Topological Recursion, Airy structures in the space of cycles.

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Abstract: Topological recursion associates to a spectral curve, a sequence of meromorphic differential forms. A tangent space to the "moduli space" of spectral curves (its space of deformations) is locally described by meromorphic 1-forms, and we use form-cycle duality to re-express it in terms of cycles (generalized cycles). This formulation allows to express the ABCD tensors of Quantum Airy Structures acting on the vector space of cycles, in an intrinsic spectral-curve geometric way.

1 Introduction

Topological recursion (T.R.) [17] associates to a spectral curve (a plane curve with extra structure, see below), a sequence of mutilinear meromorphic differential forms, called the invariants of the spectral curve. These invariants have many applications in mathematical physics, for example they compute the large N asymptotic expansion of correlation functions in random matrix theory [10, 6, 12], and they also compute Gromov-Witten invariants in enumerative geometry [5, 13, 16], Hurwitz numbers [15], Weil-Petersson volumes [11], Jones polynomials [7, 9], and many other interesting enumerative-algebro-geometric invariants happen to be the T.R. invariants of a suitable spectral curve [14].

In [20], Kontsevich and Soibelman, then [1] reformulated T.R. in a geometric setting, instead of a spectral curve, their data was a vector space V and its dual V^*, and tensors acting in them. They called it a quantum Airy structure. In this setting, they showed that T.R. is equivalent to the statement that a family of quadratic differential operators built from the tensors, annihilate a wave function.

Here we shall re–translate KSABCD formalism in the language of spectral curves, and identify the vector space and operators. The vector space V^* should be a space of deformations of the Airy structure, and therefore it should be identified with the space of deformations of spectral curves, which in turn was shown to be isomorphic to a space of cycles [8].

2 Spectral curves

This section is a short reminder of some notions from [8]. We first define spectral curves, then forms and *generalized* cycles.

2.1 Spectral curve

Definition 2.1 (Spectral curve) *The objects of the category are spectral-curves.*

A spectral curve data is

$$\mathcal{S} = (\Sigma, \overset{\circ}{\Sigma}, \mathrm{x}, \mathrm{y}, B) \tag{2-1}$$

where

- Σ *is a smooth surface, not necessarily compact nor connected (it could be just a union of smooth discs, sometimes called a "local spectral curve")*
- $\overset{\circ}{\Sigma}$ *a Riemann surface, called the base, not necessarily compact nor connected*
- $\mathrm{x} : \Sigma \to \overset{\circ}{\Sigma}$ *a* C^∞ *map. The pullback by* x *of the complex structure of* $\overset{\circ}{\Sigma}$*, gives a complex structure to* Σ*, which is then seen as a Riemann surface (but remind that it's complex structure depends on the choice of* x*).* Σ *is then a ramified cover of* $\overset{\circ}{\Sigma}$*, and let* R *the divisor of the ramification points, weighted by their order.*
- y *is a meromorphic (with the above complex structure) 1-form on* Σ*.*
- B *is a symmetric meromorphic* $1 \otimes 1$ *form on* $\Sigma \times \Sigma$ *(again with the above complex structure), with a double pole on the diagonal and no other pole, i.e* $B \in H^0(\Sigma \times \Sigma, K_\Sigma \overset{sym}{\boxtimes} K_\Sigma(2\,\mathrm{diag}))$*, normalized such that, in any chart*

$$B(z_1, z_2) \sim \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} + \textit{analytic at } z_1 = z_2. \tag{2-2}$$

A morphism

$$(\Sigma, \overset{\circ}{\Sigma}, \mathrm{x}, \mathrm{y}, B) \longrightarrow (\tilde{\Sigma}, \overset{\circ}{\Sigma}, \tilde{\mathrm{x}}, \tilde{\mathrm{y}}, \tilde{B}) \tag{2-3}$$

is a diffeomorphism $\phi : \Sigma \to \tilde{\Sigma}$*, such that* $\tilde{\mathrm{x}} = \mathrm{x} \circ \phi$*, and* $\mathrm{y} = \phi^* \tilde{\mathrm{y}}$ *and* $B = \phi^* \tilde{B}$*.*

• *if there is a morphism $\mathcal{S} \to \tilde{\mathcal{S}}$ and a morphism $\tilde{\mathcal{S}} \to \mathcal{S}$ that are inverse of each other, we say that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are isomorphic.*

• *a spectral curve $\mathcal{S}$ is an equivalence class of spectral curve datas modulo isomorphisms.*

Let us denote the moduli space of spectral curves modulo isomorphisms:

$$\mathcal{M}. \tag{2-4}$$

In all what follows, the base curve $\overset{\circ}{\Sigma}$ is kept fixed, and in particular a local coordinate x is chosen once for all in each chart.

2.2 Bundle of meromorphic forms

We have a vector bundle

$$\mathfrak{M}^1 \to \mathcal{M} \tag{2-5}$$

whose fiber is the infinite dimensional vector space

$$\mathfrak{M}^1(\mathcal{S}) = \{\text{meromorphic 1-forms on } \Sigma\}. \tag{2-6}$$

It is an infinite dimensional vector space, it is not countably generated.

Definition 2.2 (Topological recursion) *Topological recursion (see [17, 8]) associates to a spectral curve a collection of forms as follows:*

$$\omega_{0,1}(\mathcal{S}) = \mathrm{y} \qquad , \qquad \omega_{0,2}(\mathcal{S}) = B, \tag{2-7}$$

and for $2g-2+n>0$, we have that

$$\omega_{g,n}(\mathcal{S}) \in H_0(\Sigma^n, K_\Sigma^{\boxtimes n,\ sym}(*R)) \tag{2-8}$$

i.e. symmetric tensor products of n 1-forms, with poles only over ramification points R. For readibility, we postpone the definition below in def. 3.2, def. 3.9 where we will introduce appropriate notations, or also the definition can be found in the literature [17, 8].

