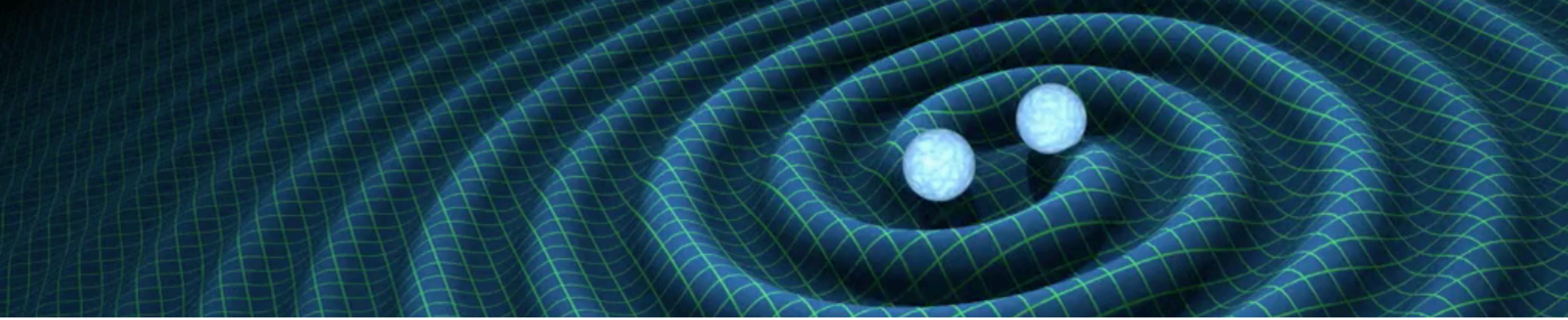
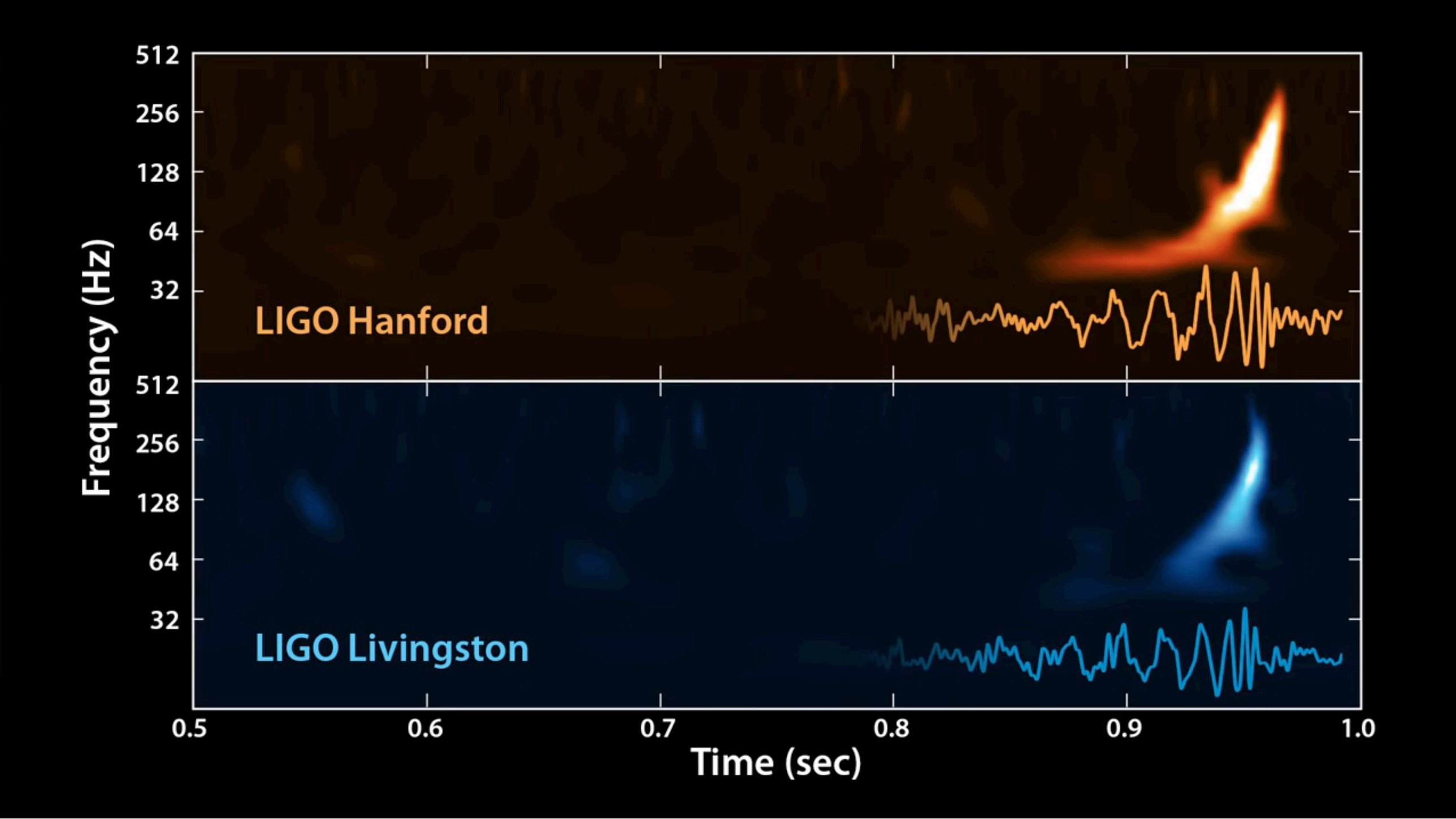


