PERTURBING ISORADIAL TRIANGULATIONS

FRANÇOIS DAVID AND JEANNE SCOTT

Abstract. We consider an infinite, planar, Delaunay graph $\mathbf{G}_\epsilon$ which is obtained by locally deforming the coordinate embedding of a general, isoradial graph $\mathbf{G}_{\mathrm{cr}}$, with respect to a real deformation parameter ϵ. This entails a careful analysis of Whitehead edge-flips induced by the deformation and the Delaunay constraints. Using R. Kenyon's exact and asymptotic results for the Green's function on an isoradial graph, we calculate the leading asymptotics of the first and second order terms in the perturbative expansion of the log-determinant of the Beltrami-Laplace operator $\Delta(\epsilon)$, the David-Eynard Kähler operator $\mathcal{D}(\epsilon)$, and the conformal Laplacian $\underline{\mathbf{\Delta}}(\epsilon)$ on the deformed Delaunay graph $\mathbf{G}_\epsilon$. We show that the scaling limits of the second order *bi-local* term for both the Beltrami-Laplace and David-Eynard Kähler operators exist and coincide, with a shared value independent of the choice of initial isoradial graph $\mathbf{G}_{\mathrm{cr}}$. Our results allow us to define a discrete analogue of the stress energy tensor for each of the three operators. Furthermore we can identify a central charge ($c = -2$) in the case of both the Beltrami-Laplace and David-Eynard Kähler operators. While the scaling limit is consistent with the stress-energy tensor and value of the central charge for the Gaussian free field (GFF), the discrete central charge value of $c = -2$ for the David-Eynard Kähler operator is, however, at odds with the value of $c = -26$ expected by Polyakov's theory of 2D quantum gravity; moreover there are problems with convergence of the scaling limit of the discrete stress energy tensor for the David-Eynard Kähler operator. The second order bi-local term for the conformal Laplacian involves anomalous terms corresponding to the creation of discrete *curvature dipoles* in the deformed Delaunay graph $\mathbf{G}_\epsilon$; we examine the difficulties in defining a convergent scaling limit in this case. Connections with some discrete statistical models at criticality are explored.

Contents

1. Introduction

1.1. **Laplacians in continuous and discrete metrics.**

Laplacian and Dirac operators in 2 dimensional metrics are interesting in physics and in mathematics: index theorems, Seeley-DeWitt heat kernel expansions and local curvature properties, trace formulas, as well as in theoretical physics: conformal field theories, quantum gravity, string theory, statistical mechanics (SAW, SLE, CLE), etc. Beyond heat-kernels and Green functions, functional Determinants can be suitably defined (for instance by QFT inspired renormalization methods of infinite dimensional Gaussian integrals).

The continuous Beltrami-Laplace operator Δ (acting on scalar functions ϕ over a Riemannian manifold M with metric $g_{\mu\nu}$) is

$$\Delta = -\frac{1}{\sqrt{g}}\partial_\mu \sqrt{g}\, g^{\mu\nu}\, \partial_\nu \tag{1.1}$$

with ∂_μ the standard derivative w.r.t. the local coordinate x^μ. Its normalized determinant can be properly defined, for example, by the functional integral for a massless scalar Gaussian Free Field ϕ (GFF), i.e. the partition function written schematically as

$$Z = \det(\Delta)^{-\frac{1}{2}} = \int \mathcal{D}[\phi]\, e^{-\phi\cdot\Delta\phi} \tag{1.2}$$

which depends explicitely on the metric g. In two dimensions, the GFF is a two dimensional conformal theory (2D CFT). The effect of varying the metric is encoded into the stress-energy tensor. For 2D CFT's its non zero components are its holomorphic components $T = T^{zz}$ and its anti-holomorphic component $\overline{T} = T^{\bar z\bar z}$ (in suitable complex coordinates), which encode the effect of metric changes under infinitesimal anti-holomorphic diffeomorphisms

$$z \to z + \epsilon\, F(\bar z) \qquad \Longrightarrow \qquad g_{zz} \to g_{zz} + \epsilon\, \partial \bar F \tag{1.3}$$

Many properties of CFTs follows from the short distance operator product expansion (OPE) of T. The OPE for the product $T(z)T(z')$ implies that the second variation of the logarithm of partition function $\log Z$ under 1.3 is

$$\frac{c}{4\,\pi^2}\iint d^2u\, d^2v\ \frac{\bar\partial F(u)\,\bar\partial F(v)}{(u-v)^4} + \frac{\partial \bar F(u)\,\partial \bar F(v)}{(\bar u-\bar v)^4} + \text{contact terms} \tag{1.4}$$

where c is the central charge of the CFT. The central charge of the GFF is $c = 1$. Another functional determinant, which is important in string theory and quantum gravity, is the Faddeev-Popov determinant of the differential operator associated to the conformal gauge fixing gauge in Polyakov's theory of two dimensional quantum gravity (a.k.a. non critical string theory). It is associated to a 2D CFT involving (b, c) ghost-antighost field system, with central charge $c = -26$, and is related to the celebrated Liouville CFT.

Discretizations of continuous geometries by simplicial graphs or lattices (locally s in 2D) are ubiquitous in pure and applied mathematics, in computer science as well as in theoretical physics, from classical and quantum gravity to condensed matter. Thus defining and studying discretized analogs of differential operators such as the Laplacian Δ or the Dirac operator $\not{D}$, on such graphs, and of their functional determinants, is important and there is a large literature on these problems.

In two dimensions a specific but especially important class of planar graphs is the class of *isoradial regular graphs embeddings*. On such graphs, the concept of discrete analyticity and discrete analytic functions can be properly defined (this allows to define discrete analogs of 2D CFT's). In particular, in [Ken02], Kenyon showed how these properties allow to compute explicitely the determinant and the Green function (the inverse) of the discrete ∂-operator (the Dirac operator), as well as for the discrete critical Laplacian $\Delta = \partial\bar{\partial}$. As we shall see the class of isoradial graphs is an analogue of flat metrics.

In this work we study (for reason to be explained later) the deformation of these isoradial lattices into Delaunay triangulations, and the effect of such deformations on several discretized Laplacian-like differential operators and their determinants.

1.2. **The random Delaunay triangulation model.**

Delaunay graphs. Delaunay triangulations in the plane are models of discrete space which has been studied by many authors, in particular in high energy physics [CFL82] and well as in statistical physics, condensed matter and soft matter physics. Anticipating the precise definitions and details given in section 1.2, we recall that

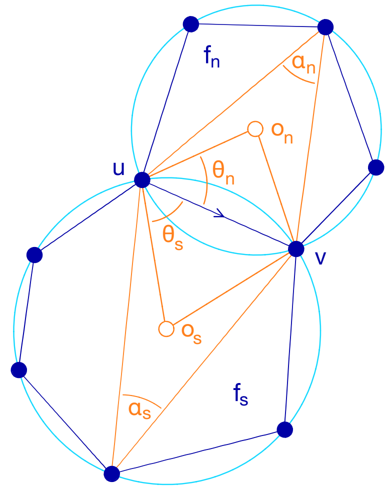

FIGURE 1. To be explained, maybe redrawn

a *polyhedral graph* $\mathtt{G}$ is a planar graph (with finitely or infinitely many vertices) equipped with an embedding $z : \mathrm{V}(\mathtt{G}) \longrightarrow \mathbb{C}$ of its vertex set $\mathrm{V}(\mathtt{G})$ such that edges are mapped to straight line segments and faces are mapped to convex, cyclic polygons. Accordingly we can associate with each face $\mathtt{f}$ of $\mathtt{G}$ the circumcircle $C_\mathtt{f}$, the circumdisk $D_\mathtt{f}$, and the corresponding circumradius $R(\mathtt{f})$ of its cyclic polygon with respect to the embedding. A polyhedral graph $\mathtt{G}$ is a *Delaunay graph* if under the embedding, (1) the interior of the circumdisk of each face of $\mathtt{G}$ contains no vertices, and (2) no two faces share the same circumdisk. Equivalently the dual of Delaunay graph $\mathtt{G}$ is the Voronoi complex $\mathtt{V}$ associated to the set of (embedded) vertices of $\mathtt{G}$. We say that a polyhedral graph $\mathtt{G}$ is a *weak Delaunay graph* as long as condition (1) is satisfied. A (weak) Delaunay triangulation $\mathtt{T}$ is just a (weak) Delaunay graph whose faces are all triangles.

Each edge $\mathtt{e}$ in a polyhedral graph $\mathtt{G}$ has an associated *conformal angle* $\theta(\mathtt{e})$ defined as follows [DE14]. To an oriented edge $\vec{\mathtt{e}} = (\mathtt{u}, \mathtt{v})$ of $\mathtt{G}$ we associate "north" and "south" faces $\mathtt{f}_{\mathrm{n}}$ and $\mathtt{f}_{\mathrm{s}}$ together with angles $\theta_{\mathrm{n}}(\vec{\mathtt{e}}) = \angle \mathtt{vuo}_{\mathrm{n}}$ and $\theta_{\mathrm{s}}(\vec{\mathtt{e}}) = \angle \mathtt{o}_{\mathrm{s}}\mathtt{uv}$ where $\mathtt{o}_{\mathrm{n}}$ and $\mathtt{o}_{\mathrm{s}}$ are the circumcenters of $\mathtt{f}_{\mathrm{n}}$ and $\mathtt{f}_{\mathrm{s}}$ respectively, as depicted in fig. 1. Reversing the orientation of $\vec{\mathtt{e}}$ interchanges the roles of north and south. The conformal angle associated to the unoriented edge $\mathtt{e}$ is $\theta(\mathtt{e}) = (\theta_{\mathrm{n}}(\vec{\mathtt{e}})+\theta_{\mathrm{s}}(\vec{\mathtt{e}}))/2$. The Delaunay condition ensures that $0 < \theta(\mathtt{e}) < \pi$ while the weak Delaunay condition ensures that $0 \le \theta(\mathtt{e}) < \pi$.

Finally, as explained in [DE14] and in section 2, to each plane Delaunay graph $\mathtt{G}$ we can associate an abstract "rhombic surface" $\mathtt{S}_{\mathtt{G}}^{\Diamond}$ obtained by gluing rhombi $\Diamond(\mathtt{e})$ associated to the edges $\mathtt{e}$ of $\mathtt{G}$ according to the incidence relations of $\mathtt{G}$. Each rhomb $\Diamond(\mathtt{e})$ has unit edge length and has a corresponding rhombus angle $2\theta(\mathtt{e})$. We view $\mathtt{S}_{\mathtt{G}}^{\Diamond}$ as a discretized Riemann surface with curvature concentrated at certain vertices. This rhombic surface $\mathtt{S}_{\mathtt{G}}^{\Diamond}$ will be "flat", i.e. can be isometrically embedded in the plane, *if and only if* for each face $\mathtt{f}$ of $\mathtt{G}$, the sum of the conformal angles of the edges $\mathtt{e}$ which form the boundary of $\mathtt{f}$ equals $\pi/2$

$$\sum_{\mathtt{e}\in\partial\mathtt{f}} \theta(\mathtt{e}) = \pi/2 \tag{1.5}$$

Equivalently, the Delaunay graph $\mathtt{G}$ is *isoradial*, i.e. the circumradii $R(\mathtt{f})$ are all equal. Alternatively a Delaunay graph $\mathtt{G}$ is isoradial if and only if $\mathtt{S}_{\mathtt{G}}^{\Diamond}$ coincides with the planar bipartite *kite graph* $\mathtt{G}^{\Diamond}$ discussed in section 2.1. Isoradial Delaunay graphs are also referred to as *"flat"* or *"critical"* graphs.

The David-Eynard random Delaunay model. The David-Eynard model [DE14] is (schematically) a theory of random (finite) Delaunay graphs which are sampled (with Lebesgue measure) according to the conformal angle values of the corresponding edges. By the Voronoi construction, a configuration of $N+3$ distinct marked points in the plane $\{x_{\mathtt{v}} + iy_{\mathtt{v}} \,|\, 1 \le \mathtt{v} \le N+3\}$ is equivalent to a Delaunay graph $\mathtt{G}$ with vertex set $\mathrm{V}(\mathtt{G}) = [1, \ldots, N+3]$ and embedding $z(\mathtt{v}) := x_{\mathtt{v}} + iy_{\mathtt{v}}$. Three of these points can be fixed by the action of $\mathrm{PSL}_2(\mathbb{C})$ as the model is conformally invariant. Under this formulation the relevant measure on the space of configurations of marked points is

$$\prod_{\mathtt{v}=1}^{\mathrm{N}} dx_{\mathtt{v}}\, dy_{\mathtt{v}} \det{}'\mathcal{D} \tag{1.6}$$

where $\mathcal{D}$ is the David-Eynard discrete Kähler operator of the graph $\mathtt{G}$ as defined in 1.9 below, and $\det'\mathcal{D}$ is a reduced determinant (i.e. the leading $N \times N$ principle minor) which suppresses the effect of the zero modes of $\mathcal{D}$, see [DE14]. The Delaunay graph of a generic configuration of points will be a triangulation while the subset of non-triangulations has measure zero. For this reason we speak of the David-Eynard model as a theory of random triangulations. As shown in [CDE19], the measure in 1.6 coincides with the Weil-Petersson measure on $\mathcal{M}_{0,N}$.

The three operators on Delaunay graphs. In this paper, we are interested in the three discrete operators defined on generic polyhedral graphs $\mathtt{G}$: the Beltrami-Laplace operator Δ, the conformal Laplacian $\underline{\boldsymbol{\Delta}}$, and the David-Eynard Kähler operator $\mathcal{D}$. All three operators act on the space $\mathbb{C}^{\mathrm{V}(\mathtt{G})}$ consisting of complex valued functions supported on the vertices $\mathrm{V}(\mathtt{G})$ of the graph $\mathtt{G}$.

• The discrete *Beltrami-Laplace operator* Δ is defined for $\phi \in \mathbb{C}^{\mathrm{V}(\mathtt{G})}$ by

$$\Delta\phi(\mathtt{u}) = \sum_{\text{edges } \vec{\mathtt{e}}=(\mathtt{u},\mathtt{v})} c(\vec{\mathtt{e}})(\phi(\mathtt{u})-\phi(\mathtt{v})) \quad , \quad c(\vec{\mathtt{e}}) = \frac{1}{2}\big(\tan\theta_{\mathrm{n}}(\vec{\mathtt{e}})+\tan\theta_{\mathrm{s}}(\vec{\mathtt{e}})\big) \tag{1.7}$$

This is a standard discretization of the Laplacian in the plane, both in physics (see e.g. [CFL82]) and in mathematics.

• The *conformal Laplacian* $\underline{\mathbf{\Delta}}$, that we introduce here, is defined as

$$\underline{\mathbf{\Delta}}\phi(\mathtt{u}) = \sum_{\text{edges } \vec{\mathtt{e}}=(\mathtt{u},\mathtt{v})} \tan\theta(\mathtt{e})\big(\phi(\mathtt{u}) - \phi(\mathtt{v})\big) \tag{1.8}$$

It is invariant under global conformal transformations $z \overset{g}{\mapsto} \frac{az+b}{cz+d}$ of the graph embedding $z : \mathrm{V}(\mathtt{G}) \longrightarrow \mathbb{C}$ for $g \in \mathrm{PSL}_2(\mathbb{C})$. It's worth noting that $\underline{\mathbf{\Delta}}$ can be viewed as the discrete Laplace-Beltrami operator defined not on the planar graph $\mathtt{G}$, but rather on the image of $\mathtt{G}$ inside the rhombic surface $\mathtt{S}^{\Diamond}_{\mathtt{G}}$ (i.e. the black vertices of $\mathtt{S}^{\Diamond}_{\mathtt{G}}$ where two black vertices are joined by an edge if and only if they lie on a common rhomb. We point the reader a related construction in [Mer01]. As such, $\underline{\mathbf{\Delta}}$ is a discretization of the Beltrami-Laplace operator on a Riemann surface with respect to a non-flat metric.

• The *Kähler operator* $\mathcal{D}$ has been introduced in [DE14]. It is defined in term of the geometry of the graph $\mathtt{G}$ as

$$\mathcal{D}\phi(\mathtt{u}) = \sum_{\text{edges } \vec{\mathtt{e}}=(\mathtt{u},\mathtt{v})} \frac{1}{2}\left(\frac{\tan\theta_{\mathrm{n}}(\vec{\mathtt{e}})+i}{R^2_{\mathrm{n}}(\vec{\mathtt{e}})} + \frac{\tan\theta_{\mathrm{s}}(\vec{\mathtt{e}})-i}{R^2_{\mathrm{s}}(\vec{\mathtt{e}})}\right)\big(\phi(\mathtt{u}) - \phi(\mathtt{v})\big) \tag{1.9}$$

where $R_{\mathrm{n}}(\vec{\mathtt{e}})$ and $R_{\mathrm{s}}(\vec{\mathtt{e}})$ are the circumradii of the north and south faces $\mathtt{f}_{\mathrm{n}}$ and $\mathtt{f}_{\mathrm{s}}$ adjacent to $\vec{\mathtt{e}}$ respectively. Although not obvious from this definition 1.9, the operator $\mathcal{D}$ transforms covariantly under global conformal PSL(2,$\mathbb{C}$) transformations of the graph embedding, and defines a Kähler metric $dz_{\mathtt{u}}\mathcal{D}_{\mathtt{uv}}d\bar{z}_{\mathtt{v}}$ on the space of Delaunay graphs in the plane.

These three operators can be defined for any polyhedral graph $\mathtt{G}$. The weak Delaunay condition on $\mathtt{G}$ ensures that the three operators are positive semi-definite.

Note that if $\mathtt{G}$ is isoradial, then Δ, $\underline{\mathbf{\Delta}}$ and $R^2\mathcal{D}$ (with R the common circumradius) all coincide, and agree with the critical Laplacian considered in [Ken02], where it is shown that the Green's function on $\mathtt{G}$ (the inverse of the critical laplacian) can be written explicitely in terms of the graph local structure; furthermore, the log-determinant of the critical Laplacian can be computed as a finite sum of local contributions if in addition one assumes the graph is periodic.

1.3. **Why study deformations of critical graphs ?** This work is an extension of [DE14] and [CDE19] and aims at studying the properties of the measure 1.6 in the David-Eynard model. Ideally one would like to understand if it is possible to define precisely a continuum limit for this discrete model of random plane geometry, and what is its relation with the well known continuum models of quantum two-dimensional geometry, in particular the Liouville CFT. As a first step, we attempt to study how the measure deforms under small variations of the positions of the points in the plane, starting from initial configurations of points which form critical graphs, i.e. isoradial Delaunay graphs. Indeed for these critical configurations of points, the properties of $\mathcal{D}$ are very well known, in particular through [Ken02]. While our motivating concern is the Kähler operator $\mathcal{D}$, we study in parallel the simpler Laplace operators Δ (corresponding to a discretisation of the Laplace-Beltrami operator in the plane) and $\underline{\boldsymbol{\Delta}}$ (corresponding to the Laplace-Beltrami operator in a discretized curved metric, sharing PSL(2,ℂ) properties with $\mathcal{D}$). The operator Δ is related to a CFT, namely the GFF, and we can ask which CFT properties of Δ are shared by $\underline{\boldsymbol{\Delta}}$ and $\mathcal{D}$, in particular whether theres exists a respective stress-energy tensor and corresponding operator product expansion for each of these operators (see Appendix A).

1.4. **Results.** We consider a Delaunay graph $\mathtt{G}_\epsilon$ obtained by deforming the embedding of a fixed isoradial Delaunay graph $\mathtt{G}_{\mathrm{cr}}$, with radius R_{cr}. Specifically for an isoradial, Delaunay graph $\mathtt{G}_{\mathrm{cr}}$ we define a mapping $z_\epsilon : \mathrm{V}(\mathtt{G}_{\mathrm{cr}}) \longrightarrow \mathbb{C}$ by

$$z_\epsilon(\mathtt{v}) \;:=\; z_{\mathrm{cr}}(\mathtt{v}) \,+\, \epsilon\, F(\mathtt{v}) \tag{1.10}$$

where $\epsilon \geq 0$ is a deformation parameter, where $z_{\mathrm{cr}} : \mathrm{V}(\mathtt{G}_{\mathrm{cr}}) \longrightarrow \mathbb{C}$ is the initial embedding, and where $F : \mathrm{V}(\mathtt{G}_{\mathrm{cr}}) \longrightarrow \mathbb{C}$ is a *function with finite support* $\Omega_F \subset \mathrm{V}(\mathtt{G}_{\mathrm{cr}})$. By definition, the vertex sets $\mathrm{V}(\mathtt{G}_\epsilon)$ and $\mathrm{V}(\mathtt{G}_{\mathrm{cr}})$ coincide while the edges of the graph $\mathtt{G}_\epsilon$ are determined by imposing Delaunay constraints on the configuration of points $\big\{z_\epsilon(\mathtt{v}) \,\big|\, \mathtt{v} \in \mathrm{V}(\mathtt{G}_\epsilon)\big\}$. Finally, it will be convenient to introduce the *lattice closure* $\overline{\Omega}_F$ of Ω_F defined (for generic polyhedral graph $\mathtt{G}$) as

$$\overline{\Omega}_F = \{\mathtt{v} \in \mathrm{V}(\mathtt{G}) : \mathtt{v} \text{ shares a face } \mathtt{f} \in F(\mathtt{G}) \text{ with a vertex } \mathtt{u} \in \Omega_F\} \tag{1.11}$$

The following technical lemma allows us regulate the behaviour of the perturbation by introducing a bound on the deformation parameter. Specifically

Lemma 1. *Let* $\mathtt{G}_{\mathrm{cr}}$ *be an isoradial Delaunay graph, and* F *a deformation function as above. There is a threshold* $\tilde{\epsilon}_F > 0$ *such that whenever* $0 \leq \epsilon < \tilde{\epsilon}_F$

1. $z_\epsilon : \mathrm{V}(\mathtt{G}_{\mathrm{cr}}) \longrightarrow \mathbb{C}$ *is an embedding*
2. *there is an inclusion of edge sets* $\mathrm{E}(\mathtt{G}_{\mathrm{cr}}) \subseteq \mathrm{E}(\mathtt{G}_\epsilon)$
3. *the edge sets are stable, i.e.* $\mathrm{E}(\mathtt{G}_{\epsilon_1}) = \mathrm{E}(\mathtt{G}_{\epsilon_2})$ *whenever* $0 < \epsilon_1, \epsilon_2 < \tilde{\epsilon}_F$

Conditions (1) and (3) ensure the existence of a right-sided limit graph which is both isoradial and (weakly) Delaunay, namely

Definition 1. *The* ***isoradial refinement*** $\mathtt{G}_{0^+}$ *of* $\mathtt{G}_{\mathrm{cr}}$ *determined by* F *is the isoradial (weak) Delaunay graph with vertex set* $\mathrm{V}(\mathtt{G}_{0^+}) := \mathrm{V}(\mathtt{G}_{\mathrm{cr}})$ *and embedding* $z_{0^+} := z_{\mathrm{cr}}$ *whose edge set is given by*

$$\mathrm{E}(\mathtt{G}_{0^+}) := \lim_{\epsilon \to 0^+} \mathrm{E}(\mathtt{G}_\epsilon)$$

Note that $\mathtt{G}_{0^+}$ will be a **weak** Delaunay graph precisely when the inclusion of edge sets is strict, otherwise $\mathtt{G}_{0^+}$ and $\mathtt{G}_{\mathrm{cr}}$ will coincide. It will be convenient to complete $\mathtt{G}_{0^+}$ to a (weak) Delaunay triangulation $\widehat{\mathtt{G}}_{0^+}$ by maximally saturating $\mathrm{E}(\mathtt{G}_{0^+})$ with additional non-crossing edges. The choice of these additional edges will not affect our calculations; this is because the weights assigned to these edges (or *chords* as defined in 5) by the operators Δ, $\mathcal{D}$, and $\underline{\boldsymbol{\Delta}}$ always vanish. In particular $\mathcal{O}$ defined over $\widehat{\mathtt{G}}_{0^+}$, $\mathcal{O}$ defined over $\mathtt{G}_{0^+}$, and $\mathcal{O}$ defined over $\mathtt{G}_{\mathrm{cr}}$ all coincide whenever $\mathcal{O}$ is Δ, $\underline{\boldsymbol{\Delta}}$, or $\mathcal{D}$. We want to emphasize that $\widehat{\mathtt{G}}_{0^+} = \mathtt{G}_{0^+} = \mathtt{G}_{\mathrm{cr}}$ whenever $\mathtt{G}_{\mathrm{cr}}$ is a triangulation.

Definition 2. *In this paper we shall mostly consider* ***smooth local deformations*** *defined as follows. We restrict a smooth (in general non-holomorphic) function* $F : \mathbb{C} \longrightarrow \mathbb{C}$ *with compact support* $\Omega \subset \mathbb{C}$ *to the graph* $\mathtt{G}_{\mathrm{cr}}$ *by declaring*

$$F(\mathtt{v}) \;:=\; F\big(z_{\mathrm{cr}}(\mathtt{v})\big) \tag{1.12}$$

where $\mathtt{v} \in \mathrm{V}(\mathtt{G}_{\mathrm{cr}})$ *is a vertex. Moreover, we shall consider the family of* ***rescaled smooth local deformations*** F_ℓ *defined as*

$$F_\ell(\mathtt{v}) \;:=\; \ell\, F\big(z_{\mathrm{cr}}(\mathtt{v})/\ell\big) \tag{1.13}$$

and where $\ell > 0$ *is a scaling parameter (used for defining a continuum limit). Using the construction above, we obtain a deformed embedding* $z_{\epsilon,\ell}$ *and a Delaunay graph* $\mathtt{G}_{\epsilon,\ell}$ *together with an attending isoradial refinement* $\mathtt{G}_{0^+,\ell}$ *and completion* $\widehat{\mathtt{G}}_{0^+,\ell}$.

Let Δ_{cr} (respectively $\mathcal{D}_{\mathrm{cr}} = \Delta_{\mathrm{cr}}/R^2_{\mathrm{cr}}$) denote the critical laplacian (respectively the Kähler operator) on the initial isoradial triangulation $\mathtt{G}_{\mathrm{cr}}$. Let $\mathcal{O}$ denote either the Beltrami-Laplace operator Δ, the conformal laplacian $\underline{\boldsymbol{\Delta}}$, or the Kähler operator $\mathcal{D}$ of the perturbed triangulation $\mathtt{G}_\epsilon$ and let $\mathcal{O}_{\mathrm{cr}}$ be the corresponding operator on $\mathtt{G}_{\mathrm{cr}}$. We denote the variation $\delta\mathcal{O} = \mathcal{O} - \mathcal{O}_{\mathrm{cr}}$. It is of order $\mathrm{O}(\epsilon)$. Formally we may expand the log-determinant $\log\det\mathcal{O}$ using the Green's function $\mathcal{O}_{\mathrm{cr}}^{-1}$ of the critical operator as

$$\log\det\mathcal{O} \;=\; \log\det\mathcal{O}_{\mathrm{cr}} \;+\; \mathrm{tr}\left[\delta\mathcal{O}\cdot\mathcal{O}_{\mathrm{cr}}^{-1}\right] \;-\; \frac{1}{2}\,\mathrm{tr}\left[(\delta\mathcal{O}\cdot\mathcal{O}_{\mathrm{cr}}^{-1})^2\right] \;+\; \cdots \tag{1.14}$$

The trace terms occuring on the righthand side of equation 1.14 are well defined owing to the fact that support of the perturbation is compact; consequently the difference $\log\det\mathcal{O} - \log\det\mathcal{O}_{\mathrm{cr}}$ is a well defined value. Our main results concerns the second order term $\mathrm{tr}\left[(\delta\mathcal{O}\cdot\mathcal{O}_{\mathrm{cr}}^{-1})^2\right]$ and, more precisely, the cross term contribution coming from variations $\delta\mathcal{O}$ at two distant sites of perturbation. These results are summarized in the following theorem:

Theorem 1. *Consider two complex functions* $F_1(z)$ *and* $F_2(z)$ *whose supports* $\Omega_1 = \mathrm{supp}\, F_1$ *and* $\Omega_2 = \mathrm{supp}\, F_2$ *in the vertex set* $\mathrm{V}(\mathtt{G}_{\mathrm{cr}})$ *are* finite and disjoint *(hence at finite distance), and a bi-local deformation of the embedding*

$$z_{\mathrm{cr}}(\mathtt{v}) \;\mapsto\; z_{\mathrm{cr}}(\mathtt{v}) \;+\; \epsilon_1 F_1\big(\mathtt{v}\big) \;+\; \epsilon_2 F_2\big(\mathtt{v}\big)$$

The $\epsilon_1\epsilon_2$ *cross-term of the perturbative expansion of* $\log\det\Delta$ *is obtained from* $\mathrm{tr}\left[(\delta\Delta\cdot\Delta_{\mathrm{cr}}^{-1})^2\right]$ *and takes the asymptotic form*

$$\frac{c}{\pi^2}\sum_{\substack{\text{triangles}\\ \mathtt{x}_1,\mathtt{x}_2}} A(\mathtt{x}_1)A(\mathtt{x}_2)\left(\mathfrak{Re}\left[\frac{\overline{\nabla}F_1(\mathtt{x}_1)\,\overline{\nabla}F_2(\mathtt{x}_2)}{\big(z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big)^4}\right] \;+\; \mathrm{O}\Big(\big|z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big|^{-5}\Big)\right) \tag{1.15}$$

where $\mathtt{x}_i \in \mathrm{F}(\widehat{\mathtt{G}}_{0^+})$ *is a triangle having at least one vertex in* Ω_i*, whose center has coordinate* $z_{\mathrm{cr}}(\mathtt{x}_i)$*, and whose area is* $A(\mathtt{x}_i)$ *with* $i = 1, 2$*. Moreover* $c = -2$.

Theorem 2. *The $\epsilon_1\epsilon_2$ cross-term of the perturbative expansion of* $\log\det\mathcal{D}$ *is obtained from* $\mathrm{tr}\left[(\delta\mathcal{D}\cdot\mathcal{D}_{\mathrm{cr}}^{-1})^2\right]$ *and takes the same asymptotic form as formula 1.15. In particular $c=-2$.*

Remark 1. *The Beltrami-Laplace operator Δ and the David-Eynard Kähler operator $\mathcal{D}$ have the same central charge $c=-2$. This is expected for the Beltrami-Laplace operator. In the case of the David-Eynard Kähler operator this result conflicts with the value of -26 anticipated by the continuous theory.*

Remark 2. *As noted earlier, the left hand side of equation 1.15 is independent of the choice of triangulation $\widehat{\mathtt{G}}_{0+}$ used to refine $\mathtt{G}_{0+}$ because both graphs share $\mathtt{G}_{\mathrm{cr}}$ as a common regularization (and the three operators $\mathcal{O}$ which we consider depend only on the graph regularization). The right hand side, on the other hand, is independent of $\widehat{\mathtt{G}}_{0+}$ in light of a discretized version of Green's theorem 4 and corollary 1 as detailed in Subsection 3.2.*

Remark 3. *In general, Theorem 1 is not valid for the conformal Laplacian $\underline{\boldsymbol{\Delta}}$. Specifically, formula 1.15 fails to hold when the isoradial refinement $\mathtt{G}_{0+}$ contains* chords*; see 5 for a proper definition. "Anomalous" chord-to-edge and chord-to-chord terms must be added to Formula 1.15 in order to obtain a valid asymptotic formula for the $\epsilon_1\epsilon_2$ cross-term of* $\mathrm{tr}\left[\delta\underline{\boldsymbol{\Delta}}\cdot\Delta_{\mathrm{cr}}^{-1}\right]^2$. *See section 6.3 .*

Formula 1.15 makes use of discrete derivative operators $\nabla,\overline{\nabla}:\mathbb{C}^{\mathrm{V}(\mathtt{T})}\longrightarrow\mathbb{C}^{\mathrm{F}(\mathtt{T})}$ introduced in [DE14] for a general polyhedral triangulation $\mathtt{T}$; see Section 4 for a definition. The following estimate (see Appendix A for proof) explains why ∇ and $\overline{\nabla}$ should be considered as discrete analogues of the holomorphic and anti-holomorphic derivatives ∂ and $\overline{\partial}$.

Lemma 2. *Given a smooth function $\phi:\mathbb{C}\longrightarrow\mathbb{C}$ and a triangle $\mathtt{f}$ with vertices z_1, z_2, z_3 (listed in counter-clockwise order), circumcenter $z(\mathtt{f})$, and circumradius $R(\mathtt{f})$, we have the following estimate*

$$\left|\nabla\phi(\mathtt{f})-\partial\phi(z(\mathtt{f}))\right|\leq R(\mathtt{f})\left(\frac{3}{2}\sup_{z\in B_{\mathtt{f}}}\left|\partial^2\phi\right|+2\sup_{z\in B_{\mathtt{f}}}\left|\partial\overline{\partial}\phi\right|+\frac{1}{2}\sup_{z\in B_{\mathtt{f}}}\left|\overline{\partial}^2\phi\right|\right)\tag{1.16}$$

where $B_{\mathtt{f}}$ is the disk bounded by the circumcircle of $\mathtt{f}$

$$B_{\mathtt{f}}=\{z\,;\,|z-z(\mathtt{f})]\leq R(\mathtt{f})\}\tag{1.17}$$

Using Lemma 2 we are able to formulate a smooth version of Theorem 1 involving a scaling parameter $\ell>0$ as in equation 1.13 whose continuum limit coincides with formula 1.4. More specifically:

Theorem 3. *Consider two smooth complex functions $F_1(z)$ and $F_2(z)$ whose supports $\Omega_1=\mathrm{supp}\,F_1$ and $\Omega_2=\mathrm{supp}\,F_2$ in the plane are* compact and disjoint *(hence at finite distance), and a bi-local deformation of the embedding given by*

$$z_{\mathrm{cr}}(\mathtt{v})\;\mapsto\;z_{\mathrm{cr}}(\mathtt{v})\,+\,\epsilon_1\,\ell\,F_1\Big(\frac{z_{\mathrm{cr}}(\mathtt{v})}{\ell}\Big)\,+\,\epsilon_2\,\ell\,F_2\Big(\frac{z_{\mathrm{cr}}(\mathtt{v})}{\ell}\Big)$$

where $\ell>0$ is a scaling parameter. For $i=1,2$ set $F_{i;\ell}(z):=\ell F_i(z/\ell)$ then

(1.18)

$$\lim_{\ell\to\infty} \frac{c}{\pi^2} \sum_{\substack{\text{triangles} \\ \mathtt{x}_1,\mathtt{x}_2}} A(\mathtt{x}_1)A(\mathtt{x}_2)\left(\mathfrak{Re}\left[\frac{\overline{\nabla}F_{1;\ell}(\mathtt{x}_1)\,\overline{\nabla}F_{2;\ell}(\mathtt{x}_2)}{\big(z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big)^4}\right] + \mathrm{O}\Big(\,\big|z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big|^{-5}\Big)\right)$$

$$= \frac{c}{\pi^2}\iint_{\Omega_1\times\Omega_2} dx_1^2\,dx_2^2\;\mathfrak{Re}\left[\frac{\overline{\partial}F_1(x_1)\,\overline{\partial}F_2(x_2)}{(x_1-x_2)^4}\right]$$

We stress that the limit value in formula 1.18 is independent of the isoradial, Delaunay graph $\mathtt{G}_{\mathrm{cr}}$. *As in Theorem 1 the coefficient* c *in 1.18 is found to be equal to* $c=-2$, *both for* Δ *and* $\mathcal{D}$.

Remark 4. *The sum 1.18 is taken over pairs of triangles* $\mathtt{x}_1$ *and* $\mathtt{x}_2$ *in* $\widehat{\mathtt{G}}_{0+,\ell}$ *such that the coordinate of at least one vertex of* $\mathtt{x}_i$ *is contained in the scaled support* $\Omega_i(\ell) := \operatorname{supp} F_{i;\ell}$ *for* $i=1,2$. *These two domains are disjoints and separated by a distance of order* $\mathrm{O}(\ell)$. *Formula 1.18 is the discrete analog of 1.4.*

Remark 5. *The* $\ell\to\infty$ *scaling limit of the bi-local formula for the* $\epsilon_1\epsilon_2$ *cross-term in* $\operatorname{tr}\big[\delta\underline{\boldsymbol{\Delta}}\cdot\Delta_{\mathrm{cr}}^{-1}\big]^2$ *of the conformal Laplacian* $\underline{\boldsymbol{\Delta}}$ *(as presented in section 6.3) agrees with the limit value in formula 1.18 of Corollary 3 whenever* $\mathtt{G}_{0+,\ell}$ *contains finitely many chords. In this case the central charge of the conformal Laplacian* $\underline{\boldsymbol{\Delta}}$ *takes the expected value of* $c=-2$ *as well.*

Theorem 1 makes use of a sharpened version of a Kenyon's asymptotical formula for the long-range behaviour of the Green function $\Delta_{\mathrm{cr}}^{-1}$ of the critical Laplacian:

Proposition 1. *For any pair of vertices* $\mathtt{u}$ *and* $\mathtt{v}$ *in an isoradial Delaunay graph* $\mathtt{G}$

(1.19)

$$\Big[\Delta_{\mathrm{cr}}^{-1}\Big]_{\mathtt{u},\mathtt{v}} = -\frac{1}{2\pi}\left(\log\Big(2\big|p_1(\mathtt{u},\mathtt{v})\big|\Big) + \gamma_{\mathrm{euler}} + \frac{\mathfrak{Re}\big[p_3(\mathtt{u},\mathtt{v})\big]}{6\big|p_1(\mathtt{u},\mathtt{v})\big|^3} + \mathrm{O}\left(\frac{1}{\big|p_1(\mathtt{u},\mathtt{v})\big|^4}\right)\right)$$

where γ_{euler} *is the Euler-Mascheroni constant, where* $p_1(\mathtt{u},\mathtt{v}) = z_{\mathrm{cr}}(\mathtt{v}) - z_{\mathrm{cr}}(\mathtt{u})$, *and where* $p_3(\mathtt{u},\mathtt{v})$ *is a term defined in section 4. It depends on the local geometry of the graph* T *between* $\mathtt{u}$ *and* $\mathtt{v}$, *but is bounded uniformly and linearly by*

$$\big|p_3(\mathtt{u},\mathtt{v})\big| \;\le\; 3\,\big|z_{\mathrm{cr}}(\mathtt{u}) - z_{\mathrm{cr}}(\mathtt{v})\big|$$

Remark 6. *Proposition 1 sharpens Kenyon's Theorem 7.3 in* [Ken02] *by identifying and obtaining a uniform bound on the first non-constant (indeed quadratic) subdominant term*

$$\frac{1}{6}\,\big|z_{\mathrm{cr}}(\mathtt{u}) - z_{\mathrm{cr}}(\mathtt{v})\big|^{-3}\,\mathfrak{Re}\big[p_3(\mathtt{u},\mathtt{v})\big] \;\le\; \frac{1}{2}\,\big|z_{\mathrm{cr}}(\mathtt{u}) - z_{\mathrm{cr}}(\mathtt{v})\big|^{-2}$$

In fact in Sect. 4 we obtain explicit expressions and uniform bounds for all terms of the large distance asymptotic series expansion of the Green's function.

1.5. **Plan of the paper.** This paper is organised as follows:
Section 1 was the introduction.

Section 2 presents basic concepts about the geometry of planar graphs which are relevant to the paper. Most of the material is standard, however we introduce the notion of a *chord* (see Definition 5) which allows us to slightly broaden the definition of an isoradial triangulation (given in [Ken02]) to accomodate configurations with four or more cocyclic vertices. Section 2.1 gives definitions and sets notation for polyhedral graphs, edges and chords, (weak) Delaunay graphs, isoradial graphs, etc. and makes precise the notions of abstract rhombic surface $\mathsf{S}^{\Diamond}_{\mathsf{G}}$ associated to a polyhedral graph G alluded in Section 1.2. Section 2.2 addresses geometrical concepts and properties of rhombic graphs associated to isoradial graphs, mainly following [Ken02] and [KS14]. In order to help establish the asymptotic formula in Proposition 1 we undertake in Proposition 3 a careful analysis of the interval of possible angles taken by any path in the rhombic graph of an isoradial Delaunay graph.

In section 3 we review the ∇ and $\overline{\nabla}$ operators of [DE14] and how they are used to obtain a "local factorization" of the Laplace-Beltrami and Kähler operators Δ and $\mathcal{D}$ for a general polyhedral triangulation; see remarks 3.19 and 3.18. We remark that the conformal Laplacian $\underline{\mathbf{\Delta}}$ however does not admit a simple, local factorization. Following this, we recall two approaches used to define the (normalised) log-determinant of a Laplace-like operator such as Δ, $\mathcal{D}$, and $\underline{\mathbf{\Delta}}$ for infinite polyhedral graphs which are either (1) doubly periodic or (2) obtained as a nested limit of finite graphs each with Dirichlet boundary conditions. Formulae 3.22 and 3.23 serve respectively as definitions in these two cases. We end the section by discussing Kenyon's local formula in [Ken02] for the normalised log-determinant of the critical laplacian for doubly periodic, isoradial, (weak) Delaunay graphs, as well as its formal extension to the non-periodic case.

In section 4 we derive the long range asymptotic formula for the Green's function of the critical laplacian (associated to an isoradial Delaunay graph) stated in Proposition 1 of Section 1. We rely on the methods of [Ken02] along with some added improvements, in particular for infinite non-periodic graphs. Among other things our analysis provides uniform bounds on the coefficients of the asymptotic expansion (see 5 and 4.11) thus sharpening the results and approximations in [Ken02].

Section 5 addresses deformations of critical isoradial Delaunay graphs and corresponding operators. In section 5.1 we study the first order variation of the Laplace-Beltrami and Kähler operators, when the underlying polyhedral triangulation is subject to a *formal* deformation given by 1.10 without imposing Delaunay constraints. Results are given in propositions 5 and 6 respectively. The conformal Laplacian $\underline{\mathbf{\Delta}}$ does not admit a local factorization of the kind presented in prop. 5 and 6 and for this reason there isn't an analogous formula for its first order variation. Section 5.2 sets up notation. In sections 5.3 and 5.4 we carefully discuss the effect of a *geometric* deformation on a Delaunay graph G_{cr}, where the incidence relations of the perturbed graph G_ϵ are controlled by Delaunay constraints. We explain in lemma 8 how to regulate the deformation parameter $\epsilon \geq 0$ so that edges of initial graph G_{cr} are stable and do not undergo Whitehead "flips". A generic deformation however can break the cyclicity of faces of G_{cr} having four or more vertices and whenever this happens G_ϵ will contain "new edges" which subdivide these faces. Nevertheless these additional edges are shown to be stable as the deformation parameter varies, provided the deformation parameter is bounded appropriately. These results are detailed in lemmas 9 and 10. We show the existence of an isoradial (weak) Delaunay limit graph $\mathtt{G}_{0+}$ together with an isoradial (weak) Delaunay triangulation $\widehat{\mathtt{G}}_{0+}$ which refines $\mathtt{G}_{0+}$ and which is compatible with the deformation.

The calculations of the first and second order variations of the log-determinant for the Beltrami-Laplace operator, the Kähler operator, and the conformal Laplacian are undertaken Section 6. The first order variation formulae are entirely local, i.e. expressed as sums of weights of edges. The second order variations, on the other hand, involve long-range effects of the critical Green's function $\Delta_{\mathrm{cr}}^{-1}$ associated to pairs of distant vertices and, in principle, register aspects of the global geometry of the initial isoradial Delaunay graph $\mathtt{G}_{\mathrm{cr}}$.

In Propositions 7 and 9 of Section 6.1 we present first order variation formulas for the Beltrami-Laplace and Kähler operators which are valid uniformly for all isoradial Delaunay graphs. The first order formula for the conformal Laplacian incorporates an additional term which accounts for the effect made by chords in $\mathtt{G}_{0+}$ and is given in Proposition 8. The second order formulae for the variation of the log-determinant of the Beltrami-Laplace and Kähler operators are calculated separately in Propositions 10 and 11 of Section 6.2 respectively; this is the content of Theorems 1 and 2. In both cases, our approach relies on the asymptotics of the Green's function in Proposition 1 and Lemma 12 — the latter makes use of the operator factorisations in Propositions 5 and 6 as well as a novel estimate presented in Lemma 11.

Formula 1.15 of Theorems 1 and 2 is not valid for the conformal Laplacian and it must be modified by defect terms which take into account the effect of chords in $\mathtt{G}_{0+}$. See formulae 6.58 and 6.59. We propose that this defect is indicative of a discrete curvature anomaly arising from the perturbation; this is examined in Section 6.4.

Section 7 deals with the existence and value of the scaling limit of formula 1.15 for the Beltrami-Laplace and Kähler operators. We begin section 7.2 by addressing some technical points about bi-local deformations, scaling limits, and re-summation. In Section 7.3 we prove the existence of the scaling limit of 1.15 in the case of a continuous bi-local deformation and settle Theorem 3. The basic idea is to interpret 1.15 as a Riemann sum with a mesh controlled by the scaling parameter. The scaling limit considered in Section 7.3 is taken with respect an isoradial refinement $\widehat{\mathtt{G}}_{0+,\ell}$ associated to a (scaled) deformation of our initial, isoradial Delaunay graph $\mathtt{G}_{\mathrm{cr}}$. In effect the result is a calculation of a nested limit: First we take the deformation parameter limit $\epsilon_1, \epsilon_2 \to 0$ (bringing us to $\widehat{\mathtt{G}}_{0+,\ell}$) and then we subsequently take the $\ell \to \infty$ scaling limit.

In section 7.4 we ask whether these two limits can be interchanged. This question is related to whether the scaling limit in Theorem 3 exists for a Delaunay graph (not necessarily isoradial) which is obtained as a small deformation of an isoradial Delaunay graph. We return to this issue in Section 8.

Section 7.5 addresses issues of uniform convergence in the "flip problem" for smooth, scaled deformations. In Lemma 14 we introduce a geometrical constraint on isoradial, Delaunay triangulations ensuring that no flips occur whenever the deformation parameter ϵ is bounded above by a threshold $\check{\epsilon}_F$ which is uniform both with respect to the scaling parameter and a proper subclass of isoradial, Delaunay triangulations.

In Section 8 we drop the constraints on the deformation parameter(s) stipulated by Lemma 9 and we instead work with general geometric deformations $\mathtt{G}_\epsilon$ of an isoradial, Delaunay graph $\mathtt{G}_{\mathrm{cr}}$ which may incur flips. We look for uniform bounds on the variation of the corresponding operators $\Delta(\epsilon)$ and $\mathcal{D}(\epsilon)$ for small but non-zero values of the deformation parameters. In order to get bounds uniform with respect to the choice of initial, critical graph $\mathtt{G}_{\mathrm{cr}}$ we estimate the growth of the radius $R(\mathtt{f}_\epsilon)$ of an arbitrary triangle $\mathtt{f}_\epsilon$ of $\mathtt{G}_\epsilon$ as parameter ϵ varies in formal deformation. We deduce strong results on the uniform convergence of the scaling limit for Δ (Prop. 15) and of the scaling limit of the second order bi-local term (leading to the OPE) (Prop. 16); the later result depends on a conjectural, uniform estimate (Conj. 1) on $\nabla p_3(\mathtt{f})$ and $\bar{\nabla} p_3(\mathtt{f})$ in terms of the radius $R(\mathtt{f})$ of a face $\mathtt{f}$ and the scaling parameter. We finish the section by showing that there is a qualitative difference between Δ and $\mathcal{D}$, and we obtain a weaker but interesting "simultaneous convergence" result for the scaling limit of the second order bi-local term for $\mathcal{D}$ (Prop. 20).

Section 9 summarizes our results, and presents them from a more statistical physics point of view. After reviewing the aims of the paper in 9.1 we discuss in 9.2 the first order variation of the log-determinant for the three operators Δ, $\underline{\boldsymbol{\Delta}}$ and $\mathcal{D}$ vis-à-vis the Gaussian Free Field. We show that formula 6.4 for the Laplace-Beltrami operator Δ can be re-expressed in terms of the vacuum expectation value of a discrete stress-energy tensor T_Δ for a Grassmann free field theory (for convenience we opt for a fermionic analogue of the Massless Free Field (GFF)) supported on $\mathtt{G}_{\mathrm{cr}}$ and whose scaling limit coincides with the standard continuous free field. This is not a surprise. Our results for $\mathcal{D}$ and $\underline{\boldsymbol{\Delta}}$ are similarly expressed using discrete stress-energy tensors $\mathrm{T}_\mathcal{D}$ and $\mathrm{T}_{\underline{\boldsymbol{\Delta}}}$ however neither formulae 6.8 nor 6.5 have a simple/obvious continuous limit relating it to the continuous free field.

In 9.3 we discuss the bi-local second order variation formula and the universal form of its scaling limit for Δ and $\mathcal{D}$ in terms of their respective discrete stress-energy tensors. Furthermore we address the (in general) non-existence of a scaling limit for $\underline{\boldsymbol{\Delta}}$.

In 9.4 we discuss the relation and differences between: (i) the model and the questions addressed for Delaunay graphs in our work, and (ii) previous studies made by Chelkak et al. on the O(n) model and by Hongler et al. on the GFF and the Ising model on the hexagonal and square lattices respectively.

Finally in 9.5 we briefly list some open questions and some possible extensions of this work.

Some standard material, technical derivations of results and matters not central to this work are relegated to appendices.

Appendix A present some standard reminders about the stress-energy tensor in QFT and CFT.

Appendix B gives the derivation of Lemma 2, which is instrumental for Theorem 3 and the scaling limit.

Appendix C examines the conformal Laplacian $\underline{\boldsymbol{\Delta}}$ on a particular critical Delaunay graph $\mathtt{G}$ as well as the anomalous terms associated to chords in $\mathtt{G}_{0^+}$ which arise in the second order variation of the log-determinant formula for $\underline{\boldsymbol{\Delta}}$ addressed in 6.3. The graph G is sufficiently regular and G_{0^+} has a sufficient density of chords to insure that these anomalous terms have a convergent scaling limits, which are computed explicitly in Claim 2.

2. Planar graphs and rhombic graphs

2.1. The basic objects.

Notations.
In this work we shall deal with plane triangulations, and their extensions: embedded planar graphs (or plane graphs) whose faces are cyclic polygons. Let us first introduces the notations that we shall use (most are standard).

Definition 3.

An **embedded planar graph** will be – for the purpose of this article – a graph $\mathtt{G}$, given by a set of vertices $\mathrm{V}(\mathtt{G})$ and set of edges $\mathrm{E}(\mathtt{G})$, together with an injective map $z : \mathrm{V}(\mathtt{G}) \longrightarrow \mathbb{C}$. For a vertex $\mathtt{v} \in \mathrm{V}(\mathtt{G})$ we shall denote its complex coordinate by $z(\mathtt{v})$; if there is no risk of confusion we shall sometimes denote the complex coordinate by the vertex label $\mathtt{v}$ itself. Each edge $\mathtt{e} = \overline{\mathtt{uv}}$ is embedded as a *straight line segment* joining its end-points $z(\mathtt{u})$ and $z(\mathtt{v})$ while the oriented edge $\vec{\mathtt{e}} = (\mathtt{u}, \mathtt{v})$ corresponds to the displacement vector $z(\mathtt{v}) - z(\mathtt{u})$. We require that for any pair of edges the corresponding line segments are non-crossing (i.e. do not share any interior points). The embedding determines an abstract set of faces $\mathrm{F}(\mathtt{G})$ and we require that each face $\mathtt{f} \in \mathrm{F}(\mathtt{G})$ is embedded as a *convex polygon* endowed with a counter-clockwise orientation (so that no face is folded onto an adjacent face). Furthermore the set of faces must cover the plane and they must not accumulate in any finite region of the plane (i.e. each open disk must contain only finitely many faces). We shall sometimes elide between the description of $\mathtt{G}$ as a abstract combinatorial entity (i.e. vertices, edges, faces and their incidence relations) and its description as an embedding object in the plane (points, segments, and polygons with the geometrical restrictions described above).

Definition 4.

A **polyhedral graph** will be an embedded planar graph such that each face is a *cyclic polygon*, i.e. all the vertices of the face lie on a circle (the circumcircle $C_{\mathtt{f}}$ of the face $\mathtt{f}$), in *cyclic order*. At that stage two faces may have the same circumcircle.

Definition 5.

An edge $\mathtt{e} \in \mathrm{E}(\mathtt{G})$ of a polyhedral graph $\mathtt{G}$ is a **chord** if the two faces $\mathtt{f}$ and $\mathtt{g}$ of $\mathtt{G}$ adjacent to $\mathtt{e}$ share the same circumcircle (i.e. the circumcenters of $\mathtt{f}$ and $\mathtt{g}$ coincide). An edge which is not a chord is said to be a **regular edge** of $\mathtt{G}$. If no ambiguity arise, we shall use the term edge for regular edges only, and chords for the others.

Definition 6.

A **chordless polyhedral graph** is a polyhedral graph without chords, i.e. no pair of faces share the same circumcircle. Obviously chordless polyhedral graphs correspond to a special class of circle patterns in the plane. In a general polyhedral graph, a face which does not share its circumcircle with another face will be said to be a **chordless face**.

Definition 7.

A **weak Delaunay graph** is a polyhedral graph $\mathtt{G}$ such that for any face $\mathtt{f}$, the *interior* of the circumdisk $D_{\mathtt{f}}$ (the closed disk whose boundary is the circumcircle

$C_{\mathtt{f}}$) contains no vertex of $\mathtt{G}$. The circumcircle itself contains the vertices of $\mathtt{f}$, and possibly other vertices. A **Delaunay graph** is a chordless weak Delaunay graph.

Definition 8.

A **triangulation** is an embedded planar graph $\mathtt{T}$ such that each face is a triangle. Obviously, a triangulation is a polyhedral graph. A **(weak) Delaunay triangulation** is a triangulation which is a (weak) Delaunay graph.

Definition 9.

An **isoradial graph** is a polyhedral graph $\mathtt{G}$ such that the *circumradii* $R(\mathtt{f})$ (the radius of the circumcircle $C_{\mathtt{f}}$ of $\mathtt{f}$) of all the faces of $\mathtt{G}$ are equal.

Definition 10.

Following [Ken02], a face $\mathtt{f}$ whose circumcenter is inside or on the border of $\mathtt{f}$ (considered as a cyclic polyhedron) is called a **regular face**. A polyhedral graph such that all its faces are regular is called a **regular graph**.

Remark 7. *Given an oriented edge $\vec{\mathtt{e}}$ of a polyhedral graph we define the corresponding* **north** *and* **south angles** *$\theta_{\mathrm{n}}(\vec{\mathtt{e}})$ and $\theta_{\mathrm{s}}(\vec{\mathtt{e}})$ through figure* **??** *in the introductory section 1.2. By the inscribed angle theorem $\theta_{\mathrm{n}}(\vec{\mathtt{e}})$ does not depend upon the choice of vertex $\mathtt{n} \in \mathtt{f}_{\mathrm{n}}$ in the north face. Likewise $\theta_{\mathrm{s}}(\vec{\mathtt{e}})$ is independent of the vertex $\mathtt{s} \in \mathtt{f}_{\mathrm{s}}$ in the south face. Note that reversing the orientation of $\vec{\mathtt{e}}$ exchanges the roles of north and south and so the* **conformal angle** *$\theta(\mathtt{e}) := (\theta_{\mathrm{n}}(\vec{\mathtt{e}})+\theta_{\mathrm{s}}(\vec{\mathtt{e}}))/2$ independent of the choice of edge orientation, hence the notation $\theta(\mathtt{e})$.*

Remark 8. *If $\mathtt{e} = \overline{\mathtt{uv}}$ for vertices $\mathtt{u}, \mathtt{v} \in \mathrm{V}(\mathtt{G})$ then the value of the conformal angle $\theta(\mathtt{e})$ equals the argument of the following cross-ratio involving the (coordinates of the) vertices $\mathtt{u}$, $\mathtt{v}$, $\mathtt{n}$, $\mathtt{s}$:*

$$\begin{aligned} \theta(\mathtt{e}) &= \log\Big[z(\mathtt{u}), z(\mathtt{v})\,; z(\mathtt{n}), z(\mathtt{s})\Big] \quad \textit{where} \\ \big[z_1, z_2\,; z_3, z_4\big] &= \frac{(z_1 - z_3)\cdot(z_2 - z_4)}{(z_1 - z_4)\cdot(z_2 - z_3)} \end{aligned} \tag{2.1}$$

Consequently the conformal angle is $\mathrm{SL}_2(\mathbb{C})$-invariant owing to the fact that cross-ratio is.

Remark 9. *We want to reiterate the comments in the introductory section 1.2, and stress that the Delaunay condition as stated in def. 7 is equivalent to the condition that for any oriented edge $\vec{\mathtt{e}}$ of a polyhedral graph the corresponding north and south angles $\theta_{\mathrm{n}}(\vec{\mathtt{e}})$ and $\theta_{\mathrm{s}}(\vec{\mathtt{e}})$ are both strictly positive. Alternatively, the Delaunay condition holds for a polyhedral graph if and only if for any edge $\mathtt{e}$ of the graph $0 < \theta(\mathtt{e}) < \pi$. The weak Delaunay condition, on the other hand, holds if and only if for any oriented edge $\vec{\mathtt{e}}$ of a polyhedral graph either $\theta_{\mathrm{n}}(\vec{\mathtt{e}})$ and $\theta_{\mathrm{s}}(\vec{\mathtt{e}})$ are both strictly positive or else $\theta_{\mathrm{n}}(\vec{\mathtt{e}}) = -\theta_{\mathrm{s}}(\vec{\mathtt{e}})$. Equivalently the weak Delaunay condition holds for a polyhedral graph if and only if for any edge $\mathtt{e}$ of the graph $0 \le \theta(\mathtt{e}) < \pi$.*

Remark 10. *Note that $\pi - 2\theta(\mathtt{e})$ is the intersection angle between the c.w. oriented north and south circumcircles C_{n} and C_{s}. We have $\theta(\mathtt{e}) > 0$ iff $z(\mathtt{n})$ is outside C_{s}, or equivalently if $z(\mathtt{s})$ is outside the circumcircle C_{n}, while $\theta(\mathtt{e}) = 0$ iff $z(\mathtt{n}) \in C_{\mathrm{s}}$ or equivalently iff $z(\mathtt{s}) \in C_{\mathrm{n}}$.*

Some properties. Regular graphs will be useful when discussing the discussion with the rhombic graphs of [Ken02] discussed in next section 2.1, thanks to the following simple result.

Lemma 3. *Let $\mathtt{G}_{\mathrm{cr}}$ be a planar, isoradial Delaunay graph with common circumradius R_{cr}. Then $\mathtt{G}_{\mathrm{cr}}$ is regular.*

Proof. Suppose by contradiction there exists an irregular face $\mathtt{f} \in \mathrm{F}(\mathtt{G}_{\mathrm{cr}})$. There exits an edge $\mathtt{e} \in \partial\mathtt{f}$ with an orientation $\vec{\mathtt{e}}$ such that $\mathtt{f} = \mathtt{f}_{\mathrm{s}}$ and such that face $\mathtt{f}_{\mathrm{s}}$ is contained in the intersection of the disks of circles C_{s} and C where C is the circle of radius R_{cr} obtained by reflecting C_{s} about the line determined by the edge $\mathtt{e}$. In virtue of isoradiality, the vertices $\mathtt{v} \in \partial\mathtt{f}_{\mathrm{n}}$ with $\mathtt{v} \notin \partial\mathtt{e}$ must all lie either (1) on the portion of the circle C residing in the interior of the disk of circumcircle $C_{\mathtt{f}}$ or else (2) on the circumcircle C_{s}. Case (1) is impossible because then any vertex $\mathtt{v}$ of this kind would violate the Delaunay property with respect to the face $\mathtt{f}_{\mathrm{s}}$ because edge $\mathtt{e}$ would form a chord between faces $\mathtt{f}_{\mathrm{n}}$ and $\mathtt{f}_{\mathrm{s}}$. Likewise case (2) is impossible because edge $\mathtt{e}$ would form a chord between faces $\mathtt{f}_{\mathrm{n}}$ and $\mathtt{f}_{\mathrm{s}}$. So $\mathtt{G}_{\mathrm{cr}}$ must be regular. □

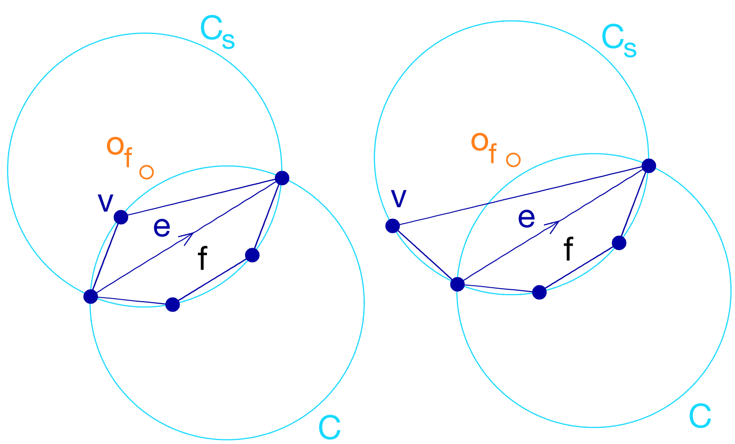

FIGURE 2. Cases (1) and (2) in the proof of Lemma 3

To any polyhedral graph $\mathtt{G}$ we associate a chordless polyhedral graph $\mathtt{G}^{\bullet}$ obtained by removing its chords. This is particularly interesting for Delaunay graphs. Hence we define

Definition 11 (**Regularized graph**). *Given a polyhedral graph $\mathtt{G}$, let $\mathtt{G}^{\bullet}$ be the graph with the same vertex set $\mathrm{V}(\mathtt{G}^{\bullet}) = \mathrm{V}(\mathtt{G})$, the same embedding $z^{\bullet} = z$, and with edge set $\mathrm{E}(\mathtt{G}^{\bullet}) = \mathrm{E}(\mathtt{G}) - \mathrm{chords}(\mathtt{G})$, where $\mathrm{chords}(\mathtt{G})$ is the set of all chords in $\mathtt{G}$. We call $\mathtt{G}^{\bullet}$ the **regularized graph** of $\mathtt{G}$. Clearly $\mathtt{G}^{\bullet}$ is made of the regular edges of $\mathtt{G}$. By Lemma 3 the graph $\mathtt{G}^{\bullet}$ is an isoradial, regular Delaunay graph whenever $\mathtt{G}$ is isoradial and weakly Delaunay.*

Rhombic graphs and abstract rhombic surfaces. We now consider the bipartite kite graph built from the vertices and the face centers of a Delaunay graph, as well as the associated concept of rhombic surface.

