(h_i - \frac{k}{2}\omega_i - 1 + j_i\right) \left\langle V^{\omega_i}_{j_i,h_i-1}(x_i,z_i) \prod_{j\neq i} V^{\omega_j}_{j_j h_j}(x_j,z_j) \right\rangle . \end{aligned} \tag{3.38}$$

The OPEs in Eq. (2.112) allow us to expand correlators involving a current insertion as

$$\left\langle J^+(z) \prod_{j=1}^n V^{\omega_j}_{j_j h_j}(x_j,z_j) \right\rangle = \sum_{i=1}^n \left[\frac{\partial_{x_i} F}{z - z_i} + \sum_{m=1}^{\omega_i} \frac{F^i_m}{(z-z_i)^{m+1}} \right] + \cdots , \tag{3.39a}$$

$$\left\langle J^3(z) \prod_{j=1}^n V^{\omega_j}_{j_j h_j}(x_j,z_j) \right\rangle = \sum_{i=1}^n \left[\frac{(h_i + x_i\partial_{x_i})F}{z - z_i} + \sum_{m=1}^{\omega_i} \frac{x_i F^i_m}{(z-z_i)^{m+1}} \right] + \cdots , \tag{3.39b}$$

$$\left\langle J^-(z) \prod_{j=1}^n V^{\omega_j}_{j_j h_j}(x_j,z_j) \right\rangle = \sum_{i=1}^n \left[\frac{(2h_i x_i + x_i^2\partial_{x_i})F}{z - z_i} + \sum_{m=1}^{\omega_i} \frac{x_i^2 F^i_m}{(z-z_i)^{m+1}} \right] + \cdots \tag{3.39c}$$

These can be combined into

$$G_j(z) \equiv \left\langle J^-(x_j,z) \prod_{l=1}^n V^{\omega_l}_{j_l h_l}(x_l,z_l) \right\rangle = \sum_{i\neq j} \left[\frac{(2h_i x_{ij} + x_{ij}^2\partial_{x_i})F}{z - z_i} + \sum_{n=1}^{\omega_i} \frac{x_{ij}^2 F^i_n}{(z-z_i)^{n+1}} \right] + \cdots \tag{3.40}$$

where $x_{ij} = x_i - x_j$ and $J^-(x_j,z) = J^-(z) - 2x_j J^3(z) + 2x_j^2 J^+(z)$. The important point is that Eq. (2.116b) then imposes stringent restrictions on the behavior of $G_j(z)$ when we take $z \to z_j$, as it must satisfy

$$(z - z_j)^{1-\omega_j} G_j(z) \to F^{j,-}_{-\omega_i} . \tag{3.41}$$

Even though $G_j(z)$ as given in (3.40) is clearly regular in the limit $z \to z_j$, this is much more restrictive, as we have found that the first $\omega_j - 1$ regular terms in the Taylor expansion of $G_j(z)$ around z_j must vanish, while the term of order $(z-z_j)^{\omega_j - 1}$ must have coefficient $F^{j,-}_{-\omega_i}$. After doing this for every $j = 1,\ldots,n$, we end up with a total of $\sum_{i=1}^n \omega_i$ *linear* constraints, which involve the primary correlators as well as all of the F^i_m. This provides, first, a complicated linear system which allows one to solve for the unknown F^i_m with $m = 1,\ldots,\omega_i - 1$. We are then left with exactly n recursion relations, involving only the original correlator together with its x_i derivatives, but also primary n-point functions where one of the h_i is shifted upwards or downwards by one unit. The latter are precisely the correlators we have denoted as $F^i_{\omega_i}$ and $F^{i,-}_{-\omega_i}$. Hence, these relations are recursive, and fully characterize the h_i-dependence of the flowed correlators.

However, in practice it is quite difficult to solve this system in full generality. As we now describe, this has recently been achieved for the case of three-point functions [51, 52, 54]. For higher point functions it is not even known how to derive the general expressions of the recursion relations, although important progress was achieved in [53] for $n = 4$.

3.3.2 The y-basis solution for three-point functions

As will be discussed in Sec. 4 below, these recursion relations admit a very simple solution in the tensionless string limit, which corresponds to $k = 3$ in the bosonic set-up. For more general values of k the structure of the recursion relations derived from the local Ward identities is quite involved.

Nevertheless, we note that there is an important similarity with the case of global Ward identities and their implications for unflowed correlators in the m-basis, where they imply somewhat analogous recursions between correlators with shifted eigenvalues $m_i \to m_i \pm 1$. The solutions are the complicated expressions obtained in [106] in terms of hypergeometric functions of the ${}_3F_2$ type. We now that these m-basis recursions are transformed into differential equations when working with x-basis operators, on which the zero modes J^a_0 act as the differential operators (2.74). This drastically simplifies the analysis, and allows one to quickly deduce the relevant x-dependence. The idea put forward in [51, 53] was to proceed analogously, i.e. to define yet another complex variable y in terms of which the modes $\tilde{J}^3_0 = J^3_0 - \frac{k}{2}\omega$ and $\tilde{J}^\pm_0 = J^\pm_{\pm\omega}$ act as differential operators as well.

How should we define the relevant (linear combinations of) vertex operators? The answer is already hidden in Eq. (3.13). Indeed, by rescaling $y \to y\varepsilon^\omega$, we obtain

$$V^\omega_{jh}(x,z) = \int d^2y\, y^{\frac{k}{2}\omega+j-h-1}\bar{y}^{\frac{k}{2}\omega+j-\bar{h}-1} V^\omega_j(x,y,z)\,, \tag{3.42}$$

where we have defined

$$V^\omega_j(x,y,z) \equiv \lim_{\varepsilon,\bar{\varepsilon}\to 0} |\varepsilon|^{2j\omega} V_j(x+y\varepsilon^\omega, z+\varepsilon) V^{\omega-1}_{\frac{k}{2}\frac{k}{2}\omega}(x,z). \tag{3.43}$$

Indeed, Eq. (3.42) is the exact analogue of the Melling-type transform (2.41) for spectrally flowed operators. From this one can also deduce the reflection property for long strings, namely

$$V^\omega_{1-j}(x,y,z) = B(1-j)\int d^2y'\,|y-y'|^{4j-4} V^\omega_j(x,y',z)\,. \tag{3.44}$$

While the zero modes still act as in (2.113), we now have

$$\left(J^+_\omega V^\omega_j\right)(x,y,z) = \partial_y V^\omega_j(x,y,z)\,, \tag{3.45a}$$

$$\left(J^3_0 V^\omega_j\right)(x,y,z) = \left(y\partial_y + j + \frac{k}{2}\omega\right) V^\omega_j(x,y,z)\,, \tag{3.45b}$$

$$\left(J^-_{-\omega} V^\omega_j\right)(x,y,z) = \left(y^2\partial_y + 2jy\right) V^\omega_j(x,y,z)\,. \tag{3.45c}$$

As for unflowed operators, we can invert the y-transform and argue that, roughly speaking, the operators $V^\omega_j(x,y,z)$ are the sum of $V^\omega_{jh}(x,z)$ over all allowed values of h. We should note that these linear combinations are quite non-standard from the CFT point of view since the different contributions have distinct spacetime and worldsheet weights!

As a warm-up, let us see how this can leads to an alternative method for computing the flowed two-points $\langle V^\omega_{j_1h_1}(x_1,z_1)V^\omega_{j_2h_2}(x_2,z_2)\rangle$ without going through the m-basis. We start by considering the corresponding y-basis correlator $\langle V^\omega_{j_1}(x_1,y_1,z_1)V^\omega_{j_2h_2}(x_2,y_2,z_2)\rangle$, and inserting an operator involving the currents of the model:

$$\oint_{z_1} dz \frac{(z-z_2)^\omega}{(z-z_1)^\omega}\langle J^-(x_1,z)V^\omega_{j_1}(x_1,y_1,z_1)V^\omega_{j_2h_2}(x_2,y_2,z_2)\,. \tag{3.46}$$

The overall powers ensure that we pick up the residue at z_1 where the current acts on $V^{\omega_1}_{j_1}(x_1,y_1,z_1)$ as $\left(J^-_{-\omega_1}V^{\omega_1}_{j_1}\right)(x_1,y_1,z_1)$. On the other hand, we can also turn the contour around and act on $V^{\omega_2}_{j_2}(x_2,y_2,z_2)$. Using that $J^-(x_1,z) = J^-(x_2,z) - 2x_{12}J^3(x_2,z) + x_{12}^2J^+(z)$, we see that there we pick up the action of the mode $J^+_{\omega_2}$. More explicitly, this gives the differential equation

$$\left(z^\omega_{12}(y_1^2\partial_{y_1}+2j_1y_1)+\frac{x_{12}^2}{z^\omega_{21}}\partial_{y_2}\right)\langle V^\omega_{j_1}(x_1,y_1,z_1)V^\omega_{j_2h_2}(x_2,y_2,z_2)\rangle = 0\,. \tag{3.47}$$

An analogous equation with $1\leftrightarrow 2$ can be obtained by starting with $J^-(x_2,z)$. We now combine this with the constraints coming from the global Ward identities, which imply that

$$\langle V^\omega_{j_1}(x_1,y_1,z_1)V^\omega_{j_2h_2}(x_2,y_2,z_2)\rangle = x_{12}^{-h_1^0-h_2^0}z_{12}^{-\Delta_1^0-\Delta_2^0}F\left(y_1\frac{z^\omega_{12}}{x_{12}},y_2\frac{z^\omega_{12}}{x_{12}}\right)\,, \tag{3.48}$$

where

$$h_i^0 = j_i+\frac{k}{2}\omega_i\,,\qquad \Delta_i^0 = -\frac{j_i(j_i-1)}{k-2}-j_i\omega_i-\frac{k}{4}\omega_i^2\,. \tag{3.49}$$

This can be understood by noting that $V_{j_i}(x_i,y_i=0,z_i) = V^{\omega_i}_{j_ih_i^0}(x_i,z_i)$ and thinking in terms of a power series expansion around $y_i=0$, such that, due to Eqs. (3.42) and (2.105), each additional power of y_i shifts $h_i\to h_i+1$ and $\Delta_i\to\Delta_i-\omega_i$. Hence, we get

$$\left(y_1^2\partial_{y_1}+2j_1y_1)+(-1)^\omega\partial_{y_2}\right)F[y_1,y_2] = 0 = \left(y_2^2\partial_{y_2}+2j_2y_2)+(-1)^\omega\partial_{y_1}\right)F[y_1,y_2]\,. \tag{3.50}$$

Up to constant rescalings (and omitting the anti-holomorphic sector), this has two independent solutions, one of them distributional:

$$F[y_1,y_2] = \delta^{(2)}(j_1-j_2)\left[(-1)^w+y_1y_2\right]^{-j_1-j_2}\,, \tag{3.51}$$

and

$$F[y_1,y_2] = \delta^{(2)}(j_1+j_2-1)y_1^{-j_1}y_2^{-j_2}\delta\left((-1)^w+y_1y_2\right)\,. \tag{3.52}$$

As usual, the overall normalization for the solution is conventional, while the relative coefficient between the two terms can be fixed by means of the reflection identity (3.44). Upon including the antiholomorphic contributions and integrating over y_1 and y_2 as in (3.42), this leads precisely to the two terms in Eq. (3.7).

We now focus on three-point functions. Working with y-basis operators will allows us to re-write the recursion relations derived formally above as differential equations for correlators of the form

$$F_y \equiv \langle V_{j_1}^{\omega_1}(x_1,y_1,z_1)V_{j_2}^{\omega_2}(x_2,y_2,z_2)V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle\,. \tag{3.53}$$

The original x-basis correlators in Eq. (3.35) can then be obtained from these by means of (3.42), although the integrals are somewhat non-trivial, see [51]. It is useful to fix $x_1 = z_1 = 0$, $x_2 = z_2 = 1$ and $x_3 = z_3 = \infty$, and consider

$$\hat{F}_y = \langle V_{j_1}^{\omega_1}(y_1)V_{j_2}^{\omega_2}(y_2)V_{j_3}^{\omega_3}(y_3)\rangle \equiv \langle V_{j_1}^{\omega_1}(0,y_1,0)V_{j_2}^{\omega_2}(1,y_2,1)V_{j_3}^{\omega_3}(\infty,y_3,\infty)\rangle\,. \tag{3.54}$$

In this case the global Ward identities imply

$$\begin{aligned}\langle V_{j_1}^{\omega_1}(x_1,y_1,z_1)V_{j_2}^{\omega_2}(x_2,y_2,z_2)V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle = \frac{x_{21}^{h_3^0-h_1^0-h_2^0}x_{31}^{h_2^0-h_1^0-h_3^0}x_{32}^{h_1^0-h_2^0-h_3^0}}{z_{21}^{\Delta_1^0+\Delta_2^0-\Delta_3^0}z_{31}^{\Delta_1^0+\Delta_3^0-\Delta_2^0}z_{32}^{\Delta_2^0+\Delta_3^0-\Delta_1^0}}\times\\ \left\langle V_{j_1}^{\omega_1}\left(0,y_1\frac{x_{32}z_{21}^{\omega_1}z_{31}^{\omega_1}}{x_{21}x_{31}z_{32}^{\omega_1}},0\right)V_{j_2}^{\omega_2}\left(1,y_2\frac{x_{31}z_{21}^{\omega_2}z_{32}^{\omega_2}}{x_{21}x_{32}z_{31}^{\omega_2}},1\right)V_{j_3}^{\omega_3}\left(\infty,y_3\frac{x_{21}z_{31}^{\omega_3}z_{32}^{\omega_3}}{x_{31}x_{32}z_{21}^{\omega_3}},\infty\right)\right\rangle\,.\end{aligned} \tag{3.55}$$

The procedure outlined above implies that, whenever the system is compatible and the F_n^i can be solved for, the y-basis correlator $\hat{F}_y$ will satisfy recursion relations of the following schematic form [54]:

$$\left[y_i(y_i\partial_{y_i}+2j_i)+\sum_{j=1}^{3}(A_{ij}y_j-B_{ij})\partial_{y_j}+C_i\right]\langle V_{j_1}^{\omega_1}(y_1)V_{j_2}^{\omega_2}(y_2)V_{j_3}^{\omega_3}(y_3)\rangle = 0\,, \tag{3.56}$$

where A_{ij}, B_{ij} and C_i are constants, but depend on the spins j_i and the charges ω_i. Moreover, generically the way in which these equations are derived only depends on the charges ω_i involved in the correlator, hence the recursion relations are do *not* dependent on whether the corresponding vertex operators belong to spectrally flowed discrete or continuous representations. Interestingly, using the definition given in Eq. (3.43), these differential equations can be shown to be equivalent to the null-state conditions associated to the generalized spectral flow operators $V_{\frac{k}{2}\frac{k}{2}\omega}^{\omega-1}(x,z)$ [52].

We now state the general solution, first conjectured in [51]. For *odd parity* correlators, namely when $\omega_1+\omega_2+\omega_3 \in 2\mathbb{Z}+1$, one has

$$\left\langle V_{j_1}^{\omega_1}(0,y_1,0)\,V_{j_2}^{\omega_2}(1,y_2,1)\,V_{j_3}^{\omega_3}(\infty,y_3,\infty)\right\rangle = C_{\boldsymbol{\omega}}(j_i)X_{123}^{\frac{k}{2}-j_1-j_2-j_3}\prod_{i=1}^{3}X_i^{-\frac{k}{2}+j_1+j_2+j_3-2j_i}\,, \tag{3.57}$$

while for the *even parity* case, i.e. when $\omega_1+\omega_2+\omega_3 \in 2\mathbb{Z}$,

$$\left\langle V_{j_1}^{\omega_1}(0,y_1,0)\,V_{j_2}^{\omega_2}(1,y_2,1)\,V_{j_3}^{\omega_3}(\infty,y_3,\infty)\right\rangle = C_{\boldsymbol{\omega}}(j_i)X_{\emptyset}^{j_1+j_2+j_3-k}\prod_{i<\ell}X_{i\ell}^{j_1+j_2+j_3-2j_i-2j_\ell}\,. \tag{3.58}$$

Here $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, and for any subset $I \subset \{1, 2, 3\}$,

$$X_I(y_1, y_2, y_3) \equiv \sum_{i\in I:\ \varepsilon_i=\pm 1} P_{\boldsymbol{\omega}+\sum_{i\in I}\varepsilon_i e_i} \prod_{i\in I} y_i^{\frac{1-\varepsilon_i}{2}} \,. \tag{3.59}$$

with $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. The numbers $P_{\boldsymbol{\omega}}$ are defined as

$$P_{\boldsymbol{\omega}} = 0 \qquad \text{for} \qquad \sum_j \omega_j < 2\text{Max}(\omega_i) \quad \text{or} \quad \sum_i \omega_i \in 2\mathbb{Z}+1 \tag{3.60}$$

and

$$P_{\boldsymbol{\omega}} = S_{\boldsymbol{\omega}} \frac{G\left(\frac{-\omega_1+\omega_2+\omega_3}{2}+1\right) G\left(\frac{\omega_1-\omega_2+\omega_3}{2}+1\right) G\left(\frac{\omega_1+\omega_2-\omega_3}{2}+1\right) G\left(\frac{\omega_1+\omega_2+\omega_3}{2}+1\right)}{G(\omega_1+1)G(\omega_2+1)G(\omega_3+1)} \,, \tag{3.61}$$

otherwise, where $G(n)$ is the Barnes G function $G(n) = \prod_{i=1}^{n-1} \Gamma(i)$, while $S_{\boldsymbol{\omega}}$ is a phase which will not be too important for our purposes. Finally, the h_i-independent normalizations are defined in terms of the unflowed ones, and read

$$C_{\boldsymbol{\omega}}(j_1, j_2, j_3) = \begin{cases} C(j_1, j_2, j_3), & \text{if} \quad \omega_1+\omega_2+\omega_3 \in 2\mathbb{Z}, \\ \mathcal{N}(j_1) C\left(\frac{k}{2}-j_1, j_2, j_3\right), & \text{if} \quad \omega_1+\omega_2+\omega_3 \in 2\mathbb{Z}+1, \end{cases} \tag{3.62}$$

with

$$\mathcal{N}(j) = \sqrt{\frac{B(j)}{B\left(\frac{k}{2}-j\right)}} \,. \tag{3.63}$$

It can beseen from the two-point function that $\mathcal{N}$ is precisely the relative coefficient discussed above Eq. (2.106) when discussin the series identifications. The following sections are devoted to explaining how derive these results, following [54].

3.3.3 Holomorphic covering maps

We would like to turn the constraints (3.41) into differential equations for the correlators $\hat{F}_y$. The usual method for this type of computation involves writing the current insertions in terms of a contour integrals, and then turning the contour around to make the current act on the rest of the vertex operators. As in the derivation of the selection rules, the problem is that J^- acts differently near each of the insertion point x_i.

In order to deal with this, it would be extremely useful to have a (holomorphic) function $\Gamma(z)$ capable of making the operator

$$J^-(\Gamma(z), z) = J^-(z) - 2\Gamma(z)J^3(z) + \Gamma^2(z)J^+(z) \tag{3.64}$$

act as the mode $J^-_{-\omega_i}(x_i)$ on each of the spectrally flowed insertions. More explicitly, we need $\Gamma(z)$ to satisfy

$$\Gamma(z) \sim x_i + a_i(z-z_i)^{\omega_i} + \cdots \qquad \text{near} \quad z = z_i \,, \qquad i = 1, 2, 3 \,, \tag{3.65}$$

where the ellipsis indicates higher order terms in $(z - z_i)$. As it turns out, such functions do exist, at least sometimes. They are known as holomorphic covering map. For a given three-point function, the covering map exists whenever

$$\sum_{i=1}^{3} \omega_i \in 2\mathbb{Z} + 1 \, , \qquad \sum_{i=1}^{3} \omega_i > 2\text{Max}(\omega_i) - 1 \, , \qquad \omega_i > 0 \, , \; \forall \, i, \tag{3.66}$$

and it is unique. The second of these conditions fits almost perfectly with the selection rule in Eq. (3.34), i.e. it only excludes the case where the latter inequality is saturated. These *edge* cases will be treated separately. However, the first condition in (3.66) tells us that the following derivation works only for the odd parity cases. The explicit construction of the covering map is carried out for instance in [55], and we will review it below. For the computation of genus zero worldsheet correlators it will be enough to establish the following properties:

- $\Gamma(z)$ is a rational function which, assuming there is no insertion at infinity, satisfies $\Gamma(z \to \infty) = \Gamma_\infty$ for some constant Γ_∞,
- the coefficients a_i appearing in Eq. (3.65) take the form

$$a_i = \begin{pmatrix} \frac{\omega_i + \omega_{i+1} + \omega_{i+2} - 1}{2} \\ \frac{-\omega_i + \omega_{i+1} + \omega_{i+2} - 1}{2} \end{pmatrix} \begin{pmatrix} \frac{-\omega_i + \omega_{i+1} - \omega_{i+2} - 1}{2} \\ \frac{\omega_i + \omega_{i+1} - \omega_{i+2} - 1}{2} \end{pmatrix}^{-1} \frac{x_{i,i+1} x_{i+2,i}}{x_{i+1,i+2}} \left(\frac{z_{i+1,i+2}}{z_{i,i+1} z_{i+2,i}} \right)^{\omega_i} , \tag{3.67}$$

  where the subscripts are understood mod 3, and the last two factors trivialize upon setting $x_1 = z_1 = 0$, $x_2 = z_2 = 1$ and $x_3 = z_3 = \infty$[18],
- at generic points $\Gamma(z)$ is an N-to-1 map, where N is defined by the Riemann-Hurwitz formula

$$N = 1 + \frac{1}{2} \sum_{i=1}^{n} (\omega_i - 1) \, , \tag{3.68}$$

- $\Gamma(z)$ has N simple poles, whose locations will be denoted z_a^*, with $a = 1, \ldots, N$,
- the derivative of the map $\partial\Gamma(z)$ has the simple form

$$\partial\Gamma(z) \sim \frac{\prod_{i=1}^{3} (z - z_i)^{\omega_i - 1}}{\prod_{a=1}^{N} (z - z_a^*)^2} \, , \tag{3.69}$$

  up to an overall factor independent of z which will not be important for us.

Let us now discuss how these covering maps are obtained [55]. For simplicity, we will fix the insertions points at $(z_1, z_2, z_3) = (x_1, x_2, x_3) = (0, 1, \infty)$; the generic case can be obtained similarly. This singles out the vertex operator at infinity, where we must require that

$$\Gamma(z \to \infty) = (-1)^{\omega_3 + 1} a_3^{-1} z^{\omega_3} + \cdots \, , \tag{3.70}$$

[18] In a slight abuse of notation, later on we will use the same notation "a_i" for the resulting purely numerical coefficients.

as follows from the transformations $z \to -1/z$ and $x \to -1/x$. Moreover, we focus on genus zero covering maps since we work on the sphere, hence the number of pre-images at a generic point is given by N, defined in Eq. (3.63). The covering map is then a ratio of polynomials $\Gamma(z) = p(z)/q(z)$. The condition at infinity fixes their degrees as

$$\deg(p) = N\,, \qquad \deg(p) = N - \omega_3\,. \tag{3.71}$$

We construct a second order differential equation whose independent solutions, which we denote as $f(z)$, are linear combinations of $p(z)$ and $q(z)$. This reads

$$0 = -\det \begin{pmatrix} f & f' & f'' \\ p & p' & p'' \\ q & q' & q'' \end{pmatrix} = (p'q - pq')f'' - (p''q - pq'')f' + (p''q' - p'q'')f\,. \tag{3.72}$$

Now, consider the function $q^2 \partial\Gamma = p'q - pq'$. This is a polynomial of degree $\deg(p) + \deg(q) - 1 = \omega_1 + \omega_2 - 1$, which we require to have zeros of order $\omega_1 - 1$ and $\omega_2 - 1$ at $z = 0$ and $z = 1$, respectively, see Eq. (3.69). Hence, it must take the form

$$p'q - pq' = Az^{\omega_1 - 1}(z-1)^{\omega_2 - 1}\,, \tag{3.73}$$

for some constant A. This also tells us that

$$p''q - pq'' = (p'q - pq')' = Az^{\omega_1 - 2}(z-1)^{\omega_2 - 2}[(\omega_1 + \omega_2 - 2)z + 1 - \omega_1]\,. \tag{3.74}$$

Similarly, it follows from the fact that p and q are solutions (with no common roots) that

$$p''q' - p'q'' = Bz^{\omega_1 - 2}(z-1)^{\omega_2 - 2} \tag{3.75}$$

for some other constant B. Inserting these expressions into (3.72) and dividing by $Az^{\omega_1-2}(z-1)^{\omega_2-2}$ then gives

$$z(z-1)f'' + [(\omega_1 - 1) - (\omega_1 + \omega_2 - 2)z]f' + Cf = 0\,, \tag{3.76}$$

with $C = -B/A$. This is a differential equation of the hypergeometric type. At large z, we can write $f \sim z^n$ for some $n > 0$. We then get the cuadratic condition $n(n-1) - n(\omega_1 + \omega_2 - 2) + C = 0$. Imposing that the two solutions $n_\pm$ satisfy $n_+ - n_- = \omega_3$ then fixes $C = N(N - \omega_3)$. Consequently, the general solution reads

$$f = \alpha\, {}_2F_1(-N, \omega_3 - N, 1 - \omega_1; z) + \beta z^{\omega_1}\, {}_2F_1(\omega_1 - N, \omega_1 + \omega_3 - N, 1 + \omega_1; z)\,, \tag{3.77}$$

where α, β are arbitrary constants. Taking the first and second terms to be proportional to the polynomials q and p, respectively, and fixing the normalization by imposing $\Gamma(1) = 1$ finally gives

$$\Gamma(z) = \frac{N!(N-\omega_3)!\Gamma(1-\omega_1)}{w_1! f(N-\omega_1)!\Gamma(N+1-\omega_1-\omega_3)} \frac{z^{\omega_1}\, {}_2F_1(\omega_1 - N, \omega_1 + \omega_3 - N, 1 + \omega_1; z)}{{}_2F_1(-N, \omega_3 - N, 1 - \omega_1; z)}\,. \tag{3.78}$$

Here the symbol Γ appearing on the LHS of this expression denotes the covering map, while on the RHS it simply refers to the usual Gamma-function.

Before coming back to the correlators, let us present a few examples. When all ω_i satisfy $\omega_i = 1$ we simply have the identity map $\Gamma(z) = z$. The case with $\omega_1 = \omega_3 = \omega$ and $\omega_2 = 1$ should give the map relevant for computing the spectrally flowed two-point function. Indeed, the map takes the form $\Gamma(z) = z^\omega$. As for a more non-trivial example, for $\omega_1 = \omega_2 = \omega_3 = 3$ we get $\Gamma(z) = \frac{z^3(2-z)}{2z-1}$.

As pointed out above, the existence of a covering map with the above properties is related with the spectral flow selection rules of the worldsheet correlators[19]. These covering maps are precisely those used in the computation of correlators in symmetric orbifold theories [55], where the charges ω_i should be identified with the *twists* of the corresponding operators. In such theories, three-point functions can only be non-zero if an appropriate covering map exists, i.e. when the conditions given in Eq. (3.66) are satisfied. If the holographically dual theory to our worldsheet model was a symmetric orbifold, the same would be true for the string correlators in AdS_3. However, we find that the conditions (3.66) are similar, but not identical to the selection rules obtained in Eq. (3.34), with $n = 3$. More precisely, the latter allow for correlators with $w_i + w_j \geq \omega_k - 1$ (for all i, j, k), while covering maps exist only when the strict inequalities $w_i + w_j > \omega_k - 1$ and $\omega_i > 0$, and the parity constraint $\sum_{i=1}^{3} \omega_i \in 2\mathbb{Z} + 1$ hold. A similar discussion holds for higher-point functions, although in those cases the conditions on the spectral flow charges are only necessary (but not sufficient) for the existence of the corresponding covering maps.

Actually, we have already computed the simplest examples of non-zero correlators in situations where the covering map does not exist, i.e. correlators where all insertions are unflowed, and also those where $\omega_1 = 1$ and $\omega_2 = \omega_3 = 0$. One of the reasons for the existence of these additional correlators comes from the fact that, as we know, in the SL(2,$\mathbb{R}$) WZW model the spectral flow charges are not uniquely defined, at least for short strings. We conclude that the holographic CFT is *not* a symmetric orbifold, although in some sense it appears to be quite close. We will come back to this discussion several times in the following sections.

3.3.4 Derivation for odd parity correlators

We now show how the existence of the covering maps $\Gamma(z)$ allows us to derive the y-basis differential equations satisfied by odd parity correlators F_y without having to worry about all the unknown F^i_m [43, 51, 54]. Consider the following correlator:

$$G(z) = \left\langle J^-(\Gamma(z), z)\, V^{\omega_1}_{j_1}(x_1, y_1, z_1) V^{\omega_2}_{j_2}(x_2, y_2, z_2) V^{\omega_3}_{j_3}(x_3, y_3, z_3) \right\rangle . \tag{3.79}$$

We will compute the contour integral

$$\oint_{z_i} \frac{dz\, G(z)}{(z - z_i)^{\omega_i}} \tag{3.80}$$

[19] In general, the non-existence of a map does not imply that the flowed correlators vanish. As will be discussed in section 4 below, the relation becomes more precise in the tensionless string limit.

in two different ways. On the one hand, by means of (3.65), near $z = z_i$ we can write

$$J^-(\Gamma(z), z) \sim J^-(x_i, z) - 2a_i(z - z_i)^{\omega_i} J^3(x_i, z) + a_i^2(z - z_i)^{2\omega_i} J^+(z) + \cdots . \tag{3.81}$$

Eqs. (2.116b) and (3.45) then give

$$\oint_{z_i} \frac{dz\, G(z)}{(z - z_i)^{\omega_i}} = \left[(2j_i y_i + y_i^2 \partial_{y_i}) - 2a_i \left(j_i + \frac{k}{2}\omega_i + y_i \partial_{y_i} \right) + a_i^2 \partial_{y_i} \right] F_y\,, \tag{3.82}$$

where F_y was defined in Eq. (3.53). On the other hand, by combining the definition of $J^-(\Gamma(z), z)$ and the OPEs in Eq. (2.112) we get

$$G(z) = \sum_{j=1}^{3} \left\{ -\frac{2[\Gamma(z) - x_j](y_j \partial_{y_j} + j_j + \frac{k}{2}\omega_j)}{z - z_j} F_y + \sum_{m=0}^{\omega_j} \frac{[\Gamma(z) - x_j]^2}{(z - z_j)^{m+1}} F_{y,m}^j \right\}, \tag{3.83}$$

where $F_{y,m}^i$ stands for the y-basis version of the F_m^i in Eq. (3.36). Let us look carefully at the RHS of (3.83). This is a rational function of z which, as implied by (3.65) in combination with the constraint equations — the equality between the RHS of Eqs. (3.40) and (3.41) — has zeros of order $\omega_i - 1$ at all z_i. It also has double poles at the N simple poles of $\Gamma(z)$ due to the $\Gamma(z)^2$ factors. Moreover, it further goes to zero as z^{-2} for $z \to \infty$ since the coefficient of the putative $\mathcal{O}\left(z^{-1}\right)$ contribution is given by

$$\begin{aligned}
&\sum_{j=1}^{3} \left\{ -\left[2[\Gamma_\infty - x_j] \left(y_j \partial_{y_j} + j_j + \frac{k}{2}\omega_j \right) \right] F_y + [\Gamma_\infty - x_j]^2 F_{y,0}^j \right\} \\
&= \sum_{j=1}^{3} \left\{ -\left[2[\Gamma_\infty - x_j] \left(y_j \partial_{y_j} + j_j + \frac{k}{2}\omega_j \right) \right] + [\Gamma_\infty - x_j]^2 \partial_{x_j} \right\} F_y \\
&= \sum_{j=1}^{3} \left\{ \Gamma_\infty^2 \partial_{x_j} - 2\Gamma_\infty \left(y_j \partial_{y_j} + j_j + \frac{k}{2}\omega_j + x_j \partial_{x_j} \right) + x_j^2 \partial_{x_j} + 2 \left(y_j \partial_{y_j} + j_j + \frac{k}{2}\omega_j \right) x_j \right\} F_y
\end{aligned} \tag{3.84}$$

which vanishes due to the global Ward identities (recall that $\left(y_j \partial_{y_j} + j_j + \frac{k}{2}\omega_j\right)$ is the y-basis analogue of h_j). These are exactly the properties that, up to a multiplicative factor, uniquely define the derivative of the covering map in Eq. (3.69). Hence, we find

$$G(z) = \alpha\, \partial\Gamma(z)\,, \tag{3.85}$$

where α must be independent of z. Since $\partial\Gamma(z \sim z_i) = a_i \omega_i (z - z_i)^{w_i - 1}$, it follows that

$$\oint_{z_i} \frac{dz\, G(z)}{(z - z_i)^{\omega_i}} = \alpha\, a_i \omega_i\,. \tag{3.86}$$

The coefficient α can be computed explicitly [43] as follows. The function $-\Gamma(z)^{-1}\partial\Gamma(z)$ clearly has unit residues at all z_a^*. This implies that

$$\alpha N = -\sum_{a=1}^{N} \oint_{z_a^*} dz \frac{G(z)}{\Gamma(z)} = -\sum_{a=1}^{N} \sum_{j=1}^{3} \sum_{m=0}^{\omega_j} \oint_{z_a^*} dz \frac{\Gamma(z)}{(z - z_i)^{m+1}} F_{y,m}^j\,, \tag{3.87}$$

where we have used that only the terms proportional to $\Gamma(z)^2$ in (3.83) contribute. By turning the contour around and using the defining properties of the covering map (there is no residue at infinity since $\Gamma^{-1}G \sim z^{-2}$ there), we get

$$\alpha N = \sum_{j=1}^{3}\sum_{m=0}^{\omega_j}\left[\oint_{z_j} dz \frac{x_j + a_j(z-z_j)^{\omega_j}}{(z-z_j)^{m+1}} F^j_{y,m}\right] = \sum_{j=1}^{3}\left[x_j F^j_{y,0} + a_j F^j_{y,\omega_j}\right]. \tag{3.88}$$

Consequently, using the global Ward identities once more we find

$$\alpha = \frac{1}{N}\sum_{j=1}^{3}\left(x_j\partial_{x_j} + a_j\partial_{y_j}\right) F_y = \frac{1}{N}\sum_{j=1}^{3}\left[(a_j - y_j)\partial_{y_j} - j_j - \frac{k}{2}\omega_j\right] F_y\,. \tag{3.89}$$

By combining Eqs. (3.82) and (3.86) and fixing the insertion points as in (3.54) we finally conclude that

$$\left\{(y_i - a_i)^2\partial_{y_i} + 2j_i(y_i - a_i) + \frac{a_i\omega_i}{N}\left[\sum_{j=1}^{3}\left((y_j - a_j)\partial_{y_j} + j_j\right) + \frac{k}{2}\right]\right\}\hat{F}_y = 0, \tag{3.90}$$

for $i = 1, 2, 3$, and where now the a_i are purely numerical.

This system of first order equations, which fits the structure given in Eq.(3.56), is easily solved and fixes the dependence $\hat{F}_y$ on y_1, y_2 and y_3 completely. Ignoring the overall y_i-independent normalization for now, we obtain

$$\begin{aligned}\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle_{\text{odd}} \sim{}& (y_1 - a_1)^{-2j_1}(y_2 - a_2)^{-2j_2}(y_3 - a_3)^{-2j_3} \\ &\times\left(\omega_1\frac{y_1 + a_1}{y_1 - a_1} + \omega_2\frac{y_2 + a_2}{y_2 - a_2} + \omega_3\frac{y_3 + a_3}{y_3 - a_3} - 1\right)^{\frac{k}{2} - j_1 - j_2 - j_3}.\end{aligned} \tag{3.91}$$

To make the connection with Eq. (3.57) we note that, while the $P_{\boldsymbol{\omega}}$ defined in (3.61) are somewhat complicated, their ratios are actually very simple. For instance, consider the X_1 term appearing in (3.57). From Eqs. (3.59) and (3.61), and up to an overall sign, we have

$$X_1 = P_{\boldsymbol{\omega}-e_1} y_1 + P_{\boldsymbol{\omega}+e_1} = P_{\boldsymbol{\omega}-e_1}\left(y_1 + \frac{P_{\boldsymbol{\omega}+e_1}}{P_{\boldsymbol{\omega}-e_1}}\right) = P_{\boldsymbol{\omega}-e_1}\left(y_1 - a_1\right). \tag{3.92}$$

A similar result holds for X_1 and X_2, while X_{123} can also be simplified in this manner. The y-dependence of the expression in (3.91) is precisely that of Eq. (3.57).

We note that, as a consequence of the result in Eq. (3.91), y-basis three-point functions diverge whenever a variable y_i approaches the corresponding coefficient a_i. Indeed, the a_i are very special points in the y-plane, which signal the existence of an appropriate holomorphic covering map. We will come back to this shortly in Sec. 6 when discussing the comparison with the correlators of the putative holographically dual theory. An even more extreme situation takes place in the tensionless limit, which corresponds to $k = 3$ in the bosonic language [43, 61–63], where spectrally flowed correlators are non-vanishing *only* when $y_i = a_i$ for all i.

3.3.5 Series identification and even parity correlators

Let us now consider even parity correlators, satisfying

$$\sum_{i=1}^{3} \omega_i \in 2\mathbb{Z}\,, \qquad \sum_{i=1}^{3} \omega_i > 2\text{Max}(\omega_i) - 1\,, \qquad \omega_i > 0\,,\ \forall i. \tag{3.93}$$

For such correlators it is not possible to construct a holomorphic covering map satisfying all the relevant properties.

As discussed below Eq.(3.56), for deriving the y-basis differential equations it is enough to focus on the discrete sector of the theory. As it turns out, in this sector even and odd parity cases are very closely related. This is due to the series identifications, see Eq.(2.106). For y-basis operators, this leads to the identities [52]

$$V_j^{\omega}(x, y = 0, z) = \mathcal{N}(j) \lim_{y\to\infty} y^{k-2j} V_{\frac{k}{2}-j}^{\omega+1}(x, y, z), \tag{3.94}$$

and

$$\lim_{y\to\infty} y^{2j} V_j^{\omega}(x, y, z) = \mathcal{N}(j) V_{\frac{k}{2}-j}^{\omega-1}(x, y = 0, z), \tag{3.95}$$

where $\mathcal{N}(j)$ was defined in Eq. (3.63). This means that all even parity correlators can be linked to (at least) three *different* situations where a covering map *does* exists. More explicitly, given a set of charges $(\omega_1, \omega_2, \omega_3)$ satisfying (3.93), it follows that the adjacent assignments $(\omega_1 + 1, \omega_2, \omega_3)$, $(\omega_1, \omega_2 + 1, \omega_3)$ and $(\omega_1, \omega_2, \omega_3 - 1)$ satisfy all conditions in (3.66). We denote the corresponding covering maps as follows:

$$\Gamma_1^+ \equiv \Gamma[\omega_1{+}1, \omega_2, \omega_3]\,, \qquad \Gamma_2^+ \equiv \Gamma[\omega_1, \omega_2{+}1, \omega_3]\,, \qquad \Gamma_3^- \equiv \Gamma[\omega_1, \omega_2, \omega_3{-}1]\,. \tag{3.96}$$

Thus, we have

$$\langle V_{j_1}^{\omega_1}(0) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(y_3)\rangle = \lim_{y_1\to\infty} y_1^{k-2j_1} \mathcal{N}(j_1) \langle V_{\frac{k}{2}-j_1}^{\omega_1+1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(y_3)\rangle, \tag{3.97a}$$

$$\langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(0) V_{j_3}^{\omega_3}(y_3)\rangle = \lim_{y_2\to\infty} y_2^{k-2j_2} \mathcal{N}(j_2) \langle V_{j_1}^{\omega_1}(y_1) V_{\frac{k}{2}-j_2}^{\omega_2+1}(y_2) V_{j_3}^{\omega_3}(y_3)\rangle, \tag{3.97b}$$

$$\lim_{y_3\to\infty} y_3^{2j_3} \langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(y_3)\rangle = \mathcal{N}(j_3) \langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{\frac{k}{2}-j_3}^{\omega_3-1}(0)\rangle. \tag{3.97c}$$

In many cases we can also shift the spectral flow charges in the opposite direction, leading to the maps

$$\Gamma_1^- \equiv \Gamma[\omega_1{-}1, \omega_2, \omega_3]\,, \qquad \Gamma_2^- \equiv \Gamma[\omega_1, \omega_2{-}1, \omega_3]\,, \qquad \Gamma_3^+ \equiv \Gamma[\omega_1, \omega_2, \omega_3{+}1]\,, \tag{3.98}$$

so that

$$\lim_{y_1\to\infty} y_1^{2j_1} \langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(y_3)\rangle = \mathcal{N}(j_1) \langle V_{\frac{k}{2}-j_1}^{\omega_1-1}(0) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(y_3)\rangle \tag{3.99a}$$

$$\lim_{y_2\to\infty} y_2^{2j_2} \langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(y_3)\rangle = \mathcal{N}(j_2) \langle V_{j_1}^{\omega_1}(y_1) V_{\frac{k}{2}-j_2}^{\omega_2-1}(0) V_{j_3}^{\omega_3}(y_3)\rangle, \tag{3.99b}$$

$$\langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(0)\rangle = \lim_{y_3\to\infty} y_3^{k-2j_3} \mathcal{N}(j_3) \langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{\frac{k}{2}-j_3}^{\omega_3+1}(y_3)\rangle. \tag{3.99c}$$

All expressions on the RHS of Eqs. (3.97) and (3.99) are limits of correlators we have already computed (up to an overall constant) in the previous section. In particular, we know that they satisfy the appropriate limits of the differential equations given in (3.90). For instance, $\langle V^{\omega_1-1}_{\frac{k}{2}-j_1}(0)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle$ is annihilated by the differential operators

$$\begin{aligned}&y_2(y_2\partial_{y_2}+2j_2)+(\omega_1-\omega_2-\omega_3)^{-1}\big\{(\omega_1+\omega_2-\omega_3)a_2[\Gamma_1^-]^2\partial_{y_2}\\&-2a_2[\Gamma_1^-]\left[(\omega_1-\omega_3)(j_2+y_2\partial_{y_2})+\omega_2(j_3+y_3\partial_{y_3}-a_3[\Gamma_1^-]\partial_{y_3}+j_1+j_3)\right]\big\}\end{aligned}\tag{3.100}$$

and

$$\begin{aligned}&y_3(y_3\partial_{y_3}+2j_3)+(\omega_1-\omega_2-\omega_3)^{-1}\big\{(\omega_1-\omega_2+\omega_3)a_3[\Gamma_1^-]^2\partial_{y_3}\\&-2a_3[\Gamma_1^-]\left[(\omega_1-\omega_2)(j_3+y_3\partial_{y_3})+\omega_3(j_2+y_2\partial_{y_2}-a_2[\Gamma_1^-]\partial_{y_2}+j_1+j_2)\right]\big\},\end{aligned}\tag{3.101}$$

where $a_i[\Gamma_1^-]$ denotes the coefficient a_i of the map Γ_1^-. The main point is that these operators must coincide with the appropriate limits of those provided in Eqs. (3.56) for original even parity correlator. In our example, these read

$$\begin{aligned}&y_2(y_2\partial_{y_2}+2j_2)+A_{22}y_2\partial_{y_2}+A_{23}y_3\partial_{y_3}-B_{22}\partial_{y_2}-B_{23}\partial_{y_3}-2A_{21}j_1+C_2,\\&y_3(y_3\partial_{y_3}+2j_3)+A_{33}y_3\partial_{y_3}+A_{32}y_2\partial_{y_2}-B_{33}\partial_{y_3}-B_{32}\partial_{y_2}-2A_{31}j_1+C_3.\end{aligned}\tag{3.102}$$

Proceeding similarly with the rest of the 12 relations, one finds a total of 60 conditions. 21 of these 60 conditions can be used to solve explicitly for all the coefficients A_{ij}, B_{ij} and C_i in (3.56). The fact that the remaining 39 identities are then satisfied provides a non-trivial check of procedure. This self-consistency follows from the identities relating the a_i coefficients of the different covering maps involved.