For $n=0$, $\omega_{g,0}(\mathcal{S})$ is a 0-form i.e. a scalar and is denoted

$$\omega_{g,0}(\mathcal{S}) = F_g(\mathcal{S}) \in \mathbb{C}. \tag{2-9}$$

A property that will be useful to us is the homogeneity

Proposition 2.1 (Homogeneity) *For $\lambda \in \mathbb{C}^*$ define the rescaling of a spectral curve $\mathcal{S} = (\Sigma, \overset{\circ}{\Sigma}, \mathrm{x}, \mathrm{y}, B)$:*

$$\lambda\mathcal{S} = (\Sigma, \overset{\circ}{\Sigma}, \mathrm{x}, \lambda\mathrm{y}, B). \tag{2-10}$$

We have [17]

$$\omega_{g,n}(\lambda\mathcal{S}) = \lambda^{2-2g-n}\omega_{g,n}(\mathcal{S}). \tag{2-11}$$

2.3 Cycles

Let $\mathcal{S}$ a spectral curve with $(\Sigma, \overset{\circ}{\Sigma}, \mathrm{x}, \mathrm{y}, B)$ a representent.

By Poincarré duality, a cycle can be viewed as an element of the dual of $\mathfrak{M}^1(\mathcal{S})$, by the integration pairing:

$$< \gamma, \omega >= \int_\gamma \omega. \tag{2-12}$$

Since B is a $1 \boxtimes 1$ form, integrating the second projection, produces a 1-form of the first projection, we define the 1-form $\hat{B}(\gamma)$ as:

$$\hat{B}(\gamma)(z_1) = \int_{z_2 \in \gamma} B(z_1, z_2). \tag{2-13}$$

If $\gamma \in H_1(\Sigma, \mathbb{Z})$, then $\hat{B}(\gamma)$ is a holomorphic 1-form on Σ. However, we can also pair the 2nd projection in B with any element of $\mathfrak{M}^1(\mathcal{S})^*$, the result will be a 1-form, but often this 1-form will not be meromorphic, it will not even be C^∞, neither C^0. We thus consider the subset of $\mathfrak{M}^1(\mathcal{S})^*$, for which the result of integrating B yields a meromorphic 1-form, we call it the space of generalized cycles[1]:

Definition 2.3 (Generalized cycles)

$$\mathfrak{M}_1(\mathcal{S}) = \{\gamma \in \mathfrak{M}^1(\mathcal{S})^* \mid \hat{B}(\gamma) \in \mathfrak{M}^1(\mathcal{S})\}. \tag{2-14}$$

By definition we have a map $\hat{B} : \mathfrak{M}_1(\mathcal{S}) \to \mathfrak{M}^1(\mathcal{S})$. It is proved in [8] that this map is surjective, but not injective. We have the exact sequence

$$0 \to \mathrm{Ker}\ \hat{B} \to \mathfrak{M}_1(\mathcal{S}) \overset{\hat{B}}{\to} \mathfrak{M}^1(\mathcal{S}) \to 0. \tag{2-15}$$

Notice that ordinary cycles are in the space of generalized cycles

$$H_1(\Sigma, \mathbb{Z}) \subset H_1(\Sigma, \mathbb{C}) \subset \mathfrak{M}_1(\mathcal{S}). \tag{2-16}$$

It is customary to say that γ is

- *a 1st kind cycle if $\hat{B}(\gamma)$ is a 1st kind form, i.e. has no poles, and thus 1st kind cycles are ordinary cycles $\in H_1(\Sigma, \mathbb{C})$.*
- *a 3rd kind cycle if $\hat{B}(\gamma)$ is a 3rd kind form, i.e. has at most simple poles. For example if γ is a chain with boundary $\partial\gamma = D = \sum_i \alpha_i [p_i]$ then $\hat{B}(\gamma)$ is a 3rd kind cycle with simple poles p_i of residues α_i.*
- *a 2nd kind cycle if $\hat{B}(\gamma)$ is a 2nd kind form, i.e. has some poles of degree ≥ 2.*

[1] This can be viewed as a Hodge decomposition of the dual space of meromorphic forms.

Definition 2.4 (Intersection and symplectic structure) *We define the intersection of generalized cycles as*

$$\gamma_1 \cap \gamma_2 = \frac{1}{2\pi \mathrm{i}} \left(\int_{\gamma_1} \hat{B}(\gamma_2) - \int_{\gamma_2} \hat{B}(\gamma_1) \right). \tag{2-17}$$

It is proved in [8] that the intersection defines a non–degenerate symplectic form on $\mathfrak{M}_1(\mathcal{S})$*. This intersection matches the usual intersection on* $H_1(\Sigma, \mathbb{C})$*.*

Definition 2.5 (Cycle bundle) *The infinite dimensional vector bundle*

$$\mathfrak{M}_1 \to \mathcal{M} \tag{2-18}$$

whose fiber is the space of generalized cycles, is a flat bundle. It admits a flat connection [8].

The flat connection is somehow a pullback of cycles from the base curve $\overset{\circ}{\Sigma}$, tensored with pullbacks of local meromorphic functions on neighborhood of the cycle in $\overset{\circ}{\Sigma}$. Since it depends only on $\overset{\circ}{\Sigma}$, it is flat (cf the Gauss-Manin connection).

The map $\hat{B}$ that sends cycles to forms, is not invertible, it has a huge kernel (Ker $\hat{B}$ is a Lagrangian of $\mathfrak{M}_1(\mathcal{S})$), however there is a right inverse of $\hat{B}$ as follows.