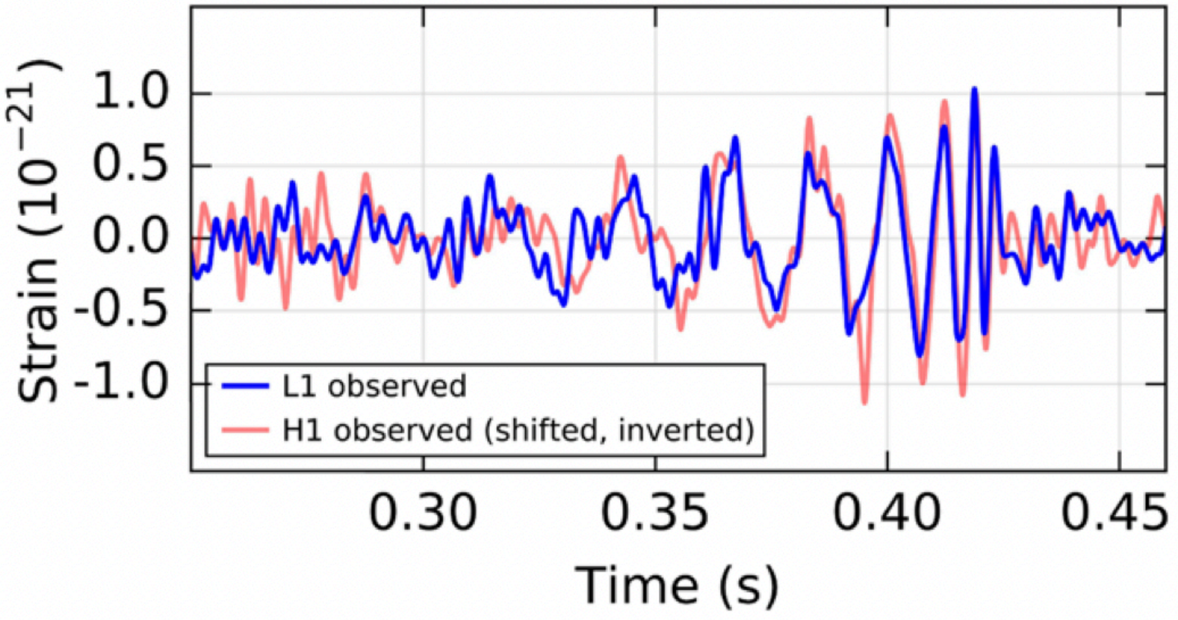
Gravitational waves in a nutshell—1

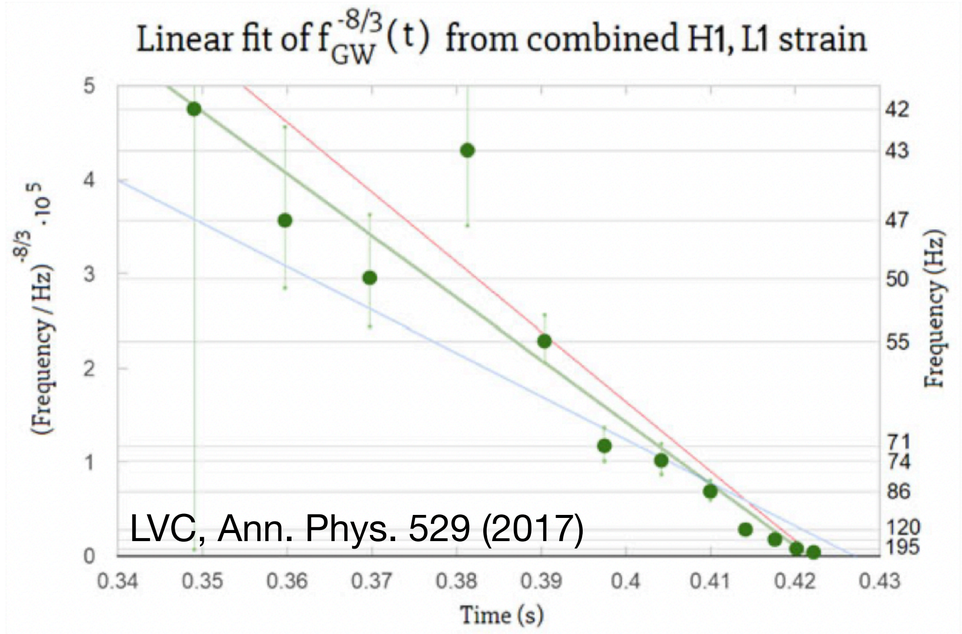
The *Event*. GWs four ways. Falling particles. Universal theory of detectors (plus a few real ones)

Michele Vallisneri, Jet Propulsion Laboratory

CEA/IPhT, June 3 2022


512
256
128
64
32
LIGO Hanford
512
256
128
64
32
LIGO Livingston
Frequency (Hz)
0.5
0.6
0.7
0.8
0.9
1.0
Time (sec)





- What is this?
- Simple arguments on the (minimally processed) signal indicate that it was emitted by **two inspiraling and then merging black holes**, with total mass ~ 70 $M_\odot$, at a luminosity distance of a few hundred Mpc, radiating ~ 3 $M_\odot c^2$ of energy.
- This conclusion is robust to theoretical complications.
- [blackboard]

Four ways to gravitational waves

1. In analogy to EM ✅
2. By the linearized field equations
3. By the dynamics of the Riemann tensor
4. From the linearized action

2. By the linearized field equations

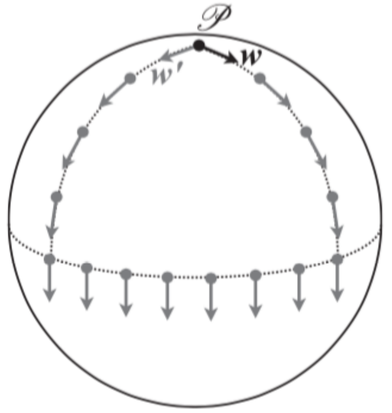
(remember GR?)

1. Metric tensor $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$

2. Covariant derivative $\nabla_b A^a = \partial_b A^a + \Gamma^a_{cb}A^c$ $\Gamma^a_{bc} = \frac{1}{2}g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d})$

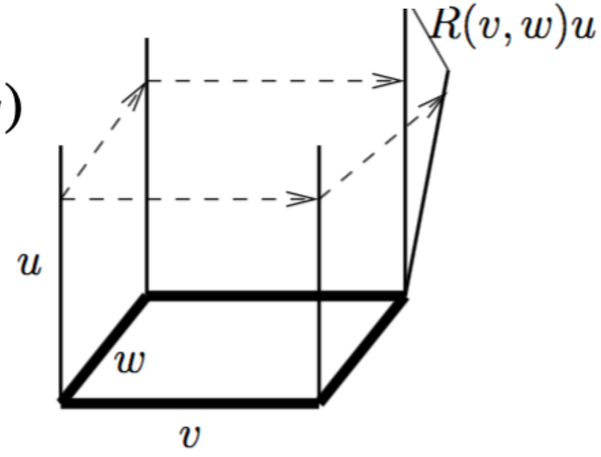
3. Parallel transport, geodesic equation $u \cdot \nabla u = u^b \nabla_b u^a = 0$

$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}$



4. Riemann tensor $(\nabla_c\nabla_d - \nabla_d\nabla_c)A^a = R^a_{bcd}A^b$

$R_{abcd} = \frac{1}{2}(\partial_c\partial_b g_{ad} + \partial_d\partial_a g_{bc} - \partial_d\partial_b g_{ac} - \partial_c\partial_a g_{bd})$



5. Geodesic deviation $\frac{\mathrm{D}^2\xi^a}{\mathrm{D}\tau^2} = -R^a_{dbc}u^d\xi^b u^c$

6. Einstein equations $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab}$

2. By the linearized field equations

1. Metric perturbation $g_{ab} = \eta_{ab} + \epsilon h_{ab}, \qquad g^{ab} = \eta^{ab} - \epsilon h^{ab}$

2. Riemann $R_{abcd} = \frac{1}{2}\epsilon\left(\partial_c\partial_b h_{ad} - \partial_c\partial_a h_{bd} - \partial_d\partial_b h_{ac} + \partial_d\partial_a h_{bc}\right)$

3. Contract twice $R_{ab} = -\frac{1}{2}\epsilon\left(\Box h_{ab} + \partial_a\partial_b h - \partial_c\partial_b h^c{}_a - \partial_a\partial_c h^c{}_b\right) \qquad R = -\epsilon\left(\Box h - \partial_c\partial_d h^{cd}\right)$