There are many equivalent ways to write the resulting coefficients. The simplest one leads to the following differential equations for even parity correlators:

$$\begin{aligned}&\Big\{\left(y_1-a_1[\Gamma_3^-]\right)^2\partial_{y_1}+2j_1\left(y_1-a_1[\Gamma_3^-]\right)+\frac{2a_1[\Gamma_3^-]\omega_1}{\omega_1+\omega_2-\omega_3}\Big[(y_1-a_1[\Gamma_3^-])\partial_{y_1}+j_1\\&+(y_2-a_2[\Gamma_3^-])\partial_{y_2}+j_2-(y_3-a_3[\Gamma_2^-])\partial_{y_3}-j_3\Big]\Big\}\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle=0\,,\end{aligned}\tag{3.103}$$

$$\begin{aligned}&\Big\{\left(y_2-a_2[\Gamma_3^-]\right)^2\partial_{y_2}+2j_2\left(y_2-a_2[\Gamma_3^-]\right)+\frac{2a_2[\Gamma_3^-]\omega_2}{\omega_1+\omega_2-\omega_3}\Big[(y_1-a_1[\Gamma_3^-])\partial_{y_1}+j_1\\&+(y_2-a_2[\Gamma_3^-])\partial_{y_2}+j_2-(y_3-a_3[\Gamma_1^-])\partial_{y_3}-j_3\Big]\Big\}\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle=0\,,\end{aligned}\tag{3.104}$$

$$\begin{aligned}&\Big\{\left(y_3-a_3[\Gamma_1^-]\right)^2\partial_{y_3}+2j_3\left(y_3-a_3[\Gamma_1^-]\right)+\frac{2a_3[\Gamma_1^-]\omega_3}{\omega_1-\omega_2-\omega_3}\Big[(y_1-a_1[\Gamma_2^-])\partial_{y_1}+j_1\\&-(y_2-a_2[\Gamma_1^-])\partial_{y_2}-j_2-(y_3-a_3[\Gamma_1^-])\partial_{y_3}-j_3\Big]\Big\}\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle=0\,.\end{aligned}\tag{3.105}$$

Up to an overall constant, the general solution then takes the form

$$\begin{aligned}\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle \sim &\left(1-\frac{y_2}{a_2[\Gamma_3^+]}-\frac{y_3}{a_3[\Gamma_2^+]}+\frac{y_2y_3}{a_2[\Gamma_3^-]a_3[\Gamma_2^+]}\right)^{j_1-j_2-j_3}\\ &\times\left(1-\frac{y_1}{a_1[\Gamma_3^+]}-\frac{y_3}{a_3[\Gamma_1^+]}+\frac{y_1y_3}{a_1[\Gamma_3^-]a_3[\Gamma_1^+]}\right)^{j_2-j_3-j_1}\\ &\times\left(1-\frac{y_1}{a_1[\Gamma_2^+]}-\frac{y_2}{a_2[\Gamma_1^+]}+\frac{y_1y_2}{a_1[\Gamma_2^+]a_2[\Gamma_1^-]}\right)^{j_3-j_1-j_2}.\end{aligned} \tag{3.106}$$

As in the odd case, one can check that this exactly reproduces the y-dependence of (3.58).

3.3.6 Edge cases, unflowed insertions, and normalization

Given that all non-zero correlation functions must satisfy the selection rules in Eq. (3.34), it only remains to consider edge cases, namely the three-point functions with

$$\omega_3=\omega_1+\omega_2 \quad \text{or} \quad \omega_3=\omega_1+\omega_2+1\,, \qquad \omega_i\geq 1\,,\,\forall\, i\,, \tag{3.107}$$

where we have assumed $\omega_3 \geq \omega_{1,2}$ for simplicity. We refer to these as the even and odd edge cases, respectively. Below we will also discuss correlators with unflowed insertions.

Edge cases must be treated carefully as several of the a_i coefficients either vanish or diverge in this limit. Fortunately, alternative techniques involving current insertions are available in these situations [54]. In order to see how this works, we start by unfixing the middle insertion. By means of the global Ward identities (3.55) we have

$$\begin{aligned}&\langle V^{\omega_1}_{j_1}(0,y_1,0)V^{\omega_2}_{j_2}(x,y_2,1)V^{\omega_3}_{j_3}(\infty,y_3,\infty)\rangle\\ &= x^{-j_1-j_2+j_3+\frac{k}{2}(-\omega_1-\omega_2+\omega_3)}\left\langle V^{\omega_1}_{j_1}\left(0,\frac{y_1}{x},0\right)V^{\omega_2}_{j_2}\left(1,\frac{y_2}{x},1\right)V^{\omega_3}_{j_3}(\infty,y_3x,\infty)\right\rangle\,.\end{aligned} \tag{3.108}$$

In the limit $x\to 0$, all vertex operators become m-basis flowed primaries. Given that we have $|\omega_1+\omega_2-\omega_3|\leq 1$, the edge cases correspond precisely to the configurations for which we expect to have a finite, non-zero m-basis correlator! This corresponds to the three-point functions studied in [15, 101], where they were denoted as spectral flow *conserving* and spectral flow *violating* three-point functions, depending on the overall parity of the spectral flow charges. We will derive some of the relevant differential equations satisfied by the edge correlators in the above collision limit, and only recover the full correlators at the end.

The connection can be made more explicit by deriving a constraint that will be satisfied in both edge cases. We set $x_1=x_2=x$ and consider the integral

$$\oint_{\mathcal{C}}\langle J^3(x,z)V^{\omega_1}_{j_1}(x,y_1,z_1)V^{\omega_2}_{j_2}(x,y_2,z_2)V^{\omega_3}_{j_3}(x_3,y_3,z_3)\rangle\, dz\,, \tag{3.109}$$

where $\mathcal{C}$ denotes a contour encircling all three insertion points. This vanishes since there is no residue at infinity. Turning the countour around, using the OPEs (3.45), fixing the worldsheet insertions to $(0,1,\infty)$, and further consider the limit $(x,x_3)\to(0,\infty)$, we find

$$\begin{aligned}&\Big[y_1\partial_{y_1}+y_2\partial_{y_2}-y_3\partial_{y_3}+j_1+j_2-j_3\\ &\quad+\frac{k}{2}(\omega_1+\omega_2-\omega_3)\Big]\langle V^{\omega_1}_{j_1}(0,y_1,0)V^{\omega_2}_{j_2}(0,y_2,1)V^{\omega_3}_{j_3}(\infty,y_3,\infty)\rangle=0\,.\end{aligned} \tag{3.110}$$

This is nothing but the usual m-basis charge conservation condition written y-basis language.

For the even edge cases, we can proceed similarly with the integrals

$$\oint_{\mathcal{C}} \langle J^-(x_3,z)V^{\omega_1}_{j_1}(x_1,y_1,z_1)V^{\omega_2}_{j_2}(x_2,y_2,z_2)V^{\omega_3}_{j_3}(x_3,y_3,z_3)\rangle \frac{(z-z_1)^{\omega_1}(z-z_2)^{\omega_2}}{(z-z_3)^{\omega_3}}dz \quad (3.111)$$

and

$$\oint_{\mathcal{C}} \langle J^-(x,z)V^{\omega_1}_{j_1}(x,y_1,z_1)V^{\omega_2}_{j_2}(x,y_2,z_2)V^{\omega_3}_{j_3}(x_3,y_3,z_3)\rangle \frac{(z-z_3)^{\omega_3}}{(z-z_1)^{\omega_1}(z-z_2)^{\omega_2}}dz\,. \quad (3.112)$$

Setting $(z_1,z_2,z_3)=(0,1,\infty)$ while sending $x\to 0$ and $x_3\to\infty$ as before, we obtain

$$\begin{aligned} 0 &= \Big[(-1)^{\omega_1}\partial_{y_1} + (-1)^{\omega_3}\partial_{y_2} + (y_3^2\partial_{y_3}+2j_3y_3)\Big]\langle\ldots\rangle\,, \\ 0 &= \Big[(-1)^{\omega_1}(y_1^2\partial_{y_1}+2j_1y_1) + (-1)^{\omega_3}(y_2^2\partial_{y_2}+2j_2y_2) + \partial_{y_3}\Big]\langle\ldots\rangle\,. \end{aligned} \quad (3.113)$$

where $\langle\ldots\rangle$ stands for $\langle V^{\omega_1}_{j_1}(0,y_1,0)V^{\omega_2}_{j_2}(0,y_2,1)V^{\omega_3}_{j_3}(\infty,y_3,\infty)\rangle$. Up to an overall constant, the solution reads [51]

$$\begin{aligned} &\langle V^{\omega_1}_{j_1}(0,y_1,0)V^{\omega_2}_{j_2}(0,y_2,1)V^{\omega_3}_{j_3}(\infty,y_3,\infty)\rangle \\ &\sim ((-1)^{\omega_1}y_1-(-1)^{\omega_3}y_2)^{j_3-j_1-j_2}(1+(-1)^{\omega_3}y_2y_3)^{j_1-j_2-j_3}(1+(-1)^{\omega_1}y_1y_3)^{j_2-j_1-j_3}\,. \end{aligned} \quad (3.114)$$

For the odd edge cases, we make use of the integrals

$$\oint_{\mathcal{C}} \langle J^-(x_3,z)V^{\omega_1}_{j_1}(x_1,y_1,z_1)V^{\omega_2}_{j_2}(x_2,y_2,z_2)V^{\omega_3}_{j_3}(x_3,y_3,z_3)\rangle \frac{(z-z_1)^{\omega_1+1}(z-z_2)^{\omega_2}}{(z-z_3)^{\omega_3}}dz\,, \quad (3.115)$$

and

$$\oint_{\mathcal{C}} \langle J^-(x_3,z)V^{\omega_1}_{j_1}(x_1,y_1,z_1)V^{\omega_2}_{j_2}(x_2,y_2,z_2)V^{\omega_3}_{j_3}(x_3,y_3,z_3)\rangle \frac{(z-z_1)^{\omega_1}(z-z_2)^{\omega_2+1}}{(z-z_3)^{\omega_3}}dz\,, \quad (3.116)$$

leading to

$$\begin{aligned} 0 &= \Big[(-1)^{\omega_3}\partial_{y_2} + (y_3^2\partial_{y_3}+2j_3y_3)\Big]\langle\ldots\rangle\,, \\ 0 &= \Big[(-1)^{\omega_1}\partial_{y_1} + (y_3^2\partial_{y_3}+2j_3y_3)\Big]\langle\ldots\rangle\,. \end{aligned} \quad (3.117)$$

with $\langle\ldots\rangle = \langle V^{\omega_1}_{j_1}(0,y_1,0)V^{\omega_2}_{j_2}(0,y_2,1)V^{\omega_3}_{j_3}(\infty,y_3,\infty)\rangle$ as before. In this case we find

$$\begin{aligned} &\langle V^{\omega_1}_{j_1}(0,y_1,0)V^{\omega_2}_{j_2}(0,y_2,1)V^{\omega_3}_{j_3}(\infty,y_3,\infty)\rangle \\ &\sim y_3^{j_1+j_2-j_3-\frac{k}{2}}(1+(-1)^{\omega_1}y_1y_3+(-1)^{\omega_3}y_2y_3)^{\frac{k}{2}-j_1-j_2-j_3}\,, \end{aligned} \quad (3.118)$$

The precise form of the edge correlators away from the collision limit can be obtained by using the global Ward identities. The explicit expressions can be found in [54]. One can check that they exactly match the y-dependence given Eqs. (3.58) and (3.57).

By now, we have derived the y-dependence of all correlators where all insertions are spectrally flowed. By using the series identifications once more, we can also derive three-point functions with unflowed insertions. For instance, for $\omega_1 = 0$ we have

$$\langle V_{j_1} V^{\omega}_{j_2}(y_2) V^{\omega+1}_{j_3}(y_3)\rangle = \mathcal{N}(j_1) \lim_{y_1\to\infty} y_1^{k-2j_1} \langle V^1_{\frac{k}{2}-j_1}(y_1) V^{\omega}_{j_2}(y_2) V^{\omega+1}_{j_3}(y_3)\rangle \,, \tag{3.119}$$

and

$$\langle V_{j_1} V^{\omega}_{j_2}(y_2) V^{\omega}_{j_3}(y_3)\rangle = \mathcal{N}(j_1) \lim_{y_1\to\infty} y_1^{k-2j_1} \langle V^1_{\frac{k}{2}-j_1}(y_1) V^{\omega}_{j_2}(y_2) V^{\omega}_{j_3}(y_3)\rangle \,, \tag{3.120}$$

where we have abbreviated $V_{j_1}(0,0) \equiv V_{j_1}$. Hence, up to the overall constant we find

$$\langle V_{j_1} V^{\omega}_{j_2}(y_2) V^{\omega+1}_{j_3}(y_3)\rangle \sim y_3^{j_1+j_2-j_3-\frac{k}{2}} \left(1 - y_3 - (-1)^{\omega} y_2 y_3\right)^{\frac{k}{2}-j_1-j_2-j_3} . \tag{3.121}$$

and

$$\langle V_{j_1} V^{\omega}_{j_2}(y_2) V^{\omega}_{j_3}(y_3)\rangle \sim (y_2 + (-1)^{\omega})^{j_3-j_1-j_2} (y_3 - 1)^{j_2-j_1-j_3} \left((-1)^{\omega} + y_2 y_3\right)^{j_1-j_2-j_3} . \tag{3.122}$$

Finally, for correlators with two unflowed insertions we get

$$\begin{aligned}\langle V_{j_1} V_{j_2} V^1_{j_3}(y_3)\rangle &= \mathcal{N}(j_2) \lim_{y_2\to\infty} y_2^{k-2j_2} \langle V_{j_1} V^1_{\frac{k}{2}-j_2}(y_2) V^1_{j_3}(y_3)\rangle \\ &\sim y_3^{j_1+j_2-j_3-\frac{k}{2}} (y_3 - 1)^{\frac{k}{2}-j_1-j_2-j_3} \,, \end{aligned}\tag{3.123}$$

Upon integrating over y_i, one finds that these results reproduce the original computations of [15, 101].

We end this section by fixing the overall y-independent normalizations [52]. Once again, the argument relies on the SL(2,$\mathbb{R}$) series identifications. Indeed, the relations given in Eqs. (3.97) and (3.99) must hold including the overall factors. This allows us to determine the structure constants recursively, starting from the unflowed three-point functions of [15, 49]. For instance, we first focus on the identity

$$\lim_{y_3\to\infty} y_3^{2j_3} \left\langle V^{\omega_1}_{j_1}(y_1)\, V^{\omega_2}_{j_2}(y_2)\, V^{\omega_3}_{j_3}(y_3)\right\rangle = \mathcal{N}(j_3) \left\langle V^{\omega_1}_{j_1}(y_1)\, V^{\omega_2}_{j_2}(y_2)\, V^{\omega_3-1}_{\frac{k}{2}-j_3}(0)\right\rangle . \tag{3.124}$$

Since we are interested in the y-independent factors we are free to set $y_1 = y_2 = 0$. According to Eqs.(3.57) and (3.58), the product of X_I factors on the left- an right-hand sides of (3.124) then give either

$$P_{\boldsymbol{\omega}}^{j_1+j_2+j_3-k}\, P_{\boldsymbol{\omega}+e_1+e_2}^{j_3-j_1-j_2}\, P_{\boldsymbol{\omega}+e_2-e_3}^{j_1-j_2-j_3}\, P_{\boldsymbol{\omega}+e_1-e_3}^{j_2-j_3-j_1} , \tag{3.125}$$

or

$$P_{\boldsymbol{\omega+e_1+e_2-e_3}}^{\frac{k}{2}-j_1-j_2-j_3}\, P_{\boldsymbol{\omega}+e_1}^{-j_1+j_2+j_3-\frac{k}{2}}\, P_{\boldsymbol{\omega}+e_2}^{j_1-j_2+j_3-\frac{k}{2}}\, P_{\boldsymbol{\omega}-e_3}^{j_1+j_2-j_3-\frac{k}{2}} , \tag{3.126}$$

depending on the overall parity. In both cases, this implies

$$C_{\boldsymbol{\omega}}(j_1, j_2, j_3) = \mathcal{N}(j_3) C_{\boldsymbol{\omega}-e_3}\left(j_1, j_2, \frac{k}{2} - j_3\right) . \tag{3.127}$$

Analogous statements can be derived by shifting ω_1 and ω_2 instead,consistent with the identity

$$\mathcal{N}(j_1)C\left(\frac{k}{2}-j_1,j_2,j_3\right)=\mathcal{N}(j_2)C\left(j_1,\frac{k}{2}-j_2,j_3\right)=\mathcal{N}(j_3)C\left(j_1,j_2,\frac{k}{2}-j_3\right)\,. \quad (3.128)$$

Since $\mathcal{N}(j)\mathcal{N}(\frac{k}{2}-j)=1$, we conclude that the normalization is precisely that given in Eq. (3.62), i.e. $C_{\boldsymbol{\omega}}(j_1,j_2,j_3)$ can only be $C(j_1,j_2,j_3)$, i.e. the unflowed three-point function, or $\mathcal{N}(j_1)C(\frac{k}{2}-j_1,j_2,j_3)$. This argument is valid for discrete representations, although we expect that it holds also for the continuous series by analytic continuation in j [15, 86].

In the previous pages we have finally succeeded in deriving all non-zero three-point functions with arbitrary spectral flow charges. For the third time, we ask: are we done now? Well... almost! The more precise statement is that we have derived all y-basis three-point functions. In order to obtain the full h-dependence of the x-basis three-point functions one actually needs to carry out the integration over the y variables, where the relevant h_i appear in the exponents of the prefactors, see Eq. (3.42). This computation is beyond the scope of these notes. It was carried out in detail in [51] for the case of three long-string states. The result can be written in terms of the so-called Lauricella hypergeometric functions of type A [108].

3.4 The conjecture for four-point functions

This analysis was extended to four-point functions in [53]. However, an important complication comes from the fact that the corresponding conditions on the spectral flow charges ω_i are necessary for the existence of the appropriate covering maps, but they are not sufficient. Indeed, the values of ω_i fix the orders of the different ramification points, but the putative map Γ is then fixed by the conditions (3.65) near z_i for $i=1,2,3$. The resulting function Γ will only be a covering map behaving as (3.67) for all four insertions if the additional condition $\Gamma(z_4)=x_4$ is satisfied.

The authors of [53] put forward the following conjectured solution:

$$\begin{aligned}
&\left\langle V^{w_1}_{j_1,h_1,\bar h_1}(0;0)V^{w_2}_{j_2,h_2,\bar h_2}(1;1)V^{w_3}_{j_3,h_3,\bar h_3}(\infty;\infty)V^{w_4}_{j_4,h_4,\bar h_4}(x;z)\right\rangle\\
&\quad=\int\prod_{i=1}^{4}\mathrm{d}^2y_i\;y_i^{\frac{kw_i}{2}+j_i-h_i-1}\bar y_i^{\frac{kw_i}{2}+j_i-\bar h_i-1}|X_\emptyset|^{2(j_1+j_2+j_3+j_4-k)}\\
&\qquad\times|X_{12}|^{2(-j_1-j_2+j_3-j_4)}\,|X_{13}|^{2(-j_1+j_2-j_3+j_4)}|X_{23}|^{2(j_1-j_2-j_3+j_4)}|X_{34}|^{-4j_4}\\
&\qquad\times\left\langle V^0_{j_1}(0;0)V^0_{j_2}(1;1)V^0_{j_3}(\infty;\infty)V^0_{j_4}\left(\frac{X_{23}X_{14}}{X_{12}X_{34}};z\right)\right\rangle \qquad (3.129a)
\end{aligned}$$

for $\sum_i w_i \in 2\mathbb{Z}$ and

$$\begin{aligned}
&\left\langle V^{w_1}_{j_1,h_1,\bar{h}_1}(0;0) V^{w_2}_{j_2,h_2,\bar{h}_2}(1;1) V^{w_3}_{j_3,h_3,\bar{h}_3}(\infty;\infty) V^{w_4}_{j_4,h_4,\bar{h}_4}(x;z)\right\rangle \\
&\quad = \mathcal{N}(j_3) \int \prod_{i=1}^{4} \mathrm{d}^2 y_i \; y_i^{\frac{kw_i}{2}+j_i-h_i-1} \bar{y}_i^{\frac{kw_i}{2}+j_i-\bar{h}_i-1} |X_{123}|^{2(\frac{k}{2}-j_1-j_2-j_3-j_4)} \\
&\qquad\quad \times |X_1|^{2(-j_1+j_2+j_3+j_4-\frac{k}{2})} |X_2|^{2(j_1-j_2+j_3+j_4-\frac{k}{2})} |X_3|^{2(j_1+j_2-j_3+j_4-\frac{k}{2})} |X_4|^{-4j_4} \\
&\qquad\quad \times \left\langle V^0_{j_1}(0;0) V^0_{j_2}(1;1) V^0_{\frac{k}{2}-j_3}(\infty;\infty) V^0_{j_4}\left(\frac{X_2 X_{134}}{X_{123} X_4}; z\right)\right\rangle
\end{aligned} \tag{3.129b}$$

for $\sum_i w_i \in 2\mathbb{Z}+1$. Here the factors X_I are defined similarly to the three-point case, see Eq. (3.59). However, now the numbers $P_{\boldsymbol{\omega}}$ are promoted to polynomials $P_{\boldsymbol{\omega}}(x,z)$ in the worldsheet and spacetime cross-ratios. The precise definitions can be found in [53]. The values of x and z where one of these polynomials vanish coincide precisely with the situations where some covering map associated to the original correlator or to one of the adjacent ones (where one or more of the ω_i are shifted by one unit) actually exists.

We see that, as it was the case for three-point functions, the X_I behave in many ways as generalized differences (although $X_i - X_j \neq X_{ij}$!). Moreover, these integral expressions depend on the *unflowed* four-point functions, evaluated at the corresponding generalized cross-ratio, as follows from combining the recursion relations with the KZ equations for the spectrally flowed correlators. This is a direct generalization of what happens for three-point functions, where we have seen that the unflowed structure constants $C(j_1, j_2, j_3)$ appear in the normalization.

4 The tensionless string limit

In the previous chapters we have studied the propagation of strings in AdS_3 without specifying the value of the parameter k, which sets the value of the affine level. In general, k is not restricted to be integer since the target space of our WZW model is non-compact.

In geometric terms, k is identified with the AdS_3 radius in string units. Although we have drawn important lessons from the semiclassical limit of the model, where k is taken to be very large, working in this regime is by no means necessary. The WZW description is *exact* in k, at least as long as $k > 2$ so that the central charge SL(2,$\mathbb{R}$) $c = \frac{3k}{k-2}$ makes sense. In the supersymmetric setting with target space $\mathrm{AdS}_3 \times S^3 \times T^4$ (or K3) the worldsheet anomaly cancellation condition sets $k = n_5 + 2$ for the bosonic AdS_3 sector, where n_5 the number of NS5-brane sources, which is of course quantized, hence one must restrict to $k \geq 3$, $k \in \mathbb{Z}$. We now discuss what happens in the limit where k takes the smallest possible value, $k = 3$, which corresponds to the background source by n_1 fundamental strings and a single NS5-brane. This defines the tensionless limit of the model, and we will see that the dual in this regime the holographic dual simplifies drastically.

4.1 Higher spin massless spectrum

We start by looking more closely at the massless sector of the theory. Although so far we have concentrated on operators that are spacetime scalars, here we include fields with arbitrary (spacetime) spin. The mass-squared formulas for symmetric traceless tensors of rank s and p-form fields in AdS_{d+1} read

$$m^2_{\mathrm{AdS}_{d+1}}(s) = (\Delta_{\mathrm{st}} - s + 2 - d)(\Delta_{\mathrm{st}} + s - 2)\,, \tag{4.1}$$

$$m^2_{\mathrm{AdS}_{d+1}}(p) = (\Delta_{\mathrm{st}} + p - d)(\Delta_{\mathrm{st}} - p)\,, \tag{4.2}$$

respectively, in units of the AdS radius. For AdS_3, we identify $\Delta_{\mathrm{st}} = h + \bar{h}$ and s or p with $|l| = |h - \bar{h}|$, giving

$$m^2_{\mathrm{AdS}_3} = (h + \bar{h} - |h - \bar{h}|)(h + \bar{h} + |h - \bar{h}| - 2)\,, \tag{4.3}$$

in both cases. This reproduces $m^2 = 4j(j-1)$ when $h = \bar{h} = j$. Massless states appear when either $h = 0$ or $\bar{h} = 0$, such that, as per the AdS/CFT dictionary, they correspond to conserved spin l currents of the boundary theory.

Which massless modes can we construct from the worldsheet WZW model? Recall that the mass-shell conditions read

$$-\frac{j(j-1)}{k-2} - \omega h + \frac{k}{4}\omega^2 + h_{\mathrm{int}} + N - 1 = 0 = -\frac{j(j-1)}{k-2} - \omega \bar{h} + \frac{k}{4}\omega^2 + \bar{h}_{\mathrm{int}} + \bar{N} - 1\,, \tag{4.4}$$

and upon subtracting both constraints we get the level-matching condition

$$\omega(h - \bar{h}) = N - \bar{N} + h_{\mathrm{int}} - \bar{h}_{\mathrm{int}}\,. \tag{4.5}$$

The universal solutions are the dilaton and the graviton modes. These include the *identity* operator $\mathcal{I}$ (2.176) and the energy-momentum tensor defined in Eq. (2.178), together with its anti-holomorphic counterpart. These are solutions with $j=1$, $\omega=0$, $h_{\rm int}=\bar{h}_{\rm int}=0$ and $N=\bar{N}=1$, while h and $\bar{h}$ are either zero or two. Of course, we also have the B-field modes. In the bosonic model we would not expect to have any additional solutions, at least for generic values of k. Let us prove this.

From now on we set $h_{\rm int}=\bar{h}_{\rm int}=0$ for simplicity, although the arguments below are easily extended to the general case. In the unflowed sector we only need to consider discrete states in lowest-weight representations. In order to get, say, $h=0$, we need to solve

$$-j(j-1)+(k-2)(N-1)=0\,. \tag{4.6}$$

Now, at level N, the eigenvalue of J^3_0 is restricted to the range $m\geq j-N$. This is because $m\geq j$ for the affine primary, which can be lowered by N units (at most) by applying $(J^-_{-1})^N$. The lowest energy mode then corresponds to $m=0$, i.e. $j=N$, for which (4.6) becomes $-j(j-1)+(k-2)(j-1)=0$. The only solutions with $j\neq 1$ correspond to $j=k-2$, which is outside the physical range (2.93).

Moving to short strings with $\omega>0$, we now ask for solutions with, say, $\bar{h}=0$, for which

$$-\frac{j(j-1)}{k-2}=1-\bar{N}-\frac{k}{4}\omega^2\,. \tag{4.7}$$

Imposing $j\leq\frac{k}{2}$ gives $-\frac{j(j-1)}{k-2}\geq-\frac{k}{4}$, which translates into

$$1-\bar{N}-\frac{k}{4}(\omega^2-1)\geq 0\,. \tag{4.8}$$

This only allows for $\omega=1$ and $\bar{N}=0,1$. As before, the lowest energy state has $\bar{m}\geq j-\bar{N}\geq\frac{1}{2}-\bar{N}\geq-\frac{1}{2}$, which is incompatible with $\bar{h}=0$ (i.e. $\bar{m}=-\frac{k}{2}<-1$) for $k>2$.

Finally, we also consider long string states. Upon setting $\bar{h}=0$, level-matching implies $N=\omega h+\bar{N}$, hence the holomorphic mass-shell condition becomes

$$\frac{1}{4}+s^2=(k-2)\left(1-\bar{N}-\frac{k}{4}\omega^2\right)\,. \tag{4.9}$$

For $k>2$ and $s\in\mathbb{R}$, there is no solution *unless* $s=0$, $\bar{N}=0$, $\omega=1$ and, crucially, $k=3$.

We conclude that for $k>2$, $k\neq 3$ the only massless states are the usual supergravity modes. On the other hand, at $k=3$ we have an infinite tower of higher-spin massless states with $l=h=N$, which are singly-wound long strings. The interpretation is that, roughly speaking, this constitutes the AdS$_3$ version of the $\alpha'\to 0$ limit for strings in flat space, where excited string states become massless. As a consequence, the AdS$_3\times S^3\times T^4$ model with $n_5=1$ is understood as describing the *tensionless* string limit[20].

[20]Note that, for $k=3$ the physical range for the discrete sector becomes

$$\frac{1}{2}<j<1, \tag{4.10}$$

so that the $j=1$ supergravity modes naively drop out. Nevertheless, by means of the SL(2,$\mathbb{R}$) series identifications (2.106) we can re-interpret them as states with $\omega=1$ and unflowed spin $\tilde{\jmath}=\frac{3}{2}-1=\frac{1}{2}$, which merge with the massless long strings described above.

4.2 Reproducing the symmetric orbifold spectrum

Although work mostly with the bosonic SL(2,$\mathbb{R}$) WZW model, in this section we are ultimately interested in the $\text{AdS}_3 \times S^3 \times T^4$ supersymmetric case. When $n_5 = 1$ we are in the tensionless limit: the level of the bosonic AdS_3 subsector is $k = n_5 + 2 = 3$, and a tower of higher-spin modes become massless. As it turns out, an even more drastic simplification occurs in the supersymmetric setting. This was discussed in [109], where the authors showed that

- All short-string states drop out of the spectrum.
- Out of all long-string states, only those with $j = \frac{1}{2}$ survive.

These statements are not that easy to derive precisely in our language since the RNS formalism breaks down at $n_5 = 1$. This is because the bosonic SU(2) sector has level $k' = n_5 - 2$, which becomes negative at $n_5 = 1$. Due to this technical complication one needs to work with the so-called hybrid formalism [110]. Here we simply quote the most relevant results, which follow from the representation theory of the relevant supergroup, namely PSU(1,1|2) at level 1, for which the maximal bosonic subgroup is $\text{SL}(2,\mathbb{R})_1 \times \text{SU}(2)_1$ [63]. Nevertheless, we provide the following heuristic motivation: for $n_5 = 1$ the corresponding AdS space is string-size, and as a consequence there is simply no space for the discrete bound states, while long strings with non-zero radial momentum have nowhere to go!

We now describe how these additional restrictions affect the physical spectrum in holographic terms. For bosonic strings in $\text{AdS}_3 \times M_{\text{int}}$, the worldsheet model is built by combining the SL(2,$\mathbb{R}$) WZW model at level k and an *internal* CFT with central charge

$$c_{\text{int}} = 26 - \frac{3k}{k-2}\,, \tag{4.11}$$

together with the usual bc-system of ghosts. For long string (Virasoro) primary states, the physical state condition is solved by

$$h = \frac{1}{\omega}\left[-\frac{j(j-1)}{k-2} + \frac{k}{4}\omega^2 + h_{\text{int}} - 1\right]. \tag{4.12}$$

By defining[21]

$$Q = \frac{k-3}{\sqrt{k-2}}\,, \qquad \beta = \frac{\frac{k}{2} - 1 - j}{\sqrt{k-2}}\,, \tag{4.13}$$

we can write

$$h = \frac{k(\omega^2-1)}{4\omega} + \frac{h_{\text{int}}}{\omega} + \frac{\beta(Q-\beta)}{\omega}\,. \tag{4.14}$$

The first two terms reproduce exactly the spectrum of a symmetric orbifold theory

$$\text{Sym}^N\left(\mathbb{R}_Q \times M_{\text{int}}\right)\,, \tag{4.15}$$

[21] This is related to the conventions employed in [76] by $j \to 1 - j$. The distinction between the β used here and the Wakimoto field for which we have used the same notation should be clear from the context.

where $\mathbb{R}_Q$ denotes a Liouville-type factor, i.e. a linear dilaton theory with background charge Q, such that

$$c_{\text{seed}} = 1 + 6Q^2 + c_{\text{int}} = 1 + \frac{6(k-3)^2}{k-2} + 26 - \frac{3k}{k-2} = 6k\,. \tag{4.16}$$

Indeed, for an operator living on a sector of twist ω, the first factor on the RHS of Eq. (4.14) gives the weight of the corresponding twist operator σ_ω, while the second and third factors give the weight of the relevant operator of the seed theory as viewed form the associated covering space. This leads us to identify the worldsheet spectral flow sector with the holographic twist sector. It is then tempting to identify N, the number of copies, with n_1.

Even though the spectrum reproduces that of the symmetric orbifold model, we cannot conclude that this *is* precisely the holographic CFT at the same point in moduli space where the worldsheet theory is defined, at least not for generic values of k. This cannot be the case, since the structure of the holographic correlators as computed from the worldsheet is not that of a symmetric orbifold: we have obtained a full family of correlators which are non-zero even when the corresponding covering map does not exist, namely the even parity correlators of Sec. 3. It was proposed recently in [75, 76] that the correct CFT can be defined, at least perturbatively, by starting from the above symmetric orbifold and including a particular marginal deformation by a twist-two operator, which moreover is trivial in the M_{int} sector but contains a Liouville-type potential for the non-compact scalar ϕ. We will come back to this in Sec. 6 below, when discussing the holographic theory away from the tensionless point.

For now, we note that at $k = 3$ the background charge Q vanishes, and moreover we get $\beta = \frac{1}{2} - j$. Hence, upon restricting to long strings with $j = \frac{1}{2}$ we trivialize the sector of the non-compact scalar, and we are left with a sub-sector describing the states of

$$\text{Sym}^N\left(M_{\text{int}}\right). \tag{4.17}$$

Moreover, it can be shown that correlation functions of operators belonging to this sub-sector are unaffected by the marginal deformation for $k = 3$. In the following sections we describe how the corresponding worldsheet correlators simplify drastically. This is particularly relevant for the supersymmetric setting of $\text{AdS}_3 \times S^3 \times T^4$ at $n_5 = 1$, where it was shown in [43, 44, 63, 65–67, 111] that the full HCFT collapses to the following symmetric orbifold:

$$\text{Sym}^N\left(T^4\right) \qquad \text{with} \qquad \mathcal{N} = (4,4) \quad \text{supersymmetry}\,. \tag{4.18}$$

Here the S^3 provides the geometrical realization of the R-symmetry group [41], as usual in holography.

4.3 Localization of worldsheet correlators

Let us now discuss what happens with string correlators in the tensionless limit of the theory. As argued above, we are ultimately interested in computing n-point functions of spectrally flowed operators where $j_i = \frac{1}{2}$ for $i = 1, \ldots, n$ when $k = 3$. For any value of n, this is a particular case of a the more general condition given by

$$\sum_{i=1}^{n} j_i = \frac{k}{2}(n-2) - (n-3)\,, \tag{4.19}$$

under which spectrally flowed correlators simplify drastically [43]. This is because the constraint equations derived from the local Ward identities in Sec. 3 admit a generic solution of the following form:

$$\langle \prod_{i=1}^{n} V^{\omega_i}_{j_i h_i}(x_i, z_i)\rangle = \sum_{\Gamma} \prod_{i=1}^{n} a_i^{-h_i} \bar{a}_i^{-\bar{h}_i} \prod_{j=4}^{n} \delta^{(2)}\left(x_j - \Gamma(z_j)\right) W_\Gamma(z_1, \ldots, z_4)\,, \tag{4.20}$$

where we have fixed $(z_1, z_2, z_3) = (x_1, x_2, x_3) = (0, 1, \infty)$ for simplicity (otherwise the coefficients a_i, $\bar{a}_i$ depend explicitly on these insertion points).

Before showing that this is actually a solution, let us briefly unravel this formula. For a given set of spectral flow charges ω_i, worldsheet insertion points z_i, and boundary insertion points x_i, one must ask if an appropriate holomorphic covering map $\Gamma(z)$ exists. When the ω_i fail to satisfy either of the conditions

$$\sum_{i=1}^{n} w_i \geq 2\,\mathrm{Max}(\omega_i) + n - 2\,, \qquad \sum_{i=1}^{n} (w_i - 1) \in 2\mathbb{Z}\,, \tag{4.21}$$

the answer is negative. In such situations the solution (4.20) simply vanishes. On the other hand, if the conditions in Eq. (4.21) hold, the situation is similar to that of the generic four-point function considered above: one can always find a holomorphic map with appropriate branchings, namely

$$\Gamma(z) - \Gamma(z_i) = a_i (z - z_i)^{\omega_i} + \ldots\,, \qquad \forall \quad i = 1, \ldots, n\,. \tag{4.22}$$

However, enforcing $\Gamma(z_i) = x_i$ for $i = 1, 2, 3$ is as far as we can go. The covering map needed to compute our correlator will only exist if the remaining pairs (z_i, x_i) are chosen so that x_i actually coincides with $\Gamma(z_i)$ for $i = 4, \ldots, n$. In the solution given in Eq.(4.20), the x-basis worldsheet correlators localize on the locus defined by the existence of such covering maps. Although for the three-point case the map is unique (at genus zero), generically for $n \geq 3$ there is a discrete set of possible maps, hence the sum over Γ in (4.20). When this correlator is non-vanishing, the dependence on h_i, $\bar{h}_i$ and $x_{i\geq 4}$ is captured in a very simple manner by the overall prefactor involving the coefficients a_i of the corresponding map. For this to be a solution of the recursion relations, the function W_Γ must be independent of the spacetime weights h_i and of all x_i, but otherwise generic.