Definition 12 (**Kite graphs** $\mathtt{G}^\Diamond$)**.** *For a Delaunay graph* $\mathtt{G}$ *let* $\mathtt{G}^\Diamond$ *denote the bipartite graph whose vertex set consists of all vertices* $\mathtt{v}$ *of* $\mathtt{G}$ *(the black vertices* $\bullet$*) together with all circumcenters* $\mathtt{o}_\mathtt{f}$ *of faces* $\mathtt{f}$ *of* $\mathtt{G}$ *(the white vertices* $\circ$*), and whose edges correspond precisely to those pairs* $\{\mathtt{v}, \mathtt{o}_\mathtt{f}\}$ *for which* $\mathtt{v} \in \partial\mathtt{f}$*. We extend the embedding* $z : \mathrm{V}(\mathtt{G}) \longrightarrow \mathbb{C}$ *to* $\mathrm{V}(\mathtt{G}^\Diamond)$ *by setting* $z(\mathtt{v}) := z(\mathtt{v})$ *for each vertex* $\mathtt{v} \in \mathrm{V}(\mathtt{G})$ *and* $z(\mathtt{o}_\mathtt{f}) := z(\mathtt{f})$ *for each face* $\mathtt{f} \in \mathrm{F}(\mathtt{G})$ *where*
(2.2)

$$z(\mathtt{f}) := \frac{1}{4\mathrm{i}} \frac{|z(\mathtt{u})|^2(z(\mathtt{v}) - z(\mathtt{w})) + |z(\mathtt{v})|^2(z(\mathtt{w}) - z(\mathtt{u})) + |z(\mathtt{w})|^2(z(\mathtt{u}) - z(\mathtt{v}))}{z(\mathtt{v})\overline{z}(\mathtt{u}) - z(\mathtt{u})\overline{z}(\mathtt{v}) + z(\mathtt{w})\overline{z}(\mathtt{v}) - z(\mathtt{v})\overline{z}(\mathtt{w}) + z(\mathtt{u})\overline{z}(\mathtt{w}) - z(\mathtt{w})\overline{z}(\mathtt{u})}$$

is the complex coordinate of the circumcenter of the face $\mathtt{f} \in \mathrm{F}(\mathtt{G})$ *with* $\mathtt{u}, \mathtt{v}, \mathtt{w} \in \partial\mathtt{f}$ *any choice of three vertices appearing in counter-clockwise order. As constructed, each face of the graph* $\mathtt{G}^\Diamond$ *is quadrilateral (in fact a* kite*)* $\Diamond(\overline{\mathtt{uv}}) = (\mathtt{u}, \mathtt{o}_\mathtt{s}, \mathtt{v}, \mathtt{o}_\mathtt{n})$ *corresponding to a unique unoriented edge* $\overline{\mathtt{uv}}$ *of the graph.*

Remark 11. *For any weak Delaunay graph* $\mathtt{G}$ *we define* $\mathtt{G}^\Diamond := (\mathtt{G}^\bullet)^\Diamond$*. Clearly* $\mathtt{G}_1^\Diamond = \mathtt{G}_2^\Diamond$ *if and only if* $\mathtt{G}_1^\bullet = \mathtt{G}_2^\bullet$ *for any two weak Delaunay graphs* $\mathtt{G}_1$ *and* $\mathtt{G}_2$*.*

Definition 13 (**Rhombic surface** $\mathtt{S}_\mathtt{G}^\Diamond$)**.** *Following* [DE14]*, a rhombic surface* $\mathtt{S}_\mathtt{G}^\Diamond$ *can be constructed from a Delaunay graph* $\mathtt{G}$ *in the following way: Assign to each unoriented edge* $\mathtt{e} = \overline{\mathtt{uv}}$ *a rhombus* $\Diamond(\mathtt{e}) = \tilde{\mathrm{u}}\tilde{\mathrm{o}}_\mathrm{s}\tilde{\mathrm{v}}\tilde{\mathrm{o}}_\mathrm{n}$ *with unit edge lengths* $l = 1$ *and rhombus angle* $\angle\tilde{\mathrm{o}}_\mathrm{s}\tilde{\mathrm{u}}\tilde{\mathrm{o}}_\mathrm{n} = 2\theta(\mathtt{e})$ *as depicted in fig. 3.*

If two edges $\mathtt{e}_1$ *and* $\mathtt{e}_2$ *of the graph share a common vertex and simultaneously belong to a common face then rhombi* $\Diamond(\mathtt{e}_1)$ *and* $\Diamond(\mathtt{e}_2)$ *are glued together along their common edge. In this way we obtain an abstract rhombic surface* $\mathtt{S}_\mathtt{G}^\Diamond$*.*

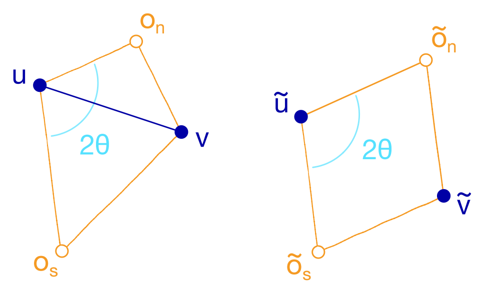

FIGURE 3. An edge $\mathtt{e} = \mathtt{uv}$ of $\mathtt{G}$ and the associated kite in the plane (left), and the associated rhombus $\Diamond(\mathtt{e})$ of $\mathtt{S}_\mathtt{G}^\Diamond$ (right)

A simple example is depicted in the figure 4 below. In this example an explicit isometric embedding as a tesselated rhombic surface in $\mathbb{R}^3$ is possible. (a) is a piece of an Delaunay graph $\mathtt{G}$, in blue, with the kites associated to each edge (in orange); (b) is the associated kite graph $\mathtt{G}^\Diamond$ (in orange). (c) is an isometric embedding in $\mathbb{R}^3$ of the associated rhombic surface $\mathtt{S}_\mathtt{G}^\Diamond$. In this particular example, the conformal angles $\theta(\mathtt{e})$ for each edge of $\mathtt{G}$ equals $\pi/2$, and so the faces of $\mathtt{S}_\mathtt{G}^\Diamond$ are in fact squares,

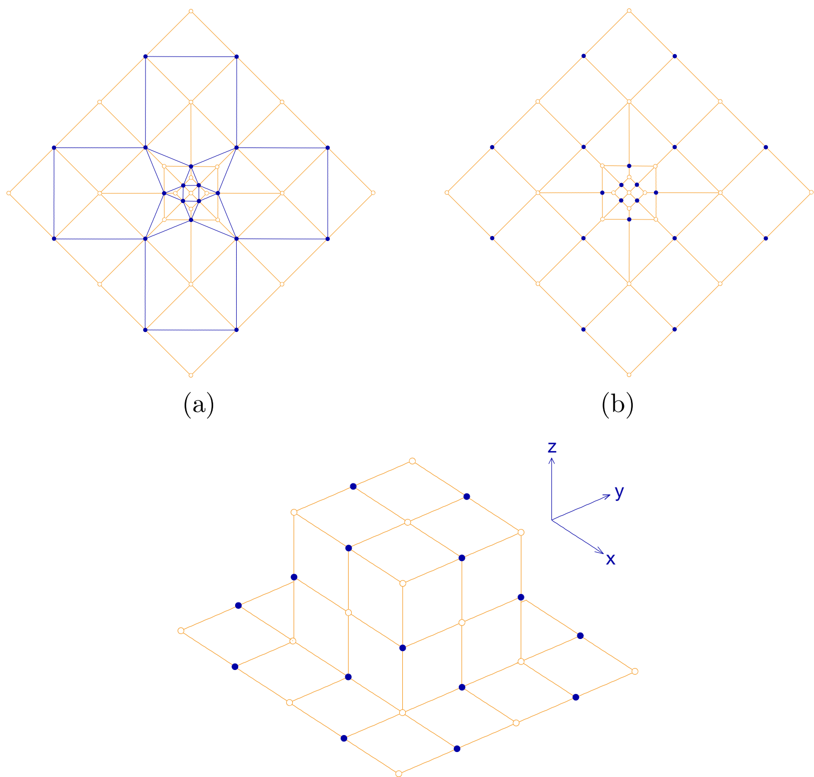

FIGURE 4. An example (in blue) of a Delaunay graph $\mathtt{G}$ (a) the associated kite graph $\mathtt{G}^{\diamondsuit}$ (b) the rhombic surface $\mathtt{S}^{\diamondsuit}_{\mathtt{G}}$ consisting of square rhombs embedding in $\mathbb{R}^3$ (c) as discussed in the text.

and the embedding (c) is a surface in $\mathbb{Z}^3$. In general the rhombic surface $\mathtt{S}^{\diamondsuit}_{\mathtt{G}}$ cannot be embedded isometrically and rigidly into $\mathbb{R}^3$.

A rhombic surface is flat at each vertex $\tilde{\mathtt{u}}$ associated to a vertex $\mathtt{u}$ of $\mathtt{G}$ but has a potential curvature defect at each vertex $\tilde{\mathtt{o}}_{\mathtt{f}}$ corresponding to a circumcenters $\mathtt{o}_{\mathtt{f}}$ of a face $\mathtt{f}$ of $\mathtt{G}$, with scalar (Ricci) curvature R_{scal} defined by

$$R_{\mathrm{scal}}(\tilde{\mathtt{o}}_{\mathtt{f}}) := \; 4\pi - 2 \sum_{\mathtt{e}\in\partial\mathtt{f}} \big(\pi - 2\theta(\mathtt{e})\big) \tag{2.3}$$

If $R_{\mathrm{scal}}(\tilde{\mathtt{o}}_{\mathtt{f}}) = 0$ for every face $\mathtt{f}$ of the graph, $\mathtt{G}$ is said to be *flat*. It is easy to see that this is equivalent to saying that the Delaunay graph is *isoradial*, namely that all circumradii are equal to some R. Note that for every oriented edge $\vec{\mathtt{e}}$ of an isoradial, polyhedral graph either $\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) = \theta_{\mathrm{s}}(\vec{\mathtt{e}}\,) = \theta(\mathtt{e}) > 0$ or $\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) = -\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)$ in which case $\theta(\mathtt{e}) = 0$.

When $\mathtt{G}$ is isoradial (with common circumradius R) each kite $\Diamond(\overline{\mathtt{uv}})$ will be a rhombus with side length R; in this case we shall refer to $\mathtt{G}^{\Diamond}$ as a **rhombic graph**. Up to a global rescaling $R \to 1$ we have $\mathtt{G}^{\Diamond} = \mathtt{S}^{\Diamond}_{\mathtt{G}}$. This corresponds to the rhombic graphs discussed in [Ken02].

Remark 12. *Isoradial Delaunay graphs are in bijection with the rhombic graphs of* [Ken02].

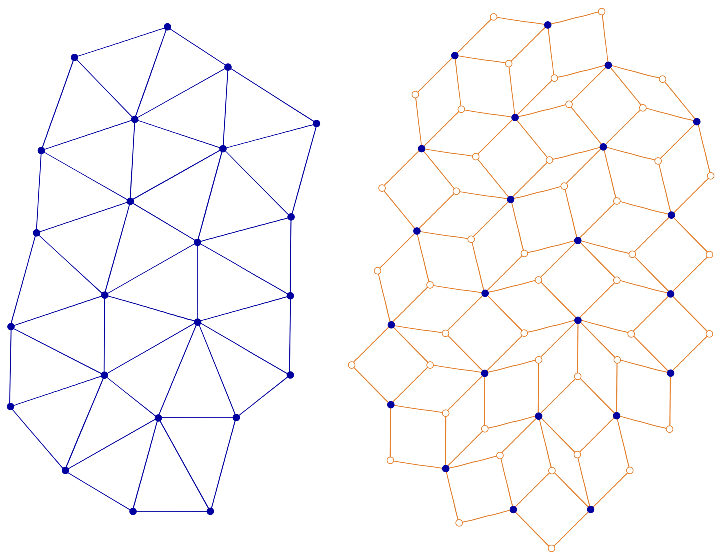

FIGURE 5. Fragments of an isoradial Delaunay graph $\mathtt{G}_{\mathrm{cr}}$ (on the left) and its rhombic graph $\mathtt{G}^{\Diamond}_{\mathrm{cr}}$ (on the right).

2.2. **Geometry on rhombic graphs.** In the following discussion $\mathtt{G}_{\mathrm{cr}}$ will be an isoradial Delaunay graph with embedding $z_{\mathrm{cr}} : \mathrm{V}(\mathtt{G}_{\mathrm{cr}}) \longrightarrow \mathbb{C}$ and, if not specified otherwise, we shall assume for simplicity that the value of the common circumradius is $R_{\mathrm{cr}} = 1$.

Let us recall some geometrical concepts of [Ken02] and [KS14], with some more material needed in this paper.

Paths on rhombic graphs. A path in $\mathtt{G}^{\Diamond}_{\mathrm{cr}}$ is a finite sequence of vertices $\mathtt{v} = \big(\mathtt{v}_0, \dots, \mathtt{v}_k\big)$ such that for each $1 \le j \le k$ the vertices $\mathtt{v}_{j-1}$ and $\mathtt{v}_j$ are joined by an edge $\mathtt{e}_j$ of $\mathtt{G}^{\Diamond}_{\mathrm{cr}}$; in this case we say $\mathtt{v}$ is a path of length k from $\mathtt{v}_0$ to $\mathtt{v}_k$. Let $\vec{\mathtt{e}}_j = (\mathtt{v}_{j-1}, \mathtt{v}_j)$ be the oriented edge corresponding to $\mathtt{e}_j$, let $\vec{\mathbb{E}}(\mathtt{v}) = (\vec{\mathtt{e}}_1, \cdots, \vec{\mathtt{e}}_k)$ be the sequence of oriented edges of $\mathtt{v}$, and $E\big(\mathtt{v}\big) = \bigcup_j \{\mathtt{e}_j\}$ the set of edges of $\mathtt{v}$. To each edge $\vec{\mathtt{e}}_j$ of $\mathtt{v}$ is associated a phase $e^{\mathrm{i}\theta_j} := z_{\mathrm{cr}}(\mathtt{v}_j) - z_{\mathrm{cr}}(\mathtt{v}_{j-1})$. We denote by $\underline{\theta}(\mathtt{v}) = \big(\theta_1, \dots, \theta_k\big)$ the sequence of angles of these phases.

We can regard the rhombic graph $\mathtt{G}^{\Diamond}_{\mathrm{cr}}$ as a cellular decomposition of the plane; accordingly vertices, oriented edges, and oriented faces of $\mathtt{G}^{\Diamond}_{\mathrm{cr}}$ can be viewed respectively as 0, 1, and 2-chains of a cellular complex $\mathcal{X}$ with $\mathbb{Z}$-coefficients. For a

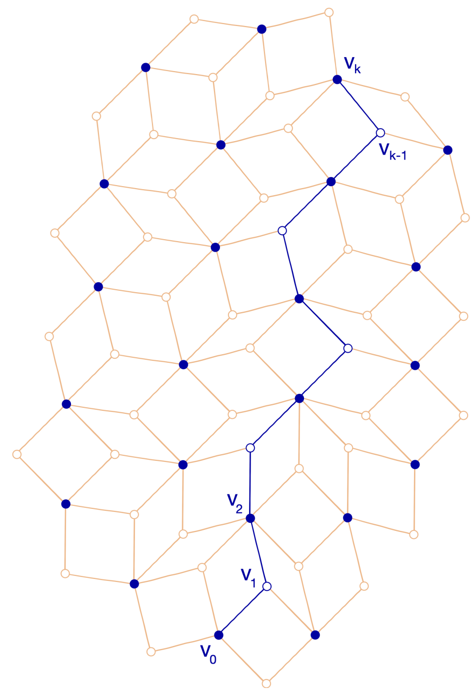

FIGURE 6. Path $\mathbb{v} = (\mathtt{v}_0, \ldots, \mathtt{v}_k)$ in the rhombic graph $\mathtt{G}_{\mathrm{cr}}^{\Diamond}$

path $\mathbb{v}$, let $\vec{\mathbb{v}}$ denote the 1-chain $\vec{\mathtt{e}}_1 + \cdots + \vec{\mathtt{e}}_k$ in $\mathrm{C}_1(\mathcal{X}; \mathbb{Z})$. Two paths $\mathbb{v}_1$ and $\mathbb{v}_2$ are said to differ by an oriented rhomb $\Diamond^{\circlearrowleft}$ if $\vec{\mathbb{v}}_2 = \vec{\mathbb{v}}_1 + \partial \Diamond^{\circlearrowleft}$; see figure 7 for an example. The vanishing of $\mathrm{H}_1(\mathcal{X}; \mathbb{Z})$ is equivalent to the fact that any two paths $\vec{\mathbb{v}}_1$ and $\vec{\mathbb{v}}_2$ both from a vertex $\mathtt{u}$ to a vertex $\mathtt{v}$ must differ by a sum of oriented rhombs.

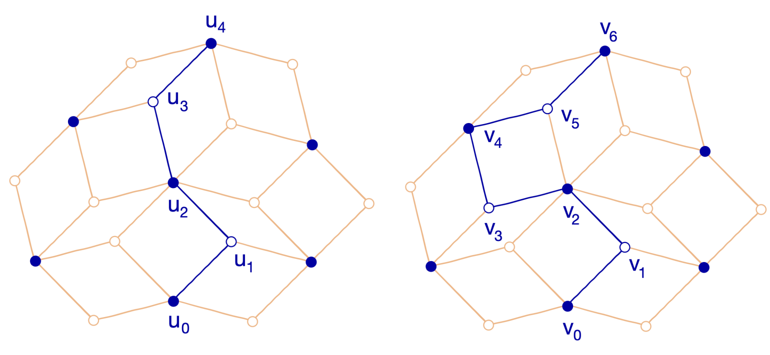

FIGURE 7. Paths $\mathbb{u} = (\mathtt{u}_0, \ldots, \mathtt{u}_4)$ and $\mathbb{v} = (\mathtt{v}_0, \ldots, \mathtt{v}_6)$ differ by a rhomb.

For an integer n together with an oriented edge $\vec{\mathtt{e}}$ joining a vertex $\mathtt{u}$ to a vertex $\mathtt{v}$ set $\lfloor \vec{\mathtt{e}} \rfloor_n := e^{in\theta}$ where $e^{\mathrm{i}\theta} = z_{\mathrm{cr}}(\mathtt{v}) - z_{\mathrm{cr}}(\mathtt{u})$ is the phase of the difference of the coordinates of the vertices; extend this by linearity to 1-chains in $\mathrm{C}_1(\mathcal{X}; \mathbb{Z})$, and thus define $\left\lfloor \sum_j a_j \vec{\mathtt{e}}_j \right\rfloor_n := \sum_j a_j \lfloor \vec{\mathtt{e}}_j \rfloor_n$. Notice that $\lfloor \lozenge^{\circlearrowleft} \rfloor_n = 0$ for any oriented rhomb $\lozenge^{\circlearrowleft}$ whenever n is an odd integer. It follows that for any path $\mathtt{v}$, and for any odd integer $n = 2d+1$, $\lfloor \vec{\mathtt{v}} \rfloor_n$ depends only on the two end-points $(\mathtt{v}_0, \mathtt{v}_k)$ of $\mathtt{v}$.

Definition 14. *For any pair of vertices* $\mathtt{u}$ *and* $\mathtt{v}$ *of* $\mathbf{G}^{\lozenge}_{\mathrm{cr}}$ *and for any odd integer* $n = 2d+1$*, we define* $p_n(\mathtt{u}, \mathtt{v}) := \lfloor \vec{\mathtt{v}} \rfloor_n$ *where* $\mathtt{v}$ *is any path from* $\mathtt{u}$ *to* $\mathtt{v}$.

Note that $p_1(\mathtt{u}, \mathtt{v}) = z_{\mathrm{cr}}(\mathtt{v}) - z_{\mathrm{cr}}(\mathtt{u})$. In addition $p_n(\mathtt{u}, \mathtt{v}) = -p_n(\mathtt{v}, \mathtt{u})$.

Proposition 2. *For any pair of vertices* $\mathtt{u}$ *and* $\mathtt{v}$ *in* $\mathbf{G}^{\lozenge}_{\mathrm{cr}}$*, there is a finite set of angles* $\Theta \subset \big(\theta_0 - \pi, \theta_0 + \pi\big)$ *where* $\theta_0 = \arg\big(z_{\mathrm{cr}}(\mathtt{v}) - z_{\mathrm{cr}}(\mathtt{u})\big)$ *together with multiplicities* $m_\vartheta \in \mathbb{Z}_{>0}$ *for* $\vartheta \in \Theta$ *such that for any odd integer* $n = 2d+1$

$$p_n\big(\mathtt{u}, \mathtt{v}\big) = \sum_{\vartheta \in \Theta} m_\vartheta \, e^{in\vartheta} \tag{2.4}$$

Moreover Θ *is contained in an open subinterval whose length is smaller than* π*, i.e.* $\max \Theta - \min \Theta < \pi$.

In fact the phases $\big\{e^{\mathrm{i}\vartheta} \,\big|\, \vartheta \in \Theta\big\}$ form a subset of the set of phases $\big\{e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_k}\big\}$ of any path $\mathtt{v}$ going from $\mathtt{u}$ to $\mathtt{v}$. These results can probably be found in the litterature, at least implicitly. For completeness we give a short derivation, which relies on the essential concept of train-tracks on rhombic graphs.

Train-tracks.

Definition 15 (train-track)**.** *A* ***train-track*** *in the rhombic graph* $\mathbf{G}^{\lozenge}_{\mathrm{cr}}$ *is an infinite sequence of rhombs* $\mathbb{t} = \big(\lozenge_n \,\big|\, n \in \mathbb{Z}\big)$ *whose consecutive rhombs* $\lozenge_n$ *and* $\lozenge_{n+1}$ *are incident along a common edge* $\mathtt{e}_n$ *for each* $n \in \mathbb{Z}$ *and for which the edges* $\mathtt{e}_n$ *and* $\mathtt{e}_{n+1}$ *are parallel for each* $n \in \mathbb{Z}$*. We shall denote these parallel edges "train-track tie", or in short "**tie**". We consider train-tracks up to shift and inversion, i.e.* $\mathbb{t}^{(1)} = \big\{\lozenge^{(1)}_n \,\big|\, n \in \mathbb{Z}\big\}$ *is equivalent to* $\mathbb{t}^{(2)} = \big\{\lozenge^{(2)}_n \,\big|\, n \in \mathbb{Z}\big\}$ *if* $\lozenge^{(2)}_n = \lozenge^{(1)}_{\pm n+d}$ *for some* $d \in \mathbb{Z}$*. Let* $\mathtt{Ties}(\mathbb{t}) = \big\{\mathtt{e}_n \,\big|\, n \in \mathbb{Z}\big\}$ *denote this set of edges. A train-track* $\mathbb{t}$ *has* ***inclination*** $\theta_{\mathbb{t}} \in [0, \pi)$ *if the ties* $\mathtt{e}_n$ *are parallel to the phase* $\exp\big(\mathrm{i}\theta_{\mathbb{t}}\big)$.

Clearly any train-track is determined (but not uniquely) by an initial rhomb $\lozenge_0$ together with a choice of one of its edges $\mathtt{e}_0$. For any choice of initial edge $\mathtt{e}_0$ in $\mathbb{t}$ the distance of each edge $\mathtt{e}_n$ from the axis determined by $\mathtt{e}_0$ is monotonically increasing with n; i.e. the train-track must move forward in the axis perpendicular to $\mathtt{e}_0$.

We say two train-tracks $\mathbb{t}^{(1)} = \big\{\lozenge^{(1)}_n \,\big|\, n \in \mathbb{Z}\big\}$ and $\mathbb{t}^{(2)} = \big\{\lozenge^{(2)}_n \,\big|\, n \in \mathbb{Z}\big\}$ **intersect** if $\lozenge^{(1)}_m = \lozenge^{(2)}_n$ for some $m, n \in \mathbb{Z}$. Two important features of any rhombic-graph $\mathbf{G}^{\lozenge}_{\mathrm{cr}}$ are

Fact 1. *No train-track can intersect itself, i.e. if* $\mathbb{t} = \big\{\lozenge_n \,\big|\, n \in \mathbb{Z}\big\}$ *then* $\lozenge_m \neq \lozenge_n$ *for all integers* $m \neq n$.

Fact 2. *Any two distinct train-tracks are either disjoint or else intersect once.*

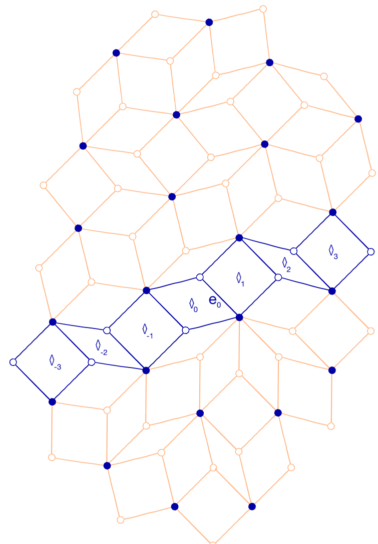

FIGURE 8. Train-track $\mathbb{t}$.

The notion of train-track is amenable to any quad-graph (a planar graph consisting entirely of quadrilateral faces) and these two properties characterize rhombic graphs within the broader class of quad-graphs; specifically any quad-graph satisfying these two properties is a deformation of a rhombic graph (see [KS14]).

Intersections of train-tracks with paths.
A train-track $\mathbb{t}$ partitions the vertex set $\mathrm{V}\big(\mathtt{G}^{\diamondsuit}_{\mathrm{cr}}\big)$ into two disjoint subsets V_1 and V_{-1}. Specifically, the edge set $E\big(\mathtt{G}^{\diamondsuit}_{\mathrm{cr}}\big) - \mathtt{Ties}(\mathbb{t})$ defines a disconnected subgraph of $\mathtt{G}^{\diamondsuit}_{\mathrm{cr}}$ with two disjoint components; V_{-1} and V_1 are the respective vertex sets of these components. Accordingly, we say that two vertices $\mathtt{u}$ and $\mathtt{v}$ are **separated by** $\mathbb{t}$ if they lie in different components; furthermore we say $\mathbb{t}$ **separates** the path $\mathbb{v}$ if the end-points of the path $\mathtt{v}_0$ and $\mathtt{v}_k$ are separated by $\mathbb{t}$.

Given a path $\mathbb{v} = \big(\mathtt{v}_0, \dots \mathtt{v}_k\big)$ and a train-track $\mathbb{t}$ let $I(\mathbb{v};\mathbb{t}) := \big\{1 \le j \le k \,\big|\, \mathtt{e}_j \in \mathrm{Ties}(\mathbb{t})\big\}$ be the set of indices of edges common to both $\mathbb{v}$ and $\mathbb{t}$. If $\mathbb{t}$ separates $\mathbb{v}$ then its cardinality $\big|I(\mathbb{v};\mathbb{t})\big|$ must be odd due to the fact the path must begin on one side of $\mathbb{t}$ and end on the other. If on the other hand $\mathbb{t}$ does not separate $\mathbb{v}$ then $\big|I(\mathbb{v};\mathbb{t})\big|$ is even (and may in fact be zero if there is no intersection at all).

The edges $\mathtt{e}_j$ for $j \in I(\mathbb{v};\mathbb{t})$ are clearly parallel (since they all inhabit the train-track $\mathbb{t}$) but the oriented edges $\vec{\mathtt{e}}_j$ for $j \in I(\mathbb{v};\mathbb{t})$ must alternate in direction and so their phases $e^{\mathrm{i}\theta_j}$ for $j \in I(\mathbb{v};\mathbb{t})$ must must alternate in sign. Consequently, if $I(\mathbb{v};\mathbb{t}) = \big\{j_1 < \cdots < j_d\big\}$ and n is odd then

$$\sum_{s=1}^{d} e^{\mathrm{i}n\theta_{j_s}} = \begin{cases} e^{\mathrm{i}n\theta_{j_1}} & \text{whenever } \mathbb{t} \text{ separates } \mathbb{v} \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

If $\mathbb{t}$ separates $\mathbb{v}$ their *intersection angle* is defined as $\vartheta(\mathbb{v},\mathbb{t}) := \theta_{j_1}$ and $\Theta(\mathbb{v}) = \big\{\vartheta(\mathbb{v},\mathbb{t}) \,\big|\, \mathbb{t} \text{ intersects } \mathbb{v}\big\}$ is the set of intersections angles of all train-tracks that separate the path $\mathbb{v}$. For $\vartheta \in \Theta(\mathbb{v})$ define its multiplicity as the cardinality of set $m_\vartheta := \left|\left\{\mathbb{t} \text{ separates } \mathbb{v} \,\middle|\, \vartheta = \vartheta(\mathbb{v}, \mathbb{t})\right\}\right|$.

It follows from equation 2.5 that for odd n

$$\lfloor \vec{\mathbb{v}} \rfloor_n = \sum_{j=1}^{k} e^{in\theta_j} = \sum_{\substack{\text{train-tracks } \mathbb{t} \\ \text{separating } \mathbb{v}}} e^{in\vartheta(\mathbb{v},\mathbb{t})} = \sum_{\vartheta \in \Theta(\mathbb{v})} m_\vartheta e^{in\vartheta} \tag{2.6}$$

For obvious topological reasons the set $\left\{\mathbb{t} \text{ separates } \mathbb{v} \,\middle|\, \vartheta = \vartheta(\mathbb{v}, \mathbb{t})\right\}$ only depends on the end-points $\mathtt{v}_0$ and $\mathtt{v}_k$ of the path $\mathbb{v}$. As immediate consequence of Proposition 3 is that if $\vartheta \in \Theta(\mathbb{v})$ then $\vartheta + \pi \notin \Theta(\mathbb{v})$ — this means that if two distinct train-tracks $\mathbb{t}_1$ and $\mathbb{t}_2$ share the same inclination and both separate $\mathbb{v}$ then $\vartheta(\mathbb{v}, \mathbb{t}_1) = \vartheta(\mathbb{v}, \mathbb{t}_2)$. Consequently the set $\Theta(\mathbb{v})$ together with the multiplicities m_ϑ for $\vartheta \in \Theta(\mathbb{v})$ must only depend on the end-points $\mathtt{v}_0$ and $\mathtt{v}_k$ of the path $\mathbb{v}$ as well. This observation is consonant with the fact that the value of $\lfloor \vec{\mathbb{v}} \rfloor_n$ depends only on end-points of $\mathbb{v}$.

Let $\mathbb{v}$ be a path in $\mathbf{G}^\diamondsuit_{\mathrm{cr}}$ with starting and ending points $\mathtt{u}$ and $\mathtt{v}$ respectively. Let $\mathbb{t}$ be a train-track separating $\mathbb{v}$, and $\theta = \theta(\mathbb{v}, \mathbb{t})$ their angle of intersection. Let $R^{\mathtt{u}}_\theta = z_{\mathrm{cr}}(\mathtt{u}) + \mathbb{R}_{>0} e^{\mathrm{i}\theta}$ be the ray (half-line) starting from $z_{\mathrm{cr}}(\mathtt{u})$ in the direction θ, and $R^{\mathtt{v}}_{\theta+\pi} = z_{\mathrm{cr}}(\mathtt{v}) + \mathbb{R}_{>0} e^{\mathrm{i}(\theta+\pi)}$ be the ray starting from $z_{\mathrm{cr}}(\mathtt{v})$ in the direction $\theta + \pi$. It is geometrically clear, as depicted in fig. 9, that $\mathbb{t}$ must intersect the righthand sides of rays $R^{\mathtt{u}}_\theta$ and $R^{\mathtt{v}}_{\theta+\pi}$, without back tracking in the direction orthogonal to $R^{\mathtt{u}}_\theta$ (and without intersecting the opposite rays $R^{\mathtt{u}}_{\theta+\pi}$ and $R^{\mathtt{v}}_\theta$). See fig. 9.

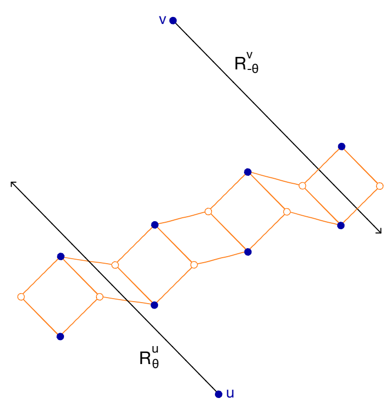

FIGURE 9. Vertices $\mathtt{u}$ and $\mathtt{v}$ separated by a train-track $\mathbb{t}$.

For completeness, one should consider the case where the lozenges in $\mathbb{t}$ becomes infinitely flat, so that $\mathbb{t}$ goes to infinity in the θ direction before intersecting $R^{\mathtt{u}}_\theta$ (see Fig. 2.2). Then one can consider that $\mathbb{t}$ crosses $R^{\mathtt{u}}_\theta$ at infinity.

Proposition 3. *Let* $\mathbb{v} = \left(\mathtt{v}_0, \ldots, \mathtt{v}_k\right)$ *be a path in* $\mathbf{G}^\diamondsuit_{\mathrm{cr}}$*, let the direction of the path be* $\theta_0 = \arg\left(z_{\mathrm{cr}}(\mathtt{v}_k) - z_{\mathrm{cr}}(\mathtt{v}_0)\right)$*. Let us fix the determinations of the angles* $\vartheta \in \Theta(\mathbb{v})$ *as real numbers in*

$$\vartheta \in (\theta_0 - \pi, \theta_0 + \pi] \tag{2.7}$$

and let

$$\alpha = \max\{\vartheta \in \Theta(\mathbb{v})\}\ , \quad \beta = \min\{\vartheta \in \Theta(\mathbb{v})\} \tag{2.8}$$

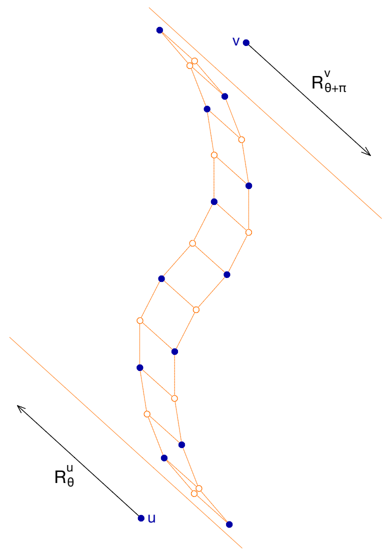

FIGURE 10. Vertices $\mathtt{u}$ and $\mathtt{v}$ asymptotically separated by a train-track.

FIGURE 11. Semicircle

Then

$$\alpha - \beta < \pi \quad \textit{and} \quad \beta \leq \theta_0 \leq \alpha \tag{2.9}$$

In other words, the set $\Theta(\mathtt{v})$ *and the angle* θ_0 *are contained in the open subinterval* $\big(\theta_{\mathtt{v}} - \frac{\pi}{2}, \theta_{\mathtt{v}} + \frac{\pi}{2}\big)$ *where* $\theta_{\mathtt{v}} = \frac{1}{2}(\alpha - \beta)$.

Proof. Set $\theta_0 = \arg\big(z_{\mathrm{cr}}(\mathtt{v}_k) - z_{\mathrm{cr}}(\mathtt{v}_0)\big) \in \big[0, \pi\big)$. Each $\vartheta \in \Theta(\mathtt{v})$ is the intersection angle of at least one train-track $\mathbb{t}$ whose inclination equals ϑ (modulo π) and which separates the vertices $\mathtt{v}_0$ and $\mathtt{v}_k$.

First let us note that the angle $\theta_0 + \pi$ cannot be an element of $\Theta(\mathtt{v})$. Were this the case, there would be train-track joining the righthand sides of the rays $R^{\mathtt{u}}_{\theta_0+\pi}$ and $R^{\mathtt{v}}_{\theta_0}$ without backtracking. This is impossible, as depicted in figure 12.

Consequently the angles in $\Theta(\mathtt{v})$ are in the interval $(\theta_0 - \pi, \theta_0 + \pi)$. Consider $\alpha = \max\Theta(\mathtt{v})$ and $\beta = \min\Theta(\mathtt{v})$. It is enough to prove that $\alpha - \beta \leq \pi$. Indeed, suppose instead that $\alpha - \beta > \pi$. Both α and β are intersection angles for two respective train-tracks $\mathbb{t}_1$ and $\mathbb{t}_2$ which separate $\mathtt{u} := \mathtt{v}_0$ and $\mathtt{v} := \mathtt{v}_k$. If we attempt to draw $\mathbb{t}_1$ and $\mathbb{t}_2$ bearing in mind monotonicity and their requisite intersections with the rays $R^{\mathtt{u}}_{\alpha}$, $R^{\mathtt{u}}_{\beta}$, $R^{\mathtt{v}}_{\alpha+\pi}$, and $R^{\mathtt{v}}_{\beta+\pi}$ we will observe that the two train-tracks will be forced to intersect at least three times (as depicted on figure 13). Since two distinct train-tracks may intersect at most once we are forced to conclude that $\alpha - \beta \leq \pi$.

Finally, by equation 2.6, the difference $z_{\mathrm{cr}}(\mathtt{v}) - z_{\mathrm{cr}}(\mathtt{u})$ can be written as

$$z_{\mathrm{cr}}(\mathtt{v}) - z_{\mathrm{cr}}(\mathtt{u}) = \sum_{\vartheta \in \Theta(\mathtt{v})} m_{\vartheta}\, e^{\mathrm{i}\vartheta} \quad \text{with} \qquad m_{\vartheta} \in \mathbb{Z}_{>0}$$

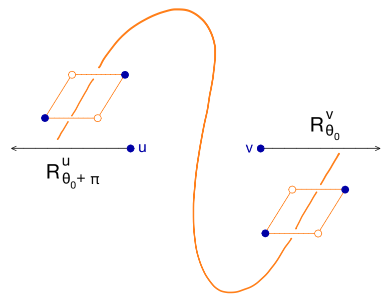

FIGURE 12. A track separating $\mathtt{u}$ and $\mathtt{v}$ with orientation $\theta_0 + \pi$ must backtrack

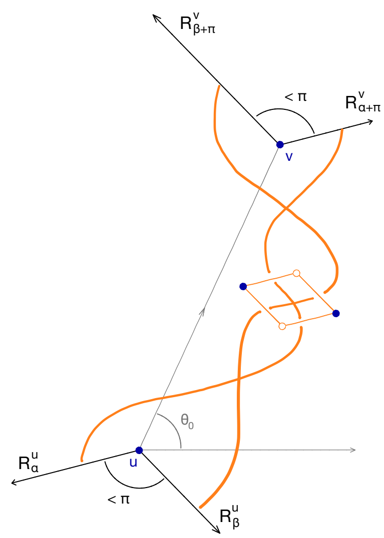

FIGURE 13. Two train-tracks separating u and v cannot have separating angles differing by more that π

Any positive combination of phases $e^{\mathrm{i}\vartheta}$ for $\vartheta \in \Theta(\mathtt{v})$ must lie in the positive cone $\left\{ae^{\mathrm{i}\alpha} + be^{\mathrm{i}\beta} \,\middle|\, a, b \in \mathbb{R}_{>0}\right\}$ because $\alpha - \beta < \pi$. It follows that $\beta < \theta_0 < \alpha$. $\square$

3. Laplacians and their determinants

3.1. **Laplacians and the critical laplacian.**

The laplacians.
Given a polyhedral graph $\mathtt{G}$, we denote by $\mathbb{C}^{\mathrm{V}(\mathtt{G})}$, $\mathbb{C}^{\mathrm{E}(\mathtt{G})}$, and $\mathbb{C}^{\mathrm{F}(\mathtt{G})}$ the vector spaces of complex-valued functions supported respectively on the vertices, edges, and faces of $\mathtt{G}$.

The operators associated to a general planar triangulation $\mathtt{G}$ have been introduced in section 1, namely: The Laplace-Beltrami operator Δ, the conformal laplacian $\underline{\boldsymbol{\Delta}}$ and the Kähler operator $\mathcal{D}$. Each operator is a linear map $\mathbb{C}^{\mathrm{V}(\mathtt{G})} \to \mathbb{C}^{\mathrm{V}(\mathtt{G})}$ defined respectively by

$$\Delta\phi(\mathtt{u}) = \sum_{\text{edge } \vec{\mathtt{e}}=(\mathtt{u},\mathtt{v})} c(\vec{\mathtt{e}})\big(\phi(\mathtt{u})-\phi(\mathtt{v})\big) \quad , \quad c(\vec{\mathtt{e}}) = \frac{1}{2}\big(\tan\theta_{\mathrm{n}}(\vec{\mathtt{e}}) + \tan\theta_{\mathrm{s}}(\vec{\mathtt{e}})\big) \tag{3.1}$$

$$\underline{\boldsymbol{\Delta}}\phi(\mathtt{u}) = \sum_{\text{edge } \vec{\mathtt{e}}=(\mathtt{u},\mathtt{v})} \tan\theta(\mathtt{e})\big(\phi(\mathtt{u})-\phi(\mathtt{v})\big) \tag{3.2}$$

and

$$\mathcal{D}\phi(\mathtt{u}) = \sum_{\text{edge } \vec{\mathtt{e}}=(\mathtt{u},\mathtt{v})} \frac{1}{2}\left(\frac{\tan\theta_{\mathrm{n}}(\vec{\mathtt{e}})+i}{R_n^2(\vec{\mathtt{e}})} + \frac{\tan\theta_{\mathrm{n}}(\vec{\mathtt{e}})-i}{R_{\mathrm{s}}^2(\vec{\mathtt{e}})}\right)\big(\phi(\mathtt{u})-\phi(\mathtt{v})\big) \tag{3.3}$$

$\theta_{\mathrm{n}}(\vec{\mathtt{e}})$, $\theta_{\mathrm{s}}(\vec{\mathtt{e}})$ and $\theta(\mathtt{e})$ are respectively the north, south and conformal angles associated to the oriented edge $\vec{\mathtt{e}} = (\mathtt{u},\mathtt{v})$ while $R_{\mathrm{n}}(\vec{\mathtt{e}})$ and $R_{\mathrm{s}}(\vec{\mathtt{e}})$ are the circumradii of the respective north $\mathtt{f}_{\mathrm{n}}$ and south $\mathtt{f}_{\mathrm{s}}$ faces associated to $\vec{\mathtt{e}}$ (see figure 1).

Remark 13. *The definitions of the Beltrami-Laplace operator Δ, of the conformal laplacian $\underline{\boldsymbol{\Delta}}$, and of the David-Eynard Kähler operator $\mathcal{D}$ given in 1.2 by 1.7, 1.8 and 1.9 for generic triangulations extend naturally to case of a polyhedral graph $\mathtt{G}$. Moreover, the operators thus defined coincide with the operator defined on its chordless regularized graph $\mathtt{G}^\bullet$.*

Indeed, we can start from a polyhedral graph $\mathtt{G}$, and "fill up" its faces which have $k > 3$ edges by chords, until we get a triangulation $\mathtt{T}$. Similarity, we can remove the chords of $\mathtt{G}$ until we get $\mathtt{G}^\bullet$. If an edge $\mathtt{e} = \overline{\mathtt{uv}}$ of $\mathtt{T}$ is a chord of $\mathtt{G}$ then $\theta_{\mathrm{n}}(\vec{\mathtt{e}}) + \theta_{\mathrm{s}}(\vec{\mathtt{e}}) = \theta(\mathtt{e}) = 0$. Moreover the circumradii of the north and south faces are equal $R_{\mathrm{n}}(\vec{\mathtt{e}}) = R_{\mathrm{s}}(\vec{\mathtt{e}})$. This implies from 1.7, 1.8 and 1.9 that the matrix elements $\Delta_{\mathtt{u},\mathtt{v}}$ and $[\underline{\boldsymbol{\Delta}}]_{\mathtt{u},\mathtt{v}}$ and $\mathcal{D}_{\mathtt{u},\mathtt{v}}$ are zero.

Areas, angles and circumradii.

We recall some basic geometrical formula for these quantities. Let $\mathtt{f} = (\mathtt{v}_1,\mathtt{v}_2,\mathtt{v}_3)$ be a c.c.w. oriented triangle with vertices labelled $\mathtt{v}_1$, $\mathtt{v}_2$, $\mathtt{v}_3$ and respective coordinates z_1, z_2, z_3 then the area $A(\mathtt{f})$ of the triangle is

$$A(\mathtt{f}) = \frac{1}{4\mathrm{i}}(z_2\bar{z}_1 - z_1\bar{z}_2 + z_3\bar{z}_2 - z_2\bar{z}_3 + z_1\bar{z}_3 - z_3\bar{z}_1) \tag{3.4}$$

The circumcenter $z(\mathtt{f})$ of the triangle is given by

$$z(\mathtt{f}) = \frac{z_1\bar{z}_1(z_2 - z_3) + z_2\bar{z}_2(z_3 - z_1) + z_3\bar{z}_3(z_1 - z_2)}{4\mathrm{i}A(\mathtt{f})} \tag{3.5}$$

and the circumradius $R(\mathtt{f})$ of the triangle is given by the trigonometric relation

$$R(\mathtt{f}) = \frac{|z_1 - z_2||z_2 - z_3||z_3 - z_1|}{4\,A(\mathtt{f})} \tag{3.6}$$

while the north angle associated to the oriented edge $\vec{\mathtt{e}} = (\mathtt{v}_1, \mathtt{v}_2)$ is

$$\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) = \frac{1}{2\mathrm{i}} \log\left(-\frac{(\bar{z}_2 - \bar{z}_3)(z_1 - z_3)}{(z_2 - z_3)(\bar{z}_1 - \bar{z}_3)} \right) \tag{3.7}$$

Furthermore $\tan^2\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)$ can be written explicitly in coordinates as

$$\begin{aligned} \tan^2\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) &= \frac{2 + \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3}\frac{\bar{z}_1 - \bar{z}_3}{z_1 - z_3} + \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}\frac{\bar{z}_2 - \bar{z}_3}{z_2 - z_3}}{2 - \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3}\frac{\bar{z}_1 - \bar{z}_3}{z_1 - z_3} - \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}\frac{\bar{z}_2 - \bar{z}_3}{z_2 - z_3}} \\ &= 4\,\frac{|z(\mathtt{f}) - z_{\overline{12}}|^2}{|z_2 - z_1|^2} \qquad \text{with} \quad z_{\overline{12}} = \frac{z_2 + z_1}{2} \end{aligned} \tag{3.8}$$

The derivatives of $A(\mathtt{f})$, $R(\mathtt{f})$ and of the angles $\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)$ under a variation of a vertex coordinate are easy to calculate, using for instance

$$\partial_{z_1} A(\mathtt{f}) = \frac{1}{4\mathrm{i}}(\bar{z}_3 - \bar{z}_2)\ , \quad \partial_{z_1}|z_1 - z_2| = \frac{1}{2}\frac{\bar{z}_1 - \bar{z}_2}{|z_1 - z_2|} \quad \text{with} \quad \partial_{z_1} = \frac{\partial}{\partial z_1} \tag{3.9}$$

and will be discussed later.

The critical Laplacian.

Definition 16. *Let $\mathtt{G}_{\mathrm{cr}}$ be an isoradial Delaunay graph. The Beltrami-Laplace operator Δ, the conformal laplacian $\underline{\boldsymbol{\Delta}}$, and the normalized David-Eynard Kähler operator $R^2\mathcal{D}$ of $\mathtt{G}_{\mathrm{cr}}$ coincide. This common operator is called the* **critical Laplacian** *of $\mathtt{G}_{\mathrm{cr}}$ and is denoted Δ_{cr}.*

3.2. Factorization of laplacians using ∇ and $\overline{\nabla}$ operators.

If these explicit representations in term of angles and circumradii are sufficient, an alternate representation of the operators Δ and $\mathcal{D}$ is convenient for the calculations. We follow the definition and the notations of [DE14].

Definition 17. *The operators ∇ and $\overline{\nabla}$ are linear operators from the space of complex-valued functions over the set of vertices $\mathrm{V}(\mathtt{T})$ of $\mathtt{T}$, onto the space of complex-valued functions over the set of triangles (faces) $\mathrm{F}(\mathtt{T})$ of $\mathtt{T}$.*

$$\mathbb{C}^{\mathrm{V}(\mathtt{T})} \xrightarrow{\nabla} \mathbb{C}^{\mathrm{F}(\mathtt{T})}\ , \quad \mathbb{C}^{\mathrm{V}(\mathtt{T})} \xrightarrow{\overline{\nabla}} \mathbb{C}^{\mathrm{F}(\mathtt{T})}$$

∇ is defined as follows. Given a triangle $\mathtt{f}$ (a face of the triangulation $\mathtt{T}$) with vertices $\mathtt{v}_1$, $\mathtt{v}_2$, $\mathtt{v}_3$ (listed in ccw order) and complex coordinates $z_j := z(\mathtt{v}_j)$ for $1 \le j \le 3$ together with a function $\phi \in \mathbb{C}^{\mathrm{V}(\mathtt{T})}$ define

$$\nabla\phi(\mathtt{f}) = \mathrm{i}\,\frac{\phi(\mathtt{v}_1)(\bar{z}_2 - \bar{z}_3) + \phi(\mathtt{v}_2)(\bar{z}_3 - \bar{z}_1) + \phi(\mathtt{v}_3)(\bar{z}_1 - \bar{z}_2)}{4\,A(\mathtt{f})} \tag{3.10}$$

∇ corresponds to a discrete linear derivative w.r.t. the embedding $\mathtt{v} \mapsto z(\mathtt{v})$ because

$$\nabla z = 1\ , \quad \nabla\bar{z} = 0 \tag{3.11}$$

Similarily, its conjugate $\overline{\nabla}$ *is defined as*

$$\overline{\nabla}\phi(\mathtt{f}) = -\mathrm{i}\,\frac{\phi(\mathtt{v}_1)(z_2-z_3)+\phi(\mathtt{v}_2)(z_3-z_1)+\phi(\mathtt{v}_3)(z_1-z_2)}{4\,A(\mathtt{f})} \tag{3.12}$$

and satisfies

$$\overline{\nabla} z = 0\ , \quad \overline{\nabla}\bar{z} = 1 \tag{3.13}$$

The transpose of these operators are defined accordingly:

$$\mathbb{C}^{\mathrm{F}(\mathtt{T})} \xrightarrow{\nabla^{\top}} \mathbb{C}^{\mathrm{V}(\mathtt{T})}\ , \quad \mathbb{C}^{\mathrm{F}(\mathtt{T})} \xrightarrow{\overline{\nabla}^{\top}} \mathbb{C}^{\mathrm{V}(\mathtt{T})}$$

Remark 14. *It follows from definitions 3.10 and 3.12 and the area formula 3.4 that*

$$\phi(\mathtt{v}_1)-\phi(\mathtt{v}_2) \;=\; (z_1-z_2)\,\nabla\phi(\mathtt{f}) \;+\; (\overline{z}_1-\overline{z}_2)\overline{\nabla}\phi(\mathtt{f}) \tag{3.14}$$

where $\phi\in\mathbb{C}^{\mathrm{V}(\mathtt{T})}$ *is a function.*

Note that the discrete derivatives ∇ and $\overline{\nabla}$ are defined for general triangulations. Even when to triangulation is isoradial, ∇ and $\overline{\nabla}$ *do not coincide* with the discrete holomorphic and discrete antiholomorphic derivatives ∂ and $\bar{\partial}$ considered in [Ken02] for isoradial bipartite graphs. Indeed ∇ and $\overline{\nabla}$ do not even act on the same space of functions than ∂ and $\bar{\partial}$.

Nevertheless, we shall need to bound the difference between the $\nabla\phi$ and the ordinary continuous derivative $\partial\phi$ in the case of a smooth complex-valued function $\phi:\mathbb{C}\longrightarrow\mathbb{C}$ with compact support and its restriction to $\mathrm{V}(\mathtt{T})$ given by $\phi(\mathtt{v}) := \phi(z(\mathtt{v}))$ where $z:\mathrm{V}(\mathtt{T})\longrightarrow\mathbb{C}$ is the embedding of $\mathtt{T}$. This estimate is explained in Lemma 2 of the introduction and proven in Appendix B.

In addition, the ∇-operator satisfies a discrete analogue of Green's Theorem

$$\iint_{\Omega}\partial\phi(z,\bar{z})\,dz\,d\bar{z} \;=\; \oint_{\partial\Omega}\phi(z,\bar{z})\,d\bar{z}$$

in complex coordinates, namely:

Lemma 4. *Let* $\mathtt{T}$ *be a polyhedral triangulation with embedding* $z:\mathrm{V}(\mathtt{T})\longrightarrow\mathbb{C}$*, let* $\Omega\subset\mathrm{F}(\mathtt{T})$ *be a finite collection of triangular faces (each taken with a counter-clockwise orientation), let* $\partial\Omega\subset\mathrm{E}(\mathtt{T})$ *be the finite subset of (oriented) edges corresponding to the boundary of* Ω*, and let* $\phi\in\mathbb{C}^{\mathrm{V}(\mathtt{T})}$ *be a complex-valued function, then*

$$\sum_{\mathrm{x}\,\in\,\Omega} A(\mathrm{x})\nabla\phi(\mathrm{x}) \;=\; \sum_{(\mathrm{u},\mathrm{v})\,\in\,\partial\Omega}\big(\overline{z}(\mathtt{v})-\overline{z}(\mathtt{u})\big)\,\frac{\phi(\mathtt{v})+\phi(\mathtt{u})}{4\mathrm{i}} \tag{3.15}$$

The polyhedral condition can in fact be dropped but we assume it to keep the exposition simple. Lemma 4 implies the following corollary which is relevant to our results.

Corollary 1. *Let* $\mathtt{T}_1$ *and* $\mathtt{T}_2$ *be two polyhedral triangulation which share a common regularized graph* $\mathtt{G}:=\mathtt{T}_1^{\bullet}=\mathtt{T}_2^{\bullet}$*. Given a face* $\mathtt{f}\in\mathrm{F}(\mathtt{G})$ *with vertex set* $\mathrm{V}(\mathtt{f})$ *let* $\Omega_i(\mathtt{f})$ *be the set of triangular faces of* $\mathtt{T}_i$ *each of whose vertices are in* $\mathrm{V}(\mathtt{f})$ *then*

$$\sum_{x_1\,\in\,\Omega_1(\mathtt{f})} A(\mathrm{x}_1)\nabla\phi(\mathrm{x}_1) \;=\; \sum_{x_2\,\in\,\Omega_2(\mathtt{f})} A(\mathrm{x}_2)\nabla\phi(\mathrm{x}_2) \tag{3.16}$$

for any complex-valued function $\phi\in\mathbb{C}^{\mathrm{V}(\mathtt{G})}$.

Definition 18. *The diagonal operators* $A = \operatorname{diag}(\{A(\mathtt{f});\, \mathtt{f} \in \mathrm{F}(\mathtt{G})\})$ *and* $R = \operatorname{diag}(\{R(\mathtt{f});\, \mathtt{f} \in \mathrm{F}(\mathtt{G})\})$ *map* $\mathbb{C}^{\mathrm{F}(\mathtt{G})} \to \mathbb{C}^{\mathrm{F}(\mathtt{G})}$ *and are defined as*

$$A\psi(\mathtt{f}) = A(\mathtt{f})\psi(\mathtt{f}) \;, \quad R\psi(\mathtt{f}) = R(\mathtt{f})\psi(\mathtt{f}) \tag{3.17}$$

Remark 15. *The Laplace-Beltrami operator* Δ *can be factored as*

$$\Delta = 2\left(\overline{\nabla}^{\top} A\, \nabla + \nabla^{\top} A\, \overline{\nabla}\right) \tag{3.18}$$

The derivation is left to the reader.

Remark 16. *The* $\mathcal{D}$ *can be factored as*

$$\mathcal{D} = 4\, \overline{\nabla}^{\top} \frac{A}{R^2} \nabla \tag{3.19}$$

See [DE14] for details.

Remark 17. *No similar decomposition holds for the conformal Laplacian* $\underline{\boldsymbol{\Delta}}$*, since the weight associated to an oriented edge* $\vec{\mathtt{e}}$ *depends non-additively on the north and south angles* $\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)$ *and* $\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)$*.*

3.3. Making sense of the log-determinant for infinite lattices.

The problems. As explained in the introduction, we are interested in studying the variation of the $\log\det\mathcal{O}$ under a variation of the coordinates of the triangulation $\mathtt{T}$, where $\mathcal{O}$ is any of the laplace-like operators Δ, $\underline{\boldsymbol{\Delta}}$ and $\mathcal{D}$. Two potential dangers arise: (1) These operators have zero modes and some care is needed in imposing boundary conditions in order to exclude them. (2) We consider infinite polygonal graphs — and so by any naive account, the log-determinant will infinite. There is a host of standard methods used to handle these issues; below we discuss two situations where problem (1) and (2) can be side stepped.

Using periodic triangulations: Consider a polyhedral graph $\mathtt{G}$ which is periodic with respect to a lattice $\mathbb{Z} + \tau\mathbb{Z}$ with $\mathfrak{Im}\,\tau > 0$. This means there is an action of the additive group $\Lambda = \mathbb{Z}^2$ on $\mathrm{V}(\mathtt{G})$ denoted $\mathtt{v} \mapsto \mathtt{v} + (a,b)$ such that

(1) $z\big(\mathtt{v} + (a,b)\big) = z(\mathtt{v}) + a + \tau b$
(2) $\mathtt{u} + (a,b)$ and $\mathtt{v} + (a,b)$ are joined by an edge whenever $\mathtt{u}$ and $\mathtt{v}$ are joined by an edge (moreover the weights of these edges agree)

for all $\mathtt{u}, \mathtt{v} \in \mathrm{V}(\mathtt{G})$ and $(a,b) \in \Lambda$. Given a choice of an additive subgroup $\Lambda_{mn} := m\mathbb{Z} \times n\mathbb{Z}$ of Λ with $m, n \in \mathbb{Z}_{>0}$ form the quotient graph $\mathtt{G}/\Lambda_{mn}$, which we can view as a finite graph embedded in the torus $\mathbb{T}_{mn} := \mathbb{C}/(m\mathbb{Z} + \tau n\mathbb{Z})$. Since the edge weights are periodic, the operator $\mathcal{O}$ descends to an operator $\mathcal{O}_{mn}$ on the quotient graph $\mathtt{G}/\Lambda_{mn}$; moreover if we identify the vertices of $\mathtt{G}/\Lambda_{mn}$ with the subset V_{mn} consisting of vertices $\mathtt{v} \in \mathrm{V}(\mathtt{G})$ for which $z(\mathtt{v}) \in \big\{s + t\tau \,\big|\, (s,t) \in [0,m) \times [0,n)\big\}$ then $\mathcal{O}_{mn}$ is a finite dimensional operator acting on vector space of dimension $\big|\mathrm{V}_{mn}\big|$.

We define the reduced log-determinant $\log\det'\mathcal{O}_{mn}$ as the sum of the logarithms of the non-zero eigenvalues of $\mathcal{O}$ (the non-zero part of the spectrum is real and positive since $\mathcal{O}$ will be a positive operator in the cases we consider). The normalized reduced log-determinant $\log\det'\mathcal{O}$ is defined as

$$\log\det{}'_{*}\mathcal{O}_{mn} = \frac{1}{|\mathrm{V}_{mn}|}\log\det{}'\mathcal{O}_{mn} \tag{3.20}$$

The normalized log-determinant of $\mathcal{O}$, acting on the entire graph $\mathtt{G}$ (see discussion in 1) can be defined simply as

$$\log\det{}_*\mathcal{O} = \lim_{m,n\to\infty} \log\det{}'_*\mathcal{O}_{mn} \tag{3.21}$$

So $\log\det_*\mathcal{O}$ corresponds to an "effective action" density (free energy density) per vertex on the infinite lattice. In fact the limit in formula 3.21 exists and coincides with the following description in terms of matrix-valued symbols: Choose complex parameters z and w and define the space of quasi-periodic functions

$$\mathcal{F}_{mn}(z,w) = \left\{ \phi : \mathrm{V}(\mathtt{G}) \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} \phi\big(\mathtt{v} + (am, bn)\big) = z^a w^b \phi(\mathtt{v}) \\ \text{for all } \mathtt{v} \in \mathrm{V}(\mathtt{G}) \text{ and } a, b \in \mathbb{Z} \end{array} \right\}$$

This is a finite dimensional vector space of dimension $\dim \mathcal{F}_{mn}(z,w) = \big|\mathrm{V}_{mn}\big|$. Clearly $\mathcal{O}\phi \in \mathcal{F}_{mn}$ whenever $\phi \in \mathcal{F}_{mn}$. and consequently the operator $\mathcal{O}$ restricts to a finite dimensional linear operator $\sigma^{\mathcal{O}}_{mn}$ on $\mathcal{F}_{mn}(z,w)$ which is called the symbol of $\mathcal{O}$. As a matrix the entries of $\sigma^{\mathcal{O}}_{mn}$ are Laurent polynomials in z and w and for generic values of z and w it will be invertible; indeed work of Kassel and Kenyon [KK12] implies that its determinant $\det \sigma^{\mathcal{O}}_{mn}$ is non-negative for values of z and w each having unit modulus. One checks that the average value of the log-determinant of this symbol agrees with normalized log-determinant of $\mathcal{O}$:

$$\log\det{}_*\mathcal{O} = \frac{1}{4\pi^2}\,\frac{1}{\big|\mathrm{V}_{mn}\big|} \int_0^{2\pi}\!\!\int_0^{2\pi} d\zeta\, d\omega\, \log\det \sigma^{\mathcal{O}}_{mn}\big(e^{\mathrm{i}\zeta}, e^{\mathrm{i}\omega}\big) \tag{3.22}$$

Remark 18. *The value of the right hand side of 3.22 can be evaluated using Jensen's formula (twice) and is independent of the choice of $m, n \in \mathbb{Z}_{>0}$.*

Using Dirichlet boundary conditions: Alternatively, for a arbitrary polygonal graph $\mathtt{G}$ (not necessarily periodic) one can consider a sequence of truncated operators $\mathcal{O}_n$ obtained from a nested sequence of domains $\Omega_1 \subset \cdots \Omega_n \subset \Omega_{n+1} \subset \cdots$ whose union is $\mathbb{C}$. For instance, the sequence of $2n \times 2n$ squares $\Omega_n = \{z;\, |\mathrm{Re}(z)| < n, |\mathrm{Im}(z)| < n\}$ where $\mathcal{O}_n$ is the restriction of the operator $\mathcal{O}$ to the subset of vertices $\mathrm{V}_n = \{\mathtt{v} \in \mathrm{V}(\mathtt{G}) \,|\, z(\mathtt{v}) \in \Omega_n\}$ with Dirichlet boundary conditions imposed on the complement of Ω_n; this amounts to setting all matrix elements of $\mathcal{O}$ to zero which involve vertices $\mathtt{v}$ with $z(\mathtt{v}) \notin \Omega_n$. As a matrix $\mathcal{O}_n$ will zero outside a $|\mathrm{V}_n| \times |\mathrm{V}_n|$ submatrix without zero modes. The normalized ∞-volume log-determinant is expected to be equal to the limit

$$\log\det{}_*\mathcal{O} = \lim_{n\to\infty} \frac{1}{|\mathrm{V}_n|} \log\det \mathcal{O}_n \tag{3.23}$$

of such a nested family of graphs (with Dirichlet conditions) in the case of non-periodic graph $\mathtt{G}$ which is sufficiently "regular/homogeneous" (e.g. a quasi-periodic lattice).

Local variation of ∞-volume determinants. Defining properly the finite variation of ∞-volume determinants (by themselves infinite) under local deformation can be done in the two schemes that we have presented above. Let's explain the idea in the Dirichlet boundary scheme. We begin with a polyhedral graph $\mathtt{G}$ and make perturbation $\mathrm{G} \to \mathrm{G}'$ by moving some of its vertices inside a finite size compact domain Ω. The operator $\mathcal{O}$ changes accordingly

$$\mathcal{O} \to \mathcal{O}' = \mathcal{O} + \delta\mathcal{O}$$

If the incidence relations of G do not change, the variation $\delta\mathcal{O}$ will be an operator supported on the finite set $\bar{\Omega}$ consisting of all vertices in Ω plus their nearest neighbouring vertices (any vertex which shares a common face with a vertex in Ω) Considering a nested sequence of domains $\Omega_1 \subset \Omega_2 \cdots \subset \Omega_n \subset \cdots \to \mathbb{C}$ such that $\bar{\Omega} \subset \Omega_1$, it is clear that one can write the variation series expansion for the restriction of $\mathcal{O}$ in each Ω_n

$$\log\det\mathcal{O}'_n = \log\det\mathcal{O}_n + \mathrm{tr}\Big[\delta\mathcal{O}_n\cdot\mathcal{O}_n^{-1}\Big] - \frac{1}{2}\,\mathrm{tr}\Big[\big(\delta\mathcal{O}\cdot\mathcal{O}^{-1}\big)^2\Big] + \cdots \tag{3.24}$$

In the $n \to \infty$ limit, since the $\delta\mathcal{O}_n$ extended to G are equals to $\delta\mathcal{O}$, each term in the expansion will converge to its ∞-volume limit

$$\mathrm{tr}\Big[\big(\delta\mathcal{O}_n\cdot\mathcal{O}_n^{-1}\big)^K\Big] \to \mathrm{tr}\Big[\big(\delta\mathcal{O}\cdot\mathcal{O}^{-1}\big)^K\Big] \tag{3.25}$$

so that, although $\log\det\mathcal{O}'$ and $\log\det\mathcal{O}$ are formally infinite, the difference is finite and one can write

$$\log\det\mathcal{O}' = \log\det\mathcal{O} + \mathrm{tr}\Big[\delta\mathcal{O}\cdot\mathcal{O}^{-1}\Big] - \frac{1}{2}\,\mathrm{tr}\Big[\big(\delta\mathcal{O}\cdot\mathcal{O}^{-1}\big)^2\Big] + \cdots \tag{3.26}$$

We shall study the perturbation around an isoradial, Delaunay graph G_{cr}, where we have seen that $\mathcal{O}_{\mathrm{cr}}^{-1}$ (the Greens function) can be expressed in a simple contour integral form. Moreover we shall consider infinitesimal transformations 5.1, and study the general form of the first order term in 3.26, and some especially interesting terms in the second order term.