Definition 2.6 (forms to cycles [8]) *There is a linear map*

$$\check{B} : \mathfrak{M}^1(\mathcal{S}) \to \mathfrak{M}_1(\mathcal{S}), \tag{2-19}$$

such that

$$\hat{B} \circ \check{B} = Id \qquad , \quad \check{B} \circ \hat{B} = \Pi \qquad \Pi^2 = \Pi. \tag{2-20}$$

Π *is the projection onto* Im $\check{B}$*, parallel to* Ker $\hat{B}$*.*

Ker $\hat{B}$ *and* Im $\check{B}$ *are Lagrangian submanifolds, and*

$$\mathfrak{M}_1(\mathcal{S}) = \text{Ker } \hat{B} \oplus \text{Im } \check{B}. \tag{2-21}$$

We leave the reader see the actual definition of $\check{B}$ in [8], but to get an idea, imagine that we would have $\{\mathcal{A}_i\}_{i \in I}$ a basis of $\mathfrak{M}_1(\mathcal{S})$, and $I_{i,j} = \mathcal{A}_i \cap \mathcal{A}_j$ its intersection matrix, the definition of $\check{B}(\omega)$ for a meromorphic 1-form ω would be

$$\check{B}(\omega) = \frac{1}{2\pi \mathrm{i}} \sum_{i,j} \left((I^{-1})_{i,j} \int_{\mathcal{A}_j} \omega \right) \mathcal{A}_i. \tag{2-22}$$

In fact $\dim \mathfrak{M}_1(\mathcal{S}) = \infty$ and is not countably generated, so it seems that this definition involving infinite sums would be ill-defined. The actual geometric definition, given in [8] is based on the Riemann-bilinear identity. However, (2-22) is morally correct and can be used in practice. Indeed (2-22) is invariant under change of basis, and for a given meromorphic 1-form ω, it is always possible to find a basis in which only finitely many terms in (2-22) are non-vanishing, or in which the sum is absolutely convergent. The true definition of [8] will however not be needed in the rest of this article.

2.4 Tangent moduli space

In [8] it was shown that the tangent space $T_{\mathcal{S}}\mathcal{M}$ (i.e. the space of deformations of spectral curves) is in fact isomorphic to a space of meromorphic forms, and thanks to the dualities above, it can be embedded in a space of cycles:

$$T_{\mathcal{S}}\mathcal{M} \hookrightarrow \mathfrak{M}^1(\mathcal{S}) \oplus (\mathfrak{M}^1(\mathcal{S}) \overset{\text{sym}}{\otimes} \mathfrak{M}^1(\mathcal{S})), \tag{2-23}$$

with the following map:

Definition 2.7 (cycles to tangent vectors [8]) *We define the map* $\partial : \mathfrak{M}_1(\mathcal{S}) \oplus (\mathfrak{M}_1(\mathcal{S}) \overset{sym}{\otimes} \mathfrak{M}_1(\mathcal{S})) \to T_{\mathcal{S}}\mathcal{M}$ *as follows:*

for $\gamma \in \mathfrak{M}_1(\mathcal{S})$*:*

$$\begin{aligned} \partial_\gamma \mathrm{x} &= 0 \\ \partial_\gamma \mathrm{y} &= \hat{B}(\gamma) \\ \partial_\gamma B(z_1, z_2) &= \int_{z\in\gamma} \omega_{0,3}(\mathcal{S}; z_1, z_2, z) \end{aligned} \tag{2-24}$$

and for $\gamma_1 \overset{sym}{\otimes} \gamma_2 = \frac{1}{2}(\gamma_1 \otimes \gamma_2 + \gamma_2 \otimes \gamma_1) \in \mathfrak{M}_1(\mathcal{S}) \overset{sym}{\otimes} \mathfrak{M}_1(\mathcal{S})$*:*

$$\begin{aligned} \partial_{\gamma_1 \overset{sym}{\otimes} \gamma_2} \mathrm{x} &= 0 \\ \partial_{\gamma_1 \overset{sym}{\otimes} \gamma_2} \mathrm{y} &= 0 \\ \partial_{\gamma_1 \overset{sym}{\otimes} \gamma_2} B &= \frac{1}{2}\left(\hat{B}(\gamma_1) \otimes \hat{B}(\gamma_2) + \hat{B}(\gamma_2) \otimes \hat{B}(\gamma_1)\right) \end{aligned} \tag{2-25}$$

Remark that the mapping between tangent space and cycles is very similar to the Goldman cycles, and indeed the intersection symplectic form, is mapped to the Goldman bracket, see [18, 2].

The following theorem [8] follows immediately from [17]

Theorem 2.1 *For* $(g, n) \neq (0, 0)$

$$\partial_\gamma \omega_{g,n} = \int_\gamma \omega_{g,n+1} \tag{2-26}$$

$$\partial_{\gamma_1 \overset{sym}{\otimes} \gamma_2} \omega_{g,n}(z_1,\dots,z_n) = \int_{\gamma_1}\int_{\gamma_2} \Big(\omega_{g-1,n+2}(z_1,\dots,z_n,z,z') + \sum_{g_1+g_2=g, I_1\sqcup I_2=\{z_1,\dots,z_n\}} \omega_{g_1,1+|I_1|}(I_1,z)\omega_{g_2,1+|I_2|}(I_2,z')\Big) \tag{2-27}$$

2.5 Hirota derivative

Definition 2.8 (Hirota derivative) *Let $z\in\Sigma$ a generic point, and let $\mathcal{B}_{z,1}\in\mathfrak{M}_1(\mathcal{S})$ be the linear form defined by $\int_{\mathcal{B}_{z,1}}\omega = \mathrm{Res}_{p\to z}\frac{1}{\mathrm{x}(p)-\mathrm{x}(z)}\omega$. We define*

$$\Delta_z = d\mathrm{x}(z)\otimes\partial_{\mathcal{B}z,1}. \tag{2-28}$$

It is the tensor product of a 1-form by a tangent vector

$$\Delta\in H^0(\Sigma, K_\Sigma\otimes T_{\mathcal{S}}\mathcal{M}). \tag{2-29}$$

We have $\partial_\gamma =<\gamma,\Delta>$, in the sense that if f is a function on $\mathcal{M}$ we have

$$\partial_\gamma f = \int_\gamma \Delta f. \tag{2-30}$$

It is called the insertion operator because of the following property:

Theorem 2.2 (Insertion operator [17]) *For $(g,n)\neq(0,0)$*

$$\Delta_z\omega_{g,n}(z_1,\dots,z_n) = \omega_{g,n+1}(z,z_1,\dots,z_n). \tag{2-31}$$

3 Quantum Airy structures

3.1 Topological recursion

Let $\mathcal{L}$ a flat Lagrangian in the total space of the bundle $\mathfrak{M}_1\to\mathcal{M}$, we assume that it is generically transverse to Ker $\hat{B}$. Let $\mathcal{S}$ a spectral curve and $V=\mathcal{L}\cap\mathrm{fiber}(\mathcal{S})$. V is an infinite dimensional vector space, it is a Lagrangian in the fiber $\mathfrak{M}_1(\mathcal{S})$. We assume (generically true)

$$\mathfrak{M}_1(\mathcal{S}) = V\oplus \mathrm{Ker}\ \hat{B}. \tag{3-1}$$

We have the map

$$\hat{B}: V\to\mathfrak{M}^1(\mathcal{S}), \tag{3-2}$$

which is an isomorphism. Moreover the inclusion map $\mathfrak{M}_1(\mathcal{S})\subset\mathfrak{M}^1(\mathcal{S})^*$ restricts to an isomorphism $V^*\sim\mathfrak{M}^1(\mathcal{S})$ so that $\hat{B}$ can be viewed as the dualizing map

$$\hat{B}: V\to V^*. \tag{3-3}$$

We also have the Hirota operator $\Delta \in \mathfrak{M}^1(\mathcal{S}) \otimes T_{\mathcal{S}}\mathcal{M}$.

We shall use the vector space V and its dual V^* as the vector space of the Airy structure of [20], and we shall define some tensors acting on them.

First, observe that the projection $\mathrm{x} : \Sigma \to \overset{\circ}{\Sigma}$ has ramification points, let $R = \sum_a (\mathrm{order}_a(\mathrm{x} - \mathrm{x}(a)) - 1).a$ the divisor of ramification points. In this section assume that all ramification points are generic, of order $r_a = \mathrm{order}_a(\mathrm{x}-\mathrm{x}(a)) = 2$. We postpone higher order cases to section 3.5.

There are 2 sheets meeting at a. Let $\sigma_a \neq \mathrm{Id}$ the unique holomorphic involution in a simply connected neighborhood of a exchanging the 2 sheets, i.e. such that $\mathrm{x}(\sigma_a(z)) = \mathrm{x}(z)$ and $\sigma_a(a) = a$. If ω is a 1-form holomorphic in a simply connected neighborhood of a, we define $d_a^{-1}\omega = \int_{z'=a}^{z} \omega(z')$, the unique primitive of ω that vanishes at $z = a$.

Definition 3.1 (Recursion kernels [17]) *Define $K_2 : V^* \otimes V^* \to V^*$*

$$K_2(\omega,\omega') = -\sum_{a\in R} \operatorname*{Res}_{a} \frac{d_a^{-1} B}{\mathrm{y} - \sigma_a^* \mathrm{y}} \omega \otimes \sigma_a^* \omega', \tag{3-4}$$

where the d_a^{-1} and the residue act only on the second projection of B, i.e. K_2 returns a 1-form in the 1st projection of B. In formulas [17] this reads

$$K_2(\omega,\omega')(z_1) = -\sum_{a\in R} \operatorname*{Res}_{z_2\to a} \frac{\int_{z'=a}^{z_2} B(z_1,z')}{\mathrm{y}(z_2) - \mathrm{y}(\sigma_a(z_2))} \omega(z_2) \otimes \omega'(\sigma_a(z_2)). \tag{3-5}$$

Definition 3.2 (Topological recursion [17]) *The $\omega_{g,n} \in (V^*)^{\otimes n}$ are defined by*

$$\omega_{0,1} = \mathrm{y}, \quad , \quad \omega_{0,2} = B, \tag{3-6}$$

and by the recursion for $2g-2+n+1>0$ and $n \geq 0$

$$\begin{aligned} \omega_{g,n+1}(z_0,z_1,\dots,z_n) &= K_2\Big(\omega_{g-1,n+2}(.,.,z_1,\dots,z_n) \\ &\quad + \sum_{g_1+g_2=g, I_1\sqcup I_2}^{no\ (0,1)} \omega_{g_1,n_1+1}(.,I_1) \otimes \omega_{g_2,n_2+1}(.,I_2)\Big) \end{aligned} \tag{3-7}$$

where K_2 acts on the dots variables. and for $g \geq 2$

$$\omega_{g,0} = F_g = \frac{1}{2-2g} < \hat{\eta}, \omega_{g,1} > \tag{3-8}$$

We leave the reader look the definition of $F_1 = \omega_{1,0}$ in [17, 19], and F_0 will not be needed here.

Theorem 3.1 ([6, 17]) *We have*

$$\partial_\gamma K_2(\omega,\omega') = K_2(\hat{B}(\gamma), K_2(\omega,\omega')) + K_2(\partial_\gamma \omega, \omega') + K_2(\omega, \partial_\gamma \omega'). \tag{3-9}$$

3.2 Tensors and Airy structure

Following KS [20], we define

Definition 3.3 (ABCD) *We define the following tensors*

- $A = \omega_{0,3} \in (V^* \otimes V^* \otimes V^*)^{sym}$, *i.e.*

$$A(\gamma_1, \gamma_2, \gamma_3) = \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} \omega_{0,3}. \tag{3-10}$$

- $D = \omega_{1,1} \in V^*$, *i.e.*

$$D(\gamma) = \int_{\gamma} \omega_{1,1}. \tag{3-11}$$

- $C \in V^* \otimes (V \otimes V)^{sym}$, *is the dual of the recursion kernel,* $C = 2K_2^*$*:*

$$C(\gamma, \omega, \omega') = 2 \int_{\gamma} K_2(\omega, \omega'). \tag{3-12}$$