Let us see how this proposal comes about. We start with the simpler case of three-point functions, for which there are no delta functions, although we still require (4.21). We already know from the analysis of Sec.3 that using the covering maps in combination with the JV^w OPEs leads to recursion relations of the form

$$\sum_{i=1}^{3} \left(a_i F_+^i - h_i F\right) = \frac{N}{a_j \omega_j} \left(F_-^j - 2 a_j h_j + a_j^2 F_+^j\right) \qquad j = 1, 2, 3\,, \tag{4.23}$$

with $F = \langle \prod_{i=1}^{3} V^{\omega_i}_{j_i h_i}(x_i, z_i) \rangle$ and $F^j_{\pm} = \langle \left(J^{\pm}_{\pm\omega_j} V^{\omega_j}_{j_j h_j} \right)(x_j, z_j) \prod_{i \neq j} V^{\omega_i}_{j_i h_i}(x_i, z_i) \rangle$, while $2N = 2 + \sum_{i=1}^{3}(\omega_i - 1)$. It not difficult to see that, when $\sum_{i=1}^{3} j_i = \frac{k}{2}$, this is satisfied by Eq. (4.20) since the $F^j_{\pm}$ are proportional to the correlator with the shift $h_j \pm 1$, which simply adds or removes a power of a_j.

The perhaps more surprising statement is that this structure actually generalizes to higher point functions. Indeed, since we are sitting at the locus where the covering map is available, the recursion relations can be derived directly by a procedure almost analogous to what we did for three-point functions. Nevertheless, one must be careful when replacing $\Gamma(z_i) = x_i$ in several places since the correlators with current insertions might involve derivatives of the derivative of the correlator with respect to x_i; when this derivative hits the delta function, one must uses the usual identity $x\delta'(x) = -\delta(x)$, see [43] for the details. The only subtlety comes towards the end of the computation of the coefficient α appearing in the relation $G(z) = \langle J^-(\Gamma(z)) \prod_{i=1}^{n} V^{\omega_i}_{j_i h_i}(x_i, z_i) \rangle = \alpha \partial \Gamma(z)$. We get

$$\begin{aligned} \alpha N &= \sum_{i=1}^{n} \left[a_i F^i_+ + \Gamma(z_i) \partial_{x_i} F \right] \\ &= \sum_{i=1}^{n} \left[a_i F^i_+ + x_i \partial_{x_i} F \right] + (n-3)F = \sum_{i=1}^{n} \left[a_i F^i_+ - h_i F \right] + (n-3)F \,, \end{aligned} \tag{4.24}$$

where in the middle step we have used the ansatz (4.20), and in the last one the global Ward identities. As a result, the presence of the $(n-3)$ delta functions leads to recursion relations of the form

$$\sum_{i=1}^{n} \left(a_i F^i_+ - h_i F \right) + (n-3)F = \frac{N}{a_j \omega_j} \left(F^j_- - 2a_j h_j + a_j^2 F^j_+ \right) \qquad j = 1, 2, 3 \,, \tag{4.25}$$

which are indeed solved by (4.20).

For $k = 3$ and $j_i = \frac{1}{2}$, the constraint (4.19) is always satisfied. This suggests that such correlators are only non-zero when the appropriate covering map can be constructed. Moreover, in order to compute the holographic CFT correlator from the string perspective theory we are instructed to integrate the insertion points $z_{i \geq 4}$ over the full complex plane parametrizing the string worldsheet. But this can be done trivially for the solution in Eq. (4.20) due to the presence of the delta functions $\delta^{(2)}(x_i - \Gamma(z_i))$! We simply get a sum over the pre-images $\Gamma^{-1}(x_i)$. This is exactly the behavior we expect for the correlation functions of a symmetric orbifold theory on the boundary, see [55, 59]. An explicit comparison would of course require the computation of the function W_Γ.

4.4 Focusing on spacetime twist operators

In this section we provide the y-basis picture of the discussion related to the localization of worldsheet correlators, and provide the precise matching with the symmetric orbifold for a set of three- and four-point functions. Note that upon setting $k = 3$ and $j = \frac{1}{2}$ and $h_{\text{int}} = 0$, we select operators which are precisely the worldsheet avatars of the spacetime twist operators, which create the ground states of the corresponding twisted sectors.

Let us start with three-point functions. For a given odd parity correlator, the y-basis differential equations become

$$\left\{(y_i-a_i)^2\partial_{y_i}+(y_i-a_i)+\frac{a_i\omega_i}{N}\left[3+\sum_{j=1}^{3}(y_j-a_j)\partial_{y_i}\right]\right\}F_y=0\,. \tag{4.26}$$

In the original solution (3.57), the power of X_{123} vanishes, hence the putative expression for the y-dependence is of the form

$$[X_1X_2X_3]^{-1}\sim[(y_1-a_1)\,(y_2-a_2)\,(y_3-a_3)]^{-1}\ , \tag{4.27}$$

where we have ignored the overall numerical coefficients and structure constants and suppressed the antiholomorphic contributions. However, in this special case there exist an additional distributional solution given by the replacements

$$(y_i-a_i)^{-1}\to\delta(y_i-a_i)\,,\qquad i=1,2,3\,. \tag{4.28}$$

This is the correct solution for $j=\frac{1}{2}$ and $k=3$, and we get an additional localization: the y-variables explicitly become precisely the covering map coefficients. While generically the SL(2,$\mathbb{R}$)$_k$ correlators have divergences coming from the integrals over the y-variables associated to contributions near the locus where some of the generalized differences X_I vanish, for instance near $y_i=a_i$, the situation is different for the distributional solution we have just found. Here the y-integrals leading to the x-basis correlators can be realized explicitly, and lead to a finite answer, with all factors of y_i replaced by a_i. More explicitly, these integrals take the form

$$\int d^2y_i y_i^{\frac{k}{2}\omega_i+j_i-h_i-1}\bar{y}_i^{\frac{k}{2}\omega_i+j_i-\bar{h}_i-1}\delta^{(2)}(y_i-a_i)=a_i^{-h_i-\bar{h}_i+3\omega_i-1}\,, \tag{4.29}$$

and similarly for the antiholomorphic sector. This reproduces the h_i-dependence anticipated in Eq. (4.20). Moreover, for this particular choice of spins we can be more explicit and fix the dependence on the spectral flow charges ω_i and the covering map data completely. The unflowed structure constant $\mathcal{N}\left(\frac{1}{2}\right)C\left(1,\frac{1}{2},\frac{1}{2}\right)$ gives a trivial numerical factor, hence according to Eq. (3.57) we are left with

$$\langle V_{h_1}^{\omega_1}(0,0)V_{h_2}^{\omega_2}(1,1)V_{h_3}^{\omega_3}(\infty,\infty)\rangle_{k=3}\sim\prod_{i=1}^{3}a_i^{-h_i-\bar{h}_i+3\omega_i-1}P^2_{\boldsymbol{\omega}-e_i}\,, \tag{4.30}$$

where we have omitted the j_i since they are all set to $\frac{1}{2}$. As it turns out, by using the definitions of the numbers $P_{\boldsymbol{\omega}}$ as well as various identities related to the covering maps, one can re-write this as

$$\langle V_{h_1}^{\omega_1}(0,0)V_{h_2}^{\omega_2}(1,1)V_{h_3}^{\omega_3}(\infty,\infty)\rangle_{k=3}\sim\prod_{i=1}^{3}a_i^{-h_i-\bar{h}_i+\frac{3}{2}(\omega_i-1)}\omega_i^{-\frac{3}{2}(\omega_i+1)+1}\Pi^{-3}\,, \tag{4.31}$$

where $\Pi=\prod_{a=1}^{N}r_a$ stands for the product of all the residues of the covering map $\Gamma(z)$, namely $r_a=\lim_{z\to z_a^*}(z-z_a^*)\Gamma(z)$, which was computed in [55]. This precisely reproduces the symmetric orbifold three-point functions [55, 76] with $c_{\rm seed} = 6k = 18$, see Appendix D.

We now move to the more interesting case of four-point function. For $j_i = \frac{1}{2}$ and $k = 3$ the conjecture of [53] for the y-basis correlator reads

$$\begin{aligned}\langle V^{\omega_1}_{h_1}(0,y_1,0)V^{\omega_2}_{h_2}(1,y_2,1)V^{\omega_3}_{h_3}(\infty,y_3,\infty)V^{\omega_4}_{h_4}(x,y_4,z)\rangle_{k=3}& \\ = |X_\emptyset X_{12}X_{34}|^{-2}\langle V_{\frac{1}{2}}(0,0)V_{\frac{1}{2}}(1,1)V_{\frac{1}{2}}(\infty,\infty)V_{\frac{1}{2}}(X,z)\rangle_{k=3}\,,&\end{aligned} \tag{4.32}$$

where the X_I are the generalized differences introduced in Sec. 3.4, while X is the generalized cross-ratio. In particular, the factor $X_\emptyset$ is proportional to the polynomial $P_{\boldsymbol{\omega}}(x,z)$, which defines the necessary condition for the existence of the relevant covering map, i.e. $P_{\boldsymbol{\omega}}(x,z) = 0$. In other words, $P_{\boldsymbol{\omega}}(x,z)$ vanishes whenever $\Gamma(z) = x$. Here it appears with power -1, signaling that there is an alternative solution where we replace

$$|X_\emptyset|^{-2} \to \delta^{(2)}(X_\emptyset) \propto \delta^{(2)}\left(x - \Gamma(z)\right)\,, \tag{4.33}$$

as expected from the general discussion of the previous section. Moreover, at the locus $P_{\boldsymbol{\omega}}(x,z) = 0$ the rest of Eq. (4.32) simplifies considerably since [77]

$$X_{12} = (1-a_1^{-1}y_1)(1-a_2^{-1}y_2)\,, \quad ,X_{34} = (1-a_3^{-1}y_3)(1-a_4^{-1}y_4) \quad X = z\,, \tag{4.34}$$

where we have ignored some overall signs. The behavior of the unflowed correlator in the limit $X \to z$ can be derived directly from the KZ equation (2.168) by inserting $\mathcal{F} = |X-z|^{2\delta}$, giving $a = 0$ or $a = k - \sum_{i=1}^4 j_i = 1$. We keep only the former solution, otherwise the divergence of the prefactor would cancel since $X - z$ is actually proportional to $X_\emptyset$. Hence, the dependence on the spacetime cross-ratio x is reduced that of the overall delta function. One can also derive the powers of z and $z - 1$ in this limit analogously. In fact, for this particular choice of spins the unflowed correlator can even be computed exactly from Liouville theory [77]. When the dust settles, all explicit powers of z and $z - 1$ cancel out (although the covering map coefficients implicitly depend on z), giving

$$\langle V^{\omega_1}_{h_1}(0,y_1,0)V^{\omega_2}_{h_2}(1,y_2,1)V^{\omega_3}_{h_3}(\infty,y_3,\infty)V^{\omega_4}_{h_4}(x,y_4,z)\rangle_{k=3} \sim \delta^{(2)}\left(P_{\boldsymbol{\omega}}(x,z)\right)\prod_{i=1}^4(1-a_i^{-1}y_i)^{-1}\,. \tag{4.35}$$

Once again, this suggests the possibility of an alternative solution where the last factors on the RHS are replaced by delta functions[22] $\delta(y_i - a_i)$. Indeed, as in the previous section, we could have derived this from first principles since we do have a covering map, i.e. without starting from the general form conjectured in [53]. For our four-point functions the y-basis differential equations read

$$\left\{(y_i-a_i)^2\partial_{y_i} + 2j_i(y_i-a_i) + \frac{a_i\omega_i}{N}\left[4 + \sum_{j=1}^4(y_j-a_j)\partial_{y_i}\right]\right\}F_y = 0\,, \tag{4.36}$$

[22]The reader might worry about the difference between integrating these two solutions over y_i. For the solution with delta functions this is trivial, while for the other one this can be done using (2.42). The latter integral gives various γ functions which combine into the reflection coefficients in (3.44) associated to each of the vertex operators. However, as the flip $j_i \to 1 - j_i$ leaves the condition $\sum_{i=1}^4 j_i = k - 1$ invariant for $k = 3$, these coefficients must be trivial.

which is solved by both

$$\prod_{i=1}^{4}(y_i - a_i)^{-1}\,, \qquad \text{and} \qquad \prod_{i=1}^{4}\delta(y_i - a_i). \tag{4.37}$$

where the first solution gives the usual form $X_{1234}^{k-1-\sum j_i}\prod_i X_i^{-2j_i}$ four our particular choice of spins j_i and level k. For the second solution, we find that, upon inserting the factors $y_i^{\frac{k}{2}\omega_i+j_i-h_i-1} = y_i^{\frac{3}{2}\omega_i-h_i-\frac{1}{2}}$ and integrating over worldsheet cross-ratio z and over all y_i, one obtains a string correlator of the form [77]

$$\sum_{\Gamma}|\Pi|^{-3}\prod\omega_i^{-\frac{3(\omega_i+1)}{2}}a_i^{-h_i+\frac{3(\omega_i-1)}{4}}\bar{a}_i^{-\bar{h}_i+\frac{3(\omega_i-1)}{4}}. \tag{4.38}$$

Once again, this reproduces the known answer for the four-point functions of dressed twist operators in symmetric orbifold theories [55, 58, 59] with $c_{\text{seed}} = 6k = 18$. Such correlators are computed by using covering-space methods, where one goes to a covering of the physical space — here the boundary of AdS$_3$ — where the boundary conditions enforced by the twist operator insertions trivialize. The string computation shows that, in the tensionless limit, the string worldsheet *is* precisely this covering space. As anticipated above, the derivation shows that the worldsheet spectral flow charges must be identified with the twists of the corresponding boundary insertions.

4.5 Back to (stringy) geometry

First of all, what geometry? We are working in the tensionless limit, where the AdS$_3$ target space is string-size. In principle, there is no such thing as a an interpretation in terms of particles moving on the base manifold subject to semi-classical gravitational interactions. Nevertheless, we can follow [43] to try and use the Wakimoto fields to provide a geometrical intuition for the correlation functions we have computed in the previous sections.

Let us recall our two main conclusions so far:

- the only states we consider are long strings with $j = \frac{1}{2}$, and
- their correlators localize on configurations for which an appropriate holomorphic covering map $\Gamma(z)$ from the worldsheet to the boundary exists.

As discussed above, we can think of these long strings as having zero radial momentum. Hence, once they are created by an insertion on the boundary, they will remain pinned to it. Since they remain far from the interior[23], we expect the Wakimoto free-field description to be accurate.

Now, consider the Wakimoto field γ. As discussed around Eq. (2.128), near a spectrally flowed vertex operator we have

$$\gamma(z)V_{h_i}^{\omega_i}(x_i,z_i) \sim x_i V_{h_i}^{\omega_i}(x_i,z_i) + (z-z_i)^{\omega_i}V_{h_i-1}^{\omega_i}(x_i,z_i)\,, \tag{4.39}$$

[23]See however the recent discussion in [112] about the large ω limit.

In terms of correlators, we know that, for the solution (4.20), shifting $h_i \to h_i - 1$ amounts to multiplying the original n-point function by a_i. Hence, we get

$$\langle \gamma(z \sim z_i) \prod_{j=1}^{n} V_{h_j}^{\omega_j}(x_j, z_j) \rangle \sim (x + a_i (z - z_i)^{\omega_i} + \cdots) \langle \prod_{j=1}^{n} V_{h_j}^{\omega_j}(x_j, z_j) \rangle \,, \tag{4.40}$$

for all $i = 1, \ldots, n$. Given that $\gamma(z)$ is holomorphic, has weight zero and remains finite in the $z \to \infty$ limit, it is natural to propose that

$$\langle \gamma(z) \prod_{j=1}^{n} V_{h_j}^{\omega_j}(x_j, z_j) \rangle = \Gamma(z) \langle \prod_{j=1}^{n} V_{h_j}^{\omega_j}(x_j, z_j) \rangle \,. \tag{4.41}$$

At first sight, this makes perfect sense since the semiclassical solutions are located at $\gamma = x$ on the boundary! This also justifies, at least heuristically, choosing the distributional solutions such as (4.20) over any alternative. Indeed, the LHS of (4.41) is always well defined, and is a holomorphic function with all the right properties to be a covering map, hence the primary correlator should vanish unless the map $\Gamma(z)$ actually exists.

We can proceed similarly with the non-compact field ϕ, for which the derivative satisfies

$$\partial \phi(z) V_{h_i}^{\omega_i}(x_i, z_i) \sim -\sqrt{2} \left(\frac{\omega_i - 1}{2(z - z_i)} \right) V_{h_i}^{\omega_i}(x_i, z_i) \,, \tag{4.42}$$

where we have used Eq. (2.129) with $j_i = \frac{1}{2}$ and $k = 3$. Given that $\partial \Gamma(z \sim z_i) \sim A(z - z_i)^{\omega_i - 1}$ for some constant A, we have

$$\frac{\partial^2 \Gamma}{\partial \Gamma} = \sum_{i=1}^{n} \frac{\omega_i - 1}{z - z_i} - \sum_{a=1}^{N} \frac{2}{z - z_a^*} \,, \tag{4.43}$$

hence the candidate relation with the covering map then reads

$$\langle \partial \phi(z) \prod_{j=1}^{n} V_{h_j}^{\omega_j}(x_j, z_j) \rangle = \sqrt{2} \left[- \frac{\partial^2 \Gamma(z)}{2 \partial \Gamma(z)} \right] \langle \prod_{j=1}^{n} V_{h_j}^{\omega_j}(x_j, z_j) \rangle \,. \tag{4.44}$$

In other words, this suggests that inside tensionless string correlators the following identities hold:

$$\gamma(z) = \Gamma(z) \ , \qquad \phi(z, \bar{z}) \sim \phi_0 - \sqrt{2} \log |\partial \Gamma| \,, \tag{4.45}$$

where ϕ_0 a constant we can formally take to be very large in order to be close to the boundary. This precisely reproduces what we expect from the classical solutions. For instance, we can look at a two-point functions with insertions of spectral flow charge $\omega_1 = \omega_2 = \omega$ at $z_1 = x_1 = 0$ and $z_2 = x_2 \to \infty$, for which the classical long string configuration reads

$$\gamma(z) = z^{\omega} \,, \qquad \phi(z) = \phi_0 - \sqrt{2}(\omega - 1) \log |z| \,, \tag{4.46}$$

consistent with the corresponding covering map being $\Gamma(z) = z^w$.

However, a word of caution is in order: there is an important subtlety related to the identifications provided in Eqs. (4.41) and (4.44). Both the covering map $\Gamma(z)$ and the function $(\partial\Gamma)^{-1}\partial^2\Gamma$ have additional simple poles at poles at z_a^*, i.e. away form the original insertion points. More precisely, there are $N = 1 + \frac{1}{2}\sum_{i=1}^{n}(\omega_i - 1)$ such poles. How can they be there on the LHS of (4.41) and (4.44) if there are no insertions at these points on the RHS?

Although the picture is not completely clear, a possible resolution to this puzzle is as follows[24]. In our derivation of the recursion relations we have worked in the exact theory, and only made use of the OPEs between vertex operators and symmetry currents. On the other hand, in the free-field formalism involving the Wakimoto variables the correlators are defined perturbatively. Indeed, one needs to cancel the background charge Q_ϕ by including screening operators. The simplest screening operator is given by the interaction term in (2.63), namely (for $k = 3$)

$$\mathcal{S} = \oint dz e^{-\sqrt{2}\phi}\beta\bar{\beta}\,. \tag{4.47}$$

These has weight $\Delta = 0$ (after the integration) but charge -1 w.r.t. ϕ. In our conventions, and taking into account the background charge $Q_\phi = -1$, inserting (the integrand of) s screening operators leads to the neutrality condition

$$-s + \frac{1}{2}\sum_{i=1}^{n}(\omega_i - 1) = Q_\phi\,. \tag{4.48}$$

This means that the number of extra insertions we need is precisely $s = N$. Moreover, the presence of the field β in the integrand of $\mathcal{S}$ ensures that this will generate exactly N simple poles with the $\gamma(z)$ insertion in (4.41), and similarly for $\partial\phi(z)$.

Nevertheless, this is not fully satisfactory since, so far, there is no apparent reason why these poles should be located at the points z_a^*, which are the poles of the corresponding covering map. For this, we consider the correlator

$$\langle\prod_{i=1}^{n} V_{h_i}^{\omega_i}(x_i, z_i)\prod_{a=1}^{N}\left(e^{-\sqrt{2}\phi}\beta\bar{\beta}\right)(w_a)\rangle\,, \tag{4.49}$$

go back to our derivation of the recursion relations, and repeat the exercise carried out in Sec. 3.3.3, albeit making use of the current

$$J^-(\gamma, z) = J^-(z) - 2\gamma J^3(z) + \gamma^2 J^+(z)\,, \tag{4.50}$$

as opposed to using directly $J^-(\Gamma, z)$. This immediately implies that the correlator must be proportional to the derivative of the covering map, provided the additional insertion points w_a coincide with the location of the poles of $\Gamma(z)$, namely z_a^*, and vanishes otherwise. Of course, this argument relies on the Wakimoto field γ being holomorphic, which only holds near the boundary. In other words, from the free-field perspective the localization of correlation functions on the locus where the covering maps exist implies that the correlator (4.49) must be proportional to $\prod_{a=1}^{N}\delta^{(2)}(w_a - z_a^*)$. This trivializes the integration over the additional insertions, explaining why Eqs. (4.41) and (4.44) make sense.

[24] A similar argument was put forward in [43]. See also [113].

4.6 Comments on tensionless superstrings on $\mathbf{AdS}_3 \times S^3 \times T^4$

Before moving on, let us make a few comments on the supersymmetric case. If we insist in working with the RNS formalism at $n_5 = 1$, the model is of the form

$$\mathrm{SL}(2,\mathbb{R})_3 \times \mathrm{SU}(2)_{-1} \times \mathrm{U}(1)^4 \quad + \quad 10 \text{ free fermions}\,. \tag{4.51}$$

The problem is that the bosonic SU(2) has negative level and central charge, such that quantization becomes problematic and there are no unitary representations. Nevertheless, one can use the available free-field realization in terms of symplectic bosons which effectively behave as fermionic ghosts [61]. These then cancel four fermionic degrees of freedom, generating a consistent theory albeit with only four physical bosons and four physical fermions. In other words, the only remaining fields are those of the T^4 model, the internal CFT. Combined with the above discussion, we conclude that in the tensionless limit the holographic dual to strings in $\mathrm{AdS}_3 \times S^3 \times T^4$ *is* the symmetric orbifold $\mathrm{Sym}^{n_1}\left(T^4\right)$ itself.

Alternatively, one can work directly in the hybrid formalism, i.e. with the super WZW model based on $\mathrm{PSU}(1,1|2)_1$, as was done in originally in [63]. This leads to a beautiful story, which unfortunately lies beyond the scope of these notes. We simply mention that, in this context, the shortening conditions on the spectrum and the localization of worldsheet correlators[25] were proven [63, 111]. Moreover, the discussion was extended to higher genus, and the matching of the full string partition function was demonstrated explicitly [45, 46, 46, 65, 112, 114, 115].

Finally, we stress that the non-conservation of spectral flow shows that n_1, the number of fundamental strings in the background, is strictly speaking not constant. This is related to the operator $\mathcal{I}$ in (2.176) acting differently depending on the spectral flow sector. It was argued in [116] that one should think of the worldsheet theory as describing the system in the grand canonical ensemble. Indeed, the string partition function was matched with that of the grand canonical ensemble of symmetric orbifold models over n_1 [65].

[25] The argument is essentially the same as the one given above, except that for the $\mathrm{PSU}(1,1|2)_1$ model the free-field description is exact.

5 The exact chiral ring for strings in $\mathrm{AdS}_3 \times S^3 \times T^4$

We now continue with our holographic applications of the spectrally flowed worldsheet correlators derived in the Sec. 3. As a first step in going beyond the tensionless limit and work with arbitrary $k \geq 3$, we describe the propagation of superstrings in $\mathrm{AdS}_3 \times S^3 \times T^4$ and focus on the protected BPS sector. We present the worldsheet operators corresponding to chiral primaries of the spacetime theory, and derive their three-point functions. These are constant along the moduli space, and we show that they reproduce exactly the chiral ring as computed in the D1D5 CFT at the symmetric orbifold point [56, 74]. Our presentation draws heavily from [74].

5.1 Worldsheet model and short string spectrum

String propagation in $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ with NSNS fluxes is characterized by the supersymmetric WZW model based on $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)\times\mathrm{U}(1)^4$. As shown in [117–119], a field redefinition allows allows one to rewrite any $\mathcal{N}=1$ super WZW model at level k in terms of the bosonic WZW model with a shifted level $k-\mathfrak{c}$, where $\mathfrak{c}$ is the dual Coxeter number, and a system of free fermions. For the case at hand, we the relevant OPEs read

$$J^a(z)\psi^b(w) \sim \frac{i\epsilon^{ab}{}_c\psi^c(w)}{z-w}, \quad K^a(z)\chi^b(w) \sim \frac{i\epsilon^{ab}{}_c\chi^c(w)}{z-w}, \tag{5.1}$$

$$\psi^a(z)\psi^b(w) \sim \frac{\frac{n_5}{2}\eta^{ab}}{z-w}, \qquad \chi^a(z)\chi^b(w) \sim \frac{\frac{n_5}{2}\delta^{ab}}{z-w}, \tag{5.2}$$

where K^a are the currents of the SU(2) model studied in Sec. A.2, while ψ^a and χ^a are the $\mathrm{SL}(2,\mathbb{R})$ and SU(2) fermions, respectively. In both cases, the supersymmetric level is set by n_5, the number of NS5-brane sources, hence the AdS_3 and S^3 factors have identical radii. The supersymmetric currents split as

$$J^a = j^a + \hat{\jmath}^a\,, \qquad K^a = k^a + \hat{k}^a, \tag{5.3}$$

where j^a and k^a generate bosonic $\mathrm{SL}(2,\mathbb{R})_k$ and $\mathrm{SU}(2)_{k'}$ current algebras with $k = n_5 + 2$ and $k' = n_5 - 2$, and

$$\hat{\jmath}^a = -\frac{i}{n_5}\epsilon^a{}_{bc}\psi^b\psi^c\,, \qquad \hat{k}^a = -\frac{i}{n_5}\epsilon^a{}_{bc}\chi^b\chi^c\,, \tag{5.4}$$

generate fermionic $\mathrm{SL}(2,\mathbb{R})_{-2}$ and $\mathrm{SU}(2)_2$ algebras. In other words, at the algebraic level the decomposition reads

$$\mathcal{N}=1 \quad \mathfrak{sl}(2,\mathbb{R})_{n_5} \oplus \mathfrak{su}(2)_{n_5} \simeq \mathfrak{sl}(2,\mathbb{R})_{n_5+2} \oplus \mathfrak{su}(2)_{n_5-2} \oplus 6\,\text{free fermions}\,. \tag{5.5}$$

We also have the free bosons and fermions associated with the torus directions, namely Y^i and λ^i, with $i = 6,\dots,9$. It will be convenient to bosonize the fermions by introducing canonically normalized scalars H_I, $I = 1,\dots 5$, such that[26]

$$\psi^{\pm} = \sqrt{n_5}\,e^{\pm iH_1}\,, \quad \chi^{\pm} = \sqrt{n_5}\,e^{\pm iH_2}\,, \quad \lambda^6 \pm i\lambda^7 = e^{\pm iH_4}\,, \quad \lambda^8 \pm i\lambda^9 = e^{\pm iH_5}\,, \tag{5.6a}$$

$$\psi^0 = \frac{\sqrt{n_5}}{2}\left(e^{iH_3} - e^{-iH_3}\right)\,, \qquad \chi^0 = \frac{\sqrt{n_5}}{2}\left(e^{iH_3} + e^{-iH_3}\right)\,. \tag{5.6b}$$

[26] One should refine this in order to keep track of the cocycle factors, replacing H_I by $\hat{H}_I$ with $\hat{H}_I = H_I + \pi\sum_{J<I} N_J$, where $N_J \equiv \oint i\partial H_J$ so that $e^{ia\hat{H}_I}e^{ib\hat{H}_J} = e^{ib\hat{H}_J}e^{ia\hat{H}_I}\,e^{i\pi ab}$ if $I > J$ [71].

with $H_I^\dagger = H_I$ for $I \neq 3$ and $H_3^\dagger = -H_3$.

The stress tensor T and supercurrent G describing the matter sector of the worldsheet CFT are

$$T = \frac{1}{n_5}\left(j^a j_a - \psi^a \partial \psi_a + k^a k_a - \chi^a \partial \chi_a\right) + \frac{1}{2}\left(\partial Y^i \partial Y_i - \lambda^i \partial \lambda_i\right), \tag{5.7}$$

$$G = \frac{2}{n_5}\left(\psi^a j_a + \frac{2i}{n_5}\psi^0\psi^1\psi^2 + \chi^a k_a - \frac{2i}{n_5}\chi^0\chi^1\chi^2\right) + i\,\lambda^i \partial Y_i\,. \tag{5.8}$$

so that the matter central charge gives

$$c = \frac{3(n_5+2)}{n_5} + \frac{3}{2} + \frac{3(n_5-2)}{n_5} + \frac{3}{2} + 4 + 2 = 15\,. \tag{5.9}$$

This is cancelled by the contribution from the standard bc and $\beta\gamma$ ghost systems. The BRST charge then reads

$$\mathcal{Q} = \oint dz \left[c\,(T + T_{\beta\gamma}) - \gamma\, G + c(\partial c)b - \frac{1}{4}b\gamma^2\right]. \tag{5.10}$$

The $\beta\gamma$ system (not to be confused with the Wakimoto fields) is further bosonized as $\beta = e^{-\varphi}\partial\xi$, $\gamma = e^{\varphi}\eta$, where φ has a background charge $Q_\varphi = -2$, while $\xi(z)\eta(w) \sim (z-w)^{-1}$. The spacetime supercharges can be constructed as [10]

$$Q_\varepsilon = \oint dz\, e^{-\varphi/2} S_\varepsilon\,, \qquad S_\varepsilon = \exp\left(\frac{i}{2}\sum_{I=1}^{5}\varepsilon_I H_I\right), \qquad \varepsilon_I = \pm 1\,. \tag{5.11}$$

BRST-invariance and mutual locality constraints impose $\varepsilon_1\varepsilon_2\varepsilon_3 = \varepsilon_4\varepsilon_5 = 1$, thus giving precisely the spacetime $\mathcal{N} = (4,4)$ superconformal algebra. Moreover, the zero-modes of the worldsheet SU(2) currents are identified with those of the spacetime R-symmetry.

Let us describe the short string sector of the spectrum. We first focus on the NSNS sector of the theory. The (holomorphic part of) vertex operators holographically dual to the low-lying chiral primaries of the holographic CFT is given by [70, 71]

$$\mathcal{V}_j(x,u,z) \;= e^{-\varphi(z)}\psi(x,z)V_j(x,z)W_{j-1}(u,z), \tag{5.12a}$$

$$\mathcal{W}_j(x,u,z) \;= e^{-\varphi(z)}V_j(x,z)\chi(u,z)W_{j-1}(u,z). \tag{5.12b}$$

in the canonical (-1) ghost picture. Here x is the holographic coordinate, u is the SU(2) isospin variable, z is the worldsheet coordinate, and $W_l(u,z)$ are the SU(2) primary fields. Given the weights $\Delta_j = -j(j-1)/n_5$ and $\Delta_l' = l(l+1)/n_5$, the Virasoro condition $\Delta_j + \Delta_l = 0$ is solved by setting $l = j-1$. As the supersymmetric zero-modes J_0^a are identified the holographic Virasoro modes at level $0, \pm 1$, the spacetime weight H, identified with the spin of the corresponding $\mathrm{SL}(2,\mathbb{R})_{n_5}$ may be shifted by the fermions. Indeed, one has $H = j$ for $\mathcal{W}_j$, while $H = j-1$ for $\mathcal{V}_j$, showing that we are indeed dealing with states belonging to the chiral multiplets of the HCFT.

On the other hand, states associated to spacetime chiral primaries with $H > n_5/2$ belong to the spectrally-flowed sectors of the worldsheet theory [72]. Bosonic flowed primaries where studied at length in the previous sections. A similar story holds for the SL(2,$\mathbb{R}$) fermions, for which, in the m-basis, we can write

$$\psi^{-,\omega}(z) = \sqrt{n_5} e^{-i(1+\omega)H_1(z)} \,, \quad \psi^{+,\omega}(z) = \sqrt{n_5} e^{i(1-\omega)H_1(z)} \,, \quad \psi^{0,\omega}(z) = \psi^0(z) e^{-i\omega H_1(z)}. \tag{5.13}$$

Their x-basis cousins are defined as usual, i.e. $\psi^\omega(x,z) = e^{x\hat{\jmath}_0^+}\psi^{-,\omega}(z)e^{-x\hat{\jmath}_0^+}$, and we also have $\hat{\jmath}^\omega(x,z) = e^{x\hat{\jmath}_0^+}\hat{\jmath}^{-,\omega}(z)e^{-x\hat{\jmath}_0^+}$.

Now, although including spectral flow in the SU(2) sector is, strictly speaking, not necessary, for short string states it is actually quite useful. Hence, we also introduce the operators $W^\omega_{l,q}(u,z)$, $\chi^\omega(u,z)$, $\hat{k}^\omega(u,z)$, which are analogous to their SL(2,$\mathbb{R}$) counterparts. The short-string flowed supersymmetric vertex operators then read

$$\mathcal{V}^\omega_j(x,u,z) = \frac{1}{\sqrt{n_5}} e^{-\varphi(z)}\psi^\omega(x,z)V^\omega_j(x,z)\chi^{\omega-1}(u,z)W^\omega_{j-1}(u,z), \tag{5.14a}$$

$$\mathcal{W}^\omega_j(x,u,z) = \frac{1}{\sqrt{n_5}} e^{-\varphi(z)}\psi^{\omega-1}(x,z)V^\omega_j(x,z)\chi^{\omega}(u,z)W^\omega_{j-1}(u,z). \tag{5.14b}$$

Indeed, flowing the SU(2) sector allows us to cancel the extra contributions to the worldsheet weights generated by the SL(2,$\mathbb{R}$) spectral flow, so that flowed states also satisfy $l = j-1$. Here we have also introduced the shorthands

$$V^\omega_j(x,z) \equiv V^\omega_{j,h=j_\omega}(x,z)\,, \quad W^\omega_l(u,z) \equiv W^\omega_{l,q=l_\omega}(u,z) \tag{5.15}$$

with $j_\omega = j + k\omega/2$ and $l_\omega = l + k'\omega/2$. The resulting spacetime weights are of the form

$$H\left[\mathcal{V}^\omega_j\right] = j - 1 + n_5\omega/2\,, \qquad H\left[\mathcal{W}^\omega_j\right] = j + n_5\omega/2\,. \tag{5.16}$$

We now move to the Ramond-Ramond (RR) sector. Here vertex operators are constructed from the spin fields defined in Eq. (5.11). We denote the AdS$_3\times$S^3 chirality as $\varepsilon = \varepsilon_1\varepsilon_2\varepsilon_3$, such that the GSO projection imposes $\varepsilon_4\varepsilon_5 = \varepsilon$. The relevant unflowed primaries involve SL(2,$\mathbb{R}$)$_{-2}\times$SU(2)$_2$ fields of spins $(-1/2, 1/2)$, namely[27]

$$s_\varepsilon(x,u,z) = e^{u\hat{k}_0^0}e^{x\hat{\jmath}_0^0}e^{\frac{i}{2}(-H_1(z)-H_2(z)+\varepsilon H_3(z))}e^{-x\hat{\jmath}_0^0}e^{-u\hat{k}_0^0}\,, \qquad \varepsilon = \pm 1. \tag{5.17}$$

BRST-invariant unflowed states have total spins $(j-1/2, l+1/2 = j-1/2)$, and take the form [68, 71]

$$\mathcal{Y}^\epsilon_j(x,u,z) = e^{-\frac{1}{2}\varphi(z)}s_-(x,u,z)V_j(x,z)W_{j-1}(u,z)e^{i\frac{\epsilon}{2}(H_4(z)-H_5(z))}, \tag{5.18}$$

where we have renamed $\varepsilon_4 \to \epsilon$. The corresponding flowed operators read

$$\mathcal{Y}^{\epsilon,\omega}_j(x,u,z) = e^{-\frac{1}{2}\varphi(z)}s^\omega_-(x,u,z)V^\omega_j(x,z)W^\omega_{j-1}(u,z)e^{i\frac{\epsilon}{2}(H_4(z)-H_5(z))}, \tag{5.19}$$

where $s^\omega_-(x,u)$ is built upon the flowed primary $s^\omega_{---} = e^{-i\left(\frac{1}{2}+\omega\right)H_1(z)-i\left(\frac{1}{2}+\omega\right)H_2(z)-\frac{i}{2}H_3(z)}$, the extremal state in a spin $(-1/2-\omega, 1/2+\omega)$ representation of the fermionic zero-mode algebra. Consequently, the spacetime weights are

$$H\left[\mathcal{Y}^{\epsilon,\omega}_j\right] = j - 1/2 + n_5\omega/2. \tag{5.20}$$

[27] Cocycle factors are important for computing the RHS of (5.17).

Let us now describe what each of these operators corresponds to in terms of the D1D5 CFT at the symmetric orbifold point, namely $\text{Sym}^N\left(T^4\right)$ with $N = n_1 n_5$, following [69, 71, 72]. At large N, we should identify single-string states in the bulk with single-cycle fields of the dual CFT. As discussed in Appendix D, each twist sector contains four chiral primary multiplets [56]. We will denote them as $O_n^-(x)$, $O_n^\epsilon(x)$ and $O_n^+(x)$, respectively. Their holomorphic weights are given by

$$H\left[O_n^-\right] = \frac{n-1}{2}, \quad H\left[O_n^\epsilon\right] = \frac{n}{2}, \quad H\left[O_n^+\right] = \frac{n+1}{2}; \quad n = 1, 2, \dots. \tag{5.21}$$

These local operators of the HCFT the boundary correspond to z-integrated x-basis operators of the worldsheet theory. Focusing on chiral primary operators that are spacetime scalars, the worldsheet operators constructed above are promoted to NSNS and RR vertex operators by including the anti-holomorphic polarizations, giving, for instance,

$$\begin{aligned}
&\mathbb{V}_j^\omega(x,\bar{x},u,\bar{u},z,\bar{z}) \equiv \\
&\frac{1}{n_5} e^{-\varphi(z)-\bar{\varphi}(\bar{z})} \psi^\omega(x,z)\bar{\psi}^\omega(\bar{x},\bar{z}) V_j^\omega(x,\bar{x},z,\bar{z}) \chi^{\omega-1}(u,z)\bar{\chi}^{\omega-1}(\bar{u},\bar{z}) W_{j-1}^\omega(u,\bar{u},z,\bar{z}),
\end{aligned} \tag{5.22}$$

where we have momentarily reinstated the dependence of the bosonic primaries in the anti-holomorphic variables. $\mathbb{W}_j^\omega$ and $\mathbb{Y}_j^{\epsilon,\omega}$ are defined similarly. Hence, up to the normalization (to be discussed below) the holographic dictionary is given by

$$\begin{aligned}
O_n^-(x,\bar{x},u,\bar{u}) &\leftrightarrow \mathbb{V}_j^\omega(x,\bar{x},u,\bar{u},z,\bar{z}), \\
O_n^\epsilon(x,\bar{x},u,\bar{u}) &\leftrightarrow \mathbb{Y}_j^{\epsilon,\omega}(x,\bar{x},u,\bar{u},z,\bar{z}), \\
O_n^+(x,\bar{x},u,\bar{u}) &\leftrightarrow \mathbb{W}_j^\omega(x,\bar{x},u,\bar{u},z,\bar{z}),
\end{aligned} \tag{5.23}$$

together with the identification

$$n \;=\; 2j - 1 + n_5\,\omega\,. \tag{5.24}$$

From the worldsheet point of view, the allowed ranges are

$$j = 1, \frac{3}{2}, \dots, \frac{n_5}{2}, \qquad \omega = 0, 1, \dots\,, \tag{5.25}$$

which shows that the worldsheet theory accounts for all chiral primaries of the holographic CFT, except for those in the twisted sectors where n is a (non-zero) multiple of n_5 [71, 72]. These would sit exactly at the lower boundary of the allowed range for j. However, at this point, the spectrum degenerates due to the presence of the zero-momentum states belonging to the continuous representations [34, 48]. This indicates that the NS5-F1 model sits at a singular point in moduli space, see [78]. Upon (perturbatively) including RR fluxes, the long-string sector is lifted, and this issue is resolved [120].