3.4. Kenyon's local formula for $\log\det\Delta_{\mathrm{cr}}$.

Kenyon's formula for a periodic infinite lattice. Kenyon derived an explicit formula for the normalized log-determinant of Δ_{cr} for periodic, isoradial, Delaunay triangulations T_{cr}. The proof of this result relies only on the structure of the corresponding rhombic graph $\mathrm{T}_{\mathrm{cr}}^{\Diamond}$ and indeed works for any rhombic graph. For this reason Kenyon's formula implicitly extends to all periodic, isoradial, Delaunay graphs G_{cr}.

$$\log\det{}_*\Delta_{\mathrm{cr}} = \frac{2}{\pi\,|\mathrm{V}_{11}|}\sum_{\substack{\text{edges } \mathsf{e}\\ \text{of } \mathrm{G}_{\mathrm{cr}}/\Lambda}} Л\Big(\theta(\mathsf{e})\Big) + Л\Big(\frac{\pi}{2}-\theta(\mathsf{e})\Big) + \theta(\mathsf{e})\log\tan\theta(\mathsf{e}) \tag{3.27}$$

Extension to general isoradial (weak) Delaunay graphs. Kenyon's formula can be formally extended to express the un-normalized log-determinant $\log\det\Delta_{\mathrm{cr}}$ for a general isoradial, Delaunay graph G_{cr} as a sum over all edges $\mathsf{e} \in \mathrm{E}(\mathrm{G}_{\mathrm{cr}})$, namely:

$$\log\det\Delta_{\mathrm{cr}} = \frac{2}{\pi}\sum_{\mathsf{e}\in\mathrm{E}(\mathrm{G}_{\mathrm{cr}})}\mathcal{L}(\theta(\mathsf{e})) \tag{3.28}$$

with the function $\mathcal{L}$ of the conformal angles $\theta(\mathsf{e})$ given by

$$\mathcal{L}(\theta(\mathsf{e})) = Л\Big(\theta(\mathsf{e})\Big) + Л\Big(\frac{\pi}{2}-\theta(\mathsf{e})\Big) + \theta(\mathsf{e})\,\log\tan\theta(\mathsf{e}) \tag{3.29}$$

where Л is the Lobachevsky function (related to the Clausen function Cl_2)

$$\text{Л}(x) = -\int_0^x dy\,|2\,\log(y)| = \mathrm{Cl}_2(2x)/2 \tag{3.30}$$

We may further generalize this formula to any isoradial *weak* Delaunay graph $\mathsf{G}_{\text{д}}$ obtained from G_{cr} by adding chords inside the faces of G_{cr}, i.e. any graph such that $\mathsf{G}_{\text{д}}^{\bullet} = \mathsf{G}_{\mathrm{cr}}$. Indeed, if e is a chord in $\mathsf{G}_{\text{д}}$ then $\theta_{\mathrm{n}}(\vec{\mathsf{e}}\,) = -\theta_{\mathrm{s}}(\vec{\mathsf{e}}\,)$ and $\mathcal{L}(\theta_{\mathrm{n}}(\vec{\mathsf{e}}\,)) = -\mathcal{L}(\theta_{\mathrm{s}}(\vec{\mathsf{e}}\,))$ where the function $\mathcal{L}(\theta)$ is analytically extended to an *odd function* of θ over $(-\pi, \pi)$. For any isoradial weak Delaunay graph $\mathsf{G}_{\text{д}}$ of this kind, formula 3.28 becomes

$$\log\det\Delta_{\mathrm{cr}} = \frac{1}{\pi}\sum_{\mathsf{e}\,\in\,\mathrm{E}(\mathsf{G}_{\text{д}})} \mathcal{L}(\theta_{\mathrm{n}}(\vec{\mathsf{e}}\,)) + \mathcal{L}(\theta_{\mathrm{s}}(\vec{\mathsf{e}}\,)) \tag{3.31}$$

since the contribution of any chord is zero. This is true in particular for the isoradial, weak Delaunay graphs G_{0+} and $\widehat{\mathsf{G}}_{0+}$ mentioned in definition 1 of the introduction. Note that the derivative of $\mathcal{L}$ is

$$\mathcal{L}'(\theta) = \frac{d}{d\theta}\mathcal{L}(\theta) = \frac{\theta}{\sin\theta\cos\theta} \tag{3.32}$$

4. The critical Green's function and its asymptotics

4.1. Kenyon's formula for the critical Green's function.

The Green's function $\Delta_{\mathrm{cr}}^{-1}$ studied by Kenyon in [Ken02] is a right-inverse of the critical laplacian Δ_{cr} characterized uniquely by the following three conditions

1) $\Delta_{\mathrm{cr}}\,\Delta_{\mathrm{cr}}^{-1} = \mathbb{1}$
2) $\big[\Delta_{\mathrm{cr}}^{-1}\big]_{\mathtt{u},\mathtt{v}} = \mathrm{O}\big(\log|z_{\mathrm{cr}}(\mathtt{u}) - z_{\mathrm{cr}}(\mathtt{v})|\big)$ for $|z_{\mathrm{cr}}(\mathtt{u}) - z_{\mathrm{cr}}(\mathtt{v})| \gg 0$
3) $\big[\Delta_{\mathrm{cr}}^{-1}\big]_{\mathtt{u},\mathtt{u}} = 0$

Here $\mathtt{G}_{\mathrm{cr}}$ is an isoradial Delaunay graph with embedding z_{cr} and $\mathtt{G}_{\mathrm{cr}}^{\Diamond}$ its associated rhombic graph (its embedding is also denoted z_{cr}). Kenyon showed that this critical Green function $\Delta_{\mathrm{cr}}^{-1}$ on $\mathtt{G}_{\mathrm{cr}}$ is expressed by the explicit integral

$$\big[\Delta_{\mathrm{cr}}^{-1}\big]_{\mathtt{u},\mathtt{v}} \;=\; -\frac{1}{8\pi^2\mathrm{i}}\oint_{\mathcal{C}}\frac{dw}{w}\,\log(w)\,\mathrm{E}_{\underline{\theta}(\mathbb{v})}(w) \tag{4.1}$$

where $\mathbb{v} = \big(\mathtt{v}_0,\ldots,\mathtt{v}_k\big)$ is any choice of path from $\mathtt{v}_0 = \mathtt{u}$ to $\mathtt{v}_k = \mathtt{v}$ on $\mathtt{G}_{\mathrm{cr}}^{\Diamond}$ and where $\underline{\theta}(\mathbb{v}) = \big(\theta_1,\ldots,\theta_k\big)$ is the associated sequence of angles. $\mathrm{E}_{\underline{\theta}}(w)$ is the meromorphic function in w

$$\mathrm{E}_{\underline{\theta}}(w) \;:=\; \prod_{j=1}^{k}\frac{w+e^{\mathrm{i}\theta_j}}{w-e^{\mathrm{i}\theta_j}} \tag{4.2}$$

The value of $\mathrm{E}_{\underline{\theta}}(w)$ depends only on the end points $\mathtt{v}_0$ and $\mathtt{v}_k$ of the path; this follows from an argument similar to the proof in demonstrating that the value of $\lfloor\mathbb{v}\rfloor_n$ for odd positive integers n also depends only on the end points $\mathtt{v}_0$ and $\mathtt{v}_k$ of the path. If we fix $\mathtt{v}_0$ and allow the end point $\mathtt{v} = \mathtt{v}_k$ of the path to vary then the mapping $\mathtt{v} \mapsto \mathrm{E}_{\underline{\theta}}(w)$ is an example of a discrete analytic function on $\mathtt{G}_{\mathrm{cr}}^{\Diamond}$ as discussed in [Ken02]. By Lemma 6 the restriction of this mapping to vertices $\mathtt{v} \in \mathrm{V}(\mathtt{G}_{\mathrm{cr}})$ may be viewed as a lattice approximation of the continuous exponential function

$$z \;\mapsto\; \exp\big\{2w\,\big[\overline{z} - \overline{z}_{\mathrm{cr}}(\mathtt{v}_0)\big]\big\}$$

provided $|w| < 1$. For this reason $\mathrm{E}_{\underline{\theta}}(w)$ is referred to as *discrete exponential function*. Finally $\mathcal{C}$ is any closed, counter-clockwise oriented contour enclosing the finite set of phases $\Phi(\mathbb{v}) := \big\{e^{\mathrm{i}\vartheta}\,\big|\,\vartheta \in \Theta(\mathbb{v})\big\}$. As explained in Proposition 3 the set of angles $\Theta(\mathbb{v})$, and thus $\Phi(\mathbb{v})$, are finite and depend only on the end-points $\mathtt{u}$ and $\mathtt{v}$ of the path $\mathbb{v}$. The set of poles of the integrand in formula 4.1 is precisely $\Phi(\mathbb{v})$ and $e^{-\mathrm{i}\theta_0} \notin \Phi(\mathbb{v})$, so a contour C can be choosen which avoids the branch cut $-\theta_0 = \arg\big(z_{\mathrm{cr}}(\mathtt{u}) - \mathrm{cr}(\mathtt{v})\big)$ of the logarithm; see subsection 4.3 below for details.

Remark 19. *Formula (12) is invariant under both global translation and rotation of the graph* $\mathtt{G}_{\mathrm{cr}}$.

Remark 20. *As a special case, Kenyon obtains a simple expression of the Green function for neighbouring vertices* $\mathtt{u}$ *and* $\mathtt{v}$ *of* $\mathtt{G}_{\mathrm{cr}}$ *sharing a common rhombus* $\Diamond$ *of* $\mathtt{G}_{\mathrm{cr}}^{\Diamond}$ *(i.e. two "black" vertices joined by a diagonal of the lozenge). It takes the form*

$$\big[\Delta_{\mathrm{cr}}^{-1}\big]_{\mathtt{u},\mathtt{v}} = -\frac{1}{\pi}\,\theta(\mathtt{e})\cot\theta(\mathtt{e}) \tag{4.3}$$

with $\theta(\mathtt{e}) = \theta(\overline{\mathtt{uv}})$ the angle associated to the edge $\mathtt{e} = \overline{\mathtt{uv}}$. This is a crucial step in the derivation of 3.27. This result is in fact valid for any pair of vertices $\mathtt{u}, \mathtt{v}$ sharing a common cyclic face $\mathtt{f}$ of an isoradial Delaunay graph $\mathtt{G}_{\mathrm{cr}}$, even when the vertices $\mathtt{u}, \mathtt{v}$ are not joined by an edge of $\mathtt{G}_{\mathrm{cr}}$, as depicted in fig 14. The result reads

$$[\Delta_{\mathrm{cr}}^{-1}]_{\mathtt{u},\mathtt{v}} = -\frac{1}{\pi}\,\theta_{\mathrm{n}}(\overline{\mathtt{uv}}) \cot\theta_{\mathrm{n}}(\overline{\mathtt{uv}}) = -\frac{1}{\pi}\,\theta_{\mathrm{s}}(\overline{\mathtt{uv}}) \cot\theta_{\mathrm{s}}(\overline{\mathtt{uv}}) \tag{4.4}$$

This result is valid whenever $\mathtt{u}, \mathtt{v}$ are joined by an edge so that $\theta_{\mathrm{n}}(\overline{\mathtt{uv}}) = \theta_{\mathrm{s}}(\overline{\mathtt{uv}})$ or else joined by a chord of the cyclic face $\mathtt{f}$ so that $\theta_{\mathrm{n}}(\overline{\mathtt{uv}}) = -\theta_{\mathrm{s}}(\overline{\mathtt{uv}}) = \theta(\overline{\mathtt{uv}})$ (the former is the case one recover Kenyon's result).

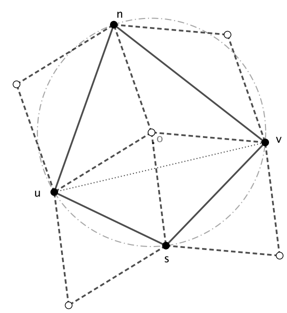

FIGURE 14.

Proof. Select the path $\mathtt{v} = (\mathtt{u}, \mathtt{o_f}, \mathtt{v})$ with $\mathtt{o_f}$ the circumcenter of $\mathtt{f}$ (see fig 14). Set $e^{\mathrm{i}\theta_1} = z_{\mathrm{cr}}(\mathtt{f}) - z_{\mathrm{cr}}(\mathtt{u})$ and $e^{\mathrm{i}\theta_2} = z_{\mathrm{cr}}(\mathtt{v}) - z_{\mathrm{cr}}(\mathtt{f})$ (with $R_{\mathrm{cr}} = 1$) and note that $\theta_{\mathrm{n}}(\overline{\mathtt{uv}}) = (\theta_1 - \theta_2)/2$. Then

$$\begin{aligned}
[\Delta_{\mathrm{cr}}^{-1}]_{\mathtt{u},\mathtt{v}} &= -\frac{1}{8\pi^2 \mathrm{i}} \oint_{\mathcal{C}} \frac{dw}{w}\, \log(w)\, \frac{w + e^{\mathrm{i}\theta_1}}{w - e^{\mathrm{i}\theta_1}}\, \frac{w + e^{\mathrm{i}\theta_2}}{w - e^{\mathrm{i}\theta_2}} \\
&= -\frac{1}{4\pi}\left(2e^{\mathrm{i}\theta_1}\, \frac{e^{\mathrm{i}\theta_1} + e^{\mathrm{i}\theta_2}}{e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2}}\, \frac{\log\left(e^{\mathrm{i}\theta_1}\right)}{e^{\mathrm{i}\theta_1}} + 2e^{\mathrm{i}\theta_2}\, \frac{e^{\mathrm{i}\theta_2} + e^{\mathrm{i}\theta_1}}{e^{\mathrm{i}\theta_2} - e^{\mathrm{i}\theta_1}}\, \frac{\log\left(e^{\mathrm{i}\theta_2}\right)}{e^{\mathrm{i}\theta_2}} \right) \\
&= -\frac{1}{4\pi}\left(2\mathrm{i}(\theta_1 - \theta_2)\, \frac{e^{\mathrm{i}\theta_1} + e^{\mathrm{i}\theta_2}}{e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2}} \right) = -\frac{1}{\pi}\, \frac{\theta_1 - \theta_2}{2} \cot\left(\frac{\theta_1 - \theta_2}{2}\right)
\end{aligned} \tag{4.5}$$

□

4.2. Expansion and bounds for the discrete exponential.

Lemma 5. *Consider a finite sequence of angles $(\theta_1, \ldots, \theta_k)$ contained in the closed interval of the form $\left[\vartheta - \frac{\pi}{2}, \vartheta + \frac{\pi}{2}\right]$ centered about some fixed angle ϑ. Let (by similarity to Def. 14 and Prop. 2)*

$$p_{2n+1} := \sum_{j=1}^{k} e^{\mathrm{i}(2n+1)\theta_j} \tag{4.6}$$

Then we have the uniform bound

$$\left| p_{2n+1} \right| \le (2n+1) \left| p_1 \right|. \tag{4.7}$$

Proof. Clearly it is enough to verify the lemma in the case of $\vartheta = 0$, otherwise we have $p_{2n+1} = e^{-\mathrm{i}\vartheta}\tilde{p}_{2n+1}$ where $\tilde{p}_{2n+1} = \sum_{j=1}^{k} e^{\mathrm{i}(2n+1)\tilde{\theta}_j}$ and where $\tilde{\theta}_j = \theta_j - \vartheta \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$.

Begin with the following polynomial $q_{2n+1}(w) := 2w^2\Big(w^{2n} - (-1)^n\Big)\big(w^2+1\big)^{-1}$ and notice that

$$\begin{aligned} q_{2n+1}(iw) := \quad & 2\big(iw\big)^2\left(\frac{\big(iw\big)^{2n} - (-1)^n}{\big(iw\big)^2+1}\right) = \big(-1\big)^n\, 2w^2\,\frac{w^{2n}-1}{w^2-1} \\ = \quad & \big(-1\big)^n\, 2\left(w^{2n} + w^{2n-2} + \cdots + w^2 + 1\right) \end{aligned}$$

therefore $q_{2n+1}(w) = \big(-1\big)^n 2\Big(1 - w^2 + w^4 - w^6 + \cdots + \big(-1\big)^n w^{2n}\Big)$

For $w = e^{\mathrm{i}\theta}$ with $\theta \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$ the function $\theta \mapsto q_{2n+1}\big(e^{\mathrm{i}\theta}\big)$ is clearly continuous and its modulus takes maximal value $\Big|q_{2n+1}\big(\pm i\big)\Big| = 2n$ and so $\big|q_{2n+1}\big|_\infty = 2n$. By construction $e^{\mathrm{i}(2n+1)\theta} = \cos(\theta)\, q_{2n+1}\big(e^{\mathrm{i}\theta}\big) + (-1)^n e^{\mathrm{i}\theta}$ and so $p_{2n+1} = \sum_{j=1}^{k}\cos(\theta_j)\, q_{2n+1}\big(e^{\mathrm{i}\theta_j}\big) + (-1)^n p_1$. We now proceed with a yoga of inequalities:

$$\begin{aligned} \big|p_{2n+1}\big| \;&\le\; \Big|\sum_{1\le j\le k}\cos(\theta_j)\, q_{2n+1}\big(e^{\mathrm{i}\theta_j}\big)\Big| \;+\; \big|p_1\big| \\ &\le\; \sum_{1\le j\le k}\Big|\cos(\theta_j)\, q_{2n+1}\big(e^{\mathrm{i}\theta_j}\big)\Big| \;+\; \big|p_1\big| \\ &\le\; \sum_{1\le j\le k}\cos(\theta_j)\Big|q_{2n+1}\big(e^{\mathrm{i}\theta_j}\big)\Big| \;+\; \big|p_1\big| \qquad \left(\begin{array}{l}\text{Note that } \cos(\theta_j)\ge 0 \\ \text{because } -\frac{\pi}{2}\le\theta_j\le\frac{\pi}{2}\end{array}\right) \\ &\le\; \sum_{1\le j\le k}\cos(\theta_j)\,\big|q_{2n+1}\big|_\infty \;+\; \big|p_1\big| \\ &\le\; 2n\,\mathfrak{Re}\big[p_1\big] \;+\; \big|p_1\big| \\ &\le\; \big(2n+1\big)\big|p_1\big| \quad \Big(\text{since } 0\le\mathfrak{Re}\big[p_1\big]\le\big|p_1\big|\Big) \end{aligned}$$

□

Lemma 6. *Given a finite sequence of angles* $\underline{\theta} = \big(\theta_1,\ldots,\theta_k\big)$ *and* $|w| < 1$ *the following expansions of* $\mathrm{E}_{\underline{\theta}}(w)$ *are valid:*

$$\mathrm{E}_{\underline{\theta}}(w) \;=\; \big(-1\big)^k\prod_{n\,\mathrm{odd}}\exp\left(\frac{2}{n}w^n\,\overline{p}_n\right) \quad \textit{where } p_n = \sum_{1\le j\le k} e^{\mathrm{i}n\theta_j}. \tag{4.8}$$

Proof.

$$\begin{aligned}
\prod_{j=1}^{k} \frac{w+e^{\mathrm{i}\theta_j}}{w-e^{\mathrm{i}\theta_j}} &= \big(-1\big)^k \prod_{j=1}^{k} \frac{1+we^{-\mathrm{i}\theta_j}}{1-we^{-\mathrm{i}\theta_j}} \\
&= \big(-1\big)^k \exp \sum_{j=1}^{k} \log\left(\frac{1+we^{-\mathrm{i}\theta_j}}{1-we^{-\mathrm{i}\theta_j}}\right) \\
&= \big(-1\big)^k \exp \sum_{j=1}^{k} 2\left(we^{-\mathrm{i}\theta_j} + \frac{1}{3}\,w^3 e^{-3i\theta_j} + \frac{1}{5}\,w^5 e^{-5i\theta_j} \cdots\right) \\
&= \big(-1\big)^k \exp\left(2w\sum_{j=1}^{k} e^{-\mathrm{i}\theta_j} + \frac{2}{3}\,w^3 \sum_{j=1}^{k} e^{-3i\theta_j} + \frac{2}{5}\,w^5 \sum_{j=1}^{k} e^{-5i\theta_j} \cdots\right) \\
&= \big(-1\big)^k \prod_{n\,\mathrm{odd}} \exp\left(\frac{2}{n} w^n\, \overline{p}_n\right) \\
&= \big(-1\big)^k \exp\big(2w\,\overline{p}_1\big)\cdot\left(1+\sum_{N\geq 3} w^N\,\overline{\mathrm{c}}_N\right)
\end{aligned}$$

□

Remark 21. *Let* $\underline{\theta} = \big(\theta_1, \ldots, \theta_n\big)$ *be a finite sequence of angles contained in an interval of the form* $\big[\vartheta - \frac{\pi}{2}, \vartheta + \frac{\pi}{2}\big]$ *where* n *is a positive odd integer. Define*

$$u_n \;=\; \frac{1}{n}\,\frac{p_n}{p_1} \quad \textit{and} \quad \mathfrak{u}(w) = \sum_{\substack{\mathrm{odd}\\ n\geq 3}} u_n w^n \tag{4.9}$$

By Lemma 5 each $|u_n| \leq 1$ *and* $\mathfrak{u}(w)$ *is analytic in the unit disk and* $\mathrm{E}_{\underline{\theta}}(w) = \big(-1\big)^k \cdot \exp\big(2\overline{p}_1 w\big)\cdot\exp\big(2\overline{p}_1\mathfrak{u}(w)\big)$*. Furthermore we have, through the standard combinatorial vinyasas,*

$$\mathrm{E}_{\underline{\theta}}(w) \;=\; \big(-1\big)^k \cdot \exp\big(2\overline{p}_1 w\big)\cdot\left(1+\sum_{m=1}^{\infty}\sum_{d=1}^{m} w^{2m+d}\,\big(2\overline{p}_1\big)^d\,\overline{\mathrm{c}}_{m,d}\right) \tag{4.10}$$

with the coefficients $\mathrm{c}_{m,d}$ *given by*

$$\mathrm{c}_{m,d} = \sum_{\substack{\mathbb{r}\vdash m\\ \#(\mathbb{r})=d}} \prod_{s\geq 1} \frac{1}{(r_s)!}\,(u_{1+2s})^{r_s} \tag{4.11}$$

and where the sum is taken over infinite tuples $\mathbb{r} = \big(r_1, r_2, r_3, \ldots\big) \in \mathbb{Z}_{\geq 0}^{\mathbb{N}}$ *with* $\sum_{s\geq 1} r_s = d$ *and such that* $\sum_{s\geq 1} s\,r_s = m$*.*

Let $\mathtt{u}$ and $\mathtt{v}$ be distinct vertices of $\mathbf{G}_{\mathrm{cr}}$ and let $\mathbb{v} = \big(\mathtt{v}_0, \ldots, \mathtt{v}_k\big)$ be a path from $\mathtt{u}$ to $\mathtt{v}$. Translation and rotation invariance of the Green's function allows us to assume without loss of generality that $\mathtt{u}$ is situated at the origin and that the phases $e^{\mathrm{i}\theta_j} := z_{\mathrm{cr}}(\mathtt{v}_j) - z_{\mathrm{cr}}(\mathtt{v}_{j-1})$ of the path lie in the open interval $\big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$; if not the embedding of $\mathbf{G}_{\mathrm{cr}}$ may be shifted $z \mapsto z - z_{\mathrm{cr}}(\mathtt{u})$ and rotated $z \mapsto z\exp\big(-i\theta_{\mathbb{v}}\big)$ to achieve these features; see Proposition 3 for a definition of $\theta_{\mathbb{v}}$.

4.3. **Contour integral for the expansion.** In [Ken02] Kenyon handles the asymptotic behaviour of the Green's function with respect to the distance $|\mathtt{u}-\mathtt{v}|$ using a *keyhole* contour C with a corridor of width $\epsilon > 0$ avoiding the cut of the logarithm $\arg(\theta) = -\pi$. Paraphrasing Kenyon, this contour C_ϵ runs counter-clockwise along the circle of radius R about the origin (connecting $-R \pm i\epsilon$), then travels horizontally above the x-axis from $-R + i\epsilon$ to $-r + i\epsilon$, runs clockwise along the circle of radius r about the origin (connecting $-r \pm i\epsilon$), and finally returns horizontally from $-r - i\epsilon$ to $-R - i\epsilon$ below the x-axis. Here $R \gg |u - v|$ and $r \ll |\mathtt{u} - \mathtt{v}|^{-1}$; see figure 15. The following lemma allows us to compute the Green's function by integrating along the cut of the logarithm provided we subtract off the logarithmic divergences.

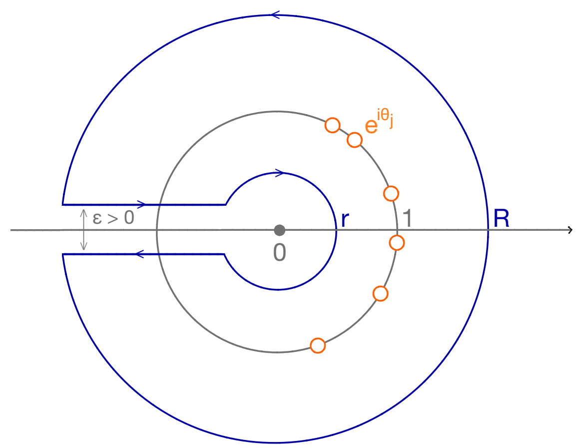

FIGURE 15. keyhole contour C.

Lemma 7. *Let* $\mathrm{F}(w)$ *be a function which is holomorphic on the extended complex plane* $\mathbb{C} \cup \{\infty\}$ *outside a subset* S *contained in the interior of keyhole contour* C *for some values of* R, r, *and* ϵ, *and such that* $F(0) = F(\infty) = 1$ *then*

$$\oint_C \frac{dw}{w} \log(w)\,\mathrm{F}(w) \;=\; -2\pi i \int_0^\infty \Big(\mathrm{F}(-t) - 1\Big)\,\frac{dt}{t} \tag{4.12}$$

Proof.

$$\oint_C \frac{dw}{w}\,\log(w)\,\mathrm{F}(w) \;=\; \lim_{\substack{r\to 0\\ R\to\infty}} \lim_{\epsilon\to 0} \oint_C \frac{dw}{w}\,\log(w)\,\mathrm{F}(w)$$

$$= \lim_{\substack{r\to 0\\ R\to\infty}} \begin{cases} i\displaystyle\int_{\pi}^{-\pi} \log\big(re^{\mathrm{i}\phi}\big)\,\mathrm{F}\big(re^{\mathrm{i}\phi}\big)\,d\phi & \text{integral (1): contribution of circle radius } r \\ \quad + & \\ 2\pi i \displaystyle\int_{-R}^{-r} \mathrm{F}(t)\,\frac{dt}{t} & \text{integral (2): contribution along the cut} \\ \quad + & \\ i\displaystyle\int_{-\pi}^{\pi} \log\big(Re^{\mathrm{i}\phi}\big)\,\mathrm{F}\big(Re^{\mathrm{i}\phi}\big)\,d\phi & \text{integral (3): contribution of circle radius } R \end{cases}$$

$$= \lim_{\substack{r\to 0\\ R\to\infty}} \begin{cases} -2\pi i\log(r) - 2\pi i \displaystyle\sum_{N\geq 1} \frac{1}{N}\big(-r\big)^N a_N \\ \quad + \\ -2\pi i \displaystyle\int_r^R \mathrm{F}(-t)\,\frac{dt}{t} \\ \quad + \\ 2\pi i\log(R) + 2\pi i \displaystyle\sum_{N\geq 1} \frac{1}{N}\big(-R\big)^{-N} b_N \end{cases}$$

$$= -2\pi i \int_0^{\infty} \frac{dt}{t}\,\Big(\mathrm{F}(-t) - 1\Big)$$

where $1 + \sum_{N\geq 1} a_N w^N$ and $1 + \sum_{N\geq 1} b_N w^N$ are the power series expansions of $\mathrm{F}(w)$ at 0 and ∞ respectively. $\square$

Corollary 2. *For vertices* $\mathtt{u}$ *and* $\mathtt{v}$ *in* $\mathbf{G}_{\mathrm{cr}}$ *the value of the Green's function is*

$$\big[\Delta_{\mathrm{cr}}^{-1}\big]_{\mathtt{u},\mathtt{v}} \;=\; \frac{1}{2\pi}\,\mathfrak{Re}\int_0^1 \Big(\mathrm{E}_{\underline{\theta}(\mathtt{v})}(-t) \;-\; 1\Big)\,\frac{dt}{t} \tag{4.13}$$

Proof. We begin with the observation that $\mathrm{E}_{\underline{\theta}}\big(w^{-1}\big) = \big(-1\big)^k\,\overline{\mathrm{E}}_{\underline{\theta}}(w)$ for any finite sequence of angles $\underline{\theta} = \big(\theta_1,\ldots,\theta_k\big)$. Since $\mathtt{u}$ and $\mathtt{v}$ are vertices in $\mathbf{G}_{\mathrm{cr}}$, the length k of any path $\mathtt{v} = \big(\mathtt{v}_0,\ldots,\mathtt{v}_k\big)$ from $\mathtt{v}_0 = \mathtt{u}$ to $\mathtt{v}_k = \mathtt{v}$ in $\mathbf{G}_{\mathrm{cr}}^{\Diamond}$ must be even. Thus $\mathrm{E}_{\underline{\theta}}\big(w^{-1}\big) = \overline{\mathrm{E}}_{\underline{\theta}(\mathtt{v})}(w)$.

$$
\begin{aligned}
\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{u},\mathtt{v}} &= -\frac{1}{8\pi^2 i}\oint_C \frac{dw}{w}\,\log(w)\,\mathrm{E}_{\underline{\theta}(\mathtt{v})}(w)\\
&= \frac{1}{4\pi}\int_0^{\infty}\Big(\mathrm{E}_{\underline{\theta}(\mathtt{v})}(-t)-1\Big)\frac{dt}{t}\\
&= \frac{1}{4\pi}\int_0^{1}\Big(\mathrm{E}_{\underline{\theta}(\mathtt{v})}(-t)-1\Big)\frac{dt}{t} \;+\; \frac{1}{4\pi}\int_1^{\infty}\Big(\mathrm{E}_{\underline{\theta}(\mathtt{v})}(-t)-1\Big)\frac{dt}{t}\\
&= \frac{1}{4\pi}\int_0^{1}\Big(\mathrm{E}_{\underline{\theta}(\mathtt{v})}(-t)-1\Big)\frac{dt}{t} \;+\; \frac{1}{4\pi}\int_0^{1}\Big(\overline{\mathrm{E}}_{\underline{\theta}(\mathtt{v})}(-t)-1\Big)\frac{dt}{t}\\
&= \frac{1}{2\pi}\,\mathfrak{Re}\left[\int_0^{1}\Big(\mathrm{E}_{\underline{\theta}(\mathtt{v})}(-t)-1\Big)\frac{dt}{t}\right]
\end{aligned}
$$

□

Remark 22. *Since $|t|<1$ in formula 4.13 we may use the presentation of $\mathrm{E}_{\underline{\theta}(\mathtt{v})}(t)$ given in Remark 21 and write*

$$
\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{u},\mathtt{v}} \;=\; \frac{1}{2\pi}\,\mathfrak{Re}\int_0^1 \Big(\exp\big(-2p_1 t\big)\cdot\exp\big(2p_1\mathfrak{u}(-t)\big) \;-\; 1\Big)\frac{dt}{t}
$$

We shall adopt the view that p_1 and $\overline{p}_1$ are an independent variables on the plane and that $\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{u},\mathtt{v}}$ is a smooth function of p_1 and $\overline{p}_1$.

4.4. **The general asymptotics.**

Proposition 4. *The Green's function $\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{u},\mathtt{v}}$ has a series expansion at ∞ given by:*

$$
(4.14)\quad -\frac{1}{2\pi}\left(\log\big(2|p_1|\big)+\gamma_{\mathrm{euler}}-\sum_{m\ge d\ge 1}(-1)^d\big(2m+d-1\big)!\,\mathfrak{Re}\Big[\mathrm{c}_{m,d}\,\big(2p_1\big)^{-2m}\Big]\right)
$$

where the coefficients $\mathrm{c}_{m,d}$ are defined in equation 4.11 in terms of the u_{1+2s} defined by 4.9 and bounded by 5.

Proof.

$$
\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{u},\mathtt{v}} = \frac{1}{2\pi}\,\mathfrak{Re}\left[\int_0^1\Big(\mathrm{E}_{\underline{\theta}(\mathtt{v})}(-t)-1\Big)\frac{dt}{t}\right]
$$

$$
= \begin{cases}
\dfrac{1}{2\pi}\,\mathfrak{Re}\left[\displaystyle\int_0^1\Big(\exp\big(-2p_1t\big)-1\Big)\frac{dt}{t}\right]\\[2ex]
+\\[2ex]
\dfrac{1}{2\pi}\displaystyle\sum_{m\ge d\ge 1}\mathfrak{Re}\left[\mathrm{c}_{m,d}\big(2p_1\big)^d\int_0^1 -\big(-t\big)^{2m+d-1}\exp\big(-2p_1t\big)\,dt\right]
\end{cases}
$$

$$
= \left\{ \begin{array}{c} -\dfrac{1}{2\pi}\,\mathfrak{Re}\left[\log\left(2p_1\right) + \gamma_{\text{euler}} + \underbrace{\int_{2p_1}^{\infty} \exp(-t)\,\frac{dt}{t}}_{\substack{\text{null power series}\\ \text{development at } \infty}}\right] \\ + \\ -\dfrac{1}{2\pi}\displaystyle\sum_{m\geq d\geq 1}\mathfrak{Re}\left[\mathrm{c}_{m,d}\left(-1\right)^{d}\left(2p_1\right)^{-2m}\sum_{i=0}^{2m+d-1}\frac{\left(2m+d-1\right)!}{i!}\underbrace{\left(2p_1 t\right)^{i}\exp\left(-2p_1 t\right)}_{\substack{\text{null power series}\\ \text{development at } \infty}}\right]\Bigg|_0^1 \end{array} \right.
$$

$$
= -\frac{1}{2\pi}\left(\log\left(2|p_1|\right) + \gamma_{\text{euler}} - \sum_{m\geq d\geq 1}\left(-1\right)^{d}\left(2m+d-1\right)!\,\mathfrak{Re}\left[\mathrm{c}_{m,d}\left(2p_1\right)^{-2m}\right]\right) \tag{4.15}
$$

□

5. Variations of operators under small deformations

5.1. **Formal deformations of triangulations and variations of operators.** We now consider a formal deformation of the embedding $z : \mathrm{V}(\mathtt{T}) \longrightarrow \mathbb{C}$ of a polygonal triangulation $\mathtt{T}$ and the corresponding deformations of the Laplace-like operators Δ, $\mathcal{D}$, and $\underline{\boldsymbol{\Delta}}$ which are induced. Let $F \in \mathbb{C}^{\mathrm{V}(\mathtt{T})}$ be a complex-valued function on the set of vertices of $\mathtt{T}$,

$$F: \mathrm{V}(\mathtt{T}) \to \mathbb{C}$$

and ϵ a formal parameter. The deformed embedding is defined by

$$z_\epsilon(\mathtt{v}) := z(\mathtt{v}) + \epsilon\, F(\mathtt{v}) , \quad \mathtt{v} \in \mathrm{V}(\mathtt{T}) \tag{5.1}$$

The combinatorics of the triangulation $\mathtt{T}$ is not affected, i.e. the vertex, edge, and face sets of $\mathtt{T}$ remain unchanged. We may introduce deformed discrete differential operators $\nabla_\epsilon, \overline{\nabla}_\epsilon : \mathbb{C}^{\mathrm{V}(\mathtt{T})} \longrightarrow \mathbb{C}^{\mathrm{F}(\mathtt{T})}$ as well as deformed area and radius operators $A_\epsilon, R_\epsilon : \mathbb{C}^{\mathrm{F}(\mathtt{T})} \longrightarrow \mathbb{C}^{\mathrm{F}(\mathtt{T})}$ simply by making the substitution $z \mapsto z_\epsilon$ in formulae 3.10, 3.12, 3.4, and 3.6 respectively. This allows us to unambiguously define deformed versions $\Delta(\epsilon)$ and $\mathcal{D}(\epsilon)$ of the Beltrami-Laplace and discrete Kähler operators using the factorizations 3.18 and 3.19, namely:

$$\Delta(\epsilon) = 2\left(\overline{\nabla}_\epsilon^\top A_\epsilon\, \nabla_\epsilon + \nabla_\epsilon^\top A_\epsilon\, \overline{\nabla}_\epsilon\right) \quad \text{and} \quad \mathcal{D}(\epsilon) = 4\, \overline{\nabla}_\epsilon^\top \frac{A_\epsilon}{R_\epsilon^2}\, \nabla_\epsilon \tag{5.2}$$

We emphasize the purely formal nature of this deformation: Indeed the mapping $\mathtt{v} \mapsto z_\epsilon(\mathtt{v})$ may fail to be an embedding of $\mathtt{T}$ when the deformation parameter ϵ is specialized to any positive real number. Furthermore, upon specialization, the deformed coordinates $z_\epsilon(\mathtt{u})$, $z_\epsilon(\mathtt{v})$, and $z_\epsilon(\mathtt{w})$ of a given triangle $\mathtt{f} = (\mathtt{u}, \mathtt{v}, \mathtt{w})$ in $\mathtt{T}$ may become colinear, causing the area to vanish and rendering $\nabla_\epsilon(\mathtt{f})$ and $\bar{\nabla}_\epsilon(\mathtt{f})$ ill-defined. In spite of these difficulties, we may work over the power series ring $\mathbb{C}[[\epsilon]]$ and expand all the relevant operators as series in ϵ. Up to first order in ϵ, the terms in these developments can be compactly expressed using the discrete derivatives ∇F and $\overline{\nabla} F$ with respect to the triangulation $\mathtt{T}$.

Proposition 5. *The variation of the Laplace-Beltrami operator is*

$$\Delta(\epsilon) \;=\; \Delta - 4\epsilon\left(\nabla^\top (A\, \overline{\nabla} F)\, \nabla + \overline{\nabla}^\top (A\, \nabla \bar{F})\, \overline{\nabla}\right) + \mathrm{O}(\epsilon^2) \tag{5.3}$$

Proposition 6. *The variation of the Kähler operator is*

$$\begin{aligned} \mathcal{D}(\epsilon) \;=\; \mathcal{D} - 4\epsilon \Big[& \overline{\nabla}^\top \frac{A}{R^2} \left(\nabla F + \overline{\nabla} \bar{F} + C\, \overline{\nabla} F + \bar{C}\, \nabla \bar{F}\right) \nabla \\ & + \nabla^\top \frac{A}{R^2} (\overline{\nabla} F) \nabla + \overline{\nabla}^\top \frac{A}{R^2} (\nabla \bar{F}) \overline{\nabla} \Big] + \mathrm{O}(\epsilon^2) \end{aligned} \tag{5.4}$$

with the diagonal function $C \in \mathbb{C}^{\mathrm{F}(\mathtt{T})}$ and its conjugate $\bar{C}$ are defined for a triangle $\mathtt{f} = (\mathtt{u}, \mathtt{v}, \mathtt{w})$ by

$$C(\mathtt{f}) = \left(\frac{\bar{z}(\mathtt{u}) - \bar{z}(\mathtt{v})}{z(\mathtt{u}) - z(\mathtt{v})} + \frac{\bar{z}(\mathtt{v}) - \bar{z}(\mathtt{w})}{z(\mathtt{v}) - z(\mathtt{w})} + \frac{\bar{z}(\mathtt{w}) - \bar{z}(\mathtt{u})}{z(\mathtt{w}) - z(\mathtt{u})}\right) , \quad \bar{C}(\mathtt{f}) = \overline{C(\mathtt{f})} \tag{5.5}$$

Proof: From 3.14, for a pair of vertices $\mathtt{u}$ and $\mathtt{v}$ of a triangle $\mathtt{f}=(\mathtt{u},\mathtt{v},\mathtt{w})$ in $\mathrm{F}(\mathtt{T})$

$$\begin{aligned} z_\epsilon(\mathtt{u})-z_\epsilon(\mathtt{v}) &= z(\mathtt{u})-z(\mathtt{v})+\epsilon\Big((z(\mathtt{u})-z(\mathtt{v}))\nabla F(\mathtt{f})+(\bar z(\mathtt{u})-\bar z(\mathtt{v}))\overline{\nabla} F(\mathtt{f})\Big)\\ \bar z_\epsilon(\mathtt{u})-\bar z_\epsilon(\mathtt{v}) &= \bar z(\mathtt{u})-\bar z(\mathtt{v})+\epsilon\Big((z(\mathtt{u})-z(\mathtt{v}))\nabla \bar F(\mathtt{f})+(\bar z(\mathtt{u})-\bar z(\mathtt{v}))\overline{\nabla} \bar F(\mathtt{f})\Big)\end{aligned} \tag{5.6}$$

Inserting this in 3.4 gives the variation of the area of the triangle $\mathtt{f}$

$$A_\epsilon(\mathtt{f})=A(\mathtt{f})+\epsilon\, A(\mathtt{f})\big(\nabla F(\mathtt{f})+\overline{\nabla}\bar F(\mathtt{f})\big)+\mathrm{O}(\epsilon^2) \tag{5.7}$$

that we rewrite in compact form with the diagonal operators A, ∇F and $\overline{\nabla}\bar F$, acting as $\mathbb{C}^{\mathrm{F}(\mathtt{T})}\to\mathbb{C}^{\mathrm{F}(\mathtt{T})}$

$$A_\epsilon = A+\epsilon\, A(\nabla F+\overline{\nabla}\bar F)+\mathrm{O}(\epsilon^2) \tag{5.8}$$

Note that the exact formula is in Appendix 8. Using 3.6 we can write the variation of the circumradius $R_{\mathtt{f}}$) of the face $\mathtt{f}$. We write only the leading term of order $\mathrm{O}(\epsilon)$ with the same compact notation and with C, and $\bar C$ defined by 5.5

$$\frac{A_\epsilon}{R_\epsilon^2}=\frac{A}{R^2}-\epsilon\,\frac{A}{R^2}\left(C\,\overline{\nabla}F+\bar C\,\nabla\bar F\right)+\mathrm{O}(\epsilon^2) \tag{5.9}$$

Similarily, we get the variation of the matrix elements of the ∇ operator. It is at first order

$$\big[\nabla_\epsilon\big]_{\mathtt{f},\mathtt{v}}=\nabla_{\mathtt{f},\mathtt{v}}-\epsilon\,\left(\nabla F(\mathtt{f})\,\nabla_{\mathtt{f},\mathtt{v}}+\nabla\bar F(\mathtt{f})\,\overline{\nabla}_{\mathtt{f},\mathtt{v}}\right)+\mathrm{O}(\epsilon^2) \tag{5.10}$$

This, and its complex conjugate, read in compact operator form

$$\begin{aligned}\nabla_\epsilon &= \nabla-\epsilon\,\left(\nabla F\,\nabla+\nabla\bar F\,\overline{\nabla}\right)+\mathrm{O}(\epsilon^2)\\ \overline{\nabla}_\epsilon &= \overline{\nabla}-\epsilon\,\left(\overline{\nabla}\bar F\,\overline{\nabla}+\overline{\nabla}F\,\nabla\right)+\mathrm{O}(\epsilon^2)\end{aligned} \tag{5.11}$$

Combining this with 5.2 and the Leibnitz product rule we get 5.3 and 5.4. □

Remark 23. *There is no such a compact expression for the variation of the conformal Laplacian* $\boldsymbol{\Delta}$ *in the general case. In particular, the variation of the weight associated to an edge* $\mathtt{e}=\overline{\mathtt{uv}}$ *will depend on the discrete derivatives of* F *both at the north triangle* $\mathtt{f}_{\mathrm{n}}$ *and the south triangle* $\mathtt{f}_{\mathrm{s}}$ *of the oriented edge* $\vec{\mathtt{e}}=(\mathtt{u},\mathtt{v})$*, which are a priori independent (see figure 1).*

We can of course make the substitution $z\mapsto z_\epsilon$ in formula 3.7 for the north south angles and formally define

$$\begin{aligned}\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon) &:= \frac{1}{2\mathrm{i}}\,\log\left(-\frac{(\bar z_\epsilon(\mathtt{v})-\bar z_\epsilon(\mathtt{n}))(z_\epsilon(\mathtt{u})-z_\epsilon(\mathtt{n}))}{(z_\epsilon(\mathtt{v})-z_\epsilon(\mathtt{n}))(\bar z_\epsilon(\mathtt{u})-\bar z_\epsilon(\mathtt{n}))}\right)\\ \theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon) &:= \frac{1}{2\mathrm{i}}\,\log\left(-\frac{(\bar z_\epsilon(\mathtt{u})-\bar z_\epsilon(\mathtt{s}))(z_\epsilon(\mathtt{v})-z_\epsilon(\mathtt{s}))}{(z_\epsilon(\mathtt{u})-z_\epsilon(\mathtt{s}))(\bar z_\epsilon(\mathtt{v})-\bar z_\epsilon(\mathtt{s}))}\right)\end{aligned} \tag{5.12}$$

whose power series development up to first order in ϵ are

$$\begin{aligned}\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon) &= \theta_{\mathrm{n}}(\vec{\mathtt{e}})+\epsilon\,\frac{\mathrm{i}}{2}\Big(\overline{\nabla}F(\mathtt{f}_{\mathrm{n}})\,\mathcal{E}_{\mathrm{n}}(\vec{\mathtt{e}})-\nabla\bar F(\mathtt{f}_{\mathrm{n}})\,\overline{\mathcal{E}}_{\mathrm{n}}(\vec{\mathtt{e}})\Big)+\mathrm{O}(\epsilon^2)\\ \theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon) &= \theta_{\mathrm{s}}(\vec{\mathtt{e}})+\epsilon\,\frac{\mathrm{i}}{2}\Big(\overline{\nabla}F(\mathtt{f}_{\mathrm{s}})\,\mathcal{E}_{\mathrm{s}}(\vec{\mathtt{e}})-\nabla\bar F(\mathtt{f}_{\mathrm{s}})\,\overline{\mathcal{E}}_{\mathrm{s}}(\vec{\mathtt{e}})\Big)+\mathrm{O}(\epsilon^2)\end{aligned} \tag{5.13}$$

where $\mathtt{n} \in \mathtt{f}_{\mathrm{n}}$ and $\mathtt{s} \in \mathtt{f}_{\mathrm{s}}$ are the respective north and south vertices and where

$$\begin{aligned} \mathcal{E}_{\mathrm{n}}(\vec{\mathtt{e}}) &:= \frac{\overline{z}(\mathtt{v})-\overline{z}(\mathtt{n})}{z(\mathtt{v})-z(\mathtt{n})} - \frac{\overline{z}(\mathtt{u})-\overline{z}(\mathtt{n})}{z(\mathtt{u})-z(\mathtt{n})} &= \frac{-4A(\mathtt{f}_{\mathrm{n}})}{\big(z(\mathtt{v})-z(\mathtt{n})\big)\big(z(\mathtt{u})-z(\mathtt{n})\big)} \\ \mathcal{E}_{\mathrm{s}}(\vec{\mathtt{e}}) &:= \frac{\overline{z}(\mathtt{u})-\overline{z}(\mathtt{s})}{z(\mathtt{u})-z(\mathtt{s})} - \frac{\overline{z}(\mathtt{v})-\overline{z}(\mathtt{s})}{z(\mathtt{v})-z(\mathtt{s})} &= \frac{-4A(\mathtt{f}_{\mathrm{s}})}{\big(z(\mathtt{v})-z(\mathtt{s})\big)\big(z(\mathtt{u})-z(\mathtt{s})\big)} \end{aligned} \tag{5.14}$$

5.2. **Generic notation for variations/derivatives under graph deformations.** Further on, we shall use the following compact notation for the derivatives and the variations of general objects $\mathtt{Obj}$ associated to the deformation of a triangulation $\mathtt{T} \to \mathtt{T}_\epsilon$ (or more generally of a polygonal graph $\mathtt{G} \to \mathtt{G}_\epsilon$) under the deformation of the embedding $z \to z_\epsilon = z + \epsilon\, F$ of the vertices of the graph (as in 5.1), without changing the connectivity of the triangulation, i.e. keeping its edges unchanged $\mathrm{E}(\mathtt{T}_\epsilon) = \mathrm{E}(\mathtt{T})$.

The objects $\mathtt{Obj}$ can be local quantities such as the angles θ, θ_{n}, θ_{s} associated to oriented edge $\vec{\mathtt{e}}$ of $\mathtt{T}$ or the area A and circumradius R of a face $\mathtt{f}$. Other objects of interest include the Laplace-like operator Δ, $\underline{\boldsymbol{\Delta}}$ and $\mathcal{D}$.

If $\mathtt{Obj}$ is the object defined on the unperturbed graph $\mathtt{G}$, the corresponding object on the perturbed graph $\mathtt{G}(\epsilon)$ is denoted

$$\mathtt{Obj}(\epsilon) \quad \text{or sometimes} \quad \mathtt{Obj}_\epsilon \ \ \text{(for clarity or compactness)} \tag{5.15}$$

This is consistent with the notations of the previous section 5.1. The variation of $\mathtt{Obj}$ for finite ϵ is denoted

$$\delta\mathtt{Obj}(\epsilon) = \mathtt{Obj}(\epsilon) - \mathtt{Obj} \tag{5.16}$$

The first derivatives w.r.t. ϵ are denoted

$$\frac{\partial}{\partial\epsilon}\mathtt{Obj}(\epsilon) = \eth_\epsilon\mathtt{Obj}(\epsilon)\ , \quad \frac{\partial^2}{\partial\epsilon^2}\mathtt{Obj}(\epsilon) = \eth_{\epsilon\epsilon}\mathtt{Obj}(\epsilon)\ , \quad \text{etc.} \tag{5.17}$$

so that

$$\frac{\partial}{\partial\epsilon}\mathtt{Obj}(\epsilon)\bigg|_{\epsilon=0} = \eth_\epsilon\mathtt{Obj}\ , \quad \frac{\partial^2}{\partial\epsilon^2}\mathtt{Obj}(\epsilon)\bigg|_{\epsilon=0} = \eth_{\epsilon\epsilon}\mathtt{Obj}\ , \quad \text{etc.} \tag{5.18}$$

Obviously the Taylor expansion of $\mathtt{Obj}$ reads

$$\mathtt{Obj}(\epsilon) = \mathtt{Obj} + \epsilon\ \eth_\epsilon\mathtt{Obj} + \frac{1}{2}\epsilon^2\ \eth_{\epsilon\epsilon}\mathtt{Obj} + \mathrm{O}(\epsilon^3) \tag{5.19}$$

The terms of order ϵ obtained in the previous section 5.1 give the explicit formula of the first derivatives $\eth_\epsilon$ for the objects considered there. We do not rewrite them explicitly.

5.3. **Setup and problems for deformations of isoradial Delaunay graphs.** Here we consider a *geometric deformation* of a fixed, initial *isoradial, Delaunay graph* $\mathtt{G}_{\mathrm{cr}}$, and the induced deformations of the Laplace-like operators $\mathtt{Op} = \Delta$, $\underline{\boldsymbol{\Delta}}$ and $\mathcal{D}$ defined on this initial graph $\mathtt{G}_{\mathrm{cr}}$. The vertex coordinates are transformed according to 5.1

$$z_\epsilon(\mathtt{v}) \ := \ z_{\mathrm{cr}}(\mathtt{v}) \ + \ \epsilon\, F(\mathtt{v}) \tag{5.20}$$

except that the displacements $F(\mathtt{v}) \in \mathbb{C}$ are implemented by a complex-valued function $F \in \mathbb{C}^{\mathrm{V}(\mathtt{G}_{\mathrm{cr}})}$ with **finite support**, i.e. a finite subset $\Omega_F \subset \mathrm{V}(\mathtt{G}_{\mathrm{cr}})$ such that $\mathtt{v} \in \Omega_F \iff F(\mathtt{v}) \neq 0$.

Unlike subsection 5.1, the deformation parameter ϵ is not a formal variable, but rather is allowed to take sufficiently small non-negative real values. The deformed graph $\mathtt{G}_\epsilon$ is the Delaunay graph built out of the set of embedded vertices $\mathtt{v}$ in the plane with coordinates $z_\epsilon(\mathtt{v})$. The deformed operators Δ, $\underline{\boldsymbol{\Delta}}$, and $\mathcal{D}$ are now defined with respect to the Delaunay graph $\mathtt{G}_\epsilon$. There are two issues that must addressed: (i) the deformed mapping $z_\epsilon : \mathrm{V}(\mathtt{G}_{\mathrm{cr}}) \longrightarrow \mathbb{C}$ may fail to be an injective map; (ii) the deformed edge set $\mathrm{E}(\mathtt{G}_\epsilon)$ may differ from the initial critical edge set $\mathrm{E}(\mathtt{G}_{\mathrm{cr}})$. As we shall now argue, these two issues can be controlled for values of ϵ which are small enough.

Lemma 8 of the next section proves that the requirement that F has finite support insures there is a strictly positive bound $\epsilon'_F > 0$, depending on F and on $\mathtt{G}_{\mathrm{cr}}$, which guarantees that the mapping $z_\epsilon : \mathrm{V}(\mathtt{G}_{\mathrm{cr}}) \longrightarrow \mathbb{C}$ is an embedding (i.e. an injective map) as long as $0 \leq \epsilon \leq \epsilon'_F$.

The second issue is more tricky. By construction, the vertex sets of the deformed and critical graphs agree, i.e. $\mathrm{V}(\mathtt{G}_\epsilon) = \mathrm{V}(\mathtt{G}_{\mathrm{cr}})$. On the other hand, the edge set $\mathrm{E}(\mathtt{G}_\epsilon)$ is uniquely and independently determined by the Delaunay construction as applied to the deformed coordinates $z_\epsilon(\mathtt{v})$ for vertices $\mathtt{v} \in \mathrm{V}(\mathtt{G}_\epsilon)$. Consequently $\mathrm{E}(\mathtt{G}_\epsilon)$ may in fact differ from the critical edge set $\mathrm{E}(\mathtt{G}_{\mathrm{cr}})$.

We show in Lemma 9 that there is a second strictly positive bound ϵ_F such that for values of the deformation parameter which are "small enough" $0 \leq \epsilon < \epsilon_F \leq \epsilon'_F$ the deformation 5.20 will not force any Lawson edge flips to occur, and so the edges of $\mathtt{G}_{\mathrm{cr}}$ will remain edges in $\mathtt{G}_\epsilon$, i.e. $\mathrm{E}(\mathtt{G}_{\mathrm{cr}}) \subset \mathrm{E}(\mathtt{G}_\epsilon)$. In particular the edge sets $\mathrm{E}(\mathtt{G}_\epsilon)$ and $\mathrm{E}(\mathtt{G}_{\mathrm{cr}})$ are equal in the case that $\mathtt{G}_{\mathrm{cr}}$ is a triangulation. Complications arise however if $\mathtt{G}_{\mathrm{cr}}$ is not a triangulation, and instead contains cyclic polygonal faces which are not triangles. For *arbitrarily small* $\epsilon > 0$ the deformation 5.20 may introduce new edges which subdivide each of these non-triangular faces. For displacements $\mathtt{v} \mapsto F(\mathtt{v})$ which are sufficiently generic, each subdivision induced by the deformation will form a triangulation of the corresponding face; however it can happen that a subdivision consists of faces some of which are still non-triangular cyclic polygons. An example of such a deformation is depicted in Fig. 16.

It turns out that for values of the deformation parameter $\epsilon > 0$ regulated by the bounds given in Lemma 9, chordal flips inside these non-triangular cyclic faces $\mathtt{f} \in \mathrm{F}(\mathtt{G}_{\mathrm{cr}})$ may still occur. This means that for $\epsilon_1 > 0$ and $\epsilon_2 > 0$ which are both small but distinct, it may happens that $\mathrm{E}(\mathtt{T}_{\epsilon_1}) \neq \mathrm{E}(\mathtt{T}_{\epsilon_2})$, even though both contain $\mathrm{E}(\mathtt{G}_{\mathrm{cr}})$ as a subset. As we shall see in Lemma 10, there is a tighter bound $\tilde{\epsilon}_F > 0$ under which no Lawson flips occur, and thus $\mathrm{E}(\mathtt{T}_\epsilon)$ does not change for values of the deformation parameter within the range $0 < \epsilon < \tilde{\epsilon}_F$. This implies that as $\epsilon \to 0^+$ from the right, the Delaunay graph $\mathtt{G}_\epsilon$ continuously transforms into a graph $\mathtt{G}_{0^+}$, defined properly in lemma 10. This graph $\mathtt{G}_{0^+}$ shares the same vertices with the original Delaunay graph T_{cr}, but its edge set $\mathrm{E}(\mathtt{G}_{0^+})$ will be in general larger than $\mathrm{E}(\mathtt{T}_{\mathrm{cr}})$, since it may contains chords of $\mathtt{T}_{\mathrm{cr}}$. Thus $\mathtt{G}_{0^+}$ will be a weak Delaunay graph, which can substitute as an initial graph to perform a perturbative expansion in $\epsilon > 0$. Furthermore any triangulation $\widehat{\mathtt{G}}_{0^+}$ which completes $\mathtt{G}_{0^+}$ in the sense of Remark 27 is subject to the results of section 5 concerning the variation

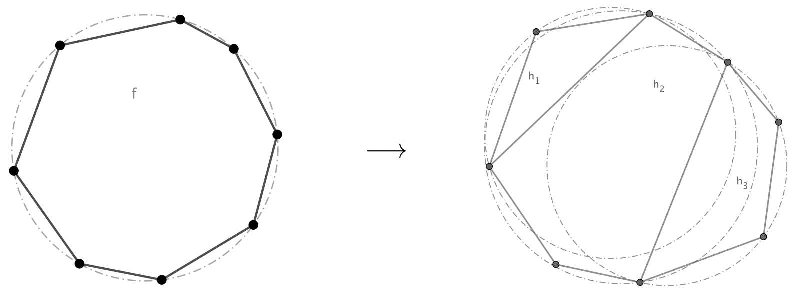

FIGURE 16. Example of deformation of a general cyclic face of a Delaunay graph $\mathtt{T}_{\text{д}}$ into several cyclic faces; here a cyclic octogon f ($n = 8$) splits into 3 cyclic polygons h_1, h_2 an h_3, a triangle ($n_1 = 3$), a pentagon ($n_2 = 5$) and a quadrilateral ($n_3 = 4$).

of the operators Δ, $\underline{\boldsymbol{\Delta}}$ and $\mathcal{D}$ in the linearized approximation (first order in $\epsilon > 0$), without having to consider flips.

In the next section 5.4, we made these assertions precise, and get explicit values for the bounds over $\epsilon > 0$, when such bounds exist.

In appendix 8, we extend the discussion to the cases where flips can occurs for small values of the perturbation parameter $\epsilon > 0$, so that the linear approximation in ϵ is not valid anymore. In some cases one may still expect some control on the variation of the operators. This will be useful when discussing the scaling limit.

5.4. ϵ bounds for keeping control on deformations.

Some care is needed to insure that vertices do not collide under the perturbation; otherwise the mapping $\mathtt{v} \mapsto z_\epsilon(\mathtt{v})$ will not be an embedding. This can be achieved by placing an upper bound on the deformation parameter $\epsilon > 0$, through this simple lemma.

Lemma 8. *For any for pair of distinct vertices* $\mathtt{u}, \mathtt{v} \in \mathtt{G}_{\text{cr}}$ *the corresponding perturbed coordinates* $z_\epsilon(\mathtt{u})$ *and* $z_\epsilon(\mathtt{v})$ *will always remain distinct provided*

$$0 \leq \epsilon < \epsilon'_F = M_F^{-1} \tag{5.21}$$

where

$$M_F = \max_{\mathtt{u} \neq \mathtt{v}} \left| dF\big(\overline{\mathtt{uv}}\big) \right| \quad \text{and} \quad dF\big(\mathtt{u}, \mathtt{v}\big) = \frac{F(\mathtt{u}) - F(\mathtt{v})}{z_{\text{cr}}(\mathtt{u}) - z_{\text{cr}}(\mathtt{v})} \tag{5.22}$$

Proof. The mapping $\mathtt{v} \mapsto F(\mathtt{v})$ has finite support, so M_F is well-defined. The coordinates $z_\epsilon(\mathtt{u})$ and $z_\epsilon(\mathtt{v})$ are distinct provided $1 + \epsilon\, dF(\mathtt{u}, \mathtt{v})$ is non-zero; the later is clearly the case whenever $\epsilon \leq M_F^{-1}$. □

Unlike the vertex set, the edge set $\mathrm{E}(\mathtt{G}_\epsilon)$ evolves with the deformation parameter $\epsilon > 0$ according to the Delaunay condition. Nevertheless, we may restrict the deformation parameter by an additional upper bound $\epsilon_F > 0$ to insure that the edge set of the critical graph is stable under deformation, i.e. to force $\mathrm{E}(\mathtt{G}_{\text{cr}}) \subset \mathrm{E}(\mathtt{G}_\epsilon)$ whenever $0 \leq \epsilon < \epsilon_F$.

Recall that two vertices $\mathtt{u}, \mathtt{v} \in \mathrm{V}(\mathsf{G})$ in a Delaunay graph G with embedding $z : \mathrm{V}(\mathsf{G}) \longrightarrow \mathbb{C}$ are joined by an edge $\overline{\mathtt{uv}} \in \mathrm{E}(\mathsf{G})$ if and only if there exists a circle $\mathrm{C}_{\overline{\mathtt{uv}}}$ passing through $z(\mathtt{u})$ and $z(\mathtt{v})$ such that the closed disk of $\mathrm{C}_{\overline{\mathtt{uv}}}$ does not contain $z(\mathtt{w})$ for any third vertex $\mathtt{w} \neq \mathtt{u}, \mathtt{v}$. Equivalently, there exists an auxiliary point $\mathfrak{z}(\overline{\mathtt{uv}}) \neq z(\mathtt{u}), z(\mathtt{v})$ such that

$$\begin{gathered} \mathfrak{Im}\big[z(\mathtt{u}), z(\mathtt{v})\,; z(\mathtt{w}), \mathfrak{z}(\overline{\mathtt{uv}})\big] > 0 \quad \text{for all vertices } \mathtt{w} \notin \{\mathtt{u}, \mathtt{v}\} \\ \text{— or —} \\ \mathfrak{Im}\big[z(\mathtt{u}), z(\mathtt{v})\,; \mathfrak{z}(\overline{\mathtt{uv}}), z(\mathtt{w})\big] > 0 \quad \text{for all vertices } \mathtt{w} \notin \{\mathtt{u}, \mathtt{v}\} \end{gathered} \tag{5.23}$$

where $\big[z_1, z_2\,; z_3, z_4\big]$ denotes the cross-ration defined by

$$\big[z_1, z_2\,; z_3, z_4\big] \;:=\; \frac{\big(z_1 - z_3\big)\cdot\big(z_2 - z_4\big)}{\big(z_1 - z_4\big)\cdot\big(z_2 - z_3\big)}$$

The edge condition 5.23 simplifies in the case of the critical graph G_{cr} in virtue of the isoradial property. Specifically we may choose the circle $\mathrm{C}_{\overline{\mathtt{uv}}}$ centered at the midpoint of the line segment joining $z_{\mathrm{cr}}(\mathtt{u})$ and $z_{\mathrm{cr}}(\mathtt{v})$ which passes through $z_{\mathrm{cr}}(\mathtt{u})$ and $z_{\mathrm{cr}}(\mathtt{v})$. In term of the cross-ratio this means

$$\mathfrak{Im}\big[z_{\mathrm{cr}}(\mathtt{u}), z_{\mathrm{cr}}(\mathtt{v})\,; z_{\mathrm{cr}}(\mathtt{w}), \mathfrak{z}(\overline{\mathtt{uv}})\big] \;=\; \mathfrak{Re}\left[\frac{z_{\mathrm{cr}}(\mathtt{u}) - z_{\mathrm{cr}}(\mathtt{w})}{z_{\mathrm{cr}}(\mathtt{v}) - z_{\mathrm{cr}}(\mathtt{w})}\right] > 0$$

for any third vertex $\mathtt{w} \neq \mathtt{u}, \mathtt{v}$ where $\mathfrak{z}(\overline{\mathtt{uv}})$ is any point on the circle $\mathrm{C}_{\overline{\mathtt{uv}}}$, for example

$$\mathfrak{z}(\overline{\mathtt{uv}}) \;:=\; \frac{z_{\mathrm{cr}}(\mathtt{v}) + z_{\mathrm{cr}}(\mathtt{u})}{2} \;+\; \mathrm{i}\,\frac{z_{\mathrm{cr}}(\mathtt{v}) - z_{\mathrm{cr}}(\mathtt{u})}{2}$$

For $\epsilon \geq 0$, an edge $\overline{\mathtt{uv}} \in \mathrm{E}(\mathsf{G}_{\mathrm{cr}})$ and a vertex $\mathtt{w} \neq \mathtt{u}, \mathtt{v}$ define

$$\kappa_\epsilon\big(\overline{\mathtt{uv}}\,; \mathtt{w}\big) := \mathfrak{Re}\left[\frac{z_\epsilon(\mathtt{u}) - z_\epsilon(\mathtt{w})}{z_\epsilon(\mathtt{v}) - z_\epsilon(\mathtt{w})}\right] \tag{5.24}$$

which is clearly positive when $\epsilon = 0$ for all vertices $\mathtt{w} \neq \mathtt{u}, \mathtt{v}$.

Lemma 9. *Let $F : \mathrm{V}(\mathsf{G}_{\mathrm{cr}}) \longrightarrow \mathbb{C}$ be a non-zero, complex-valued function on the set of vertices having finite support $\bar{\Omega}_F$. Define the extrema*

$$\vartheta_F \;:=\; \min\left\{\theta(\overline{\mathtt{uv}}) \,\middle|\, \overline{\mathtt{uv}} \in \mathrm{E}(\mathsf{G}_{\mathrm{cr}}) \cap \bar{\Omega}_F\right\}$$

which exists and is positive since F has finite support and is non-zero. Define the following deformation threshold

$$\epsilon_F = \frac{\sin\big(\vartheta_F\big)}{2M_F\big(1 + M_F\big)} \tag{5.25}$$

then $\mathrm{E}(\mathsf{G}_{\mathrm{cr}}) \subset \mathrm{E}(\mathsf{G}_\epsilon)$ *whenever* $0 \leq \epsilon < \epsilon_F$.

Proof. We need to show that $\kappa_\epsilon\big(\overline{\mathtt{uv}}\,; \mathtt{w}\big) > 0$ for all edges $\overline{\mathtt{uv}} \in \mathrm{E}(\mathsf{G}_{\mathrm{cr}})$ and all vertices $\mathtt{w} \neq \mathtt{u}, \mathtt{v}$ provided the deformation is within the range $0 \leq \epsilon < \epsilon_F$. Let us begin by assuming (provisionally) that $\epsilon < (1 + M_F)^{-1}$ then

$$
\begin{aligned}
\left|\kappa_\epsilon\big(\overline{\mathtt{uv}}\,;\mathtt{w}\big)-\kappa_0\big(\overline{\mathtt{uv}}\,;\mathtt{w}\big)\right| &= \epsilon\left|\mathfrak{Re}\left[\frac{z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{w})}{z_{\mathrm{cr}}(\mathtt{v})-z_{\mathrm{cr}}(\mathtt{w})}\cdot\frac{dF(\mathtt{u},\mathtt{w})-dF(\mathtt{v},\mathtt{w})}{1+\epsilon dF(\mathtt{v},\mathtt{w})}\right]\right| \\
&\le \epsilon\left|\frac{z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{w})}{z_{\mathrm{cr}}(\mathtt{v})-z_{\mathrm{cr}}(\mathtt{w})}\right|\cdot\left|\frac{dF(\mathtt{u},\mathtt{w})-dF(\mathtt{v},\mathtt{w})}{1+\epsilon dF(\mathtt{v},\mathtt{w})}\right| \\
&\le \epsilon\left|\frac{z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{w})}{z_{\mathrm{cr}}(\mathtt{v})-z_{\mathrm{cr}}(\mathtt{w})}\right|\cdot\frac{|dF(\mathtt{u},\mathtt{w})|+|dF(\mathtt{v},\mathtt{w})|}{1-\epsilon|dF(\mathtt{v},\mathtt{w})|} \\
&\le \epsilon\left|\frac{z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{w})}{z_{\mathrm{cr}}(\mathtt{v})-z_{\mathrm{cr}}(\mathtt{w})}\right|\cdot\frac{2M_F}{1-\epsilon M_F} \\
&\le \epsilon\left|\frac{z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{w})}{z_{\mathrm{cr}}(\mathtt{v})-z_{\mathrm{cr}}(\mathtt{w})}\right|\cdot 2M_F\big(1+M_F\big)
\end{aligned} \tag{5.26}
$$

Clearly $\kappa_\epsilon\big(\overline{\mathtt{uv}}\,;\mathtt{w}\big)>0$ whenever $\left|\kappa_\epsilon\big(\overline{\mathtt{uv}}\,;\mathtt{w}\big)-\kappa_0\big(\overline{\mathtt{uv}}\,;\mathtt{w}\big)\right|<\kappa_0\big(\overline{\mathtt{uv}}\,;\mathtt{w}\big)$ the later being achieved (in view of the preceding chain of inequalities) provided

$$
\begin{aligned}
\epsilon &< \kappa_0\big(\overline{\mathtt{uv}}\,;\mathtt{w}\big)\cdot\left|\frac{z_{\mathrm{cr}}(\mathtt{v})-z_{\mathrm{cr}}(\mathtt{w})}{z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{w})}\right|\,\Big(2M_F\big(1+M_F\big)\Big)^{-1} \\
&< \cos\Big(\sphericalangle\mathtt{vwu}\Big)\cdot\Big(2M_F\big(1+M_F\big)\Big)^{-1}
\end{aligned}
$$

By the inscribed angle theorem we know that the angle measure $\sphericalangle\mathtt{vwu}$ can exceed neither $\pi/2-\theta_{\mathrm{n}}(\vec{\mathtt{uv}})$ nor $\pi/2-\theta_{\mathrm{s}}(\vec{\mathtt{uv}})$. Accordingly we many simultaneously guarantee the positivity of $\kappa_\epsilon\big(\overline{\mathtt{uv}}\,;\mathtt{w}\big)$ for for all edges $\overline{\mathtt{uv}}\in\mathrm{E}(\mathbf{G}_{\mathrm{cr}})$ and all vertices $\mathtt{w}\neq\mathtt{u},\mathtt{v}$ by bounding the deformation parameter by

$$
\begin{aligned}
&\epsilon < \cos\Big(\frac{\pi}{2}-\vartheta_F\Big)\cdot\Big(2M_F\big(1+M_F\big)\Big)^{-1} \quad \text{where} \\
&\vartheta_F := \min\Big\{\theta(\overline{\mathtt{uv}})\,\Big|\,\overline{\mathtt{uv}}\in\mathrm{E}(\mathbf{G}_{\mathrm{cr}})\cap\bar{\Omega}_F\Big\} \quad \text{which is non zero}
\end{aligned}
$$

□

Any additional edge $\overline{\mathtt{uv}}\in\mathrm{E}(\mathbf{G}_\epsilon)$ which is not present in $\mathrm{E}(\mathbf{G}_{\mathrm{cr}})$ for values of the deformation parameter within the range $0<\epsilon<\epsilon_F$ corresponds to a pair of non-adjacent vertices $\mathtt{u},\mathtt{v}\in\mathrm{V}(\mathbf{G}_{\mathrm{cr}})$ which reside on a common cyclic face; i.e. such an edge $\overline{\mathtt{uv}}$ would form a chord were it included in the critical edge set $\mathrm{E}(\mathbf{G}_{\mathrm{cr}})$. The following result establishes that these additional edges are stable for all values of the deformation parameter $\epsilon>0$ which are sufficiently small.