- *and composing with* $\hat{B}$ *we define* $B \in V^* \otimes V^* \otimes V$*:*

$$B(\gamma_1, \gamma_2, \omega) = 2C(\gamma_1, \hat{B}(\gamma_2), \omega). \tag{3-13}$$

Here in the context of topological recursion for spectral curves we have the relations

$$A(\gamma_1, \gamma_2, \gamma_3) = C(\gamma_1, \hat{B}(\gamma_2), \hat{B}(\gamma_3)), \tag{3-14}$$

which we write

$$B = 2C^{.*.} \qquad , \qquad A = C^{.**}, \tag{3-15}$$

and notice that $\omega_{0,2} \in V^* \otimes V^*$, *and* D *can be written*

$$D(\gamma) = C(\gamma, \omega_{0,2}). \tag{3-16}$$

These tensors are such that

$$\omega_{0,3} = A \tag{3-17}$$

$$\omega_{1,1} = D \tag{3-18}$$

and $\omega_{g,n}$ with $2g - 2 + n > 1$ is given by recursively applying the tensors C and B

$$2 \int_{z_0 \in \gamma} \omega_{g,n+1}(z_0, z_1, \ldots, z_n) = C(\gamma, \omega_{g-1,n+2}(.,., z_1, \ldots, z_n)) + \sum_{g_1+g_2=g,\ I_1 \sqcup I_2 = \{z_1,\ldots,z_n\}}^{\text{stable}} C(\gamma, \omega_{g_1,1+|I_1|}(., I_1), \omega_{g_2,1+|I_2|}(., I_2))$$

$$+2\sum_{j=1}^{n} B(\gamma,\omega_{0,2}(.,z_j),\omega_{g,n}(..,\hat{z}_j)) \tag{3-19}$$

where the dot–variables are the ones on which the tensors act, and $\hat{z}_j$ means $\{z_1,\dots,z_n\}\setminus z_j$, and stable means that we exclude $(g_i,n_i)=(0,1),(0,2)$ from the sum.

These are thus the tensors of the Quantum Airy Structure of [20, 1].

3.3 Wave function

Definition 3.4 (Tautological cycle) *We define the tautological cycle as the dual of $\omega_{0,1}$ in V, i.e.*

$$\hat{\eta}=\omega_{0,1}^{*}=\Pi_V^{\|\mathrm{Ker}\ \hat{B}}\check{B}(\omega_{0,1}). \tag{3-20}$$

where $\Pi_V^{\|\mathrm{Ker}\ \hat{B}}$ is the projection on V parallel to Ker $\hat{B}$*.*

Some of its properties are:

$$\forall\gamma\in V,\qquad \partial_\gamma\hat{\eta}=\gamma, \tag{3-21}$$

$$\hat{B}(\hat{\eta})=\omega_{0,1}, \tag{3-22}$$

and the dilaton equation [17] amounts to

$$\forall\ 2-2g-n<0\ ,\qquad \partial_{\hat{\eta}}\omega_{g,n}=(2-2g-n)\omega_{g,n}. \tag{3-23}$$

Definition 3.5 *We define F_0 as*

$$F_{0,\mathcal{L}}(\mathcal{S})=\frac{1}{2}<\hat{\eta},\omega_{0,1}>. \tag{3-24}$$

Notice that F_0 depends on our choice of Lagrangian $\mathcal{L}$.

Remark that if we wouldn't project $\check{B}(\omega_{0,1})$ on V, we would have

$$<\check{B}(\omega_{0,1}),\omega_{0,1}>=0. \tag{3-25}$$

Definition 3.6 (Wave function) *Define for $\gamma'\in V$:*

$$Z'(\hbar^{-1}\mathcal{S},\gamma') = e^{\sum_{(g,n)\neq(0,0)}\frac{\hbar^{2g-2+n}}{n!}<\gamma'^{\otimes n},\omega_{g,n}(\mathcal{S})>}. \tag{3-26}$$

It is defined as a formal power series of $\hbar$, namely

$$\hbar\ln Z'\in\mathbb{C}[[\hbar]]. \tag{3-27}$$

and all equations we are going to write from now on, are understood in $\mathbb{C}[[\hbar]]$.

The notation $\hbar^{-1}\mathcal{S}$ comes from the homogeneity prop. 2.1.

Remark that formally we have the Sato formula: wave function = shifted partition function, i.e.

$$Z'(\hbar^{-1}\mathcal{S},\gamma') = Z(\hbar^{-1}\mathcal{S}+\gamma')\ e^{-\hbar^{-2}F_0(\mathcal{S})} = Z(e^{\partial_{\gamma'}}\hbar^{-1}\mathcal{S})\ e^{-\hbar^{-2}F_0(\mathcal{S})} \tag{3-28}$$

where $e^{\partial_{\gamma'}}$ is the exponential of the flow of the tangent vector $\partial_{\gamma'}$, and

$$Z(\hbar^{-1}\mathcal{S}) = \mathrm{e}^{\sum_{g=0}^{\infty}\hbar^{2g-2}F_g(\mathcal{S})}. \tag{3-29}$$

In the ratio Z', we see that F_0 cancels.

3.4 Quadratic PDE

Let us revisit Kontsevich-Soibelman in this language.