4. Trace reverse + Lorenz gauge $\bar{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h \qquad \partial_a \bar{h}^a{}_b = 0$

$\left(\bar{h}_{ab} \to \bar{h}'_{ab} = \bar{h}_{ab} - \partial_b\xi_a - \partial_a\xi_b + \eta_{ab}\partial_c\xi^c \qquad \partial^a\partial_a\xi_b = \partial^a\bar{h}_{ab}\right)$

5. Plug into EFE; vacuum solution $\boxed{\Box\bar{h}_{ab} = -\frac{16\pi G}{c^4}T_{ab}} \qquad h_{ab}(t,z) = h_{ab}\mathrm{e}^{i\omega(t-z)}$

6. Set TT gauge $\bar{h} = 0 \qquad h_{00} = h_{0x} = h_{0y} = 0$ $\left(h^{\mathrm{TT}}_{jk} = \left(\perp^l_j\perp^m_k - \frac{1}{2}\perp_{jk}\perp^{lm}\right)h_{lm}\right)$

7. Geodesic deviation $\frac{\partial^2\xi^j}{\partial t^2} = -R^j{}_{0k0}\xi^k = \frac{1}{2}\frac{\partial^2 h_{jk}}{\partial t^2}\xi^k$

2. By the linearized field equations

- In linearized gravity, the Einstein field equations turn into a wave equation for the metric perturbation h_{ab}. The wave equation that looks especially simple in the Lorenz (harmonic) gauge. The remaining gauge freedom can be used to reduce nonzero components to $h_{xx} = -h_{yy} = h_{+}$ and $h_{xy} = h_{yx} = h_{+\mathrm{x}}$ (for a plane wave propagating along z).

3. By the dynamics of the Riemann tensor

3. By the dynamics of the Riemann tensor

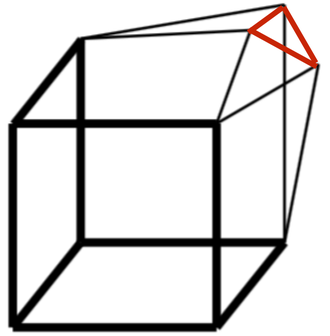
1. Riemann has a clear physical interpretation and is invariant to infinitesimal coordinate transformations

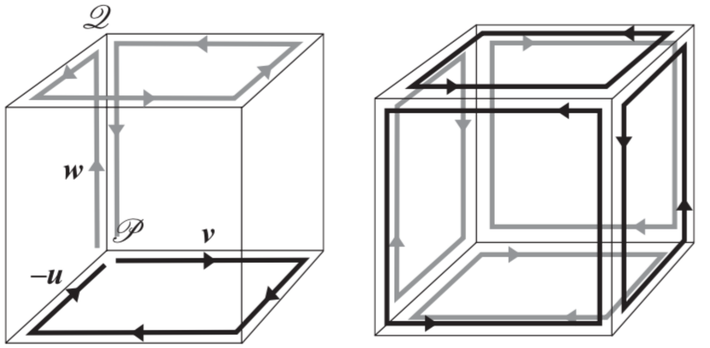
2. Riemann symmetries: $R_{abcd} = -R_{acbd}$ (definition)
   $R_{abcd} = -R_{dbca}$ (conserves metric)
   $R_{a[bcd]} = 0$ (first Bianchi identity)

3. Then second Bianchi identity $R_{abcd,e} + R_{abde,c} + R_{abec,d} = 0$
   plus $R_{bd} = 0$ in vacuum
   implies $\boxed{\Box R_{abcd} = 0}$

4. Symmetries + vanishing Ricci also imply that for a plane wave $R_{abcd}(t-z) = R_{abcd}\, e^{i\omega(t-z)}$ the only nonzero components are $R_{0x0x} = -R_{0y0y}$ and $R_{0x0y} = R_{0y0x}$.

5. One may define $\partial^2 h^{\mathrm{GW}}_{jk}/\partial t^2 = -2R_{0j0k}$.





$$(\delta a^\alpha)_{\text{top+bot}} = \epsilon^2 \left[R_{\mu\nu\rho}{}^\alpha(\mathcal{Q}) - R_{\mu\nu\rho}{}^\alpha(\mathcal{P})\right] u^\mu v^\nu a^\rho$$
$$= \epsilon^3 \left(w^\sigma \nabla_\sigma R_{\mu\nu\rho}{}^\alpha\right) u^\mu v^\nu a^\rho \,,$$

$$0 = (\delta a^\alpha)_{\text{all faces}}$$
$$= \epsilon^3 \left(\nabla_\sigma R_{\mu\nu\rho}{}^\alpha + \nabla_\mu R_{\nu\sigma\rho}{}^\alpha + \nabla_\nu R_{\sigma\mu\rho}{}^\alpha\right) u^\mu v^\nu w^\sigma a^\rho$$

3. By the dynamics of the Riemann tensor

- In linearized gravity, the components of the Riemann tensor are gauge invariant, can be interpreted as tidal forces, and they satisfy the wave equation with the partial-derivative Dalembertian. A similar derivation is possible for small perturbation of a curved background, using a two-length scale expansion.

4. From the linearized action

$$S_{\rm PP} + S_{\rm EH} = -m\int \mathrm{d}\lambda \sqrt{g_{\mu\nu}u^\mu u^\nu} - \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\,R(g_{\mu\nu})$$

$$= -\frac{m}{2}\int \mathrm{d}\tau\, u^\mu u^\nu \left(h_{\mu\nu} + O(h^2)\right) - \frac{1}{64\pi}\int_x \left(P^{\alpha\beta\gamma\delta}\partial_\mu h_{\alpha\beta}\partial^\mu h_{\gamma\delta} + O(h^3)\right)$$

in harmonic gauge,
for trace-reversed perturbations

$$P_{\alpha\beta\gamma\delta} \equiv \frac{1}{2}\left(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta}\right)$$

$$P_{\alpha\beta\gamma\delta}H^{\gamma\delta} = H_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}H$$

$$-\left[\frac{\delta^2 S^{\rm free}_{\rm rad}}{\delta h_{\alpha\beta}(k^\mu)\delta h_{\gamma\delta}(-k^\mu)}\right]^{-1}_{h_{\mu\nu}=0} = \frac{32\pi}{k^2}P^{\alpha\beta\gamma\delta}$$

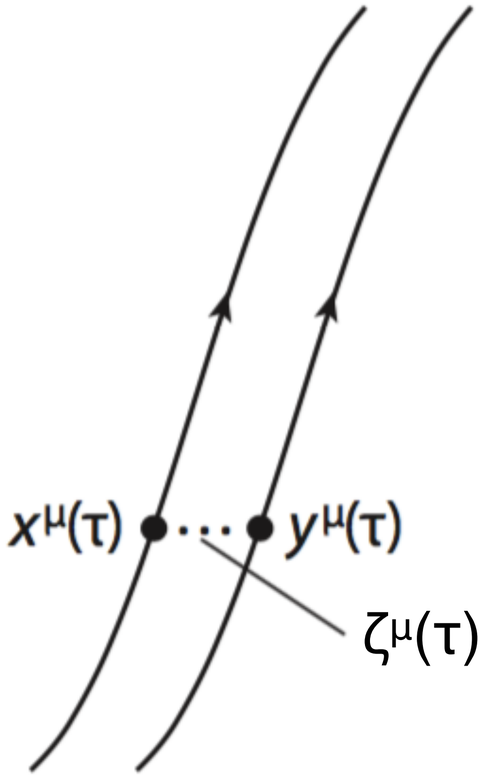
$$= \alpha\beta \quad k^\mu \quad \gamma\delta \qquad x^\mu \quad x'^\mu$$

4. From the linearized action

- The perturbative expansion of the Einstein-Hilbert action (in the harmonic gauge) leads immediately to the action for a free spin-2 massless field, coupled minimally to matter.