Lemma 10. *There exists a deformation threshold $\tilde{\epsilon}_F>0$ such that $\mathrm{E}(\mathbf{G}_{\epsilon_1})=\mathrm{E}(\mathbf{G}_{\epsilon_2})$ for all $0<\epsilon_1,\epsilon_2<\tilde{\epsilon}_F$. As a consequence, the limit when $\epsilon\to 0^+$ of the Delaunay graph $\mathbf{G}_\epsilon$ is unambiguously defined, and is denoted $\mathbf{G}_{0^+}$*

$$
\mathbf{G}_{0^+} = \lim_{\epsilon\to 0^+}\mathbf{G}_\epsilon \tag{5.27}
$$

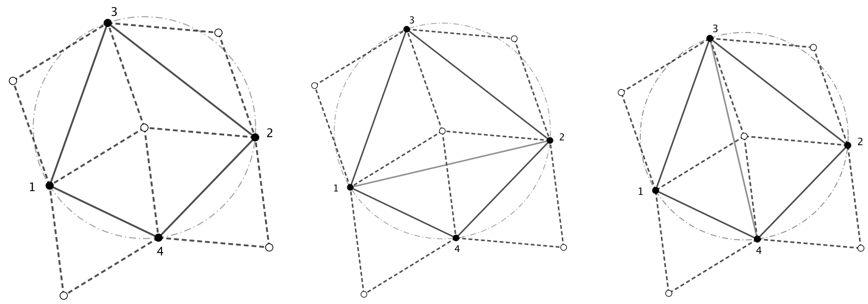

FIGURE 17. A cyclic quadrilateral face Q of G_{cr}, as considered in the proof of Lem. 10 (left), and the two possible triangulations arising from generic deformations (right).

G_{0+} *is a weak-Delaunay graph with the same set of vertices as* G_{cr}*, and such that* $\mathsf{G}_{0+}^{\bullet} = \mathsf{G}_{\mathrm{cr}}$.

Proof. Let $Q = \big(\mathtt{u},\mathtt{s},\mathtt{v},\mathtt{n}\big)$ be a quadrilateral inscribed in a face of G_{cr}, i.e. a quadruple of four distinct vertices in a face of G_{cr} listed in cyclic (counter-clockwise) order in the face; in particular neither the vertices $\mathtt{u}$, $\mathtt{v}$ nor the vertices $\mathtt{n}$, $\mathtt{s}$ are joined by an edge in the critical graph (see fig. 17).

For $\epsilon \geq 0$ real we consider the deformed conformal angle for the edge $\overline{\mathtt{uv}}$ in the quadrangle Q, given by

$$\begin{aligned}\theta_\epsilon\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big) =&\ \operatorname{Arg}\left([z_\epsilon(\mathtt{u}),z_\epsilon(\mathtt{v});z_\epsilon(\mathtt{n}),z_\epsilon(\mathtt{s})]\right)\\ =&\ \mathfrak{Im}\,\log\left[\frac{\Big(1+\epsilon\,dF\big(\mathtt{u},\mathtt{n}\big)\Big)\cdot\Big(1+\epsilon\,dF\big(\mathtt{v},\mathtt{s}\big)\Big)}{\Big(1+\epsilon\,dF\big(\mathtt{u},\mathtt{s}\big)\Big)\cdot\Big(1+\epsilon\,dF\big(\mathtt{v},\mathtt{n}\big)\Big)}\right]\end{aligned} \tag{5.28}$$

which vanishes when $\epsilon = 0$ since then $\overline{\mathtt{uv}}$ is a chord (Q is a cyclic quadrangle). $\theta_\epsilon\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)$ can clearly be extended from a function of real ϵ into an analytic function of complex ϵ in the disc $|\epsilon| < M_F^{-1}$. Therefore, either $\theta_\epsilon\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)$ vanishes identically as a function of $\epsilon \geq 0$, or there exists a positive integer $k \geq 1$ such that it's first derivatives at the origin vanishes, but the k^{th} is non-zero, namely $\theta_0^{(k)}\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big) \neq 0$ and $\theta_0^{(j)}\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big) = 0$ for all $1 \leq j < k$ where

$$\theta_0^{(k)}\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big) := \left.\frac{d^k}{d\varepsilon^k}\right|_{\varepsilon=0} \theta_\varepsilon\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big) \tag{5.29}$$

In fact this k is $1 \leq k \leq 4$ (see Remark 24). By continuity and the positivity of $\big|\theta_0^{(k)}\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)\big|$ we can find $\bar{\epsilon}_F\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big) > 0$ such that $\theta_\epsilon\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)$ is either entirely non-negative or else entirely non-positive within the range $0 \leq \epsilon < \tilde{\epsilon}_F\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)$. The same result can be made uniform by replacing $\bar{\epsilon}_F\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)$ with the minimum $\bar{\epsilon}_F > 0$ of the thresholds $\tilde{\epsilon}_F\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)$ as $\big(\mathtt{u},\mathtt{s},\mathtt{v},\mathtt{n}\big)$ varies over all quadruples of vertices within all cyclic faces of the critical graph G_{cr}, such that $\theta_\epsilon\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)$ is not identically zero (this implies that at least one of the four vertices lies in $\mathrm{Supp}(F)$, so this set is finite). Now we consider

$$\tilde{\epsilon}_F = \min(\epsilon_F, \bar{\epsilon}_F) \tag{5.30}$$

Let $\mathrm{E}_\epsilon(\mathtt{f}) := \{\overline{\mathtt{uv}} \in \mathrm{E}(\mathtt{G}_\epsilon) \,|\, \mathtt{u},\mathtt{v} \in \mathrm{V}(\mathtt{f})\}$ and $\mathrm{E}^c_\epsilon(\mathtt{f}) := \mathrm{E}_\epsilon(\mathtt{f}) - \mathrm{E}(\mathtt{f})$. For $0 \le \epsilon < \epsilon_F$ the Lawson flip algorithm implies there exists a triangulation $\mathscr{T}_\epsilon(\mathtt{f})$ of each face $\mathtt{f} \in \mathrm{F}(\mathtt{G}_{\mathrm{cr}})$ such that: (1) $\mathrm{E}(\mathtt{f}) \subset \mathrm{E}(\mathscr{T}_\epsilon(\mathtt{f}))$ and (2) whenever $\triangle^\circlearrowleft_{\mathrm{n}} = (\mathtt{u},\mathtt{v},\mathtt{n})$ and $\triangle^\circlearrowleft_{\mathrm{s}} = (\mathtt{v},\mathtt{u},\mathtt{s})$ are adjacent triangles in $\mathrm{F}(\mathscr{T}_\epsilon(\mathtt{f}))$ the deformed conformal angle $\theta_\epsilon\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)$ is non-negative. In principle a triangulation $\mathscr{T}_\epsilon(\mathtt{f})$ of this kind need not be unique; nevertheless the subset of edges $\overline{\mathtt{uv}} \in \mathrm{E}(\mathscr{T}_\epsilon(\mathtt{f})) - \mathrm{E}(\mathtt{f})$ for which $\theta_\epsilon\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)$ is positive will always equal $\mathrm{E}^c_\epsilon(\mathtt{f})$. We have established that each conformal angle $\theta_\epsilon\big(\mathtt{u},\mathtt{v}\,;\mathtt{n},\mathtt{s}\big)$ is either identically zero or else remains positive for any value of the deformation parameter within the range $0 \le \epsilon < \tilde\epsilon_F$. Consequently if $\mathscr{T}_\epsilon(\mathtt{f})$ satisfies conditions (1) and (2) for a particular value of $\epsilon = \epsilon_1$ within the range $0 < \epsilon_1 < \tilde\epsilon_F$ then it will clearly satisfy these conditions for any other choice $\epsilon = \epsilon_2$ in this range. Therefore $\mathrm{E}^c_{\epsilon_1}(\mathtt{f}) = \mathrm{E}^c_{\epsilon_2}(\mathtt{f})$ for each face $\mathtt{f} \in \mathrm{F}(\mathtt{G}_{\mathrm{cr}})$ for any pair of deformation values $0 < \epsilon_1, \epsilon_2 < \tilde\epsilon_F$. Clearly $\mathrm{E}^c(\mathtt{G}_\epsilon) = \bigcup_{\mathtt{f}\in\mathrm{F}(\mathtt{G}_{\mathrm{cr}})} \mathrm{E}^c_\epsilon(\mathtt{f})$ and so the claim is settled. □

Remark 24. *In the proof of lemma 10, it it mentioned that for a quadrilateral* $(\mathtt{u},\mathtt{s},\mathtt{v},\mathtt{n})$ *if the conformal angle* $\theta_\epsilon(\mathtt{u},\mathtt{v};\mathtt{n},\mathtt{s})$ *and its first fourth derivatives w.r.t.* ϵ *vanishes at* $\epsilon = 0$*, then* $\theta_\epsilon(\mathtt{u},\mathtt{v};\mathtt{n},\mathtt{s}) = 0$ *for all* ϵ*. Let* $\mathcal{C}_+$ *be the subset of* $\mathbb{C}$*,* $\{\epsilon : \theta_\epsilon(\mathtt{u},\mathtt{v};\mathtt{n},\mathtt{s}) = 0\}$*. It is a subset of the curve* $\mathfrak{C} = \{\epsilon :\, [z_\epsilon(\mathtt{u}), z_\epsilon(\mathtt{v}); z_\epsilon(\mathtt{n}), z_\epsilon(\mathtt{s})] \in \mathbb{R}\}$*. Using 5.28 this condition can be written*

$$\mathfrak{Re}\left((1+\epsilon\, dF(\mathtt{u},\mathtt{n}))(1+\epsilon\, dF(\mathtt{v},\mathtt{s}))(1+\bar\epsilon\,\overline{dF}(\mathtt{u},\mathtt{s}))(1+\bar\epsilon\,\overline{dF}(\mathtt{v},\mathtt{n}))\right) = 0$$

so that $\mathcal{C}$ *is a quadric curve in the real plane* $(x,y) \in \mathbb{R}^2$*,* $z = x + \mathrm{i}y$*, solution of equation*

$$\mathfrak{P}_4(x,y) = 0\ , \quad \mathfrak{P}_4 \text{ a degree 4 real polynomial} \tag{5.31}$$

The quartic $\mathfrak{C}$ *contains the origin* $(0,0)$ *(i.e.* $\epsilon = 0$*), and can cross the real axis* $\mathfrak{R} = \{(x,y);\, y = 0\}$ *(i.e.* ϵ *real) at at most 4 points. If* θ_ϵ *and its first 4 derivatives vanishes at the origin,* $x = 0$ *must be a zero of degree* ≥ 5 *of the equation* $\mathfrak{P}_4(x,0) = 0$*. This is only possible if* $\mathfrak{P}_4(x,0) \equiv 0$*.* □

Remark 25. *As discussed in appendix 8, the bound* $\tilde\epsilon_F$ *(which defines an interval* $0 < \epsilon < \tilde\epsilon_F$ *where no flips occur) may be much smaller than* ϵ_F*. In fact even for a fixed* $\mathtt{G}_{\mathrm{cr}}$ *and a generic deformation* F*, the threshold* $\tilde\epsilon_F$ *may be arbitrarily small w.r.t.* ϵ_F*. This point will become relevant when discussing the scaling limit and the problem of obtaining uniform bounds with respect to the choice of* $\mathtt{G}_{\mathrm{cr}}$*.*

Remark 26. *Since for* $0 \le \epsilon < \tilde\epsilon_F$*, the set of edges of* $\mathtt{G}_\epsilon$ *is unchanged* $\mathrm{E}(\mathtt{G}_\epsilon) = \mathrm{E}(\mathtt{G}_{0^+})$*, and since the faces of* $\mathtt{G}_{0^+}$ *stay cyclic polygons, the conformal angle* $\theta_\epsilon(\mathtt{e})$ *of any edge* $\mathtt{e} \in \mathrm{E}(\mathtt{G}_\epsilon)$ *is unambiguously defined and strictly positive in the interval, although* $\mathtt{G}_\epsilon$ *need not be a triangulation.*

Remark 27. *When* $\mathtt{G}_\epsilon$ *is not a triangulation, it will be convenient to construct a* ***triangular completion*** $\widehat{\mathtt{G}}_\epsilon$ *by maximally adjoining pairwise non-crossing chords inside each of the non-triangular cyclic faces of* $\mathtt{G}_\epsilon$*. By design* $\widehat{\mathtt{G}}_{0^+}$ *is a weak Delaunay triangulation such that* $\widehat{\mathtt{G}}^{\bullet}_\epsilon = \mathtt{G}_\epsilon$*. Clearly such a completion need not be unique, as it will depend on the choice of chords. Nevertheless the choice can be made consistently so that* $\mathrm{V}(\widehat{\mathtt{G}}_\epsilon)$ *and* $\mathrm{E}(\widehat{\mathtt{G}}_\epsilon)$ *are stable for all values of the deformation parameter within the range* $0 < \epsilon < \tilde\epsilon_F$*. As before, the right sided limit*

$$\widehat{\mathtt{G}}_{0^+} = \lim_{\epsilon\to 0^+} \widehat{\mathtt{G}}_\epsilon$$

makes sense and is a triangular completion of G_{0^+} *which is isoradial and weakly Delaunay. Clearly the formulation of the deformed operators* Δ, $\underline{\boldsymbol{\Delta}}$ *and* $\mathcal{D}$ *in term of a triangulation in sect. 3.2 do not depend on the choice of completion.*

6. Variations of log-determinants

6.1. **First order variations of determinants.**

The setup. Here we compute the first order term in the expansion in $\epsilon > 0$ of the (formally infinite) logarithm of the determinant of $\mathcal{O}$. This first order term is on general grounds

$$\delta \operatorname{tr} \log \mathcal{O} = \operatorname{tr} \left[\delta\mathcal{O} \cdot \mathcal{O}_{\mathrm{cr}}^{-1}\right] \tag{6.1}$$

The results are each expressed as a sum of local terms over the weak Delaunay graph G_{0+} arising from the critical graph G_{cr} and the displacement function F. For both the Laplace-Beltrami and Kähler operators, there is a local term associated to each edge of G_{0+}; there is an additional local term attached to each face of G_{0+} for the Kähler operator. In the case of the conformal Laplacian the local terms associated to chords of G_{0+} differ from local terms of the regular edges of G_{0+}. For this reason formula 6.5 is expressed as two sums: one over the regular edges $\mathtt{e} \in \mathrm{E}(\mathsf{G}^{\bullet}_{0+}) = \mathrm{E}(\mathsf{G}_{\mathrm{cr}})$ and another over the set of chords $\mathtt{e} \in \mathrm{C}(\mathsf{G}_{0+}) = \mathrm{E}(\mathsf{G}_{0+})\backslash\mathrm{E}(\mathsf{G}_{\mathrm{cr}})$.

The results for first order variations. We first give the results. Their derivation is given in the next sections.

Proposition 7. ***Laplace-Beltrami.*** *For the Laplace-Beltrami Laplace operator Δ, the first order variation of* $\operatorname{tr}\log\Delta$ *under the deformation 5.20 can be expressed simply in terms of the variations of the north and south angles $\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon)$ and $\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)$ of edges $\mathtt{e} \in \mathrm{E}(\mathsf{G}_{0+})$.*

$$\operatorname{tr}\left[\delta\Delta \cdot \Delta_{\mathrm{cr}}^{-1}\right] = \frac{\epsilon}{\pi} \sum_{\substack{\text{edges}\\ \mathtt{e}\in\mathsf{G}_{0+}}} \eth_{\epsilon}\theta_{\mathrm{n}}(\vec{\mathtt{e}})\,\mathcal{L}'(\theta_{\mathrm{n}}(\vec{\mathtt{e}})) + \eth_{\epsilon}\theta_{\mathrm{s}}(\vec{\mathtt{e}})\,\mathcal{L}'(\theta_{\mathrm{s}}(\vec{\mathtt{e}})) \quad + \quad \mathrm{O}(\epsilon^2) \tag{6.2}$$

The function $\mathcal{L}'$, given by 3.32, is the derivative of the function $\mathcal{L}$ given by 3.29. $\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon)$ and $\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)$ are given by 5.13.

Remark 28. *Owing to the extended form 3.31 of Kenyon's result for* $\log\det\Delta_{\mathrm{cr}}$, *it is interesting to note that, up to terms of order ϵ^2,* $\log\det\Delta$ *for the deformed lattice can still be written as a sum of local terms involving the local geometry of the deformed Delaunay graph G_{ϵ}, similar to Kenyon's result although the graph is not isoradial*

$$\log\det\Delta = \frac{1}{\pi} \sum_{\substack{\text{edges}\\ \mathtt{e}\in\mathsf{G}_{0+}}} \mathcal{L}\big(\theta_{\mathrm{n}}(\mathtt{e},\epsilon)\big) + \mathcal{L}\big(\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)\big) \quad + \quad \mathrm{O}(\epsilon^2) \tag{6.3}$$

Remark 29. *Equivalently, it can be written in terms of the displacement function F as a sum over the triangles $\mathtt{f}$ of any triangulation $\widehat{\mathsf{G}}_{0+}$ which is a* completion *of G_{0+} (see Remark 27).*

$$\operatorname{tr}\left[\delta\Delta \cdot \Delta_{\mathrm{cr}}^{-1}\right] = -4\,\epsilon \sum_{\substack{\text{faces}\\ \mathtt{f}\in\widehat{\mathsf{G}}_{0+}}} A(\mathtt{f})\Big(\overline{\nabla}F(\mathtt{f})Q(\mathtt{f}) + c.c.\Big) \quad + \quad \mathrm{O}(\epsilon^2) \tag{6.4}$$

where $Q(\mathtt{f}) = [\nabla\Delta_{\mathrm{cr}}^{-1}\nabla^{\top}]_{\mathtt{ff}}$ is a diagonal matrix entry. The result is independent of the completion.

Proposition 8. ***Conformal Laplacian.*** *For the conformal Laplacian $\underline{\boldsymbol{\Delta}}$, the first order variation of* $\operatorname{tr}\log\underline{\boldsymbol{\Delta}}$ *under the deformation 5.20 can also be expressed simply in terms of the variations of the north and south angles $\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon)$ and $\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)$ of edges $\mathtt{e}\in\mathrm{E}(\mathbf{G}_{0+})$. However, we must distinguish between the contributions made by regular edges versus chords in $\mathbf{G}_{0+}$. Keep in mind that the set of regular edges $\mathrm{E}(\mathbf{G}^{\bullet}_{0+})$ coincides with the edge set $\mathrm{E}(\mathbf{G}_{\mathrm{cr}})$ of the critical graph.*

(6.5)

$$\begin{aligned}\operatorname{tr}\left[\delta\underline{\boldsymbol{\Delta}}\cdot\Delta_{\mathrm{cr}}^{-1}\right] = {} & \frac{2\,\epsilon}{\pi}\sum_{\substack{\text{edges}\\ \mathtt{e}\in\mathbf{G}_{\mathrm{cr}}}}\eth_\epsilon\theta(\mathtt{e})\mathcal{L}'\big(\theta(\mathtt{e})\big)\\ & +\frac{\epsilon}{\pi}\sum_{\substack{\text{chords}\\ \mathtt{e}\in\mathbf{G}_{0+}}}\eth_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}})\mathcal{H}'\big(\theta_{\mathrm{n}}(\vec{\mathtt{e}})\big)+\eth_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}})\mathcal{H}'\big(\theta_{\mathrm{s}}(\vec{\mathtt{e}})\big) \quad + \quad \mathrm{O}(\epsilon^2)\end{aligned}$$

with $\mathcal{H}'(\theta)=\theta\cot\theta$ the derivative of the function

$$\mathcal{H}(\theta)=2\,\theta\log(2\sin\theta)+\text{Л}(\theta) \tag{6.6}$$

Remember that $\theta(\mathtt{e})=(\theta_{\mathrm{n}}(\vec{\mathtt{e}})+\theta_{\mathrm{s}}(\vec{\mathtt{e}}))/2$ is the conformal edge angle for general triangulations.

Remark 30. *Up to order ϵ^2, $\log\det\underline{\boldsymbol{\Delta}}$ for the deformed lattice can still be written as a sum of local terms involving the local geometry of the weak Delaunay graph $\mathbf{G}_{0+}$. (see 5.3).*

$$\begin{aligned}\log\det\underline{\boldsymbol{\Delta}} \;&=\frac{2}{\pi}\sum_{\substack{\text{edges}\\ \mathtt{e}\in\mathbf{G}_{\mathrm{cr}}}}\mathcal{L}\big(\theta(\mathtt{e},\epsilon)\big)\\ &+\frac{1}{\pi}\sum_{\substack{\text{chords}\\ \mathtt{e}\in\mathbf{G}_{0+}}}\mathcal{H}\big(\theta_{\mathrm{n}}(\mathtt{e},\epsilon)\big)+\mathcal{H}\big(\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)\big)\;+\;\mathrm{O}(\epsilon^2)\end{aligned} \tag{6.7}$$

Proposition 9. ***Kähler operator.*** *For the Kähler operator $\mathcal{D}$, a local formula also holds at order ϵ. It involves the variations of the angles $\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon)$ and $\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)$ for edges $\mathtt{e}\in\mathrm{E}(\mathbf{G}_{0+})$, but also the variations of the circumradii $R(\mathtt{f},\epsilon)$ for faces $\mathtt{f}\in\mathrm{F}(\mathbf{G}_{0+})$. We note that $R(\mathtt{f},\epsilon)=R_{\mathrm{cr}}+\delta R(\mathtt{f})=R_{\mathrm{cr}}+\epsilon\,\delta_\epsilon R(\mathtt{f})+\mathrm{O}(\epsilon^2)$.*

$$\begin{aligned}\operatorname{tr}\left[\delta\mathcal{D}\cdot\mathcal{D}_{\mathrm{cr}}^{-1}\right] \;&=\frac{\epsilon}{\pi}\sum_{\substack{\text{edges}\\ \mathtt{e}\in\mathbf{G}_{0+}}}\eth_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}})\,\mathcal{L}'(\theta_{\mathrm{n}}(\vec{\mathtt{e}}))+\eth_\epsilon\theta_{\mathrm{s}}(\mathtt{e})\,\mathcal{L}'(\theta_{\mathrm{s}}(\vec{\mathtt{e}}))\\ &-\epsilon\sum_{\substack{\text{faces}\\ \mathtt{f}\in\mathbf{G}_{0+}}}\frac{\eth_\epsilon R(\mathtt{f})}{R_{\mathrm{cr}}}\quad+\quad\mathrm{O}(\epsilon^2)\end{aligned} \tag{6.8}$$

Remark 31. *Up to order ϵ^2, $\log\det\mathcal{D}$ for the deformed lattice can still be written as a sum of local terms involving the local geometry of $\mathbf{G}_{0+}$*

$$\begin{aligned}\log\det\mathcal{D}=\frac{1}{\pi}&\sum_{\substack{\text{edges}\\ \mathtt{e}\in\mathbf{G}_{0+}}}\mathcal{L}\big(\theta_{\mathrm{n}}(\mathtt{e},\epsilon)\big)+\mathcal{L}\big(\theta_{\mathrm{s}}(\mathtt{e},\epsilon)\big)\\ -&\sum_{\substack{\text{faces}\\ \mathtt{f}\in\mathbf{G}_{0+}}}\log R(\mathtt{f},\epsilon)\quad+\quad\mathrm{O}(\epsilon^2)\end{aligned} \tag{6.9}$$

Proof of Proposition 7. We first consider the variation of the Laplace-Beltrami operator Δ under a deformation of the form 5.20. One can use 5.3 to compute explicitly the first order variation of $\log\det\Delta$, but it is simpler to start from its definition in terms of angles 1.7. For an edge $\mathtt{e}=\overline{\mathtt{uv}}$ of G_ϵ

$$\Delta_{\mathtt{uv}} = -c(\mathtt{e},\epsilon) = -\frac{\tan\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon)+\tan\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)}{2} \tag{6.10}$$

This implies that the variation is

$$\delta\Delta_{\mathtt{uv}} = -\frac{\epsilon}{2}\Big(\partial_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}})\,\sec^2\theta_{\mathrm{n}}(\vec{\mathtt{e}})+\partial_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}})\,\sec^2\theta_{\mathrm{s}}(\vec{\mathtt{e}})\Big) \;+\; \mathrm{O}(\epsilon^2) \tag{6.11}$$

where $\partial_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}})$ and $\partial_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}})$ are of order $\mathrm{O}(1)$. The limit graph G_{0^+} is weakly Delaunay and isoradial so either $\theta_{\mathrm{n}}(\vec{\mathtt{e}})=\theta_{\mathrm{s}}(\vec{\mathtt{e}})$ or $\theta_{\mathrm{n}}(\vec{\mathtt{e}})=-\theta_{\mathrm{s}}(\vec{\mathtt{e}})$. In both case $\sec^2\theta_{\mathrm{n}}(\vec{\mathtt{e}})=\sec^2\theta_{\mathrm{s}}(\vec{\mathtt{e}})$ so that at first order

$$\begin{aligned}\delta\Delta_{\mathtt{uv}} &= -\epsilon\,\frac{\partial_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}})+\partial_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}})}{2}\sec^2\theta_{\mathrm{n}}(\vec{\mathtt{e}})+\mathrm{O}(\epsilon^2)\\ &= -\epsilon\,\frac{\partial_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}})+\partial_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}})}{2}\sec^2\theta_{\mathrm{s}}(\vec{\mathtt{e}})+\mathrm{O}(\epsilon^2)\end{aligned} \tag{6.12}$$

It remains to combine this with the propagator $\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{vu}}$ which for regular edges $\mathtt{e}=\overline{\mathtt{uv}}$ of $\mathsf{G}^\bullet_{0^+}=\mathsf{G}_{\mathrm{cr}}$ is

$$\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{vu}} = -\frac{1}{\pi}\,\theta(\mathtt{e})\cot\theta(\mathtt{e}) \tag{6.13}$$

A similar relation is in fact valid for chords of G_{0^+}

$$\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{vu}} = -\frac{1}{\pi}\,\theta_{\mathrm{n}}(\vec{\mathtt{e}})\cot\theta_{\mathrm{n}}(\vec{\mathtt{e}}) = -\frac{1}{\pi}\,\theta_{\mathrm{s}}(\vec{\mathtt{e}})\cot\theta_{\mathrm{s}}(\vec{\mathtt{e}}) \tag{6.14}$$

Thus the first order variation is

(6.15)

$$\begin{aligned}\mathrm{tr}\left[\partial_\epsilon\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\right] &= \sum_{\substack{\text{vertices}\\ \mathtt{u},\mathtt{v}\in\mathsf{G}_{0^+}}}\partial_\epsilon\Delta_{\mathtt{uv}}\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{vu}}\\ &= \frac{1}{\pi}\sum_{\substack{\text{edges}\\ \mathtt{e}\in\mathsf{G}_{0^+}}}\partial_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}})\theta_{\mathrm{n}}(\vec{\mathtt{e}})\cot\theta_{\mathrm{n}}(\vec{\mathtt{e}})\sec^2\theta_{\mathrm{n}}(\vec{\mathtt{e}})+\delta\theta_{\mathrm{s}}(\vec{\mathtt{e}})\theta_{\mathrm{s}}(\vec{\mathtt{e}})\cot\theta_{\mathrm{s}}(\vec{\mathtt{e}})\sec^2\theta_{\mathrm{s}}(\vec{\mathtt{e}})\\ &= \frac{1}{\pi}\sum_{\substack{\text{edges}\\ \mathtt{e}\in\mathsf{G}_{0^+}}}\partial_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}})\frac{\theta_{\mathrm{n}}(\vec{\mathtt{e}})}{\sin\theta_{\mathrm{n}}(\vec{\mathtt{e}})\cos\theta_{\mathrm{n}}(\vec{\mathtt{e}})}+\partial_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}})\frac{\theta_{\mathrm{n}}(\vec{\mathtt{e}})}{\sin\theta_{\mathrm{s}}(\vec{\mathtt{e}})\cos\theta_{\mathrm{s}}(\vec{\mathtt{e}})}\\ &= \frac{1}{\pi}\sum_{\substack{\text{edges}\\ \mathtt{e}\in\mathsf{G}_{0^+}}}\partial_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}})\mathcal{L}'\big(\theta_{\mathrm{n}}(\vec{\mathtt{e}})\big)+\partial_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}})\mathcal{L}'\big(\theta_{\mathrm{n}}(\vec{\mathtt{e}})\big)\\ &= \partial_\epsilon\left[\frac{1}{\pi}\sum_{\substack{\text{edges}\\ \mathtt{e}\in\mathsf{G}_{0^+}}}\mathcal{L}\big(\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon)\big)+\mathcal{L}\big(\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)\big)\right]\end{aligned}$$

This, together with 3.31, leads to 6.3, and this ends the proof of Proposition 7.

Again, we obtain a nice local expression involving the local angles $\theta_{\mathrm{n}}(\vec{\mathtt{e}})$ and $\theta_{\mathrm{s}}(\vec{\mathtt{e}})$ and the local circumradii $R(\mathtt{f})$. As for the conformal Laplacian, the global conformal invariance properties of the Kähler operator are no transparent in the result. Hovever, cocyclic configurations and chords do not play any special role.

Proof of Proposition 8. For an edge $\mathtt{e} = \overline{\mathtt{uv}}$ of $\mathtt{G}_\epsilon$ the matrix element the conformal Laplacian $\underline{\boldsymbol{\Delta}}$ is

$$\text{(6.16)}\qquad \big[\underline{\boldsymbol{\Delta}}\big]_{\mathtt{uv}} = -\tan\theta(\mathtt{e},\epsilon)\ ,\quad \theta(\mathtt{e},\epsilon) = \frac{\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon)+\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)}{2}$$

This implies that the variation is

$$\text{(6.17)}\qquad \big[\delta\underline{\boldsymbol{\Delta}}\big]_{\mathtt{uv}} = -\epsilon\,\frac{\eth_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)+\eth_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)}{2}\ \sec^2\left(\frac{\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)+\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)}{2}\right)\ +\ \mathrm{O}(\epsilon^2)$$

Keep in mind that the limit graph $\mathtt{G}_{0+}$ is weakly Delaunay and isoradial and so either $\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) = \theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)$ or $\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) = -\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)$. The first case corresponds to a regular edge, while the second case corresponds to a chord. Thus to first order in ϵ the matrix entry is

$$\text{(6.18)}\qquad \begin{aligned}
\big[\eth_\epsilon\underline{\boldsymbol{\Delta}}\big]_{\mathtt{uv}} &= \begin{cases} -\dfrac{\eth_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)\sec^2\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)+\eth_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)\sec^2\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)}{2} & \text{if } \theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)=\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,) \\[2ex] -\dfrac{\eth_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)+\eth_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)}{2} & \text{if } \theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)=-\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,) \end{cases} \\
&= \begin{cases} \big[\eth_\epsilon\Delta\big]_{\mathtt{uv}} & \text{if } \theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)=\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,) \\[2ex] \big[\eth_\epsilon\Delta\big]_{\mathtt{uv}} + \dfrac{\eth_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)\tan^2\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)+\eth_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)\tan^2\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)}{2} & \text{if } \theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)=-\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,) \end{cases}
\end{aligned}$$

The first order variation of the log-determinant reads as a sum over the edges of $\mathtt{G}_{0+}$, but it is different for the edges in $\mathtt{G}^\bullet_{0+} = \mathtt{G}_{\mathrm{cr}}$ and the chords of $\mathtt{G}_{0+}$. Combining with 6.13 we get at first order

$$\text{(6.19)}\qquad \begin{aligned}
\mathrm{tr}\,\big[\eth_\epsilon\underline{\boldsymbol{\Delta}}\cdot\Delta_{\mathrm{cr}}^{-1}\big] &= \sum_{\substack{\text{vertices}\\ \mathtt{u},\mathtt{v}\,\in\,\mathtt{G}_{0+}}} \big[\eth_\epsilon\underline{\boldsymbol{\Delta}}\big]_{\mathtt{uv}}\,\big[\Delta_{\mathrm{cr}}^{-1}\big]_{\mathtt{vu}} \\
&= \frac{2}{\pi}\sum_{\substack{\text{edges}\\ \mathtt{e}\,\in\,\mathtt{G}_{\mathrm{cr}}}} \eth_\epsilon\theta(\mathtt{e})\mathcal{L}'(\theta(\mathtt{e})) + \frac{1}{\pi}\sum_{\substack{\text{chords}\\ \mathtt{e}\,\in\,\mathtt{G}_{0+}}} \eth_\epsilon\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)\mathcal{H}'(\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)) + \eth_\epsilon\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)\mathcal{H}'(\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,))
\end{aligned}$$

with the function $\mathcal{H}(\theta)$ given by

$$\text{(6.20)}\qquad \mathcal{H}(\theta) = \int_0^\theta dt\,t\,\cot(t) = 2\theta\log(2\sin\theta) + \text{Л}(\theta)$$

This leads to 6.7 and the proof of Proposition 8.

Proof of Proposition 9. The variation of the Kähler operator $\mathcal{D}$ starts from the expression of the matrix elements $\mathcal{D}_{\mathtt{u},\mathtt{v}}$ of an edge $\vec{\mathtt{e}} = (\mathtt{u},\mathtt{v})$ in terms of the angles $\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon)$ and $\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)$ and of the circumradii $R_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon)$ and $R_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)$ given by 1.8, namely

$$\mathcal{D}_{\mathtt{uv}} = -\frac{1}{2}\left(\frac{\tan\theta_{\mathrm{n}}(\vec{\mathtt{e}},\epsilon)+i}{R_{\mathrm{n}}^2(\vec{\mathtt{e}},\epsilon)} + \frac{\tan\theta_{\mathrm{s}}(\vec{\mathtt{e}},\epsilon)-i}{R_{\mathrm{s}}^2(\vec{\mathtt{e}},\epsilon)}\right) \tag{6.21}$$

Its variation is therefore

$$\begin{aligned} \delta\mathcal{D}_{\mathtt{uv}} &= \eth_\epsilon\mathcal{D}^{(1)}_{\mathtt{uv}} + \eth_\epsilon\mathcal{D}^{(2)}_{\mathtt{uv}} \ + \ \mathrm{O}(\epsilon^2) \\ \eth_\epsilon\mathcal{D}^{(1)}_{\mathtt{uv}} &= \ -\left(\frac{1}{2\,R_{\mathrm{n}}^2(\vec{\mathtt{e}})}\,\eth_\epsilon\tan\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) + \frac{1}{2\,R_{\mathrm{s}}^2(\vec{\mathtt{e}})}\,\eth_\epsilon\tan\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)\right) \\ \eth_\epsilon\mathcal{D}^{(2)}_{\mathtt{uv}} &= \ \left(\frac{\tan\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)+i}{R_{\mathrm{n}}^3(\vec{\mathtt{e}}\,)}\,\eth_\epsilon R_{\mathrm{n}}(\vec{\mathtt{e}}\,) + \frac{\tan\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)-i}{R_{\mathrm{s}}^3(\vec{\mathtt{e}}\,)}\,\eth_\epsilon R_{\mathrm{s}}(\vec{\mathtt{e}}\,)\right) \end{aligned} \tag{6.22}$$

For an isoradial triangulation (critical case), $R_{\mathrm{n}}(\vec{\mathtt{e}}\,) = R_{\mathrm{s}}(\vec{\mathtt{e}}\,) = R_{\mathrm{cr}}$, therefore one has $\mathcal{D}_{\mathrm{cr}} = R_{\mathrm{cr}}^{-2}\Delta_{\mathrm{cr}}$. Thus in the critical case, the first term in 6.22 contributes to the variation as

$$\eth_\epsilon\mathcal{D}^{(1)}_{\mathtt{uv}} = R_{\mathrm{cr}}^{-2}\,\eth_\epsilon\Delta_{\mathtt{uv}} \quad\Longrightarrow\quad \mathrm{tr}\left[\eth_\epsilon\mathcal{D}^{(1)}\cdot\mathcal{D}_{\mathrm{cr}}^{-1}\right] = \mathrm{tr}\left[\eth_\epsilon\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\right] \tag{6.23}$$

The second term contributions can be reorganized as a sum over faces of $\widehat{\mathfrak{G}}_{0+}$, i.e. counter-clockwise oriented triangles $\mathtt{f} = (\mathtt{u},\mathtt{v},\mathtt{w})$

(6.24)
$$\begin{aligned} \mathrm{tr}\left[\eth_\epsilon\mathcal{D}^{(2)}\cdot\mathcal{D}_{\mathrm{cr}}^{-1}\right] &= \sum_{\substack{\text{vertices}\\ \mathtt{u},\mathtt{v}\,\in\,\mathfrak{G}_{0+}}} \eth_\epsilon\mathcal{D}^{(2)}_{\mathtt{uv}}\ \left[\mathcal{D}_{\mathrm{cr}}^{-1}\right]_{\mathtt{vu}} \\ &= \sum_{\substack{\text{triangles}\\ \mathtt{f}=(\mathtt{u},\mathtt{v},\mathtt{w})\\ \text{in } \widehat{\mathfrak{G}}_{0+}}} \frac{\eth_\epsilon R(\mathtt{f})}{R_{\mathrm{cr}}^3} \begin{pmatrix} (\tan\theta_{\mathrm{n}}(\overrightarrow{\mathtt{uv}})+\mathrm{i})\left[\mathcal{D}_{\mathrm{cr}}^{-1}\right]_{\mathtt{vu}} & +(\tan\theta_{\mathrm{s}}(\overrightarrow{\mathtt{vu}})-\mathrm{i})\left[\mathcal{D}_{\mathrm{cr}}^{-1}\right]_{\mathtt{uv}} \\ +(\tan\theta_{\mathrm{n}}(\overrightarrow{\mathtt{vw}})+\mathrm{i})\left[\mathcal{D}_{\mathrm{cr}}^{-1}\right]_{\mathtt{wv}} & +(\tan\theta_{\mathrm{s}}(\overrightarrow{\mathtt{wv}})-\mathrm{i})\left[\mathcal{D}_{\mathrm{cr}}^{-1}\right]_{\mathtt{vw}} \\ +(\tan\theta_{\mathrm{n}}(\overrightarrow{\mathtt{wu}})+\mathrm{i})\left[\mathcal{D}_{\mathrm{cr}}^{-1}\right]_{\mathtt{wu}} & +(\tan\theta_{\mathrm{s}}(\overrightarrow{\mathtt{uw}})-\mathrm{i})\left[\mathcal{D}_{\mathrm{cr}}^{-1}\right]_{\mathtt{wu}} \end{pmatrix} \end{aligned}$$

Using the fact that $\theta_{\mathrm{n}}(\overrightarrow{\mathtt{uv}}) = \theta_{\mathrm{s}}(\overrightarrow{\mathtt{vu}})$ and that for the critical case

$$\left[\mathcal{D}_{\mathrm{cr}}^{-1}\right]_{\mathtt{vu}} = \left[\mathcal{D}_{\mathrm{cr}}^{-1}\right]_{\mathtt{uv}} = -\frac{1}{\pi}\,R_{\mathrm{cr}}^2\,\theta_{\mathrm{n}}(\overrightarrow{\mathtt{uv}})\cot\theta_{\mathrm{n}}(\overrightarrow{\mathtt{uv}})$$

and the fact that for a triangle $\mathtt{f} = (\mathtt{u},\mathtt{v},\mathtt{w})$, one has

$$\theta_{\mathrm{n}}(\overrightarrow{\mathtt{uv}}) \ + \ \theta_{\mathrm{n}}(\overrightarrow{\mathtt{vw}}) \ + \ \theta_{\mathrm{n}}(\overrightarrow{\mathtt{wu}}) \ = \ \pi/2$$

we obtain

$$\mathrm{tr}\left[\eth_\epsilon\mathcal{D}^{(2)}\cdot\mathcal{D}_{\mathrm{cr}}^{-1}\right] = -\sum_{\substack{\text{faces}\\ \mathtt{f}\,\in\,\widehat{\mathfrak{G}}_{0+}}} \frac{\eth_\epsilon R(\mathtt{f})}{R_{\mathrm{cr}}} = -\sum_{\substack{\text{faces}\\ \mathtt{f}\,\in\,\widehat{\mathfrak{G}}_{0+}}} \eth_\epsilon\log R(\mathtt{f}) \tag{6.25}$$

This leads to 6.9 and to Proposition 9.

6.2. **Second order variations.**

Principle of the calculation. The second order term of the variation is of the general form

$$-\frac{1}{2}\operatorname{tr}\Big[\big(\delta\mathcal{O}\cdot\mathcal{O}_{\mathrm{cr}}^{-1}\big)^2\Big] \text{ with } \mathcal{O}_{\mathrm{cr}}^{-1} \text{ the critical Green's function} \tag{6.26}$$

This term is bilinear in $\delta\mathcal{O}$ hence it is bi-local. As explained in the introduction, we shall mainly be interested in the large distance part of this bilocal term. To isolate this term, as stated in Theorem 1, we shall consider a two-parameter deformation of the isoradial, Delaunay graph of the form

$$z_\epsilon(\mathtt{v}) := z_{\mathrm{cr}}(\mathtt{v}) + \epsilon_1 F_1(\mathtt{v}) + \epsilon_2 F_2(\mathtt{v}) \tag{6.27}$$

where $F_1, F_2 \in \mathbb{C}^{\mathrm{V}(\mathtt{G}_{\mathrm{cr}})}$ are two functions with disjoint compact support $\Omega_1 := \Omega_{F_1}$ and $\Omega_2 := \Omega_{F_2}$ in $\mathbb{C}$. We will assume that the *distance* d between the two supports is large, i.e. $d \gg R_{\mathrm{cr}}$ where

$$d = \mathtt{dist}\big(\Omega_1,\Omega_2\big) := \inf\Big\{\big|z_{\mathrm{cr}}(\mathtt{w}_1)-z_{\mathrm{cr}}(\mathtt{w}_2)\big|\,\Big|\,\mathtt{w}_1\in\Omega_1\,,\,\mathtt{w}_2\in\Omega_2\Big\} \tag{6.28}$$

We consider here the coefficient of $\epsilon_1\,\epsilon_2$ in the perturbative expansion of $\log\det\mathcal{O}$ which can be expressed as the following trace:

$$-\operatorname{tr}\Big[\eth_{\epsilon_1}\mathcal{O}\cdot\mathcal{O}_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\mathcal{O}\cdot\mathcal{O}_{\mathrm{cr}}^{-1}\Big] = -\sum_{\substack{\mathtt{u},\mathtt{v}\in\Omega_1\\ \mathtt{p},\mathtt{q}\in\Omega_2}}\big[\eth_{\epsilon_1}\mathcal{O}\big]_{\mathtt{uv}}\big[\mathcal{O}_{\mathrm{cr}}^{-1}\big]_{\mathtt{vp}}\big[\eth_{\epsilon_2}\mathcal{O}\big]_{\mathtt{pq}}\big[\mathcal{O}_{\mathrm{cr}}^{-1}\big]_{\mathtt{qu}} \tag{6.29}$$

where $\eth_{\epsilon_1}\mathcal{O}$ and $\eth_{\epsilon_2}\mathcal{O}$ are the first order variations of the Laplace-like operator $\mathcal{O}$ as defined in Sect. 5.1 with respect to the independent displacements F_1 or F_2 respectively, using the notations of Sect. 5.2. The sum on the left hand side is taken over vertices $\mathtt{u}$, $\mathtt{v}$, $\mathtt{p}$, $\mathtt{q}$ in the weak Delaunay graph $\mathtt{G}_{0^+}$ such that both matrix entries $\big[\eth_{\epsilon_1}\mathcal{O}\big]_{\mathtt{uv}}$ and $\big[\eth_{\epsilon_2}\mathcal{O}\big]_{\mathtt{pq}}$ are non-zero. In particular this implies that $\overline{\mathtt{uv}}$ is an edge in $\mathtt{G}_{0^+}$ with vertices $\mathtt{u},\mathtt{v}\in\overline{\Omega}_1$. Likewise $\overline{\mathtt{pq}}$ must be an edge in $\mathtt{G}_{0^+}$ with vertices $\mathtt{p},\mathtt{q}\in\overline{\Omega}_2$.

Provided the two zones of support Ω_1 and Ω are far enough apart, the matrix entries $\big[\mathcal{O}_{\mathrm{cr}}^{-1}\big]_{\mathtt{vp}}$ and $\big[\mathcal{O}_{\mathrm{cr}}^{-1}\big]_{\mathtt{qu}}$ of the critical Green's function will only involve pairs of vertices with $|z_{\mathrm{cr}}(\mathtt{v})-z_{\mathrm{cr}}(\mathtt{p})|\simeq d$ and $|z_{\mathrm{cr}}(\mathtt{q})-z_{\mathrm{cr}}(\mathtt{u})|\simeq d$. Under these circumstances we may estimate the contributions made by these matrix entries using the asymptotic expansion 1.19 for the Green's function.

It will be convenient to complete the deformed Delaunay graph $\mathtt{G}_\epsilon$ to a weak Delaunay triangulation $\widehat{\mathtt{G}}_\epsilon$ as discussed in Remark 27. This will allow us to use the variational formulae 5 and 6 for the Laplace-Beltrami and Kähler operators. In general such a completion $\widehat{\mathtt{G}}_\epsilon$ will not be unique. Nevertheless $\widehat{\mathtt{G}}_\epsilon^{\bullet} = \mathtt{G}_\epsilon$ and $\widehat{\mathtt{G}}_{0^+}^{\bullet} = \mathtt{G}_{0^+}$ regardless of the choice of completion, and by Remark 27 the Laplace-Beltrami operator, Kähler operator, and the conformal Laplacian will not be affected this choice.

The Laplace-Beltrami operator.
The simplest case is the Laplace-Beltrami Δ operator. We shall need two intermediate results.

Lemma 11. *Let* $\mathtt{f} = (\mathtt{v}_1,\mathtt{v}_2,\mathtt{v}_3)$ *be a c.c.w. oriented triangle with circumcenter* $z_{\mathrm{cr}}(\mathtt{f})$ *and circumradius* $R=1$. *Define* $e^{\mathrm{i}\theta_j} := z_{\mathrm{cr}}(\mathtt{v}_j) - z_{\mathrm{cr}}(\mathtt{f})$ *for* $j=1,2,3$. *Let*

$\nabla_{\mathtt{f}\mathtt{v}_j}$ *be the matrix elements of the discrete derivative operator* ∇*, restricted to the triangle* $\mathtt{f}$*. For any integer* $m \in \mathbb{Z}$ *one has the uniform bound*

$$\left| \sum_{j=1}^{3} \nabla_{\mathtt{f}\mathtt{v}_j}\, e^{\mathrm{i}m\theta_j} \right| \leq \frac{m(m+1)}{2} \quad , \qquad m \in \mathbb{Z} \tag{6.30}$$

Proof. Using the definition of ∇ 3.10, one can rewrite

$$\sum_{j=1}^{3} \nabla_{\mathtt{f}\mathtt{v}_j}\, e^{\mathrm{i}m\theta_j} = \det \begin{pmatrix} 1 & e^{-\mathrm{i}\theta_1} & e^{\mathrm{i}m\theta_1} \\ 1 & e^{-\mathrm{i}\theta_2} & e^{\mathrm{i}m\theta_2} \\ 1 & e^{-\mathrm{i}\theta_3} & e^{\mathrm{i}m\theta_3} \end{pmatrix} \Big/ \det \begin{pmatrix} 1 & e^{-\mathrm{i}\theta_1} & e^{\mathrm{i}\theta_1} \\ 1 & e^{-\mathrm{i}\theta_2} & e^{\mathrm{i}\theta_2} \\ 1 & e^{-\mathrm{i}\theta_3} & e^{\mathrm{i}\theta_3} \end{pmatrix} \tag{6.31}$$

For $m > 0$ rewrite the numerator as

$$e^{-\mathrm{i}(\theta_1+\theta_2+\theta_3)} \det \begin{pmatrix} 1 & e^{\mathrm{i}\theta_1} & e^{\mathrm{i}(m+1)\theta_1} \\ 1 & e^{\mathrm{i}\theta_2} & e^{\mathrm{i}(m+1)\theta_2} \\ 1 & e^{\mathrm{i}\theta_3} & e^{\mathrm{i}(m+1)\theta_3} \end{pmatrix} \tag{6.32}$$

The last determinant is a generalization of the Vandermonde determinant, which is explicitely (in its general form)

$$\det \begin{pmatrix} 1 & z_1 & z_1^{m+1} \\ 1 & z_2 & z_2^{m+1} \\ 1 & z_3 & z_3^{m+1} \end{pmatrix} = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1) S_{m-1}(z_1, z_2, z_3) \tag{6.33}$$

with S_n the homogeneous symmetric polynomial of degree n (a Schur polynomial),

$$S_n(z_1, z_2, z_3) = \sum_{\substack{p_1,p_2,p_3 \in \mathbb{N} \\ p_1+p_2+p_3=n}} z_1^{p_1} z_2^{p_2} z_3^{p_3} \tag{6.34}$$

which is made of $(n+1)(n+2)/2$ monomials. The denominator in the r.h.s. of 6.31 is the same object for $m = 1$. Hence we get when $m > 0$

$$\sum_{j=1}^{3} \nabla_{\mathtt{f}\mathtt{v}_j}\, e^{\mathrm{i}m\theta_j} = S_{m-1}(e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2}, e^{\mathrm{i}\theta_3}) \tag{6.35}$$

and it is clear that we have the bound.

$$|S_{m-1}(e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2}, e^{\mathrm{i}\theta_2})| \leq m(m+1)/2 \tag{6.36}$$

which is saturated when $\theta_1 = \theta_2 = \theta_3$.

When $m < 0$, we can rewrite by the same trick

$$\sum_{j=1}^{3} \nabla_{\mathtt{f}\mathtt{v}_j}\, e^{\mathrm{i}m\theta_j} = S_{|m|-2}(e^{-\mathrm{i}\theta_1}, e^{-\mathrm{i}\theta_2}, e^{-\mathrm{i}\theta_3}) \tag{6.37}$$

with a similar bound. We thus get 6.30. ☐

We can now get uniform asymptotics estimates for the discrete derivatives of the Green function.

Lemma 12. *Let* $\Delta_{\mathrm{cr}}^{-1}$ *be the critical Green's function on an isoradial, Delaunay graph* $\mathtt{G}_{\mathrm{cr}}$*, let* $\mathtt{f}$ *and* $\mathtt{g}$ *be two faces (triangles) of a triangulation* $\widehat{\mathtt{G}}_{0^+}$ *completing the limit graph* $\mathtt{G}_{0^+}$*, and let* $z_{\mathrm{cr}}(\mathtt{f})$ *and* $z_{\mathrm{cr}}(\mathtt{g})$ *be the complex coordinate of their respective circumcenters* $\mathtt{o}_\mathtt{f}$ *and* $\mathtt{o}_\mathtt{g}$*. Let* $d = |z_{\mathrm{cr}}(\mathtt{f}) - z_{\mathrm{cr}}(\mathtt{g})|$ *be the distance between* *the centers. Then the discrete double derivatives of the Green function have the following large distance asymptotics*

(6.38)

$$\left[\nabla\ \Delta_{\rm cr}^{-1}\ \overline{\nabla}^{\top}\right]_{\tt fg} \;=\; \frac{1}{4\pi}\left(\frac{\prod_{\tt v\in g}\, e^{{\rm i}\theta_{\tt v}}}{\big(z_{\rm cr}({\tt f})-z_{\rm cr}({\tt g})\big)^3} \;-\; \frac{\prod_{\tt u\in f}\, e^{-{\rm i}\theta_{\tt u}}}{\big(\bar z_{\rm cr}({\tt f})-\bar z_{\rm cr}({\tt g})\big)^3}\right) \;+\; {\rm O}(d^{-4})$$

$$\left[\overline{\nabla}\ \Delta_{\rm cr}^{-1}\ \nabla^{\top}\right]_{\tt fg} \;=\; \frac{1}{4\pi}\left(\frac{\prod_{\tt v\in g}\, e^{-{\rm i}\theta_{\tt v}}}{\big(\bar z_{\rm cr}({\tt f})-\bar z_{\rm cr}({\tt g})\big)^3} \;-\; \frac{\prod_{\tt u\in f}\, e^{{\rm i}\theta_{\tt u}}}{\big(z_{\rm cr}({\tt f})-z_{\rm cr}({\tt g})\big)^3}\right) \;+\; {\rm O}(d^{-4})$$

and

$$\text{(6.39)}\qquad \begin{aligned}\left[\nabla\ \Delta_{\rm cr}^{-1}\ \nabla^{\top}\right]_{\tt fg} \;&=\; -\frac{1}{4\pi}\frac{1}{(z_{\rm cr}({\tt f})-z_{\rm cr}({\tt g}))^2} \;+\; {\rm O}(d^{-3})\\ \left[\overline{\nabla}\ \Delta_{\rm cr}^{-1}\ \overline{\nabla}^{\top}\right]_{\tt fg} \;&=\; -\frac{1}{4\pi}\frac{1}{(\bar z_{\rm cr}({\tt f})-\bar z_{\rm cr}({\tt g}))^2} \;+\; {\rm O}(d^{-3})\end{aligned}$$

Proof. Let $\tt f$ = (123) and $\tt g$ = (456) be the vertices of $\tt f$ and $\tt g$ respectively. $\tt G$ is isoradial, hence denote $z_{\rm cr}({\tt u}) - z_{\rm cr}({\tt f}) = e^{{\rm i}\theta_{\tt u}}$ $({\tt u} = 1,2,3)$ and $z_{\rm cr}({\tt v}) - z_{\rm cr}({\tt g}) = e^{{\rm i}\theta_{\tt v}}$ $({\tt v} = 4,5,6)$. Use 1.19 to separate the Green function $\left[\Delta_{\rm cr}^{-1}\right]_{\tt uv}$ into its leading large distance term (continuous limit term) of order $\log d$, its subleading large distance correction of order d^{-2}, and the rest of its large distance expansion of order d^{-4}.

$$\text{(6.40)}\qquad \left[\Delta_{\rm cr}^{-1}\right]_{\tt uv} = G^{(0)}_{\tt uv} + G^{(2)}_{\tt uv} + G^{(4)}_{\tt uv}$$

with

$$\text{(6.41)}\qquad \begin{aligned} G^{(0)}_{\tt uv} &= -\,\frac{1}{2\pi}\ \left(\log(2|z_{\rm cr}({\tt u})-z_{\rm cr}({\tt v})|)+\gamma_{\rm euler}\right)\\ G^{(2)}_{\tt uv} &= -\,\frac{1}{24\pi}\left(\frac{p_3({\tt u},{\tt v})}{(z_{\rm cr}({\tt u})-z_{\rm cr}({\tt v}))^3}+\frac{\bar p_3({\tt u},{\tt v})}{(\bar z_{\rm cr}({\tt u})-\bar z_{\rm cr}({\tt v}))^3}\right)\\ G^{(4)}_{\tt uv} &= {\rm O}\left(|z_{\rm cr}({\tt u})-z_{\rm cr}({\tt v})|^{-4}\right)\end{aligned}$$

Begin by writing

$$\text{(6.42)}\qquad (z_{\rm cr}({\tt u})-z_{\rm cr}({\tt v})) = (z_{\rm cr}({\tt f})-z_{\rm cr}({\tt g})) + e^{{\rm i}\theta_{\tt u}} - e^{{\rm i}\theta_{\tt v}}$$

and expand the logs and powers of $(z_{\rm cr}({\tt u})-z_{\rm cr}({\tt v}))$ and $(\bar z_{\rm cr}({\tt u})-\bar z_{\rm cr}({\tt v}))$ in formulae 6.41 as asymptotic series in $(z_{\rm cr}({\tt f})-z_{\rm cr}({\tt g}))$ and $(\bar z_{\rm cr}({\tt f})-\bar z_{\rm cr}({\tt g}))$ where $d = |z_{\rm cr}({\tt f})-z_{\rm cr}({\tt g})| \gg 1$ is large. For example:

$$G^{(0)}_{\tt uv} \;=\; -\frac{1}{2\pi}\left(\log\left(2|z_{\rm cr}({\tt f})-z_{\rm cr}({\tt g})|\right)+\gamma_{\rm euler}\right) \;+\; \frac{1}{2\pi}\,\mathfrak{Re}\sum_{r\geq 1}\frac{1}{r}\,\left(\frac{e^{{\rm i}\theta_{\tt v}}-e^{{\rm i}\theta_{\tt u}}}{z_{\rm cr}({\tt f})-z_{\rm cr}({\tt g})}\right)^r$$

The coefficients in these asymptotic expansions are expressed as a Laurent polynomials in $e^{{\rm i}\theta_{\tt u}}$ and $e^{{\rm i}\theta_{\tt v}}$ and so the matrix entries in formulae 6.38 and 6.39 can be computed using the basic identities

$$\text{(6.43)}\qquad \sum_{\tt u\in f}\nabla_{\tt fu}\,e^{{\rm i}\theta_{\tt u}} = 1 \quad\text{and}\quad \sum_{\tt u\in f}\nabla_{\tt fu}\,e^{-{\rm i}\theta_{\tt u}} = \sum_{\tt u\in f}\nabla_{\tt fu} = 0$$

along with values of $\nabla_{\tt fu}$, $\overline{\nabla}_{\tt fu}$ and $\nabla^{\top}_{\tt vg} = \nabla_{\tt gv}$, $\nabla^{\dagger}_{\tt vg} = \overline{\nabla}_{\tt gv}$ explicitly given in 3.10 and 3.12). As illustration:

$$
\begin{aligned}
\left[\nabla G^{(0)}\overline{\nabla}^{\top}\right]_{\mathtt{fg}} &= \sum_{\mathtt{u}\in\mathtt{f}}\sum_{\mathtt{v}\in\mathtt{g}} \nabla_{\mathtt{fu}}\overline{\nabla}_{\mathtt{gv}}\, G^{(0)}_{\mathtt{uv}} \\
&= \begin{cases} \dfrac{1}{4\pi}\left(\dfrac{\prod_{\mathtt{u}\in\mathtt{f}} e^{\mathrm{i}\theta_{\mathtt{u}}}}{\big(z_{\mathrm{cr}}(\mathtt{f})-z_{\mathrm{cr}}(\mathtt{g})\big)^3} - \dfrac{\prod_{\mathtt{v}\in\mathtt{g}} e^{-\mathrm{i}\theta_{\mathtt{v}}}}{\big(\bar z_{\mathrm{cr}}(\mathtt{f})-\bar z_{\mathrm{cr}}(\mathtt{g})\big)^3}\right) + \\[2ex]
\dfrac{1}{2\pi}\displaystyle\sum_{r\geq 4}\sum_{\mathtt{u}\in\mathtt{f}}\sum_{\mathtt{v}\in\mathtt{g}} \nabla_{\mathtt{fu}}\overline{\nabla}_{\mathtt{gv}}\,\frac{1}{r}\,\mathfrak{Re}\Big(\frac{e^{\mathrm{i}\theta_{\mathtt{v}}}-e^{\mathrm{i}\theta_{\mathtt{u}}}}{z_{\mathrm{cr}}(\mathtt{f})-z_{\mathrm{cr}}(\mathtt{g})}\Big)^r \end{cases}
\end{aligned}
$$

The vanishing of the coefficients of order $r \leq 2$ is straight forward. We present the calculation of the coefficient of $(z_{\mathrm{cr}}(\mathtt{f}) - z_{\mathrm{cr}}(\mathtt{g}))^{-3}$ occurring in $\left[\nabla G^{(0)}\overline{\nabla}^{\top}\right]_{\mathtt{fg}}$ here:

$$
\frac{1}{3}\sum_{\mathtt{u}\in\mathtt{f}}\sum_{\mathtt{v}\in\mathtt{g}} \nabla_{\mathtt{fu}}\overline{\nabla}_{\mathtt{gv}}\left(e^{\mathrm{i}\theta_{\mathtt{v}}}-e^{\mathrm{i}\theta_{\mathtt{u}}}\right)^3 = \begin{cases}
\dfrac{1}{3}\ \overbrace{\Big(\displaystyle\sum_{\mathtt{u}\in\mathtt{f}} \nabla_{\mathtt{fu}}\Big)}^{\text{vanishes}}\cdot\Big(\displaystyle\sum_{\mathtt{v}\in\mathtt{g}} \overline{\nabla}_{\mathtt{gv}}\, e^{3i\theta_{\mathtt{v}}}\Big) \\[3ex]
-\ \overbrace{\Big(\displaystyle\sum_{\mathtt{u}\in\mathtt{f}} \nabla_{\mathtt{fu}}\, e^{\mathrm{i}\theta_{\mathtt{u}}}\Big)}^{\text{equals } 1}\cdot\overbrace{\Big(\displaystyle\sum_{\mathtt{v}\in\mathtt{g}} \overline{\nabla}_{\mathtt{gv}}\, e^{2i\theta_{\mathtt{v}}}\Big)}^{-\prod_{\mathtt{v}\in\mathtt{g}} e^{\mathrm{i}\theta_{\mathtt{v}}}} \\[3ex]
+\ \Big(\displaystyle\sum_{\mathtt{u}\in\mathtt{f}} \nabla_{\mathtt{fu}}\, e^{2i\theta_{\mathtt{u}}}\Big)\cdot\overbrace{\Big(\displaystyle\sum_{\mathtt{v}\in\mathtt{g}} \overline{\nabla}_{\mathtt{gv}}\, e^{\mathrm{i}\theta_{\mathtt{v}}}\Big)}^{\text{vanishes}} \\[3ex]
-\ \dfrac{1}{3}\Big(\displaystyle\sum_{\mathtt{u}\in\mathtt{f}} \nabla_{\mathtt{fu}}\, e^{3i\theta_{\mathtt{u}}}\Big)\cdot\overbrace{\Big(\displaystyle\sum_{\mathtt{v}\in\mathtt{g}} \overline{\nabla}_{\mathtt{gv}}\Big)}^{\text{vanishes}}
\end{cases}
$$

and thanks to the general Lemma 11 (or in this case through a direct estimate) its norm is uniformly bounded by a constant independently of the shape of the faces.

For $G^{(0)}$, which is a smooth function of the vertex coordinates, these calculations amount to replacing ∇ and $\overline{\nabla}$ by their corresponding continuous derivatives ∂ and $\bar\partial$, up to subdominant terms of order $0(d^{-3})$. This is in agreement with the general Lemma 2. The result is that the asymptotics 6.38 and 6.39 are valid for $G^{(0)}$ alone.