Definition 3.7 *Define the quadratic differential operators*

$$L = \hbar\Delta - \hat{B}(\gamma') - \hbar\left(\omega_{1,1} + \hbar^2 K_2(\Delta\otimes\Delta)\right) \tag{3-30}$$

and for any $\gamma\in V$

$$L_\gamma =< \gamma, L >= \hbar\partial_\gamma - \int_\gamma \hat{B}(\gamma') - \hbar\left(D(\gamma) + \frac{\hbar^2}{2}C(\gamma,\Delta\otimes\Delta)\right) \tag{3-31}$$

Remark that if we assign an "order" $\hbar^{-1}$ to derivatives, we see that L is a $O(\hbar)$ deformation of the Hirota operator shifted by $\hat{B}(\gamma')$:

$$L = \hbar\Delta - \hat{B}(\gamma') + O(\hbar). \tag{3-32}$$

The following is the main theorem:

Theorem 3.2 (Annihilating the wave function) *We have* $\forall\gamma'\in V$

$$L.Z'(\hbar^{-1}\mathcal{S},\gamma') = 0 \tag{3-33}$$

or equivalently $\forall\gamma\in V$

$$L_\gamma.Z'(\hbar^{-1}\mathcal{S},\gamma') = 0. \tag{3-34}$$

proof: This is merely a way of rewriting topological recursion eq 3-7.

$$\Delta\ln Z' \;=\; \hbar^{-1}\hat{B}(\gamma') + \sum_{2g-2+n+1>0}\frac{\hbar^{2g-2+n}}{n!} < \gamma'^{\otimes n}, \omega_{g,n+1} >$$

$$
\begin{aligned}
&= \hbar^{-1}\hat{B}(\gamma') + \sum_{2g-2+n+1>0} \frac{\hbar^{2g-2+n}}{n!} K_2(<\gamma'^{\otimes n}, \omega_{g-1,n+2}>) \\
&\quad + \sum_{2g-2+n+1>0} \sum_{g_1+g_2=g} \sum_{n_1+n_2=n} \frac{\hbar^{2g-2+n}}{n_1!n_2!} K_2(<\gamma'^{\otimes n_1}, \omega_{g_1,n_1+1}>, <\gamma'^{\otimes n_2}, \omega_{g_2,n_2+1}>) \\
&= \hbar^{-1}\hat{B}(\gamma') + \omega_{1,1} + \hbar^2 K_2(\Delta^2 \ln Z') + \hbar^2 K_2(\Delta \ln Z', \Delta \ln Z') \\
&= \hbar^{-1}\hat{B}(\gamma') + \omega_{1,1} + \hbar^2 K_2(Z'^{-1}\Delta^2 Z') \qquad (3\text{-}35)
\end{aligned}
$$

□

3.5 Higher order branchpoints

At a ramification point a of order $r_a \geq 2$, there are r_a sheets meeting, x is locally like $x(z) \sim x(a) + c_a z^{r_a}$, and there is a local Galois group

$$G_a = \mathbb{Z}_{r_a} \qquad (3\text{-}36)$$

exchanging the sheets, acting by multiplication of z by a r_ath root of unity. Let $G_a^* = G_a \setminus \{\mathrm{Id}\}$ (for $r_a = 2$, we recognize $G_a^* = \{\sigma_a\}$ the local involution.) For $k \geq 2$, we shall consider the set of all possible $(k-1)-$uples of G_a^*, denoted $\sigma = (\sigma_2, \sigma_3, \dots, \sigma_k)$. If $k > r_a$ this set is empty.

Definition 3.8 *For $k \geq 2$, define $K_k : V^{\otimes k} \to V^*$*

$$K_k(\omega_1, \dots, \omega_k) = -\sum_a \sum_{\sigma \subset_k G_a^*} \operatorname*{Res}_a \frac{d_a^{-1} B}{\prod_{i=2}^k (\mathrm{y} - \sigma_i^* \mathrm{y})} \omega_1 \otimes \sigma_2^* \omega_2 \otimes \cdots \otimes \sigma_k^* \omega_k, \qquad (3\text{-}37)$$

and after dualizing, define $C_k = K_k^* \in V^* \otimes V^{\otimes k}$:

$$C_k(\gamma, \omega_1, \dots, \omega_k) = \int_\gamma K_k(\omega_1, \dots, \omega_k) \qquad (3\text{-}38)$$

Definition 3.9 (Topological recursion [4]) *For $2g - 2 + n + 1 > 0$:*

$$
\begin{aligned}
\omega_{g,n+1}(z_0, z_1, \dots, z_n) &= \sum_{k=2}^{\max r_a} \sum_{\mu \vdash k} \sum_{J_1 \sqcup \cdots \sqcup J_{\ell(\mu)} = \{z_1, \dots, z_n\}} \sum_{g_1, \dots, g_{\ell(\mu)},\ \sum_i g_i = g-k+\ell(\mu)} \\
&\quad K_k\Big(\omega_{g_i, |\mu_i| + |J_i|}(J_i, \mu_i)\Big) \qquad (3\text{-}39)
\end{aligned}
$$

where K_k acts on the μ_i variables. and for $g \geq 2$

$$\omega_{g,0} = F_g = \frac{1}{2-2g} <\hat{\eta}, \omega_{g,1}> \qquad (3\text{-}40)$$

Theorem 3.3 *The following differential operator (whose order is* $\max_{a\in R} r_a$*)*

$$L = \hbar\Delta - \hbar\sum_{k\geq 2}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}K_k(\hbar^l\Delta^{\otimes l}, U_{k-l}) \tag{3-41}$$

where U_k *is given by*

$$U_k = e^{-F_0}\Delta^{k,\ no\ discs}e^{F^0} = \left(e^{-F_0}\Delta^k e^{F^0}\right)_{\omega_{0,1}\to 0}. \tag{3-42}$$

$$U_1 = 0 \quad , \quad U_2 = \omega_{0,2} \quad , \quad U_3 = \omega_{0,3} \quad , \quad U_4 = \omega_{0,4} + 3\omega_{0,2}\omega_{0,2} \ , \dots \tag{3-43}$$

annihiliates Z'

$$L.Z' = 0. \tag{3-44}$$

proof: Again this is a mere rewriting of topological recursion.

$$\Delta \ln Z' = \hat{B}(\gamma') + \sum_k \sum_{\mu,\ |\mu|=k} c_\mu K_k(\prod_i \Delta^{\mu_i}\ln Z' + \delta_{\mu_i\geq 2}\Delta^{\mu_i}F_0) \tag{3-45}$$

$$\Delta Z' = \hat{B}(\gamma')Z' + \sum_k\sum_{a=0}^{k}\frac{k!}{a!(k-a)!}K_k(\Delta^{k-a}Z', e^{-F_0}\Delta^{a,\text{ no discs}}e^{F_0}) \tag{3-46}$$