Falling test particles

Falling test particles (any geometry)



Newtonian

$$a = \frac{d^2 x}{dt^2} = -\nabla \Phi(x)$$

$$\Delta a_i = -\zeta^j \underbrace{\frac{\partial^2 \Phi}{\partial x^i \partial x^j}}_{\mathcal{E}_{ij}} = -\mathcal{E}_{ij}\zeta^j$$

$$a^\alpha = -R_{\mu\rho\nu}{}^\alpha u^\mu \zeta^\rho u^\nu$$

$$\mathcal{E}_{ij} = R_{0i0j}$$ for particle at rest in proper reference frame

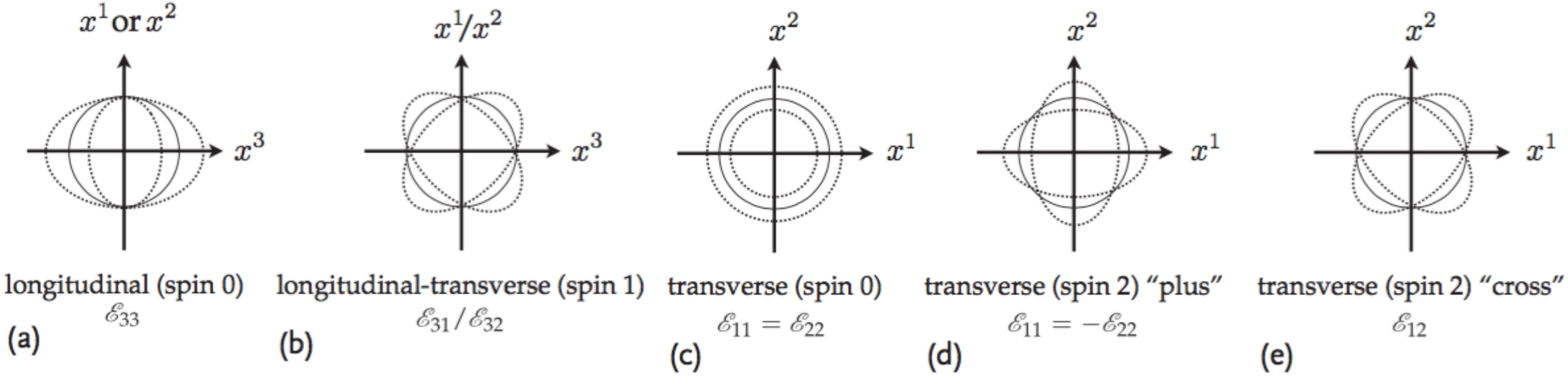
In any metric theory...

$$\frac{\partial^2 h^{\rm gw}_{jk}}{\partial t^2} = -2R_{0j0k} \qquad \delta x^j = \frac{1}{2} h^{\rm gw}_{jk} x^k$$

Integrating, local inertial frame

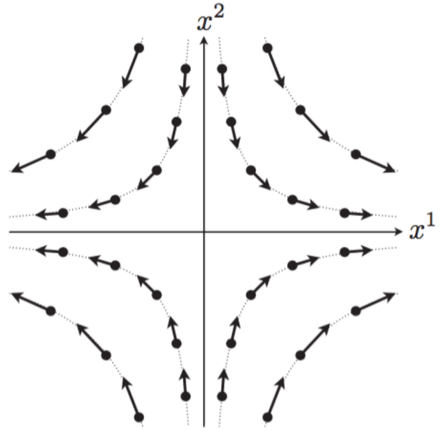
$$R_{0j0k}(t-z)$$

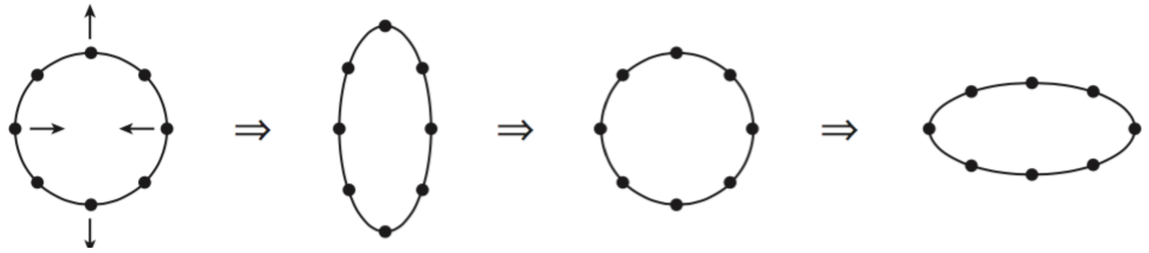
plane wave along *z*

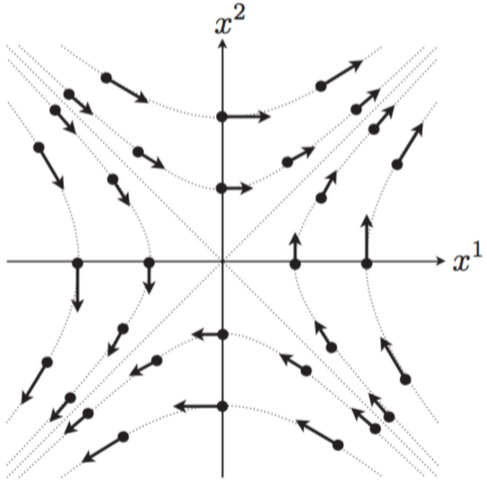


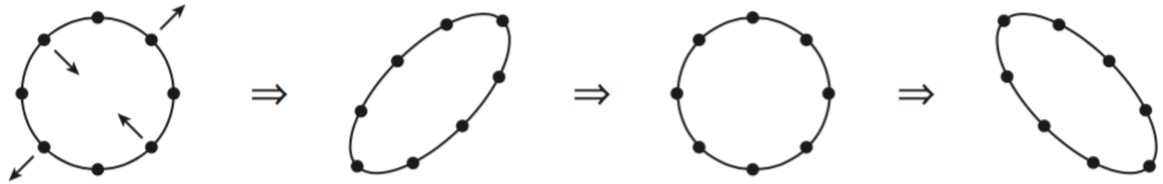
six independent degrees of freedom

...while in GR









$$\mathscr{E}_{ij} = -\frac{1}{2}\ddot{h}_{+}\mathbf{e}_{+} - \frac{1}{2}\ddot{h}_{\times}\mathbf{e}_{\times}$$