To end the proof of the lemma, one must show that the corresponding derivative terms for $G^{(2)}$ and $G^{(4)}$ are $\mathrm{O}(d^{-3})$. This is clear for $G^{(4)}$, which is itself $\mathrm{O}(d^{-4})$, hence its discrete derivatives are also $\mathrm{O}(d^{-4})$. But this is not obvious for $G^{(2)}$ which is only $\mathrm{O}(d^{-2})$. We must use the explicit form of $G^{(2)}$. Let us consider the term

$$
\sum_{\mathtt{u}\in\mathtt{f}}\sum_{\mathtt{v}\in\mathtt{g}} \nabla_{\mathtt{fu}}\left(\frac{p_3(\mathtt{u},\mathtt{v})}{(z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{v}))^3}\right)\nabla^{\top}_{\mathtt{vg}}
$$

which appears in $\nabla G^{(2)}\nabla^{\top}$. One has

$$
p_3(\mathtt{u},\mathtt{v}) = p_3(\mathtt{o}_{\mathtt{f}},\mathtt{o}_{\mathtt{g}}) + e^{-3i\theta_{\mathtt{u}}} - e^{-3i\theta_{\mathtt{v}}}
$$

So we have to consider three terms. The first term is

$$\begin{aligned}
&\sum_{\mathtt{u}\in\mathtt{f}}\sum_{\mathtt{v}\in\mathtt{g}}\nabla_{\mathtt{fu}}\left(\frac{p_3(\mathtt{o_f},\mathtt{o_g})}{(z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{v}))^3}\right)\nabla^\top_{\mathtt{vg}}\\
&= p_3(\mathtt{o_f},\mathtt{o_g})\sum_{\mathtt{u}\in\mathtt{f}}\sum_{\mathtt{v}\in\mathtt{g}}\nabla_{\mathtt{fu}}\left(\frac{1}{(z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{v}))^3}\right)\nabla^\top_{\mathtt{vg}}\\
&= p_3(\mathtt{o_f},\mathtt{o_g})\left(\frac{-12}{(z_{\mathrm{cr}}(\mathtt{f})-z_{\mathrm{cr}}(\mathtt{g}))^5}+\mathcal{O}(d^{-6})\right)\\
&= \mathrm{O}(d^{-4})
\end{aligned}$$

In the last step we used the uniform bound from Lemma 5

$$|p_3(\mathtt{o_f},\mathtt{o_g})|\le 3\,|z_{\mathrm{cr}}(\mathtt{f})-z_{\mathrm{cr}}(\mathtt{g})| = 3d$$

The second term is

$$\begin{aligned}
&\sum_{\mathtt{u}\in\mathtt{f}}\sum_{\mathtt{v}\in\mathtt{g}}\nabla_{\mathtt{fu}}\left(\frac{e^{-3i\theta_{\mathtt{u}}}}{(z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{v}))^3}\right)\nabla^\top_{\mathtt{vg}} = \sum_{\mathtt{u}\in\mathtt{f}}\nabla_{\mathtt{fu}}\left(\frac{3\,e^{-3i\theta_{\mathtt{u}}}}{(z_{\mathrm{cr}}(\mathtt{f})-z_{\mathrm{cr}}(\mathtt{g}))^4}+\mathrm{O}(d^{-5})\right)\\
&= 3\left(\sum_{\mathtt{u}\in\mathtt{f}}\nabla_{\mathtt{fu}}\,e^{-3i\theta_{\mathtt{u}}}\right)\frac{1}{(z_{\mathrm{cr}}(\mathtt{f})-z_{\mathrm{cr}}(\mathtt{g}))^4}+\mathrm{O}(d^{-5})
\end{aligned}$$

From Lemma 11

$$\left|\sum_{\mathtt{u}\in\mathtt{f}}\nabla_{\mathtt{fu}}\,e^{-3i\theta_{\mathtt{u}}}\right|\le 6$$

hence the second term is of order $\mathrm{O}(d^{-4})$. By the same argument, the third term is

$$-\sum_{\mathtt{u}\in\mathtt{f}}\sum_{\mathtt{v}\in\mathtt{g}}\nabla_{\mathtt{fu}}\left(\frac{e^{-3i\theta_{\mathtt{v}}}}{(z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{v}))^3}\right)\nabla^\top_{\mathtt{vg}} = \mathrm{O}(d^{-4})$$

This ends the derivation of 6.39 (the second equation is the c.c.). The derivation of 6.38 goes along the same line. $\square$

We are now in a position to state the main result.

Proposition 10. *The second order variation for the Laplace-Beltrami operator* Δ *on an isoradial, Delaunay graph* $\mathtt{G}_{\mathrm{cr}}$ *is*

(6.44)

$$\begin{aligned}
&\mathrm{tr}\left[\eth_{\epsilon_1}\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\right] =\\
&\qquad\frac{1}{\pi^2}\sum_{\mathtt{f}\in\Omega_1}\sum_{\mathtt{g}\in\Omega_2}A(\mathtt{f})\,A(\mathtt{g})\left[\frac{\overline{\nabla}F_1(\mathtt{f})\overline{\nabla}F_2(\mathtt{g})}{\big(z_{\mathrm{cr}}(\mathtt{f})-z_{\mathrm{cr}}(\mathtt{g})\big)^4}+\frac{\nabla\bar{F}_1(\mathtt{f})\nabla\bar{F}_2(\mathtt{g})}{\big(\bar{z}_{\mathrm{cr}}(\mathtt{f})-\bar{z}_{\mathrm{cr}}(\mathtt{g})\big)^4}\right]\ +\ \mathrm{O}(d^{-5})
\end{aligned}$$

where the double sum it taken over pairs of faces $\mathtt{f},\mathtt{g}\in\mathrm{F}(\widehat{\mathtt{G}}_{0^+})$ *such that* $z_{\mathrm{cr}}(\mathtt{u})\in\Omega_1$ *and* $z_{\mathrm{cr}}(\mathtt{v})\in\Omega_2$ *at least one vertex* $\mathtt{u}\in\mathtt{f}$ *and one vertex* $\mathtt{v}\in\mathtt{g}$.

Proof. We start from the local form of the Δ operator 3.18, which implies that the first order variation on Δ is

$$\eth_\epsilon\Delta = 2\left(\eth_\epsilon\overline{\nabla}^\top A\,\nabla+\overline{\nabla}^\top\eth_\epsilon A\,\nabla+\overline{\nabla}^\top A\,\eth_\epsilon\nabla+\eth_\epsilon\nabla^\top A\,\overline{\nabla}+\nabla^\top\eth_\epsilon A\,\overline{\nabla}+\nabla^\top A\,\eth_\epsilon\overline{\nabla}\right)$$

We use the formula for the variation of A

$$\eth_\epsilon A = A(\nabla F+\overline{\nabla}\bar{F})$$

and for the variations of the ∇ and $\overline{\nabla}$ operators given by 5.11, which read

$$\eth_\epsilon \nabla = -\left(\nabla F\,\nabla + \nabla\bar F\,\overline{\nabla}\right)$$
$$\eth_\epsilon \overline{\nabla} = -\left(\overline{\nabla}\bar F\,\overline{\nabla} + \overline{\nabla} F\,\nabla\right)$$

to get

$$\eth_\epsilon \Delta = -4\left(\overline{\nabla}^\top(\nabla\bar F)A\,\overline{\nabla} + \nabla^\top(\overline{\nabla}F)A\,\nabla\right) \tag{6.45}$$

One uses this and the cyclicity of the trace to rewrite the second order variation as

$$\begin{aligned}\mathrm{tr}\left[\eth_{\epsilon_1}\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\right] = 16\ \Big[&\mathrm{tr}\left(A\,\nabla\bar F_1\cdot\overline{\nabla}\Delta_{\mathrm{cr}}^{-1}\overline{\nabla}^\top\cdot A\,\nabla\bar F_2\cdot\overline{\nabla}\Delta_{\mathrm{cr}}^{-1}\,\overline{\nabla}^\top\right)\\ &+\mathrm{tr}\left(A\,\overline{\nabla}F_1\cdot\nabla\Delta_{\mathrm{cr}}^{-1}\overline{\nabla}^\top\cdot A\,\nabla\bar F_2\cdot\overline{\nabla}\Delta_{\mathrm{cr}}^{-1}\,\nabla^\top\right)\\ &+\mathrm{tr}\left(A\,\nabla\bar F_1\cdot\overline{\nabla}\Delta_{\mathrm{cr}}^{-1}\nabla^\top\cdot A\,\overline{\nabla}F_2\cdot\nabla\Delta_{\mathrm{cr}}^{-1}\,\overline{\nabla}^\top\right)\\ &+\mathrm{tr}\left(A\,\overline{\nabla}F_1\cdot\nabla\Delta_{\mathrm{cr}}^{-1}\nabla^\top\cdot A\,\overline{\nabla}F_2\cdot\nabla\Delta_{\mathrm{cr}}^{-1}\,\nabla^\top\right)\Big]\end{aligned}$$

Note that the trace on the l.h.s. is a sum over vertices, while the trace on the r.h.s. is a sum over faces (triangles) Using the large distances asymptotics 6.38 and 6.39, and writing the trace explicitely as a double sum over faces $\mathtt{f}$ and $\mathtt{g}$ gives the theorem.

□

We now consider the other operators. The case of the conformal laplacian is more complicated, so let us first discuss the Kähler operator.

The Kähler operator $\mathcal{D}$.

Proposition 11. *The second order variation for the Kähler operator $\mathcal{D}$ on an isoradial, Delaunay graph $\mathtt{G}_{\mathrm{cr}}$ is of the same form as for the laplacian Δ*

(6.46)

$$\begin{aligned}&\mathrm{tr}\left[\eth_{\epsilon_1}\mathcal{D}\cdot\mathcal{D}_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\mathcal{D}\cdot\mathcal{D}_{\mathrm{cr}}^{-1}\right] =\\ &\frac{1}{\pi^2}\sum_{f\in\Omega_1}\sum_{g\in\Omega_2}A(\mathtt{f})\,A(\mathtt{g})\left[\frac{\overline{\nabla}F_1(\mathtt{f})\overline{\nabla}F_2(\mathtt{g})}{\left(z_{\mathrm{cr}}(\mathtt{f})-z_{\mathrm{cr}}(\mathtt{g})\right)^4}+\frac{\nabla\bar F_1(\mathtt{f})\nabla\bar F_2(\mathtt{g})}{\left(\bar z_{\mathrm{cr}}(\mathtt{f})-\bar z_{\mathrm{cr}}(\mathtt{g})\right)^4}\right]\ +\ \mathrm{O}(d^{-5})\end{aligned}$$

where the double sum it taken over pairs of faces $\mathtt{f},\mathtt{g}\in\mathrm{F}(\widehat{\mathtt{G}}_{0^+})$ such that $z_{\mathrm{cr}}(\mathtt{u})\in\Omega_1$ and $z_{\mathrm{cr}}(\mathtt{v})\in\Omega_2$ at least one vertex $\mathtt{u}\in\mathtt{f}$ and one vertex $\mathtt{v}\in\mathtt{g}$.

Proof. The derivation goes along the same line. We start from Prop. 6 which gives the explicit form 5.4 of the first order variation of $\mathcal{D}$ and from the fact that all circumradii are equal $R(\mathtt{f}) = R_{\mathrm{cr}}$ on an isoradial graph.

$$\mathcal{D}_{\mathrm{cr}} = R_{\mathrm{cr}}^{-2}\Delta_{\mathrm{cr}}$$

This implies that for the deformation of $\mathtt{G}_{\mathrm{cr}}$ the first order variation of $\mathcal{D}$ has a special form

$$\eth_\epsilon\mathcal{D} = R_{\mathrm{cr}}^{-2}\eth_\epsilon\Delta - 4R_{\mathrm{cr}}^{-2}\overline{\nabla}^\top\left(A(\nabla F+\overline{\nabla}\bar F) + C\,\overline{\nabla}F + \bar C\,\nabla\bar F\right)\nabla \tag{6.47}$$

with C and $\bar C$ defined by 5.5. Repeating the analysis done for Theorem 10, which relies on the asymptotics of Lemma 12, one can check that the new term does not change the asymptotics 6.44 obtained for Δ, and leads to the theorem. □

6.3. **The case of the conformal Laplacian: the anomalous term.**

Second order variation for the conformal laplacian $\underline{\boldsymbol{\Delta}}$.
When deforming an isoradial, Delaunay graph, we have seen in the proof 6.1 of Proposition 8 that the contribution made by regular edges $\mathtt{e} \in \mathrm{E}(\mathtt{G}^{\bullet}_{0+})$ to the first order variation $\eth_\epsilon \underline{\boldsymbol{\Delta}}$ of the conformal Laplacian is identical to the variation $\eth_\epsilon \Delta$ of the Laplace-Beltrami laplacian. There is, however, an additional term in the first order variation $\eth_\epsilon \underline{\boldsymbol{\Delta}}$ coming from the chords of $\mathtt{G}_{0+}$. We call this the "anomalous" term and denote it $\delta\mathbb{A}$:

$$\eth_\epsilon \underline{\boldsymbol{\Delta}} = \eth_\epsilon \Delta + \eth_\epsilon \mathbb{A} \tag{6.48}$$

The non-diagonal elements of $\eth_\epsilon \mathbb{A}$ are non-zero only for chords. From 6.18, for vertices $\mathtt{u} \neq \mathtt{v}$, they are
(6.49)

$$\eth_\epsilon \mathbb{A}(\vec{\mathtt{e}}\,) := \big[\eth_\epsilon \mathbb{A}\big]_{\mathtt{uv}} = \begin{cases} \frac{1}{2}\big(\eth_\epsilon \theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)\tan^2\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) + \eth_\epsilon \theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)\tan^2\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)\big) & \text{if } \mathtt{e} = \overline{\mathtt{uv}} \text{ is a chord in } \mathrm{E}(\mathtt{G}_{0+}) \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathtt{e} = \overline{\mathtt{uv}}$ is an edge of $\mathtt{G}_{0+}$ and $\vec{\mathtt{e}} = (\mathtt{u}, \mathtt{v})$ is an orientation. Bear in mind that $\mathtt{G}_{0+}$ since the graph is isoradial and weakly Delaunay and so $\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) = \pm\,\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)$ for any edge $\mathtt{e}$ in $\mathtt{G}_{0+}$. In particular $\tan^2\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) = \tan^2\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)$ and so $\eth_\epsilon \mathbb{A}(\vec{\mathtt{e}}\,) = \eth_\epsilon \mathbb{A}(\vec{\mathtt{e}}^{\,*})$ where $\vec{\mathtt{e}}^{\,*} = (\mathtt{v}, \mathtt{u})$ is the opposite orientation. For the diagonal terms

$$\big[\eth_\epsilon \mathbb{A}\big]_{\mathtt{uu}} = -\sum_{\mathtt{v}\neq\mathtt{u}} \big[\eth_\epsilon \mathbb{A}\big]_{\mathtt{uv}} \tag{6.50}$$

In the case of a chord $\vec{\mathtt{e}}$ we may use 5.13 for the angle variations $\eth_\epsilon \theta_{\mathrm{n}}(\vec{\mathtt{e}}\,)$ and $\eth_\epsilon \theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)$ and re-express the anomalous term $\eth_\epsilon \mathbb{A}(\vec{\mathtt{e}}\,)$ given in formula 6.49 as

$$\eth_\epsilon \mathbb{A}(\vec{\mathtt{e}}\,) = \frac{1}{2}\,\mathfrak{Im}\Big[\overline{\nabla} F(\mathtt{f}_{\mathrm{n}})\,\mathscr{E}_{\mathrm{n}}(\vec{\mathtt{e}}\,)\tan^2\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) + \overline{\nabla} F(\mathtt{f}_{\mathrm{s}})\,\mathscr{E}_{\mathrm{s}}(\vec{\mathtt{e}}\,)\tan^2\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)\Big] \tag{6.51}$$

where the functions $\mathscr{E}_{\mathrm{n}}(\vec{\mathtt{e}})$ and $\mathscr{E}_{\mathrm{s}}(\vec{\mathtt{e}})$ are defined in 5.14 and where $\mathtt{f}_{\mathrm{n}}$ and $\mathtt{f}_{\mathrm{s}}$ are the respective north and south triangles abutting $\vec{\mathtt{e}}$ in the triangulation $\widehat{\mathtt{G}}_{0+}$ which completes $\mathtt{G}_{0+}$.

The second order variation

$$\mathrm{tr}\,\big[\eth_{\epsilon_1}\underline{\boldsymbol{\Delta}}\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\underline{\boldsymbol{\Delta}}\cdot\Delta_{\mathrm{cr}}^{-1}\big] \tag{6.52}$$

is the sum of the second order variation made by the Laplace-Beltrami laplacian, namely

$$\mathrm{tr}\,\big[\eth_{\epsilon_1}\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\big] \tag{6.53}$$

along with three anomalous trace terms which we can express (in light of 6.50) as follows:

(6.54)

$$\underbrace{\operatorname{tr}\left[\eth_{\epsilon_1}\mathbb{A}\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\right]}_{\text{chord-edge term}} = \sum_{\substack{\text{chords } \vec{\mathtt{e}}_1\in\mathsf{G}_{0+}\\ \text{edges } \vec{\mathtt{e}}_2\in\widehat{\mathsf{G}}_{0+}}} \eth_{\epsilon_1}\mathbb{A}(\vec{\mathtt{e}}_1)\,K(\vec{\mathtt{e}}_1,\vec{\mathtt{e}}_2)\,\eth_{\epsilon_2}\Delta(\vec{\mathtt{e}}_2)\,K(\vec{\mathtt{e}}_2,\vec{\mathtt{e}}_1)$$

$$\underbrace{\operatorname{tr}\left[\eth_{\epsilon_1}\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\mathbb{A}\cdot\Delta_{\mathrm{cr}}^{-1}\right]}_{\text{edge-chord term}} = \sum_{\substack{\text{edges } \vec{\mathtt{e}}_1\in\widehat{\mathsf{G}}_{0+}\\ \text{chords } \vec{\mathtt{e}}_2\in\mathsf{G}_{0+}}} \eth_{\epsilon_1}\Delta(\vec{\mathtt{e}}_1)\,K(\vec{\mathtt{e}}_1,\vec{\mathtt{e}}_2)\,\eth_{\epsilon_2}\mathbb{A}(\vec{\mathtt{e}}_2)\,K(\vec{\mathtt{e}}_2,\vec{\mathtt{e}}_1)$$

$$\underbrace{\operatorname{tr}\left[\eth_{\epsilon_1}\mathbb{A}\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\mathbb{A}\cdot\Delta_{\mathrm{cr}}^{-1}\right]}_{\text{chord-chord term}} = \sum_{\substack{\text{chords}\\ \vec{\mathtt{e}}_1,\,\vec{\mathtt{e}}_2\in\mathsf{G}_{0+}}} \eth_{\epsilon_1}\mathbb{A}(\vec{\mathtt{e}}_1)\;K(\vec{\mathtt{e}}_1,\vec{\mathtt{e}}_2)\;\eth_{\epsilon_2}\mathbb{A}(\vec{\mathtt{e}}_2)\;K(\vec{\mathtt{e}}_2,\vec{\mathtt{e}}_1)$$

where $\vec{\mathtt{e}}_1=(\mathtt{u}_1,\mathtt{v}_1)$ and $\vec{\mathtt{e}}_2=(\mathtt{u}_2,\mathtt{v}_2)$ are oriented edges of the triangulation $\widehat{\mathsf{G}}_{0+}$ whose vertices $\mathtt{u}_1,\mathtt{v}_1$ and $\mathtt{u}_2,\mathtt{v}_2$ lie in $\overline{\Omega}_1$ and $\overline{\Omega}_2$ respectively and where

$$K(\vec{\mathtt{e}}_1,\vec{\mathtt{e}}_2) := \left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{v}_1\mathtt{v}_2}-\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{u}_1\mathtt{v}_2}-\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{v}_1\mathtt{u}_2}+\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{u}_1\mathtt{u}_2} \tag{6.55}$$

Note that $K(\vec{\mathtt{e}}_1,\vec{\mathtt{e}}_2)=K(\vec{\mathtt{e}}_2,\vec{\mathtt{e}}_1)=-K(\vec{\mathtt{e}}_1^*,\vec{\mathtt{e}}_2)$ where $\vec{\mathtt{e}}_1^*=(\mathtt{v}_1,\mathtt{u}_1)$ has the reverse orientation. Applying two rounds of formula 3.14 we obtain

$$\begin{aligned} K(\vec{\mathtt{e}}_1,\vec{\mathtt{e}}_2) &= \left\{\begin{array}{c} p_1(\mathtt{u}_2,\mathtt{v}_2)\left[\Delta_{\mathrm{cr}}^{-1}\nabla^\top\right]_{\mathtt{u}_1\mathtt{f}_2} \;-\; p_1(\mathtt{u}_2,\mathtt{v}_2)\left[\Delta_{\mathrm{cr}}^{-1}\nabla^\top\right]_{\mathtt{v}_1\mathtt{f}_2}\\ + \\ \overline{p}_1(\mathtt{u}_2,\mathtt{v}_2)\left[\Delta_{\mathrm{cr}}^{-1}\overline{\nabla}^\top\right]_{\mathtt{u}_1\mathtt{f}_2} \;-\; \overline{p}_1(\mathtt{u}_2,\mathtt{v}_2)\left[\Delta_{\mathrm{cr}}^{-1}\overline{\nabla}^\top\right]_{\mathtt{v}_1\mathtt{f}_2} \end{array}\right. \\ &= 2\,\mathfrak{Re}\left[\begin{array}{c} p_1(\mathtt{u}_1,\mathtt{v}_1)\,p_1(\mathtt{u}_2,\mathtt{v}_2)\left[\nabla\Delta_{\mathrm{cr}}^{-1}\nabla^\top\right]_{\mathtt{f}_1\mathtt{f}_2}\\ + \\ p_1(\mathtt{u}_1,\mathtt{v}_1)\,\overline{p}_1(\mathtt{u}_2,\mathtt{v}_2)\left[\nabla\Delta_{\mathrm{cr}}^{-1}\overline{\nabla}^\top\right]_{\mathtt{f}_1\mathtt{f}_2} \end{array}\right] \end{aligned} \tag{6.56}$$

where $\mathtt{f}_i$ is a triangle of $\widehat{\mathsf{G}}_{0+}$, north or south, containing the edge $\vec{\mathtt{e}}_i$ for $i=1,2$. By assumption $\overline{\Omega}_1$ and $\overline{\Omega}_2$ are separated by a large distance $d\gg R_{\mathrm{cr}}$ and so we can estimate $K(\vec{\mathtt{e}}_1,\vec{\mathtt{e}}_2)$ as presented in formula 6.56 using asymptotic expansions 6.38 and 6.39 of Lemma 12. We end up with

$$K(\vec{\mathtt{e}}_1,\vec{\mathtt{e}}_2) = \frac{1}{2\pi}\,\mathfrak{Re}\left[\frac{p_1(\mathtt{u}_1,\mathtt{v}_1)\,p_1(\mathtt{u}_2,\mathtt{v}_2)}{\big(z_{\mathrm{cr}}(\mathtt{f}_1)-z_{\mathrm{cr}}(\mathtt{f}_2)\big)^2}\right] + \mathrm{O}\left(\frac{1}{\big|z_{\mathrm{cr}}(\mathtt{f}_1)-z_{\mathrm{cr}}(\mathtt{f}_2)\big|^3}\right) \tag{6.57}$$

where $p_1(\mathtt{u},\mathtt{v})=z_{\mathrm{cr}}(\mathtt{v})-z_{\mathrm{cr}}(\mathtt{u})$ as introduced in Definition 14.

The chord-chord term. Let's begin by examining the chord-chord term of 6.54. It involves the contribution of two (oriented) chords $\vec{\mathtt{e}}_1=(\mathtt{u}_1,\mathtt{v}_1)$ and $\vec{\mathtt{e}}_2=(\mathtt{u}_2,\mathtt{v}_2)$ whose vertices of $\mathtt{u}_1,\mathtt{v}_1$ and $\mathtt{u}_2,\mathtt{v}_2$ are contained in $\overline{\Omega}_1$ and $\overline{\Omega}_2$ respectively. Since

$\vec{\mathtt{e}}_i = (\mathtt{u}_i, \mathtt{v}_i)$ is a chord for $i = 1, 2$ in $\mathtt{G}_{0^+}$ the corresponding north and south triangles $\mathtt{f}_{i\mathrm{n}}$ and $\mathtt{f}_{i\mathrm{s}}$ in $\widehat{\mathtt{G}}_{0^+}$ are concyclic and therefore share a common circumcenter whose complex coordinate we denote $\mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_i) = z(\mathtt{f}_{i\mathrm{n}}) = z(\mathtt{f}_{i\mathrm{s}})$. This is depicted in Fig 18. Putting things together, we see that the contribution made by a pair of (oriented) chords $(\vec{\mathtt{e}}_1, \vec{\mathtt{e}}_2)$ to the chord-chord anomalous trace term in 6.54 is

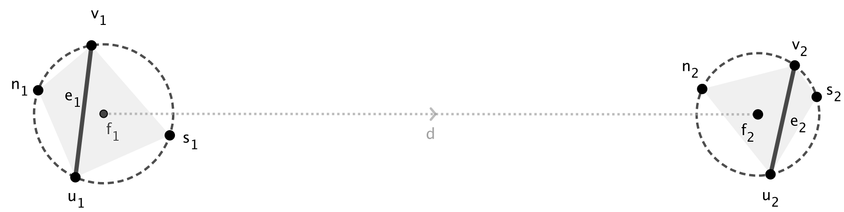

FIGURE 18. Two far apart chords $\vec{\mathtt{e}}_1 = (\mathtt{u}_1\mathtt{v}_1)$ and $\vec{\mathtt{e}}_2 = (\mathtt{u}_2\mathtt{v}_2)$ at distance $d \gg 1$

(6.58)

$$\frac{1}{16\,\pi^2}\,\eth_{\epsilon_1}\mathbb{A}(\vec{\mathtt{e}}_1)\,\eth_{\epsilon_2}\mathbb{A}(\vec{\mathtt{e}}_2)\left(\mathfrak{Re}\left[\frac{p_1(\mathtt{u}_1,\mathtt{v}_1)\,p_1(\mathtt{u}_2,\mathtt{v}_2)}{\big(\mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_1) - \mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_2)\big)^2}\right]\right)^2 + \mathrm{O}\left(\frac{1}{\big|\mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_1) - \mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_2)\big|^5}\right)$$

with $\eth_{\epsilon_1}\mathbb{A}(\vec{\mathtt{e}}_1)$ and $\eth_{\epsilon_2}\mathbb{A}(\vec{\mathtt{e}}_2)$ given by 6.51, that we recall for completeness.

$$\eth_{\epsilon}\mathbb{A}(\vec{\mathtt{e}}\,) = \frac{1}{2}\,\mathfrak{Im}\Big[\overline{\nabla}F(\mathtt{f}_{\mathrm{n}})\,\mathscr{E}_{\mathrm{n}}(\vec{\mathtt{e}}\,)\tan^2\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) + \overline{\nabla}F(\mathtt{f}_{\mathrm{s}})\,\mathscr{E}_{\mathrm{s}}(\vec{\mathtt{e}}\,)\tan^2\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)\Big]$$

with

$$\mathscr{E}_{\mathrm{n}}(\vec{\mathtt{e}}\,) = \frac{\overline{z}(\mathtt{v}) - \overline{z}(\mathtt{n})}{z(\mathtt{v}) - z(\mathtt{n})} - \frac{\overline{z}(\mathtt{u}) - \overline{z}(\mathtt{n})}{z(\mathtt{u}) - z(\mathtt{n})} = \frac{-4A(\mathtt{f}_{\mathrm{n}})}{\big(z(\mathtt{v}) - z(\mathtt{n})\big)\big(z(\mathtt{u}) - z(\mathtt{n})\big)}$$

and a similar form for $\mathcal{E}_{\mathrm{s}}(\vec{\mathtt{e}}\,)$. Any triangulation $\widehat{\mathtt{G}}_{0^+}$ which completes the limit graph $\mathtt{G}_{0^+}$ is itself isoradial and weakly Delaunay consequently $\tan^2\theta_{\mathrm{n}}(\vec{\mathtt{e}}\,) = \tan^2\theta_{\mathrm{s}}(\vec{\mathtt{e}}\,)$ the value of which is given by 3.8.

The result 6.58 for the anomalous chord-chord contribution to the variation of $\log\det\underline{\boldsymbol{\Delta}}$ does not have the same for than the "regular" contribution 6.53 which is similar to the variation of the Laplace-Beltrami operator Δ, which is a sum over triangles of terms

$$A(\mathtt{f}_1)A(\mathtt{f}_2)\frac{\overline{\nabla}F_1(\mathtt{f}_1)\cdot\overline{\nabla}F_2(\mathtt{f}_2)}{(z_{\mathrm{cr}}(\mathtt{f}_1) - z_{\mathrm{cr}}(\mathtt{f}_2))^4} \;+\; \text{c.c.}$$

First, besides harmonic terms in the coordinate of the circumcenters of the form

$$\big(\mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_1) - \mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_2)\big)^{-4} \quad\text{and}\quad \big(\overline{\mathscr{Z}}_{\mathrm{cr}}(\vec{\mathtt{e}}_1) - \overline{\mathscr{Z}}_{\mathrm{cr}}(\vec{\mathtt{e}}_2)\big)^{-4}$$

it contains non harmonic term of the form

$$\big|\mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_1) - \mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_2)\big|^{-4}$$

problematic with conformal invariance and an interpretation in term of CFT, as will be discussed in Sect. 9.

Second, from the form of $\eth_{\epsilon_1}\mathbb{A}(\vec{\mathtt{e}}_1)$ and $\eth_{\epsilon_2}\mathbb{A}(\vec{\mathtt{e}}_2)$, it does not contains only terms of the form

$$\overline{\nabla} F_1(\mathtt{f}_1)\cdot\overline{\nabla} F_2(\mathtt{f}_2) \quad\text{and}\quad \nabla \bar{F}_1(\mathtt{f}_2)\cdot\nabla\bar{F}_2(\mathtt{f}_2)$$

but also terms of the form

$$\overline{\nabla} F_1(\mathtt{f}_1)\cdot\nabla \bar{F}_2(\mathtt{f}_2) \quad\text{and}\quad \nabla \bar{F}_1(\mathtt{f}_2)\cdot\overline{\nabla} F_2(\mathtt{f}_2)$$

Third, the geometric terms associated to the faces (the triangles $\mathtt{f}_1$ and $\mathtt{f}_2$) are not simply the area terms $A(\mathtt{f}_1)$ and $A(\mathtt{f}_2)$, but they depend of the detailled geometry and orientation of the chords and the triangles through the terms $\mathcal{E}_{\mathrm{n/s}}(\vec{\mathtt{e}})$ and $\tan^2\theta_{\mathrm{n/s}}(\vec{\mathtt{e}})$.

The chord-edge term. We now discuss briefly the chord-edge term present in 6.54 which involves the anomalous variation term $[\eth_{\epsilon_1}\mathbb{A}]_{\mathtt{u}_1\mathtt{v}_1}$ of a chord $\vec{\mathtt{e}}_1 = (\mathtt{u}_1,\mathtt{v}_1)$ and the ordinary variation term $[\eth_{\epsilon_2}\Delta]_{\mathtt{u}_2\mathtt{v}_2}$ of an edge $\vec{\mathtt{e}}_2 = (\mathtt{u}_2,\mathtt{v}_2)$. It will be simpler to group together the terms made by a single chord $\vec{\mathtt{e}}_1 = \vec{\mathtt{e}} = (\mathtt{u},\mathtt{v})$ and the edges $\vec{\mathtt{e}}_2$ forming the boundary of a fixed (counter-clockwise oriented) triangle $\mathtt{f}$ and then sum the contributions as the chord $\vec{\mathtt{e}}$ in $\mathfrak{G}_{0+}$ and triangle $\mathtt{f}$ in $\widehat{\mathfrak{G}}_{0+}$ both vary; see the illustration in Fig. 19. Accordingly, the contribution made by a chord-triangle pair $(\vec{\mathtt{e}},\mathtt{f})$ is found to be

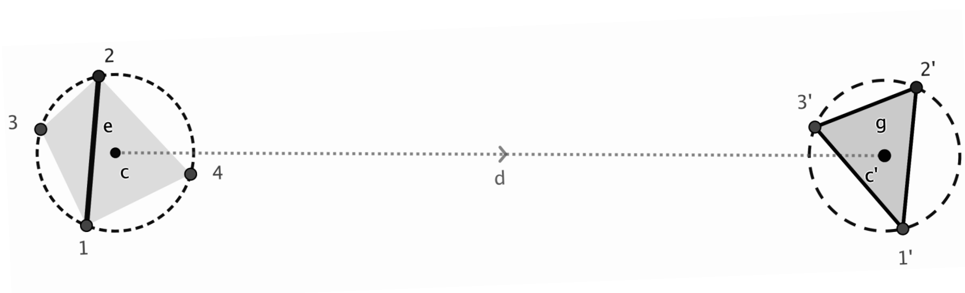

FIGURE 19. A chord $\mathtt{e} = (12)$ and a triangle $\mathtt{g} = (1'2'3')$ at distance d

$$\frac{1}{4\pi^2}\,\eth_{\epsilon_1}\mathbb{A}(\vec{\mathtt{e}})\,\mathfrak{Re}\left[\frac{p_1^2(\mathtt{u},\mathtt{v})\,A(\mathtt{f})\,\overline{\nabla}F_2(\mathtt{f})}{\big(\mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}})-z_{\mathrm{cr}}(\mathtt{f})\big)^4}\right] + \mathrm{O}\left(\frac{1}{\big|\mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}})-z_{\mathrm{cr}}(\mathtt{f})\big|^5}\right) \tag{6.59}$$

This term is again different from the regular term. Now it is harmonic in the coordinates ov the circumcenters, since it does not contain the non harmonic term

$$\big|\mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_1)-\mathscr{Z}_{\mathrm{cr}}(\vec{\mathtt{e}}_2)\big|^{-4}$$

However, it still contains the terms of the form

$$\overline{\nabla} F_1(\mathtt{f}_1)\cdot\nabla \bar{F}_2(\mathtt{f}_2) \quad\text{and}\quad \nabla \bar{F}_1(\mathtt{f}_2)\cdot\overline{\nabla} F_2(\mathtt{f}_2)$$

and it depends on the detailled geometry and orientation of the chord, as for the chord-chord term discussed previously.

A simplification for specific deformations.
Finally, let us note that the anomalous term $\eth_\epsilon \mathbb{A}(\vec{\mathtt{e}})$ for a chord $\vec{\mathtt{e}}$ 6.51 takes a simpler form in the special case where the discrete derivatives of F coincides on the north and south triangles $\mathtt{f}_{\mathrm{n}}(\vec{\mathtt{e}})$ and $\mathtt{f}_{\mathrm{s}}(\vec{\mathtt{e}})$ thanks to the following lemma,

Lemma 13. *Consider two triangles* $\mathtt{N} = (\mathtt{v}_1, \mathtt{v}_2, \mathtt{v}_3)$ *and* $\mathtt{S} = (\mathtt{v}_2, \mathtt{v}_1, \mathtt{v}_4)$ *sharing the edge* $\overline{\mathtt{v}_1\mathtt{v}_2}$ *and the flipped triangles* $\mathtt{E} = (\mathtt{v}_3, \mathtt{v}_4, \mathtt{v}_2)$ *and* $\mathtt{W} = (\mathtt{v}_4, \mathtt{v}_3, \mathtt{v}_1)$ *sharing the edge* $\overline{\mathtt{v}_3\mathtt{v}_4}$ *as depicted on Fig. 6.3. Let* $\mathtt{v} \mapsto F(\mathtt{v})$ *be a function defined on the vertices. Then the four following equalities are equivalent*

$$\nabla F(\mathtt{N}) = \nabla F(\mathtt{S}) \ , \quad \nabla F(\mathtt{E}) = \nabla F(\mathtt{W}) \ , \quad \overline{\nabla} F(\mathtt{N}) = \overline{\nabla} F(\mathtt{S}) \ , \quad \overline{\nabla} F(\mathtt{E}) = \overline{\nabla} F(\mathtt{W}) \tag{6.60}$$

Note that the four points are not necessarily concyclic.

Proof. The proof follows from the definitions 3.10 and 3.12, and it is left to the reader. It has a simple geometric interpretation. Again, note that this is valid for any pair of triangles sharing an edge. □

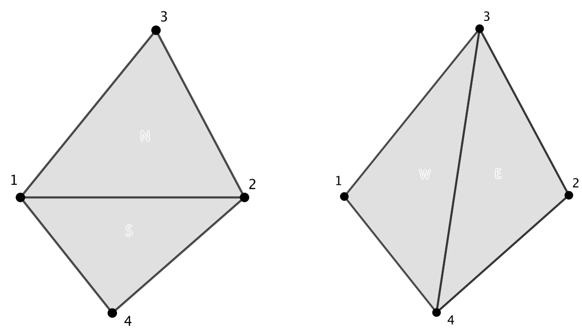

FIGURE 20. N, S, E and W triangles

In this case, to a cocyclic configuration of points, namely a simple cyclic polygon $P = (z_1, z_2, \cdots z_k)$ $(k \geq 4)$, is associated a single pair of discrete derivatives of F attached to the circumcenter $\mathtt{c}$ of P, $(\nabla F(\mathtt{c}), \overline{\nabla} F(\mathtt{c}))$. Then the variation $\eth_{\epsilon_1} \mathbb{A}(\vec{\mathtt{e}})$ for a chord $\vec{\mathtt{e}}$ is simply

$$\eth_{\epsilon_1} \mathbb{A}(\vec{\mathtt{e}}) = \frac{1}{2} \,\mathfrak{Im}\Big[\overline{\nabla} F(\mathtt{c}) \left(\mathscr{E}_{\mathrm{n}}(\vec{\mathtt{e}}) + \mathscr{E}_{\mathrm{s}}(\vec{\mathtt{e}})\right)\Big] \tan^2 \theta_{\mathrm{n/s}}(\vec{\mathtt{e}}) \tag{6.61}$$

6.4. **Curvature dipoles and the anomalous chord term.** Let us discuss a possible explanation of the anomalous terms corresponding to deformations of cocyclic configurations of points which undergo flips (the chords) of the triangulation. The adjective "anomalous" reflects the face that these contributions are not present for either the Laplace-Beltrami operator Δ or the Kähler operator $\mathcal{D}$, both of which admits a smooth continuum limit consistent with the predictions of conformal invariance.

As discussed in [DE14] and Subsection 1.2 the conformal laplacian $\underline{\boldsymbol{\Delta}}$ for a Delaunay graph G can be considered as the discretized *Laplace-Beltrami operator* on a *curved surface* $\mathsf{S}^{\Diamond}_{\mathsf{G}}$. The construction of $\mathsf{S}^{\Diamond}_{\mathsf{G}}$ is illustrated in Fig. 21 for an isoradial Delaunay graph G and in Fig. 22 for a generic (non-isoradial) Delaunay triangulation G.

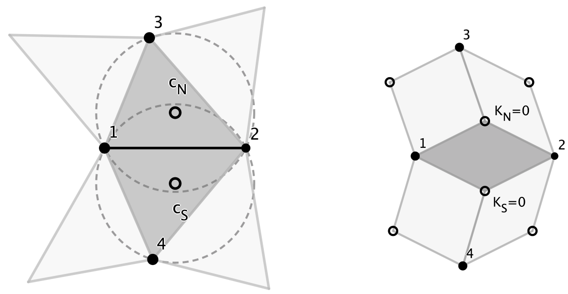

FIGURE 21. A regular edge $\mathtt{e} = (12)$ of a critical triangulation $\mathtt{G}$ (left) and its associated rhombic lattice $\mathtt{G}^\lozenge$ (right), the curvature K associated to each face of $\mathtt{G}$, i.e. its white $\tilde{\mathtt{o}}$-vertices of $\mathtt{G}^\lozenge$, is zero.

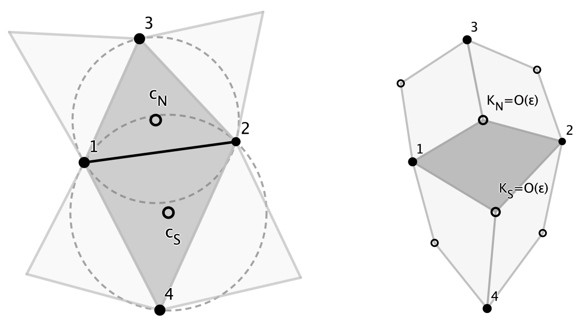

FIGURE 22. A $\mathrm{O}(\epsilon)$ deformation of $\mathtt{G}$. The Gauss curvatures K of the N and S faces are non-zero, but of order $\mathrm{O}(\epsilon)$.

It is easy to that the surface $\mathtt{S}_{\mathtt{G}}^\lozenge$ is piecewise flat, with curvature defects (i.e. conical singularities) localized at the vertices $\tilde{\mathtt{o}}_{\mathtt{f}}$ associated to circumcenters of faces $\mathtt{f}$ in $\mathtt{G}$. The defect angle $K(\mathtt{f})$ corresponds to a localized curvature defect at $\tilde{\mathtt{o}}_{\mathtt{f}}$ and its value is given in term if the conformal angles $\theta(\mathtt{e})$ of the edges forming the boundary of the face $\mathtt{f}$. Recall from 2.3 that the associated scalar curvature $R_{\text{scal}}(\tilde{\mathtt{o}}_{\mathtt{f}})$ at a vertex $\tilde{\mathtt{o}}_{\mathtt{f}}$ is twice the measure of the defect angle around the circumcenter $\mathtt{o}_{\mathtt{f}}$ of the face $\mathtt{f}$, namely

$$R_{\text{scal}}(\tilde{\mathtt{o}}_{\mathtt{f}}) = 4\pi - 2\sum_{\mathtt{e}\in\partial\mathtt{f}} \big(\pi - 2\theta(\mathtt{e})\big) \tag{6.62}$$

or equivalently the Gauss curvature

$$\underbrace{K(\mathtt{f}) \ := \ 2\pi - \sum_{\mathtt{e}\in\mathtt{f}} \big(\pi - 2\theta(e)\big)}_{\text{discrete Gauss curvature}} \tag{6.63}$$

For an isoradial Delaunay graph $\mathtt{G}$ the rhombic surface $\mathtt{S}_{\mathtt{G}}^\lozenge$ coincides with the planar kite graph $\mathtt{G}^\lozenge$ whose faces, in this case, are all rhombs. Furthermore, the scalar curvature $R_{\text{scal}}(\tilde{\mathtt{o}}_{\mathtt{f}})$ associated to each face $\mathtt{f}$ in $\mathtt{G}$ is zero. For a generic Delaunay graph $\mathtt{G}$ the scalar curvature $R_{\text{scal}}(\tilde{\mathtt{o}}_{\mathtt{f}})$ will be non-zero (see Fig. 22). Indeed, consider a cyclic quadrilateral face $\mathtt{f}$ in an isoradial triangulation $\mathtt{G}_{\text{cr}}$ depicted in Fig. 23 and the effects of a generic deformation $\mathtt{G}_{\text{cr}} \to \mathtt{G}_{\epsilon}$ depicted in Fig. 24. In $\mathtt{S}^{\diamondsuit}_{\mathtt{G}_{\text{cr}}}$ four lozenges meet at $\tilde{\mathtt{o}}_{\mathtt{f}}$ where the scalar curvature $R_{\text{scal}}(\tilde{\mathtt{o}}_{\mathtt{f}})$ vanishes.

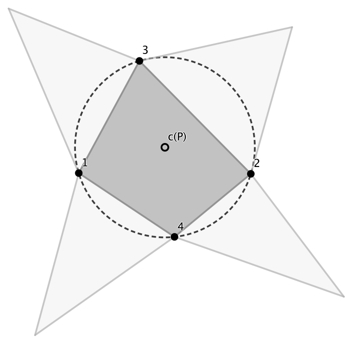

FIGURE 23. A cocylic face $P = (1423)$ of a critical triangulation $\mathtt{G}_{\text{cr}}$ (left) and its associated rhombic lattice $\mathtt{G}^{\diamondsuit}$ (right), the curvature K associated to each face of $\mathtt{G}$, i.e. its white $\mathtt{o}$-vertices of $\mathtt{G}^{\diamondsuit}$, is zero.

However, as soon as we deform this cyclic quadrilateral, generically a diagonal edge $\mathtt{e}$ emerges in $\mathtt{G}_{\epsilon}$ (infinitesimally a chord $\mathtt{e}$ in $\mathtt{G}_{0+}$) which subdivides the quadrilateral $\mathtt{f}$ into two triangles $\mathtt{f}_{\text{n}}$ and $\mathtt{f}_{\text{s}}$, while the circumcenter $\mathtt{o}_{\mathtt{f}}$ splits into two circumcenters $\mathtt{o}_{\text{n}}$ and $\mathtt{o}_{\text{s}}$. In the deformed rhombic surface $\mathtt{S}^{\diamondsuit}_{\mathtt{G}_{\epsilon}}$ a new lozenge appears between $\tilde{\mathtt{o}}_{\text{n}}$ and $\tilde{\mathtt{o}}_{\text{s}}$. However now this new lozenge is "flat" i.e. its angles at first order in $(\epsilon\ 0,\pi,0,\pi)$! To first order in ϵ the defect saclar curvatures $R_{\text{scal}}(\tilde{\mathtt{o}}_{\text{n}})$ and $R_{\text{scal}}(\tilde{\mathtt{o}}_{\text{s}})$ for these two faces are found to be of order $\mathrm{O}(1)$ and opposite, not of order $\mathrm{O}(\epsilon)$

$$K(\mathtt{f}_{\text{n}}) = -2\theta_{\text{n}}(\vec{\mathtt{e}}\,) \ + \ \mathrm{O}(\epsilon) \quad , \qquad K(\mathtt{f}_{\text{s}}) = -2\theta_{\text{s}}(\vec{\mathtt{e}}\,) \ + \ \mathrm{O}(\epsilon) \tag{6.64}$$

Thus the chord $\mathtt{e}$ is associated to a *curvature dipole*, i.e. neighboring curvature defects with non-zero but opposite signs. In this way the smooth deformation $\mathtt{G}_{\text{cr}} \to \mathtt{G}_{\epsilon}$ manifests itself as a *discontinuity* of the curvature. with respect to a smooth deformation. Generically when one deforms a cyclic face $\mathtt{f}$ of $\mathtt{G}_{\text{cr}}$ with four or more vertices, i.e. when the deformation implies a flip for the triangulation, a curvature dipole appears, and the smooth deformation of $\mathtt{G}_{\text{cr}} \to \mathtt{G}_{\epsilon}$ corresponds to a *non-smooth deformation* of curvatures.

Finally, let us stress that a curvature dipole appears if the anomalous term $\eth_{\epsilon}\mathbb{A}(\vec{\mathtt{e}}\,)$ discussed above in 6.3 is non-zero. Indeed this anomalous term is proportional to $\tan^2\theta_{\text{n}}(\vec{\mathtt{e}}\,)$, while the dipole is proportional to $\theta_{\text{n}}(\vec{\mathtt{e}}\,)$. Thus if for the chord $\theta_{\text{n}}(\vec{\mathtt{e}}\,) = \theta_{\text{s}}(\vec{\mathtt{e}}\,) = 0$, no anomalous term is present and no curvature dipole appears at first order in the deformation. The variation term for the chord $\vec{\mathtt{e}}$ is the same for Δ and for $\underline{\boldsymbol{\Delta}}$. One should note that this occurs iff the circumcenter $\mathtt{o}_{\mathtt{f}}$ of the face $\mathtt{f}$ lies on the edge $\mathtt{e}$. If $\mathtt{f}$ is a quadrilateral (as on Fig. 24) and if both $\mathtt{e} = (12)$ and the flipped edge $\mathtt{e}^* = (34)$ share this property $(\theta_{\text{n}}(\vec{\mathtt{e}}^*\,) = \theta_{\text{s}}(\vec{\mathtt{e}}^*\,) = 0,$

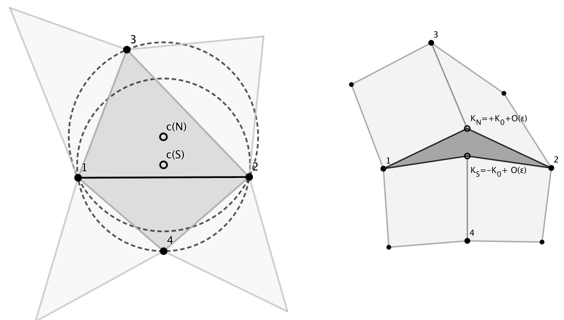

FIGURE 24. A $\mathrm{O}(\epsilon)$ deformation of the cocyclic configuration. The Delaunay condition select a chord $\mathtt{e} = (12)$, which splits the face $P = (1423)$ into two triangles $N = (123)$ and $S = (214)$. A flat lozenge $(1S2N)$ appear in the rhombic lattice $\mathtt{G}^{\diamond}$. The curvatures K of the N and S faces are non-zero, but of order $\mathrm{O}(1)$ and opposite. N and S form a "curvature dipole".

then the face $\mathtt{f}$ is a rectangle. Then to first order in the deformation parameter ϵ, the deformation is $\mathtt{isoradial} \to \mathtt{isoradial}$, not $\mathtt{isoradial} \to \mathtt{non-isoradial}$. These $\mathtt{isoradial} \to \mathtt{isoradial}$ deformations are the one considered by Kenyon in the seminal paper [Ken02].

Of course these considerations extend to concyclic configurations involving more that four points.

7. The scaling limit of variations

Rescaling the deformations.

7.1. **Rescaling smooth deformations.** As explained in the introduction we incorporate a scaling factor factor $\ell > 0$ into the deformation in order to define and study a continuum limit. We may view the scaling parameter $\ell > 0$ as imparting a *resolution* on the critical graph, i.e. we get a rescaled embedding $z_{\rm cr}^{1/\ell} := z_{\rm cr}/\ell$ of $\mathtt{G}_{\rm cr}$, under which vertices become closer and denser in any compact region of the plane as $\ell > 0$ increases. In particular the area $A(\mathtt{f})$ of a face $\mathtt{f} \in \mathrm{F}(\mathtt{G}_{\rm cr})$ shrinks by a factor of $1/\ell^2$ under the rescaled embedding while its circumcenter coordinate $z_{\rm cr}(\mathtt{f})$ is rescaled by a factor of $1/\ell$. In this way, the scaling parameter $\ell > 0$ allows us to interpret the critical graph as a planar partition and can be used to define a Riemann sum. More specifically, given any continuous complex-valued function $H : \mathbb{C} \longrightarrow \mathbb{C}$ with compact support $\Omega = \operatorname{supp} H$ then

$$\lim_{\ell\to\infty} \sum_{\mathtt{x}\in \mathrm{F}(\mathtt{G}_{\rm cr})} A(\mathtt{x})/\ell^2 \cdot H\big(z_{\rm cr}(\mathtt{x})/\ell\big) \;=\; \int_\Omega d^2x\, H(x) \tag{7.1}$$

Given a smooth complex-valued function $F : \mathbb{C} \longrightarrow \mathbb{C}$ with compact support, and $\ell > 0$ a scaling real parameter, we set $F_\ell(z) := \ell F(z/\ell)$. When deforming a critical isoradial Delaunay graph $\mathtt{G}_{\rm cr}$ (with unit circumradius $R_{\rm cr} = 1$), we shall consider the restriction of F_ℓ to (the coordinates of) the vertices of the critical graph. By abuse of notation we shall write $F_\ell(\mathtt{v}) := \ell\, F\big(z_{\rm cr}(\mathtt{v})/\ell\big)$ for each vertex $\mathtt{v} \in \mathrm{V}(\mathtt{G}_{\rm cr})$. We use F_ℓ to displace the coordinates of the critical graph and define a deformed embedding, namely

$$z_{\epsilon,\ell}(\mathtt{v}) :=\; z_{\rm cr}(\mathtt{v}) \;+\; \epsilon\, F_\ell\big(\mathtt{v}\big)$$

7.2. **Rescaling bi-local deformations.** Our analysis of second order variations (for the log-determinants which we consider) involve a bi-local deformation implemented by two smooth, complex-valued functions F_1 and $F_2 : \mathbb{C} \longrightarrow \mathbb{C}$ whose respective supports Ω_1 and Ω_2 are *compact* and have closures $\overline{\Omega}_1$ and $\overline{\Omega}_2$ which are *disjoint*. Set

$$d := \mathtt{dist}(\Omega_1, \Omega_2) = \inf\big\{|w_1 - w_2| \,\big|\, w_i \in \Omega_i\big\}$$

to be the distance between the supports Ω_1 and Ω_2. Obviously $0 < d < \infty$. The corresponding deformed embedding $z_{\underline{\epsilon},\ell} : \mathrm{V}(\mathtt{G}_{\rm cr}) \longrightarrow \mathbb{C}$ of the critical lattice is given by

$$z_{\underline{\epsilon},\ell}(\mathtt{v}) :=\; z_{\rm cr}(\mathtt{v}) \;+\; \epsilon_1 F_{1;\ell}(\mathtt{v}) \;+\; \epsilon_2 F_{2;\ell}(\mathtt{v})$$

where $\underline{\epsilon} = (\epsilon_1, \epsilon_2)$ is a pair of deformation parameters $\epsilon_1, \epsilon_2 \geq 0$ and where we use the notation $F_{i;\ell}(z) := \ell F_i(z/\ell)$ and by abuse of notation $F_{i;\ell}(\mathtt{v}) := F_{i;\ell}\big(z_{\rm cr}(\mathtt{v})\big)$ for a vertex $\mathtt{v} \in \mathrm{V}(\mathtt{G}_{\rm cr})$ and $i = 1, 2$. The results of Lemma 10 still hold for the bi-local deformed embedding $z_{\underline{\epsilon},\ell}$; simply apply the Lemma to F_1 and F_2 independently and take $\tilde{\epsilon}_F = \min(\tilde{\epsilon}_{F_1}, \tilde{\epsilon}_{F_2})$. Let us denote by $\mathtt{G}_{\underline{\epsilon},\ell}$ the Delaunay graph uniquely determined by the vertex set $\mathrm{V}(\mathtt{G}_{\underline{\epsilon},\ell}) := \mathrm{V}(\mathtt{G}_{\rm cr})$ together with the deformed embedding $z_{\underline{\epsilon},\ell}$. As we have seen, the one-sided limit $\epsilon_i \to 0^+$ for $i = 1, 2$ induces the structure of a weak Delaunay graph $\mathtt{G}_{0+,\ell}$ on the vertex set $\mathrm{V}(\mathtt{G}_{\rm cr})$ with respect to the critical embedding $z_{\rm cr}$. In general, the edge set $\mathrm{E}(\mathtt{G}_{0+,\ell})$ will vary as the scaling parameter $\ell > 0$ evolves; nevertheless $\mathrm{E}(\mathtt{G}_{\rm cr}) \subseteq \mathrm{E}(\mathtt{G}_{0+,\ell})$ for all $0 < \ell \leq \infty$. For each value of $\ell > 0$ select a weak Delaunay triangulation $\widehat{\mathtt{G}}_{0+,\ell}$ which completes $\mathtt{G}_{0+,\ell}$. Because $\mathrm{E}(\mathtt{G}_{\mathrm{cr}}) \subseteq \mathrm{E}(\mathtt{G}_{0+,\ell}) \subseteq \mathrm{E}(\widehat{\mathtt{G}}_{0+,\ell})$ for each $0 < \ell \leq \infty$ we may always perform the following resummation

$$\sum_{\mathtt{x}\in\mathrm{F}(\widehat{\mathtt{G}}_{0+,\ell})} A(\mathtt{x})\, H(z_{\mathrm{cr}}(\mathtt{x})) \;=\; \sum_{\mathtt{y}\in\mathrm{F}(\mathtt{G}_{\mathrm{cr}})} A(\mathtt{y})\, H\big(z_{\mathrm{cr}}(\mathtt{y})\big) \tag{7.2}$$

where we combine terms on the left hand side involving triangles of $\widehat{\mathtt{G}}_{0+,\ell}$ which share a common circumcenter and where $H\big(\mathtt{x}\big)$ is any quantity which depends only upon the circumcenter $z_{\mathrm{cr}}(\mathtt{x})$ of $\mathtt{x} \in \mathrm{F}(\widehat{\mathtt{G}}_{0+,\ell})$. Consequently the choice of triangulation $\widehat{\mathtt{G}}_{0+,\ell}$ completing $\mathtt{G}_{0+,\ell}$ will not affect our calculations.

7.3. **Scaling limit and derivation of Theorem 3.** We now are in a position to study the scaling limit of the bilocal deformation terms 6.44 (Prop. 10) and 6.46 (Prop. 11)) and to derive Theorem 3. For $\mathtt{G} = \mathtt{G}_{\mathrm{cr}}$ or $\mathtt{G} = \widehat{\mathtt{G}}_{0+,\ell}$ let $\mathrm{F}_{\Omega_i(\ell)}(\mathtt{G})$ denote the subset of faces $\mathtt{x}$ of $\mathtt{G}$ each of which contains at least one vertex whose coordinate lies inside $\Omega_i(\ell) := \operatorname{supp} F_{i;\ell}$ for $i = 1, 2$.

The initial ℓ finite term. Let $\mathcal{O}$ denote either the Laplace-Beltrami operator Δ or the Kähler operator $\mathcal{D}$ on the Delaunay graph $\mathtt{G}_{\underline{\epsilon},\ell}$. From prop. 10 and 11 the $\epsilon_1\epsilon_2$ cross-term of the variation of $\det\log[\mathcal{O}]$ is given by the trace term

$$\eth_{\epsilon_1\epsilon_2}\det\log[\mathcal{O}] \;=\; -\operatorname{tr}\big[\eth_{\epsilon_1}\mathcal{O}\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\mathcal{O}\cdot\Delta_{\mathrm{cr}}^{-1}\big] \tag{7.3}$$

which can be expressed as the following double sum over triangles in $\widehat{\mathtt{G}}_{0+,\ell}$

$$-\frac{2}{\pi^2}\sum_{\substack{\mathtt{x}_1\in\,\mathrm{F}_{\Omega_1(\ell)}\,(\widehat{\mathtt{G}}_{0+,\ell})\\ \mathtt{x}_2\in\,\mathrm{F}_{\Omega_2(\ell)}\,(\widehat{\mathtt{G}}_{0+,\ell})}} A(\mathtt{x}_1)A(\mathtt{x}_2)\left(\mathfrak{Re}\left[\frac{\overline{\nabla}F_{1;\ell}(\mathtt{x}_1)\,\overline{\nabla}F_{2;\ell}(\mathtt{x}_2)}{\big(z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big)^4}\right]+\mathrm{O}\Big(\big|z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big|^{-5}\Big)\right) \tag{7.4}$$

where $z_{\mathrm{cr}}(\mathtt{x}_i)$ is the circumcenter of $\mathtt{x}_i$ for $i = 1, 2$. Both F_1 and F_2 have compact support so by Lemma 2 we have that $\overline{\nabla}F_{i;\ell}(\mathtt{x}) = \overline{\partial}F_i\big(z_{\mathrm{cr}}(\mathtt{x})/\ell\big) + R_{\mathrm{cr}}/\ell\cdot E_i(\mathtt{x})$ where $\big|E_i(\mathtt{x})\big|$ is bounded by a constant $B_i > 0$ independent of both $\mathtt{x}$ and $\ell > 0$. We begin by breaking 7.4 into two pieces and evaluate their large ℓ limits separately.

The subleading term. The large ℓ limit of the second part of 7.4, vanishes as the following computation shows:

$$
\begin{aligned}
&\Big|\sum_{\substack{\mathtt{x}_1\in \mathrm{F}_{\Omega_1(\ell)}\,(\widehat{\mathbf{G}}_{0+,\ell})\\ \mathtt{x}_2\in \mathrm{F}_{\Omega_2(\ell)}\,(\widehat{\mathbf{G}}_{0+,\ell})}} A(\mathtt{x}_1)A(\mathtt{x}_2)\cdot \mathrm{O}\Big(\big|z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big|^{-5}\Big)\Big|\\
&\le \sum_{\substack{\mathtt{x}_1\in \mathrm{F}_{\Omega_1(\ell)}\,(\widehat{\mathbf{G}}_{0+,\ell})\\ \mathtt{x}_2\in \mathrm{F}_{\Omega_2(\ell)}\,(\widehat{\mathbf{G}}_{0+,\ell})}} A(\mathtt{x}_1)A(\mathtt{x}_2)\cdot \Big|\mathrm{O}\Big(\big|z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big|^{-5}\Big)\Big|\\
&\le \sum_{\substack{\mathtt{x}_1\in \mathrm{F}_{\Omega_1(\ell)}\,(\mathbf{G}_{\mathrm{cr}})\\ \mathtt{x}_2\in \mathrm{F}_{\Omega_2(\ell)}\,(\mathbf{G}_{\mathrm{cr}})}} A(\mathtt{x}_1)A(\mathtt{x}_2)\cdot \Big|\mathrm{O}\Big(\big|z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big|^{-5}\Big)\Big|\\
&\le \frac{1}{d}\,\frac{1}{\ell}\sum_{\substack{\mathtt{x}_1\in \mathrm{F}_{\Omega_1(\ell)}\,(\mathbf{G}_{\mathrm{cr}})\\ \mathtt{x}_2\in \mathrm{F}_{\Omega_2(\ell)}\,(\mathbf{G}_{\mathrm{cr}})}} A(\mathtt{x}_1)/\ell^2\, A(\mathtt{x}_2)/\ell^2\cdot \Big|\mathrm{O}\Big(\big|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell-z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\big|^{-4}\Big)\Big|
\end{aligned}
\tag{7.5}
$$

In the large ℓ limit the sum over the triangles becomes a standard Riemann integral

$$
\begin{aligned}
&\le \lim_{\ell\to\infty}\sum_{\substack{\mathtt{x}_1\in \mathrm{F}_{\Omega_1(\ell)}\,(\mathbf{G}_{\mathrm{cr}})\\ \mathtt{x}_2\in \mathrm{F}_{\Omega_2(\ell)}\,(\mathbf{G}_{\mathrm{cr}})}} A(\mathtt{x}_1)/\ell^2\, A(\mathtt{x}_2)/\ell^2\cdot \Big|\mathrm{O}\Big(\big|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell-z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\big|^{-4}\Big)\Big|\\
&= \iint_{\Omega_1\times\Omega_2} d^2x_1\, d^2x_2\cdot \Big|\mathrm{O}\Big(\big|x_1-x_2\big|^{-4}\Big)\Big| \;=\; \mathrm{O}(1)
\end{aligned}
\tag{7.6}
$$

Hence

$$
\lim_{\ell\to\infty}\sum_{\substack{\mathtt{x}_1\in \mathrm{F}_{\Omega_1(\ell)}\,(\mathbf{G}_{\mathrm{cr}})\\ \mathtt{x}_2\in \mathrm{F}_{\Omega_2(\ell)}\,(\mathbf{G}_{\mathrm{cr}})}} A(\mathtt{x}_1)A(\mathtt{x}_2)\cdot \mathrm{O}\Big(\big|z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big|^{-5}\Big) \;=\; 0
\tag{7.7}
$$

The leading term. To evaluate the first part in 7.4 we consider the norm of the difference between the original term with discrete derivative and the corresponding term with continuous derivatives, and use the previous results to get the bounds

$$
\left|\sum_{\substack{\mathtt{x}_1\in \mathrm{F}_{\Omega_1(\ell)}\,(\widehat{\mathbf{G}}_{0+,\ell})\\ \mathtt{x}_2\in \mathrm{F}_{\Omega_2(\ell)}\,(\widehat{\mathbf{G}}_{0+,\ell})}} A(\mathtt{x}_1)A(\mathtt{x}_2)\,\mathfrak{Re}\left[\frac{\overline{\nabla}F_{1;\ell}(\mathtt{x}_1)\,\overline{\nabla}F_{2;\ell}(\mathtt{x}_2)-\overline{\partial}F_1\big(z_{\mathrm{cr}}(\mathtt{x}_1)/\ell\big)\,\overline{\partial}F_2\big(z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\big)}{\big(z_{\mathrm{cr}}(\mathtt{x}_1)-z_{\mathrm{cr}}(\mathtt{x}_2)\big)^4}\right]\right|
$$

$$\leq \begin{cases} \dfrac{R_{\mathrm{cr}}}{\ell} \displaystyle\sum_{\substack{\mathtt{x}_1 \in \mathrm{F}_{\Omega_1(\ell)}(\widehat{\mathtt{G}}_{0+,\ell}) \\ \mathtt{x}_2 \in \mathrm{F}_{\Omega_2(\ell)}(\widehat{\mathtt{G}}_{0+,\ell})}} A(\mathtt{x}_1)/\ell^2 \, A(\mathtt{x}_2)/\ell^2 \, \dfrac{\left|E_1(\mathtt{x}_1)\right| \cdot \left|\overline{\partial} F_2\big(z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\big)\right|}{\left|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right|^4} \\ + \dfrac{R_{\mathrm{cr}}}{\ell} \displaystyle\sum_{\substack{\mathtt{x}_1 \in \mathrm{F}_{\Omega_1(\ell)}(\widehat{\mathtt{G}}_{0+,\ell}) \\ \mathtt{x}_2 \in \mathrm{F}_{\Omega_2(\ell)}(\widehat{\mathtt{G}}_{0+,\ell})}} A(\mathtt{x}_1)/\ell^2 \, A(\mathtt{x}_2)/\ell^2 \, \dfrac{\left|\overline{\partial} F_1\big(z_{\mathrm{cr}}(\mathtt{x}_1)/\ell\big)\right| \cdot \left|E_2(\mathtt{x}_2)\right|}{\left|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right|^4} \\ + \dfrac{R_{\mathrm{cr}}^2}{\ell^2} \displaystyle\sum_{\substack{\mathtt{x}_1 \in \mathrm{F}_{\Omega_1(\ell)}(\widehat{\mathtt{G}}_{0+,\ell}) \\ \mathtt{x}_2 \in \mathrm{F}_{\Omega_2(\ell)}(\widehat{\mathtt{G}}_{0+,\ell})}} A(\mathtt{x}_1)/\ell^2 \, A(\mathtt{x}_2)/\ell^2 \, \dfrac{\left|E_1(\mathtt{x}_1)\right| \cdot \left|E_2(\mathtt{x}_2)\right|}{\left|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right|^4} \end{cases}$$

$$\leq \begin{cases} \frac{R_{\mathrm{cr}}}{\ell} \sum_{\substack{\mathtt{x}_1 \in \mathrm{F}_{\Omega_1(\ell)}(\widehat{\mathtt{G}}_{0+,\ell}) \\ \mathtt{x}_2 \in \mathrm{F}_{\Omega_2(\ell)}(\widehat{\mathtt{G}}_{0+,\ell})}} A(\mathtt{x}_1)/\ell^2 \, A(\mathtt{x}_2)/\ell^2 \, \frac{B_1 \cdot \left|\overline{\partial} F_2\big(z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\big)\right|}{\left|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right|^4} \\ + \dfrac{R_{\mathrm{cr}}}{\ell} \displaystyle\sum_{\substack{\mathtt{x}_1 \in \mathrm{F}_{\Omega_1(\ell)}(\widehat{\mathtt{G}}_{0+,\ell}) \\ \mathtt{x}_2 \in \mathrm{F}_{\Omega_2(\ell)}(\widehat{\mathtt{G}}_{0+,\ell})}} A(\mathtt{x}_1)/\ell^2 \, A(\mathtt{x}_2)/\ell^2 \, \dfrac{\left|\overline{\partial} F_1\big(z_{\mathrm{cr}}(\mathtt{x}_1)/\ell\big)\right| \cdot B_2}{\left|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right|^4} \\ + \dfrac{R_{\mathrm{cr}}^2}{\ell^2} \displaystyle\sum_{\substack{\mathtt{x}_1 \in \mathrm{F}_{\Omega_1(\ell)}(\widehat{\mathtt{G}}_{0+,\ell}) \\ \mathtt{x}_2 \in \mathrm{F}_{\Omega_2(\ell)}(\widehat{\mathtt{G}}_{0+,\ell})}} A(\mathtt{x}_1)/\ell^2 \, A(\mathtt{x}_2)/\ell^2 \, \dfrac{B_1 \cdot B_2}{\left|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right|^4} \end{cases}$$

(7.8)

$$\leq \begin{cases} \dfrac{R_{\mathrm{cr}}}{\ell} \displaystyle\sum_{\substack{\mathtt{x}_1 \in \mathrm{F}_{\Omega_1(\ell)}(\mathtt{G}_{\mathrm{cr}}) \\ \mathtt{x}_2 \in \mathrm{F}_{\Omega_2(\ell)}(\mathtt{G}_{\mathrm{cr}})}} A(\mathtt{x}_1)/\ell^2 \, A(\mathtt{x}_2)/\ell^2 \, \dfrac{B_1 \cdot \left|\overline{\partial} F_2\big(z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\big)\right|}{\left|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right|^4} \\ + \dfrac{R_{\mathrm{cr}}}{\ell} \displaystyle\sum_{\substack{\mathtt{x}_1 \in \mathrm{F}_{\Omega_1(\ell)}(\mathtt{G}_{\mathrm{cr}}) \\ \mathtt{x}_2 \in \mathrm{F}_{\Omega_2(\ell)}(\mathtt{G}_{\mathrm{cr}})}} A(\mathtt{x}_1)/\ell^2 \, A(\mathtt{x}_2)/\ell^2 \, \dfrac{\left|\overline{\partial} F_1\big(z_{\mathrm{cr}}(\mathtt{x}_1)/\ell\big)\right| \cdot B_2}{\left|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right|^4} \\ + \dfrac{R_{\mathrm{cr}}^2}{\ell^2} \displaystyle\sum_{\substack{\mathtt{x}_1 \in \mathrm{F}_{\Omega_1(\ell)}(\mathtt{G}_{\mathrm{cr}}) \\ \mathtt{x}_2 \in \mathrm{F}_{\Omega_2(\ell)}(\mathtt{G}_{\mathrm{cr}})}} A(\mathtt{x}_1)/\ell^2 \, A(\mathtt{x}_2)/\ell^2 \, \dfrac{B_1 \cdot B_2}{\left|z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right|^4} \end{cases}$$

In the large ℓ limit the sums over triangles becomes Riemann integrals, hence the large ℓ limit of the l.h.s. of 7.8 is bounded by

$$
\begin{aligned}
(7.9)\qquad &\leq \left\{
\begin{aligned}
& \lim_{\ell\to\infty} \frac{c}{2\pi^2} \frac{B_1 R_{\mathrm{cr}}}{\ell} \cdot \iint_{\Omega_1\times\Omega_2} \frac{dx_1\, dx_2}{\left|x_1 - x_2\right|^4} \left|\overline{\partial} F_2(x_2)\right| \\
& + \lim_{\ell\to\infty} \frac{c}{2\pi^2} \frac{B_2 R_{\mathrm{cr}}}{\ell} \cdot \iint_{\Omega_1\times\Omega_2} \frac{dx_1\, dx_2}{\left|x_1 - x_2\right|^4} \left|\overline{\partial} F_1(x_1)\right| \\
& + \lim_{\ell\to\infty} \frac{c}{2\pi^2} \frac{B_1 B_2 R_{\mathrm{cr}}^2}{\ell^2} \cdot \iint_{\Omega_1\times\Omega_2} \frac{dx_1\, dx_2}{\left|x_1 - x_2\right|^4}
\end{aligned}
\right. \\
&= 0
\end{aligned}
$$

Summing up. From this it follows that

$$
\begin{aligned}
& \lim_{\ell\to\infty} \sum_{\substack{\mathtt{x}_1\in\,\mathrm{F}_{\Omega_1(\ell)}\,(\widehat{\mathbf{G}}_{0+,\ell}) \\ \mathtt{x}_2\in\,\mathrm{F}_{\Omega_2(\ell)}\,(\widehat{\mathbf{G}}_{0+,\ell})}} A(\mathtt{x}_1)A(\mathtt{x}_2)\ \mathfrak{Re}\left[\frac{\overline{\nabla} F_{1;\ell}(\mathtt{x}_1)\,\overline{\nabla} F_{2;\ell}(\mathtt{x}_2)}{\left(z_{\mathrm{cr}}(\mathtt{x}_1) - z_{\mathrm{cr}}(\mathtt{x}_2)\right)^4}\right] \\
& \lim_{\ell\to\infty} \sum_{\substack{\mathtt{x}_1\in\,\mathrm{F}(\widehat{\mathbf{G}}_{0+,\ell}) \\ \mathtt{x}_2\in\,\mathrm{F}(\widehat{\mathbf{G}}_{0+,\ell})}} A(\mathtt{x}_1)A(\mathtt{x}_2)\ \mathfrak{Re}\left[\frac{\overline{\nabla} F_{1;\ell}(\mathtt{x}_1)\,\overline{\nabla} F_{2;\ell}(\mathtt{x}_2)}{\left(z_{\mathrm{cr}}(\mathtt{x}_1) - z_{\mathrm{cr}}(\mathtt{x}_2)\right)^4}\right] \\
(7.10)\qquad & = \lim_{\ell\to\infty} \sum_{\substack{\mathtt{x}_1\in\,\mathrm{F}(\widehat{\mathbf{G}}_{0+,\ell}) \\ \mathtt{x}_2\in\,\mathrm{F}(\widehat{\mathbf{G}}_{0+,\ell})}} A(\mathtt{x}_1)/\ell^2\, A(\mathtt{x}_2)/\ell^2\ \mathfrak{Re}\left[\frac{\overline{\partial} F_1\big(z_{\mathrm{cr}}(\mathtt{x}_1)/\ell\big)\,\overline{\partial} F_2\big(z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\big)}{\left(z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right)^4}\right] \\
& = \lim_{\ell\to\infty} \sum_{\substack{\mathtt{x}_1\in\,\mathrm{F}(\mathbf{G}_{\mathrm{cr}}) \\ \mathtt{x}_2\in\,\mathrm{F}(\mathbf{G}_{\mathrm{cr}})}} A(\mathtt{x}_1)/\ell^2\, A(\mathtt{x}_2)/\ell^2\ \mathfrak{Re}\left[\frac{\overline{\partial} F_1\big(z_{\mathrm{cr}}(\mathtt{x}_1)/\ell\big)\,\overline{\partial} F_2\big(z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\big)}{\left(z_{\mathrm{cr}}(\mathtt{x}_1)/\ell - z_{\mathrm{cr}}(\mathtt{x}_2)/\ell\right)^4}\right] \\
& = \iint_{\Omega_1\times\Omega_2} dx_1\, dx_2\ \mathfrak{Re}\left[\frac{\overline{\partial} F_1(x_1)\,\overline{\partial} F_2(x_2)}{(x_1 - x_2)^4}\right]
\end{aligned}
$$

Thus we have

$$
(7.11)\qquad \lim_{\ell\to\infty} \mathrm{tr}\big[\boldsymbol{\partial}_{\epsilon_1}\mathcal{O}\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\boldsymbol{\partial}_{\epsilon_2}\mathcal{O}\cdot\Delta_{\mathrm{cr}}^{-1}\big] = \frac{2}{\pi^2}\iint_{\Omega_1\times\Omega_2} dx_1\, dx_2\ \mathfrak{Re}\left[\frac{\overline{\partial} F_1(x_1)\,\overline{\partial} F_2(x_2)}{(x_1 - x_2)^4}\right]
$$

This settles the proof of Theorem 3 by establishing eq. 1.18 .