□

Example with $r=3$ (and written with $\hbar = 1$):

$$\begin{aligned}
\Delta \ln Z' = {} & \hat{B}(\gamma') \\
& + K_2(\Delta^2\ln Z' + \Delta^2 F_0) + K_2(\Delta\ln Z', \Delta\ln Z') \\
& + K_3(\Delta^3 \ln Z' + \Delta^3 F_0) \\
& + 3K_3(\Delta^2\ln Z' + \Delta^2 F_0, \Delta \ln Z') \\
& + K_3(\Delta\ln Z', \Delta\ln Z', \Delta\ln Z') \\
= {} & \hat{B}(\gamma') + \omega_{1,1} + K_3(\omega_{0,3}) \\
& + K_2(\Delta^2\ln Z') + K_2(\Delta\ln Z', \Delta\ln Z') \\
& + K_3(\Delta^3\ln Z') \\
& + 3K_3(\Delta^2\ln Z', \Delta\ln Z') \\
& + 3K_3(\omega_{0,2}, \Delta\ln Z') \\
& + K_3(\Delta\ln Z', \Delta\ln Z', \Delta\ln Z')
\end{aligned} \tag{3-47}$$

i.e. the operator that annihilates Z' is

$$L = \Delta - \hat{B}(\gamma') - \omega_{1,1} - K_2(\Delta^2) - 3K_3(\omega_{0,2}\otimes\Delta) - K_3(\Delta^3). \tag{3-48}$$

3.6 Local times formulation

Let us show how to recover the usual KS Airy-structures formulation [20, 1]. This is in some sense a mere "change of basis" in the space of cycles, however many subtleties arise because the local cycles, depending on the branchpoints $x(a)$ are not flat sections of the cycle-bundle. The flat connection is not trivial in that non-flat basis.

3.6.1 Local cycles and local times

In order to describe the $\omega_{g,n}$, which have poles at ramification points, we introduce the following family of cycles (called local cycles). Let a a ramification point of order r_a, and let the generalized cycle $\Gamma_{a,k} \in \mathfrak{M}_1(\mathcal{S})$ be defined as a a linear form $\in \mathfrak{M}^1(\mathcal{S})^*$, which acts on any meromorphic 1-form ω as

$$\int_{\Gamma_{a,k}} \omega =< \Gamma_{a,k}, \omega >= \operatorname*{Res}_{z\to a} (x - x(a))^{-k/r_a} \omega(z). \tag{3-49}$$

They intersect as

$$\Gamma_{a,k} \cap \Gamma_{b,j} = \frac{|k|}{2\pi \mathrm{i}} \ \delta_{a,b} \ \delta_{k,-j}. \tag{3-50}$$

Define the local times for $k \geq 1$ (in fact they are 0 if $k \leq 0$)

$$t_{a,k} = \oint_{\Gamma_{a,k}} \omega_{0,1} \tag{3-51}$$

they are the coefficients of the Taylor expansion of $\omega_{0,1}$ at a

$$\omega_{0,1} \sim \frac{1}{r_a} \sum_{k=1}^{\infty} t_{a,k} (x(z) - x(a))^{\frac{k-r_a}{r_a}} dx(z) \ + \text{analytic at } a. \tag{3-52}$$

It is important to keep in mind that ramification points move when we deform the spectral curve, and these cycles are not flat. We have

$$\Delta x(a) = -\frac{r_a}{t_{a,r+1}} \hat{B}(\Gamma_{a,1}) \tag{3-53}$$

$$\Delta \Gamma_{a,k} = -\frac{k}{t_{a,r+1}} \hat{B}(\Gamma_{a,1}) \otimes \Gamma_{a,k+r_a}. \tag{3-54}$$

For $k \geq 1$, these local times are not flat coordinates:

$$\Delta t_{a,k} = \left(\hat{B}(\Gamma_{a,k}) - k \frac{t_{a,r+k}}{t_{a,r+1}} \hat{B}(\Gamma_{a,1}) \right). \tag{3-55}$$

3.6.2 Wave function

In def. 3.6, let us choose a cycle γ', written

$$\Gamma' = \sum_a \sum_{k\geq r_a+1} \frac{1}{k} t'_{a,k} \Gamma_{a,-k} \quad , \quad \gamma' = \Pi_V^{\|\mathrm{Ker}\ \hat{B}} \Gamma'. \tag{3-56}$$

Denoting pairs (a_j, k_j) as i_j, and $\mathcal{B}_{a,k} = \frac{1}{k}\Gamma_{a,-k}$, we define for all (g,n) such that $2g-2+n>0$

$$F_{g,n}[i_1,\dots,i_n] = \int_{\mathcal{B}_{i_1}} \cdots \int_{\mathcal{B}_{i_n}} \omega_{g,n}, \tag{3-57}$$

$$F_{g,n}(t') = \sum_{i_1,\dots,i_n} F_{g,n}[i_1,\dots,i_n] t'_{i_1}\dots t'_{i_n} = \int_{\Gamma'} \cdots \int_{\Gamma'} \omega_{g,n} = \int_{\gamma'} \cdots \int_{\gamma'} \omega_{g,n}. \tag{3-58}$$

and, as a formal $\hbar$ series

$$\ln Z(t') = \sum_{(g,n),\ 2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(t'), \tag{3-59}$$

so that we have

$$\ln Z' = \hbar^{-1} \int_{\gamma'} \omega_{0,1} + \frac{1}{2} \int_{\gamma'} \int_{\gamma'} \omega_{0,2} + \ln Z(t'). \tag{3-60}$$