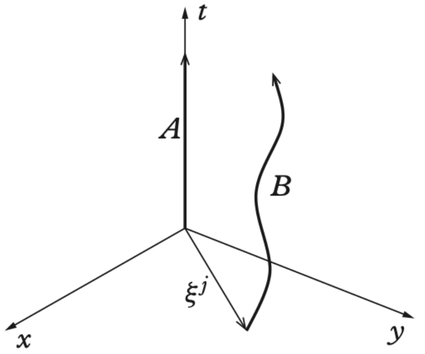
$$\mathbf{e}_{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_{\times} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{e}_{+} \rightarrow \mathbf{e}'_{+} = \mathbf{e}_{+}\cos 2\phi - \mathbf{e}_{\times}\sin 2\phi$$
$$\mathbf{e}_{\times} \rightarrow \mathbf{e}'_{\times} = \mathbf{e}_{\times}\cos 2\phi + \mathbf{e}_{+}\sin 2\phi$$

In a proper reference frame...

$$ds^2 = -(1 + R_{j0k0}x^j x^k)dt^2 - \frac{4}{3}R_{jkl0}x^k x^l dt dx^j + (\delta_{ij} - \frac{1}{3}R_{ijlm}x^l x^m)dx^i dx^j$$

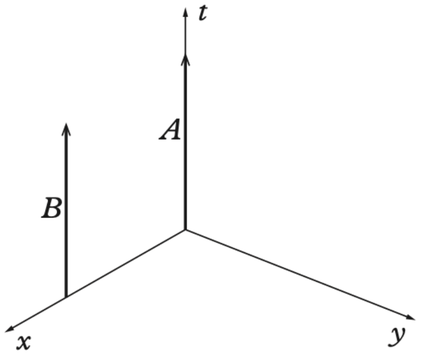
$$\frac{\partial^2 \xi^j}{\partial t^2} = -R^j{}_{0k0}\xi^k = \frac{1}{2}\ddot{h}^j_k \xi^k \rightarrow (\Delta s)^2 = (\Delta x)^2 = \left(L + \frac{1}{2}h_+ L\right)^2 \rightarrow \Delta s \approx \left(1 + \frac{h_+}{2}\right)L$$

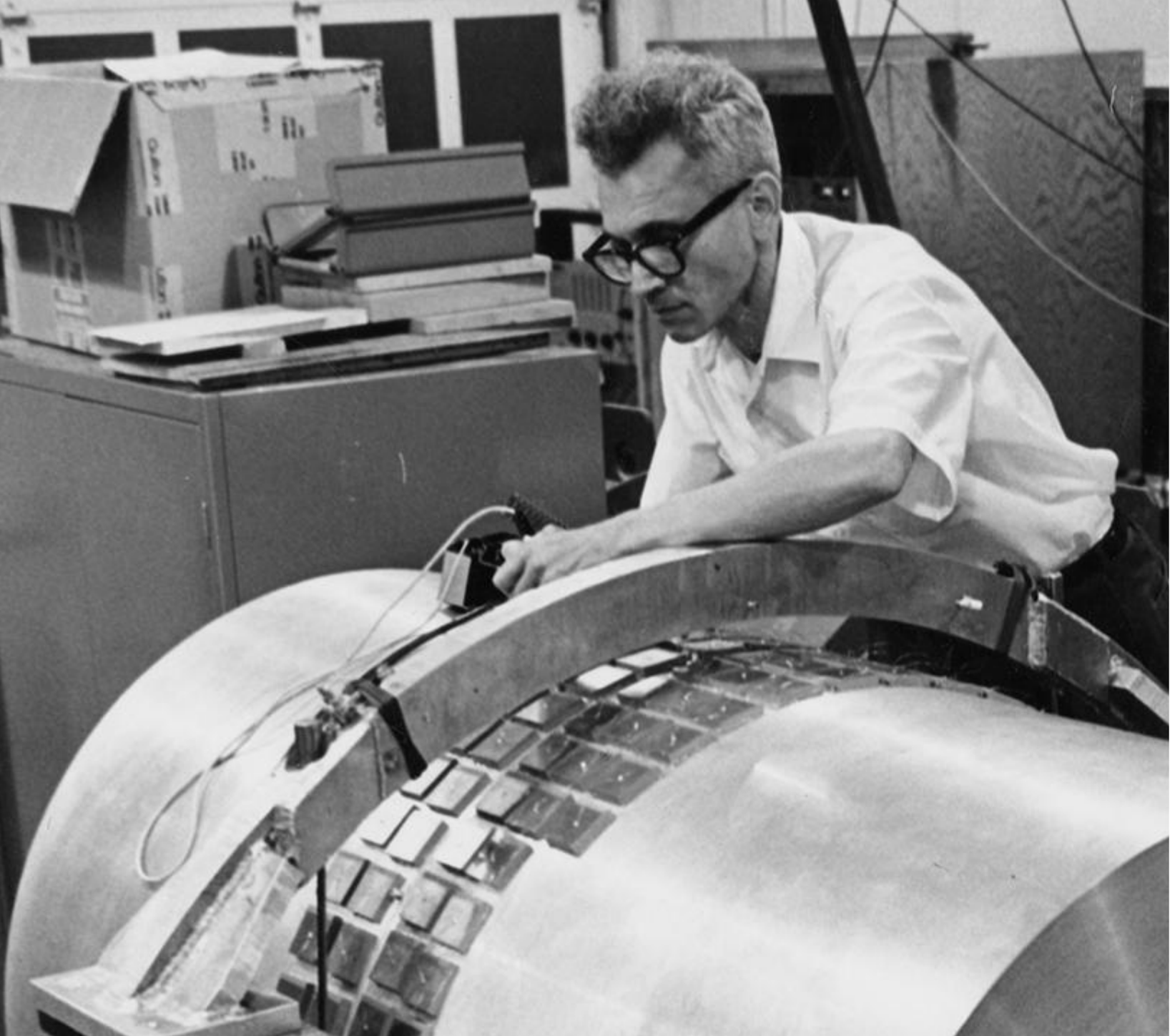


...while in the global TT frame

$$ds^2 = -dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + 2h_\times dx dy + dz^2$$

$$\frac{\partial^2 \xi^j}{\partial t^2} = 0 \qquad \rightarrow (\Delta s)^2 = (1 + h^{\mathrm{TT}}_{xx})(\Delta x)^2 \rightarrow \Delta s \approx \left(1 + \frac{h_+}{2}\right)L$$