Finally, we also need to discuss picture changing. Due to the presence of the ghost background charge, for three-point functions it is necessary to derive the ghost picture (0) version of the NSNS operators constructed above. These are obtained as

$$\mathcal{O}^{(0)}(z) = \lim_{w\to z}\left[e^{\varphi(w)}G(w)\right]\mathcal{O}^{(-1)}(z)\,. \tag{5.26}$$

We obtain

$$\mathcal{V}_j^{\omega,(0)}(x,u,z) = \mathcal{A}_j^{\omega,1}(x,u,z) + (-1)^\omega \mathcal{A}_j^{\omega,2}(x,u,z)\,, \tag{5.27}$$

with

$$\mathcal{A}_j^{\omega,1}(x,u,z) = \left[j^-_{-1-\omega}(x,z) - H\hat{\jmath}^-_{-1-\omega}(x,z)\right]\hat{\psi}^\omega(x,z)V_j^\omega(x,z)\hat{\chi}^\omega(u,z)W^\omega_{j-1}(u,z) \tag{5.28}$$

$$\mathcal{A}_j^{\omega,2}(x,u,z) = -\frac{1}{n_5}\left[k^+_\omega(u,z) - H\hat{k}^+_w(u,z)\right]\psi^\omega(x,z)V_j^\omega(x,z)\chi^\omega(u,z)W^\omega_{j-1}(u,z), \tag{5.29}$$

and

$$\mathcal{W}_j^{\omega,(0)}(x,u,z) = \mathcal{B}_j^{\omega,1}(x,u,z) + (-1)^\omega \mathcal{B}_j^{\omega,2}(x,u,z)\,, \tag{5.30}$$

with

$$\mathcal{B}_j^{\omega,1}(x,u,z) = \left[k^-_{-1-\omega}(u,z) - H\hat{k}^-_{-1-\omega}(u,z)\right]\hat{\psi}^\omega(x,z)V_j^\omega(x,z)\hat{\chi}^\omega(u,z)W^\omega_{j-1}(u,z), \tag{5.31}$$

$$\mathcal{B}_j^{\omega,2}(x,u,z) = \frac{1}{n_5}\left[j^+_\omega(x,z) + H\hat{\jmath}^+_\omega(x,z)\right]\psi^\omega(x,z)V_j^\omega(x,z)\chi^\omega(u,z)W^\omega_{j-1}(u,z). \tag{5.32}$$

In the RR sector, we will also need the field (5.19) in its $(-3/2)$ ghost picture version for computing two-point functions. This gives

$$\mathcal{Y}_j^{\epsilon,\omega,(-\frac{3}{2})}(x,u,z) = -\frac{\sqrt{n_5}}{2j-1+n_5\omega}e^{-\frac{3}{2}\varphi}s_+^\omega(x,u,z)V_j^\omega(x,z)W^\omega_{j-1}(u,z)e^{i\frac{\epsilon}{2}(H_4-H_5)}\,, \tag{5.33}$$

where $s_+^\omega(x,u)$ is the x- and u-basis version of $s^\omega_{--+} = e^{-i\left(\frac{1}{2}+\omega\right)H_1 - i\left(\frac{1}{2}+\omega\right)H_2 + \frac{i}{2}H_3}$.

5.2 Flowed primary correlators in the SU(2) and fermionic sectors

In the SU(2) sector, spectrally-flowed correlators are merely complicated linear combinations of primary and descendant unflowed correlators. Nevertheless, we can use techniques analogous to those of Sec. 3 to derive them directly. For this, we introduce the v-basis operators

$$\tilde{W}_l(u,v,z) = \sum_{n,\bar{n}=-l}^{l} v^{l+n}\bar{v}^{l+\bar{n}} W_{l,n+\frac{k'}{2}\omega,\bar{n}+\frac{k'}{2}\omega}(u,z)\,, \tag{5.34}$$

in analogy with the SL(2,$\mathbb{R}$) y-basis construction. By using the same covering maps, it is easy to see that three-point functions of flowed SU(2) primaries must satisfy the same recursion relations as those of the SL(2,$\mathbb{R}$) model, albeit with the replacements

$$k \to -k'\,, \qquad j_i \to -l_i\,. \tag{5.35}$$

It follows that, after fixing the overall dependence in the worldsheet, one gets

$$\begin{aligned}\langle \tilde{W}^{\omega_1}_{l_1}(v_1)\tilde{W}^{\omega_2}_{l_2}(v_2)\tilde{W}^{\omega_3}_{l_3}(v_3)\rangle &= N'_{\text{odd}}\,(v_1-a_1)^{2l_1}(v_2-a_2)^{2l_2}(v_3-a_3)^{2l_3} \\ &\times\left(\omega_1\frac{v_1+a_1}{v_1-a_1} + \omega_2\frac{v_2+a_2}{v_2-a_2} + \omega_3\frac{v_3+a_3}{v_3-a_3} - 1\right)^{-\frac{k'}{2}+l_1+l_2+l_3},\end{aligned} \tag{5.36}$$

for odd parity correlators, and

$$\langle \tilde{W}^{\omega_1}_{l_1}(v_1)\tilde{W}^{\omega_2}_{l_2}(v_2)\tilde{W}^{\omega_3}_{l_3}(v_3)\rangle = N'_{\rm even}\left(1-\frac{v_2}{a_2[\Gamma_3^+]}-\frac{v_3}{a_3[\Gamma_2^+]}+\frac{v_2v_3}{a_2[\Gamma_3^-]a_3[\Gamma_2^+]}\right)^{-l_1+l_2+l_3} \times\left(1-\frac{v_1}{a_1[\Gamma_3^+]}-\frac{v_3}{a_3[\Gamma_1^+]}+\frac{v_1v_3}{a_1[\Gamma_3^-]a_3[\Gamma_1^+]}\right)^{-l_2+l_3+l_1} \times\left(1-\frac{v_1}{a_1[\Gamma_2^+]}-\frac{v_2}{a_2[\Gamma_1^+]}+\frac{v_1v_2}{a_1[\Gamma_2^+]a_2[\Gamma_1^-]}\right)^{-l_3+l_1+l_2}, \tag{5.37}$$

for even parity correlators. The normalizations are given by

$$N'_{\rm even}(l_i,\omega_i) = C'\left(l_1,l_2,l_3\right)\tilde{N}'_{\rm even}\,, \quad N'_{\rm odd}(l_i,\omega_i) = C'\left(k'/2-l_1,l_2,l_3\right)\tilde{N}'_{\rm odd}\,, \tag{5.38}$$

where $C'(l_1,l_2,l_3)$ are the unflowed SU(2) three-point functions computed in [42], while

$$\tilde{N}'_{\rm even}(l_i,\omega_i) = P^{-l_1-l_2-l_3+k'}_{(\omega_1,\omega_2,\omega_3)}P^{-l_3+l_2+l_1}_{(\omega_1+1,\omega_2+1,\omega_3)}P^{-l_1+l_2+l_3}_{(\omega_1,\omega_2+1,\omega_3+1)}P^{-l_2+l_3+l_1}_{(\omega_1+1,\omega_2,\omega_3+1)} \tag{5.39}$$

and

$$\tilde{N}'_{\rm odd}(l_i,\omega_i) = \left(\frac{P_{(\omega_1-1,\omega_2-1,\omega_3-1)}}{\omega_1+\omega_2+\omega_3-1}\right)^{-\frac{k'}{2}+l_1+l_2+l_3}P^{-l_3-l_2+l_1+\frac{k'}{2}}_{(\omega_1-1,\omega_2,\omega_3)}P^{-l_3+l_2-l_1+\frac{k'}{2}}_{(\omega_1,\omega_2-1,\omega_3)}P^{l_3-l_2-l_1+\frac{k'}{2}}_{(\omega_1,\omega_2,\omega_3-1)}\,. \tag{5.40}$$

We will also need correlators involving flowed fermions and spin fields. They can be deduced by noting that $\psi^\omega(x,z)$ ($\chi^\omega(u,z)$) is the flowed version of a spin $\hat{\jmath}=-1$ ($\hat{l}=1$) unflowed fermion, and belongs to a spin $\hat{\jmath}_\omega=-1-\omega$ ($\hat{l}_\omega=1+\omega$) representation of the zero-mode algebra of $\mathrm{SL}(2,\mathbb{R})_{-2}$ (respectively $\mathrm{SU}(2)_2$). Similarly, the fermionic field $\hat{\psi}^\omega(x,z)$ ($\hat{\chi}^\omega(u,z)$) can be seen as the flowed version of the fermionic identity, with flowed spin $\hat{\jmath}_\omega=-\omega$ ($\hat{l}_\omega=\omega$). Finally, the spin fields $s^\omega_\pm(x,u,z)$ belong to $\mathrm{SL}(2,\mathbb{R})_{-2}\times\mathrm{SU}(2)_2$ representations with flowed spins $(\hat{\jmath}_\omega,\hat{l}_\omega)=(-1/2-\omega,1/2+\omega)$, obtained by the application of spectral flow on a spin $(\hat{\jmath},\hat{l})=(-1/2,1/2)$ state. Consequently, all relevant spectrally-flowed fermionic correlators are obtained from the bosonic formulas given above by inserting the corresponding spins and levels.

5.3 Spectrally-flowed correlators involving current insertions

Let us consider NS-NS-NS short-string three-point functions, and focus on those involving only states polarized in the AdS_3 directions for concreteness. These correlators are computed by inserting two vertex operators with ghost picture (-1) and one with ghost picture (0). We see from Eq. (5.27) that it is not enough to know the SL(2,$\mathbb{R}$) primary correlators, as one must also compute descendant correlators of the form

$$\langle V^{\omega_1}_{j_1}(x_1,z_1)V^{\omega_2}_{j_2}(x_2,z_2)\left(j^{\omega_3}V^{\omega_3}_{j_3}\right)(x_3,z_3)\rangle\,, \tag{5.41}$$

where we use the shorthand defined in Eq. (5.15), such that $V^\omega_j(x,z)\equiv V^\omega_{j,h}(x,z)$ with $h=j+\frac{k}{2}\omega$, which belongs to the $\mathcal{D}^{+,\omega}_j$ representation, while $\left(j^\omega V^\omega_j\right)(x,z)$ stands for

$$\left(j^\omega V^\omega_j\right)(x,z) = e^{xj_0^+}\oint_z dw\frac{j^-(w)V^\omega_{j,j}(z)}{(w-z)^{1+\omega}}e^{-xj_0^+}\,. \tag{5.42}$$

Here $V^{\omega}_{j,j}(z)$ is a flowed primary m-basis operator, derived from an unflowed lowest-weight state, as opposed to an x-basis operator. Correlators of the form (5.41) were identified one of the main obstacles for the worldsheet computation of structure constants of the holographic chiral ring by the authors of [72, 73].

The relevant bosonic three-point functions involve the action of the mode $j^-_{-1-\omega}$ on a vertex operator $V^{\omega}_j(x,z)$, which is a negative mode of the current j^- in the corresponding spectrally-flowed frame. The problem is that, since these are not affine primary states, one cannot use the usual contour integration techniques to compute this. Nevertheless, it turns out that by using the series identification (2.106) one gets

$$\left(j^{\omega}V^{\omega}_j\right)(x,z) = \mathcal{N}(j)e^{xj_0^+}\oint_z dw\frac{1}{(w-z)^{1+\omega}}j^-(w)V^{\omega+1}_{\frac{k}{2}-j,-(\frac{k}{2}-j)}(z)e^{-xj_0^+}\,, \tag{5.43}$$

where $V^{\omega+1}_{\frac{k}{2}-j,-(\frac{k}{2}-j)}(z)$ is the highest-weight m-basis operator of the $\mathcal{D}^{-,\omega}_{\frac{k}{2}-j}$ representation. Crucially, the current mode $j^-_{-1-\omega}$ can be seen as a zero-mode in the spectrally-flowed frame associated to an operator with charge $\omega+1$. Hence, this leads to

$$\left(j^{\omega}V^{\omega}_j\right)(x,z) = -\mathcal{N}(j)e^{xj_0^+}V^{\omega+1}_{\tilde{j},-\tilde{j}-1}(z)e^{-xj_0^+} = -\mathcal{N}(j)V^{\omega+1}_{\tilde{j},\tilde{h}-1}(x,z), \tag{5.44}$$

where

$$\tilde{j} = k/2 - j\,, \quad \tilde{h} = -\tilde{j} + k(\omega+1)/2\,. \tag{5.45}$$

In Eq. (5.44), $V^{\omega+1}_{\tilde{j},-\tilde{j}-1}(z)$ is an m-basis operator with spin projection $m=-\tilde{j}-1$ belonging to the $\mathcal{D}^{-,\omega+1}_{\tilde{j}}$, while $V^{\omega+1}_{\tilde{j},\tilde{h}-1}(x,z)$ is its x-basis counterpart.

We can then express the relevant correlation functions with current insertions in terms of flowed primary correlators, namely

$$\begin{aligned}\langle V^{\omega_1}_{j_1}(x_1,z_1)V^{\omega_2}_{j_2}(x_2,z_2)\left(j^{\omega_3}V^{\omega_3}_{j_3}\right)(x_3,z_3)\rangle &= \\ &-\mathcal{N}(j_3)\,\langle V^{\omega_1}_{j_1}(x_1,z_1)V^{\omega_2}_{j_2}(x_2,z_2)V^{\omega_3+1}_{\tilde{j}_3,\tilde{h}_3-1}(x_3,z_3)\rangle\,.\end{aligned} \tag{5.46}$$

In y-basis language, the correlator appearing on the RHS of (5.46) reads

$$\langle V^{\omega_1}_{j_1}V^{\omega_2}_{j_2}V^{\omega_3+1}_{\tilde{j}_3,\tilde{h}_3-1}\rangle = \lim_{y_3\to\infty} y_3^{k-2j_3}\,\langle \tilde{V}^{\omega_1}_{j_1}(y_1=0)\tilde{V}^{\omega_2}_{j_2}(y_2=0)D^-_{y_3,\tilde{j}_3}\tilde{V}^{\omega_3+1}_{\tilde{j}_3}(y_3)\rangle\,, \tag{5.47}$$

where we have fixed the insertions at $(z_1,z_2,z_3)=(x_1,x_2,x_3)=(0,1,\infty)$, and used that for half-BPS states we are dealing with lowest-weight vertex operators. By using Eqs. (3.91) and (3.106) we obtain

$$\langle V^{\omega_1}_{j_1}V^{\omega_2}_{j_2}V^{\omega_3+1}_{\tilde{j}_3,\tilde{h}_3-1}\rangle = \alpha_{\boldsymbol{\omega}}\lim_{y_3\to\infty} y_3^{k-2j_3}\,\langle \tilde{V}^{\omega_1}_{j_1}(y_1=0)\tilde{V}^{\omega_2}_{j_2}(y_2=0)\tilde{V}^{\omega_3+1}_{\tilde{j}_3}(y_3)\rangle\,, \tag{5.48}$$

where the coefficient is given by

$$\alpha_{\boldsymbol{\omega}} \equiv \begin{cases} \dfrac{2a_3[\Gamma^{++}_{13}]\left[(\omega_1-\omega_2)(j_1-j_2)+(\omega_3+1)(\frac{k}{2}-j_3)\right]}{\omega_1+\omega_3-\omega_2+1} & \text{if } \displaystyle\sum_{i=1}^3\omega_i\in 2\mathbb{Z}+1, \\[2ex] \dfrac{2a_3[\Gamma^{+}_{3}]\left[(1+\omega_1+\omega_2)j_3-(1+\omega_3)(j_1+j_2)-\frac{k}{2}(\omega_1+\omega_2-\omega_3)\right]}{\omega_3-\omega_2-\omega_1} & \text{if } \displaystyle\sum_{i=1}^3\omega_i\in 2\mathbb{Z}. \end{cases} \tag{5.49}$$

Here $a_3[\Gamma_{13}^{++}]$ denotes the coefficient a_3 of the covering map $\Gamma[\omega_1+1,\omega_2,\omega_3+1](z)$. Finally, by using the series identification once more we conclude that

$$\langle V_{j_1}^{\omega_1} V_{j_2}^{\omega_2} \left(j^{\omega_3} V_{j_3}^{\omega_3}\right)\rangle = \alpha_{\boldsymbol{\omega}} \langle V_{j_1}^{\omega_1} V_{j_2}^{\omega_2} V_{j_3}^{\omega_3}\rangle \,. \tag{5.50}$$

Recall that this result holds only for the discrete lowest-weight states. Analogous formulas hold for the SL(2,$\mathbb{R}$) fermionic sector, and also for the SU(2) bosons and fermions.

5.4 Supersymmetric three-point functions

We now have all the necessary ingredients for computing the NS-NS-NS short-string correlators with arbitrary spectral flow charges. This includes not only the extremal correlators (in the holographic CFT language, i.e. those with $H_1 + H_2 = H_3$), some of which were briefly discussed in [72], but also the non-extremal ones.

We continue with correlators involving three vertex operators polarized in the AdS$_3$ directions, namely $\langle \mathcal{V}_{j_1}^{\omega_1} \mathcal{V}_{j_2}^{\omega_2} \mathcal{V}_{j_3}^{\omega_3,(0)}\rangle$. We take the picture (0) operator to be the one with the largest spectral flow charge, say ω_3, and set $(x_1,x_2,x_3) = (u_1,u_2,u_3) = (z_1,z_2,z_3) = (0,1,\infty)$. All y-basis (and v-basis) three-point functions involved in the supersymmetric computation[28] turn out to be regular in the limits $y_i \to 0$ (and $v_i \to 0$). We can thus select the relevant residues by setting $y_i = v_i = 0$ in all relevant y-basis and v-basis correlators. The vertex operators involved in $\langle \mathcal{V}_{j_1}^{\omega_1} \mathcal{V}_{j_2}^{\omega_2} \mathcal{V}_{j_3}^{\omega_3,(0)}\rangle$ were given in Eqs. (5.14) and (5.27). For even parity correlators only the first term in (5.27) contributes, while only the second one is relevant in the odd-parity cases. In the former case, we have

$$\langle \mathcal{V}_{j_1}^{\omega_1} \mathcal{V}_{j_2}^{\omega_2} \mathcal{V}_{j_3}^{\omega_3,(0)}\rangle = \langle \mathcal{V}_{j_1}^{\omega_1} \mathcal{V}_{j_2}^{\omega_2} \mathcal{A}_{j_3}^{\omega_3,1}\rangle \,, \qquad \sum_i \omega_i \in 2\mathbb{Z}. \tag{5.51}$$

This factorizes into ghost, bosonic and fermionic correlators, the latter including a current insertion, namely

$$\langle \psi^{\omega_1} \psi^{\omega_2} \left(\hat{\jmath}^{\omega_3} \hat{\psi}^{\omega_3}\right)\rangle = -\frac{2a_3[\Gamma_3^{+}](2+\omega_1+\omega_2+\omega_3)}{\omega_3-\omega_2-\omega_1} \langle \psi^{\omega_1} \psi^{\omega_2} \hat{\psi}^{\omega_3}\rangle, \tag{5.52}$$

where we have used (5.50) with $k \to \hat{k} = -2$, $j_{1,2} \to \hat{\jmath}_{1,2} = -1$ and $j_3 \to \hat{\jmath}_3 = 0$. Hence,

$$\begin{aligned} \langle \mathcal{V}_{j_1}^{\omega_1} \mathcal{V}_{j_2}^{\omega_2} \mathcal{V}_{j_3}^{\omega_3,(0)}\rangle &= (h_1+h_2+h_3-2) \\ &\times \frac{2(1+\omega_3)a_3[\Gamma_3^{+}]}{\omega_1+\omega_2-\omega_3} \langle V_{j_1}^{\omega_1} V_{j_2}^{\omega_2} V_{j_3}^{\omega_3}\rangle \langle \psi^{\omega_1} \psi^{\omega_2} \hat{\psi}^{\omega_3}\rangle \langle W_{j_1-1}^{\omega_1} W_{j_2-1}^{\omega_2} W_{j_3-1}^{\omega_3}\rangle \langle \hat{\chi}^{\omega_1} \hat{\chi}^{\omega_2} \hat{\chi}^{\omega_3}\rangle \,, \end{aligned} \tag{5.53}$$

where $h_i = j_i + n_5\omega_i/2 = H_i + 1$. The different spectrally-flowed primary three-point can be evaluated explicitly. Although the individual factors look somewhat complicated, the final result becomes extremely simple:

$$\langle \mathcal{V}_{j_1}^{\omega_1} \mathcal{V}_{j_2}^{\omega_2} \mathcal{V}_{j_3}^{\omega_3,(0)}\rangle = (h_1+h_2+h_3-2) n_5 \mathcal{C}(j_i). \tag{5.54}$$

where $\mathcal{C}(j_i)$ is the product of the SL$(2,\mathbb{R})_k$ and SU$(2)_{k'}$ *unflowed* three-point functions.

[28] The edge cases must be treated separately. It was shown in [74] these correlation functions vanish.

It was shown in [71] that for short strings the relation between the SL(2,$\mathbb{R}$) and SU(2) spins stemming from the Virasoro condition leads to important cancellations for the product of unflowed three-point functions contained in $\mathcal{C}(j_i)$, whose final expression takes the form

$$\mathcal{C}(j_i) \equiv C(j_i)C'(j_i-1) = \sqrt{\frac{b^2\gamma(-b^2)}{4\pi\nu}}\prod_{i=1}^{3}\sqrt{B(j_i)} \equiv Q\prod_{i=1}^{3}\sqrt{B(j_i)}. \tag{5.55}$$

We have now found that such cancellations extend highly non-trivially to the spectrally-flowed sectors of the theory. As a result of the structural similarities between spectrally-flowed correlators for SL(2,$\mathbb{R}$) and SU(2), all combinatorial factors coming from the different contributions in the second line of (5.53) exactly cancel. The only dependence of the final expression (5.54) on the spectral flow charges ω_i is contained in the overall prefactor.

Moreover, the result in Eq. (5.54) actually holds also for odd parity correlators, for which

$$\langle \mathcal{V}^{\omega_1}_{j_1}\mathcal{V}^{\omega_2}_{j_2}\mathcal{V}^{\omega_3,(0)}_{j_3}\rangle = \langle \mathcal{V}^{\omega_1}_{j_1}\mathcal{V}^{\omega_2}_{j_2}\mathcal{A}^{\omega_3,2}_{j_3}\rangle. \tag{5.56}$$

By using that

$$\begin{aligned}\langle W^{\omega_1}_{l_1}W^{\omega_2}_{l_2}\left(k^+_{\omega_3}W^{\omega_3}_{l_3}\right)\rangle &= \langle \tilde{W}^{\omega_1}_{l_1}(v_1=0)\tilde{W}^{\omega_2}_{l_2}(v_2=0)\partial_{v_3}\tilde{W}^{\omega_3}_{l_3}(v_3)|_{v_3=0}\rangle \\ &= -\frac{2\left[\omega_3(l_1+l_2-\frac{k'}{2})-(1+\omega_1+\omega_2)l_3\right]}{(1+\omega_1+\omega_2+\omega_3)a_3}\langle W^{\omega_1}_{l_1}W^{\omega_2}_{l_2}W^{\omega_3}_{l_3}\rangle\,,\end{aligned} \tag{5.57}$$

where a_3 is from the map $\Gamma[\omega_1,\omega_2,\omega_3]$, and

$$\langle \hat{\chi}^{\omega_1}\hat{\chi}^{\omega_2}\left(\hat{k}^+_{\omega_3}\chi^{\omega_3}\right)\rangle = \frac{2}{a_3}\langle \hat{\chi}^{\omega_1}\hat{\chi}^{\omega_2}\chi^{\omega_3}\rangle\,, \tag{5.58}$$

we find

$$\begin{aligned}\langle \mathcal{V}^{\omega_1}_{j_1}\mathcal{V}^{\omega_2}_{j_2}\mathcal{A}^{\omega_3,2}_{j_3}\rangle = &-(h_1+h_2+h_3-2) \\ &\times 2\frac{\langle V^{\omega_1}_{j_1}V^{\omega_2}_{j_2}V^{\omega_3}_{j_3}\rangle\langle W^{\omega_1}_{j_1-1}W^{\omega_2}_{j_2-1}W^{\omega_3}_{j_3-1}\rangle\langle\psi^{\omega_1}\psi^{\omega_2}\psi^{\omega_3}\rangle\langle\hat{\chi}^{\omega_1}\hat{\chi}^{\omega_2}\chi^{\omega_3}\rangle}{(1+\omega_1+\omega_2+\omega_3)a_3}.\end{aligned} \tag{5.59}$$

One readily obtains (5.54) after inserting the explicit expressions for each factor. A similar computation can be carried out for NS-NS-NS correlators involving one, two or three states polarized in the SU(2) directions. The final results for these cases will be given in Section 5.5 below.

We now consider the R-R-NS three-point functions. The three-point functions

$$\langle \mathcal{Y}^{\epsilon_1,\omega_1}_{j_1}\mathcal{Y}^{\epsilon_2,\omega_2}_{j_2}\mathcal{V}^{\omega_3}_{j_3}\rangle \qquad \text{and} \qquad \langle \mathcal{Y}^{\epsilon_1,\omega_1}_{j_1}\mathcal{Y}^{\epsilon_2,\omega_2}_{j_2}\mathcal{W}^{\omega_3}_{j_3}\rangle \tag{5.60}$$

are technically simpler since no picture changing is necessary. The only new pieces of information we need are the fermionic correlators involving spectrally-flowed spin fields, namely $\langle s^{\omega_1}_- s^{\omega_2}_- \psi^{\omega_3}\hat{\chi}^{\omega_3}\rangle\langle e^{\frac{i\epsilon_1}{2}(H_4-H_5)}e^{\frac{i\epsilon_2}{2}(H_4-H_5)}\rangle$. The different sectors factorize up to an overall phase coming from the cocycle factors, which can be ignored since it will cancel out upon including the contributions from the anti-holomorphic sector. The torus correlators involving H_4 and H_5 impose $\epsilon_1 = -\epsilon_2$. On the other hand, the SL(2,$\mathbb{R}$) and

SU(2) contributions give a product of a flowed three-point function with $\mathrm{SL}(2,\mathbb{R})_{-2}$ spins $(\hat{\jmath}_1,\hat{\jmath}_2,\hat{\jmath}_3) = (-1/2,-1/2,-1)$ and $\mathrm{SU}(2)_2$ spins $(\hat{l}_1,\hat{l}_2,\hat{l}_3) = (1/2,1/2,0)$. As for the NS-NS-NS cases, we find that all combinatorial factors related to spectral flow cancel out, leading to

$$\langle \mathcal{Y}_{j_1}^{\epsilon_1,\omega_1}\mathcal{Y}_{j_2}^{\epsilon_2,\omega_2}\mathcal{V}_{j_3}^{\omega_3}\rangle = \langle \mathcal{Y}_{j_1}^{\epsilon_1,\omega_1}\mathcal{Y}_{j_2}^{\epsilon_2,\omega_2}\mathcal{W}_{j_3}^{\omega_3}\rangle = \sqrt{n_5}\,\mathcal{C}(j_i)\xi^{\epsilon_1,\epsilon_2}\,, \tag{5.61}$$

where

$$\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.62}$$

5.5 Normalization and holographic matching

As it was argued in [15, 72] and proved in [73], in order to obtain a precise holographic matching, the worldsheet vertex operators must be properly normalized. Local operators of the CFT living on the AdS_3 boundary are given by vertex operators such as $\mathcal{V}(x,z)$ integrated over their worldsheet insertion point z. Note that the worldsheet two-point functions contain a divergent factor $\delta(h_1-h_2)$. As discussed in [15, 72], this divergence is cancelled by the integration over z_1 and z_2, or alternatively by fixing the insertion points at 0 and ∞ and dividing by the remaining conformal volume. However, this cancellation produces an additional finite but non-trivial multiplicative factor depending on h and ω.

This constant factor can be obtained by using the spacetime Ward identities associated with the R-symmetry currents $\mathcal{K}^a(x)$, see Sec. 2.5 above. In the supersymmetric version of the model

$$\mathcal{K}^a(x) = -\int d^2z\,\frac{1}{n_5 c_\nu}K^a\bar{j}(\bar{x})V_1(x,\bar{x},z,\bar{z}), \quad c_\nu = \frac{\pi\gamma(1-b^2)}{\nu b^2}, \tag{5.63}$$

provide the worldsheet representatives for the R-currents, with $b^2 = n_5^{-1}$. Hence a generic vertex operator of the form $V_h^\omega(x)\Phi_{\text{int}}$, where Φ_{int} stands for the internal, fermionic and ghost contributions, must satisfy

$$\langle \mathcal{K}^3(x_1)V_j^\omega(x_2)\Phi_{\text{int},2}V_j^\omega(x_3)\Phi_{\text{int},3}\rangle = \left[\frac{q_2}{x_{12}}+\frac{q_3}{x_{13}}\right]\langle V_j^\omega(x_2)\Phi_{\text{int},2}V_j^\omega(x_3)\Phi_{\text{int},3}\rangle\,, \tag{5.64}$$

where $q_3 = -q_2$ denote the corresponding R-charges. We can evaluate both sides of Eq. (5.64) independently. Using the three-point functions derived above, the LHS becomes

$$\frac{-q_2}{n_5c_\nu}\frac{\bar{x}_{12}\bar{x}_{13}}{\bar{x}_{23}}\left|\frac{z_{23}}{z_{12}z_{13}}\right|^2[2\omega+1-2h]\,\langle V_1(x_1)V_j^\omega(x_2)V_j^\omega(x_3)\rangle\,\langle\Phi_{\text{int},2}\Phi_{\text{int},3}\rangle\,, \tag{5.65}$$

Moreover, (5.55) gives

$$\langle V_1(x_1)V_j^\omega(x_2)V_j^\omega(x_3)\rangle = \mathcal{C}(1,j,j) = 2Q^2B(j). \tag{5.66}$$

The comparison with the RHS then shows that the string two-point function differs from the spacetime one by a factor

$$[2h-1-2\omega]\,n_5c_\nu^{-1}2Q^2B(j). \tag{5.67}$$

This leads to

$$\mathbb{O}_j^{\omega}(x,\bar{x},u,\bar{u},z,\bar{z}) = \frac{\mathcal{O}_j^{\omega}(x,\bar{x},u,\bar{u},z,\bar{z})}{\sqrt{2c_\nu^{-1} n_5 Q^2 \,(2h-1)\, B(j) v_4}}, \tag{5.68}$$

where $h = j + n_5\omega/2$, while v_4 is the T^4 volume, and $\mathbb{O}$ stands for either $\mathbb{V}$ or $\mathbb{W}$. A similar computation shows that, due to the extra factor appearing in Eq. (5.33), for vertex operators in the RR sector of the worldsheet theory we have

$$\mathbb{Y}_j^{\epsilon,\bar{\epsilon},\omega}(x,\bar{x},u,\bar{u},z,\bar{z}) = \sqrt{\frac{2h-1}{2c_\nu^{-1} n_5^2 Q^2 B(j) v_4}}\,\mathcal{Y}_j^{\epsilon,\bar{\epsilon},\omega}(x,\bar{x},u,\bar{u},z,\bar{z}). \tag{5.69}$$

In these expressions we have re-inserted the anti-holomorphic dependence. String three-point functions are then obtained directly from the results of the previous sections, and one must also include a factor of the string coupling $g_s = \sqrt{\frac{n_5 v_4}{n_1}}$ together with an additional factor of v_4. Consequently, we find that the full set of normalized (spacetime) chiral primary three-point functions takes the following form:

$$\langle \mathbb{V}_{j_1}^{\omega_1}\mathbb{V}_{j_3}^{\omega_2}\mathbb{V}_{j_3}^{\omega_3,(0)}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(h_1+h_2+h_3-2)^4}{(2h_1-1)(2h_2-1)(2h_3-1)}\right]^{1/2}, \tag{5.70a}$$

$$\langle \mathbb{W}_{j_1}^{\omega_1}\mathbb{V}_{j_2}^{\omega_2}\mathbb{V}_{j_3}^{\omega_3,(0)}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(1+h_1-h_2-h_3)^4}{(2h_1-1)(2h_2-1)(2h_3-1)}\right]^{1/2}, \tag{5.70b}$$

$$\langle \mathbb{W}_{j_1}^{\omega_1}\mathbb{W}_{j_2}^{\omega_2}\mathbb{V}_{j_3}^{\omega_3,(0)}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(h_1+h_2-h_3)^4}{(2h_1-1)(2h_2-1)(2h_3-1)}\right]^{1/2}, \tag{5.70c}$$

$$\langle \mathbb{W}_{j_1}^{\omega_1}\mathbb{W}_{j_2}^{\omega_2}\mathbb{W}_{j_3}^{\omega_3,(0)}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(h_1+h_2+h_3-1)^4}{(2h_1-1)(2h_2-1)(2h_3-1)}\right]^{1/2}, \tag{5.70d}$$

and

$$\langle \mathbb{Y}_{j_1}^{\epsilon_1,\bar{\epsilon}_1,\omega_1}\mathbb{Y}_{j_2}^{\epsilon_2,\bar{\epsilon}_2,\omega_2}\mathbb{V}_{j_3}^{\omega_3}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(2h_1-1)\,(2h_2-1)}{(2h_3-1)}\right]^{1/2}\delta^{\epsilon_1,\epsilon_2}\delta^{\bar{\epsilon}_1,\bar{\epsilon}_2}, \tag{5.71a}$$

$$\langle \mathbb{Y}_{j_1}^{\epsilon_1,\bar{\epsilon}_1,\omega_1}\mathbb{Y}_{j_2}^{\epsilon_2,\bar{\epsilon}_2,\omega_2}\mathbb{W}_{j_3}^{\omega_3}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(2h_1-1)\,(2h_2-1)}{(2h_3-1)}\right]^{1/2}\xi^{\epsilon_1,\epsilon_2}\xi^{\bar{\epsilon}_1,\bar{\epsilon}_2}, \tag{5.71b}$$

where $h_i = j_i + n_5\omega_i/2$. The overall scaling with $N = n_1 n_5$ is obtained from [71]

$$\frac{1}{\sqrt{N}} = \frac{g_s}{n_5\sqrt{v_4}}\sqrt{\frac{2\pi^5}{\nu b^4\gamma(1+b^2)}}. \tag{5.72}$$

Note that the six-dimensional coupling $g_6 = g_s/\sqrt{v_4}$ is independent of the T^4 volume. Finally, the selection rules on the SU(2) spins l_i and spectral flow charges ω_i can be summarized as follows:

$$l_i \leq l_k + l_j \quad \text{and} \quad \omega_i \leq \omega_k + \omega_j \quad \forall \quad i,j,k = 1,2,3. \tag{5.73}$$

The fusion rules (5.73) and the final expressions for the structure constants of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ chiral ring presented in Eqs. (D.52) and (D.53) precisely reproduce the holographic CFT computations at the symmetric orbifold point [55, 56, 71, 72, 121], see Appendix D.

6 Holography theory for other values of k

So far we have derived a precise holographic matching for

1. generic correlators at the tensionless point, namely $k = 3$, and

2. correlators which are protected by supersymmetry for $k > 3$.

In both cases, the dual derivation was performed at the symmetric orbifold point in moduli space. In this section, we discuss what is known about the holographic theory away from the tensionless point beyond protected quantities. The path we choose here draws heavily from [75, 76, 78]. Given that the nature of the HCFT is still under debate, our discussion will be slightly more speculative than in other sections of these notes.

6.1 An alternative description for long strings

In the previous sections we have discussed the role of the dynamical long string states in AdS_3. We have seen that they can reach the asymptotic boundary of spacetime while remaining at finite energy. In fact, in the tensionless regime long strings are actually forced to lie almost on top of the AdS_3 boundary. As a result, the string worldsheet itself becomes the covering space employed in the computation of a given correlation function. Hence, in this limit the dynamics of worldsheet excitations accurately describe the physics of the holographic CFT, which takes the form of a symmetric orbifold $\mathrm{Sym}^{n_1}(\mathbb{R}_\phi \times M_{\mathrm{int}})$. It thus makes sense to wonder what long strings can tell us about the nature of the dual theory at $k \neq 3$.

In order to explore this, we consider the theory *on* the long strings. More precisely, we study excitations on top of the classical long string solutions, employing light-cone quantization for simplicity. The simplest solution of this type corresponds to a single long string with unit winding sitting at a fixed radial distance; the latter is taken to be very large so that we remain close to the boundary, and are able to make use of the Wakimoto description. This classical configuration takes the form

$$\gamma(z) = z \qquad \phi(z) = \phi_0 \gg 1\,. \tag{6.1}$$

In other words, use the gauge freedom to fix $\gamma(z) = z$, and use the Virasoro constraint

$$0 = T_{\mathrm{AdS}} + T_{\mathrm{int}} = -\beta\partial\gamma - \frac{1}{2}\partial\phi\partial\phi - \frac{Q_\phi}{\sqrt{2}}\partial^2\phi + T_{\mathrm{int}}\,, \qquad Q_\phi = -\frac{1}{\sqrt{k-2}}\,, \tag{6.2}$$

to solve for β, giving

$$\beta = -\frac{1}{2}\partial\phi\partial\phi - \frac{Q_\phi}{\sqrt{2}}\partial^2\phi + T_{\mathrm{int}}\,. \tag{6.3}$$

We now use the currents $J^+ = \beta$ and $J^3 = (\beta\gamma) - \sqrt{\frac{k-2}{2}}\partial\phi$ to rewrite the *spacetime* Virasoro modes. We have

$$\begin{aligned}
\mathcal{L}_n^{(1)} &= \oint dz \left[(n+1)\gamma^n J^3 - n\gamma^{n+1}J^+\right] \\
&= \oint dz \left[\beta z^{n+1} - (n+1)z^n \sqrt{\frac{k-2}{2}}\partial\phi\right] \\
&= \oint dz\, z^{n+1} \left[-\frac{1}{2}\partial\phi\partial\phi - \frac{Q_\phi - Q_\phi^{-1}}{\sqrt{2}}\partial^2\phi + T_{\text{int}}\right] .
\end{aligned} \tag{6.4}$$

Hence, it is tempting to identify the spacetime stress tensor (in the unit winding sector, hence the superscript) with the term in brackets. More precisely, for this interpretation to make sense, we should be able to replace $z \to x$, but, at least heuristically, this is exactly what we expect form the fact that (1) the long string itself is, roughly speaking, on top of the boundary, and (2) in the $\omega = 1$ sector γ should behave as the appropriate covering map, i.e. the main contributions to correlators of excitations on top of the long string comes from configurations with $z = \gamma(z) \sim \Gamma(z) = x$. Hence, we find that $\mathcal{T}^{(1)}$ contains a factor analogous to the internal CFT on the worldsheet, combined with a non-compact scalar ϕ. Importantly, the latter comes with a *modified* background charge, namely

$$Q = Q_\phi - Q_\phi^{-1} = \frac{k-3}{\sqrt{k-2}}. \tag{6.5}$$

In particular, we see that this changes sign as we move from $k > 3$ to $k < 3$, and vanishes precisely at the tensionless string point. We will come back to this in the next section. For now, we note that, as expected, the central charge on this single-long-string theory gives

$$c_{\text{st}}^{(1)} = 1 + 6Q^2 + c_{\text{int}} = 1 + 6\frac{(k-3)^2}{k-2} + 26 - \frac{3k}{k-2} = 6k\,. \tag{6.6}$$

This is not the first time we have seen this formula, see Eq. (4.16).

So far we have described the theory on a single long string. The maximum number of long strings available is set by n_1, and given that such long strings are indistinguishable from each other, one is tempted to propose that the holographic CFT should be related to the following symmetric product orbifold:

$$\text{Sym}^{n_1}\left[\mathbb{R}_\phi^Q \times M_{\text{int}}\right] , \tag{6.7}$$

where M_{int} stands for the non-linear sigma-model with the internal manifold as its target space. This has $c_{\text{st}} = 6kn_1$. Of course, this description can only be an approximate one, since our considerations are only valid in regime where the free field description holds and ignore interactions between the different long strings. Nevertheless, it is encouraging that this proposal is in precise agreement with the long string spectrum derived in Sec. (4.2) above. In particular, the non-compact direction (more precisely the center-of-mass field ϕ_{com} given by the gauge-invariant sum over all the ϕ_i in different copies of the seed theory) is identified with the bulk radial direction.

We can make the connection to the symmetric orbifold theory even more explicit by extending the above computation to the sectors with spectral flow $\omega > 1$. There we can take $\gamma(z) = z^\omega$. We first consider the spacetime R-symmetry generators for simplicity, which were derived in Sec. 2:

$$\mathcal{K}_n^{a\,(\omega)} = \oint dz\, \gamma^n K^a(z) = \oint dz\, z^{\omega n} K^a(z)\,. \tag{6.8}$$

For this to be single-valued, the modes of the must take fractional values, i.e. we must have $n \in \frac{1}{\omega}\mathbb{Z}$. This is exactly what one expects in the ω-twisted sector of a symmetric orbifold theory. As for the spacetime Virasoro modes, we get

$$\mathcal{L}_n^{(\omega)} = \oint dz \left[\frac{z^{\omega n+1}}{\omega}\left(-\frac{1}{2}\partial\phi^{(\omega)}\partial\phi^{(\omega)} - \frac{Q}{\sqrt{2}}\partial^2\phi^{(\omega)} + T_{\rm int} - \frac{\delta}{z^2}\right) - \frac{z^{\omega n}(\omega-1)}{\sqrt{2}Q_\phi\omega}\partial\phi^{(\omega)}\right], \tag{6.9}$$

where we have included a possible normal ordering constant entering when solving the Virasoro condition. This was ignored in the $\omega = 1$ case, hence in order to be vanish in that particular case. It produces a shift in $\mathcal{L}_0^{(\omega)}$, which was computed in [122], giving $\delta = -\frac{1}{2}(\omega+2)(\omega-1)$. Finally, we make use of the transformation of ϕ when adding $\omega - 1$ units of spectral flow, which reads $\phi \to \phi^{(\omega)} = \phi - \frac{(w-1)}{\sqrt{2}Q_\phi}\log|z|^2$. Up to an additional shift of the zero mode, this is precisely what we need to remove term proportional to $\partial\phi$, leading to

$$\mathcal{L}_n^{(\omega)} = \oint dz \frac{z^{\omega n+1}}{\omega}\left(-\frac{1}{2}\partial\phi\partial\phi - \frac{Q}{\sqrt{2}}\partial^2\phi + T_{\rm int}\right) + \delta_{n,0}\frac{k(\omega^2-1)}{4k}\,. \tag{6.10}$$

This mirrors the usual procedure of defining the *untwisted* generators in $\hat{\mathcal{L}}_n$ in the symmetric orbifolds models with seed central charge $6k$ in terms of the fractionally-moded ones, i.e.

$$\mathcal{L}_{\frac{n}{\omega}} = \frac{1}{\omega}\hat{\mathcal{L}}_n + \frac{k(\omega^2-1)}{4\omega}\delta_{n,0}\,, \qquad n \in \mathbb{Z}\,. \tag{6.11}$$

The theory on the long strings strongly suggests that, at least in the weakly coupled region, the symmetry generators and the continuous sector of the theory should be well described by the holographic CFT given in Eq. (6.7). The worldsheet spectrum computed in Sec. 4.2 provides further evidence. In the following section we construct some of the relevant of the untwisted sector operators explicitly. We then study the appropriate deformation of the boundary theory, which takes into account the presence of a non-trivial potential for the field ϕ in the interior of AdS$_3$, and leads to the appearance of short-strings as bound states.