Using

$$\Delta \mathcal{B}_{a,k} = \frac{k-r_a}{t_{a,r+1}} \hat{B}(\Gamma_{a,1}) \otimes \mathcal{B}_{a,k-r_a} \qquad , \tag{3-61}$$

let us compute the action of Δ on $F_{g,n}$:

$$\begin{aligned} \Delta F_{g,n}[i_1,\dots,i_n] \quad = \quad & \sum_{a,k} F_{g,n+1}[i_1,\dots,i_n,(a,k)] \hat{B}(\Gamma_{a,k}) \\ & + \sum_{j=1}^n F_{g,n}[i_1,\dots,(a_j,k_j-r_{a_j}),\dots,i_n] \frac{k_j-r_{a_j}}{t_{a_j,r_{a_j}+1}} \hat{B}(\Gamma_{a_j,1}) \end{aligned} \tag{3-62}$$

i.e.

$$\Delta F_{g,n}(t') = \frac{1}{n+1} \sum_{a,k} \hat{B}(\Gamma_{a,k}) \frac{\partial}{\partial t_{a,k}} F_{g,n+1}(t') + \sum_{a,k} k \frac{t'_{a,k+r}}{t_{a,r+1}} \hat{B}(\Gamma_{a_k,1}) \frac{\partial}{\partial t_{a,k}} F_{g,n}(t'). \tag{3-63}$$

Altogether that implies

$$\hbar \Delta \ln Z' = \Delta' \ln Z + \hat{B}(\gamma') + \int_{\Delta\gamma'} \omega_{0,1} + \hbar \int_{\gamma'} \int_{\Delta\gamma'} \omega_{0,2} \tag{3-64}$$

where we have defined

$$\Delta' = \sum_{a,k} \left(\hat{B}(\Gamma_{a,k}) + \hbar k \frac{t'_{a,k+r}}{t_{a,r+1}} \hat{B}(\Gamma_{a_k,1}) \right) \frac{\partial}{\partial t'_{a,k}}. \tag{3-65}$$

Moreover notice that
$$\hbar\Delta Z = \Delta' Z - Z\frac{\hbar}{2}\int_{\gamma'}\int_{\gamma'}\omega_{0,3}. \tag{3-66}$$

In the end, theorem 3.2 can be rewritten as an operator which is quadratic in the $\partial/\partial t'_j$ and quadratic in the t'_js, annihilating $Z(t')$. It coincides with Kontsevich-Soibelman [20, 1].

More generally, we see that the equation (3-33) (resp. (3-44)) amounts to a order $\max r_a$ PDE with respect to times t'_i, and whose coefficients can themselves be polynomials of the t'_i of the same order.

3.6.3 Reminder KS method

Let us rewrite KS proof. Notice that for all (g,n) such that $2g-2+n>0$ we have

$$\omega_{g,n}(z_1,\dots,z_n) = \sum_{i_1,\dots,i_n} F_{g,n}[i_1,\dots,i_n]\ \hat{B}(\Gamma_{i_1})(z_1)\otimes\cdots\otimes B(\Gamma_{i_n})(z_n) \tag{3-67}$$

(indeed the difference between the LHS and RHS has all its $\mathcal{B}_{a,k}$ integrals vanishing, which implies that it has no poles at branchpoints, but it also can't have other poles, therefore it must be a holomorphic form, and it is easy to see that the integral on $H_1(\Sigma,\mathbb{C})\cap \mathrm{Ker}\ \hat{B})$ also vanishes, which implies that the difference is zero). Let

$$C[i_1,i_2,i_3] = 2\int_{\mathcal{B}_{i_1}} K_2(\hat{B}(\Gamma_{i_2}),\hat{B}(\Gamma_{i_3})) \tag{3-68}$$

and
$$B[i_1,i_2,i_3] = 2\int_{z\in\mathcal{B}_{i_3}}\left(\int_{\mathcal{B}_{i_1}} K_2(\hat{B}(\Gamma_{i_2}),\omega_{0,2}(z,.))\right) \tag{3-69}$$

Then, topological recursion implies that for all $(g,n)\neq(0,0),(1,0),(0,1),(0,2)$ we have

$$\begin{aligned} 2F_{g,n+1}[i_0,i_1,\dots,i_n] &= \sum_{i,j} C[i_0,i,j]\Big(F_{g-1,n+2}[i,j,i_1,\dots,i_n] \\ &\quad + \sum_{g_1+g_2=g,I_1\sqcup I_2=\{i_1,\dots,i_n\}}^{2g_i-2+n_i>0} F_{g_1,1+|I_1|}[i,I_1]F_{g_2,1+|I_2|}[j,I_2]\Big) \\ &\quad +2\sum_{k=1}^{n}\sum_j B[i_0,i_k,j]F_{g,n}[j,i_1,\dots,\widehat{i_k},\dots,i_n] \end{aligned} \tag{3-70}$$

and in addition we have

$$F_{0,3}[i,j,k] = \int_{\mathcal{B}_i}\int_{\mathcal{B}_j}\int_{\mathcal{B}_k}\omega_{0,3} = \frac{1}{2}A[i,j,k] \tag{3-71}$$

$$F_{1,1}[i] = \int_{\mathcal{B}_i} \omega_{1,1} = D[i]. \tag{3-72}$$

Then, (3-70) can be written as the quadratic PDE

$$\Big(\hbar\frac{\partial}{\partial t'_i} - \hbar D[i] - \frac{\hbar}{2}\sum_{j,k}(A[i,j,k]t'_it'_j + 2B[i,j,k]t'_j\hbar\frac{\partial}{\partial t'_k} + C[i,j,k]\hbar\frac{\partial}{\partial t'_j}\hbar\frac{\partial}{\partial t'_k}\Big)Z(t') = 0 \tag{3-73}$$

4 Conclusion

For a spectral curve (either compact or local), the space of generalized cycles is the natural vector space on which the ABCD tensors of an Airy structure act. The topological recursion is then equivalent to a differential operator annihilating a partition function. This formalism makes rather easy the generalization to more than quadratic operators. It should amount to [3] for W-algebra structures.

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