Detectors

effective length
for fundamental mode

$$\ddot{\xi} + \tau^{-1}\dot{\xi} + \omega_0\xi = \frac{1}{2}\ddot{h}\,L = \frac{1}{2}\omega^2 h\,\mathrm{e}^{i\omega t}L$$

$$\xi = \frac{1}{2}h\,\mathrm{e}^{i\omega t}L\frac{\omega^2}{\omega_0^2 - \omega^2 + i\tau^{-1}\omega}$$

$$\xi_{\max} = \frac{1}{2}\omega_0\,\tau\,h\,L$$

$$h_{\min} \lesssim \frac{1}{\omega_0 LQ}\left(\frac{15k_BT}{M}\right)^{1/2}$$

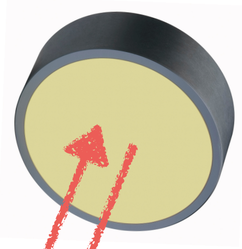
Joseph Weber, 1919–2000

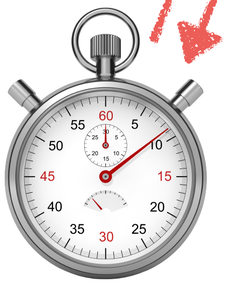
Gravitational-wave detector

pulsar timing

$$L = L_{\text{no gw}} + \frac{1}{2}\int (n \cdot h(\lambda) \cdot n)\, d\lambda$$

$$\frac{\Delta f}{f} = \frac{1}{2}\frac{n \cdot [h(z_S) - h(z_R)] \cdot n}{1 - k \cdot n}$$

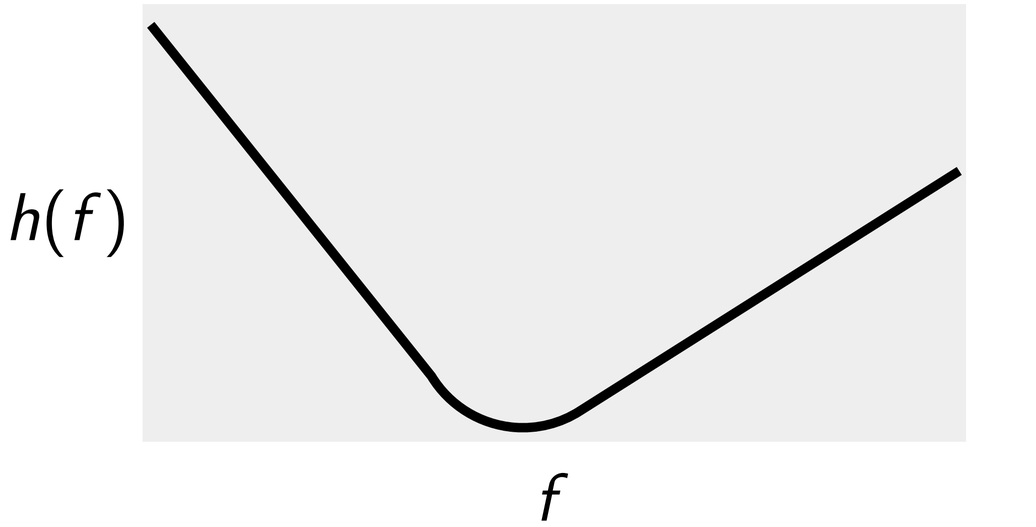




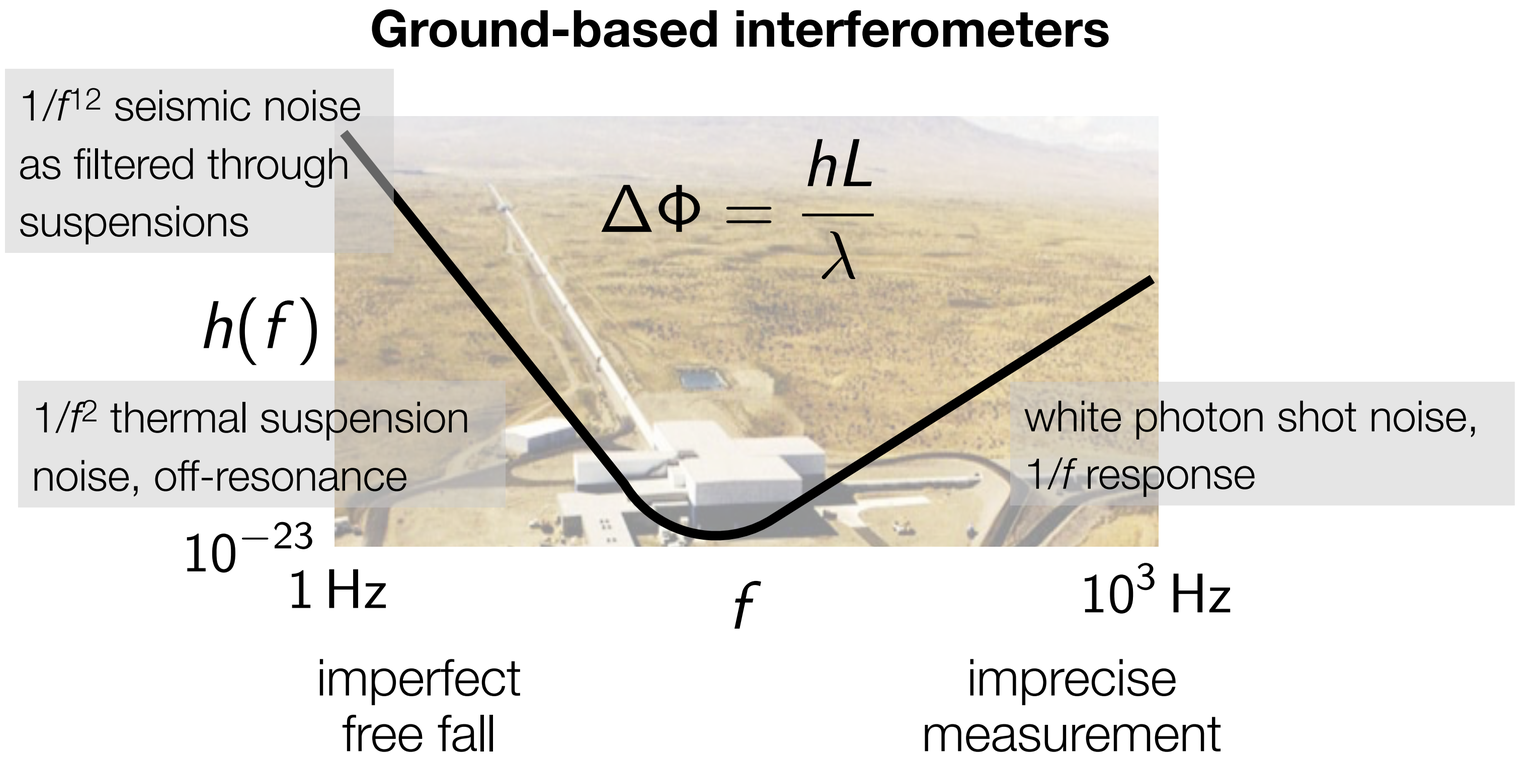
Doppler tracking,
eLISA, LIGO

Gravitational-wave detector sensitivity

Universal: “it must get better before it gets worse”