6.2 Operators of the untwisted sector of the spacetime theory

Let us first pause in order to briefly discuss the interpretation of our results from the previous section. We have focused on the radial field ϕ. From the worldsheet point of view discussed at length in Sec. 2, we know that (near the boundary) this behaves as a free boson satisfying $\phi(z,\bar{z})\phi(0) \sim -\log|z|^2$. The background charge is Q_ϕ, while the energy momentum tensor and central charge read

$$T(z) = -\frac{1}{2}\partial\phi\partial\phi - \frac{Q_\phi}{\sqrt{2}}\partial^2\phi\,, \qquad c = 1 + 6Q_\phi^2. \tag{6.12}$$

In our conventions, vertex operators $V_\alpha = e^{\sqrt{2}\alpha\phi}$ have weights

$$\Delta\left[V_\alpha\right] = \alpha(Q_\phi - \alpha). \tag{6.13}$$

The (Euclidean) path integral definition for correlation functions is given by

$$\langle\prod_{i=1}^{n} V_i(x_i)\rangle = \int D\phi\, e^{-S} \prod_{i=1}^{n} V_i(x_i). \tag{6.14}$$

By inserting the action

$$S = \frac{1}{4\pi}\int d^2x\sqrt{g}\left[2\partial\phi\bar{\partial}\phi + \frac{Q_\phi}{\sqrt{2}}R^{(2)}\phi\right], \tag{6.15}$$

together with n insertions of the form

$$V_i = \exp\left[\sqrt{2}\alpha_i\phi(x_i,\bar{x}_i)\right] = \exp\left[\int d^2x\sqrt{2}\alpha_i\phi(x,\bar{x})\delta^{(2)}(x-x_i)\right], \tag{6.16}$$

we see that the contribution from the large (constant) ϕ region scales as

$$\langle\prod_{i=1}^{n} V_i(x_i)\rangle \sim \exp\left[\sqrt{2}\left(\sum_{i=1}^{n}\alpha_i - Q_\phi\right)\phi\right], \tag{6.17}$$

where we have used that for the sphere $\int d^2x\sqrt{g}R = 8\pi$.

The first type of operator one considers are is the analog of the flat space plane waves, which have

$$\alpha = \frac{Q_\phi}{2} + ip\,, \qquad p\in\mathbb{R}\,. \tag{6.18}$$

Applying the above analysis to the two-point functions one finds that they are delta-function normalizable. In the language of [49, 123] these are the macroscopic states. On the other hand, the microscopic operators have $\alpha\in\mathbb{R}$. These are local operators when their wave-functions diverge in the weakly coupled regime. From the worldsheet action (2.63) we identify the dilaton Φ as $\Phi \sim Q_\phi\phi$ (up to a positive constant), so that when $Q_\phi < 0$ the coupling $g_s = e^\Phi$ vanishes in the limit $\phi\to+\infty$. This can also be seen by introducing a Liouville-type potential to regularize the $\phi\to-\infty$ region, which we identify as the strong coupling region in the bulk; we will come back to this type of deformation below. For now, we note that, as follows from (6.17), at large ϕ the corresponding wave-functions behave as $e^{\sqrt{2}\left(\alpha-\frac{Q_\phi}{2}\right)\phi}$, hence leading to the condition

$$\alpha > \frac{Q_\phi}{2}. \tag{6.19}$$

This should agree with the holographic perspective, where ϕ is identified with the radial direction in AdS$_3$, such that local operators of the HCFT correspond to the non-normalizable modes of the operators (2.68), see Eq. (2.36). Comparing the two expressions we get $\alpha = \frac{(j-1)}{\sqrt{k-2}}$ and $Q_\phi = -\frac{1}{\sqrt{k-2}}$, which indeed gives $\Delta = \alpha(Q_\phi-\alpha) = -\frac{j(j-1)}{k-2}$. Thus, Eq. (6.20) translates into $j>\frac{1}{2}$.

On the other hand, we have learned that from the point of view of the boundary theory we should consider an analogous non-compact boson, albeit with an effective charge $Q = Q_\phi - Q_\phi^{-1} = \frac{k-3}{\sqrt{k-2}}$. Note that for $k > 3$ this has opposite sign as compared to Q_ϕ. This means that for the HCFT the weak coupling regime is at $\phi \to -\infty$. The condition analogous to Eq. (6.20) hence reads

$$\alpha < \frac{Q}{2}. \tag{6.20}$$

In terms of the worldsheet SL(2,$\mathbb{R}$) spin, this gives $j < \frac{k-1}{2}$. We thus recover the bound (2.93) from the holographic perspective. Conversely, for $k < 3$ both weak coupling regions lie at the AdS$_3$ boundary.

Let us now consider long string n-point functions. In the worldsheet language, all insertions have $\mathrm{Re}\,\alpha_i = \frac{Q_\phi}{2}$, so that the path integral is well behaved since

$$\mathrm{Re}\,\sqrt{2}\left(\sum_{i=1}^{n}\alpha_i - Q_\phi\right)\phi = \frac{1}{\sqrt{2}}(n-2)Q_\phi\phi. \tag{6.21}$$

However, from the boundary point of view we would like to think of $e^{\sqrt{2}\beta\phi}$ as operators living in the untwisted sector of the symmetric orbifold theory. Since we want to identify the latter with the singly-flowed sector of the worldsheet WZW model, by evaluating the corresponding spacetime weight h derived in Eq. (4.12) for $\omega = 1$ and $h_{\text{int}} = 0$ and comparing with $h = \beta(Q-\beta)$ we find that we should take

$$\beta = \frac{\frac{k}{2} - j - 1}{\sqrt{k-2}}. \tag{6.22}$$

This makes sense since long strings should correspond to macroscopic operators with

$$\beta = \frac{\frac{k}{2} - (\frac{1}{2} + is) - 1}{\sqrt{k-2}} = \frac{\frac{k}{2} - (\frac{1}{2} + is) - 1}{\sqrt{k-2}} = \frac{Q}{2} + ip\,, \qquad p \equiv -\frac{s}{\sqrt{k-2}}\,. \tag{6.23}$$

In other words, the associated worldsheet operator is

$$e^{\sqrt{2}\beta\phi(x,\bar{x})} \quad \leftrightarrow \quad \int d^2z\, V^{\omega=1}_{j=\frac{1}{2}+ip\sqrt{k-2},h=\beta(Q-\beta)}(x,z). \tag{6.24}$$

The relative factor between s and p ensures that the phase of the exponential behaviors match ϕ behavior agrees since $e^{isQ_\phi\phi} = e^{ip\phi}$.

For short strings, the relation between β and the α defined above is consistent with the series identifications of Eq. (2.106), which shows that the unflowed operators can be identified as operators with $\omega = 1$ upon implementing the replacement $j \to \frac{k}{2} - j$. This tells us that our construction of the important operators $\mathcal{I}$, $\mathcal{T}(x)$ and $\mathcal{K}^a(x)$ of Sec. 2.5, which was based on the unflowed bosonic SL(2,$\mathbb{R}$) primary with $V_{j=1}(x,z)$, can be recast in terms of operators with $\omega = 1$. Indeed, for operators in discrete representations the quantum number ω is not uniquely defined due to the identifications of Eq. (2.106). For instance, the integrand of the identity operator defined in (2.176) becomes

$$\mathcal{I} = \int d^2z(J\bar{J}V_1)(x,\bar{x},z,\bar{z}) \sim \int d^2z\, V^{\omega=1}_{\tilde{j},h=-\tilde{j}-1+\frac{k}{2}=0}(x,\bar{x},z,\bar{z})\,, \qquad \tilde{j} = \frac{k}{2} - 1\,, \tag{6.25}$$

where we have used that in the frame with $\omega = 1$ we have $J^-_{-1} = \tilde{J}^-_0$ For $\mathcal{K}^a(x)$ and $\mathcal{T}(x)$ we need to work with operators which are not spacetime scalars, but the idea is analogous. Hence, we can indeed see these operators as living in the untwisted sector of the Holographic CFT. As a check, we note that taking the limit $\beta \to 0$ should give the identity operator, and from Eq. (6.22) we see that this indeed corresponds to taking $j \to \tilde{\jmath} = \frac{k}{2} - 1$.

We should perhaps provide a small clarification. From the spacetime point of view, the relevant spacetime fields should be interpreted as the (gauge-invariant) sum over the corresponding images over all copies of the seed theory. In other words, in this section whenever we write a boundary field such as $\phi(x, \bar{x})$, we really mean $\sum_{i=1}^{n_1} \phi_i$. With this notation, the rest of the untwisted spectrum is built by combining exponentials in ϕ with currents and primary fields of the sigma model with target space M_{int}. The latter will translate almost trivially to the corresponding worldsheet fields belonging to the internal sector, which is why we have focused mostly ignored them.

6.3 The difference between $k > 3$ and $k < 3$

The boundary path integral for the long-string correlator will have a factor

$$\text{Re}\,\sqrt{2}\left(\sum_{i=1}^{n} \beta_i - Q\right)\phi = \frac{1}{\sqrt{2}}(n-2)Q\phi. \tag{6.26}$$

This is well-behaved at the boundary $\phi \to +\infty$ only when $Q < 0$. Thus, such correlators can only be computed directly from the path integral formalism in models with $k \leq 3$, i.e. for stringy or even sub-stringy AdS_3 spaces. These are the configurations considered in the dualities proposed in [63, 75].

The situation is qualitatively different when we move from $k > 3$ to $k < 3$. Indeed, in the latter case, the range for defining normalizable (short string) operators in the worldsheet SL(2,$\mathbb{R}$) model reduces to $\frac{1}{2} < j < 1$. Since it is created by the operator $\mathcal{I}$ constructed in (2.176) in terms of the $j = 1$ bosonic vertex, this means that vacuum state is no longer part of the spacetime spectrum[29]. The same goes for the states created by $\mathcal{T}(x)$ and $\mathcal{K}(x)$. However, $\mathcal{I}$, $\mathcal{T}(x)$ and $\mathcal{K}(x)$ still make sense as local operators of the theory. The situation is similar to what happens in Liouville theory, where the spectrum is built solely from the macroscopic states, i.e. the long strings in our context.

Nevertheless, given that the resulting correlators are analytic in the momenta β_i and the level k, it was argued in [76] that one can define correlators at $k > 3$ by analytic continuation, which suggests that a very similar duality should hold also in this regime. This leads to the proposal of [76, 77], which is strongly supported by the (perturbative) matching between the highly non-trivial residues of the correlators of the putatitve holographic CFT and those of the worldsheet correlators at genus zero. (We briefly review the computation

[29]This has interesting implications for black holes. Given that the BTZ black holes are obtained as quotients of global AdS_3, they also become non-normalizable and drop out of the spectrum for $k < 3$ [124]. In this sense, $k = 3$ was interpreted as the string-black hole correspondence point in the AdS_3 context in [124]. The absence of black holes is another aspect which makes the the holographic duality is more tractable for $k < 3$ [75].

later on). In the following sections we will be agnostic about this discussion and consider both situations on equal footing.

6.4 The twist-2 deformation

More generally, for $k \neq 3$ we expect the holographic CFT to be a *deformed* symmetric orbifold model. By this, we mean that, at least in conformal perturbation theory, one should be able to define the boundary theory by introducing a deformation of (6.7) by some marginal operator. At the very least, we expect this deformation to

- be negligible in the large ϕ regime, where the long string description is expected to be accurate,
- be generated by an operator belonging to a twisted sector, say $\omega = 2$, in order to break the symmetric orbifold structure, which, as we know, is not respected by the worldsheet correlators,
- generate a (for instance Liouville-type) potential for the field ϕ, resulting in the appearance of bound states corresponding to the short-string sector of the worldsheet theory,
- and break the holomorphicity of the field $\partial_x\phi(x)$, since the Wakimoto fields γ, β and $\partial\phi$ are *not* holomorphic in the bulk.

We now want to identify the deforming operator, which we denote as Φ. The last point in our list of requirements is perhaps the most useful for this goal. Indeed, the equations of motion derived from the worldsheet action (2.62) is of the form

$$\bar{\partial}\partial\phi \sim \beta\bar{\beta}e^{\sqrt{2}Q_\phi\phi}\,, \tag{6.27}$$

which tells us the precise way in which we deviate from holomorphicity of $\partial\phi$ in the bulk. An analogous phenomenon will then occur in the long string theory, albeit with the replacement $Q_\phi \to Q$. Moreover, in the $\omega = 1$ sector we have $\gamma = z \sim x$, hence we have $\partial_x \sim J^+ \sim \beta \sim \partial_\gamma$, so that

$$\partial_{\bar{x}}\partial_x\phi \sim \partial_{\bar{x}}\partial_x e^{\sqrt{2}Q\phi}\,. \tag{6.28}$$

From here we derive our first proposal for the deforming operator [75]:

$$\Phi_{\rm st} \sim \partial_{\bar{x}}\phi\partial_x\phi e^{\sqrt{2}Q\phi}\,. \tag{6.29}$$

When adding this operator to the spacetime action we do so at first order in the deformation parameter, which we can call μ. It is important to note that the actual value of μ is it can be absorbed by a constant rescaling of ϕ. In other words, it only matters that μ does not vanish.

It is interesting to ask what is the avatar of $\Phi_{\rm st}$ in the worldsheet theory. For this, we can match the falloff at large ϕ with that of the vertex operator in (2.127). This means setting $\exp(\sqrt{2}Q\phi) = \exp(-\sqrt{2}Q_\phi(j-1)\phi)$, so that we get $j = k-2$. This is precisely what we need in order to have $h = \bar{h} = 1$ and spectral flow charge $\omega = 2$, see Eq. (4.12). Hence, the deforming operator should correspond to a short string with $m = h - k = 1 - k = -j - 1$, i.e.

$$\Phi_{\rm ws} = \int d^2 z V^{\omega=2}_{j=k-2,h=1}(x, \bar{x}, z, \bar{z}) \,. \tag{6.30}$$

Furthermore, by means of the series identifications (which only make sense once the deformation is turned on) we can also view the integrand of $\Phi_{\rm ws}$ as an operator in the singly wound sector, namely

$$\Phi_{\rm ws} = \int d^2 z (J\bar{J} V^{\omega=1}_{\tilde{j},h=2})(x, \bar{x}, z, \bar{z}) \,, \qquad \tilde{j} = \frac{k}{2} - j = 2 - \frac{k}{2} \,. \tag{6.31}$$

Here we have used that $\tilde{J}_0^{-(2)} = J^-_{-2} = \tilde{J}_{-1}^{-(1)}$ in the corresponding flowed sectors. Note that this operator is non-normalizable, i.e. the associated state is not part of the spectrum.

To summarize, the proposal of [75, 76] states that the holographic dual for strings in an AdS_3 space of size $\sim k$ sourced by a combination of NS5-branes and n_1 fundamental strings is given by

$$\mathrm{Sym}^{n_1}\left[\mathbb{R}^Q_\phi \times M_{\rm int}\right] \;+\; \Phi_{\rm st}-\text{deformation} \,, \tag{6.32}$$

where $(k-2)Q^2 = (k-3)$ and $\Phi_{\rm st}$ is the marginal twist-2 deformation constructed above.

6.5 Matching of correlation functions in perturbation theory

In order to have conclusive test for the proposed duality it would be necessary to match the exact correlation functions we have computed from the worldsheet with those of the putative boundary theory. Unfortunately, the available definition for the latter, namely Eq. (6.32), is only a perturbative definition.

This means that from the holographic side one needs to make use of conformal perturbation theory. The different contributions to a given correlator appearing at each order in the perturbative expansion will contain additional (integrated) insertions of the operator $\Phi_{\rm st}$, and must be computed in the undeformed theory. In particular, $\Phi_{\rm st}$ contains a factor that is an exponential in the field ϕ. The corresponding charge conservation condition (whose precise form is dictated by the background charge Q) then indicates that this method can only be used when the exponents β_i of the original operator insertions add up to a series of discrete values parameterized by an integer q. In analogy with the usual Coulomb-gas description of Liouville theory, this means that one can only access certain residues of the full correlation functions [86].

Nevertheless, the matching of all such residues would constitute a highly non-trivial test of the holographic duality, at a level of precision that is very hard to access in any other set-up. The task at hand is two-fold:

- First, one should extract the relevant residues from the string worldsheet correlators. Since the β_i define the corresponding unflowed spins j_i, we see for instance from the three-point functions in Eq.(3.57) and (3.58) (which should be integrated over the y-variables) that there are two sources for the relevant poles. On the one hand, there could be divergences originating from the *unflowed* structure constants $C(j_i)$.

This happens for $q \geq 3$. On the other hand, the j_i also appear in the exponents of the generalized exponents $X_I(y_i)$. For $q \leq 2$ some of these exponents become negative integers, hence the integrals can develop divergences localized around the regions where the corresponding X_I vanish. For example, for $q = 0$ this happens when $X_i(y_i) = 0$. This corresponds to the limit where y_i approaches the coefficient a_i of the associated covering map.

- Second, one should compute the boundary correlators. Fortunately, by charge conservation one finds that, for a given value of q, only a single term in the perturbative series gives a non-zero contribution. The latter contains the original insertions together with q insertions of the twist-2 deformation operator. This is computed in the symmetric orbifold theory by using techniques based on holomorphic covering maps, see [55, 58, 59]. Once we are in the covering space, we are simply left with a correlator involving only exponentials in ϕ, which is trivially computed. Finally, one needs to integrate over the insertion point of each interaction operator $\Phi_{\rm st}$.

The matching computation is quite involved at the technical level. This program was initiated in [76, 77] for set of three- and four-point functions, and indeed a perfect agreement was obtained for all cases under consideration. We will not discuss the details in these notes, and simply refer the interested reader to the original publications.

6.6 LSZ poles and discrete states

Assuming the long string two- and three-point functions have been matched by means of the procedure described above, one finds that the holographic propagator contains the factors

$$\frac{\gamma(j + \frac{k\omega}{2} - h)}{\gamma(1 - j + \frac{k\omega}{2} - h)} \tag{6.33}$$

contained in Eq. (3.7). This shows that the holographic CFT knows about the reflection coefficient for spectrally flowed states. This coefficient has simple poles at the locations of the discrete states, i.e. at $m = j + n$ and $m = -j - n$ with $n \in \mathbb{N}_0$, where $m = h - \frac{k}{2}\omega$. It was argued in [76] that these should be understood as LSZ-type poles, in the sense of [125], which signal the locations for which external particles go on-shell. Hence, we find that, even though the short-string states are, for the most part, not visible in the undeformed symmetric orbifold theory (they are not macroscopic states in the language of [49]), they seem to become part of the spectrum when the deformation is included. This makes sense, since we view them as bound states, which live in the bulk of AdS_3, and we have identified the boundary ϕ with the radial direction in the dual gravitational theory. From the boundary perspective, they are normalizable as long as $\beta > \frac{Q}{2}$, which gives $j > \frac{1}{2}$, as expected from the semiclassical analysis. (Combining this with the series identifications leads to the upper bound in Eq. (2.93).)

6.7 Grand canonical ensemble

We finish this section by commenting on a somewhat surprising heuristic link between the holographic picture developed over the last few pages and the discussion related to the fact that the operator $\mathcal{I}$ is not exactly the spacetime identity. For this, we note that there is reflected counterpart to the deforming operator constructed in Sec. 6.4, namely $\Phi_{\rm ws}$ as given in (6.31). As it can be inferred from Eq. (2.36), it is obtained by reflecting the spin, i.e. by taking $\tilde{\jmath} \to 1 - \tilde{\jmath}$. Since $\tilde{\jmath} = 2 - \frac{k}{2}$, we get an operator $\hat{\Phi}_{\rm ws}$ with unflowed spin $1 - \tilde{\jmath} = \frac{k}{2} - 1$. But this is almost identical to the vertex operator that appeared in the $\omega = 1$ representation (6.25) of the identity operator $\mathcal{I}$! This merely suggest a relation between both operators. A slightly more precise statement is obtained by combining Eq. (6.28) with the observation that $\mathcal{I}$ can be interpreted as the zero-mode of the dilaton, we are led to the identification [76]

$$\mathcal{I} \sim \int d^2x \int d^2z \, (J\bar{J}V^{\omega=1}_{\tilde{\jmath},h=2})(x,\bar{x},z,\bar{z}) = \int d^2x \, \hat{\Phi}_{\rm ws}(x,\bar{x}) \,. \tag{6.34}$$

This is consistent with the fact that $\mathcal{I}$ is x-independent. Further motivations for this relation can be found in [76].

What is the interpretation for the identification proposed in Eq. (6.34)? At the level of the boundary action, the deformation is such that we add a term of the form

$$S_0 \to S = S_0 - \mu \int d^2x \, \Phi_{\rm st} \sim S_0 - \mu \mathcal{I} \,. \tag{6.35}$$

But, as discussed above, the expectation value of the operator $\mathcal{I}$ is nothing but the number of strings in the background, namely n_1. Hence, we find that the deformation effectively implements a Legendre transform, which, roughly speaking, *unfixes* n_1 in favor of working with a given μ, which is related to the string coupling g_s. This justifies the interpretation of the boundary theory as a grand canonical ensemble of CFTs, where μ plays the role of the chemical potential [65, 76].

7 Application I: Three roads to little string theory

We now present an interesting application of the worldsheet formalism developed in the previous chapters that goes beyond the description of strings in AdS_3. The object of interest remains the concept of holography, but now in a more general sense. More precisely, we focus on the theory on the NS5-branes. Even though the relevant supergravity backgrounds are not asymptotically AdS, one can still define a suitable five-brane decoupling limit. Nevertheless, in a certain sense the decoupled theory remains stringy, and is known as Little String Theory. We present three alternative descriptions for the dynamics of these little strings: one based on the dual gravitational picture, a second one given by a marginal deformation of the AdS_3 worldsheet model, and a third one based on an irrelevant deformation of the corresponding dual CFT. In view of the latter, energy scales will be particularly relevant in the discussion, hence we will find it useful to re-introduce various factors α' which were set to one in previous sections for convenience.

7.1 The NS5-brane decoupling limit

Consider a stack of n_5 NS5-branes. The supergravity configuration corresponds to the $n_1 \to 0$ ($f_1(r) \to 1$) limit of (2.2) [92], namely

$$ds^2 = -dt^2 + dy^2 + f_5(r)(dr^2 + r^2 d\Omega_3^2) + dz_i dz^i\,, \quad e^{2\Phi} = g_s^2 f_5(r)\,, \quad f_5(r) = 1 + \frac{r_5^2}{r^2}\,. \tag{7.1}$$

Here $r_5^2 = \alpha' n_5$, the five-branes are extended on the t, y, z^i directions with $i = 1, \ldots, 4$, and there is also the magnetic NSNS 3-form $H_3 = 2n_5 \epsilon_{S^3}$. We take the y-circle to have radius R_y. The string coupling asymptotes to g_s at infinity and diverges at the origin. We zoom in on the near-horizon region by rescaling $r \to \varepsilon r$ and taking $\varepsilon \to 0$ while going to small coupling with[30] $g_s \to \varepsilon$. This effectively drops the "1+" in f_5, and leads to

$$ds^2 = -dt^2 + dy^2 + d\phi^2 + r_5^2 d\Omega_3^2 + dz_i dz^i\,, \quad \Phi = -\frac{1}{r_5}\phi\,. \tag{7.2}$$

where we have defined the radial coordinate $\phi = -r_5 \log\left(\frac{r}{r_5}\right)$. We find that the dilaton becomes linear in ϕ, while the radius of the 3-sphere becomes constant and proportional to $\sqrt{n_5}$ in string units, giving a geometry of the form

$$\mathrm{Mink}^{1,5} \times \mathbb{R}_\phi \times S^3, \tag{7.3}$$

where, as in the $\mathrm{AdS}_3 \times S^3 \times T^4$ case, string propagation can be studied exactly. The relation between Φ and ϕ implies that the linear dilaton worldsheet field will have charge $Q_\phi = -\frac{1}{\sqrt{n_5}}$, such that the matter central charge (including fermionic contributions) will be

$$c = (6+3) + \left(1 + 6Q_\phi^2 + \frac{1}{2}\right) + \left(\frac{3(n_5-2)}{n_5} + \frac{3}{2}\right) = 15, \tag{7.4}$$

[30] Here we could have kept a fiducial (rescaled) g_s as is sometimes done in the literature, but it would lack a precise physical meaning.

where we have accounted for the fact that the bosonic SU(2) level gets shifted to $n_5 - 2$, see Sec. 5. As usual in critical superstring theory, this is cancelled by the ghost contributions.

The above near-horizon limit is reminiscent of those used for D-branes in the context of the AdS/CFT correspondence in order to reach the gravitational dual of the conformal fixed point of the field theory describing the brane dynamics at low energy. There are however, some important differences. Most notably, for D-branes one usually takes $\alpha' \to 0$, while here we take $g_s \to 0$, which has the effect of decoupling the bulk strings, while keeping α' fixed. Nevertheless, one might ask: is there some sort of holography at play for NS5-branes? It was argued in [29, 126–128] that the answer is yes. We now briefly discuss the qualitative motivations behind this proposal, and the main features of the putative boundary theory. Although the above discussion is valid for both type II string theories, some details of the discussion depend on whether we are in type IIA or type IIB.

NS5-branes are dynamical objects. In the type IIB case, their fluctuations are described by a supersymmetric six-dimensional theory with $\mathcal{N} = (1, 1)$ whose low-energy description is given by U(n_5) Super Yang-Mills [129]. By means of S-duality, one can see this from the D5-brane perspective, where the 6d gauge coupling satisfies $g^2_{\rm D5} = \tilde{g}_s \tilde{\alpha}'$. This is weakly coupled in the IR, which is why in the dual description Φ diverges as we take $\phi \to \infty$. By starting with $g^2_{\rm D5}$ and taking $\tilde{g}_s \to g_s^{-1}$ and $\tilde{\alpha}' \to g_s \alpha'$ we get the NS5 gauge coupling $g^2_{\rm NS5} = \alpha'$. The theory on the NS5-branes thus remains interacting even in the $g_s \to 0$ limit. The six-dimensional YM description is, roughly speaking, valid for energies below l_s^{-1}. Its non-renormalizability indicates that new degrees of freedom must come into play above this scale. (More precisely, at large n_5 one should work in terms of the 't Hooft coupling $g^2_{\rm NS5} n_5$, which suggests that the breakdown rather takes place at energies $E^2 \sim 1/n_5 \alpha'$.) These UV degrees of freedom are fundamental strings, whose tension $T = (2\pi\alpha')^{-1} \sim g^{-2}_{\rm NS5}$ shows that they can also be interpreted as the non-perturbative instanton excitations of the low-energy gauge fields, which are string-like in six dimensions. It is not surprising to find dynamical strings since we have kept α' fixed. However, the former are localized on the five-brane worldvolume. Moreover, there are no massless spin-two modes in the spectrum, hence we are dealing with a non-gravitational theory.

The type IIA case is slightly more involved, as there is a non-trivial IR fixed point given by the six-dimensional (2,0) SCFT. Here S-duality leads to the M-theory description. As $\phi \to \infty$ the dual metric becomes that of n_5 coincident M5-branes, described by an AdS$_7 \times S^4$ geometry. This is the supergravity dual of the (2,0) SCFT. A further stringy feature of LST is that it inherits the T-duality from that of type II strings since this commutes with the decoupling limit described above. Hence, the type IIA and IIB versions of LST are related to each other upon compactification. At energies where the little string dynamics are relevant, the corresponding theories actually become non-local.

If the background (7.5) is to provide a gravitational dual for these somewhat elusive six-dimensional, non-gravitational string theories, one should at least be able to match the (low energy) BPS spectra on both sides. This was done in [128] (exactly in α') for the full set of BPS of chiral operators using the low-energy field theory description of the NS5-theory and the worldsheet model for strings moving in the linear dilaton geometry. On the gravity side one also finds a continuum of long string states above the gap set by the NS5 't

Hooft scale $n_5\alpha'$. One can also study the thermodynamics of the system. However, the fact that the dilaton diverges in the IR makes the holographic description somewhat unreliable.

Thankfully, we know one way[31] to fix this! The IR behavior can be regularized by including n_1 fundamental string sources. By further compactifying the z^i directions on T^4, this leads to the background of Eqs. (2.2) and (2.4). Upon implementing the five-brane decoupling limit (i.e. going to the near-horizon of the NS5-branes, but not of the F1 strings), we obtain

$$ds^2 = f_1^{-1}\left(-dt^2+dy^2\right)+d\phi^2+r_5^2 d\Omega_3^2+dz_i dz^i\,, \quad e^{2\Phi}=f_1^{-1}r_5^2 e^{-2\phi}\,. \tag{7.5}$$

with $f_1 = 1+r_1^2 e^{-2\phi}$, with $r_1^2 = \alpha' n_1/v_4$. One also has $H_3 = 2n_5(1+(1-f_1^{-1})*_6)\epsilon_{S^3}$, where the AdS$_3$ part can be re-written as

$$H_{\rm AdS} = 2r_1^2 e^{-2\phi} f_1^{-2} dt\wedge dy\wedge d\phi = \frac{1}{2}d\left[\frac{1}{1+r_1^2 e^{-2\phi}}dt\wedge dy\right]\,. \tag{7.6}$$

The dilaton is now finite in the $\phi\to-\infty$ limit, and remains small as long as we have $n_1 \gg n_5$. This geometry thus interpolates between the linear dilaton background in the UV and AdS$_3\times S^3\times T^4$ in the IR, see Eq. (2.5). The latter is obtained in the string decoupling limit where we take $r,\alpha'\to 0$ with r/α' fixed. For later convenience, we note that upon rescaling $\phi\to\phi-\frac{1}{2}\log\left(\alpha' n_1 n_5/v_4\right)$ we get

$$f_1 \to 1+n_5^{-1}e^{-2\phi} \qquad e^{2\Phi}\to \frac{v_4}{n_1}\left(1+n_5^{-1}e^{-2\phi}\right)^{-1} \tag{7.7}$$

7.2 Solvable worldsheet deformations

In the previous section we have described the step by step procedure that takes us from the asymtptotically flat NS5-F1 configuration (on T^4) to the IR AdS$_3\times S^3$ description, passing by the intermediate linear dilaton regime associated to the NS5-brane decoupling. In short, in the first step we drop the the "1+" in the NS5-brane harmonic function f_5 only, while in the second one we do the same for the f_1 harmonic function associated to the fundamental string sources. We now describe how the inverse process can be realized, at least for the second step. This is done by implementing a suitable deformation of the worldsheet model, which effectively *adds back* the "1+" in f_1 [132, 133]. Here we switch to the bosonic setting to make use of the expressions given in Sec. 2, hence some factors of n_5 are replaced by k or $k-2$. At any rate, we are mostly interested in the large k regime where the difference is negligible.

Let us recall the AdS$_3$ worldsheet model, whose action is given in terms of the Wakimoto fields by

$$S_0 = \frac{1}{2\pi}\int d^2z\left(\partial\phi\bar\partial\phi - Q_\phi R^{(2)}\phi+\bar\beta\partial\bar\gamma+\beta\bar\partial\gamma-\beta\bar\beta e^{-\frac{2}{\sqrt{k-2}}\phi}\right)\,. \tag{7.8}$$

The holomorphic null current $J^+=\beta$ has protected dimension $\Delta=1$. One can thus deform the model by the marginal current-current operator $J^+\bar J^+ = \beta\bar\beta$, leading to

$$S = \frac{1}{2\pi}\int d^2z\left[\partial\phi\bar\partial\phi - Q_\phi R^{(2)}\phi+\bar\beta\partial\bar\gamma+\beta\bar\partial\gamma-\beta\bar\beta\left(1+e^{-\frac{2}{\sqrt{k-2}}\phi}\right)\right]\,. \tag{7.9}$$

[31] Another possibility was discussed in [130, 131].

In the IR region of the spacetime geometry $\phi \to -\infty$, that is, the strongly coupled regime of the worldsheet theory, the effect of the deformation is negligible. Conversely, in the large ϕ regime the new term becomes important. In particular, it changes the asymptotics of the spacetime configuration. By integrating out the Lagrange multipliers β and $\bar{\beta}$, considering the classical large k limit and redefining $\phi \to \sqrt{k}\left(\phi + \log \frac{k}{2}\right)$ we obtain

$$S = \frac{1}{2\pi}\int d^2z \left\{ k\partial\phi\partial\phi + \frac{\bar{\partial}\gamma\partial\bar{\gamma}}{(1+k^{-1}e^{-2\phi})} + \frac{1}{4}R^{(2)}\log\left[e^{-2\phi}(1+k^{-1}e^{-2\phi})^{-1} + \text{const}\right]\right\}. \tag{7.10}$$

Here the dilaton factor, which goes like $\Phi \sim -\phi$ at large ϕ, is necessary for the worldsheet beta-functions to vanish. This fixes Φ up to a multiplicative constant. The background we have landed on is thus exactly the (t, y, ϕ) sector of that given in Eqs. (7.5) and (7.6) for $k \approx n_5$, see Eq. (7.7). In the supersymmetric setup the analogous procedure leads exactly to the full NS5-F1 background in the five-brane decoupling limit.

Once the deformation is included, the geometry is not AdS_3 anymore, and the spacetime conformal symmetry is broken. As it turns out, at the worldsheet level one can still treat the theory exactly. This is because, as we now show, the deformation turns the geometry into a coset space[32].

Coset models are obtained to gauging some of the conserved current of an underlying WZW model. The starting point is not the SL(2,$\mathbb{R}$), but rather a simple 3+2 dimensional extension where we include an auxiliary time direction $\tilde{t}$ together with an auxiliary spatial circle $\tilde{y}$, and consider the coset [79]

$$\frac{\text{SL}(2,\mathbb{R}) \times \mathbb{R}_{\tilde{t}} \times \text{U}(1)_{\tilde{y}}}{\mathbb{R} \times U(1)}\,, \tag{7.11}$$

where we gauge the null, chiral currents

$$\mathcal{J} = J^+ + i(\partial\tilde{y} - \partial\tilde{t})\,, \qquad \bar{\mathcal{J}} = \bar{J}^+ + i(\partial\tilde{y} + \partial\tilde{t})\,. \tag{7.12}$$

These correspond to the isometries

$$(x^-, \gamma) \to (x^- + \alpha, \gamma + \alpha)\,, \qquad (x^+, \bar{\gamma}) \to (x^+ + \bar{\alpha}, \bar{\gamma} + \bar{\alpha})\,, \tag{7.13}$$

respectively, and with $x^\pm = \tilde{y} \pm \tilde{t}$. Following [134, 135], the gauge invariant action reads Lagrangian takes the form

$$L_{\text{gWZW}} = k\left[\partial\phi\bar{\partial}\phi + e^{2\phi}(\bar{\partial}\gamma + \bar{A})(\partial\bar{\gamma} + A)\right] + (\partial x^+ + A)(\bar{\partial}x^- + \bar{A})\,. \tag{7.14}$$

In two dimensions the chiral gauge fields A and $\bar{A}$ act as Lagrange multipliers. By integrating them out and gauge-fixing $\tilde{y} = \tilde{t} = 0$ we obtain

$$L_{\text{gWZW}} = k\partial\phi\bar{\partial}\phi + \frac{\bar{\partial}\gamma\partial\bar{\gamma}}{(1+k^{-1}e^{-2\phi})} + \frac{1}{4}R^{(2)}\log\left[e^{-2\phi}(1+k^{-1}e^{-2\phi})^{-1} + \text{const}\right]\,. \tag{7.15}$$

[32] An alternative, equivalent approach based on Wakimoto fields with modified boundary conditions was used in [80].

Hence, in the $\phi \to -\infty$ region we simply gauge away the auxiliary directions and recover the AdS_3 configuration, while in the UV we re-obtain the interpolating background discussed above. Of course, we could have equivalently gauged-fixed $\gamma = \bar{\gamma} = 0$. Gauging away $\tilde{y}$ and $\tilde{t}$ is more natural in the IR, since there we have the undeformed AdS. On the other hand, gauging away γ and $\bar{\gamma}$ makes more sense in the UV since the geometry asymptotes to a linear dilaton background where the geometry is flat and the spatial circle has a fixed radius.

We now note that upon re-inserting α' in the above formulas by means of $k \to \alpha' k$ (this is the dimensionful squared radius of the AdS factor in the IR) one effectively gets

$$1 + k^{-1} e^{-2\phi} \to \frac{1}{\alpha'} \left(\alpha' + k^{-1} e^{-2\phi} \right) . \tag{7.16}$$

Hence, it makes sense to think of the gauging parameter, i.e. the new factor in the metric, B-field and dilaton, as parametrized by α' [79]. The fact that this has units of length^2 this is consistent with the discussion carried out in the following section, where the procedure is interpreted in terms of an $(h, \bar{h}) = (2, 2)$ irrelevant deformation of the boundary theory. It also makes sense within the holographic perspective, as we have seen that the deformation should be negligible in the (string) near-horizon limit where we take $\alpha' \to 0$.

As discussed in [135], when the currents are chiral and null the model is automatically anomaly free. The string worldsheet model is then built from the *ungauged* action (i.e. that of the *upstairs* 3+2 dimensional model, ignoring the internal directions which simply go along for the ride) combined with an additional system of $\tilde{b}\tilde{c}$ ghosts associated with the gauging procedure. The holomoprhic and anti-holomorphic BRST charges acquire additional contributions of the form $\oint dz\, \tilde{c} \mathcal{J}$ and $\oint d\bar{z}\, \bar{\tilde{c}} \bar{\mathcal{J}}$, respectively, ensuring that the coset operators are gauge-invariant.