7.4. **Controlling the geometry of the lattice for small deformations.**

The limits we considered. Let us summarize what we did previously, up to sect. 7.3. $\mathbf{G}_{\mathrm{cr}}$ is an infinite critical graph, F_2 and F_2 deformation functions with disjoint, compact supports. Through the deformation $z_{\mathrm{cr}} \to z_{\mathrm{cr}} + \epsilon_1 F_1 + \epsilon_2 F_2$ one constructs the deformed Delaunay graph $\mathbf{G}_{\underline{\epsilon}}$ along with a corresponding deformed operator $\mathcal{O}_{\underline{\epsilon}}$ where $\underline{\epsilon} = (\epsilon_1, \epsilon_2)$. Since the supports of F_1 and F_2 are disjoint, the corresponding variations of the matrix elements of $\mathcal{O}_{\underline{\epsilon}}$ are independent for small values of ϵ_1 and ϵ_2, so that one can write the total variation as the sum of the two independent variations

$\delta\mathcal{O}_{\underline{\epsilon}} = \mathcal{O}_{\underline{\epsilon}} - \mathcal{O}_{\rm cr} = \delta\mathcal{O}_{\epsilon_1} + \delta\mathcal{O}_{\epsilon_2}$ with $\delta\mathcal{O}_{\epsilon_1} = \mathcal{O}_{\epsilon_1,0} - \mathcal{O}_{\rm cr}$ and $\delta\mathcal{O}_{\epsilon_2} = \mathcal{O}_{0,\epsilon_2} - \mathcal{O}_{\rm cr}$. The bi-local deformation term tr $\left[\delta\mathcal{O}_{\epsilon_1} \cdot \mathcal{O}_{\rm cr}^{-1} \cdot \delta\mathcal{O}_{\epsilon_2} \cdot \mathcal{O}_{\rm cr}^{-1}\right]$ is replaced by its linear approximation at $\underline{\epsilon} = 0$ through $\delta\mathcal{O}_{\epsilon_1} \to \epsilon_1 \eth_{\epsilon_1}\mathcal{O}$, $\delta\mathcal{O}_{\epsilon_2} \to \epsilon_2 \eth_{\epsilon_2}\mathcal{O}$ reducing the problem to studying the bi-local term

$$\operatorname{tr}\left[\eth_{\epsilon_1}\mathcal{O} \cdot \mathcal{O}_{\rm cr}^{-1} \cdot \eth_{\epsilon_2}\mathcal{O} \cdot \mathcal{O}_{\rm cr}^{-1}\right] \tag{7.12}$$

defined on the weak Delaunay graph $\widehat{\mathbf{G}}_{0+}$ (the isoradial refinement of the initial graph $\mathbf{G}_{\rm cr}$ relative to the deformation). We then rescale the deformation by ℓ and consider the family of deformations $z_{\rm cr} \to z_{\rm cr} + \epsilon_1 F_{1;\ell} + \epsilon_2 F_{2;\ell}$ and show the scaling limit $\ell \to \infty$ of 7.12 exists and is independent of the choice of initial critical graph $\mathbf{G}_{\rm cr}$. Stated simply, we study the nested limit

$$\lim_{\ell\to\infty} \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \left(\operatorname{tr}\left[\eth_{\epsilon_1}\mathcal{O}_{\epsilon_1} \cdot \mathcal{O}_{\rm cr}^{-1} \cdot \eth_{\epsilon_2}\mathcal{O}_{\epsilon_2} \cdot \mathcal{O}_{\rm cr}^{-1}\right]\right) \tag{7.13}$$

An interesting question is whether these two limits can be interchanged. A positive answer would be a first step in understanding if one can define a continuum limit of (the total variation of) $\log\det\mathcal{O}$ starting from an infinite Delaunay graph which is not isoradial, but rather obtained by a small, smooth deformation of a Delaunay graph which is isoradial. A simpler question is the following: We know that for a given critical graph $\mathbf{G}_{\rm cr}$, the limit 7.13 makes sense when $\epsilon_1, \epsilon_2 \to 0$. Is the convergence uniform w.r.t. all critical graphs $\mathbf{G}_{\rm cr}$? We return to this issue in Section 8.

The problem with flips. The geometrical effects of a finite ϵ-deformation of a Delaunay graph $\mathbf{G}$ have already been discussed in Sections 5.3 and 5.4. Lemmas 9 and 10 ensure that, for a given initial graph $\mathbf{G}_{\rm cr}$ and a given deformation F (with compact support), there exist a strictly positive bound $0 < \tilde{\epsilon}_F$ such that no flip occurs in the interval $0 < \epsilon < \tilde{\epsilon}_F$. However $\tilde{\epsilon}_F$ depends non-trivially on F and on the geometry of $\mathbf{G}_{\rm cr}$. Furthermore it is clear that such a bound cannot be made uniform w.r.t. all critical graphs $\mathbf{G}_{\rm cr}$. This means that given any small value $\epsilon > 0$ of the deformation parameter flips will occur in $\mathbf{G}_\epsilon$ for some critical graph $\mathbf{G}_{\rm cr}$ within the class of all critical graphs. Consequently the operators (theirs matrix elements) $\eth_\epsilon\mathcal{O}_\epsilon$ are discontinuous functions of ϵ, and it will be difficult to control them as ϵ varies.

7.5. **A simple restriction to control small deformations: enforcing a global lower bound on the edge angles.** A simple but brutal way to manage the "flip problem" is to consider only a subclass of graphs $\mathbf{G}_{\rm cr}$ such that the bound $\tilde{\epsilon}_F$ can be controlled explicitely, so that no flip occurs. Similar constraints 7.16 on the geometry of $\mathbf{G}_{\rm cr}$ have been used in the literature for other problems involving isoradial lattices, see e.g. the paper by U. Bücking [Büc08]. Our solution is given by the following Lemma.

Lemma 14. *Let $F : \mathbb{C} \longrightarrow \mathbb{C}$ be a non-zero, smooth complex-valued function with compact support Ω_F. We define*

$$\check{M}_F \,=\, \max_{z\in\mathbb{C}} |\partial F(z)| + \max_{z\in\mathbb{C}} |\overline{\partial} F(z)| \tag{7.14}$$

This is a simple modification of the definition of the bound M_F in Lemmas 8 and 9, which does not depend on a specific graph. For a generic Delaunay triangulation

$\mathtt{T}$ *we define in analogy with* ϑ_F *in Lemma 9*

$$\check{\vartheta}(\mathtt{T}) \;=\; \inf\Big\{\theta(\mathtt{e})\,\Big|\,\mathtt{e}\in \mathrm{E}(\mathtt{T})\Big\} \tag{7.15}$$

For a fixed, strictly positive $\check{\vartheta} > 0$*, let us consider the set* $\mathscr{T}_{\check{\vartheta}}$ *of all isoradial Delaunay triangulations* $\mathtt{T}$ *of the plane such that*

$$\check{\vartheta}(\mathtt{T}) \;\geq\; \check{\vartheta} \;>\; 0 \tag{7.16}$$

Then, there exists a strictly positive bound on ϵ *given by*

$$\check{\epsilon}_F \;=\; \frac{\sin\check{\vartheta}}{2\check{M}_F(1+\check{M}_F)} \tag{7.17}$$

such that, for any triangulation $\mathtt{T}\in\mathscr{T}_{\check{\vartheta}}$ *and any* $\ell>0$*, the deformation* $z\to z_{\epsilon;\ell} = z+\epsilon F_\ell$ *of* $\mathtt{T}$ *preserves all the edges of* $\mathtt{T}$ *if* $0<\epsilon\leq\check{\epsilon}_F$*, i.e. no flip occurs.*

$$0<\epsilon\leq\check{\epsilon}_F\;,\quad \ell>0 \quad \textit{and} \quad \mathtt{T}\in\mathscr{T}_{\check{\vartheta}} \quad\Longrightarrow\quad \mathrm{E}(\mathtt{T}_{\epsilon;\ell})=\mathrm{E}(\mathtt{T}) \tag{7.18}$$

Proof. The mapping $z_{\epsilon,\ell}:\mathrm{V}(\mathtt{T}_{\epsilon,\ell})\longrightarrow\mathbb{C}$ is an embedding provided there are no "collisions", that is $z_{\epsilon,\ell}(\mathtt{u})\neq z_{\epsilon,\ell}(\mathtt{v})$ whenever $\mathtt{u}\neq\mathtt{v}$ are distinct vertices in $\mathrm{V}(\mathtt{T}_{\mathrm{cr}})$. Equivalently $1+\epsilon\, dF_{(\ell)}(\mathtt{u},\mathtt{v})$ must not vanish. Apply the fundamental theorem of calculus using $\gamma_{\mathtt{uv}}(\tau):=\tau z_{\mathrm{cr}}(\mathtt{u})/\ell+(1-\tau)z_{\mathrm{cr}}(\mathtt{v})/\ell$.

$$\begin{aligned}
\big|dF_\ell(\mathtt{u},\mathtt{v})\big| \;&=\; \left|\frac{F\big(z_{\mathrm{cr}}(\mathtt{u})/\ell\big)-F\big(z_{\mathrm{cr}}(\mathtt{v})/\ell\big)}{z_{\mathrm{cr}}(\mathtt{u})/\ell-z_{\mathrm{cr}}(\mathtt{v})/\ell}\right| \\
&=\; \frac{1}{|z_{\mathrm{cr}}(\mathtt{u})/\ell-z_{\mathrm{cr}}(\mathtt{v})/\ell\,|}\cdot\left|\int_0^1 d\tau\,\frac{d}{d\tau}F\big(\gamma_{\mathtt{uv}}(\tau)\big)\right| \\
&=\; \left|\int_0^1 d\tau\,\partial F(\gamma_{\mathtt{uv}}(\tau)) \;+\; \frac{\overline{z}_{\mathrm{cr}}(\mathtt{u})-\overline{z}_{\mathrm{cr}}(\mathtt{v})}{z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{v})}\int_0^1 d\tau\,\overline{\partial}F\big(\gamma_{\mathtt{uv}}(\tau)\big)\right| \\
&\leq\; \left|\int_0^1 d\tau\,\partial F\big(\gamma_{\mathtt{uv}}(\tau)\big)\right| \;+\; \left|\frac{\overline{z}_{\mathrm{cr}}(\mathtt{u})-\overline{z}_{\mathrm{cr}}(\mathtt{v})}{z_{\mathrm{cr}}(\mathtt{u})-z_{\mathrm{cr}}(\mathtt{v})}\right|\cdot\left|\int_0^1 d\tau\,\overline{\partial}F\big(\gamma_{\mathtt{uv}}(\tau)\big)\right| \\
&\leq\; \max\big|\partial F\big| \;+\; \max\big|\overline{\partial}F\big| \;=\; \check{M}_F
\end{aligned}$$

If we take $\epsilon<\epsilon_F<\check{M}_F^{-1}$, then clearly $\epsilon\,\big|dF_\ell(\mathtt{u},\mathtt{v})\big| \;<\; 1$. Therefore the quantity $1+\epsilon\,dF_{(\ell)}(\mathtt{u},\mathtt{v})$ cannot vanish. The assertion that $\overline{\mathtt{uv}}\in\mathrm{E}(\mathtt{T}_{\epsilon,\ell})$ whenever $\overline{\mathtt{uv}}\in\mathrm{E}(\mathtt{T})$ mirrors the proof of Lemma 9 except that we must carry out the same argument after replacing $\kappa_\epsilon(\overline{\mathtt{uv}}\,;\mathrm{w})$ by

$$\kappa_{\epsilon,\ell}\big(\overline{\mathtt{uv}}\,;\mathrm{w}\big) \;:=\; \mathfrak{Re}\left[\frac{z_{\epsilon,\ell}(\mathtt{u})-z_{\epsilon,\ell}(\mathrm{w})}{z_{\epsilon,\ell}(\mathtt{v})-z_{\epsilon,\ell}(\mathrm{w})}\right]$$

To avoid any dependence on the scaling parameter $\ell>0$ we also substitute the role played by ϑ_F with $\check{\vartheta}_F$, and therefore $\mathrm{E}(\mathtt{T}_{\epsilon;\ell})\subset\mathrm{E}(\mathtt{T})$. Finally, since $\mathtt{T}$ is a triangulation, no chord appears and $\mathtt{T}_{0+}=\mathtt{T}$, hence $\mathrm{E}(\mathtt{T}_{\epsilon;\ell})=\mathrm{E}(\mathtt{T})$. We stress that this bound on ϵ is valid for and independent of all values of the scaling parameter $\ell>0$ including $\ell=\infty$. $\square$

8. Finite ϵ variations, beyond the linear approximation

8.1. **Introduction.** In this section we look for uniform bounds on the variation of the operators Δ and $\mathcal{D}$ for small but finite deformation parameter ϵ. In order to get uniform bounds w.r.t. the geometry (i.e. isoradius) of the initial critical lattice $\mathtt{G}_{\mathrm{cr}}$, we need to fully examine generic deformations and take into account the situation where edges $\mathtt{G}_{\mathrm{cr}}$ may flip, unlike the linear approximation studied in the previous sections. This will leads us to uniform bounds on the variations of Δ and $\mathcal{D}$ (prop. 13). We deduce strong results on the uniform convergence of the scaling limit for Δ (prop. 15) and of the scaling limit of the second order bilocal term (leading to the OPE) (prop. 16). We show that there is a qualitative difference between Δ and $\mathcal{D}$, and we obtain a weaker but interesting result for the scaling limit of second order bilocal term for $\mathcal{D}$ (prop. 20).

8.2. **Deforming triangulations with and without flips.**

Geometric Deformations: Deforming with flips. We start from an isoradial Delaunay graph $\mathtt{G}_{\mathrm{cr}}$ and then deform its embedding $\mathtt{v} \mapsto z_{\mathrm{cr}}(\mathtt{v})$ using a smooth function $F : \mathbb{C} \longrightarrow \mathbb{C}$ with compact support to obtain a mapping

$$\mathtt{v} \mapsto z_\epsilon(\mathtt{v}) = z_{\mathrm{cr}}(\mathtt{v}) + \epsilon F(z_{\mathrm{cr}}(\mathtt{v})) \tag{8.1}$$

for vertices $\mathtt{v}$ of $\mathtt{G}_{\mathrm{cr}}$. The mapping $\mathtt{v} \mapsto z_\epsilon(\mathtt{v})$ defines an embedding of the vertex set $\mathrm{V}(\mathtt{G}_{\mathrm{cr}})$ as long as ϵ is small enough, namely:

$$|\epsilon| < (\max(|\partial F|) + \max(|\bar{\partial} F|))^{-1} \tag{8.2}$$

which ensures that $z_\epsilon(\mathtt{u}) \neq z_\epsilon(\mathtt{v})$ if $\mathtt{u} \neq \mathtt{v}$.

A Delaunay graph $\mathtt{G}_\epsilon$ is obtained by applying the Delaunay construction to the set of deformed coordinates $z_\epsilon(\mathtt{v})$ for $\mathtt{v} \in \mathtt{G}_{\mathrm{cr}}$. The vertices of $\mathtt{G}_\epsilon$ and $\mathtt{G}_{\mathrm{cr}}$ are identical by definition, however the edges and the faces of $\mathtt{G}_\epsilon$ may differ from those of $\mathtt{G}_{\mathrm{cr}}$ since the Delaunay constraints may force flips to occur during the deformation. Unlike the set-up of section 5.3, the inclusion $\mathrm{E}(\mathtt{G}_{\mathrm{cr}}) \subset \mathrm{E}(\mathtt{G}_\epsilon)$ of Lemma 9 may now fail. Generically $\mathtt{G}_\epsilon$ will be a triangulation whenever the critical graph $\mathtt{G}_{\mathrm{cr}}$ is.

Δ_ϵ, $\mathcal{D}_\epsilon$ and $\underline{\boldsymbol{\Delta}}_\epsilon$ are the Laplacian operators relative to the lattice $\mathtt{G}_\epsilon$. Note that these operators acts on the same space of functions $\mathbb{C}^{\mathrm{V}(\mathtt{G}_\epsilon)} = \mathbb{C}^{\mathrm{V}(\mathtt{G}_{\mathrm{cr}})}$ irrespective of ϵ since, by definition, the vertex sets $\mathrm{V}(\mathtt{G}_\epsilon) = \mathrm{V}(\mathtt{G}_{\mathrm{cr}})$ agree. Similarly, we denote by ∇_ϵ and $\overline{\nabla}_\epsilon$ the discrete derivative operators relative to the faces of $\mathtt{G}_\epsilon$, both of which are operators $\mathbb{C}^{\mathrm{V}(\mathtt{G}_\epsilon)} \to \mathbb{C}^{\mathrm{F}(\mathtt{G}_\epsilon)}$. Note that, in general, the set of deformed and critical faces differ, i.e. $\mathrm{F}(\mathtt{G}_\epsilon) \neq \mathrm{F}(\mathtt{G}_{\mathrm{cr}})$. Similarly we denote by A_ϵ and R_ϵ the area and circumradius functions for the faces of $\mathtt{G}_\epsilon$.

A simple but illustrative example of such a deformation of a triangulation $\mathtt{T} \to \mathtt{T}_\epsilon$ is depicted on Fig. 25. Note that the basis of the triangles b can be made arbitrary small $b \ll \epsilon$ so that an arbitrary number of flips may occur.

Formal Back-Deformation. Given a smooth function $F : \mathbb{C} \longrightarrow \mathbb{C}$ with compact support consider the formal deformation of the embedding of a (weak) Delaunay graph $\mathtt{G}$ as explained in Section 5.1, namely

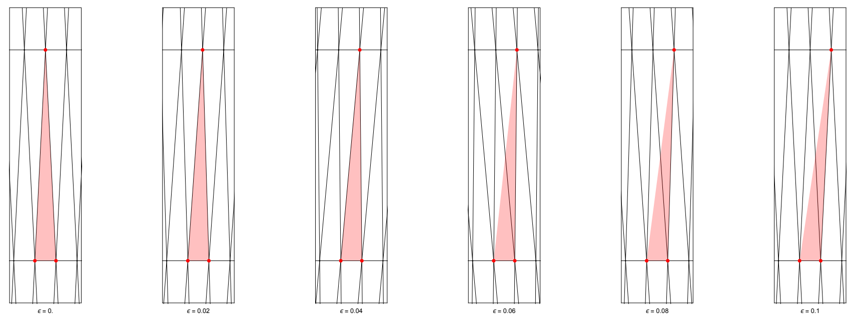

FIGURE 25. Deformation of a periodic isoradial Delaunay triangulation $\mathtt{T}_0 \to \mathtt{T}_\epsilon$ by a global shear $z \to z + \epsilon\,\mathfrak{Im} z$, keeping it Delaunay. On this example, the base and the height of the triangles are respectively $b = 1/10$ and $h = 1$, so that a flip occur for $\varepsilon = b/2 = 1/20$, and we choose $\epsilon = b = 1/10$. Since there is a a flip, an original face of $\mathtt{T}_0$ (in red) does not stay a face after the flip.

$$\mathtt{v} \mapsto z_\epsilon(\mathtt{v}) = z(\mathtt{v}) + \epsilon F(z(\mathtt{v})) \tag{8.3}$$

for $\mathtt{v} \in \mathrm{V}(\mathtt{G})$. Let $\mathtt{G}_{\epsilon:0}$ denote the graph whose vertex, edge, and face sets are identical to those of $\mathtt{G}$ but whose embedding is given by $\mathtt{v} \mapsto z_\epsilon(\mathtt{v})$. Keep in mind that when the deformation parameter ϵ is specialized $\mathtt{G}_{\epsilon:0}$ may not be (weakly) Delaunay.

Define the *back-deformation* $\mathtt{G}_{\epsilon:0}$ to be the graph whose vertex set and embedding are identical to those of our initial (weak) Delaunay graph $\mathtt{G}$ but whose edge and vertex sets coincide with those of the graph $\mathtt{G}_\epsilon$ obtained from $\mathtt{G}$ by the geometric deformation. The construction of $\mathtt{G}_{\epsilon:0}$ can be seen in two stages: First $\mathtt{G}_\epsilon$ is constructed from $\mathtt{G}$ using a geometric deformation (with $w \mapsto w + \epsilon F(w)$) and then $\mathtt{G}_{\epsilon:0}$ is subsequently constructed by a formal deformation (with $w \mapsto w - \epsilon F(w)$) of $\mathtt{G}_\epsilon$. Schematically

$$\mathtt{G} \stackrel{\text{geometric}}{\longrightarrow} \mathtt{G}_\epsilon \stackrel{\text{formal}}{\longrightarrow} \mathtt{G}_{\epsilon:0} \tag{8.4}$$

8.3. **Full variation of operators without flips.**

Variation of the area. Consider the variation of the triangulation $\mathtt{T} \to \mathtt{T}_\epsilon$ given by deforming the embedding $z(\mathtt{u}) \to z_\epsilon(\mathtt{u}) = z(\mathtt{u}) + \epsilon F(\mathtt{u})$ *without flips* (so that in fact $\mathtt{T}_\epsilon$ should be denoted $\mathtt{T}_{0:\epsilon}$ with the notations of the previous section). For a triangle $\mathtt{f}$ the full variation of its area is from 3.4 and 3.14

$$A \to A_\epsilon = A\left(1 + \epsilon(\nabla F + \bar{\nabla}\bar{F}) + \epsilon^2(\nabla F\,\bar{\nabla}\bar{F} - \bar{\nabla}F\,\nabla\bar{F})\right) \tag{8.5}$$

For brevity $D(\epsilon; F)$ will denote the scaling factor

$$D(\epsilon; F) = 1 + \epsilon(\nabla F + \bar{\nabla}\bar{F}) + \epsilon^2(\nabla F\ \bar{\nabla}\bar{F} - \bar{\nabla}F\ \nabla\bar{F}) \tag{8.6}$$

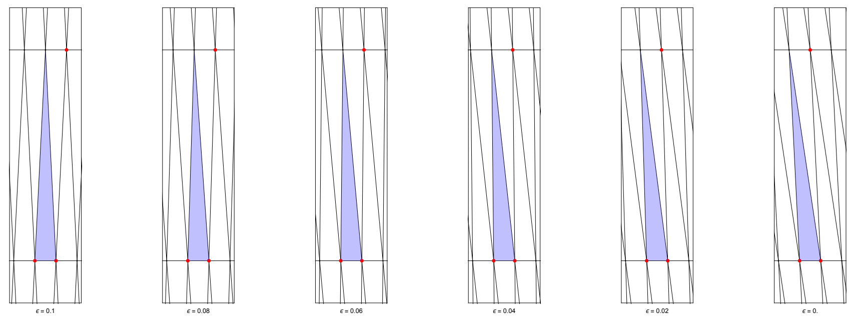

FIGURE 26. The back-deformation of this triangulation $\mathtt{T}_\epsilon \to \mathtt{T}_{\epsilon:0}$ by same shear, keeping the edges and faces of the triangulations fixed (no-flips). An original face of $\mathtt{T}_\epsilon$ (in blue) stays a face of $\mathtt{T}_{\epsilon:0}$. However $\mathtt{T}_{\epsilon:0}$ is not Delaunay.

Variation of the discrete derivatives. The vertex sets $\mathrm{V}(\mathtt{T})$ and $\mathrm{V}(\mathtt{T}_\epsilon)$ are, by definition, identical and the face sets $\mathrm{F}(\mathtt{T})$ and $\mathrm{F}(\mathtt{T}_\epsilon)$ agree so long as no flips occur in the deformation $\mathtt{T} \to \mathtt{T}_\epsilon$. Consequently the nabla operators ∇ and ∇_ϵ (and their conjugates $\bar{\nabla}$ and $\bar{\nabla}_\epsilon$) share a common range and domain. Accordingly we have:

$$
\begin{aligned}
\nabla \to \nabla_\epsilon &= \frac{1+\epsilon\bar{\nabla}\bar{F}}{D(\epsilon;F)}\,\nabla - \frac{\epsilon\nabla\bar{F}}{D(\epsilon;F)}\,\bar{\nabla} \\
\bar{\nabla} \to \bar{\nabla}_\epsilon &= \frac{1+\epsilon\nabla F}{D(\epsilon;F)}\,\bar{\nabla} - \frac{\epsilon\bar{\nabla}F}{D(\epsilon;F)}\,\nabla
\end{aligned}
\tag{8.7}
$$

A word of caution: deformations of functions. Recall that we may *restrict* a smooth, complex-valued function $G : \mathbb{C} \longrightarrow \mathbb{C}$ to the vertex set of the triangulation $\mathtt{T}$ using its graph embedding $z : \mathrm{V}(\mathtt{T}) \longrightarrow \mathbb{C}$. Bearing some abuse of notation, we define and denote this restriction by $G(\mathtt{v}) := G(z(\mathtt{v}))$ for vertices $\mathtt{v} \in \mathrm{V}(\mathtt{T})$. Some care is needed when restricting a smooth function G to the deformed triangulation $\mathtt{T}_\epsilon$. The vertex sets of $\mathtt{T}$ and $\mathtt{T}_\epsilon$ are identical but of course their respective embeddings z and z_ϵ are not, and consequently the functions $\mathtt{v} \mapsto G(z(\mathtt{v}))$ and $\mathtt{v} \mapsto G(z_\epsilon(\mathtt{v}))$ do not agree. In order to side step this discrepancy we introduce a deformed, smooth function $G_\epsilon : \mathbb{C} \longrightarrow \mathbb{C}$ defined implicitly by

$$
G_\epsilon\big(w + \epsilon F(w)\big) = G(w) \tag{8.8}
$$

for all $w \in \mathbb{C}$, where $\epsilon \geq 0$ is fixed and sufficiently small. By construction,

$$
G_\epsilon(z_\epsilon(\mathtt{v})) = G(z(\mathtt{v})) =: G(\mathtt{v}) \tag{8.9}
$$

To stress the role of the deformed embedding z_ϵ we shall define and denote $G_\epsilon(\mathtt{v}) := G_\epsilon(z_\epsilon(\mathtt{v}))$ for $\mathtt{v} \in \mathrm{V}(T_\epsilon)$. When $G = F$ this allows us to write

$$
z_{\epsilon+\epsilon'}(\mathtt{v}) = z(\mathtt{v}) + (\epsilon+\epsilon')F(z(\mathtt{v})) = z_\epsilon + \epsilon' F_\epsilon(z_\epsilon(\mathtt{v})) \tag{8.10}
$$

Variation of the circumradii. The full variation of the circumradius $R(\mathtt{f})$ of a face is more complicated. For a face with vertices labelled $1,2,3$ i.e. $\mathtt{f}=(123)$ using 3.6 we get

$$R^2 \to R_\epsilon^2 = R^2\,\frac{N_{12}(\epsilon;F)N_{23}(\epsilon;F)N_{31}(\epsilon;F)}{D(\epsilon;F)^2} \tag{8.11}$$

with

$$\begin{aligned} N_{\mathtt{uv}}(\epsilon;F) = 1 &+ \epsilon\left(\nabla F + \bar\nabla\bar F + \bar C_{\mathtt{uv}}\nabla\bar F + C_{\mathtt{uv}}\bar\nabla F\right) \\ &+ \epsilon^2\left(\nabla F\ \bar\nabla\bar F + \bar\nabla F\ \nabla\bar F + \bar C_{\mathtt{uv}}\nabla F\ \nabla\bar F + C_{\mathtt{uv}}\bar\nabla F\ \bar\nabla\bar F\right)\end{aligned} \tag{8.12}$$

where $C_{\mathtt{uv}}$ for an (unoriented) edge $\overline{\mathtt{uv}}$ denotes

$$C_{\mathtt{uv}} = \frac{\bar z(\mathtt{u}) - \bar z(\mathtt{v})}{z(\mathtt{u}) - z(\mathtt{v})} \tag{8.13}$$

Variation of the operators. Thus we get the variation of the Laplacian operators from

$$\Delta \to \Delta_\epsilon = 2\left(\nabla_\epsilon^\top\, A_\epsilon\,\nabla_\epsilon + \bar\nabla_\epsilon^\top\, A_\epsilon\,\bar\nabla_\epsilon\right) \tag{8.14}$$

$$\mathcal{D} \to \mathcal{D}_\epsilon = 4\bar\nabla_\epsilon^\top\,\frac{A_\epsilon}{R_\epsilon^2}\,\nabla_\epsilon \tag{8.15}$$

that we do not write explicitly. Note that all the expression we got are rational functions in ϵ, and that when keeping only the first order in ϵ in a series expansion, we recover the results of Sect. 5.1.

8.4. **Full variation of operators under Delaunay deformations (with flips).** Here we address the case of a critical triangulation $\mathtt{T}=\mathtt{T}_{\mathrm{cr}}$ with isoradius $R_0>0$ whose embedding undergoes a deformation

$$z \to z_\epsilon := z + \epsilon F$$

where flips are allowed, so that the deformed graph $\mathtt{T}_\epsilon$ remains Delaunay. As before the displacement function F is the (restriction) of a smooth complex-valued function on the plane with compact support. We consider the full variation of the operators associated to the deformation $\mathtt{T}_{\mathrm{cr}}\to T_\epsilon$, namely

$$\delta\Delta(\epsilon) = \Delta_\epsilon - \Delta_{\mathrm{cr}} \quad , \qquad \delta\mathcal{D}(\epsilon) = \mathcal{D}_\epsilon - \mathcal{D}_{\mathrm{cr}} \tag{8.16}$$

instead of the instantaneous, first order terms $\eth_\epsilon\Delta$ and $\eth_\epsilon\mathcal{D}$ in the respective ϵ-expansions as done in Sect. 5.1 and 6. We shall need uniform estimates for the $\epsilon\to 0$ limit of terms related to the variations $\delta\Delta(\epsilon)$ and $\delta\mathcal{D}(\epsilon)$ which are independent of the initial critical lattice $\mathtt{T}_{\mathrm{cr}}$. Furthermore uniform estimates for the $R_0\to 0$ limit will be needed, as this is synonymous with the $\ell\to\infty$ scaling limit.

Unfortunately the exact results of the previous section 8.3 cannot be directly applied, since flips generically occur within the continuous family of Delaunay graphs $\mathtt{T}_\varepsilon$ as the deformation parameter ε moves from zero to $\epsilon>0$. Nevertheless, we may write each variation as the integral of a derivative, and then try to get uniform bounds on the derivatives. This is what we discuss in the remaining part of this Appendix.

Let us first consider the simpler case of the Laplace-Beltrami operator Δ. We can write

$$\delta\Delta(\epsilon) = \int_0^\epsilon d\varepsilon\,\Delta'(\varepsilon) \quad \text{with} \qquad \Delta'(\varepsilon) = \frac{d}{d\varepsilon}\Delta_\varepsilon = \eth_\varepsilon\Delta_\varepsilon \tag{8.17}$$

Indeed, since F is smooth with compact support, there is a finite (possibly large) number of flips as ε increases, and we know that Δ_ε is a continuous function of ε, and its derivative exits and is continuous in the interval between the flips. Therefore the derivative $\Delta'(\varepsilon)$ is bounded and piecewise continuous, so that the integral 8.17 makes sense. For a given value $\varepsilon \geq 0$, the first order term in formula 5.3 extends to the case of Δ_ε defined on $\mathtt{T}_\varepsilon$ and w.r.t. the transported displacement function F_ε in the plane.

$$\Delta'(\varepsilon) \; = \; \nabla_\varepsilon^\top \cdot A_\varepsilon \cdot \mathfrak{D}_\varepsilon \cdot \nabla_\varepsilon + \overline{\nabla}_\varepsilon^\top \cdot A_\varepsilon \cdot \overline{\mathfrak{D}}_\varepsilon \cdot \overline{\nabla}_\varepsilon \tag{8.18}$$

with

$$\mathfrak{D}_\varepsilon = -4\, \overline{\nabla}_\varepsilon F_\varepsilon \quad , \qquad \overline{\mathfrak{D}}_\varepsilon = -4\, \nabla_\varepsilon \bar{F}_\varepsilon \tag{8.19}$$

Similarly, we can write the variation of the Kähler operator as

$$\delta\mathcal{D}(\epsilon) = \int_0^\epsilon d\varepsilon\, \mathcal{D}'(\varepsilon) \quad , \qquad \mathcal{D}'(\varepsilon) = \frac{d}{d\varepsilon}\mathcal{D}_\varepsilon = \boldsymbol{\delta}_\varepsilon \mathcal{D}_\varepsilon \tag{8.20}$$

The results of Section 5.1 give for the derivative of $\mathcal{D}$

$$\mathcal{D}'(\varepsilon) = \overline{\nabla}_\varepsilon^\top A_\varepsilon \mathfrak{K}_\varepsilon \nabla_\varepsilon + \nabla_\varepsilon^\top A_\varepsilon \mathfrak{Z}_\varepsilon \nabla_\varepsilon + \overline{\nabla}_\varepsilon^\top A_\varepsilon \overline{\mathfrak{Z}}_\varepsilon \overline{\nabla}_\varepsilon \tag{8.21}$$

with

$$\begin{aligned} \mathfrak{K}_\varepsilon &= -\frac{4}{R_\varepsilon^2}\left(\nabla_\varepsilon F_\varepsilon + \overline{\nabla}_\varepsilon \bar{F}_\varepsilon + C_\varepsilon\, \overline{\nabla}_\varepsilon F_\varepsilon + \bar{C}_\varepsilon\, \nabla_\varepsilon \bar{F}_\varepsilon\right) \\ \mathfrak{Z}_\varepsilon &= -\frac{4}{R_\varepsilon^2}\, \overline{\nabla}_\varepsilon F_\varepsilon \quad , \qquad \overline{\mathfrak{Z}}_\varepsilon = -\frac{4}{R_\varepsilon^2} \nabla_\varepsilon \bar{F}_\varepsilon \end{aligned} \tag{8.22}$$

and with the C_ε and $\bar{C}_\varepsilon$ defined by 5.5 for faces the triangulation T_ε, namely for a face $\mathtt{f} = (123)$,

$$C(\mathtt{f}) = C_{123} = \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} + \frac{\bar{z}_2 - \bar{z}_3}{z_2 - z_3} + \frac{\bar{z}_3 - \bar{z}_1}{z_3 - z_1} \tag{8.23}$$

Note that for such a face $\mathtt{f} = (123)$, $C(\mathtt{f})$ reads in term of the $C_{\mathtt{uv}}$ of 8.13 (associated to edges of $\mathtt{f}$) as $C(123) = C_{12} + C_{23} + C_{31}$.

8.5. **Uniform bounds under Delaunay deformations (with flips).**

Bounds on continuous derivatives. Now we study wether it is possible to give uniform bounds w.r.t. ε and $\mathtt{T}_\epsilon$ on the various coefficients A_ϵ, R_ϵ, $\mathfrak{L}_\epsilon$ and $\mathfrak{J}_\epsilon$, and on the operators ∇_ϵ and $\bar{\nabla}_\epsilon$. From now on, let $F : \mathbb{C} \to \mathbb{C}$ be a given smooth deformation function with compact support. Let

$$\begin{aligned} M_1 &= \sup_{z\in\mathbb{C}} \max\left\{|\partial F(z)|, |\bar\partial F(z)|\right\} \\ M_2 &= \sup_{z\in\mathbb{C}} \max\left\{|\partial^2 F(z)|, |\partial\bar\partial F(z)|, |\bar\partial^2 F(z)|\right\} \end{aligned} \tag{8.24}$$

We will consider the transported function F_ϵ defined by 8.10, and the transported version of 8.24

$$\begin{aligned} M_1(\epsilon) &= \sup_{z\in\mathbb{C}} \max\left[|\partial F_\epsilon(z)|, |\bar\partial F_\epsilon(z)|\right] \\ M_2(\epsilon) &= \sup_{z\in\mathbb{C}} \max\left[|\partial^2 F_\epsilon(z)|, |\partial\bar\partial F_\epsilon(z)|, |\bar\partial^2 F_\epsilon(z)|\right] \end{aligned} \tag{8.25}$$

By differentiating the functional relation 8.10 between F and F_ϵ, one gets the general inequalities

$$M_1(\epsilon) \le \overline{M}_1(\epsilon) = \frac{M_1}{1-2\epsilon\, M_1}\ , \quad M_2(\epsilon) \le \overline{M}_2(\epsilon) = \frac{M_2}{(1-2\epsilon\, M_1)^3} \tag{8.26}$$

valid as long as ϵ is small enough, namely

$$0 \le \epsilon < \check{\epsilon}_F = 1/(2M_1) \tag{8.27}$$

which ensures that F_ϵ is not multivalued (and stays smooth with compact support).

Bounds on discrete derivatives. Let $\mathtt{T}_{\mathrm{cr}}$ be a critical (Delaunay isoradial) triangulation with isoradius R_0, and $\mathtt{T}_\epsilon$ be the Delaunay triangulation $\mathtt{T}_\epsilon$ obtained by the ϵ-deformation $z \to z + \epsilon F$. We shall establish bounds on the norm of the discrete derivatives of F_ϵ on the triangulation $\mathtt{T}_\epsilon$, as well as inequalities on the radii $R(\mathtt{f})$ of the faces of $\mathtt{T}_\epsilon$.

First be define for a generic triangulation $\mathtt{T}$ and a generic smooth function with compact support G

$$B_G(\mathtt{T}) = \sup_{\text{faces}\,\mathtt{f}\in\mathtt{T}} \max\left(|\nabla G(\mathtt{f})|, |\overline{\nabla} G(\mathtt{f})|\right) \tag{8.28}$$

We use Lemma 2, which gives a bound on the difference between the discrete derivative $\nabla G(\mathtt{f})$ and the continuous derivate ∂G of G at the circumcenter of $\mathtt{f}$. This bound involves the circumradius of $\mathtt{f}$ and the max of the second derivative of G inside the circumcircle. Denote the max of the circumradii of the faces $\mathtt{f}$ of a triangulation $\mathtt{T}$

$$R_{\mathtt{max}}(\mathtt{T}) = \max_{\mathtt{f}\in\mathtt{T}}\ R(\mathtt{f}) \tag{8.29}$$

For the initial critical triangulation $\mathtt{T}_{\mathrm{cr}}$ Lemma 2 implies

$$B_F(\mathtt{T}_{\mathrm{cr}}) \le M_1 + 4\, M_2\, R_0 \tag{8.30}$$

but for $\mathtt{T}_\epsilon$ it becomes

$$B_F(\mathtt{T}_\epsilon) \le M_1(\epsilon) + 4\, M_2(\epsilon)\, R_{\mathtt{max}}(\mathtt{T}_\epsilon) \tag{8.31}$$

and we need an estimate of $R_{\mathtt{max}}(\mathtt{T}_\epsilon)$.

8.6. **Inequalities for general variations of circumradii (with or without flips).**

The problem. In order to get a bound on $R_{\mathtt{max}}(\mathtt{T}_\epsilon)$, we now derive a bound on the variation of the circumradius of the faces, of a triangulation under a deformation $z \to z + \epsilon F$.

Let us consider the following general problem. We start from a general (not isoradial) initial triangulation $\mathtt{T}_0$. We deform $\mathtt{T}_0$ by $z \to z + \varepsilon F$ from $\varepsilon = 0$ to a final ϵ. If the circumradii $R(\mathtt{f}_1)$ and $R(\mathtt{f}_2)$ of two neighboring faces $\mathtt{f}_1$ and $\mathtt{f}_2$ agree at any stage of the deformation, we may either (i) *perform an edge flip*, so that $\mathtt{f}_1$, $\mathtt{f}_2$ become two new faces $\mathtt{f}'_1$ and the graph remains Delaunay, $\mathtt{f}'_2$, or (2) *not perform the flip*, so that the local configuration ceases to be Delaunay. Thus we get a family of triangulations $\{\mathtt{T}_\varepsilon : \varepsilon \in [0, \epsilon\}$, a priori not Delaunay, but with the same set of vertex embeddings $\{z_\epsilon(\mathtt{u}) : \mathtt{u} \in V(\mathtt{T}_0)\}$ as if they were Delaunay all along.

Now consider an initial face (triangle) $\mathtt{f}_0$ of $\mathtt{T}_0$, with initial circumradius $R(0) = R_0(\mathtt{f}_0)$. When deforming $\mathtt{T}_\varepsilon$ from 0 to ϵ, we can continuously follow the face $\mathtt{f}_0$, and when it sustains a flip, we *choose one of the two faces* created by the flip. In this way we get a "continuous" family of faces $\{\mathtt{f}_\varepsilon \in \mathtt{T}_\varepsilon : \varepsilon \in [0,\epsilon]\}$, so that $\varepsilon \mapsto R(\mathtt{f}_\varepsilon)$ is a continuous, piecewise differentiable function (this is the crucial point for the following argument).

Bounds on the derivative of R and consequences. Now, in between the flips, from 5.8, 5.9 the derivative of the circumradius $R(\mathtt{f}_\varepsilon)$ of this face $\mathtt{f}_\varepsilon$ is

(8.32)

$$R'(\mathtt{f}_\varepsilon) = \frac{d}{d\varepsilon}R(\mathtt{f}_\varepsilon) = \frac{R(\mathtt{f}_\varepsilon)}{2}\left(\nabla_\varepsilon F_\varepsilon(\mathtt{f}_\varepsilon) + \overline{\nabla}_\varepsilon \bar{F}_\varepsilon(\mathtt{f}_\varepsilon) + C_\varepsilon(\mathtt{f}_\varepsilon)\overline{\nabla}_\epsilon F_\varepsilon(\mathtt{f}_\varepsilon) + \bar{C}_\varepsilon(\mathtt{f}_\varepsilon)\nabla_\varepsilon \bar{F}_\varepsilon(\mathtt{f}_\varepsilon)\right)$$

Using Lemma 2 again, for this face $\mathtt{f}_\varepsilon$ of the triangulation $\mathtt{T}_\varepsilon$ we get the bound

$$|\nabla_\varepsilon F_\varepsilon(\mathtt{f}_\varepsilon)| \text{ and } \left|\bar{\nabla}_\varepsilon F_\varepsilon(\mathtt{f}_\varepsilon)\right| \le M_1(\varepsilon) + 4\,R(\mathtt{f}_\varepsilon)\,M_2(\varepsilon) \tag{8.33}$$

and from the definition of C 8.23 we have

$$|C_\varepsilon(\mathtt{f}_\varepsilon)| \le 3 \tag{8.34}$$

We thus get the bound

$$\left|\frac{d}{d\varepsilon}R(\mathtt{f}_\varepsilon)\right| \le 4\,\overline{M}_1(\varepsilon)\,R_\varepsilon(\mathtt{f}) + 16\,\overline{M}_2(\varepsilon)\,R(\mathtt{f}_\varepsilon)^2 \tag{8.35}$$

Remember that the functions $\overline{M}_1(\varepsilon)$ and $\overline{M}_2(\varepsilon)$ are explicitly known functions of ε and the constants M_1 and M_2 associated to the displacement function F.

$$\overline{M}_1(\varepsilon) = \frac{M_1}{1-2\,\varepsilon\,M_1}\ ,\quad \overline{M}_1(\varepsilon) = \frac{M_2}{(1-2\,\varepsilon\,M_1)^3}$$

Bounds on R. We can integrate the bound 8.35 from 0 to ϵ to get explicit inequalities for the radius $R(\mathtt{f}_\varepsilon)$ of the face $\mathtt{f}_\varepsilon$. Consider the function $\bar{R}_+(\epsilon, R_0)$ defined by the ODE which saturates the upper bound 8.35

$$\frac{d\bar{R}_+(\epsilon,R_0)}{d\epsilon} = \mathcal{Q}(\bar{R}_+(\epsilon,R_0),\epsilon)\ ,\quad \bar{R}_+(0) = R_0 \tag{8.36}$$

where

$$\mathcal{Q}(R,\epsilon) = 4\,\overline{M}_1(\epsilon)\,R + 16\,\overline{M}_2(\epsilon)\,R^2 \tag{8.37}$$

whose solution is

$$\bar{R}_+(\epsilon,R_0) = \frac{R_0}{\left(1+\frac{M_2R_0}{M_1}\right)(1-2M_1\,\epsilon)^2 - \frac{M_2R_0}{M_1}\,(1-2M_1\,\epsilon)^{-2}} \tag{8.38}$$

Note that $\bar{R}_+(\epsilon)$ is a monotonously increasing convex function of ϵ for $0 \le \epsilon < \epsilon_{\mathtt{max}}(R_0)$ which diverges at $\epsilon_{\mathtt{max}}(R_0)$ given by

$$\epsilon_{\mathtt{max}}(R_0) = \frac{1}{2M_1}\left(1-\left(1+\frac{M_1}{R_0M_2}\right)^{-1/4}\right) \tag{8.39}$$

Similarily, let $\bar{R}_-(\epsilon,R_0)$ be the function whose derivative saturates the lower bound given by 8.35. It satisfies the ODE

$$\frac{d\bar{R}_-(\epsilon,R_0)}{d\epsilon} = -\mathcal{Q}(\bar{R}_-(\epsilon,R_0),\epsilon)\ ,\quad \bar{R}_-(0) = R_0 \tag{8.40}$$

$\bar{R}_-(\epsilon, R_0)$ is explicitly

$$\bar{R}_-(\epsilon, R_0) = \frac{R_0\,(1-2M_1\epsilon)^2}{1+\frac{8M_2R_0}{M_1}\log\left(\frac{1}{1-2M_1\epsilon}\right)} \tag{8.41}$$

and is a monotonous, decreasing function which vanishes at $\check{\epsilon}_F$ given by 8.27. Then we have the following general result.

Proposition 12. *The radius of the face* $\mathtt{f}_\epsilon$ *satisfy the inequalities*

$$\bar{R}_-(\epsilon, R(\mathtt{f}_0))) \;\le\; R(\mathtt{f}_\epsilon) \;\le\; \bar{R}_+(\epsilon, R(\mathtt{f}_0)) \tag{8.42}$$

valid if

$$0 \le \epsilon < \epsilon_{\mathtt{max}}(R(\mathtt{f}_0)) \tag{8.43}$$

with $\epsilon_{\mathtt{max}}(R(\mathtt{f}_0))$ *given by 8.39. Note also that in any case, one has* $0 < \epsilon_{\mathtt{max}} < \check{\epsilon}_F$*.*

Proof. Should be write it ? □

This is the main result of this section. Note that it does not require the initial triangulation to be Delaunay or isoradial. It is is also completely independent of whether we perform flips or do not perform flips during the deformation. It depends only on the deformation function F and on the initial radius of the initial face we start from.

Notice that when the initial radius of the initial face becomes very small 8.42 implies that

$$(1-2\,\epsilon\,M_1)^2 \;\le\; \lim_{R(\mathtt{f}_0)\to 0} \frac{R(\mathtt{f}_\epsilon)}{R(\mathtt{f}_0)} \;\le (1-2\,\epsilon\,M_1)^{-2} \tag{8.44}$$

Final estimates. With Prop. 12 we can complete the estimates of the previous sections 8.5. We start from an initial critical triangulation $\mathtt{T}_{\mathrm{cr}}$ with initial radius R_0, and deform it into the Delaunay triangulation $\mathtt{T}_\epsilon$. The inequality 8.42 implies that

$$R_{\mathtt{max}}(\mathtt{T}_\epsilon) = \max_{\mathtt{f}\in\mathtt{T}_\epsilon} R(\mathtt{f}) \;\le\; \bar{R}_+(\epsilon, R_0) \tag{8.45}$$

hence

$$B_F(\mathtt{T}_\epsilon) = \max_{\mathtt{f}\in\mathtt{T}_\epsilon}\left(|\nabla_\epsilon F_\epsilon|, |\bar{\nabla}_\epsilon F_\epsilon|\right) \le\; \bar{M}_1(\epsilon) + 4\,\bar{M}_2(\epsilon)\,\bar{R}_+(\epsilon, R_0) \tag{8.46}$$

We can bound the coefficients in the derivative w.r.t. ϵ of the Laplace-Beltrami operator Δ (in 8.18), and of the Kähler operator $\mathcal{D}$ (in 8.21).

$$\begin{aligned}
|\mathfrak{D}_\epsilon| \;&\le\; 4\,\bar{M}_1(\epsilon) + 16\,\bar{M}_2(\epsilon)\,\bar{R}_+(\epsilon, R_0) \\
|\mathfrak{K}_\epsilon| \;&\le\; \frac{16\bar{M}_1(\epsilon) + 64\,\bar{M}_2(\epsilon)\,\bar{R}_+(\epsilon, R_0)}{\bar{R}_-(\epsilon, R_0)^2} \\
|\mathfrak{Z}_\epsilon| \;&\le\; \frac{4\bar{M}_1(\epsilon) + 16\,\bar{M}_2(\epsilon)\,\bar{R}_+(\epsilon, R_0)}{\bar{R}_-(\epsilon, R_0)^2}
\end{aligned} \tag{8.47}$$

Using the explicit forms of $\bar{M}_1(\epsilon)$ and $\bar{M}_2(\epsilon)$ given by 8.26, and of $R_+(\epsilon, R_0)$ and $\bar{R}_-(\epsilon, R_0)$ given by 8.38 and 8.41, one deduces that $|\mathfrak{D}_\epsilon|$, $|\mathfrak{K}_\epsilon|$ and $|\mathfrak{Z}_\epsilon|$ are uniformly bounded. More precisely we can summarize the estimates we obtained into the following proposition.

Proposition 13. *Let us choose a smooth displacement function F with bounds M_1 and M_2 associated to its first and second derivatives. Let us also choose ϵ_{b} strictly smaller than $\epsilon_{\mathtt{max}}(R_0 = 1)$ given by 8.39*

$$0 < \epsilon_{\mathrm{b}} < \epsilon_{\mathtt{max}}(1) = \frac{1}{2M_1}\left(1 - \left(1 + \frac{M_1}{M_2}\right)^{-1/4}\right)$$

for instance $\epsilon_{\mathrm{b}} = \epsilon_{\mathtt{max}}(R_0 = 1)/2$. Then consider an arbitrary initial critical triangulations (isoradial and Delaunay) $\mathtt{T}_0$ with circumradius R_0, some $\epsilon > 0$, the deformed Delaunay lattice $\mathtt{T}_\epsilon$ obtained from $\mathtt{T}_0$ by the deformation $z \to z + \epsilon F(z)$, and an arbitrary face $\mathtt{f}$ of $\mathtt{T}_\epsilon$.

Then the factors $\mathfrak{D}_\epsilon(\mathtt{f})$ (given by 8.19), $\mathfrak{K}_\epsilon(\mathtt{f})$ and $\mathfrak{H}_\epsilon(\mathtt{f})$ (given by 8.22) for the face $\mathtt{f}$ are uniformly bounded over the sets of: (i) initial triangulation $\mathtt{T}_0$ with isoradius R_0 less or equal to one, (ii) deformation parameter ϵ smaller or equal to ϵ_{b}, (iii) and faces $\mathtt{f}$ of $\mathtt{T}_\epsilon$. Namely, there exist constants D_0, K_0 and H_0 which depend only of F and on the choice of ϵ_{b} such that

$$|\mathfrak{D}_\epsilon(\mathtt{f})| \leq D_0 \ , \quad |\mathfrak{K}_\epsilon(\mathtt{f})| \leq K_0 \ , \quad |\mathfrak{H}_\epsilon(\mathtt{f})| \leq H_0 \tag{8.48}$$

Similarily, there exists a constant $P_0(F;\epsilon_{\mathrm{b}})$, which depends only of F and on ϵ_{b}, which uniformly bounds the variation of the radius of the faces

$$|(R(\mathtt{f}_\epsilon) - R_0)/R_0| \leq \epsilon\, P_0(F;\epsilon_{\mathrm{b}}) \tag{8.49}$$

8.7. Consequence for the control of the scaling limit of Δ.

The Laplace-Beltrami operator Δ. To simplify, we use a 2×2 block matrix notation. The Δ operator and its ϵ-derivative Δ' on the deformed lattice $\mathtt{T}_\epsilon$ reads

(8.50)

$$\Delta(\epsilon) = 2\begin{pmatrix}\nabla_\epsilon\\ \overline{\nabla}_\epsilon\end{pmatrix}^\dagger \begin{pmatrix}A_\epsilon & 0\\ 0 & A_\epsilon\end{pmatrix}\begin{pmatrix}\nabla_\epsilon\\ \overline{\nabla}_\epsilon\end{pmatrix} \ , \quad \Delta'(\epsilon) = -4\begin{pmatrix}\nabla_\epsilon\\ \overline{\nabla}_\epsilon\end{pmatrix}^\dagger \begin{pmatrix}0 & A_\epsilon\,\nabla_\epsilon \overline{F}_\epsilon\\ A_\epsilon\,\overline{\nabla}_\epsilon F_\epsilon & 0\end{pmatrix}\begin{pmatrix}\nabla_\epsilon\\ \overline{\nabla}_\epsilon\end{pmatrix}$$

Remember that A_ϵ, $\nabla_\epsilon \overline{F}_\epsilon$ and $\overline{\nabla}_\epsilon F_\epsilon$ are defined for the faces of the deformed triangulation $\mathtt{T}_\epsilon$, whose vertices have positions $z_\epsilon = z + \epsilon F(z)$, while $\Delta(\epsilon)$ and $\Delta'(\epsilon)$ acts on the functions defined on the vertices of $\mathtt{T}_\epsilon$. Since $\mathtt{T}_\epsilon$ is obtained by deforming an initial critical lattice $\mathtt{T}_0 = \mathtt{T}_{\mathrm{cr}}$, let us rewrite them in terms on objects defined for the "back-deformed" lattice $\mathtt{T}_{\epsilon:0}$ defined by the procedure 8.4 (depicted in Fig. **??**).

$$\mathtt{T}_{\mathrm{cr}} = \mathtt{T}_0 \ \stackrel{\text{Delaunay}}{\longrightarrow}\ \mathtt{T}_\epsilon \ \stackrel{\text{no flip}}{\longrightarrow}\ \mathtt{T}_{\epsilon:0}$$

Again, $\mathtt{T}_{\epsilon:0}$ has the same vertices as $\mathtt{T}_0$, but the edges and faces of $\mathtt{T}_\epsilon$. In other word, $\mathtt{T}_\epsilon$ is obtained from $\mathtt{T}_{\epsilon:0}$ by the deformation $z \to z_\epsilon = z + \epsilon F(z)$, but without flips. We can therefore express the objects relative to the faces of $\mathtt{T}_\epsilon$ in terms of those relative to the faces of $\mathtt{T}_{\epsilon:0}$. The area A_ϵ of a face $\mathtt{f}_\epsilon$ of $\mathtt{T}_\epsilon$ is related to the area A of the corresponding face $\mathtt{f} = \mathtt{f}_{\epsilon:0}$ of $\mathtt{T}_{\epsilon:0}$ by 8.5, namely

$$A_\epsilon = D(\epsilon;F)\,A \tag{8.51}$$

with from 8.6

$$D(\epsilon;F) = 1 + \epsilon(\nabla F + \overline{\nabla}\bar{F}) + \epsilon^2(\nabla F\ \overline{\nabla}\bar{F} - \overline{\nabla}F\ \nabla\bar{F}) \tag{8.52}$$

Note that the operators ∇ and $\bar{\nabla}$ refer now to faces of $\mathtt{T}_{\epsilon:0}$. In a strict sense they should be denoted $\nabla_{\epsilon:0}$ and $\bar{\nabla}_{\epsilon:0}$. We omit the subscript to simplify notation. The discrete derivative operators on $\mathtt{T}_\epsilon$ are expressed in term of those on $\mathtt{T}_{\epsilon:0}$ by 8.7, which can be expressed in the block matrix notation as

$$\begin{pmatrix}\nabla_\epsilon\\ \overline{\nabla}_\epsilon\end{pmatrix} = \frac{1}{D(\epsilon;F)}\begin{pmatrix}1+\epsilon\,\overline{\nabla}\,\overline{F} & -\epsilon\,\nabla\overline{F}\\ -\epsilon\,\overline{\nabla}F & 1+\epsilon\,\nabla F\end{pmatrix}\begin{pmatrix}\nabla\\ \overline{\nabla}\end{pmatrix} \tag{8.53}$$

In particular

$$\begin{pmatrix}\nabla_\epsilon F_\epsilon\\ \overline{\nabla}_\epsilon F_\epsilon\end{pmatrix} = \frac{1}{D(\epsilon;F)}\begin{pmatrix}1+\epsilon\,\overline{\nabla}\,\overline{F} & -\epsilon\,\nabla\overline{F}\\ -\epsilon\,\overline{\nabla}F & 1+\epsilon\,\nabla F\end{pmatrix}\begin{pmatrix}\nabla F\\ \overline{\nabla}F\end{pmatrix} \tag{8.54}$$

Again the discrete ∇ and $\overline{\nabla}$ refer now to faces of $\mathtt{T}_{\epsilon:0}$. Including this in 8.50 one gets

$$\Delta'(\epsilon) = \begin{pmatrix}\nabla\\ \overline{\nabla}\end{pmatrix}^{\dagger} A\ \mathbb{D}(\epsilon;F)\begin{pmatrix}\nabla\\ \overline{\nabla}\end{pmatrix} \tag{8.55}$$

with $\mathbb{D}$ the 2×2 block matrix

(8.56)

$$\mathbb{D}(\epsilon;F) =$$

$$\frac{(-4)}{D(\epsilon;F)^2}\begin{pmatrix}-\epsilon\nabla\overline{F}\,\overline{\nabla}F\left(2+\epsilon(\nabla F+\overline{\nabla}\,\overline{F})\right) & \nabla\overline{F}\left((1+\epsilon\,\nabla F)^2-\epsilon^2\overline{\nabla}F\,\nabla\overline{F}\right)\\ \overline{\nabla}F\left((1+\epsilon\,\overline{\nabla}\,\overline{F})^2-\epsilon^2\nabla\overline{F}\,\overline{\nabla}F\right) & -\epsilon\nabla\overline{F}\,\overline{\nabla}F\left(2+\epsilon(\nabla F+\overline{\nabla}\,\overline{F})\right)\end{pmatrix}$$

Scaling limit for $\Delta(\epsilon)$. We can now study the scaling limit of the deformed operator $\Delta(\epsilon)$. We proceed as follows. As before, we choose a smooth displacement function F with compact support $F:\mathbb{C}\to\mathbb{C}$. For each $r\in(0,1]$ (or simply a decreasing sequence of $(r_n)_{n\in\mathbb{N}}$ converging to 0), we associate an arbitrary critical triangulation of the plane $\mathtt{T}^r_{\mathrm{cr}}=\mathtt{T}^r_0$ with isoradius r. Finally we choose a finite bound ϵ'_{b} such that

$$0<\epsilon'_{\mathrm{b}}<\frac{1}{2}\,\epsilon_{\mathtt{max}}(1) \tag{8.57}$$

for the deformation parameter ϵ where $\epsilon_{\mathtt{max}}$ is defined by 8.39 above. The calculations leading to the bounds of Prop. 13 for the deformation $\mathtt{T}_0\to\mathtt{T}_\epsilon$ can be easily repeated for the double deformations $\mathtt{T}^r_0\to\mathtt{T}^r_\epsilon\to\mathtt{T}^r_{\epsilon,0}$. In particular, the circumradius of each face $\mathtt{f}$ of $\mathtt{T}^r_{\epsilon,0}$ is bounded uniformly by

$$\epsilon\le\epsilon_{\mathrm{b}}\ ,\ \ r\le 1\ \implies\ |R(\mathtt{f})-r|\le\epsilon\,r\,P_0(F;2\,\epsilon'_{\mathrm{b}}) \tag{8.58}$$

with P_0 defined in Prop. 13. This allows us to uniformly control the $r\to 0$ limit of the discrete derivatives ∇ and $\overline{\nabla}$ by using Lemma 2 combined with the previous ingredients. Stir (do not shake)..