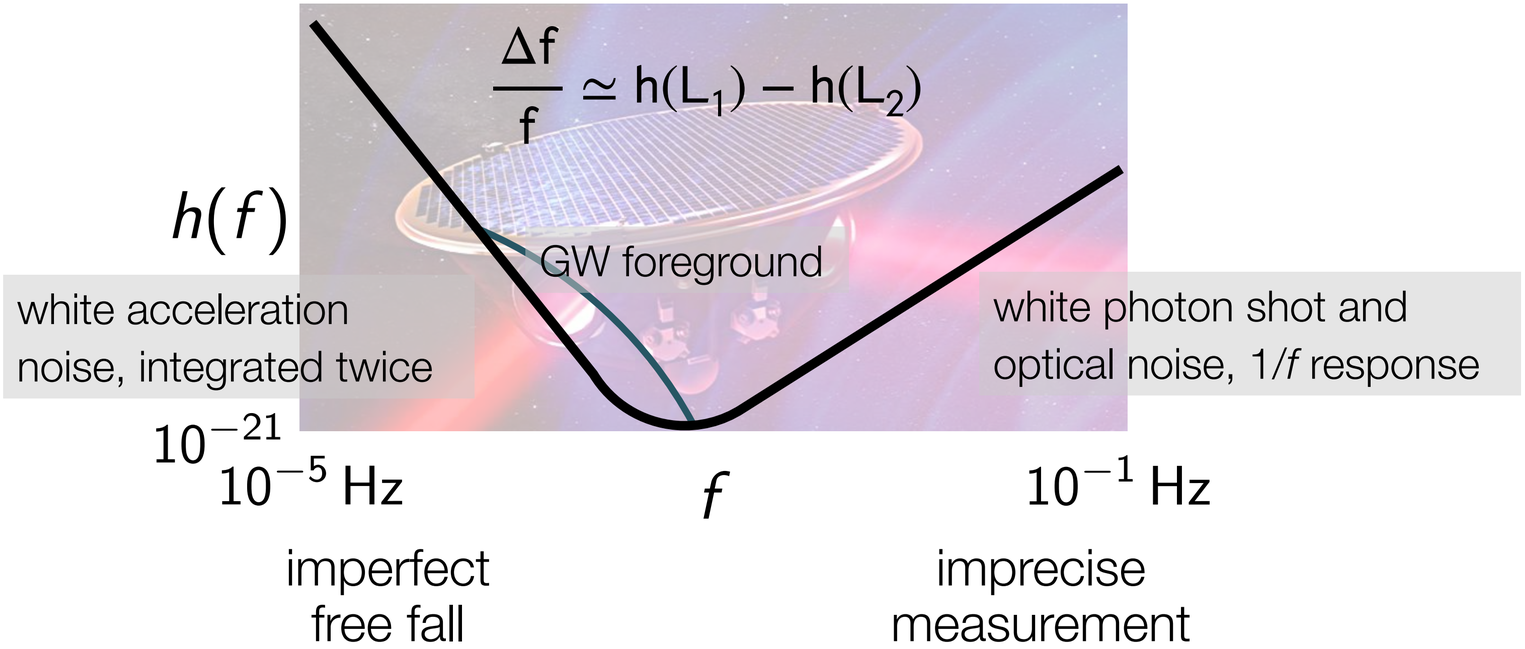


Gravitational-wave detector sensitivity

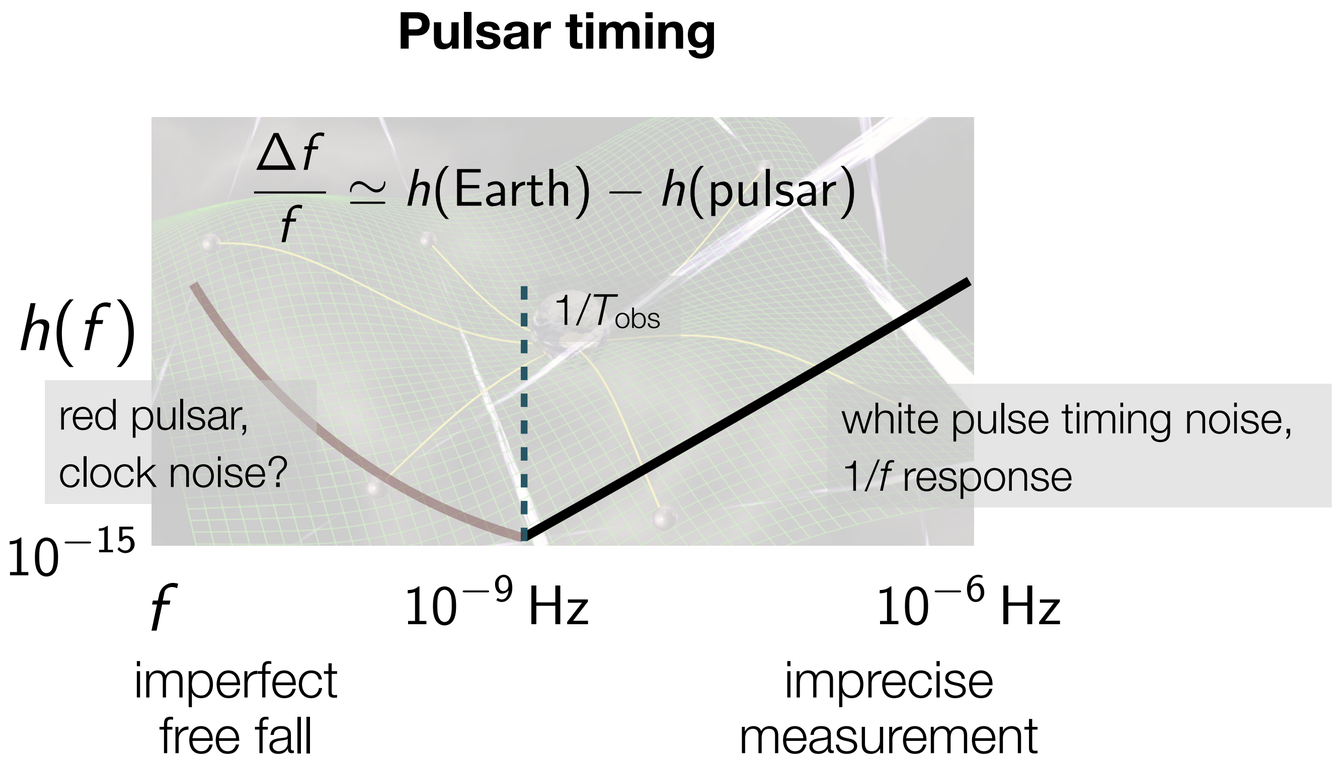


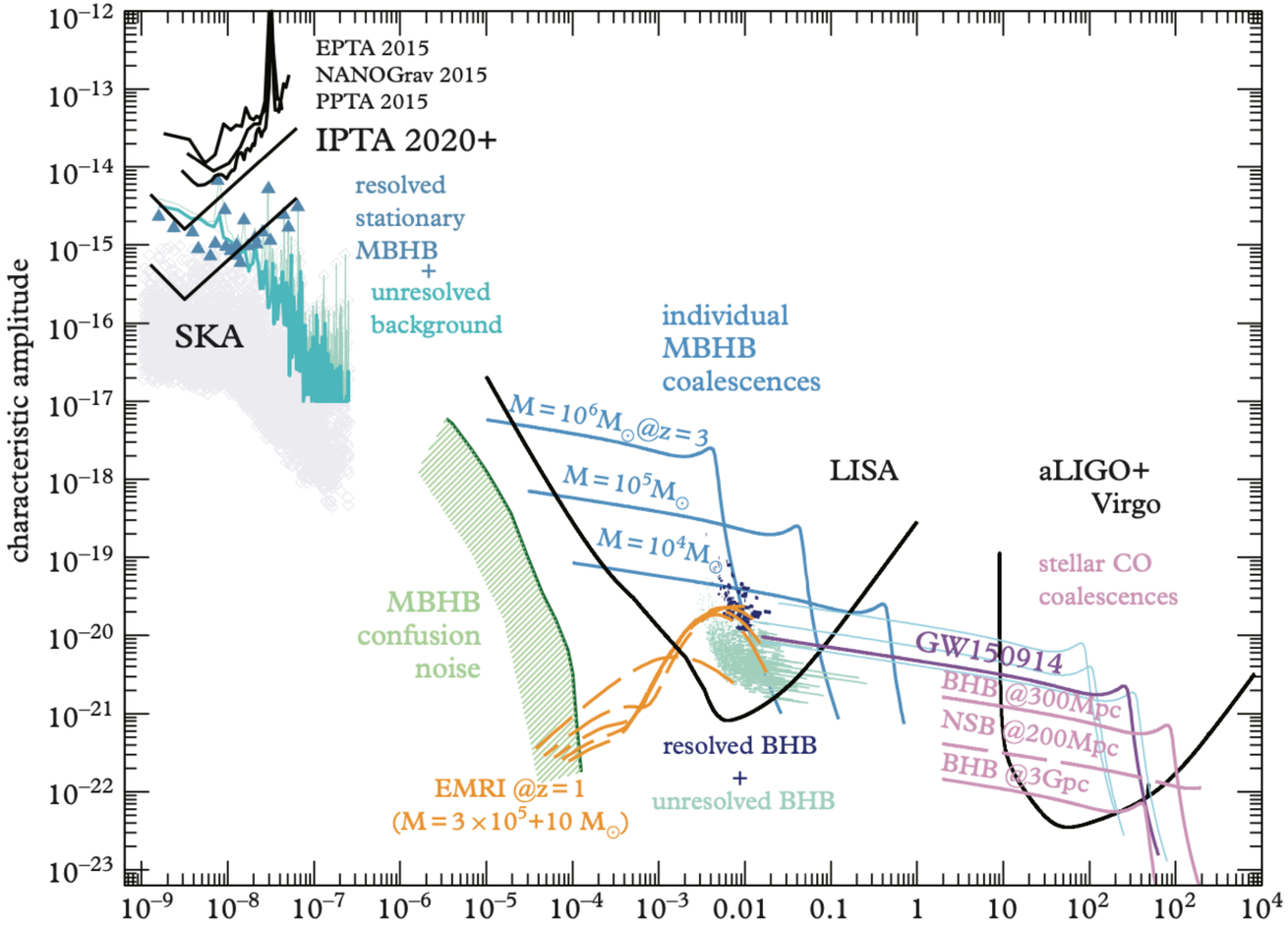
Gravitational-wave detector sensitivity

Space-based interferometers

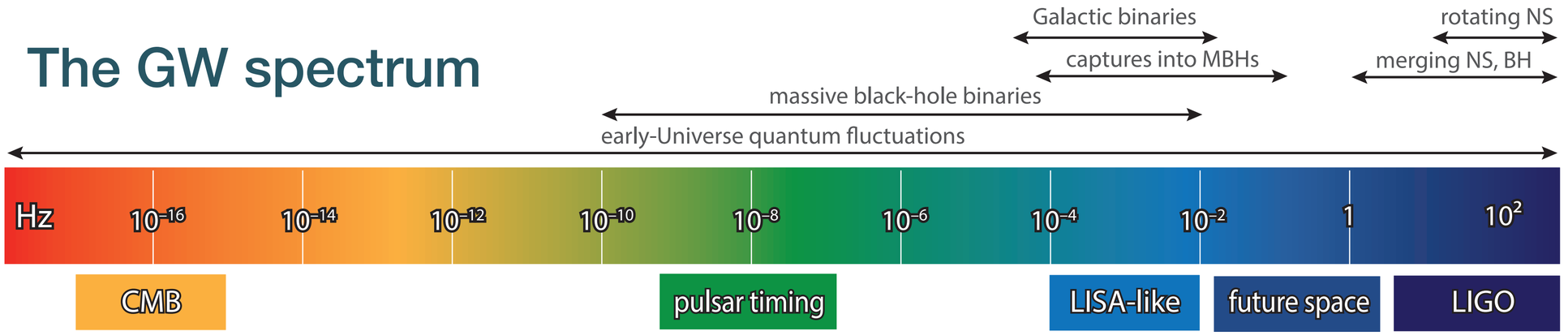