Given that our gaugings involve the SL(2,$\mathbb{R}$) currents J^+ and $\bar{J}^+$, it is useful to work directly with their eigenstates. Since $J^+ \sim \partial_x$, and $\bar{J} \sim \partial_{\bar{x}}$, this means that we consider momentum-space operators in the AdS_3 sub-sector. These are combined with exponentials in $\tilde{t}$ and $\tilde{y}$ to define

$$\mathcal{V}^{\omega}_{j,h}(p, \bar{p}) = \int d^2x e^{i(px + \bar{p}\bar{x})} V^{\omega}_{jh}(x, \bar{x}) e^{-i(E\tilde{t} + P_{\tilde{y},L}\tilde{y}_L + P_{\tilde{y},R}\tilde{y}_R)} V_{\text{int}} , \tag{7.17}$$

where we have included the possibility of having non-trivial winding around on the $\tilde{y}$-circle, so that

$$P_{\tilde{y},L/R} = \frac{n_{\tilde{y}}}{R_{\tilde{y}}} \pm \frac{\omega_{\tilde{y}} R_{\tilde{y}}}{\alpha'} . \tag{7.18}$$

Gauge invariance then implies

$$p + (E - P_{\tilde{y},L}) = 0 , \qquad \bar{p} + (E + P_{\tilde{y},R}) = 0 . \tag{7.19}$$

On the other hand, the Virasoro conditions for primary operators now receive contributions from the exponentials, giving

$$-\frac{j(j-1)}{k-2} - \omega h + \frac{k}{4}\omega^2 - \frac{\alpha'}{4}\left(E^2 - P^2_{\tilde{y},L}\right) + h_{\text{int}} - 1 = 0 , \tag{7.20}$$

$$-\frac{j(j-1)}{k-2} - \omega \bar{h} + \frac{k}{4}\omega^2 - \frac{\alpha'}{4}\left(E^2 - P^2_{\tilde{y},R}\right) + \bar{h}_{\text{int}} - 1 = 0 . \tag{7.21}$$

We have argued that $\tilde{t}$ and $\tilde{y}$ can be thought of as parametrizing the asymptotic circle, hence we interpret E as the energy in the dual theory and $n_{\tilde{y}}$ as the corresponding momentum. We now consider a state with $\omega = 0$ and $\omega_{\tilde{y}} = 1$, and also set $n_y = 0$ for simplicity, hence $\bar{h}_{\rm int} = h_{\rm int}$. After redefining the energy as that of the fluctuation above the winding contribution by replacing $E \to E + R_{\tilde{y}}$, the (sum of the) Virasoro conditions gives

$$\left(E + \frac{R_{\tilde{y}}}{\alpha'}\right)^2 - \left(\frac{R_{\tilde{y}}}{\alpha'}\right)^2 = \frac{2}{\alpha'}\left(-\frac{2j(j-1)}{k-2} + h_{\rm int} + \bar{h}_{\rm int} - 2\right) . \tag{7.22}$$

We now recall that for a singly wound long string in the *undeformed* model we had

$$h_0^{(1)} = -\frac{j(j-1)}{k-2} + \frac{k}{4} + h_{\rm int} - 1\,, \qquad \bar{h}_0^{(1)} = -\frac{j(j-1)}{k-2} + \frac{k}{4} + \bar{h}_{\rm int} - 1 \tag{7.23}$$

hence we can rewrite the energy in the *deformed* model in terms of undeformed quantities as

$$\left(E + \frac{R_{\tilde{y}}}{\alpha'}\right)^2 - \left(\frac{R_{\tilde{y}}}{\alpha'}\right)^2 = \frac{2}{\alpha'}\left(h_0^{(1)} + \bar{h}_0^{(1)} - \frac{k}{2}\right) . \tag{7.24}$$

Repeating the exercise for states with arbitrary $\omega_{\tilde{y}}$ gives

$$\left(E + \frac{\omega_{\tilde{y}} R_{\tilde{y}}}{\alpha'}\right)^2 - \left(\frac{\omega_{\tilde{y}} R_{\tilde{y}}}{\alpha'}\right)^2 = \frac{2}{\alpha'}\left(h_0^{(1)} + \bar{h}_0^{(1)} - \frac{k}{2}\right) . \tag{7.25}$$

The curious reader might wonder why we are comparing states which wind around the $\tilde{y}$-circle in the coset model with states which wind around the AdS$_3$ circle in the original theory, i.e. spectrally flowed (long string) states. The point is that although we construct the vertex operators in Eq. (7.17) by combining the elements of the auxiliary *upstairs model*, which contains both an SL(2,$\mathbb{R}$) factor and the auxiliary circle and time coordinates $\mathbb{R}_{\tilde{t}} \times S^1_{\tilde{y}}$, in the gauged theory there is only really a single physical spatial circle. Indeed, we are free to gauge away either $\tilde{t}$ and $\tilde{y}$ or t and y. This is clear in the geometrical analysis, where the UV circle becomes the AdS$_3$ boundary circle in the near-horizon limit.

In the quantum theory, this manifests as a residual discrete gauge symmetry. This was discussed in detail in for a related model in [22, 26, 28]. It implies that operators with windings (ω, ω_y) have equivalent descriptions in terms of operators with windings $(\omega + q, \omega_y - q)$ with $q \in \mathbb{Z}$. This is easily seen at the level of the spectrum. Indeed, the gauge constraints identify the energy E and momentum P_y with the AdS$_3$ ones. This suggests that we can extend the relation $E_0(h_0, \bar{h}_0)$ to an analogous relation between the quantities in the deformed theory. For instance, focusing on states with $h_0 = \bar{h}_0$ for simplicity, the Virasoro conditions for a spectrally flowed operator with $\omega = 1$ and $\omega_y = 0$ gives

$$E^2 + \frac{2}{\alpha'}\left(2h - \frac{k}{2}\right) = \frac{2}{\alpha'}\left(2h_0^{(1)} - \frac{k}{2}\right) . \tag{7.26}$$

This coincides with (7.24) upon identifying $ER_{\tilde{y}} = 2h - \frac{k}{2}$. More generally, the identification reads $ER_{\tilde{y}} = 2h - \frac{k}{2}\omega$, in agreement with the fact that we are computing the energy *above* the winding contribution.

The coset description of the NS5-F1 linear dilaton background can also be used to compute correlation functions exactly. In turn, these can be seen as providing a precise definition for the corresponding holographic observables, namely Little String Theory correlators [136–138]. The above construction also implies that these are best described in momentum space, consistent with the expectation that the holographic theory is now non-local. The characteristic scale of this non-locality is set by the deformation parameter, which we have identified with α', i.e. the fundamental length scale of the little strings. In short, for long-string operators with definite momentum p (and $\bar{p}$) the deformed correlators are obtained from the undeformed ones simply by replacing the original weight h_0 by the momentum-dependent expression $h = h_0 + \frac{\alpha'}{\omega} p\bar{p}$, and similarly for $\bar{h}$. These are nothing but the weights derived from the modified Virasoro conditions (7.20), combined with the gauge constraints which relate E and P_y with p, $\bar{p}$ and the deformation parameter.

7.3 Single-trace TTbar

We now provide yet another intepretation of the linear dilaton background directly in terms of the holographic theory. In the previous section we have seen that one can obtain the NS5-F1 linear dilaton background by a marginal deformation of the worldsheet theory associated the IR AdS$_3$ description. From the spacetime point of view, this represents a flow from the IR regime to the UV, where the geometry is not asymptotically AdS anymore. This RG flow to the UV can be understood holographically as an irrelevant deformation of the boundary CFT. The resulting two-dimensional theory is not conformal invariant. The LST picture suggests that it is not even a local quantum field theory. However, it does retain Lorentz invariance.

In general one would expect such an irrelevant deformation of a CFT to be ill-defined in the UV, at least on its own, as it might be necessary to specify which new degrees of freedom should be included and how they couple to the low-energy modes. However, in the present context, the fact that the worldsheet description of the deformation is exactly solvable suggests that our RG flow should be under control: we expect observables such as the energy spectrum and correlation functions to be computable in terms of their undeformed expressions. This is the trademark of the so-called $T\bar{T}$ deformations, which lead to UV complete theories.

Let us briefly review the basic facts about these very special deformations, following [139]. For any Lorentz-invariant two-dimensional QFT one can define a conserved and symmetric energy-momentum tensor. One can then build the $T\bar{T}$ operator, given by[33]

$$\mathcal{O}_{T\bar{T}} \equiv T\bar{T} - \Theta^2\,, \tag{7.27}$$

where $T = T_{xx}$, $\bar{T} = T_{\bar{x}\bar{x}}$, and $\Theta = T_{x\bar{x}}$. The latter vanishes in a CFT, but is non-trivial in more general situations. The operator $\mathcal{O}_{T\bar{T}}$ is well defined (up to total derivatives) since there are no short-distance singularities in $T(x+\varepsilon,\bar{x}+\bar{\varepsilon})\bar{T}(x,\bar{x}) - \Theta(x+\varepsilon,\bar{x}+\bar{\varepsilon})\Theta(x,\bar{x})$. This follows from the fact that the expectation value $\langle T(x,\bar{x})\bar{T}(y,\bar{y}) - \Theta(x,\bar{x})\Theta(y,\bar{y})\rangle$

[33] For the generalization to higher dimensions it is more useful to think of this as $\mathcal{O}_{T\bar{T}} = detT_{\mu\nu} \sim T_{\mu\nu}T_{\rho\sigma}\varepsilon^{\mu\rho}\varepsilon^{\nu\sigma}$.

is actually coordinate-independent as a consequence of translational invariance and the conservation equations. Another important property of this operator is that its expectation value factorizes. This holds even upon quantizing the theory on a cylinder, where the spatial coordinate σ is compactified on a circle of radius R. Indeed, consider the expectation value of $T(x,\bar{x})\bar{T}(y,\bar{y})-\Theta(x,\bar{x})\Theta(y,\bar{y})$ in an energy-momentum eigenstate $|n\rangle = |E_n, P_n\rangle$. Upon inserting an identity built out of a basis of similar eigenstates $|n'\rangle$, we find that the result is consistent with coordinate indepencence iff the only contribution comes from the term with $n = n'$. This implies that

$$\langle n|\mathcal{O}_{T\bar{T}}|n\rangle = -\frac{1}{4}\left(\langle n|T_{\tau\tau}|n\rangle\langle n|T_{\sigma\sigma}|n\rangle\langle n|T_{\tau\sigma}|n\rangle^2 = -\frac{1}{16\pi^2 R^2}\left[(1+R\partial_R)E_n + P_n^2\right]\right)\,, \tag{7.28}$$

where τ denotes (Euclidean) time coordinate. On the other hand, the $T\bar{T}$ operator has weight (2,2), and can be used to define an irrelevant deformation by adding it to the action, with a dimensionful parameter μ in front. In other words, the action of the $T\bar{T}$-deformed theory is defined by the flow equation

$$\partial_\mu S = -2\int d^2x\,\mathcal{O}^{(\mu)}_{T\bar{T}}, \tag{7.29}$$

where the superscript on the RHS indicates that the definition of the deforming operator must be updated at each step.

One of the remarkable properties of $T\bar{T}$-deformed QFTs is that several observables can be computed exactly for finite values of the parameter μ in terms of the corresponding the undeformed quantities. One of these is the spectrum of excitations for the theory on the cylinder. Indeed, by means of perturbation theory a small change in μ produces a variation in the energy eigenvalue of an eigenstate $|n\rangle$ of the form

$$\partial_\mu E_n = -2\int d^2x\,\langle n|\mathcal{O}^{(\mu)}_{T\bar{T}}|n\rangle\,. \tag{7.30}$$

Inserting (7.28) on the RHS and including a factor R from the integration, and further focusing on states with $P_n = 0$ for simplicity then leads to the flow equation

$$2\pi\partial_\mu E_n = E_n\partial_R E_n\,. \tag{7.31}$$

When the starting point is a CFT, we usually trade the index n for the weights h and $\bar{h}$. The solution $E_{h,\bar{h}}(\mu,R)$ must then satisfy

$$RE_{h,\bar{h}}(\mu=0,R) = \left(h+\bar{h}-\frac{c}{12}\right)\,, \tag{7.32}$$

where c is the central charged at the IR fixed point, such that $\frac{-c}{12}$ provides the Casimir energy. With this initial condition, the solution is specified by the quadratic relation

$$\left(E_{h,\bar{h}}+\frac{\pi R}{\mu}\right)^2 = \left(\frac{\pi R}{\mu}\right)^2 + \frac{2\pi}{\mu}\left(h+\bar{h}-\frac{c}{12}\right)\,. \tag{7.33}$$

We find that the energy spectrum (7.33) precisely reproduces Eq. (7.24) upon identifying

$$R = R_{\tilde{y}}, \qquad h = h_0^{(1)} \qquad \bar{h} = \bar{h}_0^{(1)} \qquad \mu = \pi\alpha' \qquad c = c_{\text{st}}^{(1)} = 6k\,. \tag{7.34}$$

Hence, the long-string spectrum in the singly-wound sector of the worldsheet theory can be interpreted as that of the $T\bar{T}$-deformation of the *seed* theory in the symmetric orbifold of Eq. (6.32).

If this interpretation is to hold, one should establish (1) how it generalizes to higher-twist sectors of the symmetric orbifold model, and (2) what is the effect of the deformation in Eq. (6.32) which accounts for the fact that the holographic CFT is actually *not* a symmetric orbifold.

For the first question, there are two possible natural answers. One of them is to simply consider the $T\bar{T}$-deformation of the full symmetric orbifold. However, we note that the operator $\mathcal{O}_{T\bar{T}}$ is a double-trace operator, i.e. it involves a double sum over all copies of the theory. At zero-th order in μ we have $T(x) = \sum_{i=1}^{n_1} T_i(x)$, hence $\mathcal{O}_{T\bar{T}}(x,\bar{x}) = T(x)\bar{T}(\bar{x}) = \sum_{i,j=1}^{n_1} T_i(x)\bar{T}_j(\bar{x})$. From the string point of view this is a multi-particle state, which seems at odds with the fact that we have been able to describe the linear dilaton background in terms of a marginal deformation of the worldsheet theory by the current-current operator $J^+(z)\bar{J}^+(\bar{z})$. From the exact AdS$_3$ description we know that the spacetime holomorphic energy-momentum tensor $\mathcal{T}(x)$ is constructed on the worldsheet as in Eq. (2.178). The double-trace $T\bar{T}$-deformation under consideration should then correspond to an insertion involving not one but two integrals over the worldsheet coordinates.

In the context of symmetric orbifolds there is, however, another possibility: one can maintain the orbifold structure and consider the symmetric orbifold theory where the *seed* is $T\bar{T}$-deformed. For the case at hand, we thus consider the orbifold model

$$\text{Sym}^{n_1}\left[\left(R_\phi^Q \times M_{\text{int}}\right)_{T\bar{T}}\right]\,. \tag{7.35}$$

Here the energies in the twisted sector will be related to those in the $T\bar{T}$-deformed seed in exactly the same way as what was described in the previouse sections, see for instance (4.14), which is in agreement with the result obtained in Eq. (7.25). This proposal was put forward in [32], and is known as single-trace $T\bar{T}$. The deforming operator is generated by the following (2,2) operator:

$$\mathcal{O}_{\text{st}\,T\bar{T}}(x,\bar{x}) = \sum_{i=1}^{n_1} T_i(x)\bar{T}_i(\bar{x})\,. \tag{7.36}$$

Upon inserting the integrated version to the action we have

$$\delta S = \mu \int d^2x\, \mathcal{O}_{\text{single-trace}\,T\bar{T}}(x,\bar{x}) \sim \mu \mathcal{L}_{-1}\bar{\mathcal{L}}_{-1}\,. \tag{7.37}$$

This makes sense because these Virasoro spacetime modes are seen from the worldsheet as $\mathcal{L}_{-1} = \partial_x = J_0^+$ and $\bar{\mathcal{L}}_{-1} = \partial_{\bar{x}} = \bar{J}_0^+$. The natural candidate for the exact deformation at the worldsheet level is obtained by extending the holomorphic structure in the definition of

$\mathcal{T}(x)$ given in Eq. (2.178) to the anti-holomorphic sector. In other words, at the worldsheet level we should add the operator

$$\mathcal{O}^{\rm ws}_{{\rm st}\,T\bar{T}}(x,\bar{x}) = \frac{1}{4}\int d^2z\,\left[(\partial_x J)\partial_x + 2(\partial_x^2 J)\right]\left[(\partial_{\bar{x}}\bar{J})\partial_{\bar{x}} + 2(\partial_{\bar{x}}^2\bar{J})\right]V_1(x,\bar{x},z,\bar{z})\,. \tag{7.38}$$

to the worldsheet action. This indeed has spacetime weights $h = \bar{h} = 2$. Moreover, integrating by parts we have

$$\begin{aligned}\int d^2x\mathcal{O}^{\rm ws}_{{\rm st}\,T\bar{T}}(x,\bar{x}) &= \frac{1}{4}\int d^2x\,d^2z\,(\partial_x^2 J)(\partial_{\bar{x}}^2\bar{J})V_1(x,\bar{x},z,\bar{z})\\ &= \int d^2xd^2zJ^+(z)\bar{J}^+(\bar{z})V_1(x,\bar{x},z,\bar{z})\,.\end{aligned} \tag{7.39}$$

The only singular term in the JV_1 OPE is a total derivative, hence we can treat this as a simple product. Since near the AdS$_3$ boundary we have $V_1 \sim \delta^{(2)}(x-\gamma)$, the integration over x trivializes, leading to

$$\int d^2x\mathcal{O}^{\rm ws}_{{\rm st}\,T\bar{T}}(x,\bar{x}) = \int d^2zJ^+(z)\bar{J}^+(\bar{z})\,, \tag{7.40}$$

which is consistent with the current-current deformation considered in the previous section.

Finally, we address the second question raised above, related to the deformation of the symmetric orbifold and its implications for the $T\bar{T}$ interpretation of the irrelevant flow of the spacetime theory. Clearly, the notion of single-trace $T\bar{T}$-deformation relies heavily on having a symmetric orbifold structure. Consequently, it is not clear how one would define it when considering a generic point in the moduli space of the holographic CFT as the starting point of the deformation. Although a precise answer is presently not known, we note that one can take the worldsheet deformation given in (7.38) as a holographic *definition* for the generalization of the single-trace deformation, at least along the NSNS locus.

As we have discussed in detail, in the supersymmetric model the symmetric orbifold point corresponds to the background with $n_5 = 1$. In such situations the spectrum only contains long strings, and we expect the coset model to be the exact dual of the symmetric orbifold built out of n_1 copies of the $T\bar{T}$-deformed (4,4) SCFT with target space T^4. On the other hand, for $n_5 \neq 1$ the precise matching holds only for the long-string spectrum, while short strings and correlation functions are not expected to preserve the symmetric orbifold structure.

7.4 Little string thermodynamics

We now discuss the thermodynamics, which provide one of the main arguments for the proposed triality between type IIB little string theory, the linear dilaton NS5-F1 background, and, roughly speaking, the (single-trace) $T\bar{T}$-deformed holographic CFT.

We start by constructing black holes in the asymptotically flat backgrounds under consideration. Focusing on the zero-momentum case for simplicity, the line element reads

$$ds^2 = f_1^{-1}\left(fdt^2 + dy^2\right) + f_5\left(f^{-1}dr^2 + r^2d\Omega^3\right) + dz_idz^i\,, \tag{7.41}$$

while the dilaton and NSNS 3-form take the form

$$H = 2q_5\varepsilon_{S^3} + +\frac{2q_1}{r^3 f_1^2} dt\wedge dy\wedge dr \ , \ e^{2\Phi} = g_s^2 f_5 f_1^{-1} \,. \tag{7.42}$$

Here the harmonic functions are

$$f = 1 - \frac{r_0^2}{r^2}\,, \qquad f_{1,5} = 1 + \frac{r_{1,5}^2}{r^2}\,, \qquad q_{1,5}^2 = r_{1,5}^2(r_{1,5}^2 + r_0^2)\,, \tag{7.43}$$

with $q_1 = n_1 g_s^2\alpha'/v_4$ and $q_5 = n_5\alpha'$, such that the horizon is located at $r = r_0$. In the $r_0 \to 0$ limit, we recover the extremal BTZ black hole, i.e. the ground state in the (spacetime) R sector considered in the previous sections. We can compute the associated entropy as a function of r_0 by evaluating the horizon area (in Einstein frame) and plugging it in the Bekenstein-Hawking formula

$$S = \frac{A_{\rm h}}{4G_N} = \frac{2\pi R_y v_4 r_0^3}{g_s^2 l_s^4}\sqrt{f_1(r_0) f_5(r_0)}\,, \tag{7.44}$$

where $G_N = 8\pi^6 g_s^2 l_s^4$ is the ten-dimensional gravitational coupling. One can also compute the Arnowitt-Deser-Misner (ADM) mass and obtain the energy above extremality

$$E = M_{\rm ADM} - E_{\rm ext}\,, \qquad E_{\rm ext} = 2\pi R_y n_1 T_{\rm F1} + (2\pi)^5 R_y v_4 l_s^4 n_5 T_{\rm NS5}\,, \tag{7.45}$$

where the tensions are $T_{\rm F1} = (2\pi\alpha')^{-1}$ and $T_{\rm NS5} = [(2\pi)^5 l_s^6 g_s^2)^{-1}]^{-1}$. This gives [140]

$$E = \frac{R_y v_4 r_0^2}{2 l_s^4 g_s^2}\left[3 + \frac{2}{r_0^2}(r_1^2 + r_5^2 - q_1 - q_5)\right]. \tag{7.46}$$

We find that in the asymptotically flat spacetime and for large r_0 we have $E \sim r_0^2$ and $S \sim r_0^3$, so that $S \sim E^{3/2}$.

This changes drastically in the five-brane decoupling limit. As before, we rescale $r \to g_s r$, $r_0 \to g_s r_0$ and take $g_s \to 0$ (with l_s fixed). Since we effectively drop the constant term in f_5, the relation between S and E becomes

$$S = 2\pi\sqrt{n_5 l_s^2 E^2 + 2n_1 n_5 R_y E}\,. \tag{7.47}$$

At low energies, we can ignore the first term on the RHS, so that

$$S \approx 2\pi\sqrt{2n_1 n_5 E R_y}\,. \tag{7.48}$$

This precisely reproduces the Cardy formula for a CFT with central charge $c = 6n_1n_5$ on a circle of radius R_y, namely $S = 2\pi\sqrt{\frac{c}{3}ER_y}$. Here we have used the relation $ER_y = h+\bar{h}-\frac{c}{12}$ and set $h = \bar{h}$ since our black holes have no angular momentum. On the other hand, for very high energies we find

$$S \approx \beta_H E\,, \beta_H = 2\pi\sqrt{n_5\alpha'}. \tag{7.49}$$

The is known as a Hagedorn regime [141, 142], while β_H defines the (inverse) Hagedorn temperature. This is characteristic of systems where the density of states grows exponentially at high energies, and can be seen as a trademark for stringy degrees of freedom.

The Hagedorn behavior dominating the high-energy regime of the theory is a strong indication that the dual LST theory is indeed a theory of strings. There is, however, an important difference with critical string theory. For critical strings in flat space, the Hagedorn temperature is set by the fundamental string tension, i.e. $\beta_H \sim \sqrt{\alpha'}$. For little strings we have found $\beta_H \sim \sqrt{n_5\alpha'}$. This suggests that the effective tension in LST is lowered by a factor of n_5. We can interpret this as coming from the fact that fundamental strings bound to the NS5-branes can fractionate their winding charge in units of n_5^{-1}, hence the name *little strings*. In the low-energy six-dimensional gauge theory description, there are indeed string-like instantons excitations of the fractional type (see [23] and references therein), and $n_5\alpha' = n_5 g^2_{\rm NS5}$ is the corresponding 't Hooft coupling.

The above picture can be reproduced exactly from the $T\bar{T}$ description. We first consider the $T\bar{T}$ deformation of a single copy of the holographic CFT. The result obtained in Eq. (7.33) describes how each of the individual energy eigenvalues evolves upon including the irrelevant deformation as a function of the deformation parameter μ. This shows that the different levels of the original CFT do not cross each other as we climb up the RG flow. Hence, we can still use h and $\bar{h}$ to label the different states, and the entropy is given by the Cardy formula $S^{(1)} = S^{(1)}(h,\bar{h},c)$. Importantly, the expression of $S^{(1)}(E,P,c)$ *does* change, as can be derived directly from (7.33). Focusing on the sector with $h=\bar{h}$, from Eq. (7.33) we get

$$S^{(1)} = 2\pi\sqrt{\frac{c^{(1)}}{3}\left(2h - \frac{c^{(1)}}{12}\right)} = 2\pi\sqrt{n_5\left(2ER + \frac{\mu}{\pi}E^2\right)}\,. \tag{7.50}$$

At least at the orbifold point, the holographic theory is interpreted as the symmetric orbifold of $T\bar{T}$-deformed CFTs, where n_1 denotes the total number of copies of the seed theory. For fixed total energy E, the entropy is extremized (and actually maximized) when each copy carries an equal fraction E/n_1. We conclude that

$$S(E,c) = n_1 S^{(1)}(E/n_1, c^{(1)}) = 2\pi\sqrt{n_5\left(2n_1ER + \frac{\mu}{\pi}E^2\right)}\,, \tag{7.51}$$

where we have used $\mu = \pi\alpha'$. Hence, the single-trace $T\bar{T}$ thermodynamics reproduce the gravitational entropy in Eq. (7.47) upon identifying $R = R_y$ and $\mu = \pi l_s^2$. The fact that we have obtained a precise matching for all values of n_1 and n_5 suggests that this entropy is dominated by the physics of long strings, otherwise the deformation of the symmetric orbifold would have had a non-trivial impact.

Before finishing, we should mention a further highly non-trivial check of the duality under consideration that was presented in [33, 81]. There, the authors studied the symmetry structure of the $T\bar{T}$-deformed theory both from the symmetric orbifold point of view and from the gravitational, asymptotically linear dilaton background. The conclusion is that the Virasoro $\times$ Virasoro symmetry of the undeformed theory survives in the deformed one, although the generators become field-dependent, such that the resulting theory is non-local.

8 Application II: black holes and their microstates

8.1 Coset models. The Cigar.

8.2 BTZ Black holes in AdS_3

8.3 Sub-stringy geometries and correspondence point for AdS_3 black holes

8.4 Black hole microstates and Heavy-Light correlators

9 Outlook

This was fun!

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A WZW models

Here we review some relevant concepts of CFT in two dimensions together with the basic aspects of WZW models. We then discuss coset theories, i.e. gauged WZW (gWZW) models. We mostly follow [82, 84, 143, 144].

A.1 Basic concepts of CFT$_2$ and WZW models

Let us consider a d-dimensional manifold $\mathcal{M}$ parametrized by coordinates X^μ, with $\mu = 1, \ldots, d$. We are interested in describing the motion of a two-dimensional object on $\mathcal{M}$, which we refer to interchangeably as the *target space* or the *spacetime*, by thinking of the coordinates on $\mathcal{M}$ as fields $X^\mu(\sigma^\alpha)$, $\alpha = 0, 1$. The timelike and spacelike coordinates on the *worldsheet* Σ are usually denoted as $\sigma^0 = \tau$ and $\sigma^1 = \sigma$, respectively.

The dynamics of the fields X^μ is captured by the action

$$S_{\text{kin}} = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \, G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \eta^{\alpha\beta}, \tag{A.1}$$

where G is a local metric on $\mathcal{M}$, and α' is a constant with units of length squared, which defines the so-called string tension. This is known as a non-linear σ-model. Here we take the worldsheet metric to be $\eta = \text{diag}(-1, 1)$. In the string theory context this can always be achieved by making use of the invariance of the Polyakov action under reparametrizations and Weyl transformations. We will mostly work with the Euclidean version of the model, where σ^0 is taken to be space-like, and we replace $\eta^{\alpha\beta} \to \delta^{\alpha\beta}$. Moreover, we make use of the complex coordinates

$$z = \sigma^0 + i\sigma^1 \,, \quad \bar{z} = \sigma^0 - i\sigma^1 \,, \quad \partial \equiv \partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \,, \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \,, \tag{A.2}$$

for which the worldsheet metric becomes $g_{\alpha\beta} = \text{diag}(1/2, 1/2)$. Results for Minkowskian worldsheets can then be obtained by analytic continuation, namely taking $\sigma^0 \to i\sigma^0$.

One can also consider additional couplings, related to the presence of other background fields. For instance, given a globally defined closed 3-form H on $\mathcal{M}$ one can include a term proportional to the Wess-Zumino action

$$S_{\rm WZ} = \frac{i}{6\pi\alpha'}\int_\Omega H_{\mu\nu\rho}(X) dX^\mu \wedge dX^\nu \wedge dX^\rho = \frac{i}{4\pi\alpha'}\int_\Sigma B_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \varepsilon^{\alpha\beta}, \tag{A.3}$$

where $\varepsilon^{01} = 1$, Ω is a 3-dimensional manifold whose boundary is Σ. The factor of i makes the Euclidean action real. The choice of Ω is irrelevant since H is closed. The final expression in (A.3) is only valid when H is exact, $H = dB$ for some two-form B, with d the exterior derivative. The inclusion of $S_{\rm WZ}$ implies that our string-like object is (electrically) charged under the potential $B_{\mu\nu}$. For non-trivial worldsheet metrics $\gamma_{\alpha\beta}$ one can also include a coupling to a target-space scalar field $\Phi(X)$, namely the dilaton:

$$S_\Phi = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \sqrt{\gamma}\,\alpha'\Phi(X)R_\gamma, \tag{A.4}$$

where R_γ is the two-dimensional Ricci scalar. The fields G, B and Φ constitute the massless fields of the universal NSNS sector of closed string theories.

Except when G is flat and both B and Φ are trivial, the non-linear σ model is an interacting theory. By expanding in powers of X, the backgrounds fields then define a (possibly infinite) set of interaction terms. Focusing on the kinetic sector, we can estimate the corresponding coupling constants by thinking of a configuration where G has a characteristic length r_G. Then, roughly speaking, the effective dimensionless coupling is given by $\sqrt{\alpha'}/r_G$. For small $\sqrt{\alpha'}/r_G$, the worldsheet theory is weakly coupled and can be treated perturbatively, and the low-energy physics are captured in terms of a gravitational model in d dimensions. In particular, the β-functions characterizing the renormalization group flow of the σ-model are identified with the equations of motion (EOMs) of the effective space-time action. In the string theory context, and at the two-derivative level, this provides a derivation of the EOMs of general relativity coupled with the relevant matter fields.

Of course, if we are interested in extremely high-energy processes it is necessary to include an ever-growing set of higher-derivative terms, known as α'-corrections. This quickly becomes quite cumbersome. Moreover, when $\sqrt{\alpha'}/r_G$ is not small the worldsheet model becomes strongly coupled. In order to address both of these problems it would be very useful to be able to study the worldsheet theory beyond perturbation theory. However, it is not known how to do this in arbitrary backgrounds.

We now introduce Wess-Zumino-Witten (WZW) models, which constitute a set of non-linear σ-models which can be treated exactly — at least in principle — and provide the crucial building blocks in the construction of non-perturbative string backgrounds. Let us take the target space $\mathcal{M}$ to be the group manifold associated to a Lie group $\mathfrak{G}$. The WZW action is then

$$S_{\rm WZW} = S_0 + k\Gamma. \tag{A.5}$$

Here S_0 is the kinetic part of the action defined in Eq. (A.1), which can be written as

$$S_0 = \frac{1}{4a^2}\int d^2\sigma \,{\rm Tr}\left[g^{-1}\partial^\alpha g g^{-1}\partial_\alpha g\right], \tag{A.6}$$

where we have included an overall constant which will be fixed shortly. For simplicity, we consider the model as defined on the Riemann sphere, $\Sigma = S^2$. The bosonic field $g(z,\bar{z})$

parametrizes the group elements of $\mathfrak{G}$, such that $g^{-1}\partial_\alpha g$ is an element of the corresponding Lie algebra $\mathfrak{g}$. The trace in (A.6) defines the Killing form, and for a suitable basis of a matrix representation of $\mathfrak{g}$, t^a, $a = 1, \ldots, \dim \mathfrak{g}$ we have

$$\mathrm{Tr}\left[t^a t^b\right] = 2g^{ab} \,, \qquad \left[t^a, t^b\right] = i f^{ab}{}_c t^c \,, \tag{A.7}$$

where $f^{ab}{}_c$ are the structure constants and g^{ab} is the Killing metric.

The second term in S_{WZW} is the WZ part of the action,

$$\Gamma = \frac{-i}{24\pi} \int_\Omega d^3 y \varepsilon_{\alpha\beta\gamma} \,\mathrm{Tr}\left[\tilde{g}^{-1}\partial^\alpha \tilde{g}\tilde{g}^{-1}\partial^\beta \tilde{g}\tilde{g}^{-1}\partial^\gamma \tilde{g}\right] . \tag{A.8}$$

One must be careful in extending the field g to $\tilde{g}$, which is defined on a given Ω, since the procedure is not unique. For small variations $\tilde{g} \to \tilde{g} + \delta\tilde{g}$, one finds that the ingrand of the corresponding variation $\delta\Gamma$ is a total derivative, hence Γ is invariant as long as $\delta\tilde{g}|_{S^2} = \delta g = 0$. On the other hand, when the variation is topologically non-trivial we can use that $(\Omega \cup \Omega')/S^2$ is topologically equivalent to a three-sphere, and compute the difference between two choices from

$$\Delta\Gamma = \frac{-i}{24\pi} \int_{S^3} d^3 y \varepsilon_{\alpha\beta\gamma} \,\mathrm{Tr}\left[\tilde{g}^{-1}\partial^\alpha \tilde{g}\tilde{g}^{-1}\partial^\beta \tilde{g}\tilde{g}^{-1}\partial^\gamma \tilde{g}\right] . \tag{A.9}$$

At least when $\mathfrak{G}$ is compact and has an SU(2) subgroup, it turns out to be enough to focus on mappings $g : S^3 \to SU(2)$, hence we can choose [82]

$$g(y) = y^0 - i y^i \sigma^i \,, \qquad (y^0)^2 + y^i y^i = 1 \quad i = 1, 2, 3 \,, \tag{A.10}$$

where σ^i are the Pauli matrices. This leads to

$$\Delta\Gamma = \frac{\pi}{6}\varepsilon_{ijk} \,\mathrm{Tr}\left[\sigma^i \sigma^j \sigma^k\right] = 2\pi i. \tag{A.11}$$

We conclude that S_{WZW} in Eq. (A.5) is invariant as long the k is quantized, $k \in \mathbb{Z}$. This is known as the *level* of the WZW model.

The model defined by Eq. (A.5) is clasically scale invariant. Moreover, it is invariant under global $\mathfrak{G}_L \times \mathfrak{G}_R$ transformations

$$g(z, \bar{z}) \to g_L g(z, \bar{z}) g_R^{-1}. \tag{A.12}$$

In the σ-model language, these come from the Killing vectors of the target space metric which also leave the background H-flux invariant. The EOMs derived from S_{WZW} read

$$\left(1 + \frac{a^2 k}{4\pi}\right) \partial \left(g^{-1}\bar{\partial} g\right) + \left(1 - \frac{a^2 k}{4\pi}\right) \bar{\partial} \left(g^{-1}\partial g\right) = 0. \tag{A.13}$$

Hence, upon taking $k > 0$ and choosing $a^2 = 4\pi/k$ we obtain

$$\partial \left(g^{-1}\bar{\partial} g\right) = 0 \,, \qquad \bar{\partial} \left(\partial g g^{-1}\right) = 0 \tag{A.14}$$

where the second identity is implied by the first one since $\bar{\partial}\left(\bar{\partial}gg^{-1}\right) = g\partial\left(g^{-1}\bar{\partial}g\right)g^{-1}$. Hence, WZW models (by which we mean those with $a^2 = 4\pi/k$ from now on) have holomorphic and antihomorphic conserved currents

$$J = -k\partial gg^{-1}\,, \qquad \bar{J} = kg^{-1}\bar{\partial}g. \tag{A.15}$$

These presence of these currents comes from the fact that the global $\mathfrak{G}_L \times \mathfrak{G}_R$ symmetry is enhanced to a local one:

$$g(z,\bar{z}) \to g_L(z)g(z,\bar{z})g_R^{-1}(\bar{z})\,. \tag{A.16}$$

Indeed, for $g_L(z) = 1 + \omega(z)$ and $g_R(\bar{z}) = 1$ we have

$$\delta_\omega S_{\mathrm{WZW}} = -\frac{1}{2\pi}\int d^2\sigma\, \partial_{\bar{z}}\,\mathrm{Tr}\left[\omega(z)J(z)\right] = 0\,. \tag{A.17}$$

From now on we focus on the holomorphic sector, although analogous formulas can be obtained for the antiholomorphic one.

Let us now move to the quantum theory. Eq. (A.17) translates into the (local) Ward identity for the correlation functions of the theory, namely

$$\delta_\omega\langle X\rangle - \oint dz\omega^a(z)\langle J^a(z)X\rangle, \qquad J(z) \equiv J^a(z)t^a, \tag{A.18}$$

where we have integrated by parts[34]. On the other hand, the variation $\delta g = \omega g$ gives that of the current gives $\delta_\omega J = [\omega, J] - k\partial\omega$. By combining this with Eq. (A.18) we find the operator product expansion (OPE)

$$J^a(z)J^b(w) \sim \frac{kg^{ab}}{(z-w)^2} + \frac{if^{ab}{}_cJ^c(w)}{z-w}, \tag{A.19}$$

which holds inside correlation functions, and defines the current algebra of the model. In terms of the corresponding modes

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}J^a_n\,, \tag{A.20}$$

this gives

$$\left[J^a_n, J^b_m\right] = \oint_0 dw \oint_w dz\, w^n z^m J^a(z)J^b(\omega) = kng^{ab}\delta_{n+m,0} + if^{ab}{}_cJ^c_{n+m}, \tag{A.21}$$

which defines an affine Lie algebra $\hat{\mathfrak{g}}$ at level k. The modes J^a_n and $\bar{J}^b_m$ commute with each other, hence the current algebra factorizes.

The presence of the two conserved currents $J(z)$ and $\bar{J}(\bar{z})$ is very powerful. In particular, it implies that WZW models are conformally invariant at the quantum level. In order to see this, we need to show that there is a conserved holomorphic tensor $T(z)$ of weight h=2 whose modes generate the Virasoro algebra, captured by the OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{A.22}$$

[34]Here and from now on we will omit the factors $(2\pi i)^{-1}$ when dealing with contour integrals.

where c is the central charge. This is known as the Sugawara construction. We can start from the classical expression for the energy-momentum tensor and propose

$$T(z) \equiv \gamma g_{ab}(J^a J^b)(z) = \gamma \oint_z \frac{dw}{w-z} g_{ab} J^a(w) J^b(z), \tag{A.23}$$

for some constant γ, and where the brackets indicate normal ordering. Note that, classically, one has $\gamma = (2k)^{-1}$. However, requiring that the currents are primary fields of weight $h = 1$ with respect to $T(z)$, namely

$$T(z) J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}, \tag{A.24}$$

imposes

$$\gamma = \frac{1}{2(k+\mathfrak{c})}\,, \qquad f^{ab}{}_c f^{cd}{}_b = 2\mathfrak{c} g^{ab}, \tag{A.25}$$

as can be seen by using the OPE (A.19). Here $\mathfrak{c}$ is the so-called dual Coxeter number. Hence, the value of γ receives a quantum correction. Indeed, $k \to \infty$ defines the classical limit, where all modes $J^a_{n\neq 0}$ decouple. As the fields $g(z,\bar{z})$ are dimensionless, $1/\sqrt{k}$ plays the role of the squared effective coupling. Once the quantum shift is taken into account, it can be seen that the modes

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n\,, \qquad L_n = \frac{g_{ab}}{2(k+\mathfrak{c})}\left[\sum_{m<0} J^a_m J^b_{n-m} + \sum_{m\geq 0} J^a_{n-m} J^b_m\right] \tag{A.26}$$

satisfy the Virasoro algebra

$$[L_n, L_m] = \frac{c}{12} n(n^2-1)\delta_{n+m,0} + (n-m)L_{n+m}\,, \qquad c = \frac{k\,\mathrm{dim}\mathfrak{g}}{k+\mathfrak{c}}\,, \tag{A.27}$$

while the commutation relations with the current modes implied by (A.24) are

$$[L_n, J^a_m] = -m J^a_{n+m}. \tag{A.28}$$

We conclude that WZW models constitute two-dimensional conformal field theories.

Let us now discuss the primary fields of the theory, which will be denoted as $V_{j,\bar{j}}(z,\bar{z})$ (and in general do *not* factorize). Here we must distinguish between the usual Virasoro primaries and the so-called affine primary fields. Affine primaries appear in CFTs with symmetry algebras of the type (A.21). They transform as in Eq. (A.16), which translates into an OPE with the currents of the form

$$J^a(z) V_{j,\bar{j}}(w,\bar{\omega}) \sim \frac{-t^a_j V_{j,\bar{j}}(w,\bar{w})}{z-w}\,, \qquad \bar{J}^a(\bar{z}) V_{j,\bar{j}}(w,\bar{\omega}) \sim \frac{V_{j,\bar{j}}(w,\bar{w}) t^a_{\bar{j}}}{\bar{z}-\bar{w}} \tag{A.29}$$

where j and $\bar{j}$ label the corresponding zero-mode representations $R_j \times R_{\bar{j}}$. Of course, in general there will be additional quantum numbers which specify the particular state we are dealing with, $V_{j,\bar{j},m,\bar{m}}(z,\bar{z})$ with $m = 1,\ldots,\dim R_j$ and $\bar{m} = 1,\ldots,\dim R_{\bar{j}}$, but we will omit them for now. For the state created by $V_{j,\bar{j}}(z,\bar{z})$ when acting on the vacuum,

$$V_j(0)|0\rangle = |j\rangle \tag{A.30}$$

the simple pole in the OPE captures the action of the current zero-modes, while the absence of higher-order poles shows that positive modes act as annihilation operators, i.e.

$$J_0^a|j\rangle = -t_j^a|j\rangle\,, \qquad J_n^a|j\rangle = 0 \quad \forall \quad n>0. \tag{A.31}$$

As before, here we have focused on the holomorphic sector of the algebra. In these notes we mainly focus on *diagonal* operators with $\bar{\jmath}=j$, which represent scalar fields. Note the minus sign in the action of the zero modes, which leads to

$$[J_0^a, J_0^b]|j\rangle = (J_0^b t_j^a - J_0^a t_j^b)|j\rangle = (t_j^a J_0^b - t_j^b J_0^a)|j\rangle = -[t_j^a, t_j^b]|j\rangle = -if^{ab}{}_c t_j^c|j\rangle = if^{ab}{}_c J_0^c|j\rangle\,,$$

as expected. Importantly, by means of (A.26), Eq. (A.31) automatically implies that

$$L_0|j\rangle = \frac{g_{ab}t_j^a t_j^b}{2(k+\mathfrak{c})}|j\rangle\,, \qquad L_n|j\rangle = 0 \quad \forall \quad n>0, \tag{A.32}$$

which shows that the conformal weight $h=h(j)$ is proportional to the quadratic Casimir of the representation. In other words,

$$T(z)V_j(w,\bar{w}) \sim \frac{hV_j(w,\bar{w})}{(z-w)^2} + \frac{\partial V_j(w,\bar{w})}{z-w}\,, \qquad h = \frac{g_{ab}t_j^a t_j^b}{2(k+\mathfrak{c})}. \tag{A.33}$$

The rest of the states in the affine module are of the form $J_{n_1}^{a_1} J_{n_1}^{a_1}\dots|jm\rangle$ with $n_i<0$. In most examples one is interested in the highest-weight representations of the zero-mode algebra $\mathfrak{g}$. These contain a so-called highest-weight state, from which all the other states can be obtained by acting with $J^a_{n\leq 0}$.