Proposition 14. *Let F be a smooth displacement function with compact support, fix ϵ, and let $\mathcal{F}=\{\mathtt{T}^r_0\}$ be a family of critical triangulations as above. To each point $z\in\mathbb{C}$ and to each r we associate the face $\mathtt{f}^r_{\epsilon:0}(z)$ of the deformed triangulation $\mathtt{T}^r_{\epsilon:0}$ which contains z. Note that the set of z which are either vertices or else belong to an edge of the triangulation is a set measure zero and can be ignored). Then in the limit $r\to 0$, the discrete derivative operators ∇ and $\overline{\nabla}$ for the face $\mathtt{f}^r_{\epsilon:0}(z)$ converge uniformly towards the continuum partial derivative ∂ and $\bar{\partial}$ at the point z. More* *precisely let ϕ be a* smooth function *(or at least of class C^2) with compact support Ω of the plane. Then*

$$\lim_{r\to 0} \nabla\phi(\mathtt{f}^r_{\epsilon:0}(z)) \;=\; \partial\phi(z)\ ,\quad \lim_{r\to 0} \overline{\nabla}\phi(\mathtt{f}^r_{\epsilon:0}(z)) \;=\; \bar\partial\phi(z) \tag{8.59}$$

Moreover, the limit is uniform, namely there is a constant $\mathtt{C}$ *independent of $z \in \Omega$, of the choice of family $\mathcal{F}$ of triangulations and of the value of $\epsilon \in [0, \epsilon'_{\mathrm{b}}]$ (but depending of F, of ϵ'_{b} and of ϕ), such that*

$$|\nabla\phi(\mathtt{f}^r_{\epsilon:0}(z)) - \partial\phi(z)| \ \textit{and}\ \left|\overline{\nabla}\phi(\mathtt{f}^r_{\epsilon:0}(z)) - \bar\partial\phi(z)\right| \;\leq\; \mathtt{C}\ r \tag{8.60}$$

It follows that the full variation of the discrete Laplace-Beltrami operator $\delta\Delta(\epsilon) = \Delta(\epsilon) - \Delta$ converges *uniformly* towards a local Laplace-like operator which depend on ϵ and F, in the following sense.

Proposition 15. *Let F, ϵ and $\mathcal{F} = \{\mathtt{T}^r_0\}$ as in Prop. 14 and ϕ be a* smooth function *(or at least of class C^2) with compact support Ω of the plane. Then*

$$\phi\cdot\delta\Delta(\epsilon)\cdot\phi = \sum_{\mathtt{u},\mathtt{v}\in\mathtt{T}^r_0} \bar\phi(\mathtt{u})\ (\delta\Delta(\epsilon))_{\mathtt{uv}}\ \phi(\mathtt{v}) \tag{8.61}$$

converges uniformly when $r \to 0$ towards the local quadratic form

$$\int_\Omega d^2z\ \begin{pmatrix}\partial\phi\\ \bar\partial\phi\end{pmatrix}^\dagger\ \mathbb{E}(\epsilon;F)\ \begin{pmatrix}\partial\phi\\ \bar\partial\phi\end{pmatrix} \tag{8.62}$$

with $\mathbb{E}(\epsilon;F)$ the 2×2 matrix

$$\begin{aligned}
\mathbb{E}(\epsilon;F) \;&=\; \int_0^\epsilon d\varepsilon\ \mathbb{E}'(\varepsilon;F) \quad \textit{with}\\
\mathbb{E}'(\varepsilon;F) \;&=\; \frac{-4}{((1+\varepsilon\,\partial F)(1+\varepsilon\,\bar\partial\bar F) - \varepsilon^2\,\bar\partial F\,\partial\bar F)^2}\ \times\\
&\begin{pmatrix} -\varepsilon\partial\bar F\,\bar\partial F\left(2+\varepsilon(\partial F+\bar\partial\,\bar F)\right) & \partial\bar F\left((1+\varepsilon\,\partial F)^2 - \varepsilon^2\bar\partial F\,\partial\bar F\right)\\ \bar\partial F\left((1+\varepsilon\,\bar\partial\,\bar F)^2 - \varepsilon^2\partial\bar F\,\bar\partial F\right) & -\varepsilon\partial\bar F\,\bar\partial F\left(2+\varepsilon(\partial F+\bar\partial\,\bar F)\right)\end{pmatrix}
\end{aligned} \tag{8.63}$$

Proof. One just writes $\delta\Delta(\epsilon)$ as

$$\delta\Delta(\epsilon) = \delta\Delta(\epsilon) = \int_0^\epsilon d\varepsilon\,\Delta'(\varepsilon)$$

and use the explicit representation 8.55 8.56 for $\Delta'(\varepsilon)$ to write

$$\phi\cdot\Delta'(\varepsilon)\cdot\phi = \sum_{\mathtt{f}\in\mathtt{T}^r_{\varepsilon:0}} A(\mathtt{f})\ \begin{pmatrix}\nabla\phi(\mathtt{f})\\ \overline{\nabla}\phi(\mathtt{f})\end{pmatrix}^\dagger\cdot\left[\mathbb{D}(\varepsilon;F)\right](\mathtt{f})\cdot\begin{pmatrix}\nabla\phi(\mathtt{f})\\ \overline{\nabla}\phi(\mathtt{f})\end{pmatrix} \tag{8.64}$$

which is a Riemann sum. Then 8.58 and Prop. 14 ensures that in the $r \to 0$ limit this converges uniformly towards an ordinary integral involving continuous derivatives of ϕ and F ($\mathbb{D}$ becoming $\mathbb{E}$). One thus recover 8.62. □

Scaling limit for the bilocal deformation of $\operatorname{tr}\log\Delta$. This can be repeated for studying the scaling limit of the bilocal term

$$\operatorname{tr}\left[\delta_1\Delta(\epsilon_1)\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\delta_2\Delta(\epsilon_2)\cdot\Delta_{\mathrm{cr}}^{-1}\right] \tag{8.65}$$

For finite deformation parameters ϵ_1 and ϵ_2. Again we consider two smooth deformation fonctions F_1 and F_2 with disjoint compact suppports Ω_1 and Ω_2. $\delta_1\Delta(\epsilon_1) = \Delta(\epsilon_1) - \Delta_{\mathrm{cr}}$ (resp. $\delta_2\Delta(\epsilon_2) = \Delta(\epsilon_2) - \Delta_{\mathrm{cr}}$) is the variation of the Laplace-Beltrami operator under the deformation $z \to z + \epsilon_1 F_1(z)$ (resp. $z \to z + \epsilon_2 F_2(z)$). As above, instead of considering a fixed initial critical lattice $\mathtt{T}_{\mathrm{cr}}$ with isoradius $R_0 = 1$, and rescaled deformation functions $F_\ell(z) = \ell\, F(z/\ell)$, with $\ell \to \infty$ a rescaling parameter, we consider a family $\mathcal{F} = \{\mathtt{T}^r\}$ of critical lattices with isodradii r, fixed deformation functions F's, and study the limit $r \to 0$. This is equivalent since by a change of variable $r \sim 1/\ell$.

For a finite $0 < r \leq 1$, deforming the initial $\mathtt{T}^r_{\mathrm{cr}}$ critical lattice, the bilocal deformation term reads as a double sum over the faces of the two non-isoradial lattices $\mathtt{T}^r_{\epsilon_1:0}$ and $\mathtt{T}^r_{\epsilon_2:0}$, which share the same vertices, but not the same faces, with $\mathtt{T}^r_{\mathrm{cr}}$, of the explicit form

(8.66)

$$\mathrm{Tr}\left[\Delta'(\epsilon_1)\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\Delta'(\epsilon_2)\cdot\Delta_{\mathrm{cr}}^{-1}\right] = \sum_{\mathtt{f}_1\in\mathtt{T}^r_{\epsilon_1:0}}\ \sum_{\mathtt{f}_2\in\mathtt{T}^r_{\epsilon_2:0}} A(\mathtt{f}_1)\ A(\mathtt{f}_2)$$

$$\mathrm{tr}\left(\left[\mathbb{D}(\epsilon_1;F_1)\right](\mathtt{f}_1)\cdot\left[\begin{pmatrix}\nabla\\ \bar\nabla\end{pmatrix}\Delta_{\mathrm{cr}}^{-1}\begin{pmatrix}\nabla\\ \bar\nabla\end{pmatrix}^{\dagger}\right]_{\mathtt{f}_1\mathtt{f}_2}\cdot\left[\mathbb{D}(\epsilon_2;F_2)\right](\mathtt{f}_2)\cdot\left[\begin{pmatrix}\nabla\\ \bar\nabla\end{pmatrix}\Delta_{\mathrm{cr}}^{-1}\begin{pmatrix}\nabla\\ \bar\nabla\end{pmatrix}^{\dagger}\right]_{\mathtt{f}_2\mathtt{f}_1}\right)$$

The trace Tr [ ] in the l.h.s. of 8.66 is the “big trace” over the infinite set of vertices of the critical lattice. The trace tr ( ) in the r.h.s of 8.66 is a finite trace over a product of 2×2 matrices. This appears again as a double Riemann discrete sum over the faces of the triangulations $\mathtt{T}^r_{\epsilon_1:0}$ and $\mathtt{T}^r_{\epsilon_2:0}$.

Studying the scaling limit $r \to 0$ might seem similar to what was done above for Δ. There is however a delicate point. $\Delta_{\mathrm{cr}}^{-1}$ is the critical propagator on the critical lattice $\mathtt{T}^r_{\mathrm{cr}}$, given by the explicit Kenyon integral representation. But its elements $\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{u},\mathtt{v}}$ are not given by the restriction of a smooth function of the positions of the vertices $G(z(\mathtt{u}), z(\mathtt{v}))$.

Indeed, the large distance asymptotics of $\Delta_{\mathrm{cr}}^{-1}$ on a critical lattice with isoradius $R_0 = 1$ given by Prop. 4 implies that the propagator $\Delta_{\mathrm{cr}}^{-1}$ on a lattice $\mathtt{T}^r_{\mathrm{cr}}$ can be separated in a dominant smooth part G_{s} and a subdominant non-smooth part G_{ns}.

$$\left[\Delta_{\mathrm{cr}}^{-1}\right]_{\mathtt{u},\mathtt{v}} = G_{\mathrm{s}}(\mathtt{u},\mathtt{v}) + G_{\mathrm{ns}}(\mathtt{u},\mathtt{v}) \tag{8.67}$$

The smooth part is the continuum propagator (note now the r dependence)

$$G_{\mathrm{s}}(\mathtt{u},\mathtt{v}) = -\frac{1}{2\pi}\left(\log\left(2\left|z(\mathtt{u})-z(\mathtt{v})\right|/r\right)+\gamma_{\mathrm{euler}}\right) \tag{8.68}$$

The non-smooth part is

(8.69)

$$G_{\mathrm{ns}}(\mathtt{u},\mathtt{v}) = \frac{1}{2\pi}\left(\sum_{m\geq d\geq 1}(-1)^d(2m+d-1)!\,\mathfrak{Re}\left(c_{m,d}(\mathtt{u},\mathtt{v})\left(\frac{r/2}{z(\mathtt{v})-z(\mathtt{u})}\right)^{2m}\right)\right)$$

with the $c_{m,d}(\mathtt{u},\mathtt{v})$ defined by 4.11, from the $u_{1+2s}(\mathtt{u},\mathtt{v})$ defined by 4.9, where the $p_{1+2s}(\mathtt{u},\mathtt{v})$ are defined by 4.6, with the θ_j's the arguments of the successive vectors (with iso-modulus r) of a path $\mathtt{v} = (\mathtt{u},\cdots,\mathtt{v})$ on the rhombic lattice $\mathtt{T}_0^{r\,\Diamond}$ associated to the critical (isoradial Delaunay) lattice $\mathtt{T}^r_0$. Note that now $p_1(\mathtt{u},\mathtt{v}) = (z(\mathtt{v})-z(\mathtt{u}))/r$. From Lemma 5 the $c_{m,d}$'s are of order O(1), so the sum of the terms given by a fixed $m > 0$ is bounded by a $\mathrm{O}(r^{2m})$ in the scaling $r \to 0$ limit, and is indeed subdominant.

In 8.66 in the scaling limit $r \to 0$, the sum over triangles becomes a Riemann integral

$$\sum_{\mathtt{f}_1 \in \mathtt{T}^r_{\epsilon_1:0}} \sum_{\mathtt{f}_2 \in \mathtt{T}^r_{\epsilon_2:0}} A(\mathtt{f}_1)\ A(\mathtt{f}_2) \quad \longrightarrow \quad \int_{\Omega_1} d^2 z_1 \int_{\Omega_2} d^2 z_2 \tag{8.70}$$

The $\mathbb{D}(\epsilon_a; F_a)(\mathtt{f}_a)$ $(a = 1, 2)$ are easy to control since as above from Pro. 8.66 we know that they converge uniformly towards the $\mathbb{E}'(\epsilon_a; F_a)(z_a)$ given by 8.63. Controling the scaling limit of the discrete derivatives of the smooth part of the propagator is also easy through Lemma 2. We get the uniform limit

$$\left[\begin{pmatrix}\nabla\\ \overline{\nabla}\end{pmatrix} G_{\mathrm{s}} \begin{pmatrix}\nabla\\ \overline{\nabla}\end{pmatrix}^{\dagger}\right]_{\mathtt{f}_1\mathtt{f}_2} \quad \xrightarrow[r\to 0]{} \quad -\frac{1}{4\pi}\begin{pmatrix} 0 & (z_1-z_2)^{-2} \\ (\bar z_1 - \bar z_2)^{-2} & 0\end{pmatrix} \tag{8.71}$$

The non-trivial point is to get a uniform bound on the scaling limit of the left+right discrete derivatives of the non-smooth part of the propagator, and to show that it is subdominant. This issue has been discussed in detail in Sect. 6.2 through Lemmas 11 and 12. However Lemma 11 relies on the fact that the discrete derivatives ∇ and $\overline{\nabla}$ are relative to the faces $\mathtt{f}$ of an isoradial triangulation $\mathtt{T}_0$. This is not the case anymore here, since the discrete derivatives are relative to the faces of a non-isoradial triangulation $\mathtt{T}^r_{\epsilon:0}$ derived from an isoradial one $\mathtt{T}^r_0$ by flips of edges, without moving the position of the vertices.

We can repeat the analysis of Sect. 6.2 for this more general case. Again the dangerous contribution which could give a term of order $|z_1 - z_2|^{-2}$ is the $m = 1$ term in 8.69, which is explicitely proportional to the real part of

$$\frac{p_3(\mathtt{u},\mathtt{v})\ r^3}{(z(\mathtt{u}) - z(\mathtt{v}))^3}$$

Again, the most dangerous contribution comes from applying left+right discrete derivatives to $p_3(\mathtt{u},\mathtt{v})$). Generically a naive dimensional analysis shows that each discrete derivative applied on p_3 will bring a term of order r^{-1}, so that we will get for a pair of triangles $\mathtt{f}_1 \in \mathtt{T}^r_{\epsilon_1:0} \cap \Omega_1$, $\mathtt{f}_2 \in \mathtt{T}^r_{\epsilon_2:0} \cap \Omega_2$

$$\sum_{\mathtt{u}_1\in\mathtt{f}_1}\sum_{\mathtt{u}_2\in\mathtt{f}_2} \begin{pmatrix}\nabla\\ \overline{\nabla}\end{pmatrix}_{\mathtt{f}_1,\mathtt{u}_1} p_3(\mathtt{u}_1,\mathtt{u}_2) \begin{pmatrix}\nabla\\ \overline{\nabla}\end{pmatrix}^{\dagger}_{\mathtt{u}_2,\mathtt{f}_1} \quad \sim \quad \mathtt{cst.}\ r^{-2}$$

However, we shall see that this estimate is genericaly *not uniform.* Namely, the $\mathtt{cst.}$ in this estimate can be arbitrarily large ! remember that from Lemma 11 if $\mathtt{f}_1$ and $\mathtt{f}_2$ are faces of the original isoradial triangulation $\mathtt{T}_0$ then this $\mathtt{cst.}$ is bounded by $\mathtt{cst.} \leq 9$.

This is a technical point which comes from the fact that generically, if we start from an isoradial Delaunay triangulation $\mathtt{T}_0$ with isoradius r, and consider an arbitrary triangle $\mathtt{t} = (\mathtt{u}_1, \mathtt{u}_2, \mathtt{u}_3)$ which is not a face $\mathtt{f}$ of $\mathtt{T}_0$, this triangle may have a circumradius $R(\mathtt{t})$ very large ($R(\mathtt{t}) \gg r$), and an area $A(\mathtt{t})$ arbitrarily small ($A(\mathtt{t}) \ll r^2$). "Experimental mathematics" studies of such singular cases and some analytical estimates leads us to make the following conjecture.

Conjecture 1. *Let $\mathtt{T}_0$ be an isoradial Delaunay triangulation $\mathtt{T}^r_0$ of the plane with isoradius r, and $p_3(\mathtt{u},\mathtt{v})$ the function defined for any pair of points $(\mathtt{u},\mathtt{v})$ of $\mathtt{T}_0$ by*

$$p_3(\mathtt{u},\mathtt{v}) = \sum_{i=1}^{2n} \mathrm{e}^{\mathrm{i}3\theta_i} \quad , \qquad \theta_i = \arg(z_i - z_{i-1})$$

where $\mathtt{v} = (z_0, z_1, \cdots, z_{2n-1}, z_{2n})$ *is a path on the rhombic lattice* $\mathtt{T}_0^{r\,\diamondsuit}$ *obtained from* $\mathtt{T}_0^r$*, going from* $\mathtt{u}$ $(z_0 = z(\mathtt{u}))$ *to* $\mathtt{v}$ $(z_{2n} = z(\mathtt{v}))$*.*

For any non degenerate triangle $\mathtt{t} = (\mathtt{u}_1, \mathtt{u}_2, \mathtt{u}_3)$ *in* $\mathtt{T}_0^r$ *, let* $\nabla p_3(\mathtt{t})$ *and* $\overline{\nabla} p_3(\mathtt{t})$ *be the discrete derivatives w.r.t.* $\mathtt{u}$ *taken at the face* $\mathtt{t}$ *(note, this is independent of* $\mathtt{v}$*). Then there is a uniform bound*

$$|\nabla p_3(\mathtt{t})| \quad \textit{and} \quad |\overline{\nabla} p_3(\mathtt{t})| \;\le\; \mathtt{cst.}\, R(\mathtt{t})/r^2 \tag{8.72}$$

with $\mathtt{cst.}$ *a number of order* $\mathrm{O}(1)$ *independent on the choice of critical triangulation* $\mathtt{T}_0^r$ *and of the triangle* $\mathtt{t}$*. On the examples we have studied, we found* $\mathtt{cst.} = 6$*.*

Assuming the validity of the conjecture, it is easy to adapt the arguments of Sect. 6.2, and to use that fact that the circumradii of the faces $\mathtt{f}_1$ and $\mathtt{f}_2$ of the deformed-back-deformed non-isoradial triangulations $\mathtt{T}^r_{\epsilon_1:0}$ and $\mathtt{T}^r_{\epsilon_2:0}$ are uniformly bounded for ϵ_1 and ϵ_2 small enough by 8.58. This leads to

Lemma 15. *Assuming Conjecture 1, the left-right discrete derivative of the non-smooth part of the propagator is* uniformly bounded *in the scaling limit* $r \to 0$ *by*

$$\left| \left[\begin{pmatrix} \nabla \\ \overline{\nabla} \end{pmatrix} G_{\mathrm{ns}} \begin{pmatrix} \nabla \\ \overline{\nabla} \end{pmatrix}^{\dagger} \right]_{\mathtt{f}_1 \mathtt{f}_2} \right| \quad \le \quad \mathtt{cst.}\ \frac{r}{|z(\mathtt{f}_1) - z(\mathtt{f}_2)|^3} \tag{8.73}$$

It is therefore subdominant when compared to the contribution of the smooth part of the propagator given by 8.71.

Combining everything, we get the final result for the scaling limit of the bilocal term

Proposition 16. *Assuming that Conjecture 1 allows to control the derivatives of the non-smooth part of the propagator, the bilocal term* $\mathrm{Tr}\left[\Delta'(\epsilon_1) \cdot \Delta_{\mathrm{cr}}^{-1} \cdot \Delta'(\epsilon_2) \cdot \Delta_{\mathrm{cr}}^{-1}\right]$ *defined on critical triangulations* $\mathtt{T}_0^r$ *converges uniformy in the scaling limit* $r \to 0$ *towards the bilocal term*

$$\begin{aligned} \int_{\Omega_1} d^2 z_1 \int_{\Omega_2} d^2 z_2 \ \ \mathrm{tr}\Bigg[\mathbb{E}'(\epsilon_1; F_1)(z_1) \cdot \begin{pmatrix} 0 & (z_1 - z_2)^{-2} \\ (\bar{z}_1 - \bar{z}_2)^{-2} & 0 \end{pmatrix} \cdot \\ \mathbb{E}'(\epsilon_2; F_2)(z_2) \cdot \begin{pmatrix} 0 & (z_1 - z_2)^{-2} \\ (\bar{z}_1 - \bar{z}_2)^{-2} & 0 \end{pmatrix} \Bigg] \end{aligned} \tag{8.74}$$

Note that this term depends on the fours derivatives ∂F_1, $\bar{\partial} F_1$, ∂F_2, $\bar{\partial} F_2$ and their c.c., and contains both the analytic term $(z_1 - z_2)^{-4}$, the anti-analytic term $(\bar{z}_1 - \bar{z}_2)^{-4}$, and the mixed term $(z_1 - z_2)^{-2} (\bar{z}_1 - \bar{z}_2)^{-2}$.

Finally, from the explicit expression 8.63, the limit $\epsilon \to 0$ of $\mathbb{E}'(\epsilon; F)$ exists and is uniform.

$$\lim_{\epsilon \to 0} \mathbb{E}'(\epsilon; F) \;=\; \begin{pmatrix} 0 & -4\, \partial \bar{F} \\ -4\, \bar{\partial} F & 0 \end{pmatrix} \tag{8.75}$$

Together with Prop. 16, this leads to the commutation of limits result for Δ.

Proposition 17. *Under the assumptions of the previous propositions, the limit* $\epsilon \to 0$ *and the scaling limit* $r \to 0$ *for the bilocal term exist, are uniform, and commute. One recovers the result obtained previously for the scaling limit of the*

OPE on the lattice for Δ.

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\lim_{r\to 0} \mathrm{Tr}\left[\Delta'(\epsilon_1)\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\Delta'(\epsilon_2)\cdot\Delta_{\mathrm{cr}}^{-1}\right]\\
&\qquad = \lim_{r\to 0}\lim_{\epsilon\to 0} \mathrm{Tr}\left[\Delta'(\epsilon_1)\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\Delta'(\epsilon_2)\cdot\Delta_{\mathrm{cr}}^{-1}\right]\\
&\qquad = \frac{1}{\pi^2}\int_{\Omega_1} d^2z_1\int_{\Omega_2} d^2z_2\ \frac{\bar\partial F_1(z_1)\ \bar\partial F_2(z_2)}{(z_1-z_2)^4}\ +\ \frac{\partial\bar F_1(z_1)\ \partial\bar F_2(z_2)}{(\bar z_1-\bar z_2)^4}
\end{aligned}
\tag{8.76}
$$

8.8. **About the scaling limit of the Kähler operator $\mathcal{D}$.** We now discuss briefly the deformations of the Kähler operator, without giving details of the calculations. In the block matrix representation, the Kähler operator $\mathcal{D}$ and its ϵ-derivative read

(8.77)

$$
\mathcal{D}(\epsilon)=4\begin{pmatrix}\nabla_\epsilon\\ \overline{\nabla}_\epsilon\end{pmatrix}^\dagger\begin{pmatrix}A_\epsilon/R_\epsilon^2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\nabla_\epsilon\\ \overline{\nabla}_\epsilon\end{pmatrix}\quad,\quad \mathcal{D}'(\epsilon)=\begin{pmatrix}\nabla_\epsilon\\ \overline{\nabla}_\epsilon\end{pmatrix}^\dagger\begin{pmatrix}A_\epsilon\,\mathfrak{R}_\epsilon & A_\epsilon\,\overline{\mathfrak{Z}}_\epsilon\\ A_\epsilon\,\mathfrak{Z}_\epsilon & 0\end{pmatrix}\begin{pmatrix}\nabla_\epsilon\\ \overline{\nabla}_\epsilon\end{pmatrix}
$$

with A_ϵ and R_ϵ the areas and circumradii of the faces of the deformed lattice $\mathtt{T}_\epsilon$, while $\mathfrak{R}_\epsilon$ and $\mathfrak{Z}_\epsilon$ are given by 8.22 and 8.23. In order to study $\mathcal{D}(\epsilon)$ at finite epsilon and to compare it to $\mathcal{D}(0)=\mathcal{D}_{\mathrm{cr}}$, and its scaling limit, one can try to repeat the argument for Δ presented in the previous section. It is enough to consider $\mathcal{D}'(\epsilon)$. We start from a critical lattice $\mathtt{T}_0^r$ with isoradius r, perform the deformation $z\to z+\epsilon F(z)$, and reexpress $\mathcal{D}'(\epsilon)$, defined on the deformed Delaunay lattice $\mathtt{T}_\epsilon^r$, on the back-deformed lattice $\mathtt{T}_{\epsilon:0}^r$. We can thus rewrite $\mathcal{D}'(\epsilon)$ under a block form similar to 8.55

$$
\mathcal{D}'(\epsilon)=\begin{pmatrix}\nabla\\ \overline{\nabla}\end{pmatrix}^\dagger A\cdot\mathbb{F}'(\epsilon;F)\begin{pmatrix}\nabla\\ \overline{\nabla}\end{pmatrix}
\tag{8.78}
$$

with $\mathbb{F}'(\epsilon;F)$ a 2×2 block matrix made of diagonal matrices relative to the faces $\mathtt{f}$ of $\mathtt{T}_{\epsilon:0}^r$, defined implicitely by 8.78. The 2×2 matrix extracted of $\mathbb{F}'$ relative to a face $\mathtt{f}$, $[\mathbb{F}'(\epsilon;F)](\mathtt{f})$, can be computed explicitly out of the $\nabla F(\mathtt{f})$ and $\overline{\nabla}F(\mathtt{f})$, and of the geometry of the face $\mathtt{f}$, but the result will be quite long and not very illuminating at this stage. The difference with the previous case of Δ is that for a face $\mathtt{f}$ (let us denote its vertices (123)) $\mathbb{F}'$ will depend explicitely of the circumradius $R(\mathtt{f})$ of the face, and of the phases $C_{\mathtt{e}}$ associated to the unoriented edges $\mathtt{e}=(12)$, (23) and (31) of $\mathtt{f}$, defined by 8.13. Indeed the coefficient $\mathfrak{Z}(\mathtt{f})$ depends explicitely of $R(\mathtt{f})$, and the coefficient $\mathfrak{R}(\mathtt{f})$ depends also of the coefficients $C(\mathtt{f})=\sum_{\mathtt{e}\in\mathtt{f}}C_{\mathtt{e}}$. Moreover the variation of these coefficients under the backdeformation $\mathtt{T}_{\epsilon:0}\leftrightarrow\mathtt{T}_\epsilon$ depends also of these $C_{\mathtt{e}}$.

We can now use Pro. 13 which bound the $\mathfrak{Z}(\mathtt{f})$ and $\mathfrak{R}(\mathtt{f})$ and $R(\mathtt{f})$, and the fact that since the $C_{\mathtt{e}}$ are phases so that $|C_{\mathtt{e}}|=1$, to bound uniformly the coefficients of the matrices $[\mathbb{F}'(\epsilon;F)](\mathtt{f})$'s w.r.t the deformation parameter ϵ (small enough) and the triangulations $\mathtt{T}_0^r$. More precisely

Proposition 18. *Let F be a deformation function, $\mathcal{F}=\{\mathtt{T}_0^r : r\in(0,1]\}$ a family of critical triangulations labelled by their isoradius r, and $\epsilon\in(0,\epsilon'_{\mathrm{b}}]$ with ϵ'_{b} defined by 8.57. There is a constant $\mathtt{Cst.}$ which depends only on F and the choice of ϵ'_{b} such that there is a uniform bound for the matrix elements of the $[\mathbb{F}'(\epsilon;F)](\mathtt{f})$ matrices*

$$
\|\,[\mathbb{F}'(\epsilon;F)]\,(\mathtt{f})\,\|\ \le\ \mathtt{Cst.}\ r^{-2}
\tag{8.79}
$$

with $\|\cdot\|$ the standard operator norm on matrices (for instance).

Proof. The proof relies on writing explicitely the matrix $\mathbb{F}'$. This is lengthy but not difficult. Note that the r^{-2} factor, where r is the isoradius of the initial lattice $\mathtt{T}_0^r$, comes from the A/R_0^2 in the initial definition of $\mathcal{D}$ 8.77. □

If we look now at the limit $r \to 0$, keeping ϵ fixed, denoting as in proposition14 $\mathtt{f}^r_{\epsilon:0}(z)$ the face of $\mathtt{T}^r_{\epsilon:0}$ which contains the point z, there is for a generic family $\mathcal{F} = \{\mathtt{T}^r_{\epsilon:0}\}$ no reason that the ratio $\bar{R}(\mathtt{f}^r_{\epsilon:0}(z)) = R(\mathtt{f}^r_{\epsilon:0}(z))/r$ and the coefficients $C_{\mathtt{e}}(\mathtt{f}^r_{\epsilon:0}(z))$ and $C(\mathtt{f}^r_{\epsilon:0}(z))$ converge towards fixed values $\bar{R}(z;\epsilon)$, $C(z;\epsilon)$, $C_{\mathtt{e}}(z;\epsilon)$ in the scaling limit $r \to 0$. Indeed, these quantities depend explicitely on the detailed local geometrical structure of the lattices $\mathtt{T}_0^r$ in the neighborhood of the point z, for each values of r. Only for some *very specific sequences* of $\mathtt{T}_0^r$, for instance iterative isoradial refinements of the initial lattice for $r = 1$, can we expect strong correlations leading to the existence of a $r \to 0$ limit for theses quantities. We can therefore state:

Proposition 19. *Under the hypothesis of Prop. 18 the matrix $\mathbb{F}'(\epsilon;F)/r^2$ has generically no local scaling limit for ϵ finite when $r \to 0$.*

$$\lim_{r\to 0} \left[\mathbb{F}'(\epsilon;F)\right] (\mathtt{f}^r_{\epsilon:0}(z))/r^2 \qquad \textit{does not exist} \tag{8.80}$$

Of course one must have $z \in \Omega = \mathtt{supp}(F)$, since othewise this limit exists and is zero. The same is obviously true for the non existence of the $r \to 0$ limit of the bilocal term at finite ϵ_1, ϵ_2

$$\lim_{r\to 0} \mathrm{Tr}\left[\mathcal{D}'(\epsilon_1)\cdot\mathcal{D}_{\mathrm{cr}}^{-1}\cdot\mathcal{D}'(\epsilon_2)\cdot\mathcal{D}_{\mathrm{cr}}^{-1}\right] \qquad \textit{does not exist} \tag{8.81}$$

Therefore, the existence of a scaling limit for $\mathcal{D}$ could make sense in a much more limited setting than for Δ. Remember that we want to compare (i) the limit $\epsilon \to 0$, which, for $\mathcal{D}'$ as well as for Δ', has the effect of keeping only the terms linear in ∇F, $\overline{\nabla} F$ and their c.c.; (ii) the scaling limit $r \to 0$; which allows to replace the discrete derivatives ∇, $\overline{\nabla}$ by continuous derivatives ∂ and $\bar\partial$, and in particular 8.71. In fact the best result we obtain so far is as follows.

Proposition 20. *Let F be a deformation function, $\mathcal{F} = \{\mathtt{T}_0^r : r \in (0,1]\}$ a family of critical triangulations labelled by their isoradius r, and ϵ'_{b} defined by 8.57. We consider the "simultaneous limit" where*

$$r \to 0 \quad , \qquad \epsilon_a = \epsilon(r) = r\,\mathtt{c}_a \quad \textit{with} \quad 0 \le \mathtt{c}_a \le \epsilon'_{\mathrm{b}} \quad \textit{for } a = 1,2 \tag{8.82}$$

Then the bilocal term of 8.81 converges uniformly towards its continuum limit given in Th. 3.

$$\begin{aligned}&\lim_{\substack{r\to 0\\ \epsilon_1/r=\mathtt{c}_1\\ \epsilon_2/r=\mathtt{c}_2}} \mathrm{Tr}\left[\mathcal{D}'(\epsilon_1)\cdot\mathcal{D}_{\mathrm{cr}}^{-1}\cdot\mathcal{D}'(\epsilon_2)\cdot\mathcal{D}_{\mathrm{cr}}^{-1}\right] = \\ &\qquad \frac{1}{\pi^2}\int_{\Omega_1} d^2 z_1 \int_{\Omega_2} d^2 z_2 \; \left(\frac{\bar\partial F_1(z_1)\,\bar\partial F_2(z_2)}{(z_1-z_2)^4} + \frac{\partial \bar{F}_(z_1)\,\partial\bar{F}_2(z_2)}{(\bar z_1 - \bar z_2)^4}\right)\end{aligned} \tag{8.83}$$

9. Discussion and Perspectives

9.1. **The aim of the study.**

Firstly, let us remind the purpose of this work: study some of the properties of the measure over triangulations of the plane introduced in [DE14], in view of a better understanding of the relations between this discrete model of random geometry on the plane, and the continuum models of random geometries on the plane given by Conformal Field Theories (CFT), in particular the Quantum Liouville Theory. The model is defined as an integral over the space of all Delaunay triangulations of the plane. This paper is devoted to a particular and somehow limited study. We do not study as a whole the global properties of this integral, and of the associated measure. We rather study the measure in the neighborhood of very specific (subspace of) triangulations, namely isoradial triangulations. Our motivation is twofold : (i) isoradial triangulations can be viewed as a discretization of flat geometry, so that this should amount to some "semiclassical limit"; (ii) deforming the geometry is one way to associate a stress-energy tensor to a statistical model, whose properties are crucial for conformal theories.

The measure of the model is a Kähler mesure (in fact equivalent to the Weil-Petersson measure) and its density can be written as the determinant of a Laplacian-like Kähler operator $\mathcal{D}$ (defined on the Delaunay traingulations), with specific global conformal invariance properties under $\mathrm{PSL}(2,\mathbb{C})$ transformations. In order to compare our result with other case, we have studied in parallel the Kähler operator $\mathcal{D}$, the ordinary discrete Laplace-Beltrami operator Δ (which is not $\mathrm{PSL}(2,\mathbb{C})$ invariant), and a variation of the discrete Δ, that we introduce here, $\underline{\boldsymbol{\Delta}}$, which shares with $\mathcal{D}$ the global $\mathrm{PSL}(2,\mathbb{C})$ invariance property.

9.2. **The first order variations and discretized CFT.**

The Laplace-Beltrami operator Δ.

The calculation for the first order variation for the discretized Laplace-Beltrami Δ is easy to discuss in the framework of discretized CFT on the lattice. We refer to Appendix A for a reminder of the definitions and properties of CFT which are needed in this discussion. Our result 6.4 in Prop. 7 is

$$\delta_\epsilon \log\det(\Delta) = -\sum_{\substack{\text{faces}\\ \mathtt{f}\in\widehat{\mathfrak{G}}_{0+}}} 4\,A(\mathtt{f})\Big(\overline{\nabla}F(\mathtt{f})\,Q(\mathtt{f}) + c.c.\Big) \tag{9.1}$$

with

$$Q(\mathtt{f}) = [\nabla\Delta^{-1}\nabla^{\top}]_{\mathtt{ff}} = \sum_{\mathtt{u},\mathtt{v}} \nabla_{\mathtt{fu}}\nabla_{\mathtt{fv}}\,[\Delta_{\mathrm{cr}}^{-1}]_{\mathtt{uv}} \tag{9.2}$$

It reads as the discretized version of the first order variation of the partition function under a diffeomorphism for a CFT (see A.10) given by

$$\delta_\epsilon \log(Z) = -\frac{1}{\pi}\int d^2x\,\Big(\partial\bar F(x)\,\langle\bar T(x)\rangle + \bar\partial F(x)\,\langle T(x)\rangle\Big)$$

the sum over faces being the discrete version of the integral over the plane, and the discrete derivatives $\nabla\bar F$ and $\overline{\nabla}F$ being the discrete versions of $\partial\bar F$ and $\bar\partial F$.

$$\sum_{\mathtt{f}} A(\mathtt{f}) \;\leftrightarrow\; \int d^2x\,,\quad \nabla\bar F \;\leftrightarrow\; \partial\bar F\,,\quad \overline{\nabla}F \;\leftrightarrow\; \bar\partial F \tag{9.3}$$

The term $Q(\mathtt{f})$ is given by the v.e.v. of the discretised stress-energy tensor T for the discretized theory with Grassmann fields $(\Phi,\bar\Phi)$ attached to the vertices of the triangulation $\mathbf{G}_{\mathrm{cr}}$, with discretized action S

$$S[\Phi,\bar\Phi]=\Phi\cdot\Delta\bar\Phi=\sum_{\substack{\text{vertices}\\ \mathtt{u},\mathtt{v}\,\in\,\mathbf{G}_{\mathrm{cr}}}}\Phi_{\mathtt{u}}\Delta_{\mathtt{uv}}\bar\Phi_{\mathtt{v}} \tag{9.4}$$

and discrete stress-energy tensor $T_{\vartriangle}$ attached to the faces (triangles) of the triangulation $\mathbf{G}_{\mathrm{cr}}$

$$T_{\vartriangle}(\mathtt{f})=-4\pi\nabla\Phi(\mathtt{f})\nabla\bar\Phi(\mathtt{f})=-4\pi\sum_{\mathtt{u},\mathtt{v}\,\in\,\mathtt{f}}\nabla_{\mathtt{fu}}\Phi_{\mathtt{u}}\,\nabla_{\mathtt{fv}}\bar\Phi_{\mathtt{v}} \tag{9.5}$$

through the relation

$$4\pi\,Q(\mathtt{f})=\langle T_{\vartriangle}(\mathtt{f})\rangle \tag{9.6}$$

Note that this definition 9.5 for the discrete stress energy tensor follows directly from 9.4 and the variation of the discrete Laplace-Beltrami operator Δ given by Prop. 5 and eq. 5.3.

The above discussion is valid regardless of whether we consider the variation of the Laplace-Bletrami operator defined on an isoradial Delaunay graph $\mathbf{G}_{\mathrm{cr}}$ or instead on a general Delaunay graph $\mathbf{G}$. Indeed, 9.5 follows from the general equation 5.3 for the variation of Δ on generic triangulations. Note also that the absence of a $\nabla F+\overline{\nabla}\bar F$ term in the variation of Δ means $\mathrm{Tr}(\mathbf{T})=T^{z\bar z}=T^{\bar z z}$ is zero, and that the discrete Laplace-Beltrami operator Δ has a discrete conformal invariance property.

The interesting result, relevant for the discussion here, is that for an isoradial Delaunay graph $\mathbf{G}_{\mathrm{cr}}$ the term $Q(\mathtt{f})$, i.e. the v.e.v. of the discretized stress energy tensor T, depends only on the local geometry of the graph, i.e. on the shape of the triangle $\mathtt{f}$, as stated in prop. 7. This is not true when $\mathbf{G}$ is not isoradial; in that case, $\langle T(\mathtt{f})\rangle$ will depend on the full geometry of the lattice.

The Kähler operator $\mathcal{D}$.

The first order variation for the Kähler operator $\mathcal{D}$ is given by 6.8 in Prop. 9. The first term in 6.8 is the same as the first order variation for Δ in 6.2, which is rewritten in 9.1 as a sum over the triangles of the lattice involving the discrete derivatives of the deformation $\overline{\nabla}F$ and $\nabla\bar F$. The second term in 6.8 involves the first order variation $\eth_\epsilon R(\mathtt{f})$ of the circumradii $R(\mathtt{f},\epsilon)$ of a face, which can be obtained from 5.8 and 5.9. The final result is

$$\eth_\epsilon\log\det(\mathcal{D})=-\sum_{\substack{\text{faces}\\ \mathtt{f}\in\widehat{\mathbf{G}}_{0+}}}\left(\left(4\,A(\mathtt{f})Q(\mathtt{f})+\frac{1}{2}C(\mathtt{f})\right)\overline{\nabla}F(\mathtt{f})+\frac{1}{2}\nabla F(\mathtt{f})+\text{c.c.}\right) \tag{9.7}$$

with the geometrical factor $C(\mathtt{f})$ for a triangle $\mathtt{f}$ given by 5.5, while $Q(\mathtt{f})$ is given by 9.2, and corresponds to the v.e.v. of the discretized stress energy tensor $T_{\vartriangle}(\mathtt{f})$ defined by 9.5 for the Laplace-Beltrami theory.

Like the Laplace-Beltrami theory, the variation 9.7 can be written in term of a discretized stress energy tensor $\mathbf{T}_{\mathcal{D}}$ for a theory with discretized action

$$S_{\mathcal{D}}[\Phi,\bar\Phi]=\Phi\cdot\mathcal{D}\bar\Phi \tag{9.8}$$

(9.9)
$$\eth_\epsilon \log\det(\mathcal{D}) = \mathrm{tr}\left[\eth_\epsilon \mathcal{D}\cdot\mathcal{D}^{-1}\right] = -\frac{1}{\pi}\sum_{\mathtt{f}} A(\mathtt{f})\left(\overline{\nabla}F(\mathtt{f})\langle T_{\mathcal{D}}(\mathtt{f})\rangle + \nabla\bar{F}(\mathtt{f})\langle\bar{T}_{\mathcal{D}}(\mathtt{f})\rangle\right) + \frac{1}{2}\sum_{\mathtt{f}} A(\mathtt{f})\left(\nabla F(\mathtt{f}) + \overline{\nabla}\bar{F}(\mathtt{f})\right)\langle\mathrm{tr}\,\mathbf{T}_{\mathcal{D}}(\mathtt{f})\rangle$$

where the components of the discretized stress energy-tensor are

$$\begin{aligned} T_{\mathcal{D}} &= -4\pi\,\frac{1}{R^2}\,\left(\nabla\Phi\,\nabla\bar{\Phi} + C\,\overline{\nabla}\Phi\ \nabla\bar{\Phi}\right)\\ \bar{T}_{\mathcal{D}} &= -4\pi\,\frac{1}{R^2}\,\left(\overline{\nabla}\Phi\,\overline{\nabla}\bar{\Phi} + \bar{C}\,\overline{\nabla}\Phi\ \nabla\bar{\Phi}\right)\\ \mathrm{tr}\,\mathbf{T}_{\mathcal{D}} &= \quad 8\,\frac{1}{R^2}\,\left(\overline{\nabla}\Phi\,\nabla\bar{\Phi}\right)\end{aligned} \tag{9.10}$$

One should note the non zero $(\overline{\nabla}F + \nabla\bar{F})/2$ term in 9.7 and the non-vanishing of the v.e.v. of the trace of a discrete stress energy tensor $\mathrm{tr}\,\mathbf{T}_{\mathcal{D}}$. This follows from the fact that the length dimension of the matrix elements of $\mathcal{D}$ is $\mathtt{length}^{-2}$.

The definition 9.10 and the variation formula 9.9 remailn valid if we replace the isoradial Delaunay graph $\mathtt{G}_{\mathrm{cr}}$ by a generic Delaunay graph $\mathtt{G}$. The additional term $C(\mathtt{f})$ in 9.10, which depends explicitely on the local geometry of the graph in the neighborhood of the triangle $\mathtt{f}$. This term cannot be written simply in the continuum limit $\ell\to\infty$ in terms of continuous derivatives ∂ and $\bar{\partial}$ of a "smooth" complex Grassmann field $\Phi(x)$ in the flat continuum plane $\mathbb{R}^2$. This implies that $\mathbf{T}_{\mathcal{D}}$ hase no direct interpretation in a continuum field theory setting, at variance with $\mathbf{T}_{\Delta}$.

Again, the interesting explicit local form given in Prop. 9 and in Remark 31 are only valid for the variation of an isoradial Delaunay graph $\mathtt{G}_{\mathrm{cr}}$.

The conformal Laplacian $\underline{\boldsymbol{\Delta}}$.
The result given by Prop. 8 for $\underline{\boldsymbol{\Delta}}$ admits a similar interpretation. Again the absence of a $\nabla F + \overline{\nabla}\bar{F}$ term signals the conformal invariance of $\underline{\boldsymbol{\Delta}}$, which in this case is ensured from start, before one takes the scaling limit. The first order variation can still be written as a sum over triangles, of the form

$$\eth_\epsilon \log\det(\underline{\boldsymbol{\Delta}}) = -\sum_{\substack{\text{faces}\\ \mathtt{f}\in\widehat{\mathtt{G}}_{0+}}} 4A(\mathtt{f})\Big(\overline{\nabla}F(\mathtt{f})Q_{\mathrm{conf}}(\mathtt{f}) + \text{c.c.}\Big) \tag{9.11}$$

but now the local face term $Q_{\mathrm{conf}}(\mathtt{f})$ differs from $Q(\mathtt{f})$ when one or several of the edges of the triangle $\mathtt{f}$ are chords, owing to the additional terms in 6.5. More precisely, the contribution for a chord can be separated into equal contributions for its adjacent "north" and "south" triangles, so that one writes

$$A(\mathtt{f})Q_{\mathrm{conf}}(\mathtt{f}) = A(\mathtt{f})Q(\mathtt{f}) + H_{\mathrm{anom.}}(\mathtt{f}) \tag{9.12}$$

with the anomalous term $H_{\mathrm{anom.}}(\mathtt{f})$ for a (counter-clockwise oriented) face $\mathtt{f}$ expressed as a sum over its (oriented) edges $\vec{\mathtt{e}}$ which are chords

$$H_{\mathrm{anom.}}(\mathtt{f}) := \sum_{\substack{\text{chords}\\ \vec{\mathtt{e}}\in\partial\mathtt{f}}} H(\vec{\mathtt{e}},\mathtt{f}) \quad \text{with} \quad H(\vec{\mathtt{e}},\mathtt{f}) := \frac{1}{8\pi\mathrm{i}}\,\theta_{\mathrm{n}}(\vec{\mathtt{e}})\cot\theta_{\mathrm{n}}(\vec{\mathtt{e}})\,\mathcal{E}_{\mathrm{n}}(\vec{\mathtt{e}}) \tag{9.13}$$

and where $\mathcal{E}_{\mathrm{n}}(\vec{\mathtt{e}})$ is defined in 5.14. These explicit results are valid when deforming an isoradial Delaunay graph G_{cr}.

Again for the deformation of a generic triangulation G_{cr}, the variation 9.11 can be written in term of a discretized stress-energy tensor $\mathbf{T}_{\underline{\blacktriangle}}$ a theory for a Grassmann field $(\Phi,\bar{\Phi})$ with action $S_{_{\mathrm{conf}}}=\Phi{\cdot}\underline{\boldsymbol{\Delta}}\bar{\Phi}$

$$\begin{aligned}\eth_{\epsilon}\log\det(\underline{\boldsymbol{\Delta}}) = & -\frac{1}{\pi}\sum_{\mathtt{f}} A(\mathtt{f})\,\left(\overline{\nabla}F(\mathtt{f})\langle T_{\underline{\blacktriangle}}(\mathtt{f})\rangle+\nabla\bar{F}(\mathtt{f})\langle\bar{T}_{\underline{\blacktriangle}}(\mathtt{f})\rangle\right)\\ & +\frac{1}{2}\sum_{\mathtt{f}} A(\mathtt{f})\,\left(\nabla F(\mathtt{f})+\overline{\nabla}\bar{F}(\mathtt{f})\right)\,\langle \mathrm{tr}(\mathbf{T}_{\underline{\blacktriangle}}(\mathtt{f}))\rangle\end{aligned} \tag{9.14}$$

One has generically $\mathrm{tr}(\mathbf{T}_{\underline{\blacktriangle}})=0$ (conformal invariance). The discretized analytic and anti-analytic components $T_{\underline{\blacktriangle}}$ and $\bar{T}_{\underline{\blacktriangle}}$ can be written explicitely, using Section 5.1 and in particular 5.13 in Remark 23. We get a generic form for $T_{\underline{\blacktriangle}}$ involving all possible binomials of discrete derivatives of the fields

$$T_{\underline{\blacktriangle}}=\mathfrak{a}\,\nabla\Phi\,\nabla\bar{\Phi}+\mathfrak{b}\,\nabla\Phi\,\overline{\nabla}\bar{\Phi}+\mathfrak{c}\,\overline{\nabla}\Phi\nabla\bar{\Phi}+\mathfrak{d}\,\overline{\nabla}\Phi\,\overline{\nabla}\bar{\Phi} \tag{9.15}$$

The coefficients $\mathfrak{a}(\mathtt{f})$, $\mathfrak{b}(\mathtt{f})$, $\mathfrak{c}(\mathtt{f})$, $\mathfrak{d}(\mathtt{f})$ for a given face (triangle) $\mathtt{f}$ of the triangulation $\widehat{\mathsf{G}}_{0+}$ turn out to depend not only of the geometry of the triangle $\mathtt{f}$, but of its three neighbours $\mathtt{f}'$, $\mathtt{f}''$ and $\mathtt{f}'''$, since they depend explicitely of the conformal angles $\theta(\mathtt{e})$ of the three edges $\mathtt{e}'$, $\mathtt{e}''$ and $\mathtt{e}'''$ of $\mathtt{f}$. See Fig. 27. So the discrete stress energy tensor $\mathbf{T}_{\underline{\blacktriangle}}$ is still local in the fields $(\Phi,\bar{\Phi})$ than $\mathbf{T}_{\vartriangle}$, but is less local in the geometry of the lattice, and less analytic, since $T_{\underline{\blacktriangle}}$ does not involves only the term $\nabla\Phi\,\nabla\bar{\Phi}$ which a simple $\mathfrak{a}(\mathtt{f})$, as for $\mathbf{T}_{\vartriangle}$. From the general discussion of Section 5.1, the anomalous terms will always be present in a discretized $T_{\underline{\blacktriangle}}$ as soon as we consider the deformation of a generic non-critical (non-isoradial) triangulation G_{cr}. This will have important effect when discussing the second order variation, owing to Appendix C.

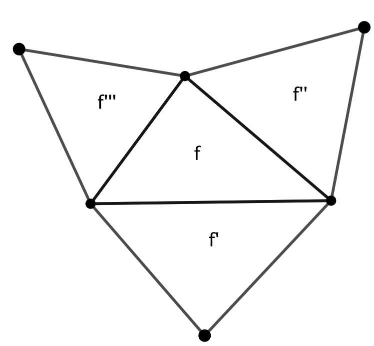

FIGURE 27. A face f (triangle) and its neighbours

9.3. **The second order variations and discretized CFT.**
We now discuss along the same line our result for the second order variation and its scaling limit.

The Laplace-Beltrami operator Δ. Here we consider the Beltrami-Laplace operator of the Delaunay graph $\mathbf{G}_\epsilon$ obtained through a bi-local deformation $z_{\underline{\epsilon}}(\mathtt{v}) := z_{\mathrm{cr}}(\mathtt{v}) + \epsilon_1 F_1(\mathtt{v}) + \epsilon_2 F_2(\mathtt{v})$ of the critical embedding of an isoradial Delaunay graph $\mathbf{G}_{\mathrm{cr}}$. The $\epsilon_1\epsilon_2$ cross-term of $\log\det\Delta$ can be calculated exactly using Proposition 5.3 and expressed using the limit graph $\mathbf{G}_{0+}$ and any weak Delaunay triangulation $\widehat{\mathbf{G}}_{0+}$ which completes it. This gives

$$
\begin{aligned}
&\eth_{\epsilon_1}\eth_{\epsilon_2}\log\det\Delta \\
&= -\operatorname{tr}\Big[\eth_{\epsilon_1}\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\eth_{\epsilon_2}\Delta\cdot\Delta_{\mathrm{cr}}^{-1}\Big] \\
&= -64\operatorname{tr}\Big[\mathfrak{Re}\Big[\overline{\nabla}^{\top}(\nabla\bar{F}_1)A\overline{\nabla}\Big]\cdot\Delta_{\mathrm{cr}}^{-1}\cdot\mathfrak{Re}\Big[\overline{\nabla}^{\top}(\nabla\bar{F}_2)A\overline{\nabla}\Big]\cdot\Delta_{\mathrm{cr}}^{-1}\Big] \\
&= \left\{\begin{array}{c}
-\displaystyle\sum_{\substack{\text{triangles}\\ \mathtt{x}_1,\mathtt{x}_2\in\widehat{\mathbf{G}}_{0+}}} 32\,A(\mathtt{x}_1)A(\mathtt{x}_2)\,\mathfrak{Re}\Big[\overline{\nabla}F_1(\mathtt{x}_1)\overline{\nabla}F_2(\mathtt{x}_2)\Big[\nabla\Delta_{\mathrm{cr}}^{-1}\nabla^{\top}\Big]^2_{\mathtt{x}_1\mathtt{x}_2}\Big] \\
+ \\
-\displaystyle\sum_{\substack{\text{triangles}\\ \mathtt{x}_1,\mathtt{x}_2\in\widehat{\mathbf{G}}_{0+}}} 32\,A(\mathtt{x}_1)A(\mathtt{x}_2)\,\mathfrak{Re}\Big[\overline{\nabla}F_1(\mathtt{x}_1)\nabla\bar{F}_2(\mathtt{x}_2)\Big[\nabla\Delta_{\mathrm{cr}}^{-1}\overline{\nabla}^{\top}\Big]^2_{\mathtt{x}_1\mathtt{x}_2}\Big]
\end{array}\right.
\end{aligned}
\tag{9.16}
$$

Using formula 9.5 for the discrete stress-energy tensor T_Δ and applying Wick's theorem we can express the two-point v.e.v.'s

$$
\begin{aligned}
\frac{1}{32\pi^2}\Big\langle T_\Delta(\mathtt{x}_1)\,T_\Delta(\mathtt{x}_2)\Big\rangle_{\mathrm{conn.}} &= \Big[\nabla\Delta_{\mathrm{cr}}^{-1}\nabla^{\top}\Big]^2_{\mathtt{x}_1\mathtt{x}_2} \\
\frac{1}{32\pi^2}\Big\langle T_\Delta(\mathtt{x}_1)\,\overline{T}_\Delta(\mathtt{x}_2)\Big\rangle_{\mathrm{conn.}} &= \Big[\nabla\Delta_{\mathrm{cr}}^{-1}\overline{\nabla}^{\top}\Big]^2_{\mathtt{x}_1\mathtt{x}_2}
\end{aligned}
\tag{9.17}
$$

and the c.c. So far we do not require the initial graph to be isoradial: We may in fact replace the critical graph $\mathbf{G}_{\mathrm{cr}}$ with any Delaunay graph $\mathbf{G}_0$ equipped with its corresponding Beltrami-Laplace operator Δ_0 and Green's function Δ_0^{-1} and the variational formula 9.16 and double correlator identity 9.17 remain valid. If, however, we incorporate a scaling parameter $\ell > 0$ and consider the bi-local smoothly deformed embedding $z_{\underline{\epsilon};\ell}(\mathtt{v}) := z_{\mathrm{cr}}(\mathtt{v}) + \epsilon_1\ell F_{1;\ell}(\mathtt{v}) + \epsilon_2\ell F_{2;\ell}(\mathtt{v})$ then the isoradial property (as manifest in the asymptotic expansion 1.19 for the critical Green's function $\Delta_{\mathrm{cr}}^{-1}$) is sufficient to establish the convergence of the scaling limit of formula 9.16 which is consistent the OPE of the CFT with the expected central charge $c = -2$, namely

$$
\lim_{\ell\to\infty}\eth_{\epsilon_1}\eth_{\epsilon_2}\log\det\Delta = \frac{c}{\pi^2}\iint_{\Omega_1\times\Omega_2} dx_1^2\,dx_2^2\;\mathfrak{Re}\left[\frac{\bar{\partial}F_1(x_1)\,\bar{\partial}F_2(x_2)}{(x_1-x_2)^4}\right]
\tag{9.18}
$$

As we have seen $\nabla\Delta_0^{-1}\nabla^\top$ and $\nabla\Delta_0^{-1}\overline{\nabla}^\top$ (and their complex conjugates) must decay in accordance with Lemma 12 in order for 9.18 to hold. Our result is of course not surprising, and should be viewed as a check of the validity of our approach.

The Kähler operator $\mathcal{D}$. Prop. 11 and its scaling limit given in Section 7.3 are the news and interesting results of the paper. They states that the scaling limit of the bilocal second order variation for $\mathrm{tr}[\log\mathcal{D}]$ is similar to the one for Δ.

$$-\frac{1}{\pi^2}\iint_{\Omega_1\times\Omega_2} dx_1\, dx_2 \left(\frac{\bar\partial F_1(x_1)\,\bar\partial F_2(x_2)}{(x_1-x_2)^4}+\frac{\partial \bar F_1(x_1)\,\partial \bar F_2(x_2)}{(\bar x_1-\bar x_2)^4}\right) \tag{9.19}$$

This is interesting for two reasons.

The $\mathcal{D}$ operator has a different form and even a different scaling dimension than Δ. Its variation 5.4 and the associated stress-energy tensor 9.10 are different. However the second order variation has exactly the same OPE form than the variatio for for Δ, and it corresponds to a CFT with the same central charge

$$c=-2\ .$$

This value for the central charge is in our opinion somehow unexpected, and this is interesting per se. Indeed it was suggesetd by one of us (F.D.) in the original paper [DE14] that the measure over triangulations given by $\det(\mathcal{D})$ (later shown in [CDE19] to coincide with the Weil-Petersson metric over marked complex curve), had a direct relation with the gauge fixing Fadeev-Popov determinant in two dimensional quantum gravity. If true, it should be related to the so called b-c ghosts system in Polyakov's formulation as Liouville theory of 2D gravity and non-critical strings (see [Fri84]). Then one could have expected a different value for the central charge, since the central charge for the b-c system is $c=-26$, and the central charge for the corresponding Liouville quantum gravity (at $Q=5/\sqrt{6}$ i.e. $\gamma=\sqrt{8/3}$) is $c=26$.

The conformal laplacian $\underline{\boldsymbol{\Delta}}$.

For the conformal Laplacian operator $\underline{\boldsymbol{\Delta}}$, we do not have such a simple result, and the corresponding OPE cannot be interpreted as coming from a CFT. There are additional contributions that comes from the chords, which have been studied in section 6.3, and are the chord-chord term given by 6.58 and the chord-edge term given by 6.59. The later chord-edge term has the expected harmonic form (depending only on $(x-x')^{-4}$ and its c.c.), but with a local geometry dependent coefficient involving both $\overline{\nabla}F_1\overline{\nabla}F_2$ and $\nabla\bar F_1\overline{\nabla}F_2$ terms. The chord-chord term is even more involved and contains a non-harmonic term, proportional to $|x-x'|^{-4}$, with a more complicated geometrical dependence in the geometry of the faces and the chords. In Appendix C we give an explicit example of a critical lattice with a finite density of chords where these addditional "anomalous" terms give a macroscopic anomalous contribution to the second order variation, which precludes an interpretation in terms of conformal field theory in the scaling limit. Of course this comes from the anomalous terms in the expression of the discretized stress-energy tensor T_{conf} (of general schematic form given in 9.15), which does not have a simple universal field theoretical interpretation in the scaling limit. This is also a new - although somehow negative - result.

9.4. **Relations and differences with other discrete models.**
The operators that we study here are defined on planar isoradial Delaunay graphs. Isoradial graph embeddings play a very important role in the study of two dimensional models of statistical mechanics in theoretical physics and in mathematics. In particular they are an essential tool in the proof of the conformal invariance of the Ising model at its critical point, and in the study of the conformal invariance of other critical models. They are very important in our study too, since they afford control of the large distance properties of the respective Greens functions.

However, we stress that there is an important difference in term of perspective. In studies of critical statistical models on such graphs, the underlying graph is fixed and the proofs of the existence of a scaling limit and of its conformal invariance are undertaken for a fixed lattice. The random triangulation model of [DE14] is a statistical model *of* planar graph*s*, rather than *on a* planar graph. The planar isoradial graphs that we consider here are just some special "semi-classical" configurations, which minimize a "local curvature functional", as discussed in the introduction in 1.2.

There are nevertheless relations between our work and some recent works, especially in regard to defining a notion of a discrete stress-energy tensor. Let us briefly discuss two of them.

Discrete stress-energy tensor in the loop model of Chelkak et al. In [CGS18] Chelkak, Glazman and Smirnov study the famous critical $\mathrm{O}(n)$ loop model [DMNS81] [Nie87] [Kos89] on abstract discrete surfaces with boundaries (denotes G_δ) made by gluing together equilateral triangles $\triangle$ and rhombs $\lozenge(\theta)$ of unit length δ where each rhomb has an independent acute angle θ selected in the range $0 < \theta \leq \frac{\pi}{2}$. See Fig. 28. The surface has in general conical singularities at all of its vertices. In general a discrete surface may admit more than one tessellation into triangles and rhombs if some vertices are flat (no conical defect). Two tessellations are equivalent (i.e. they describe the same surface) if one can be transformed into the other by applying a sequence of the following three kinds of local operations: (i) *Yang-Baxter transformations* which flips a flat hexagon made up of three rhombs sharing a common vertex, (ii) *pentagonal transformations* which interchange a triangle and a rhomb which form a flat pentagon with a triangle and two rhombs, (iii) *split transformations* which dissect a rhomb $\lozenge(\frac{\pi}{3})$ into a pair of equilateral triangles sharing a common edge. See Fig. 29.

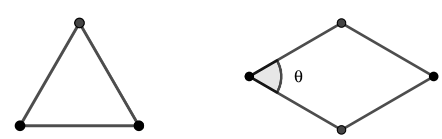

FIGURE 28. The triangles and rhombs of [CGS18]

The states of the $O(n)$ loop model for a tessellated surface G_δ are configurations γ consisting of non-crossing loops and strands (joining boundary components, if present) drawn on the surface G_δ which can be obtained by concatenating local arrangements of arcs, one for each triangle and rhomb in G_δ. A local weight $\mathrm{w}_\gamma(\mathtt{f})$ is associated to each face $\mathtt{f}$ of G_δ which depends on the configuration of the loops

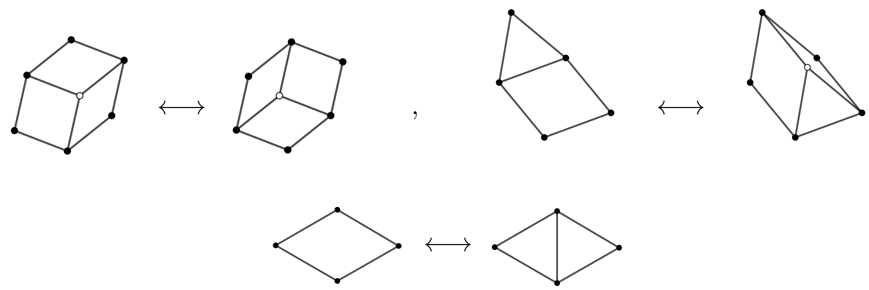

FIGURE 29. Yang-Baxter, pentagonal and split moves of [CGS18]; white vertices $\circ$ have to be flat (no conical singularity)

on $\mathtt{f}$, the geometry of the face (hence of angle θ if $\mathtt{f} = \lozenge(\theta)$ is a rhomb), and on a parameter s (related to the temperature). A factor n (loop fugacity) is associated to each closed loop. The local weight $\mathrm{w}_\gamma(\mathtt{f})$ (that we do not discuss here) are taken to have a very specific form in order to satisfy the Yang-Baxter and Pentagonal relations ensuring that the model is the same for equivalent tessellations of the surface.

The partition function $Z^{\mathfrak{b}}(G_\delta)$ for the $O(n)$ loop model on a fixed surface G_δ equipped with a boundary condition $\mathfrak{b}$ (specifying which boundary edges are joined by arcs), is given by the sum over states (loops+arcs configurations γ) by

$$Z^{\mathfrak{b}}(G_\delta) := \sum_{\mathfrak{b}\text{-configurations } \gamma} n^{\#\text{loops}(\gamma)} \prod_{\substack{\text{faces} \\ \mathtt{f} \in G_\delta}} \mathrm{w}_\gamma(\mathtt{f}) \tag{9.20}$$

In addition, when the specific relation between n (the loop fugacity) and s (the temperature parameter)

$$n = -\cos(4\pi\, s/3)$$

holds, then the loop model is critical.