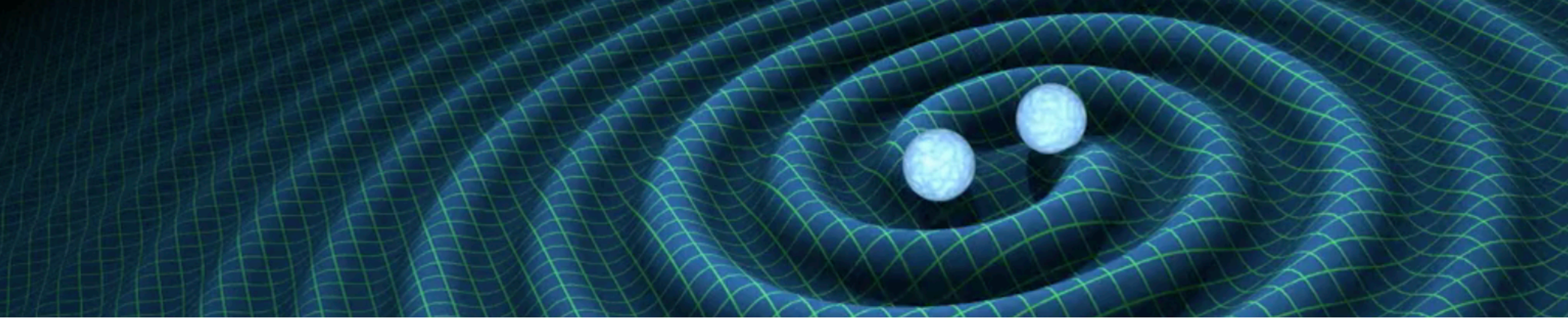
Gravitational-wave detector sensitivity





The GW spectrum





Gravitational waves in a nutshell—1

The *Event*. GWs four ways. Falling particles. Universal theory of detectors.

Michele Vallisneri, Jet Propulsion Laboratory

CEA/IPhT, June 3 2022