We have shown that, importantly, affine primary fields are also Virasoro primaries. For this reason, in most cases it is enough to focus on the physics of affine primary fields. This holds for instance for models with (Euclidean) compact target spaces, such as the SU(2) WZW model for example, which have a finite number of primary fields and are an example of what is known as *rational* CFTs. However, the converse statement is not true. As discussed in the main text, this introduces a number of important additional features when discussing strings in AdS$_3$, described by the WZW model based on the universal cover of SL(2,$\mathbb{R}$), which is both Lorentzian and non-compact. As a consequence, an important part of its spectrum is characterized by operators which are Virasoro primaries, but *not* affine primaries.

We now discuss the correlation functions of WZW models. They must be invariant under the action of both gauge and conformal transformations. For correlators of primary fields, this is captured by the (global) Ward identities, giving

$$0 = \sum_{i=1}^{n} t_{j_i}^a \langle V_{j_1}(z_1,\bar{z}_1)\dots V_{j_n}(z_n,\bar{z}_n)\rangle, \tag{A.34}$$

and

$$0 = \sum_{i=1}^{n} \partial_{z_i} \langle V_{j_1}(z_1,\bar{z}_1)\dots V_{j_n}(z_n,\bar{z}_n)\rangle, \tag{A.35a}$$

$$0 = \sum_{i=1}^{n} (z_i\partial_{z_i} + h_i) \langle V_{j_1}(z_1,\bar{z}_1)\dots V_{j_n}(z_n,\bar{z}_n)\rangle, \tag{A.35b}$$

$$0 = \sum_{i=1}^{n} (z_i^2\partial_{z_i} + 2z_i h_i) \langle V_{j_1}(z_1,\bar{z}_1)\dots V_{j_n}(z_n,\bar{z}_n)\rangle\,, \tag{A.35c}$$

respectively. As is well known, Eqs. (A.35) completely fix the coordinate dependence of two- and three-point functions. Moreover, there are additional constraints coming from the Sugawara construction itself. The most notable one is known as the Knizhnik-Zamolodchikov equation. The definition (A.26) for the mode L_{-1} in terms of the currents implies that for any affine primary state $|j\rangle$,

$$\left(L_{-1} - \frac{g_{ab}}{k+\mathfrak{c}} J^a_{-1} J^b_0\right)|j\rangle = 0. \tag{A.36}$$

From this, we obtain a differential equation for correlation functions of WZW models, namely

$$\left[\partial_{z_i} + \frac{g_{ab}}{k+\mathfrak{c}} \sum_{j\neq i} \frac{t^a_i \otimes t^b_j}{z_i - z_j}\right] \langle V_{j_1}(z_1,\bar{z}_1)\dots V_{j_n}(z_n,\bar{z}_n)\rangle, \tag{A.37}$$

where we have used that $L_{-1} \sim \partial_z$, the OPE (A.29), and

$$\begin{aligned}\langle V_{j_1}(z_1,\bar{z}_1)\dots (J^a_{-1}V_{j_i})(z_i,\bar{z}_i)\dots V_{j_n}(z_n,\bar{z}_n)\rangle &= \oint_{z_i} \frac{dz}{z-z_i} \langle J^a(z) V_{j_1}(z_1,\bar{z}_1)\dots V_{j_n}(z_n,\bar{z}_n)\rangle \\ &= \sum_{j\neq i} \frac{t^a_j}{z_i - z_j} \langle J^a(z) V_{j_1}(z_1,\bar{z}_1)\dots V_{j_n}(z_n,\bar{z}_n)\rangle,\end{aligned}$$

where in the last step we have inverted the contour of integration and used that $J^a(z \to \infty) \sim z^{-2}$, so that there is no pole at infinity. The tensor product in Eq. (A.37) is used to keep track of the action

$$\left(t^a_i \otimes t^a_j\right) V_{j_i}(z_i,\bar{z}_i) V_{j_j}(z_j,\bar{z}_j) \equiv (t^a_i V_{j_i})(z_i,\bar{z}_i)(t^a_j V_{j_j})(z_j,\bar{z}_j). \tag{A.38}$$

The existence of this crucial constraint distinguishes WZW models form generic two-dimensional CFTs. Additional constraints arise when other null vectors are present.

For some WZW models, the KZ equation leads to exact solutions for four-point functions of primary fields. This is the case, for instance, for $\mathrm{SU}(N)_k$-WZW models, with $N, k \in \mathbb{N}$. For this, one uses the conformal Ward identities (A.35) to fix the dependence in the insertions z_i and $\bar{z}_i$, with $i = 1,\dots,4$, up to an arbitrary function of the cross-ratios, defined as

$$z \equiv \frac{z_{23}z_{14}}{z_{12}z_{34}}\,, \qquad z_{ij} = z_i - z_j, \tag{A.39}$$

and similarly for $\bar{z}$. Thus, one has

$$\langle\prod_{i=1^4} V_{j_i}(z_i,\bar{z}_i)\rangle = |z_{12}^{h_3-h_1-h_2-h_4} z_{13}^{h_2+h_4-h_1-h_3} z_{23}^{h_1+h_4-h_2-h_3} z_{34}^{-2h_4}|^2 F_{14}^{23}(z,\bar{z}). \tag{A.40}$$

The KZ equations can then be written as partial differential equations for the functions $F_{14}^{23}(z,\bar{z})$, known as (combinations of) the conformal blocks, which turn out to be of the hypergeometric type. The exact solution can then be obtained by using the representation theory of $\mathrm{SU}(N)$, and further imposing that the correlator must be invariant under *crossing symmetry*, i.e. the exchange symmetry between the different fields, which reduces to the identity

$$F_{14}^{23}(z,\bar{z}) = F_{24}^{13}(1-z,1-\bar{z}). \tag{A.41}$$

The resulting expressions as well as the details of the computation can be found in [82].

A.2 The SU(2) model

We now discuss in some detail one of the prototypical examples: the WZW model based on SU(2) [42], for which the target space is S^3. We can parametrize the group elements in terms of the Euler angles,

$$g = e^{\frac{i}{2}(\psi-\varphi)\sigma^3} e^{i\theta\sigma^1} e^{\frac{i}{2}(\psi+\varphi)\sigma^3} = \begin{pmatrix} e^{i\psi}\cos\theta & ie^{-i\varphi}\sin\theta \\ ie^{i\varphi}\sin\theta & e^{-i\psi}\cos\theta \end{pmatrix}, \tag{A.42}$$

where σ^a are the Pauli matrices, $a=1,2,3$. By inserting this into the WZW action (A.5) one can read off the target space line element and B-field, namely

$$ds^2 = k\left(d\theta^2 + \sin^2\theta d\varphi^2 + \cos^2\theta d\psi^2\right), \tag{A.43}$$

$$H = k\sin 2\theta d\theta\wedge d\psi\wedge d\varphi = d\left(k\sin^2\theta d\psi\wedge d\varphi\right). \tag{A.44}$$

The generators of the Lie algebra are $t^a = \sigma^a/2$. Generic solutions of the classical EOMs $\bar\partial(\partial g g^{-1}) = 0$ take the factorized form $g(z,\bar z) = g_L(z)g_R(\bar z)$. From the holomoprhic currents, which we will denote as $k^a(z)$, one can define the ladder operators $k^\pm(z) = k^1(z)\pm ik^2(z)$. These satisfy the OPE (A.19) with $g^{ab} = \delta^{ab}/2$ and $f^{ab}{}_c = \varepsilon^{ab}{}_c$, where $\varepsilon^{012}=1$. In terms of the corresponding modes this gives

$$[k_n^3,k_m^3] = \frac{k}{2}n\delta_{n+m,0}, \qquad [k_n^3,k_m^\pm] = \pm k_{n+m}^\pm, \qquad [k_n^+,k_m^-] = 2k_{n+m}^3 + kn\delta_{n+m,0}. \tag{A.45}$$

The energy-momentum tensor and central charge are

$$T(z) = \frac{1}{k+2}(k_a k^a)(z), \qquad c = \frac{3k}{k+2}. \tag{A.46}$$

The relevant highest-weight representations of the zero-mode algebra are given by states of the form

$$\left\{|l\,n\rangle,\, l\in\frac{1}{2}\mathbb{N}_0,\, n=-l,-l+1,\ldots,l-1,l\right\}, \tag{A.47}$$

where l is the SU(2) spin and n the corresponding projection along the k^3 direction. The associated vertex operators $W_{ln\bar{n}}(z,\bar{z})$ transform in such highest-weight representations of the holomorphic and antiholomoprhic zero-mode algebras, and with $\bar{l} = l$. We work with conventions where the currents act as

$$k^3(z)W_{ln\bar{n}}(w,\bar{w}) \sim \frac{nW_{ln\bar{n}}(w,\bar{w})}{z-w}, \quad k^{\pm}(z)W_{ln\bar{n}}(w,\bar{w}) \sim \frac{(l+1\pm n))W_{l,n\pm 1,\bar{n}}(w,\bar{w})}{z-w}, \tag{A.48}$$

such that the weights are

$$\Delta_l = \bar{\Delta}_l = \frac{l(l+1)}{k+2}. \tag{A.49}$$

By restricting to diagonal states with $l = \bar{l}$ we have chosen to work with the scalar vertex operators of the WZW model.

The SU(2) Lie algebra can be described in terms of the following first order differential operators:

$$-t_0^+ \sim P_l^+ = \partial_u, \qquad -t_0^3 \sim P_l^3 = u\partial_u - l, \qquad -t_0^+ \sim P_l^- = -u^2\partial_u + 2lu. \tag{A.50}$$

The auxiliary variable u is known as the isospin coordinate. Relatedly, we have a one-parameter family of automorphisms of the affine algebra, which can be understood as translating the currents from $u = 0$ to a different point in u-space, namely

$$k^a(z) \to k^a(u,z) = e^{uk_0^+}k^a(z)e^{-uk_0^+}, \tag{A.51}$$

giving $k^+(u,z) = k^+(z)$ and

$$k^3(u,z) = k^3(z) - uk^+(z), \qquad k^-(u,z) = k^-(z) + 2uk^3(z) - u^2k^+(z). \tag{A.52}$$

By means of the holomorphic and antiholomorphic isospin variables we can encode all states in a given zero-mode representation by defining

$$W_l(u,\bar{u},z,\bar{z}) = e^{\bar{u}\bar{k}_0^+}e^{uk_0^+}W_{l,-l,-\bar{l}}(z,\bar{z})e^{-uk_0^+}e^{-\bar{u}\bar{k}_0^+} = \sum_{n,\bar{n}=-l}^{l} u^{l-n}\bar{u}^{l-\bar{n}}W_{ln\bar{n}}(z,\bar{z}) \tag{A.53}$$

where the operator at the origin is identified as $W_l(0,0,z,\bar{z}) = W_{l,-l,-\bar{l}}(z,\bar{z})$. Then, we have for these u-basis vertex operators

$$k^a(z)W_l(u,\bar{u},w,\bar{w}) \sim \frac{P_l^a W_l(u,\bar{u},w,\bar{w})}{z-w}. \tag{A.54}$$

This allows us to express the Ward identities (A.18) stemming from the action of the zero-mode algebra as partial differential equations in u, leading to

$$0 = \sum_{i=1}^{n} \partial_{u_i}\langle W_{l_1}(u_1,\bar{u}_1,z_1,\bar{z}_1)\dots W_{l_n}(u_n,\bar{u}_n,z_n,\bar{z}_n)\rangle, \tag{A.55a}$$

$$0 = \sum_{i=1}^{n}(u_i\partial_{u_i} - l_i)\langle W_{l_1}(u_1,\bar{u}_1,z_1,\bar{z}_1)\dots W_{l_n}(u_n,\bar{u}_n,z_n,\bar{z}_n)\rangle, \tag{A.55b}$$

$$0 = \sum_{i=1}^{n}(u_i^2\partial_{u_i} - 2u_il_i)\langle W_{l_1}(u_1,\bar{u}_1,z_1,\bar{z}_1)\dots W_{l_n}(u_n,\bar{u}_n,z_n,\bar{z}_n)\rangle. \tag{A.55c}$$

These equations are analogous (up to some signs) to the conformal Ward identities of Eq. (A.35), with the spin l playing the role of the holomorphic weight Δ. Hence, we find that the z and u dependence of two- and three-point functions are both fixed by the symmetries of the system. A suitable choice for the normalizations leads to

$$\langle W_{l_1}(u_1,\bar{u}_1,z_1,\bar{z}_1)W_{l_2}(u_2,\bar{u}_2,z_2,\bar{z}_2)\rangle = \delta_{l_1,l_2}\frac{|u_{12}|^{4l_1}}{|z_{12}|^{4\Delta_1}} \tag{A.56}$$

and

$$\langle\prod_{i=1}^{3} W_{l_i}(u_i,\bar{u}_i,z_i,\bar{z}_i)\rangle = C(l_1,l_2,l_3)\left|\frac{u_{12}^{l_1+l_2-l_3}u_{23}^{l_2+l_3-l_1}u_{13}^{l_1+l_3-l_2}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3}z_{23}^{\Delta_2+\Delta_3-\Delta_1}z_{13}^{\Delta_1+\Delta_3-\Delta_2}}\right|^2, \tag{A.57}$$

where $C(l_1,l_2,l_3)$ are the structure constants of the model. There explicit expressions were derived in [42]. Once they are fixed, correlators of the original operators $W_{ln\bar{n}}$ can be deduced from the u-basis ones by expanding in powers of u_i and isolating the corresponding coefficients. For three-point functions, this imposes the charge conservation condition $n_1+n_2+n_3=0$ and leads to the appearance of the well-known Clebsch-Gordan coefficients [83].

A.3 Null states in the SU(2) model and its partition function

We now discuss an important aspect of the SU(2) model at level k: the presence of null states in the affine modules. From the current algebra it follows that

$$(k_{-1}^{-})^{k-2l+1}|l,-l\rangle = 0\,, \qquad l\leq\frac{k}{2}. \tag{A.58}$$

This can be seen for instance by showing that all $J^a_{n>0}$ annihilate the state on the LHS of (A.58). Since this state is also a descendant, it must be null. Alternatively, one derive it by showing (say by induction) that [84]

$$||(k_{-1}^{-})^{N}|l,-l\rangle||^2 = \langle l,l|(k_1^{+})^N(k_{-1}^{-})^N|l,-l\rangle = \prod_{n=1}^{N} n(k-2l+1-n)\,, \tag{A.59}$$

such that the norm vanishes for $N=k-2l+1$. We will provide an alternative proof in the next section using spectral flow. For now, we describe the consequences of Eq. (A.58). As in the derivation of the KZ equation, we insert the null state

$$k_{-1}^{-}(u)^{k-2l+1}W_l(u,\bar{u},z,\bar{z}) = 0 \tag{A.60}$$

in a given correlation function, which leads to the constraint

$$0 = \left[\sum_{i=1}^{n}\frac{(u-u_i)^2\partial_{u_i}+2l_i(u-u_i)}{z-z_i}\right]^{k-2l+1}\langle W_l(u,\bar{u},z,\bar{z})\prod_{i=1}^{n}W_{l_i}(u_i,\bar{u}_i,z_i,\bar{z}_i)\rangle \tag{A.61}$$

by means of the Ward identities (A.55). Setting $n=2$ and using the expression (A.57) then gives

$$(l_1+l_2-l)(l_1+l_2-l-1)\dots(l_1+l_2+l-k)C(l,l_1,l_2) = 0. \tag{A.62}$$

From this, we see that for any $l_1, l_2 \leq k/2$, $C(l_1, l_2, l_3)$ must vanish if $l_3 > k/2$. Hence, the theory defined by including only the fields with $l \leq k/2$ is self-consistent. Moreover, we can deduce the fusion rules of the model

$$C(l_1, l_2, l_3) \neq 0 \quad \Leftrightarrow \quad l_1 + l_2 + l_3 \leq k \quad \text{and} \quad l_i \leq l_j + l_k \quad \forall\, i, j, k = 1, 2, 3. \tag{A.63}$$

In other words, the product $W_{l_1} \otimes W_{l_2}$ only contains states W_l with $|l_1 - l_2| \leq l \leq \min(l_1 + l_2, k - l_1 - l_2)$.

In order to check that this is correct, one may consider the partition function of the SU(2) model and establish modular invariance. The distribution of weights and charges in the theory is captured by the partition function of the theory defined on a torus with modular parameter τ, namely

$$Z(\tau, \bar{\tau}) = \mathrm{Tr}\left[q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}\right], \qquad q = e^{2\pi i \tau}, \tag{A.64}$$

where the trace runs over the full Hilbert space. Conformal symmetry requires $Z(\tau, \bar{\tau})$ to be modular invariant, i.e. invariant under the action of the S and T transformations,

$$T: \tau \to \tau + 1, \qquad S: \tau \to -\frac{1}{\tau}. \tag{A.65}$$

The partition function decomposes into a sum of contributions from each highest-weight representation of the left- and right-handed current algebras,

$$Z(\tau, \bar{\tau}) = \sum_{l, \bar{l}} M_{l\bar{l}} \chi_l(\tau) \chi_{\bar{l}}(\bar{\tau}), \tag{A.66}$$

where encodes the corresponding multiplicities. Although here we have focused on the diagonal case $M_{l\bar{l}} = \delta_{l,\bar{l}}$, the reader should keep in mind that there are other modular invariant constructions, which fall into the so-called ADE classification [82].

For the holomorphic sector of the SU(2) model, the corresponding characters read

$$\chi_l(\tau, z) = \mathrm{Tr}_l\left[q^{L_0 - \frac{c}{24}} y^{k_0^3}\right] = \frac{\Theta_{2l+1}^{(k+2)}(\tau, z) - \Theta_{-2l-1}^{(k+2)}(\tau, z)}{\Theta_1^{(2)}(\tau, z) - \Theta_{-1}^{(2)}(\tau, z)}, \qquad y = e^{2\pi i z}, \tag{A.67}$$

where we have restricted to the spin-l sector, introduced a chemical potential in order to keep track of the U(1) charge measured by k_0^3, and used the level-k theta functions

$$\Theta_l^k(\tau, z) = \sum_{n \in \mathbb{Z} + \frac{l}{2k}} q^{kn^2} y^{kn}. \tag{A.68}$$

It is instructive to understand how Eq. (A.67) comes about. The first ingredient that should appear is the global character,

$$\chi_l(z) = \sum_{n=-l}^{l} y^n = \frac{y^{l+\frac{1}{2}} - y^{-l-\frac{1}{2}}}{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}. \tag{A.69}$$

We should also account for the affine descendants. Those coming from the modes k^3_n for a given negative n give

$$\sum_{j=0}^{\infty} q^{jn} = (1-q^n)^{-1}, \tag{A.70}$$

while the action of $k^{\pm}_{n<0}$ induces an extra unit shift in the charges, giving analogous contributions with $q^n \to yq^n$ and $q^n \to y^{-1}q^n$. Combining these factors and taking into account the central charge (A.46) as well as the primary weights (A.49) leads to

$$\frac{q^{\frac{l(l+1)}{k+2}-\frac{k}{8(k+2)}}\chi_l(z)}{\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)}. \tag{A.71}$$

The Jacobi's triple product identity shows that the denominator coincides with that of Eq. (A.67). However, (A.71) only captures the leading term in an expansion in powers of q of the numerator in (A.67). The correct expression takes the form of an alternating series,

$$\chi_l(\tau,z) = \frac{q^{\frac{l(l+1)}{k+2}-\frac{k}{8(k+2)}}\left[\chi_l(z) - q^{k+1-2l}\chi_{k+1-l}(z) + q^{k+3+2l}\chi_{k+2+l}(z) - \cdots\right]}{\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)}. \tag{A.72}$$

Indeed, the expression given in Eq. (A.71) does not take into account the presence of null states in the affine module, which must be subtracted from the counting. Indeed, the subtraction of the null state (A.58) together with its descendants gives precisely the shift $\chi_l(z) \to \chi_l(z) - q^{k+1-2l}\chi_{k+1-l}(z)$. The following term has a similar origin: there is an additional null state among the descendants of (A.58), namely

$$(k^-_{-2})^{2l+1}(k^-_{-1})^{k-2l+1}|l,-l\rangle. \tag{A.73}$$

This means that we have to add back this state and its descendants in order to avoid overcounting. The corresponding weight and spin with respect to the zero-mode algebra imply that this is done by including the contribution $q^{k+3+2l}\chi_{k+2+l}(z)$ in (A.72). This pattern repeats *ad infinitum*, generating the full series expansion of the Theta functions appearing in the numerator of Eq. (A.67).

Finally, one can use the modular properties of the Theta functions to obtain those of the characters, and show that the full partition function is modular invariant. We will not reproduce the explicit calculation, which can be found for example in [84].

A.4 Spectral flow in the compact case

Another property of the affine SU(2) at level k algebra is the existence of the spectral flow automorphisms

$$k^{\pm}_n \to \tilde{k}^{\pm}_n = k^{\pm}_{n\pm\omega}, \quad k^3_n \to \tilde{k}^3_n = k^3_n + \frac{k\omega}{2}\delta_{n,0}\,, \qquad \omega \in \mathbb{Z}. \tag{A.74}$$

At the level of the Sugawara construction, this induces

$$L_n \to \tilde{L}_n = L_n + \omega k^3_n + \frac{k}{4}\omega^2\delta_{n,0}. \tag{A.75}$$

All relevant SU(2) states are contained in the affine highest-weight modules. The spectral flow automorphisms (A.74) merely re-shuffles primary states and their descendants with different values of j. Conversely, all states in highest-weight representations of the zero-mode algebra in the spectrally flowed frame, i.e. the one generated by $\tilde{k}_0^{\pm} = k^{\pm}_{\pm w}$ and $\tilde{k}_0^3 = k_0^3 + \frac{k\omega}{2}$, must correspond to combinations of primaries and descendants in terms of the original states. Indeed, consider a state with spin l and projection m in a flowed frame associated with a given $\omega > 0$, which we denote $|l\, n\, \omega\rangle$. This satisfies

$$k_0^3 |l\, n\, \omega\rangle = \left(n - \frac{k\omega}{2}\right) |l\, n\, \omega\rangle, \quad L_0 |l\, n\, \omega\rangle = \left(\Delta_l - wn + \frac{k\omega^2}{4}\right) |l\, n\, \omega\rangle, \tag{A.76}$$

as well as

$$k^+_{n>\omega}|l\, n\, \omega\rangle = k^-_{n>-\omega}|l\, n\, \omega\rangle = k^3_{n>0}|l\, n\, \omega\rangle = L_{n>0}|l\, n\, \omega\rangle = 0. \tag{A.77}$$

Thus, although $|l\, n\, \omega\rangle$ is clearly not an affine primary, it *is* a Virasoro primary of weight $\Delta_\omega = \Delta_l - wn + \frac{k\omega^2}{2}$. With respect to the zero-mode algebra, we see that k_0^- annihilates the state, so that we can read off the spin from (minus) the eigenvalue of k_0^3. This shows that the associated vertex operators, which we denote as $W^\omega_{ln}(z,\bar{z})$ and refer to as *spectrally flowed primaries*, transform in a spin-l_ω representation with $l_\omega = -n + \frac{\omega}{2}$. However, these operators are not always different form each other. By matching the quantum numbers one obtains the following identifications

$$W^\omega_{l,-l}(z,\bar{z}) = W^{\omega+1}_{\tilde{l},\tilde{l}}(z,\bar{z})\,, \qquad \tilde{l} = \frac{k}{2} - l, \tag{A.78}$$

where we have chosen the normalizations appropriately.

Let us pause and consider in more detail the identity (A.78) for the case $\omega = 0$, which shows that we can identify the unflowed, lowest-weight state $|l, -l\rangle$ with the highest-weight state of spin $\tilde{l}$ in the spectrally flowed frame with $\omega = 1$, namely

$$|l, -l\rangle = |\tilde{l}\, \tilde{l}\, \omega = 1\rangle\,. \tag{A.79}$$

The latter is annihilated by applying the corresponding lowering operator (at least) $2\tilde{l}+1 = k - 2l + 1$ times. Since in this frame $\tilde{k}_0^- = k^-_{-1}$, this provides an alternative proof for Eq. (A.58).

As stated above, spectrally flowed primaries are nothing but specific descendants of the original ones. We can use (A.78) to describe exactly which descendants we are dealing with:

$$\begin{aligned}
|l\, n\, \omega\rangle &\sim (k_0^{-,\omega})^{l-n}|l\, l\, \omega\rangle \\
&= (k^-_{-\omega})^{l-n}|\tilde{l}, -\tilde{l}, \omega - 1\rangle \\
&\sim (k^-_{-\omega})^{l-n}(k_0^{-,\omega-1})^{2\tilde{l}}|\tilde{l}, \tilde{l}, \omega - 1\rangle \\
&= (k^-_{-\omega})^{l-n}(k^-_{-\omega+1})^{2\tilde{l}}|l, -l, \omega - 2\rangle \;=\; \ldots\,,
\end{aligned}$$

where $k_n^{a,\omega}$ are the currents $\tilde{k}_n^a$ in a given frame with spectral flow ω. Hence, recalling that $\tilde{l} = \frac{k}{2} - l$, we have

$$|l\, n\, \omega\rangle \sim \begin{cases} (k^-_{-\omega})^{l-n}(k^-_{-\omega+1})^{k-2l}(k^-_{-\omega+2})^{2l}\ldots(k^-_{-1})^{k-2l}|l, -l\rangle & \omega \in 2\mathbb{Z}, \\ (k^-_{-\omega})^{l-n}(k^-_{-\omega+1})^{k-2l}(k^-_{-\omega+2})^{2l}\ldots(k^-_{-1})^{2l}|\frac{k}{2} - l, l - \frac{k}{2}\rangle & \omega \in 2\mathbb{Z}+1, \end{cases} \tag{A.80}$$

where we have ignored all irrelevant proportionality factors. We confirm that there are no new states in the spectrally flowed sectors of the theory. This is in stark contrast with the SL(2,$\mathbb{R}$) case studied in the main text.

Moreover, we immediately see that there are many other null states in highest-weight affine modules, which generalize Eq. (A.58):

$$0 = \begin{cases} (k^-_{-\omega+1})^{k-2l+1}(k^-_{-\omega+2})^{2l}\dots(k^-_{-1})^{k-2l}|l,-l\rangle & \omega \in 2\mathbb{Z}, \\ (k^-_{-\omega+1})^{k-2l+1}(k^-_{-\omega+2})^{2l}\dots(k^-_{-1})^{2l}|\frac{k}{2}-l,l-\frac{k}{2}\rangle & \omega \in 2\mathbb{Z}+1. \end{cases} \tag{A.81}$$

These are precisely the states we removed when constructing the partition function above.

We can also see this directly at the level of the characters. Given the spectral flow transformations, the character of a flowed highest-weight representation reads

$$\chi_l^\omega(q,y) = \mathrm{Tr}_l\left[q^{\tilde{L}_0-\frac{c}{24}}y^{\tilde{k}_0^3}\right] = \mathrm{Tr}_l\left[q^{L_0+\omega k_0^3+\frac{k}{4}\omega^2-\frac{c}{24}}y^{k_0^3+\frac{k}{2}\omega}\right] = q^{\frac{k}{4}\omega^2}y^{\frac{k}{2}\omega}\chi_l(q,yq^\omega). \tag{A.82}$$

Using (A.68) we find that for the denominator of the character (A.67) one has

$$\Theta_1^{(2)}(q,yq^\omega) - \Theta_{-1}^{(2)}(q,yq^\omega) = (-1)^\omega y^{-\omega}q^{-\frac{\omega^2}{2}}\left[\Theta_1^{(2)}(q,y) - \Theta_{-1}^{(2)}(q,y)\right]. \tag{A.83}$$

For the numerator the result depends on the parity of ω. For even ω, and up an overall factor, the effect of the replacement $y \to yq^\omega$ can be reabsorbed as a shift in the summation index of (A.68), $n \to n - \frac{\omega}{2}$. On the other hand, for odd ω we would need a half-integer shift. This can be re-interpreted as a change in the spin, since

$$l \to \frac{k}{2} - l \qquad \Rightarrow \qquad \frac{2l+1}{2(k+2)} \to \frac{1}{2} - \frac{2l+1}{2(k+2)}. \tag{A.84}$$

The additional overall factors then combine to cancel those in the final expression of Eq. (A.82), such that

$$\chi_l^\omega(q,y) = \begin{cases} \chi_l(q,y) & \omega \in 2\mathbb{Z}, \\ \chi_{\frac{k}{2}-l}(q,y) & \omega \in 2\mathbb{Z}+1. \end{cases} \tag{A.85}$$

As a result, we see that spectral flow leaves the full partition function invariant.

A.5 Gauged WZW models

B Geometry of AdS$_3$ and BTZ black holes

- Global AdS$_3$ vs Poincare patch (HCFT NSNS vs RR vacuum)
- Lorentzian vs Euclidean
- BTZ black holes as quotients

C Liouville Theory and its relation to the H_3^+ model

D Symmetric product orbifolds

In this appendix we provide some ingredients of symmetric product orbifolds CFTs. Given a conformal sigma-model with target space M, which we will refer to as the *seed* theory, one can define a family of CFTs as follows:

$$\mathrm{Sym}^N(\mathcal{M}) \equiv \mathcal{M}^N/S_N \,. \tag{D.1}$$

Here N is a natural number, $\mathcal{M}^N$ stands for the direct product of N copies of the original CFT on $\mathcal{M}$, say $\mathcal{M}^{(i)}$ with $i = 1, \ldots, N$, and S_N denotes the permutation group, which acts as

$$s : X^{(i)} \to X^{s(i)} \,, \qquad s \in S_N \,, \tag{D.2}$$

where $X^{(i)}$ is a generic field in the theory on the i-th copy.

In Eq. (D.1) the action of the permutation group is gauged. Physical operators of the symmetric orbifold CFT must be gauge-invariant. The simplest example corresponds to operators in the *untwisted* sector of the theory, such as

$$X \equiv \sum_{i=1}^{N} X^{(i)} \,, \tag{D.3}$$

where we sum over all copies of a given operator in the seed. This sector contains for instance the energy-momentum tensor of the theory, namely[35]

$$\mathcal{T}(x) = \sum_{i=1}^{N} \mathcal{T}^{(i)}(x) \,, \tag{D.4}$$

from which one derives the central charge

$$c = N c_{\text{seed}} \,. \tag{D.5}$$

If the seed theory contains other conserved bosonic or fermionic currents, they are also extended to the full symmetric orbifold CFT in a similar way.

One can also construct physical operators belonging to the twisted sectors of the theory. This is achieved by making use of the so-called (bare) twist operators σ_s. Their insertion modifies the boundary conditions for the different fields, giving

$$X^{(i)}(e^{2\pi i}x)\sigma_s(0) = X^{s(i)}(x)\sigma_s(0) \,. \tag{D.6}$$

For instance, for a twist field involving the first ω copies by means of the permutation $(12\ldots\omega)$ we have

$$\sigma_{(12\ldots\omega)} : X^{(1)} \to X^{(2)} \to \cdots \to X^{(\omega)} \to X^{(1)} \,. \tag{D.7}$$

[35]In this appendix we use x to denote the complex coordinate for the symmetric orbifold CFT in order to be consistent with the notation employed in the main text. Indeed, the symmetric orbifold CFT provides an approximate description of the holographic CFT living on the boundary of AdS_3.

The insertion of this twist operator at the origin thus imposes the modified twisted boundary conditions

$$X^{(i)}(e^{2\pi i}x) = X^{(i+1)}(x)\,, \tag{D.8}$$

where the indices are understood mod ω. This type of operators are called single-cycle twist fields. They will be our main focus since in the holographic applications they are identified with single-string states. (Of course, one can also have multi-cycle fields, but these correspond to multi-string states in the bulk picture. Their properties can be studied by fusing single-cycle operators.)

We will use the compact notation $\sigma_\omega \equiv \sigma_{(12\ldots\omega)}$. Now, consider for instance the operator σ_2 inserted at the origin $x = 0$. This implements $X^{(1)}(e^{2\pi i}x) = X^{(2)}$ and $X^{(2)}(e^{2\pi i}x) = X^{(1)}$. Hence, although we have two functions $X^{(1)}$ and $X^{(2)}$, there is no global distinction between them. We can effectively trivialize the effect of the twist field insertions by going to the covering space, which, at least locally, corresponds to the Riemann surface of the function $z = x^{1/2}$. We can also think of this in terms of the inverse transformation, given by the holomorphic map $x = \Gamma(z) = z^2$. In the z-plane, we then have a single function $X(z)$, which we identify with, say, $X^{(1)}$ in the upper-half plane and with $X^{(2)}$ in the lower-half plane. This implies that X is single-valued. We now include another insertion of σ_2 at infinity, and evaluate the expectation value of the energy momentum tensor $\langle\sigma_2|\mathcal{T}(z)|\sigma_2\rangle$ [145]. The latter is dictated by the conformal anomaly generated by the transformation to the covering space. This allows one to derive the (holomorphic) weight of the twist operator, $h[\sigma_2] = \frac{c_{\text{seed}}}{16}$. More generally, for a (single-cycle) twist operator σ_ω involving ω copies of the theory one uses $x = \Gamma(z) = z^\omega$, which leads to [55]

$$h[\sigma_\omega] = \frac{c_{\text{seed}}(\omega^2-1)}{24\omega}\,. \tag{D.9}$$

This defines the weight of the ground state in the ω-twisted sector. The spectrum of the theory is obtained by dressing these bare twist operators with excitations[36]. The latter can be constructed directly from those of the seed theory. In the ω-twisted sector we can do this by using any operator which in the seed theory with weights h_{seed} and $\bar{h}_{\text{seed}}$ satisfying $h_{\text{seed}} - \bar{h}_{\text{seed}} \in \omega\mathbb{Z}$. For scalar operators with $h_{\text{seed}} = \bar{h}_{\text{seed}}$, this leads to a spectrum where the allowed holomorphic weights take the form [76]

$$h = \frac{c_{\text{seed}}(\omega^2-1)}{24\omega} + \frac{h_{\text{seed}}}{\omega}\,. \tag{D.10}$$

The factor of ω dividing h_{seed} in the last term makes sense because (for the two-point function) the relation between the covering space coordinate and the physical one is of the form $z \sim x^{1/\omega}$.

In the twisted sector where the insertion of σ_ω at the origin defines the ground state, one can trade $X^{(1)}, \ldots, X^{(\omega)}$ for a basis of operators which diagonalize the twisted boundary

[36] To be precise, in order to complete the construction we should take gauge-invariant linear combinations of these operators. This will be discussed shortly.

conditions. These are defined (up to a phase) as

$$X^r(x) \equiv \sum_{j=0}^{\omega-1} e^{-\frac{2\pi i j r}{\omega}} X^{(j)}(x)\,, \quad r = 0,\dots,\omega-1 \qquad \Rightarrow \qquad X^r(e^{2\pi i}x) = e^{\frac{2\pi i r}{\omega}} X^r(x)\,. \tag{D.11}$$

For the case of $X = \mathcal{T}$, this shows that in the ω-twisted sector one can define the fractional Virasoro modes

$$\mathcal{L}_{\frac{n}{\omega}} = \oint dx\, x^{\frac{n}{\omega}+1}\, \mathcal{T}^r(x) \qquad n \in \omega\mathbb{Z} - r\,. \tag{D.12}$$

These fractional modes satisfy the Virasoro algebra

$$\left[\mathcal{L}_{\frac{n}{\omega}}, \mathcal{L}_{\frac{m}{\omega}}\right] = \left(\frac{n}{\omega} - \frac{m}{\omega}\right) \mathcal{L}_{\frac{n+m}{\omega}} + \frac{c_{\text{seed}}\omega}{12} \frac{n}{\omega} \left(\frac{n^2}{\omega^2} - 1\right) \delta_{n+m,0}\,, \tag{D.13}$$

hence the central charge in this sector is $c^{(\omega)} = c_{\text{seed}}\omega$. The fractional generators can be written in terms of *untwisted* ones $\hat{\mathcal{L}}_n$ with central charge c_{seed} by means of

$$\mathcal{L}_{\frac{n}{\omega}} = \frac{1}{\omega}\hat{\mathcal{L}}_n + \frac{c_{\text{seed}}(\omega^2-1)}{24\omega}\delta_{n,0}\,. \tag{D.14}$$

D.1 Correlators and large N counting

We now move to correlators. We first focus on bare twist fields for simplicity. Consider a sample n-point function of the form

$$\langle \sigma_{\omega_1}(x_1) \cdots \sigma_{\omega_n}(x_n) \rangle\,. \tag{D.15}$$

This can only be non-zero if the product of the corresponding permutations (in a given order) amounts to the identity operator. For instance, for $n = 3$ a non-trivial example with $\omega_1 = 2$, $\omega_2 = 2$ and $\omega_3 = 3$ is given by

$$\langle \sigma_{(12)}(x_1)\sigma_{(13)}(x_2)\sigma_{(123)}(x_3) \rangle\,. \tag{D.16}$$

Let us focus on connected correlators, namely those that do not factorize trivially because some of the insertions involve a subset of copies of the theory which are untouched by the others. Connected correlators can be computed via the covering space method [55]. For this one needs to construct a holomorphic map $\Gamma : \Sigma \to S^2$ satisfying

$$\Gamma(z \sim z_i) = x_i + a_i(z - z_i)^{\omega_i} + \cdots \tag{D.17}$$

for all $i = 1,\dots,n$. Here x is the complex coordinate on the physical sphere, while z is the coordinate on the covering surface Σ. Hence, Σ has ramifications of order $r_i = \omega_i - 1$ at each of the insertion points. Let us further denote the total number of *active* copies of the theory by m. For instance, $m = 3$ in the example above because all insertions involve only the copies 1, 2 and 3. The value of m sets the total number of sheets of the covering surface, i.e. the number of pre-images $\Gamma^{-1}(x)$ at a generic point x (which was denoted as

N in the main text). This is related to the genus g of Σ by the Riemann-Hurwitz formula [146]

$$g = 1 - m + \frac{1}{2}\sum_{i=1}^{n}(\omega_i - 1)\,. \tag{D.18}$$

In the example above we have $g = 0$, while for instance for $\langle \sigma_{(123)}\sigma_{(123)}\sigma_{(123)}\rangle$ we get $g = 1$.

The bare twist operators defined above are clearly not gauge-invariant. One can construct gauge-invariant twist operators $\sigma_{[\omega]}$ by summing over the corresponding S_N orbit, i.e.

$$\sigma_{[\omega]} = N_\omega \sum_{s\in S_N} \sigma_{s(12\ldots\omega)s^{-1}}\,, \qquad N_\omega = \frac{1}{\sqrt{\omega(N-\omega)!N!}}\,. \tag{D.19}$$

Here N_ω is a normalization factor ensuring that $\sigma_{[\omega]}$ has unit two-point functions (assuming this was true for the original twist fields σ_ω). Hence, one can expand a given gauge-invariant n-point function in terms correlators of bare twist operators, namely

$$\left\langle \prod_{i=1}^{n} \sigma_{[\omega_i]} \right\rangle = \left(\prod_{i=1}^{n} N_{\omega_i}\right) \sum_{g=0}^{\infty} \sum_{s_1,\ldots,s_n\in S_N} \left\langle \prod_{j=1}^{n} \sigma_{s_j(12\ldots\omega_j)s_j^{-1}} \right\rangle_g\,, \tag{D.20}$$

where we have included a sum over the genus of the covering maps employed in each case. Note that there might, and generically will be several contributions coming from different covering maps with the value of g. However, not all of the contributions on the RHS of (D.20) are independent. Indeed, for each σ_{ω_j} there are exactly $(N-\omega_j)!w_j$ choices of s_j that leave $(12\ldots\omega_j)$ invariant. Finally, by means of the Riemann-Hurwitz formula (D.18), for each value of g we have a fixed total number of active copies, denoted by $m = m(g,\omega_i)$. We are then left with a sum over contributions generating the orbit of the S_m subgroup associated with permutations of these specific copies of the theory, which is thus independent of N.