In [CGS18] Chelkak et al. consider a *planar version* without conical defects where all rhombs have angle $\theta = \frac{\pi}{3}$, and such that the discrete surface G_δ is a compact, connected domain Ω of the triangular lattice. In this planar case, they define a *discrete stress-energy tensor* as the response of the model to an infinitesimal ϵ-deformation of the original planar surface into a non-planar surface with conical defects. More precisely, two deformations are considered: (i) replacing two adjacent equilateral triangles (forming a rhomb $\lozenge(\pi/3)$) by a rhomb $\lozenge(\theta)$ with angle $\theta = \frac{\pi}{3} + \epsilon$, (ii) replacing two aligned edges by a "almost flat" rhomb $\lozenge(\epsilon)$ (see Fig. 30). The variation of the logarithm of the partition function under such ϵ-deformations defines the e.v. of discrete stress-energy tensor $\mathcal{T}_{\mathsf{e}|\mathfrak{m}}$ associated to edges e or to midlines $\mathfrak{m}$ (of the honeycomb lattice built from the original triangular lattice), and out of these related real objects, a discrete complex stress-energy tensor $\mathcal{T}$ can be associated to the vertices and the faces of the lattice (with relations). In [CGS18] it is conjectured that this object is approximately discrete-holomorphic and converges to the stress-energy tensor of the corresponding CFT in the scaling limit.

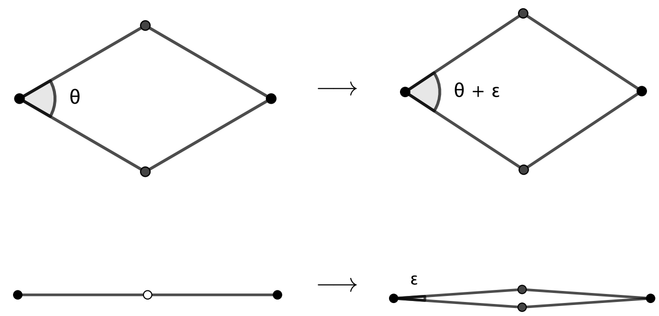

FIGURE 30. The ϵ-deformations of rhombs in [CGS18]

Similarities and differences. There are similarities but also important differences with the approach and results of our study. The discrete conformal Laplacian $\underline{\boldsymbol{\Delta}}$ defined in 1.8 is also defined with respect to a rhombic tessellated surface $\mathsf{S}_{\mathsf{G}}^{\Diamond}$ naturally associated to a Delaunay graph G in the plane (see Sect. 2.1 and especially Def. 13). However $\mathsf{S}_{\mathsf{G}}^{\Diamond}$ is constructed only out of rhombs $\Diamond(\mathsf{e})$ associated to edges e of G, and contains no equilateral triangles. Morover, the rhombic surface $\mathsf{S}_{\mathsf{G}}^{\Diamond}$ is bipartite: whose black and white vertices correspond to vertices and faces of G respectively. Finally, and most importantly, the black vertices of $\mathsf{S}_{\mathsf{G}}^{\Diamond}$ must be flat (they do not carry a conical singularity), while the white vertices carry in general a conical singularity (corresponding to a non-zero Ricci curvature given by 2.3), see Fig. 31. Thus our model considers only a subspace of the space of tessellated

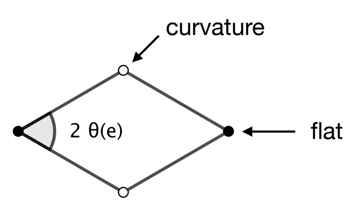

FIGURE 31. The rhombs which build the tessellated surface $\mathsf{S}_{\mathsf{G}}^{\Diamond}$ in this paper

surfaces of [CGS18].

Like [CGS18], the stress-energy tensor in our study is defined in terms of deformations. However an important difference is that we consider deformations of $\mathsf{S}_{\mathsf{G}}^{\Diamond}$ which are induced from deformations of the underlying Delaunay graph G in the plane. This space of deformations differs from those considered in [CGS18] in two respects. First, our deformations preserve the flatness of the black vertices of $\mathsf{S}_{\mathsf{G}}^{\Diamond}$. Second, and this is essential, our discrete stress-energy tensor has a specific invariance properties under *global continuous analytic transformations* of the plane, i.e.

Moebius transformations. This holds a priori, independent of the specific geometry of the Delaunay graph G.

In [CGS18] as well as in other studies, the framework is different. One looks for a discrete stress energy tensor on an isoradial critical graph G which has some specific invariance properties under the *discrete analytic and anti-analytic transformations* of G. Discrete analyticity is a very special and powerful property, but it depends explicitly on the critical graph considered. It is only in the scaling limit that discrete analyticity can be shown to "converge" (this is a crude presentation of beautiful and precise results) towards the usual analyticity in the continuum (i.e. in the complex plane $\mathbb{C}$).

Another difference is that our deformation setting include deformations of "flat rhombs" (corresponding to chords) which are not deformations of aligned edges, as the deformations considered in [CGS18] and depicted in Fig. 30. These deformations induce the appearance of the "curvature dipoles" discussed in Sect. 6.4, which complicate the analysis of the deformations of $\underline{\boldsymbol{\Delta}}$.

The overlap between our work and the results of [CGS18] is restricted to the case of the $\underline{\boldsymbol{\Delta}}$ operator, which is related to the GFF. Strictly speaking the authors of [CGS18] consider the critical $\mathrm{O}(n)$ loop model for $n \in [-2, 2]$, but it is known that the GFF can be related to the $n = 2$ model, and that there is some relation between the Laplace-Beltrami operator on a graph and the $n = -2$ model.

On the other hand, the Laplace-Beltrami operator Δ and the Kähler operator $\mathcal{D}$, which we would like to study on general Delaunay graph G, are not defined in term of the abstract rhombic surface $\mathsf{S}^{\Diamond}_{\mathsf{G}}$. We do not know how to relate precisely, and in general, their corresponding discrete stress-energy tensors to the construction of a stress-energy tensor of [CGS18].

Stress-energy tensor constructions through lattice representations of Virasoro algebra. In an approach taken by Hongler et. al. in [HKV19], a stress-energy tensor for some lattice models is defined implicitly by identifying its modes through an action of the Virasora algebra on an appropriately defined vector space $\mathfrak{F} := \mathfrak{F}^{\mathrm{loc}}/\mathfrak{F}^{\mathrm{null}}$ of *lattice local fields* (modulo *null fields*) supported on the graph. This construction avoids interpreting the stress-energy tensor as a response to a deformation of the graph embedding. Instead an intermediate action of the Heisenberg algebra is introduced using a discrete holomorphic current along with a technique of discrete contour integration and a notion of discrete half-integer power functions. Only the special cases of the discrete GFF and of the Ising model on the square lattice $\mathsf{G} = \mathbb{Z}^2_\delta$ with mesh size δ are handled in [HKV19]. However, we expect that most of their technology (e.g. the notions of medial and corner graphs, discrete power functions, and discrete contour integration) is readily adaptable to arbitrary isoradial graphs (and their rhombic graphs where the theory of discrete holomorphicity is well-behaved). The space of lattice local fields $\mathfrak{F}^{\mathrm{loc}}$ of [HKV19] depends on the translation properties of $\mathsf{G} = \mathbb{Z}^2_\delta$. Specifically $\mathfrak{F}^{\mathrm{loc}}$ consists of fields which can be constructed as polynomial expressions of elementary fields $\phi_\delta(z)$ together with their translates $\phi_\delta(z + x\delta)$ for x in some fixed, finite set $V \subset \mathbb{Z}^2$ of admissible displacements. For a general isoradial graph one would need to specify an adequate vector space of lattice local fields $\mathfrak{F}^{\mathrm{loc}}$ on which a representation of the Virasora algebra could be supported. Bearing this, it would be natural to examine whether the stress-energy tensor(s) for the operator(s) considered in our paper can be realized by such putative Virasora algebra action(s). For older references of representations of Virasoro algebra in lattice models, see the references in [HKV19].

9.5. **Open questions and possible extensions.**

1: We would like to reiterate the problem of settling Conjecture 1 of Sect. 8.6, or in lieu of that finding another adequate bound on $R(\mathtt{f})^{-1}\nabla p_3(\mathtt{f})$ uniform in the faces $\mathtt{f}$ of $\mathtt{T}_0^{(r)}$ and the scaling parameter ℓ (or $r = 1/\ell$), in order do complete the proof of props. 16 and 17 as well as 20.

2: Instead of using an isoradial Delaunay graph, we could instead begin with a Delaunay graph which is "smoothly non-isoradial", in the sense that the circumradii of the faces $R(\mathtt{f})$ vary slowly with the position of the faces in the plane. Studying the Laplace-like operators Δ, $\underline{\boldsymbol{\Delta}}$ and $\mathcal{D}$ and their deformations on such a graph is an interesting problem which might entail finding asymptotic expansions of the corresponding Green functions.

3: The properties that make an general isoradial graph $\mathtt{G}$ so useful as a starting point in our analysis are a reflection of the underlying notion of discrete analyticity supported on the lozenge graph $\mathtt{G}^{\diamondsuit}$. Chelkak, Smirnov and others [Che18] have introduced the concept of s-holomorphicity and s-embeddings of graphs, and one can try to develop a theory of deformations for such graphs and their associated operators.

4: In the scaling limit, random planar graphs are known to be related to Liouville conformal field theory. Finding a notion of discrete Liouville local field, with good properties in the scaling limit, for the model of random Delaunay triangulations is still an open problem. A solution could lead to an alternative discrete stress energy tensor on a Delaunay graph, different from the one considered here, and with different properties under geometrical deformations of the graphs; in particular having a discrete central charge different from $c = -2$ (possibly $c = -26$).

5: It should be also interesting to study the existence and description of a stress energy tensor for other discrete models on Delaunay graphs, such as Dirac Fermions, the Ising model, the O(N) model, etc. using the approach of our work. It would be fruitful to compare the results with the approaches taken in [HKV19] and [CGS18] (see section 9.4).

Acknowledgements

The authors thank for hospitality Universidad de los Andes, Bogotá, Colombia, where this work was started, as well as the Chebyshev Laboratory at St. Petersburg State University, Russia, the Perimeter Institute, Waterloo, Canada, and the Institute of the Mathematical Sciences of the Americas, U. of Miami, USA, where part of this work was done. In addition to these institutions, J.S. also thanks Institut Henri Poincaré in Paris, Institut des Hautes Études Scientifiques in Bures-sur-Yvette, and Institute de Physique Théorique in Gif-sur-Yvette for their hospitality; the later for several visits during the course of this project.

The authors thank Dmitry Chelkak for discussions at IHES hosted by Hugo Duminil-Copin. F.D. thanks Bertrand Eynard and Philippe di Francesco for discussions. J.S. would also like to thank Peter Zograf and Gaëtan Borot: Peter Zograf for the opportunity to present a short lecture course addressing preliminary results of this paper at the Chebyshev Laboratory, and Gaëtan Borot for inviting and hosting J.S. at the 2017 IHP trimester program "Combinatorics and interactions" and later at the Max-Planck-Institut, Bonn for a short visit to discuss aspects of the project.

This paper is partly a result of the ERC-SyG project, Recursive and Exact New Quantum Theory (ReNewQuantum) which received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 810573.

Appendix A. Reminders: the stress-energy tensor in QFT and the central charge in 2D CFT

A.1. **The stress-energy tensor.** For completeness, we recall some textbook material of QFT and CFT, which can be found for instance in [DFMS97]. A central concept in field theory is the *stress-energy tensor* $\mathbf{T} = (T^{\mu\nu})$ (also denoted the *energy-momentum tensor* in the literature). Firstly, $\mathbf{T}$ can be viewed (in flat space) as the conserved current $\mathbf{J}^{\nu} = (T_{\mu}{}^{\nu})$ associated to space-time translation invariance, and is defined through Noether's theorem by the action of an infinitesimal local change of coordinates

$$x^{\nu} \to x^{\nu} + \xi^{\nu}(x) \tag{A.1}$$

on the action $\mathcal{S}$ (classical or quantum) of the theory. Secondly $\mathbf{T}$ can be viewed (in a general curved space) as the "response of the theory" to an infinitesimal variation of the classical "background metric" $\mathbf{g} = (g_{\mu\nu})$

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \tag{A.2}$$

of the space-time $\mathcal{M}$ where the theory "lives". More precisely $\mathbf{T}$ is defined classically by the functional derivative of the action $\mathcal{S}$

$$T^{\mu\nu}(x) = -\frac{2}{\sqrt{g(x)}}\frac{\delta \mathcal{S}}{\delta g_{\mu\nu}(x)} \tag{A.3}$$

For a quantum theory (i.e. a local QFT), $\mathbf{T}$ is now a quantum operator. Its vacuum expectation value (the vacuum-vacuum matrix element) is given by the *first order* variation of the logarithm of the partition function Z of the QFT under an infinitesimal variation of the metric $\delta g_{\mu\nu}$

$$\delta \log Z = \frac{1}{2}\int_{\mathcal{M}} dx\,\sqrt{g(x)}\,\delta g_{\mu\nu}(x)\,\langle T^{\mu\nu}(x)\rangle \;+\; \cdots \tag{A.4}$$

Similarily the first order variation of the vacuum expectation of an observable $\mathcal{O}$, for instance a product of local operators $\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)$, gives by the connected correlator of $\mathbf{T}$ times $\mathcal{O}$

$$\delta\langle\mathcal{O}\rangle = \frac{1}{2}\int_{\mathcal{M}} dx\,\sqrt{g(x)}\;\delta g_{\mu\nu}\;\Big(\langle T^{\mu\nu}(x)\,\mathcal{O}\rangle_{\text{conn.}} \;+\; \text{contact terms}\Big) + \cdots \tag{A.5}$$

where the so-called "contact terms" are present in A.5 when the position x of $\mathbf{T}$ coincides with that of some local operators in $\mathcal{O}$.

These two definitions of the stress-energy tensor $\mathbf{T}$ are closely related, and in fact equivalent (with the proper definitions of $\mathbf{T}$), since a diffeomorphism A.1 induces a change of metric

$$\delta g_{\mu\nu} = D_{\mu}\xi_{\nu} + D_{\nu}\xi_{\mu} \tag{A.6}$$

with D_{μ} the covariant derivative and $\xi_{\nu} = g_{\nu\rho}\xi^{\rho}$.

These definitions extend to higher order terms in the expansion in the infinitesimal variation $\delta g_{\mu\nu}$, which give expectation values of products of $\mathbf{T}$ (correlators). For instance the second order term in the variation of $\log Z$ gives the two point correlator

(A.7)

$$-\frac{1}{8}\int_{\mathcal{M}} dx\,\sqrt{g(x)}\,\delta g_{\mu\nu}(x)\,\int_{\mathcal{M}} dy\,\sqrt{g(y)}\,\delta g_{\rho\sigma}(y)\,\langle T^{\mu\nu}(x)\,T^{\rho\sigma}(y)\rangle_{\text{conn.}} + \text{contact terms}$$

and so on.

A.2. **The stress-energy tensor in two dimensional CFT.** In two dimensions, it is standard to work in complex coordinates $z = x^1 + \mathrm{i}\, x^2$, $\bar{z} = x^1 - \mathrm{i}\, x^2$, so that the flat metric is

$$g_{zz} = g_{\bar{z}\bar{z}} = 0\,,\ g_{z\bar{z}} = g_{\bar{z}z} = 1/2\ .$$

An infinitesimal diffeomorphism $z \mapsto z + \epsilon\, F(z,\bar{z})$ thus amounts to a variation of the metric

$$\delta g_{zz} = \epsilon\, \partial \bar{F}\,, \quad \delta g_{\bar{z}\bar{z}} = \epsilon\, \bar{\partial} F\,, \quad \delta g_{z\bar{z}} = \delta g_{\bar{z}z} = \epsilon\, (\partial F + \bar{\partial}\bar{F})/2 \tag{A.8}$$

For QFT's in two dimension (in particular for CFT's), especially important are the holomorphic and antiholomorphic components of the stress energy tensor $\mathbf{T}$, which are denoted T and $\bar{T}$ in the litterature (see e.g. [DFMS97]). In the flat metric they are

$$T = -\frac{\pi}{2}\, T^{\bar{z}\bar{z}} = -2\pi\, T_{zz}\ , \quad \bar{T} = -\frac{\pi}{2}\, T^{zz} = -2\pi\, T_{\bar{z}\bar{z}} \tag{A.9}$$

The variation of $\log Z$ A.4 reads

$$\begin{aligned}\delta \log(Z) = &-\frac{\epsilon}{\pi} \int d^2x\, \left(\partial \bar{F}(x)\, \langle \bar{T}(x)\rangle + \bar{\partial} F(x)\, \langle T(x)\rangle\right) \\ &+ \frac{\epsilon}{2} \int d^2x\, \left(\partial F(x) + \bar{\partial}\bar{F}(x)\right) \langle \mathrm{tr}\, \mathbf{T}(x)\rangle\ +\ \cdots\end{aligned} \tag{A.10}$$

where $\mathrm{tr}\, \mathbf{T} = T^{\mu}_{\ \mu} = T^{\mu\nu} g_{\nu\mu} = T^{z\bar{z}} = T^{\bar{z}z}$.

Conformal invariance in 2D implies that $T^{z\bar{z}} = T^{\bar{z}z} = \mathrm{tr}\, \mathbf{T} = 0$ identically vanishes. For a quantum theory (a CFT) this requires a proper definition of the renormalized stress energy-tensor, and this identity is valid up to very specific contact terms. The law of conservation for the current $\partial_\mu T^{\mu\nu} = 0$ reduces to $\bar{\partial} T = 0$, $\partial \bar{T} = 0$, hence the terminology holomorphic and antiholomorphic components. This is valid for a CFT in a flat metric.

For a 2D CFT defined one a general surface with a non flat metric g, one can still use conformal coordinates where the metric reads $ds^2 = \rho(z,\bar{z})\, dz d\bar{z}$, so that the analyticity property of T and $\bar{T}$ are preserved. ρ is the conformal factor of the metric. A most important property is that the trace of the stress-energy tensor does not vanish anymore. Its expectation value is given by the *trace anomaly*

$$\langle \mathrm{tr}\, \mathbf{T}(x)\rangle = g_{\mu\nu}(x) \langle T^{\mu\nu}(x)\rangle = \frac{c}{24\,\pi}\, R_{\mathrm{scal}}(x) \tag{A.11}$$

where $R_{\mathrm{scal}}(x)$ is the local scalar curvature of the metric, with c the *central charge* of the theory. The trace anomaly is a quantum effect, caused by short distance quantum fluctuations and renormalization effects. See e.g. [Fri84] for a derivation.

Finally, another very important feature of 2D CFT is the *short distance operator product expansion* (OPE) for the stress energy tensor, which takes the form

$$T(z)\, T(z') = \frac{c}{2} \frac{1}{(z-z')^4}\ + \text{subdominant terms} \tag{A.12}$$

with again c the *central charge* of the considered CFT. A.11 and A.12 are of course not unrelated.

For a discrete statistical model, corresponding to a lattice regularized QFT, conformal invariance is expected only at a critical point and in the large distance scaling limit (a famous example is the Ising model). The scaling limit of the model corresponds to a CFT. The discretized stress-energy tensor $\mathbf{T}_{\mathrm{reg.}}$ can be defined, but it contains in general short distance UV divergent terms, proportional to negatives powers and logarithms of the short distance regulator a (the lattice mesh) or powers the high momentum/energy cut-off $\Lambda \sim 1/a$. By dimensional analysis

$$\mathbf{T}_{\mathrm{reg.}} \propto \Lambda^2 \sim a^{-2} \tag{A.13}$$

The definition of the continuum limit $a \to 0$ ($\Lambda \to \infty$) requires a renormalization prescription in order to define a renormalized stress-energy tensor $\mathbf{T}$ with the correct properties for conformal invariance (OPE, trace anomaly).

A.3. **The two dimensional boson and the Δ theory.** Finally we recall that our results for the Laplace-Beltrami operator Δ can be interpreted in the framework of the standard free boson CFT with central charge $c = 1$. Indeed, classically, the action S_{boson} and the stress energy tensor for the free boson are (on a closed Riemannian manifold)

$$S_{\text{boson}}[\phi] = \frac{1}{2}\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\partial_\mu\phi\, g^{\mu\nu}\,\partial_\nu\phi = \frac{1}{2}\int_M d^2x\,\sqrt{g}\,\phi(x)\Delta_g\phi(x), \tag{A.14}$$

with stress-energy tensor

$$T^{\mu\nu} = \Big(-\frac{1}{2}\,g^{\mu\nu}g^{\rho\sigma} + g^{\rho\mu}g^{\sigma\nu}\Big)\partial_\rho\phi\,\partial_\sigma\phi \tag{A.15}$$

In two dimensional flat space, using complex coordinates, $\Delta_g = -4\,\partial\bar\partial$. The action and the components of the stress-energy tensor are

$$S_{\text{boson}}[\phi] = 2\int d^2x\ \partial\phi\,\bar\partial\phi \tag{A.16}$$

$$T = -2\pi(\partial\phi)^2\ ,\quad \bar T = -2\pi(\bar\partial\phi)^2\ ,\quad \mathrm{tr}\,\mathbf{T} = T^{z\bar z} = T^{\bar z z} = 0 \tag{A.17}$$

The last identity shows that 2d free boson is indeed conformally invariant. The partition function for the boson is related to the determinant of Δ_g by the functional integral

$$Z_{\text{boson}} = \int \mathcal{D}[\phi]\,\mathrm{e}^{-S[\phi]} = \det(\Delta_g)^{-1/2} \tag{A.18}$$

with "det" the properly defined functional determinant (taking into account the normalization problems and the treatement of the zero mode).

Formally $\det(\Delta_g) = Z_{\text{boson}}^{-2}$ is the partition function of the "$n = -2$ components" free boson CFT, with $c = -2$. Equivalently, a standard trick is to write $\det(\Delta_g)$ as the partition function of a theory for a *scalar complex Grassmann field* (a spin zero field obeying Fermi-Dirac statistics) described by a pair of conjugate Grassmann (anti-commuting) fields $(\Phi, \bar\Phi)$, where the $\Phi(x)$'s and $\bar\Phi(x)$'s are the generators of an infinite dimensional Grassmann (or exterior) algebra. The partition function Z_Δ is given by the Berezin functional integral (see e.g. [DFMS97], [DEF+99] and more seriously [Ber66]) written (using the Berezin integration rules)

$$Z_\Delta = \det\Delta_g = \int \mathcal{D}[\Phi,\bar\Phi]\,\mathrm{e}^{-S[\Phi,\bar\Phi]}\ ,\qquad \mathcal{D}[\Phi,\bar\Phi] = \prod_x d\Phi(x)d\bar\Phi(x) \tag{A.19}$$

with the action S_Δ (here a degree 2 element of the Grassmann algebra) which is simply the Grassmann version of the action for a complex bosonic scalar field

$$S_\Delta[\Phi,\bar\Phi] = 4\int d^2x\ \partial\Phi\,\bar\partial\bar\Phi = \int d^2x\ \Phi{\cdot}\Delta_g\bar\Phi \tag{A.20}$$

Of course, unlike the bosonic case, the Berezin functional integral cannot be thought in term of probabilistic averages over random real or complex fields "living" on a space-time manifold, but as an algebraic construction. In the fermionic theory, the two point functions (the propagator) are (note the anti-commutivity)

$$\langle\bar{\Phi}(x)\Phi(y)\rangle = -\langle\Phi(x)\bar{\Phi}(y)\rangle = \left[\Delta_g^{-1}\right]_{xy} , \quad \langle\Phi(x)\Phi(y)\rangle = \langle\bar{\Phi}(x)\bar{\Phi}(y)\rangle = 0 \tag{A.21}$$

The stress energy-tensor components are

$$T_{\scriptscriptstyle\Delta} = -4\pi\, \partial\Phi\, \partial\bar{\Phi} , \quad \bar{T}_{\scriptscriptstyle\Delta} = -4\pi\, \bar{\partial}\Phi\, \bar{\partial}\bar{\Phi} , \quad \mathrm{tr}\, \mathbf{T}_{\scriptscriptstyle\Delta} = 0 \tag{A.22}$$

As explained in the discussion section 9, our results for the variations of the discretized laplacians Δ, $\underline{\boldsymbol{\Delta}}$ and the Kähler operator $\mathcal{D}$ (defined on a triangulation $\mathfrak{T}$) can be easily formulated in term of discretized stress-energy tensors attached to the faces of $\mathfrak{T}$. However, only for the Laplace-Beltrami operator Δ can the discretized stress energy tensor be given a simple continuum limit formulation as the stress energy tensor of a continuum QFT.

Appendix B. Proof of Lemma 2

Proof. For $j = 2,3$ introduce interpolations $z_j(t) := tz_j + (1-t)z_1$ between z_j and z_1. In addition set $z(s,t) := sz_3(t) + (1-s)z_2(t)$. We start from the definition of ∇

$$\nabla\phi(\mathtt{f}) \;=\; \frac{\big[\phi(z_2)-\phi(z_1)\big]\big[\overline{z}_3-\overline{z}_1\big] \;-\; \big[\phi(z_3)-\phi(z_1)\big]\big[\overline{z}_2-\overline{z}_1\big]}{-4\mathrm{i}A(\mathtt{f})} \tag{B.1}$$

where by formula 3.6 we have for the area of the triangle $\mathtt{f}$

$$4\,A(\mathtt{f}) = |z_1-z_2|\,|z_2-z_3|\,|z_3-z_1|\,/\,R(\mathtt{f}) \tag{B.2}$$

The numerator can be expressed by
(B.3)

$$\begin{aligned}
&\big[\phi(z_2)-\phi(z_1)\big]\big[\overline{z}_3-\overline{z}_1\big] \;-\; \big[\phi(z_3)-\phi(z_1)\big]\big[\overline{z}_2-\overline{z}_1\big]\\
&= \int_0^1 dt\,\frac{d}{dt}\Big[\phi\big(z_2(t)\big)\big[\overline{z}_3-\overline{z}_1\big] \;-\; \phi\big(z_3(t)\big)\big[\overline{z}_2-\overline{z}_2\big]\Big]
\end{aligned}$$

$$= \left\{\begin{array}{lr}
\displaystyle\int_0^1 dt\,\Big[\big[z_2-z_1\big]\big[\overline{z}_3-\overline{z}_1\big]\partial\phi\big(z_2(t)\big) \;-\; \big[z_3-z_1\big]\big[\overline{z}_2-\overline{z}_1\big]\partial\phi\big(z_3(t)\big)\Big] & *\text{-integral}\\
+ & \\
\displaystyle\int_0^1 dt\,\Big[\big[\overline{z}_2-\overline{z}_1\big]\big[\overline{z}_3-\overline{z}_1\big]\Big[\partial\phi\big(z_2(t)\big) \;-\; \partial\phi\big(z_3(t)\big)\Big]\Big] & **\text{-integral}
\end{array}\right.$$

Apply the fundamental theorem of calculus once again, the $*$-integral in B.3 can be expressed as a double integral
(B.4)

$$-\int_0^1\int_0^1 dt\,ds\;\frac{d}{ds}\Big[\partial\phi\big(z(s,t)\big)\Big(s[z_3-z_1]\big[\overline{z}_2-\overline{z}_1\big]+(1-s)[z_2-z_1]\big[\overline{z}_3-\overline{z}_1\big]\Big)\Big]$$

$$= \left\{\begin{array}{c}
\displaystyle\int_0^1\int_0^1 dt\,ds\;\partial\phi\big(z(s,t)\big)\underbrace{\Big([z_2-z_1]\big[\overline{z}_3-\overline{z}_1\big]-[z_3-z_1]\big[\overline{z}_2-\overline{z}_1\big]\Big)}_{=\;-4\mathrm{i}A(\mathtt{f})}\\
+\\
\displaystyle\int_0^1\int_0^1 t\,dt\,ds\;\partial\partial\phi\big(z(s,t)\big)[z_2-z_3]\Big(s[z_3-z_1]\big[\overline{z}_2-\overline{z}_1\big]+(1-s)[z_2-z_1]\big[\overline{z}_3-\overline{z}_1\big]\Big)\\
+\\
\displaystyle\int_0^1\int_0^1 t\,dt\,ds\;\partial\overline{\partial}\phi\big(z(s,t)\big)\big[\overline{z}_2-\overline{z}_3\big]\Big(s[z_3-z_1]\big[\overline{z}_2-\overline{z}_1\big]+(1-s)[z_2-z_1]\big[\overline{z}_3-\overline{z}_1\big]\Big)
\end{array}\right.$$

Dividing the $*$-integral in B.3 by $(-4\mathfrak{Im}A(\mathtt{f}))$ we obtain a first contribution to $\nabla\phi(\mathtt{f})$, namely

(B.5)

$$\begin{cases} \displaystyle\int_0^1\int_0^1 dt\,ds\ \partial\phi\big(z(s,t)\big) \\ + \\ \mathrm{i}R(\mathtt{f})\displaystyle\int_0^1\int_0^1 tdt\,ds\ \partial\partial\phi\big(z(s,t)\big)\frac{z_2-z_3}{|z_2-z_3|}\left(s\frac{z_3-z_1}{|z_3-z_1|}\frac{\overline{z}_2-\overline{z}_1}{|z_2-z_1|}+(1-s)\frac{z_2-z_1}{|z_2-z_1|}\frac{\overline{z}_3-\overline{z}_1}{|z_3-z_1|}\right) \\ + \\ \mathrm{i}R(\mathtt{f})\displaystyle\int_0^1\int_0^1 tdt\,ds\ \partial\overline{\partial}\phi\big(z(s,t)\big)\frac{\overline{z}_2-\overline{z}_3}{|z_2-z_3|}\left(s\frac{z_3-z_1}{|z_3-z_1|}\frac{\overline{z}_2-\overline{z}_1}{|z_2-z_1|}+(1-s)\frac{z_2-z_1}{|z_2-z_1|}\frac{\overline{z}_3-\overline{z}_1}{|z_3-z_1|}\right) \end{cases}$$

Again, by the fundamental theorem of calculus, we can transform the $**$-integral B.3 and obtain

(B.6)

$$\begin{aligned} &\int_0^1 dt\left(\big[\overline{z}_2-\overline{z}_1\big]\big[\overline{z}_3-\overline{z}_1\big]\Big[\partial\phi\big(z_2(t)\big)\ -\ \partial\phi\big(z_3(t)\big)\Big]\right) \\ &= -\big[\overline{z}_2-\overline{z}_1\big]\big[\overline{z}_3-\overline{z}_1\big]\int_0^1\int_0^1 dt\,ds\,\frac{d}{ds}\left(\partial\phi\big(z(s,t)\big)\right) \\ &= \big[\overline{z}_2-\overline{z}_1\big]\big[\overline{z}_3-\overline{z}_1\big]\int_0^1\int_0^1 t\,dt\,ds\left([z_2-z_3]\,\partial\overline{\partial}\phi\big(z(s,t)\big)+[\overline{z}_2-\overline{z}_3]\,\overline{\partial\partial}\phi\big(z(s,t)\big)\right) \end{aligned}$$

Dividing the $**$-integral in B.3 by $(-4\mathfrak{Im}A(\mathtt{f}))$ we obtain a second contribution to $\nabla\phi(\mathtt{f})$, namely

(B.7)

$$\mathrm{i}R(\mathtt{f})\,\frac{\overline{z}_2-\overline{z}_1}{|z_2-z_1|}\frac{\overline{z}_3-\overline{z}_1}{|z_3-z_1|}\int_0^1\int_0^1 t\,dt\,ds\left(\frac{z_2-z_3}{|z_2-z_3|}\,\partial\overline{\partial}\phi\big(z(s,t)\big)+\frac{\overline{z}_2-\overline{z}_3}{|z_2-z_3|}\,\overline{\partial\partial}\phi\big(z(s,t)\big)\right)$$

So we end up with

(B.8)

$$\nabla\phi(\mathtt{f})-\int_0^1\int_0^1 dt\,ds\,\partial\phi\big(z(s,t)\big)$$

$$= \begin{cases} \mathrm{i}R(\mathtt{f})\displaystyle\int_0^1\int_0^1 t\,dt\,ds\ \partial\partial\phi\big(z(s,t)\big)\frac{z_2-z_3}{|z_2-z_3|}\left(s\frac{z_3-z_1}{|z_3-z_1|}\frac{\overline{z}_2-\overline{z}_1}{|z_2-z_1|}+(1-s)\frac{z_2-z_1}{|z_2-z_1|}\frac{\overline{z}_3-\overline{z}_1}{|z_3-z_1|}\right) \\ + \\ \mathrm{i}R(\mathtt{f})\displaystyle\int_0^1\int_0^1 t\,dt\,ds\ \partial\overline{\partial}\phi\big(z(s,t)\big)\frac{\overline{z}_2-\overline{z}_3}{|z_2-z_3|}\left(s\frac{z_3-z_1}{|z_3-z_1|}\frac{\overline{z}_2-\overline{z}_1}{|z_2-z_1|}+(1-s)\frac{z_2-z_1}{|z_2-z_1|}\frac{\overline{z}_3-\overline{z}_1}{|z_3-z_1|}\right) \\ + \\ \mathrm{i}R(\mathtt{f})\,\dfrac{\overline{z}_2-\overline{z}_1}{|z_2-z_1|}\dfrac{\overline{z}_3-\overline{z}_1}{|z_3-z_1|}\displaystyle\int_0^1\int_0^1 t\,dt\,ds\left(\frac{z_2-z_3}{|z_2-z_3|}\,\partial\overline{\partial}\phi\big(z(s,t)\big)+\frac{\overline{z}_2-\overline{z}_3}{|z_2-z_3|}\,\overline{\partial\partial}\phi\big(z(s,t)\big)\right) \end{cases}$$

Thus we can bound the norm of the r.h.s. of B.8 by

$$R(\mathtt{f})\int_0^1\int_0^1 t\,dt\,ds\,\big(|\partial\partial\phi(z(s,t)|+2|\partial\bar{\partial}\phi(z(s,t)|+|\bar{\partial}\bar{\partial}\phi(z(s,t)|\big) \tag{B.9}$$

Thus we have

(B.10)
$$\left|\nabla\phi(\mathtt{f}) - \int_0^1\int_0^1 dt\, ds\, \partial\phi\big(z(s,t)\big)\right| \le R(\mathtt{f})\left(\frac{1}{2}\sup_{z\in\mathtt{f}}\left|\partial\partial\phi(z)\right| + \sup_{z\in\mathtt{f}}\left|\partial\bar\partial\phi(z)\right| + \frac{1}{2}\sup_{z\in\mathtt{f}}\left|\overline{\partial\partial}\phi(z)\right|\right)$$

Finally we come to bound the difference between the $\partial\phi\big(z(s,t)\big)$ and $\partial\phi(z_{\mathtt{f}})$ where $z_{\mathtt{f}}$ is the circumcenter of $\mathtt{f}$. Again, by the fundamental theorem of calculus, defining

$$z(p,s,t) = p\, z(s,t) + (1-p)z_{\mathtt{f}}$$

we write

$$\begin{aligned}\partial\phi\big(z(s,t)\big) - \partial\phi(z_{\mathtt{f}}) &= \int_0^1 dp\, \frac{d}{dp}\partial\phi\big(z(p,s,t)\big)\\ &= \int_0^1 dp\, \big((z(s,t)-z_{\mathtt{f}})\partial\partial\phi(z(p,s,t)) + (\bar z(s,t)-\bar z_{\mathtt{f}})\partial\bar\partial\phi(z(p,s,t))\big)\end{aligned} \tag{B.11}$$

Since $z(s,t)$ is inside the triangle $\mathtt{f}$, it is also in the disk $B_{\mathtt{f}}$ of radius $R(\mathtt{f})$ with center $z_{\mathtt{f}}$, hence $|z(s,t)-z_{\mathtt{f}}| \le R(\mathtt{f})$ and we get the bound

$$|\partial\phi\big(z(s,t)\big) - \partial\phi(z_{\mathtt{f}})| \le R(\mathtt{f})\left(\sup_{z\in B_{\mathtt{f}}}\left|\partial\partial\phi(z)\right| + \sup_{z\in B_{\mathtt{f}}}\left|\partial\bar\partial\phi(z)\right|\right) \tag{B.12}$$

which when averaged becomes

(B.13)
$$\left|\int_0^1\int_0^1 dt\, ds\, \partial\phi\big(z(s,t)\big) - \partial\phi(z_{\mathtt{f}})\right| \le R(\mathtt{f})\left(\sup_{z\in B_{\mathtt{f}}}\left|\partial\partial\phi(z)\right| + \sup_{z\in B_{\mathtt{f}}}\left|\partial\bar\partial\phi(z)\right|\right)$$

Combining the bounds B.10 and B.13 we get the final result of lemma 2

$$\left|\nabla\phi(\mathtt{f}) - \partial\phi(z_{\mathtt{f}})\right| \le R(\mathtt{f})\left(\frac{3}{2}\sup_{z\in B_{\mathtt{f}}}\left|\partial^2\phi\right| + 2\sup_{z\in B_{\mathtt{f}}}\left|\partial\bar\partial\phi\right| + \frac{1}{2}\sup_{z\in B_{\mathtt{f}}}\left|\bar\partial^2\phi\right|\right) \tag{B.14}$$

□

Remark 32. *For a general point $w \in B_{\mathtt{f}}$ we have $|z(s,t) - w| \le 2R(\mathtt{f})$ and after modifying our estimates by a factor of 2 we obtain*

$$\left|\nabla\phi(\mathtt{f}) - \partial\phi(w)\right| \le R(\mathtt{f})\left(\frac{5}{2}\sup_{z\in B_{\mathtt{f}}}\left|\partial^2\phi\right| + 3\sup_{z\in B_{\mathtt{f}}}\left|\partial\bar\partial\phi\right| + \frac{1}{2}\sup_{z\in B_{\mathtt{f}}}\left|\bar\partial^2\phi\right|\right) \tag{B.15}$$

Appendix C. Continuum Limits of Anomalies: Example arising from a bi-periodic tiling of the plane using a single cyclic quadrilateral

In this appendix we present an example of an isoradial Delaunay graph $\mathbf{G}_{\mathrm{cr}}$ for which the chord-to-chord $\mathbb{A}^{\mathrm{ch}\times\mathrm{ch}}$ and edge-to-chord $\mathbb{A}^{\mathrm{ed}\times\mathrm{ch}}$ anomalous terms of the associated conformal Laplacian (as explained in formulae 6.58 and 6.59 of Section 6.3) have well-defined, non-trivial $\ell \to \infty$ scaling limits. Unlike the continuum limits addressed in Corollary 1.18, the anomalous limit values computed in Claim 2 of this section reflect features of the underlying geometry of the initial critical graph $\mathbf{G}_{\mathrm{cr}}$ — specifically the choice of fundamental quadrilateral $\mathcal{Q}$ used to construct $\mathbf{G}_{\mathrm{cr}}$.

Begin with four angles $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ in the interval $[0, 2\pi)$ and construct the cyclic quadrilateral $\mathcal{Q}$ whose vertices are the unit complex numbers $\mathfrak{z}_k := \exp(\mathrm{i}\alpha_k)$ with $k \in \{1, 2, 3, 4\}$. We will require that the origin is contained in the interior of $\mathcal{Q}$; achieved whenever $\alpha_3 - \alpha_1 > \pi$ or $\alpha_4 - \alpha_2 > \pi$. This constraint is to insure that the tiling we are about to construct is Delaunay. Let $\mathcal{Q}^{\mathrm{op}}$ denote the quadrilateral obtained by rotating $\mathcal{Q}$ by 180 degrees. A cyclic quadrilateral with associated angles $\alpha_1 = \pi/3$, $\alpha_2 = 5\pi/7$, $\alpha_3 = 13\pi/9$, and $\alpha_4 = 21\pi/11$ is illustrated in Figure 32.

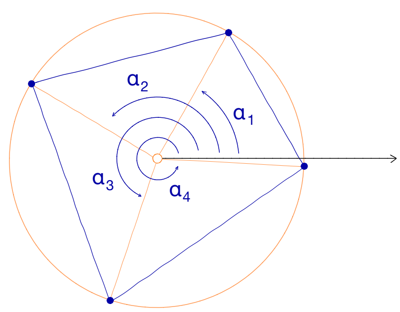

Figure 32. Fundamental quadrilateral $\mathcal{Q}$

Construct a doubly periodic, quadrilateral tiling $\mathbf{G}_{\mathrm{cr}}$ of the plane using translations of $\mathcal{Q}$ and $\mathcal{Q}^{\mathrm{op}}$. Clearly $\mathbf{G}_{\mathrm{cr}}$ will be isoradial and Delaunay in the sense of Section 2.1; by construction each face of $\mathbf{G}_{\mathrm{cr}}$ is a cyclic quadrilateral. Figure 33 depicts such a tiling.

For each quadrilateral face q of $\mathbf{G}_{\mathrm{cr}}$ let z_{q} denote the complex coordinate of its center; with respect to this center, the four vertices $\mathrm{v}_{\mathsf{q}}(k)$ of q, with $k \in \{1, 2, 3, 4\}$, have complex coordinates $z(\mathrm{v}_{\mathsf{q}}(k)) = z_{\mathsf{q}} \pm \mathfrak{z}_k$ where the sign is $+$ if q is a translation of $\mathcal{Q}$ and $-$ if q is a translation of $\mathcal{Q}^{\mathrm{op}}$. Let $\mathrm{e}_{\mathsf{q}}^{+}$ denote the chord of the quadrilateral q joining vertices $\mathrm{v}_{\mathsf{q}}(2)$ and $\mathrm{v}_{\mathsf{q}}(4)$ while $\mathrm{e}_{\mathsf{q}}^{-}$ will denote the chord joining $\mathrm{v}_{\mathsf{q}}(1)$ and $\mathrm{v}_{\mathsf{q}}(3)$. Up to a sign, the corresponding north angles are given by $\vartheta_+ := \alpha_2 - \alpha_4$ and $\vartheta_- := \alpha_1 - \alpha_3$ respectively. Define $z_+ := \mathfrak{z}_2 - \mathfrak{z}_4$ and $z_- := \mathfrak{z}_1 - \mathfrak{z}_3$. Let $A_{\mathcal{Q}}$ denote the area of $\mathcal{Q}$.

Let $F(z)$ be a smooth complex-valued function with compact support together with deformation and scaling parameter values $\epsilon > 0$ and $\ell > 0$. Let $T_{\epsilon,\ell}$ denote graph obtained by deforming the embedding of $\mathbf{G}_{\mathrm{cr}}$ by the perturbation $z \mapsto z +$

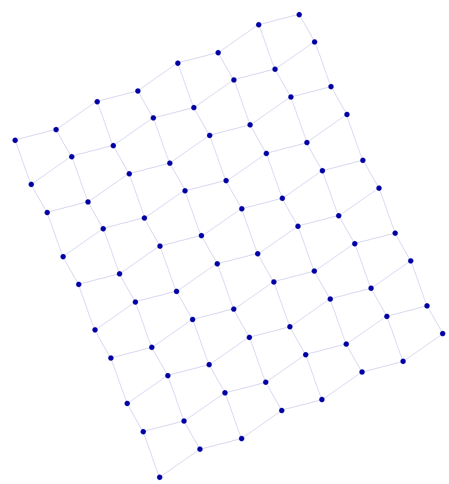

FIGURE 33. Fragment of a tiling $\mathtt{G}_{\mathrm{cr}}$ by a cyclic quadrilateral $\mathtt{q}$

$\epsilon\,\ell F(z/\ell)$ and by adjoining the edge $\mathrm{e}_{\mathtt{q}}^{+}$ or $\mathrm{e}_{\mathtt{q}}^{-}$ to each quadrilateral face $\mathtt{q}$ of $\mathtt{G}_{\mathrm{cr}}$ according to whether $\theta_{\epsilon,\ell}(\mathrm{e}_{\mathtt{q}}^{+}) > 0$ or $\theta_{\epsilon,\ell}(\mathrm{e}_{\mathtt{q}}^{-}) > 0$; these conditions are mutually exclusive, as the signs of $\theta_{\epsilon,\ell}(\mathrm{e}_{\mathtt{q}}^{+})$ and $\theta_{\epsilon,\ell}(\mathrm{e}_{\mathtt{q}}^{-})$ are opposite. Neither edge is selected if both conformal angles are zero. As long as $\epsilon > 0$ lies within the range $0 < \epsilon < \epsilon_F$ as prescribed by Claim **??** the graph $\mathtt{G}_{\epsilon,\ell}$ will remain Delaunay.

As an example consider the following "mollified" shear of $\mathtt{G}_{\mathrm{cr}}$. For simplicity we consider the case where the support of F has **one** connected component (in particular, it is a disk $\mathbb{D}$ with unit radius):

$$F(z) \;:=\; \begin{cases} \exp\left(i\phi + \dfrac{|z|^2}{|z|^2-1}\right)\mathfrak{Im}[z] & \text{if } |z| \le 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Figure 34 depicts the effect of of the corresponding deformation $z \mapsto z + \epsilon\ell F(z/\ell)$. The reader will notice that the support of $F_{(\ell)} : z \mapsto \ell F(z/\ell)$ is partitioned roughly into three "unidirectional" zones consisting of deformed quadrilaterals whose diagonals share the same alignment. In general, for any smooth compactly supported perturbation $z \mapsto z + \epsilon\,\ell F(z/\ell)$, the support of F_ℓ will be partitioned into such zones of constant alignment. If we ignore the quadrilaterals $\mathtt{q}$ for which $\theta'_{0,\ell}(\mathrm{e}_{\mathtt{q}}^{+})$ vanishes then the remaining set of quadrilaterals can be partitioned into zones over which the sign of $\theta'_{0,\ell}(\mathrm{e}_{\mathtt{q}}^{+})$ is constant. For $\ell >> 0$ large, the interfaces between these zones approximate the level curves of $\mathfrak{Im}\big[\,\overline{\partial}F_{(\ell)}\,\mathcal{E}\big] = 0$ within the disk $\mathbb{D}_\ell$ of radius ℓ where

$$\mathcal{E} := \mathfrak{e}_{12} - \mathfrak{e}_{23} + \mathfrak{e}_{34} - \mathfrak{e}_{14} \quad \text{and} \quad \mathfrak{e}_{mn} := \frac{\bar{\mathfrak{z}}_m - \bar{\mathfrak{z}}_n}{\mathfrak{z}_m - \mathfrak{z}_n} \quad \text{for } m, n \in \{1, 2, 3, 4\}.$$

This convergence is a manifestation of the existence of the scaling limit of the anomaly formalized in Claims 1 and 2. In the case of the mollified-shear example the corresponding level curves are depicted in red by Figure 34.

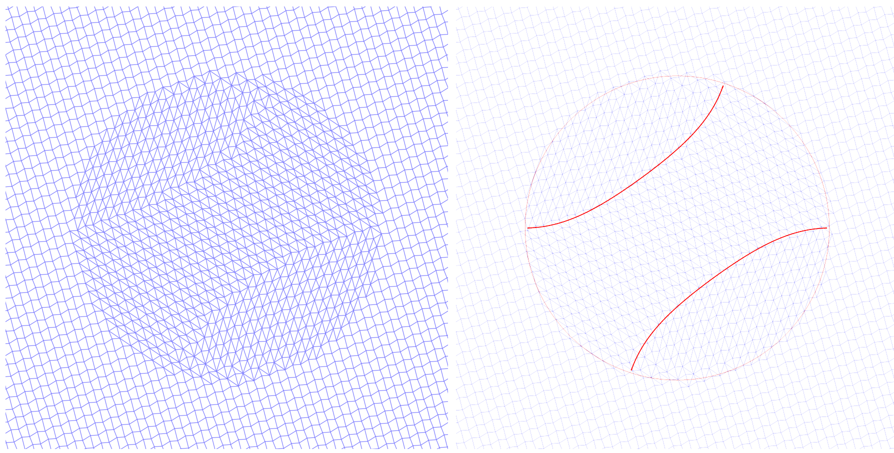

FIGURE 34. mollified-shear with angle value $\phi = -\frac{\pi}{5}$, deformation parameter value $\epsilon = 0.1$, and scaling parameter value $\ell = 22$

In order to analyze the anomalous terms arising in the second order variation of the conformal Laplacian the reader will recall that we use a bi-local perturbation: Begin with two smooth complex-valued functions $F_1(z)$ and $F_2(z)$ whose supports Ω_1 and Ω_2 are compact and disjoint together with two deformation parameters $\epsilon_1, \epsilon_2 > 0$. We consider the smooth function $F(z)$ obtained by superimposing $F_1(z)$ and $F_2(z)$, i.e.

$$F(z) := \begin{cases} F_1(z) & \text{if } z \in \Omega_1 \\ F_2(z) & \text{if } z \in \Omega_2 \\ 0 & \text{otherwise} \end{cases} \tag{C.1}$$

We economize the notation for the bi-local perturbation and write $z \mapsto z + \epsilon\ell F(z/\ell)$ where $\epsilon = \epsilon_j$ depending on whether or not $z/\ell \in \Omega_j$.

Given $p \in \mathbb{C}$ and a value of the scaling parameter $\ell > 0$ center a copy of the fundamental quadrilateral $\mathcal{Q}$ about the dialated point $\ell p \in \mathbb{C}$. The coordinates of its vertices are $q_\ell(p;k) = \ell p + \mathfrak{z}_k$ for $k \in \{1, 2, 3, 4\}$. The perturbation will displace these vertices by $q_\ell(p;k) \mapsto q_{\epsilon,\ell}(p;k)$ where $q_{\epsilon,\ell}(p;k) = q_\ell(p;k) + \epsilon\ell F\big(q_\ell(p;k)/\ell\big)$. The conformal angle $\kappa_{\epsilon,\ell}(p)$, its ϵ-derivative $\kappa'_{0,\ell}(p)$, and its infinitesimal conformal angle $\kappa_{\epsilon,\infty}(p)$ at $p \in \mathbb{C}$ are accordingly defined by:

$$\kappa_{\epsilon,\ell}(p) \;=\; \mathfrak{Im} \log \left[\frac{\big(q_{\epsilon,\ell}(p;4) - q_{\epsilon,\ell}(p;3)\big)\,\big(q_{\epsilon,\ell}(p;2) - q_{\epsilon,\ell}(p;1)\big)}{\big(q_{\epsilon,\ell}(p;4) - q_{\epsilon,\ell}(p;1)\big)\,\big(q_{\epsilon,\ell}(p;2) - q_{\epsilon,\ell}(p;3)\big)} \right] \tag{C.2}$$

$$\kappa'_{0,\ell}(p) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \kappa_{\varepsilon,\ell}(p)$$

$$= \begin{cases} \mathfrak{Im}\left[\overline{\nabla} F_{(\ell)}\Big(\ell p + \mathfrak{z}_1, \ell p + \mathfrak{z}_2, \ell p + \mathfrak{z}_4\Big)\big(\mathfrak{e}_{12} - \mathfrak{e}_{14}\big)\right] \\ + \\ \mathfrak{Im}\left[\overline{\nabla} F_{(\ell)}\Big(\ell p + \mathfrak{z}_2, \ell p + \mathfrak{z}_3, \ell p + \mathfrak{z}_4\Big)\big(\mathfrak{e}_{34} - \mathfrak{e}_{23}\big)\right] \end{cases} \tag{C.3}$$

$$= \mathfrak{Im}\Big[\overline{\partial} F(p)\,\mathcal{E}\Big] + \mathrm{O}\big(1/\ell\big)$$

$$\kappa_{\epsilon,\infty}(p) = \lim_{l\to\infty} \mathfrak{Im}\log\left[\frac{\big(q_{\epsilon,\ell}(p;4) - q_{\epsilon,\ell}(p;3)\big)\,\big(q_{\epsilon,\ell}(p;2) - q_{\epsilon,\ell}(p;1)\big)}{\big(q_{\epsilon,\ell}(p;4) - q_{\epsilon,\ell}(p;1)\big)\,\big(q_{\epsilon,\ell}(p;2) - q_{\epsilon,\ell}(p;3)\big)}\right]$$

$$= \mathfrak{Im}\,\log\left[\frac{\left(1 + \mathfrak{e}_{34}\dfrac{\epsilon\overline{\partial}F(p)}{1+\epsilon\partial F(p)}\right)\left(1 + \mathfrak{e}_{12}\dfrac{\epsilon\overline{\partial}F(p)}{1+\epsilon\partial F(p)}\right)}{\left(1 + \mathfrak{e}_{14}\dfrac{\epsilon\overline{\partial}F(p)}{1+\epsilon\partial F(p)}\right)\left(1 + \mathfrak{e}_{23}\dfrac{\epsilon\overline{\partial}F(p)}{1+\epsilon\partial F(p)}\right)}\right] \tag{C.4}$$

$$= \epsilon\,\mathfrak{Im}\Big[\overline{\partial} F(p)\,\mathcal{E}\Big] + \mathrm{O}(\epsilon^2)$$

Claim 1. *Fix a value of the scaling parameter $\ell > 0$, then for any pair of points $p, z \in \mathrm{supp}F$ with $|z-p| < 1/\ell$*

$$\left|\kappa'_{0,\ell}(z) - \mathfrak{Im}\Big[\overline{\partial} F(p)\mathcal{E}\Big]\right| \le 4/\ell\, M(z,\ell) \quad \text{where} \tag{C.5}$$

(C.6)
$$M(z,\ell) := \max_{|w-z|<1/\ell}\big|\partial^2 F(w)\big| + 2\max_{|w-z|<1/\ell}\big|\partial\overline{\partial} F(w)\big| + \max_{|w-z|<1/\ell}\big|\overline{\partial}^2 F(w)\big|$$

Definition 19. *For a fixed value of the scaling parameter $\ell > 0$ and any (continuous) function $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ let us introduce the following piecewise abridgment*

$$\langle\phi\rangle_\ell(p) := \begin{cases} \phi\big(z_{\mathsf{q}}/\ell\big) & \textit{whenever } \ell p \in \mathrm{int}\big(\mathsf{q}\big) \textit{ for a quadrilateral } \mathsf{q} \\[2ex] \dfrac{1}{2}\displaystyle\sum_{k=1}^{2}\phi\big(z_{\mathsf{q}_k}/\ell\big) & \textit{whenever } \ell p \in \mathrm{int}\big(\partial\mathsf{q}_1 \cap \partial\mathsf{q}_2\big) \textit{ for a pair of quadrilaterals } \mathsf{q}_1 \textit{ and } \mathsf{q}_2 \\[2ex] \dfrac{1}{4}\displaystyle\sum_{k=1}^{4}\phi\big(z_{\mathsf{q}_k}/\ell\big) & \textit{whenever } \ell p \in \partial\mathsf{q}_1 \cap \partial\mathsf{q}_2 \cap \partial\mathsf{q}_3 \cap \partial\mathsf{q}_4 \textit{ for quadrilaterals } \mathsf{q}_1,\ \mathsf{q}_2,\ \mathsf{q}_3, \textit{ and } \mathsf{q}_4 \end{cases} \tag{C.7}$$

Remark 33. *Let* $\chi_\ell := \big\langle \kappa'_{0,\ell} \big\rangle_\ell$ *then* $\chi_\ell \longrightarrow \mathfrak{Im}\big[\,\overline{\partial} F \cdot \mathcal{E}\big]$ *uniformly as* $\ell \to \infty$. *Furthermore* $\chi^\pm_\ell \longrightarrow \mathfrak{Im}^\pm\big[\,\overline{\partial} F \cdot \mathcal{E}\big]$ *uniformly as* $\ell \to \infty$ *where* $g^+(p) := \max\,(g(p), 0)$ *and* $g^-(p) := -\min\,(g(p), 0)$ *for any real-valued function* $g : \mathbb{C} \longrightarrow \mathbb{R}$.

Claim 2. *For signs* $\sigma, \tau \in \{+, -\}$ *define*

(C.8)

$$J^{(\sigma,\tau)} := \frac{\tan^2\vartheta_\sigma \tan^2\vartheta_\tau}{16\pi^2 A_{\mathcal{Q}}^2} \iint\limits_{\Omega_1\times\Omega_2} d^2x\, d^2y\ \mathfrak{Im}^\sigma\Big[\,\overline{\partial} F(x)\,\mathcal{E}\Big]\left[\mathfrak{Re}\,\frac{z_\sigma\, z_\tau}{\big(x-y\big)^2}\right]^2 \mathfrak{Im}^\tau\Big[\,\overline{\partial} F(y)\,\mathcal{E}\Big]$$

$$J^{(1)}_\sigma := \frac{\tan^2\vartheta_\sigma}{8\pi^2 A_{\mathcal{Q}}} \iint\limits_{\Omega_1\times\Omega_2} d^2x\, d^2y\ \mathfrak{Im}^\sigma\Big[\,\overline{\partial} F(x)\,\mathcal{E}\Big]\,\mathfrak{Re}\left[\frac{z_\sigma^2\,\overline{\partial} F(y)}{\big(x-y\big)^4}\right]$$

$$J^{(2)}_\sigma := \frac{\tan^2\vartheta_\sigma}{8\pi^2 A_{\mathcal{Q}}} \iint\limits_{\Omega_1\times\Omega_2} d^2x\, d^2y\ \mathfrak{Re}\left[\frac{\overline{\partial} F(x)\, z_\sigma^2}{\big(x-y\big)^4}\right]\mathfrak{Im}^\sigma\Big[\,\overline{\partial} F(y)\,\mathcal{E}\Big]$$

The continuum limits of the edge-to-chord $\mathbb{A}^{\mathrm{ed}\times\mathrm{ch}}_\ell$, *chord-to-edge* $\mathbb{A}^{\mathrm{ch}\times\mathrm{ed}}_\ell$, *and chord-to-chord* $\mathbb{A}^{\mathrm{ch}\times\mathrm{ch}}_\ell$ *anomalies exist and their values are:*

$$\begin{aligned} \lim_{\ell\to\infty} \mathbb{A}^{\mathrm{ed}\times\mathrm{ch}}_\ell &= J^{(2)}_+ + J^{(2)}_- \\ \lim_{\ell\to\infty} \mathbb{A}^{\mathrm{ch}\times\mathrm{ed}}_\ell &= J^{(1)}_+ + J^{(1)}_- \\ \lim_{\ell\to\infty} \mathbb{A}^{\mathrm{ch}\times\mathrm{ch}}_\ell &= J^{(+,+)} + J^{(+,-)} + J^{(-,+)} + J^{(-,-)} \end{aligned} \tag{C.9}$$

Proof. We'll verify the claim in the case of the chord-to-chord anomaly $\mathbb{A}^{\mathrm{ch}\times\mathrm{ch}}_\ell$ and leave the remaining cases to the reader. Begin with a pair of signs $\sigma, \tau \in \{\pm\}$. For $(x, y) \in \Omega_1 \times \Omega_2$ let's introduce the following step-function

(C.10)

$$\Phi^{\sigma,\tau}_\ell(x,y) := \begin{cases} \Big[\kappa'_{0,\ell}(z_{\mathtt{x}}/\ell)\Big]^\sigma \cdot \left[\mathfrak{Re}\,\dfrac{z_\sigma\, z_\tau}{\big(z_{\mathtt{x}}-z_{\mathtt{y}}\big)^2}\right]^2 \cdot \Big[\kappa'_{0,\ell}(z_{\mathtt{y}}/\ell)\Big]^\tau & \begin{array}{l}\ell x \in \mathrm{int}(\mathtt{x}) \\ \ell y \in \mathrm{int}(\mathtt{y}) \\ \mathtt{x}, \mathtt{y} \in \mathrm{F}(\mathbf{G}_{\mathrm{cr}})\end{array} \\ \\ \text{bounded noise} & \text{otherwise} \end{cases}$$

Note that $\mathbb{A}^{\mathrm{ch}\times\mathrm{cr}}_\ell = \mathbb{J}^{(+,+)}_\ell + \mathbb{J}^{(+,-)}_\ell + \mathbb{J}^{(-,+)}_\ell + \mathbb{J}^{(-,-)}_\ell$ where where

$$\mathbb{J}^{(\sigma,\tau)}_\ell = \frac{\tan^2\vartheta_\sigma \tan^2\vartheta_\tau}{16\pi^2} \sum_{\substack{X\,\in\,\mathrm{F}(\mathbf{G}_{\mathrm{cr}}) \\ X\,\cap\,\Omega_1(\ell)\neq\emptyset}} \ \sum_{\substack{\mathtt{y}\in\mathrm{F}(\mathbf{G}_{\mathrm{cr}}) \\ \mathtt{y}\,\cap\,\Omega_2(\ell)\neq\emptyset}} \Phi^{\sigma,\tau}_\ell\Big(z_{\mathtt{x}}/\ell, z_{\mathtt{y}}/\ell\Big) \tag{C.11}$$

It follows from Claim 1 that $\Phi^{\sigma,\tau}_\ell(x,y) \to \Phi^{\sigma,\tau}(x,y)$ converges uniformly on $\Omega_1\times\Omega_2$ as $\ell \to \infty$ where

$$\Phi^{\sigma,\tau}(x,y) \;:=\; \mathfrak{Im}^{\sigma}\Big[\,\overline{\partial}F(x)\,\mathcal{E}\Big]\cdot\left[\mathfrak{Re}\,\frac{z_\sigma\,z_\tau}{\big(x-y\big)^2}\right]^2\cdot\mathfrak{Im}^{\tau}\Big[\,\overline{\partial}F(y)\,\mathcal{E}\Big]$$

$$\begin{aligned}
J^{(\sigma,\tau)} \;&=\; \frac{\tan^2\vartheta_\sigma\,\tan^2\vartheta_\tau}{16\pi^2A_{\mathcal{Q}}^2}\iint\limits_{\Omega_1\times\Omega_2} d^2x\,d^2y\;\Phi^{\sigma,\tau}(x,y)\\
&=\; \frac{\tan^2\vartheta_\sigma\,\tan^2\vartheta_\tau}{16\pi^2A_{\mathcal{Q}}^2}\lim_{\ell\to\infty}\iint\limits_{\Omega_1\times\Omega_2} d^2x\,d^2y\;\Phi_\ell^{\sigma,\tau}(x,y)\\
&=\; \frac{\tan^2\vartheta_\sigma\,\tan^2\vartheta_\tau}{16\pi^2}\lim_{\ell\to\infty}\sum_{\substack{\mathtt{x}\,\in\,\mathrm{F}(\mathfrak{G}_{\mathrm{cr}})\\ \mathtt{x}\,\cap\,\Omega_1(\ell)\neq\emptyset}}\;\sum_{\substack{\mathtt{y}\in\mathrm{F}(\mathfrak{G}_{\mathrm{cr}})\\ \mathtt{y}\,\cap\,\Omega_2(\ell)\neq\emptyset}}\Phi_\ell^{\sigma,\tau}\Big(z_{\mathtt{x}}/\ell,z_{\mathtt{y}}/\ell\Big)\\
&=\; \lim_{\ell\to\infty}\,\mathbb{J}_\ell^{(\sigma,\tau)}
\end{aligned}\tag{C.12}$$

□

References

[Ber66] A. Berezin. *The Method of Second Quantization*. Academic Press, 1966.

[Büc08] U. Bücking. Approximation of conformal mappings by circle patterns. *U. Geom Dedicata*, 137:163–197, 2008.

[CDE19] S. Charbonnier, F. David, and B. Eynard. Local properties of the random Delaunay triangulation model and topological models of 2D gravity. *Ann. Inst. Henri Poincaré Comb. Phys. Interact.*, 2019.

[CFL82] N.H. Christ, R. Friedberg, and T.D. Lee. Random lattice field theory: General formulation. *Nuclear Physics B*, 202(1):89 – 125, 1982.

[CGS18] D. Chelkak, A. Glazman, and S. Smirnov. Discrete stress-energy tensor in the loop O(n) model. *arXiv:1604.06339*, 2018.

[Che18] D. Chelkak. Planar Ising model at criticality: State-of-the-art and perspectives. In S. Sirakov, P. Ney de Souza, and M. Viana, editors, *Proceedings of the International Congress of Mathematicians*, volume 2, pages 2819–2846. World Scientific, 2018.

[DE14] F. David and B. Eynard. Planar maps, circle patterns and 2d gravity. *Ann. Inst. Henri Poincaré Comb. Phys. Interact.*, 1:139–183, 2014.

[DEF+99] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhadan, W. Morgan, D. Morrisson, and E. Witten, editors. *Quantum fields and strings: A course for mathematicians*. AMS & IAS, 1999.

[DFMS97] P. Di Francesco, P. Mathieu, and D. Sénéchal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer, 1997.

[DMNS81] E. Domany, D. Mukamel, B. Nienhuis, and A. Schwimmer. Duality relations and equivalences for models with O(N) and cubic symmetry. *Nucl. Phys. B [FS3]*, B190 [FS3]:279–287, 1981.

[Fri84] D. Friedan. Introduction to Polyakov's string theory. In J. B. Zuber and R. Stora, editors, *Recent advances in field theory and statistical physics*, volume XXXIX of *Les Houches*, pages 838–868. Elsevier, 1984.

[HKV19] C. Hongler, K. Kytölä, and F. Viklund. Conformal field theory at the lattice level: discrete complex analysis and Virasoro structure. 2019.

[Ken02] R. Kenyon. The laplacian and Dirac operators on critical planar graphs. *Inventiones mathematicae*, 150(2):409–439, Nov 2002.

[KK12] A. Kassel and R. Kenyon. Random curves on surfaces induced from the laplacian determinant. *The Annals of Probability*, 45, 11 2012.

[Kos89] I. K. Kostov. O(n) vector model on a planar random lattice: Spectrum of anomalous dimensions. *Modern Physics Letters A*, 04(03):217–226, 1989.

[KS14] R. Kenyon and J. Schlenker. Rhombic embeddings of planar quad-graphs. *TRANS. AMER. MATH. SOC*, 2014.

[Mer01] C. Mercat. Discrete riemann surfaces and the Ising model. *Communications in Mathematical Physics*, 218:177–216, 2001.

[Nie87] B. Nienhuis. Coulomb gas formulation of two-dimensional phase transitions. In C. C. Domb and J. L. Lebowitz, editors, *Phase Transitions and Critical Phenomena*, volume Vol. 11. Academic Press, 1987.

François David : Université Paris-Saclay, CNRS, CEA, Institut de physique théorique; 91191 Gif-sur-Yvette, France
Email address: francois.david@ipht.fr

Jeanne Scott : North Chatham, Massachusetts, United States
Email address: jeanne@imsc.res.in