We are interested in taking the large N limit of the model. The arguments above show that for correlators of gauge-invariant twist operators, the genus g contribution scales with N as follows:

$$\left\langle \prod_{i=1}^{n} \sigma_{[\omega_i]} \right\rangle_g \sim \binom{N}{m} \prod_{i=1}^{n} \sqrt{\frac{(N-\omega_i)!\omega_i}{N!}} \sim N^{1-g-\frac{n}{2}}\,. \tag{D.21}$$

where $2m = 2 - 2g + \sum_{i=1}^{n}(\omega_i - 1)$, while in the last step we used Stirling's approximation. In the final expression we see that the exponent of $1/N$ is precisely (half of) the Euler characteristic $\chi(g,n)$ of a genus-g Riemann surface with exactly n punctures. This is precisely the string-theoretical topolgical expansion we would expect from the holographic duality, provided we identify the dual string coupling as

$$g_s^2 \sim N^{-1}\,. \tag{D.22}$$

Moreover, in the strict large N limit, corresponding to perturbative correlators where the worldsheet topology is that of a two-dimensional sphere, this shows that the relevant covering surfaces of the corresponding symmetric orbifold correlators must have $g = 0$. These are precisely the type of correlators considered in the main text.

Once the combinatorial factors are taken into account, all that remains is to actually compute the bare twist correlators. This is done by lifting to the appropriate covering space, where the twisted boundary conditions induced by these operators are trivialized. The computation can be carried out quite explicitly by following the methods of [55, 59, 145]. Here we simply quote the result, and refer the interested reader to the original publications for the details. At genus zero one obtains

$$\left\langle \prod_{i=1}^{n} \sigma_{\omega_i}(x_i) \right\rangle = \left| \prod_{i=1}^{n} \omega_i^{-\frac{c_{\text{seed}}(\omega_i+1)}{24}} a_i^{\frac{c_{\text{seed}}(\omega_i-1)}{24}} R^{-\frac{c_{\text{seed}}}{12}} \right|^2 . \tag{D.23}$$

Here the different a_i stand for the coefficients involved in the expansion of the relevant covering map around each of the z_i, see Eq. (D.17). These coefficients contain the dependence on the insertion points x_i. The $g = 0$ covering maps $\Gamma(z)$ associated with three-point functions were derived explicitly in Sec. 3 above. Finally, R represents the product of the residues of the covering map at each of its simple poles. More explicitly, for covering surfaces with $g = 0$ the number of simple poles is precisely that of active copies. There are thus m points z_a^* (with $a = 1, \ldots, s$) for which $\Gamma(z \sim z_a^*) = \frac{r_a}{z-z_a^*} + \cdots$, hence $R = \prod_{a=1}^{s} r_a$. For three-point functions this was computed in [55].

More generally, one can also compute correlators involving excitations built form the operators of the seed theory. Denoting by σ_{ω_i,h_i} the operator consisting of the bare twist field dressed with the excitation associated with a seed theory operator $\mathcal{O}_i$ with weight $h_{\text{seed}} = h_i$, similar methods to those discussed above lead to the following result [59]:

$$\left\langle \prod_{i=1}^{n} \sigma_{\omega_i,h_i}(x_i) \right\rangle = \left| \prod_{i=1}^{n} \omega_i^{-\frac{c_{\text{seed}}(\omega_i+1)}{24}} a_i^{\frac{c_{\text{seed}}(\omega_i-1)}{24} - h_i} R^{-\frac{c_{\text{seed}}}{12}} \right|^2 \left\langle \prod_{i=1}^{n} \mathcal{O}_i(z_i) \right\rangle . \tag{D.24}$$

D.2 The chiral ring for the T^4 case

We now discuss the protected sector of the supersymmetric theory for the case where the seed CFT is the $\mathcal{N} = (4,4)$ sigma-model on T^4. The Virasoro, supercurrent and R-symmetry generators, denoted as $\mathcal{L}_n$, $\mathcal{G}_n^{\alpha A}$ and J^a respectively, satisfy the following algebra:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \,, \tag{D.25}$$

$$\left[\mathcal{K}_m^a, \mathcal{K}_n^b\right] = i\varepsilon^{ab}{}_c \mathcal{K}_{m+n}^c + \frac{c}{12} m \delta^{ab} \delta_{m+n,0} \,, \tag{D.26}$$

$$\begin{aligned}\left\{\mathcal{G}_m^{\alpha A}, \mathcal{G}_n^{\beta B}\right\} = {}& (m-n)\varepsilon^{AB}\varepsilon^{\beta\gamma} (\sigma^{*a})^{\alpha}_{\gamma} \mathcal{K}_{m+n}^a - \varepsilon^{AB}\varepsilon^{\alpha\beta}\mathcal{L}_{m+n} \\ & - \frac{c}{6}(m^2 - \frac{1}{4})\varepsilon^{AB}\varepsilon^{\alpha\beta}\delta_{m+n,0} \,,\end{aligned} \tag{D.27}$$

$$[\mathcal{L}_m, \mathcal{K}_n^a] = -n\mathcal{K}_{m+n}^a \,, \tag{D.28}$$

$$[\mathcal{L}_m, \mathcal{G}_n^{\alpha A}] = \left(\frac{m}{2} - n\right) \mathcal{G}_{m+n}^{\alpha A} \,, \tag{D.29}$$

$$[\mathcal{K}_m^a, \mathcal{G}_n^{\alpha A}] = \frac{1}{2} (\sigma^{*a})^{\alpha}_{\beta} \mathcal{G}_{m+n}^{\beta A} \,. \tag{D.30}$$

Let us clarify the notation [147]. Here a, b, c are the $\mathrm{SU}(2)_L$ R-symmetry Lie algebra indices, m, n are mode indices (which are integer for bosonic currents and half-integer for the fermionic ones), and α,β,γ are spinor indices for doublets of $\mathrm{SU}(2)_L$. On the other hand, A, B are indices for doublets under the action of the $\mathrm{SU}(2)_1$ inside the outer $\mathrm{SO}(4) \sim \mathrm{SU}(2)_1{\times}\mathrm{SU}(2)_2$ that generates the T^4 rotations. Similar formulas hold in the antiholomorphic sector, where we use dotted indices $\dot\alpha,\dot\beta$ and $\dot A,\dot B$ instead. The basic fields of the T^4 model are given by the compact scalars X^i and their fermionic partners $\psi^{\alpha\dot A}$ and $\bar\psi^{\dot\alpha\dot A}$. One can also use spinor indices for the scalars by means of

$$X^{\dot A A} = \frac{1}{\sqrt{2}} X^i \left(\sigma^i\right)^{\dot A A} , \tag{D.31}$$

where $\sigma^4 \equiv i1_2$. In terms of these free fields, normalized such that their OPEs read

$$X^i(z)X^j(0) \sim -\delta^{ij}\log|z|^2 , \qquad \psi^{\alpha\dot A}(z)\psi^{\beta\dot B}(0) \sim -\frac{\varepsilon^{\alpha\beta}\varepsilon^{\dot A\dot B}}{z} , \tag{D.32}$$

the currents are realized as

$$\mathcal{G}^{\alpha A} = \psi^{\alpha\dot A}(\partial X)^{\dot B A}\varepsilon_{\dot A\dot B} , \tag{D.33}$$

$$\mathcal{K}^a = \frac{1}{4}\varepsilon_{\dot A\dot B}\varepsilon_{\alpha\beta}\psi^{\alpha\dot A}(\sigma^{*a})^\beta_\gamma\psi^{\gamma\dot B} , \tag{D.34}$$

$$T = \frac{1}{2}\varepsilon_{\dot A\dot B}\varepsilon_{AB}(\partial X)^{\dot A A}(\partial X)^{\dot B B} + \frac{1}{2}\varepsilon_{\dot A\dot B}\varepsilon_{\alpha\beta}\psi^{\alpha\dot A}\psi^{\beta\dot B} . \tag{D.35}$$

The chiral primary operators are those that create primary states $|\phi\rangle$ satisfying the usual conditions

$$\mathcal{L}_1|\phi\rangle = \mathcal{G}^{\alpha A}_{\frac{1}{2}}|\phi\rangle = \mathcal{K}^a_1|\phi\rangle = 0 , \qquad \mathcal{L}_0|\phi\rangle = h|\phi\rangle , \qquad \mathcal{K}^3_0|\phi\rangle = q|\phi\rangle , \tag{D.36}$$

where h is the holomorphic weight and q the R-symmetry charge, together with the additional extremality condition

$$h = q . \tag{D.37}$$

(Anti-chiral states have $h = -q$.) This are extremal states in the sense that the anti-commutator

$$\left\{\mathcal{G}^{-A}_{\frac{1}{2}}, \mathcal{G}^{+B}_{-\frac{1}{2}}\right\} = \varepsilon^{AB}\left(\mathcal{K}^3_0 - \mathcal{L}_0\right) , \tag{D.38}$$

combined with the condition that all norms must be greater or equal to zero imply that all states must satisfy $h \geq |q|$. Hence, chiral states satisfy

$$G^{+A}_{-\frac{1}{2}}|\phi\rangle , \qquad A = 1, 2, \tag{D.39}$$

so that they are invariant under half of the supersymmetry transformations.

In the untwisted sector of the theory there are four chiral states. They take the form

$$|0\rangle , \qquad \psi^{+\dot A}_{-\frac{1}{2}}|0\rangle , \qquad \mathcal{K}^+_{-1}|0\rangle \sim \psi^{+\dot 1}_{-\frac{1}{2}}\psi^{+\dot 2}_{-\frac{1}{2}}|0\rangle \tag{D.40}$$

such that their weights are $h = 0, \frac{1}{2}, 1$, respectively. A simple way to see that there are no chiral operators with higher weights in this sector is to note that all creation modes have

$h > q$, except for $\psi^{\alpha\dot{A}}_{-\frac{1}{2}}$ and $\mathcal{K}^{+}_{-1}$ for which $h = q$, and one cannot act with each of the fermionic modes more than once. More explicitly, this is derived from the relation

$$\left\{\mathcal{G}^{-A}_{\frac{3}{2}}, \mathcal{G}^{+B}_{-\frac{3}{2}}\right\} = \varepsilon^{AB}\left(3\mathcal{K}^{3}_{0} - \mathcal{L}_{0} - 2\right) , \tag{D.41}$$

where we have used that for a single T^4 one has $c = 6$. This leads to $0 \leq h \leq 1$ for modes with $h = q$. This extends to gauge-invariant operators in the untwisted sector of the full symmetric orbifold.

On the other hand, we must dress the bare twist operators (which have charge zero) appropriately in order to obtain the chiral operators belonging to the twisted sectors. Let us start with the case where we are in the ω-twisted sector, with ω odd. In order to add charge at minimal cost in terms of the weight we can act with the fractional modes $\mathcal{K}^{+}_{-\frac{n}{\omega}}$ (defined in analogy to the fractional Virasoro modes of Eq. (D.12)) which have charge 1 and weight $\frac{1}{\omega}$. The first chiral state is obtained by acting with the first $\frac{\omega-1}{2}$ fractional modes of $\mathcal{K}^{+}$ as

$$|\sigma^{-}_{\omega}\rangle = \mathcal{K}^{+}_{-\frac{\omega-2}{\omega}} \cdots \mathcal{K}^{+}_{-\frac{3}{\omega}}\mathcal{K}^{+}_{-\frac{1}{\omega}}\sigma_{\omega}|0\rangle \,, \tag{D.42}$$

where we have omitted the antiholomorphic modes. Indeed, this has

$$q[\sigma^{-}_{\omega}] = \frac{\omega-1}{2} \tag{D.43}$$

and

$$h[\sigma^{-}_{\omega}] = \frac{\omega^2-1}{4\omega} + \sum_{p=1}^{(\omega-1)/2}\frac{2p-1}{\omega} = \frac{\omega-1}{2}\,. \tag{D.44}$$

For ω even one must be slightly more careful because the boundary conditions for the fermions change upon going from the physical space to the covering space, hence one must include an additional spin field S^{+}_{ω} in order to go to the Ramond vacuum of the corresponding twisted sector [56]. The operators S^{+}_{ω} have $h = \frac{1}{4\omega}$ and $q = \frac{1}{2}$. The lowest-lying chiral primary in this sector is then

$$|\sigma^{-}_{\omega}\rangle = \mathcal{K}^{+}_{-\frac{\omega-2}{\omega}} \cdots \mathcal{K}^{+}_{-\frac{4}{\omega}}\mathcal{K}^{+}_{-\frac{2}{\omega}}S^{+}_{\omega}\sigma_{\omega}|0\rangle \,, \tag{D.45}$$

which again has

$$h[\sigma^{-}_{\omega}] = q[\sigma^{-}_{\omega}] = \frac{\omega-1}{2}\,. \tag{D.46}$$

As in the untwisted sector, one can obtain further chiral states by acting with the center-of-mass $\psi^{+\dot{A}}_{-\frac{1}{2}}$ and $\mathcal{K}^{+}_{-1}$. We denote the corresponding operators as $\sigma^{\dot{A}}_{\omega}$ and σ^{+}_{ω}. Their weights are

$$h[\sigma^{\dot{A}}_{\omega}] = \frac{\omega}{2}\,, \qquad h[\sigma^{+}_{\omega}] = \frac{\omega+1}{2}\,. \tag{D.47}$$

It can be shown that this completes the list of chiral operators in the theory given by the symmetric orbifold of T^4. Indeed, by using the anticommutator of the fractional modes of $\mathcal{G}^{\alpha A}_{\pm n}$ as before one can show that in the twisted sectors the bound on the chiral primary weights takes the form

$$\frac{\omega-1}{2} \leq h \leq \frac{\omega+1}{2}\,. \tag{D.48}$$

Chiral(-chiral) primary operators are the top components in the short representations of the $\mathcal{N}=(4,4)$ algebra. The spectrum and three-point functions of operators belonging to such representations are protected by supersymmetry [6, 148]. They take the same form at any point in the moduli space of the symmetric orbifold SCFT, namely the superconformal manifold generated by all possible supersymmetry-preserving marginal deformations. In particular, for the top components the corresponding structure constants define the so-called chiral ring, given by the regular contributions to the chiral primary OPE in the coincidence limit, namely

$$(\phi_a \cdot \phi_b)(0) \equiv \lim_{z\to 0} \phi_a(z)\phi_b(0) = \sum_c \mathcal{C}^c_{ab}\phi_c(0)\,. \tag{D.49}$$

Indeed, if ϕ_a and ϕ_b are chiral primaries there are no singular terms in the OPE

$$\phi_a(z)\phi_b(0) \sim \sum_c \sum_{n\geq 0} C^c_{ab} \frac{\partial^n \phi_c}{z^{h_a+h_b-h_c-n}}\,, \tag{D.50}$$

since

$$h_a + h_b - h_c - n = q_a + q_b - h_c - n \leq q_c - h_c - n \leq 0\,, \tag{D.51}$$

where we have used charge conservation together with the unitarity condition $h_c \geq q_c$. The first regular term thus corresponds to operators chiral operators with $h_c = q_c$ [144]. The chiral primary structure constants were computed in [121] for the D1D5 CFT. In the large $N(=n_1 n_5)$ limit they take the following form [71]:

$$\langle \sigma^{--}_{\omega_1}\sigma^{--}_{\omega_2}\sigma^{--}_{\omega_3}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(h_1+h_2+h_3-2)^4}{(2h_1-1)(2h_2-1)(2h_3-1)}\right]^{1/2}, \tag{D.52a}$$

$$\langle \sigma^{++}_{\omega_1}\sigma^{--}_{\omega_2}\sigma^{--}_{\omega_3}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(1+h_1-h_2-h_3)^4}{(2h_1-1)(2h_2-1)(2h_3-1)}\right]^{1/2}, \tag{D.52b}$$

$$\langle \sigma^{++}_{\omega_1}\sigma^{++}_{\omega_2}\sigma^{--}_{\omega_3}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(h_1+h_2-h_3)^4}{(2h_1-1)(2h_2-1)(2h_3-1)}\right]^{1/2}, \tag{D.52c}$$

$$\langle \sigma^{++}_{\omega_1}\sigma^{++}_{\omega_2}\sigma^{++}_{\omega_3}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(h_1+h_2+h_3-1)^4}{(2h_1-1)(2h_2-1)(2h_3-1)}\right]^{1/2}, \tag{D.52d}$$

and

$$\langle \sigma^{\dot{A}_1\dot{B}_1}_{\omega_1}\sigma^{\dot{A}_2\dot{B}_2}_{\omega_2}\sigma^{--}_{\omega_3}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(2h_1-1)\,(2h_2-1)}{(2h_3-1)}\right]^{1/2} \delta^{\dot{A}_1\dot{A}_2}\delta^{\dot{B}_1\dot{B}_2}, \tag{D.53a}$$

$$\langle \sigma^{\dot{A}_1\dot{B}_1}_{\omega_1}\sigma^{\dot{A}_2\dot{B}_2}_{\omega_2}\sigma^{++}_{\omega_3}\rangle = \frac{1}{\sqrt{N}}\left[\frac{(2h_1-1)\,(2h_2-1)}{(2h_3-1)}\right]^{1/2} \xi^{\dot{A}_1\dot{A}_2}\xi^{\dot{B}_1\dot{B}_2}, \tag{D.53b}$$

where $\xi = \sigma^1$ and $\delta = 1_2$.

References

[1] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [hep-th/9711200].

[2] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207.

[3] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99 [hep-th/9601029].

[4] A. Strominger, *Black hole entropy from near horizon microstates*, *JHEP* **02** (1998) 009 [hep-th/9712251].

[5] F. Larsen and E. J. Martinec, *U(1) charges and moduli in the D1 - D5 system*, *JHEP* **06** (1999) 019 [hep-th/9905064].

[6] J. de Boer, J. Manschot, K. Papadodimas and E. Verlinde, *The Chiral ring of AdS(3)/CFT(2) and the attractor mechanism*, *JHEP* **03** (2009) 030 [0809.0507].

[7] L. J. Romans, *New Compactifications of Chiral $N = 2d = 10$ Supergravity*, *Phys. Lett. B* **153** (1985) 392.

[8] N. Seiberg, *Observations on the Moduli Space of Superconformal Field Theories*, *Nucl. Phys. B* **303** (1988) 286.

[9] S. Ferrara, R. Kallosh and A. Strominger, *N=2 extremal black holes*, *Phys. Rev. D* **52** (1995) R5412 [hep-th/9508072].

[10] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on AdS(3)*, *Adv. Theor. Math. Phys.* **2** (1998) 733 [hep-th/9806194].

[11] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, *String theory on AdS(3)*, *JHEP* **12** (1998) 026 [hep-th/9812046].

[12] D. Kutasov and N. Seiberg, *More comments on string theory on AdS(3)*, *JHEP* **04** (1999) 008 [hep-th/9903219].

[13] J. M. Maldacena and H. Ooguri, *Strings in AdS(3) and SL(2,R) WZW model 1.: The Spectrum*, *J. Math. Phys.* **42** (2001) 2929 [hep-th/0001053].

[14] J. M. Maldacena, H. Ooguri and J. Son, *Strings in AdS(3) and the SL(2,R) WZW model. Part 2. Euclidean black hole*, *J. Math. Phys.* **42** (2001) 2961 [hep-th/0005183].

[15] J. M. Maldacena and H. Ooguri, *Strings in AdS(3) and the SL(2,R) WZW model. Part 3. Correlation functions*, *Phys. Rev.* **D65** (2002) 106006 [hep-th/0111180].

[16] O. Lunin and S. D. Mathur, *AdS / CFT duality and the black hole information paradox*, *Nucl. Phys. B* **623** (2002) 342 [hep-th/0109154].

[17] S. D. Mathur, *The Fuzzball proposal for black holes: An Elementary review*, *Fortsch. Phys.* **53** (2005) 793 [hep-th/0502050].

[18] I. Bena, E. J. Martinec, S. D. Mathur and N. P. Warner, *Fuzzballs and Microstate Geometries: Black-Hole Structure in String Theory*, 2204.13113.

[19] E. Witten, *On string theory and black holes*, *Phys. Rev.* **D44** (1991) 314.

[20] M. Banados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [hep-th/9204099].

[21] E. J. Martinec and S. Massai, *String Theory of Supertubes*, *JHEP* **07** (2018) 163 [1705.10844].

[22] E. J. Martinec, S. Massai and D. Turton, *String dynamics in NS5-F1-P geometries*, *JHEP* **09** (2018) 031 [1803.08505].

[23] E. J. Martinec, S. Massai and D. Turton, *Little Strings, Long Strings, and Fuzzballs*, *JHEP* **11** (2019) 019 [1906.11473].

[24] E. J. Martinec, S. Massai and D. Turton, *Stringy Structure at the BPS Bound*, *JHEP* **12** (2020) 135 [2005.12344].

[25] E. J. Martinec, S. Massai and D. Turton, *On the BPS sector in AdS_3/CFT_2 Holography*, 2211.12476.

[26] D. Bufalini, S. Iguri, N. Kovensky and D. Turton, *Black hole microstates from the worldsheet*, *JHEP* **08** (2021) 011 [2105.02255].

[27] D. Bufalini, S. Iguri, N. Kovensky and D. Turton, *Worldsheet Correlators in Black Hole Microstates*, *Phys. Rev. Lett.* **129** (2022) 121603 [2203.13828].

[28] D. Bufalini, S. Iguri, N. Kovensky and D. Turton, *Worldsheet computation of heavy-light correlators*, *JHEP* **03** (2023) 066 [2210.15313].

[29] D. Kutasov, *Introduction to little string theory*, *ICTP Lect. Notes Ser.* **7** (2002) 165.

[30] A. B. Zamolodchikov, *Expectation value of composite field T anti-T in two-dimensional quantum field theory*, hep-th/0401146.

[31] F. A. Smirnov and A. B. Zamolodchikov, *On space of integrable quantum field theories*, *Nucl. Phys. B* **915** (2017) 363 [1608.05499].

[32] A. Giveon, N. Itzhaki and D. Kutasov, $\mathrm{T}\overline{\mathrm{T}}$ *and LST*, *JHEP* **07** (2017) 122 [1701.05576].

[33] S. Chakraborty, S. Georgescu and M. Guica, *States, symmetries and correlators of $T\bar{T}$ and $J\bar{T}$ symmetric orbifolds*, 2306.16454.

[34] A. Giveon and D. Kutasov, *Notes on AdS(3)*, *Nucl. Phys.* **B621** (2002) 303 [hep-th/0106004].

[35] L. Dolan, *TASI lectures on perturbative string theory and Ramond-Ramond flux*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2001): Strings, Branes and EXTRA Dimensions*, pp. 161–193, 1, 2002, hep-th/0201209.

[36] V. Schomerus, *Lectures on branes in curved backgrounds*, *Class. Quant. Grav.* **19** (2002) 5781 [hep-th/0209241].

[37] C. A. Nunez, *String theory on AdS(3)*, *PoS* **WC2004** (2004) 001.

[38] V. Schomerus, *Non-compact string backgrounds and non-rational CFT*, *Phys. Rept.* **431** (2006) 39 [hep-th/0509155].

[39] P. Kraus, *Lectures on black holes and the AdS(3) / CFT(2) correspondence*, *Lect. Notes Phys.* **755** (2008) 193 [hep-th/0609074].

[40] W. McElgin, *Notes on the SL(2,R) CFT*, 1511.07256.

[41] A. Schwimmer and N. Seiberg, *Comments on the N=2, N=3, N=4 Superconformal Algebras in Two-Dimensions*, *Phys. Lett. B* **184** (1987) 191.

[42] A. Zamolodchikov and V. Fateev, *Operator Algebra and Correlation Functions in the Two-Dimensional Wess-Zumino SU(2) x SU(2) Chiral Model*, *Sov. J. Nucl. Phys.* **43** (1986) 657.

[43] L. Eberhardt, M. R. Gaberdiel and R. Gopakumar, *Deriving the AdS_3/CFT_2 correspondence*, *JHEP* **02** (2020) 136 [1911.00378].

[44] L. Eberhardt, *AdS_3/CFT_2 at higher genus*, *JHEP* **05** (2020) 150 [2002.11729].

[45] B. Knighton, *Higher genus correlators for tensionless AdS_3 strings*, *JHEP* **04** (2021) 211 [2012.01445].

[46] L. Eberhardt, *Summing over Geometries in String Theory*, *JHEP* **05** (2021) 233 [2102.12355].

[47] J. Teschner, *The Minisuperspace limit of the sl(2,C) / SU(2) WZNW model*, *Nucl. Phys. B* **546** (1999) 369 [hep-th/9712258].

[48] J. Teschner, *On structure constants and fusion rules in the SL(2,C) / SU(2) WZNW model*, *Nucl. Phys.* **B546** (1999) 390 [hep-th/9712256].

[49] J. Teschner, *Operator product expansion and factorization in the H+(3) WZNW model*, *Nucl. Phys.* **B571** (2000) 555 [hep-th/9906215].

[50] V. Fateev, A. Zamolodchikov and A. Zamolodchikov, *Unpublished notes*, .

[51] A. Dei and L. Eberhardt, *String correlators on AdS_3: three-point functions*, *JHEP* **08** (2021) 025 [2105.12130].

[52] S. Iguri and N. Kovensky, *On spectrally flowed local vertex operators in AdS_3*, *SciPost Phys.* **13** (2022) 115 [2208.00978].

[53] A. Dei and L. Eberhardt, *String correlators on AdS_3: four-point functions*, *JHEP* **09** (2021) 209 [2107.01481].

[54] D. Bufalini, S. Iguri and N. Kovensky, *A proof for string three-point functions in AdS_3*, *JHEP* **02** (2023) 246 [2212.05877].

[55] O. Lunin and S. D. Mathur, *Correlation functions for M**N / S(N) orbifolds*, *Commun. Math. Phys.* **219** (2001) 399 [hep-th/0006196].

[56] O. Lunin and S. D. Mathur, *Three point functions for M(N) / S(N) orbifolds with N=4 supersymmetry*, *Commun. Math. Phys.* **227** (2002) 385 [hep-th/0103169].

[57] A. Pakman, L. Rastelli and S. S. Razamat, *Extremal Correlators and Hurwitz Numbers in Symmetric Product Orbifolds*, *Phys. Rev. D* **80** (2009) 086009 [0905.3451].

[58] A. Pakman, L. Rastelli and S. S. Razamat, *Diagrams for Symmetric Product Orbifolds*, *JHEP* **10** (2009) 034 [0905.3448].

[59] A. Dei and L. Eberhardt, *Correlators of the symmetric product orbifold*, *JHEP* **01** (2020) 108 [1911.08485].

[60] M. R. Gaberdiel, R. Gopakumar and C. Hull, *Stringy AdS_3 from the worldsheet*, *JHEP* **07** (2017) 090 [1704.08665].

[61] M. R. Gaberdiel and R. Gopakumar, *Tensionless string spectra on AdS_3*, *JHEP* **05** (2018) 085 [1803.04423].

[62] G. Giribet, C. Hull, M. Kleban, M. Porrati and E. Rabinovici, *Superstrings on AdS_3 at $\| = 1$*, *JHEP* **08** (2018) 204 [1803.04420].

[63] L. Eberhardt, M. R. Gaberdiel and R. Gopakumar, *The Worldsheet Dual of the Symmetric Product CFT*, *JHEP* **04** (2019) 103 [1812.01007].

[64] L. Eberhardt and M. R. Gaberdiel, *Strings on $AdS_3 \times S^3 \times S^3 \times S^1$*, *JHEP* **06** (2019) 035 [1904.01585].

[65] L. Eberhardt, *Partition functions of the tensionless string*, *JHEP* **03** (2021) 176 [2008.07533].

[66] A. Dei, L. Eberhardt and M. R. Gaberdiel, *Three-point functions in AdS_3/CFT_2 holography*, 1907.13144.

[67] H. Bertle, A. Dei and M. R. Gaberdiel, *Stress-energy tensor correlators from the world-sheet*, *JHEP* **03** (2021) 036 [2012.08486].

[68] D. Kutasov, F. Larsen and R. G. Leigh, *String theory in magnetic monopole backgrounds*, *Nucl. Phys. B* **550** (1999) 183 [hep-th/9812027].

[69] R. Argurio, A. Giveon and A. Shomer, *Superstrings on AdS(3) and symmetric products*, *JHEP* **12** (2000) 003 [hep-th/0009242].

[70] M. R. Gaberdiel and I. Kirsch, *Worldsheet correlators in AdS(3)/CFT(2)*, *JHEP* **04** (2007) 050 [hep-th/0703001].

[71] A. Dabholkar and A. Pakman, *Exact chiral ring of AdS(3) / CFT(2)*, *Adv. Theor. Math. Phys.* **13** (2009) 409 [hep-th/0703022].

[72] G. Giribet, A. Pakman and L. Rastelli, *Spectral Flow in AdS(3)/CFT(2)*, *JHEP* **06** (2008) 013 [0712.3046].

[73] S. Iguri, N. Kovensky and J. H. Toro, *Spectral flow and string correlators in $AdS_3 \times S^3 \times T^4$*, *JHEP* **2023** (2023) 161 [2211.02521].

[74] S. Iguri, N. Kovensky and J. H. Toro, *Spectral flow and the exact AdS_3/CFT_2 chiral ring*, 2304.08361.

[75] B. Balthazar, A. Giveon, D. Kutasov and E. J. Martinec, *Asymptotically free AdS_3/CFT_2*, *JHEP* **01** (2022) 008 [2109.00065].

[76] L. Eberhardt, *A perturbative CFT dual for pure NS–NS AdS_3 strings*, *J. Phys. A* **55** (2022) 064001 [2110.07535].

[77] A. Dei and L. Eberhardt, *String correlators on AdS_3: Analytic structure and dual CFT*, *SciPost Phys.* **13** (2022) 053 [2203.13264].

[78] N. Seiberg and E. Witten, *The D1 / D5 system and singular CFT*, *JHEP* **04** (1999) 017 [hep-th/9903224].

[79] A. Giveon, N. Itzhaki and D. Kutasov, *A solvable irrelevant deformation of AdS_3/CFT_2*, *JHEP* **12** (2017) 155 [1707.05800].

[80] L. Apolo, S. Detournay and W. Song, *TsT, $T\bar{T}$ and black strings*, *JHEP* **06** (2020) 109 [1911.12359].

[81] S. Georgescu and M. Guica, *Infinite* $\mathrm{T}\bar{\mathrm{T}}$*-like symmetries of compactified LST*, 2212.09768.

[82] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997, 10.1007/978-1-4612-2256-9.

[83] V. Fateev and A. Zamolodchikov, *Parafermionic Currents in the Two-Dimensional Conformal Quantum Field Theory and Selfdual Critical Points in Z(n) Invariant Statistical Systems*, *Sov. Phys. JETP* **62** (1985) 215.

[84] L. Eberhardt, *Wess-Zumino-Witten models*, .

[85] H. Dorn and H. J. Otto, *Two and three point functions in Liouville theory*, *Nucl. Phys. B* **429** (1994) 375 [hep-th/9403141].

[86] A. B. Zamolodchikov and A. B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, *Nucl. Phys. B* **477** (1996) 577 [hep-th/9506136].

[87] J. Teschner, *Liouville theory revisited*, *Class. Quant. Grav.* **18** (2001) R153 [hep-th/0104158].

[88] J. Teschner, *A Lecture on the Liouville vertex operators*, *Int. J. Mod. Phys. A* **19S2** (2004) 436 [hep-th/0303150].

[89] S. Ribault, *Conformal field theory on the plane*, 1406.4290.

[90] L. Eberhardt, *Notes on crossing transformations of Virasoro conformal blocks*, 2309.11540.

[91] A. Dabholkar, G. W. Gibbons, J. A. Harvey and F. Ruiz Ruiz, *Superstrings and Solitons*, *Nucl. Phys. B* **340** (1990) 33.

[92] C. G. Callan, Jr., J. A. Harvey and A. Strominger, *Supersymmetric string solitons*, hep-th/9112030.

[93] A. A. Tseytlin, *Extreme dyonic black holes in string theory*, *Mod. Phys. Lett. A* **11** (1996) 689 [hep-th/9601177].

[94] J. Polchinski and E. Silverstein, *Large-density field theory, viscosity, and '$2k_F$' singularities from string duals*, *Class. Quant. Grav.* **29** (2012) 194008 [1203.1015].

[95] R. Blumenhagen, D. Lüst and S. Theisen, *Basic concepts of string theory*, Theoretical and Mathematical Physics. Springer, Heidelberg, Germany, 2013, 10.1007/978-3-642-29497-6.

[96] A. Hanany, N. Prezas and J. Troost, *The Partition function of the two-dimensional black hole conformal field theory*, *JHEP* **04** (2002) 014 [hep-th/0202129].

[97] D. Israel, C. Kounnas and M. P. Petropoulos, *Superstrings on NS5 backgrounds, deformed AdS(3) and holography*, *JHEP* **10** (2003) 028 [hep-th/0306053].

[98] S. Ribault and J. Teschner, *H+(3)-WZNW correlators from Liouville theory*, *JHEP* **06** (2005) 014 [hep-th/0502048].

[99] Y. Hikida and V. Schomerus, *H+(3) WZNW model from Liouville field theory*, *JHEP* **10** (2007) 064 [0706.1030].

[100] I. Gelfand and G. Shilov, *Generalized functions*. Academic Press, 1964.

[101] Y. Cagnacci and S. M. Iguri, *More AdS_3 correlators*, *Phys. Rev. D* **89** (2014) 066006 [1312.3353].

[102] G. Giribet and C. A. Nunez, *Correlators in AdS(3) string theory*, *JHEP* **06** (2001) 010 [hep-th/0105200].

[103] D. M. Hofman and C. A. Nunez, *Free field realization of superstring theory on AdS(3)*, *JHEP* **07** (2004) 019 [hep-th/0404214].

[104] S. Iguri and C. A. Nunez, *Coulomb integrals for the SL(2,R) WZW model*, *Phys. Rev.* **D77** (2008) 066015 [0705.4461].

[105] G. Giribet, *Violating the string winding number maximally in Anti-de Sitter space*, *Phys. Rev.* **D84** (2011) 024045 [1106.4191].

[106] Y. Satoh, *Three point functions and operator product expansion in the SL(2) conformal field theory*, *Nucl. Phys.* **B629** (2002) 188 [hep-th/0109059].

[107] J. Kim and M. Porrati, *On the central charge of spacetime current algebras and correlators in string theory on AdS_3*, *JHEP* **05** (2015) 076 [1503.07186].

[108] G. Lauricella, *Sulle funzioni ipergeometriche a piu variabili*, *Rendiconti del Circolo Matematico di Palermo* **111** (1893) .

[109] L. Eberhardt and M. R. Gaberdiel, *String theory on $AdS_\mathbf{3}$ and the symmetric orbifold of Liouville theory*, *Nucl. Phys.* **B948** (2019) 114774 [1903.00421].

[110] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux*, *JHEP* **03** (1999) 018 [hep-th/9902098].

[111] A. Dei, M. R. Gaberdiel, R. Gopakumar and B. Knighton, *Free field world-sheet correlators for* AdS_3, *JHEP* **02** (2021) 081 [2009.11306].

[112] B. Knighton, *Classical geometry from the tensionless string*, *JHEP* **05** (2023) 005 [2207.01293].

[113] Y. Hikida and T. Liu, *Correlation functions of symmetric orbifold from AdS_3 string theory*, *JHEP* **09** (2020) 157 [2005.12511].

[114] M. R. Gaberdiel, R. Gopakumar, B. Knighton and P. Maity, *From symmetric product CFTs to AdS_3*, *JHEP* **05** (2021) 073 [2011.10038].

[115] M. R. Gaberdiel and B. Nairz, *BPS correlators for AdS_3/CFT_2*, *JHEP* **09** (2022) 244 [2207.03956].

[116] J. Kim and M. Porrati, *On the central charge of spacetime current algebras and correlators in string theory on AdS_3*, *JHEP* **05** (2015) 076 [1503.07186].

[117] P. Di Vecchia, V. G. Knizhnik, J. L. Petersen and P. Rossi, *A Supersymmetric Wess-Zumino Lagrangian in Two-Dimensions*, *Nucl. Phys. B* **253** (1985) 701.

[118] V. G. Kac and I. T. Todorov, *SUPERCONFORMAL CURRENT ALGEBRAS AND THEIR UNITARY REPRESENTATIONS*, *Commun. Math. Phys.* **102** (1985) 337.

[119] E. B. Kiritsis and G. Siopsis, *Operator Algebra of the N=1 Super Wess-Zumino Model*, *Phys. Lett. B* **184** (1987) 353.

[120] L. Eberhardt and K. Ferreira, *Long strings and chiral primaries in the hybrid formalism*, *JHEP* **02** (2019) 098 [1810.08621].

[121] A. Jevicki, M. Mihailescu and S. Ramgoolam, *Gravity from CFT on S**N(X): Symmetries and interactions*, *Nucl. Phys. B* **577** (2000) 47 [hep-th/9907144].

[122] M. Yu and B. Zhang, *Light cone gauge quantization of string theories on AdS(3) space*, *Nucl. Phys. B* **551** (1999) 425 [hep-th/9812216].

[123] N. Seiberg, *Notes on quantum Liouville theory and quantum gravity*, *Prog. Theor. Phys. Suppl.* **102** (1990) 319.

[124] A. Giveon, D. Kutasov, E. Rabinovici and A. Sever, *Phases of quantum gravity in AdS(3) and linear dilaton backgrounds*, *Nucl. Phys. B* **719** (2005) 3 [hep-th/0503121].

[125] O. Aharony, A. Giveon and D. Kutasov, *LSZ in LST*, *Nucl. Phys. B* **691** (2004) 3 [hep-th/0404016].

[126] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large N limit of theories with sixteen supercharges*, *Phys. Rev. D* **58** (1998) 046004 [hep-th/9802042].

[127] H. J. Boonstra, K. Skenderis and P. K. Townsend, *The domain wall / QFT correspondence*, *JHEP* **01** (1999) 003 [hep-th/9807137].

[128] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS five-branes and holography*, *JHEP* **10** (1998) 004 [hep-th/9808149].

[129] N. Seiberg, *New theories in six-dimensions and matrix description of M theory on T**5 and T**5 / Z(2)*, *Phys. Lett. B* **408** (1997) 98 [hep-th/9705221].

[130] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *JHEP* **10** (1999) 034 [hep-th/9909110].

[131] A. Giveon and D. Kutasov, *Comments on double scaled little string theory*, *JHEP* **01** (2000) 023 [hep-th/9911039].

[132] S. Forste, *A Truly marginal deformation of SL(2, R) in a null direction*, *Phys. Lett. B* **338** (1994) 36 [hep-th/9407198].

[133] A. Giveon, D. Kutasov and O. Pelc, *Holography for noncritical superstrings*, *JHEP* **10** (1999) 035 [hep-th/9907178].

[134] C. Hull and B. J. Spence, *The Gauged Nonlinear σ Model With Wess-Zumino Term*, *Phys. Lett. B* **232** (1989) 204.

[135] S.-w. Chung and S. H. H. Tye, *Chiral gauged WZW theories and coset models in conformal field theory*, *Phys. Rev. D* **47** (1993) 4546 [hep-th/9202002].

[136] M. Asrat, A. Giveon, N. Itzhaki and D. Kutasov, *Holography Beyond AdS*, *Nucl. Phys. B* **932** (2018) 241 [1711.02690].

[137] G. Giribet, *$T\bar{T}$-deformations, AdS/CFT and correlation functions*, *JHEP* **02** (2018) 114 [1711.02716].

[138] W. Cui, H. Shu, W. Song and J. Wang, *Correlation Functions in the $TsT/T\bar{T}$ Correspondence*, 2304.04684.

[139] M. Guica, *TTbar deformations and holography*, *CERN Winter School on Supergravity, Strings and Gauge Theory* (2020) .

[140] J. M. Maldacena, *Black holes in string theory*, Ph.D. thesis, Princeton U., 1996. hep-th/9607235.

[141] R. Hagedorn, *Statistical thermodynamics of strong interactions at high-energies*, *Nuovo Cim. Suppl.* **3** (1965) 147.

[142] J. J. Atick and E. Witten, *The Hagedorn Transition and the Number of Degrees of Freedom of String Theory*, *Nucl. Phys. B* **310** (1988) 291.

[143] P. H. Ginsparg, *APPLIED CONFORMAL FIELD THEORY*, in *Les Houches Summer*

School in Theoretical Physics: Fields, Strings, Critical Phenomena, 9, 1988, hep-th/9108028.

[144] R. Blumenhagen and E. Plauschinn, *Introduction to conformal field theory: with applications to String theory*, vol. 779. 2009, 10.1007/978-3-642-00450-6.

[145] L. J. Dixon, D. Friedan, E. J. Martinec and S. H. Shenker, *The Conformal Field Theory of Orbifolds*, *Nucl. Phys. B* **282** (1987) 13.

[146] R. Cavalieri and E. Miles, *Riemann Surfaces and Algebraic Curves: A First Course in Hurwitz Theory*, London Mathematical Society Student Texts. Cambridge University Press, 2016, 10.1017/CBO9781316569252.

[147] S. G. Avery, *Using the D1D5 CFT to Understand Black Holes*, other thesis, 12, 2010.

[148] M. Baggio, J. de Boer and K. Papadodimas, *A non-renormalization theorem for chiral primary 3-point functions*, *JHEP* **07** (2012) 137 [1203.1036].