FROM TOPOLOGICAL RECURSION TO WAVE FUNCTIONS AND PDES QUANTIZING HYPERELLIPTIC CURVES

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Abstract. Starting from loop equations, we prove that the wave functions constructed from topological recursion on families of spectral curves with a global involution satisfy a system of partial differential equations, whose equations can be seen as quantizations of the original spectral curves. The families of spectral curves can be parametrized with the so-called times, defined as periods on second type cycles. These equations can be used to prove that the WKB solution of many isomonodromic systems coincides with the topological recursion wave function, and thus show that the topological recursion wave function is annihilated by a quantum curve. This recovers many known quantum curves for genus zero spectral curves and generalizes to hyperelliptic curves. In the particular case of a degenerate elliptic curve, apart from giving the quantum curve, we prove that the wave function satisfies the first Painlevé isomonodromic system and equation just from loop equations, making use of our system of PDEs. In general, we are able to recover the Gelfand–Dikii isomonodromic systems just from topological recursion.

Contents

1. Introduction

Topological recursion is a powerful tool, which was first discovered in the context of large size asymptotic expansions in random matrix theory [10, 7, 5, 6] and established as an independent universal theory around 2007 [15]. Its most important role was to unveil a common structure in many different topics in mathematics and physics, which helped building bridges among them and gaining general context. For instance, it has been related to fundamental structures in enumerative geometry and integrable systems, such as intersection theory on moduli spaces of curves and cohomological field theories.

A quantum curve is a Schrödinger operator-like non-commutative analogue of a plane curve that annihilates the so-called wave function, which can be seen as a WKB asymptotic solution of the corresponding differential equation. Inspired by the intuition coming from matrix models, it has been conjectured that there exists such a quantum curve associated to a spectral curve, which is the input of the topological recursion, and whose WKB asymptotic solution is reconstructed by the topological recursion output.

This claim was verified in [22] for a class of spectral curves called admissible, which are basically spectral curves whose Newton polygons have no interior point. Admissible spectral curves include a very large number of spectral curves of genus 0. Therefore they recover many cases previously studied in the literature in various algebro-geometric contexts. In the present work, we go beyond admissible spectral curves and study the quantum curve problem for spectral curves with a global involution, given by algebraic curves whose defining polynomials are of the form $y^2 = R(x)$, with R a rational function on x. This setting includes the genus 0 spectral curves with a global involution $y \mapsto -y$, such as the well-known Airy curve, which are many less that the set of admissible curves, but it also includes all genus 1 spectral curves, i.e. all elliptic curves, and all hyperelliptic curves, which are curves of genus $g > 1$ where R is a polynomial in x.

Quantum curves encode enumerative invariants in an interesting way, and help building the bridge between the geometry of integrable systems and topological recursion. One of the most celebrated applications of quantum curves is in the context of knot theory, where the quantum curve of the A-polynomial of a knot provides a conjectural constructive generalization of the volume conjecture [8, 9, 4].

1.1. **Quantum curves and topological recursion.** We start by presenting the idea of quantum curves and their relation to topological recursion. Consider $P \in \mathbb{C}[x, y]$ and let

$$\mathcal{C} = \{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0\}$$

be a plane curve.

Consider $\hbar > 0$ a formal parameter. A quantization of the plane curve $\mathcal{C}$, is a differential operator $\widehat{P}$ of the form

$$\widehat{P}(\widehat{x}, \widehat{y}; \hbar) = P_0(\widehat{x}, \widehat{y}) + O(\hbar)$$

where $\widehat{x} = x\cdot$, $\widehat{y} = \hbar \frac{\mathrm{d}}{\mathrm{d}x}$. In fact, $\hat{P}$ is a formal series (or transseries) normal–ordered operator valued polynomial (all the $\widehat{y}$ in a monomial are placed to the right of all the $\widehat{x}$) whose leading order term $P_0(\widehat{x}, \widehat{y})$ recovers the polynomial equation of the original spectral curve (normal ordered). Actually, in general $P_0(x, y)$ can be a reducible polynomial with $P(x, y)$ one of its factors. The operators $\widehat{x}$ and $\widehat{y}$ satisfy the following commutation relation which justifies the name ‘quantization”:

$$[\widehat{y}, \widehat{x}] = \hbar.$$

One can consider a Schrödinger-type differential equation

$$\widehat{P}(\widehat{x}, \widehat{y})\psi(z, \hbar) = 0, \text{ with } z \in \mathcal{C}, \tag{1}$$

whose solution can be calculated via the WKB method, that is we require ψ to have a formal series (resp. transseries) $\hbar$ expansion of the form

$$\psi(z, \hbar) = \exp\left(\sum_{m \geqslant 0} \hbar^{m-1} S_m(z)\right)$$

(resp. of the form of a formal series in powers of exponentials of inverse powers of $\hbar$ whose coefficients are formal power series of $\hbar$). The coefficients $S_k(z)$ are determined recursively via (1). One fundamental question is if the formal solution ψ can be computed directly from the original plane curve $\mathcal{C}$. The conjectural answer is provided by the topological recursion and is already established in many cases.

The input data of the topological recursion is called spectral curve. For the purpose of this paper, a *spectral curve* will be given as in [12], i.e. by the data $(\Sigma, x, y, \omega_{0,1}, \omega_{0,2})$, where Σ is a compact Riemann surface, and x and y are meromorphic functions on Σ such that the zeroes of $\mathrm{d}x$ do not coincide with the zeroes of $\mathrm{d}y$. Then x and y must be algebraically dependent, i.e. $P(x, y) = 0$ with $P \in \mathbb{C}[x, y]$. We consider $\omega_{0,1} := y\mathrm{d}x$ and $\omega_{0,2}$ a symmetric bi-differential B on Σ^2 with double poles along the diagonal and vanishing residues. The output of the topological recursion are symmetric meromorphic multi-differentials $\omega_{g,n}(z_1, \ldots, z_n)$ on Σ^n.

The perturbative wave function $\psi(z)$ constructed from topological recursion is defined (see [15]) as

$$\frac{1}{x(z)-x(o)}\exp\left(\sum_{g\geqslant 0,n\geqslant 1}\frac{\hbar^{2g-2+n}}{n!}\int_o^z\cdots\int_o^z\left(\omega_{g,n}(z_1,\ldots,z_n)-\delta_{g,0}\delta_{n,2}\frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\right)\right),$$

where $o\in\Sigma$ is a chosen base-point for integration. In general the quantum curve is obtained by choosing $o=\zeta$ such that $x(\zeta)=\infty$, and one may need to regularize the $(g,n)=(0,1)$ term in the limit $o\to\zeta$. We choose a regularization for $(g,n)=(0,2)$ which slightly differs from some part of the literature and produces the first factor of the expression. We emphasize that our definition transforms as a spinor $\frac{1}{2}$-form under change of coordinates. Actually, we will generalize the definition of the wave-function by allowing integration over any divisor as in [12].

A further question is if the differential operator $\widehat{P}$ can be directly constructed from the topological recursion. This has also been answered affirmatively in many cases. In this article, we actually construct an operator that we believe is a more fundamental object, which appears more naturally for curves of genus $g>0$, and provides a PDE which also allows to reconstruct the wave function ψ. Our system of PDEs will also imply a quantum curve in the more classical sense considered in this section.

1.2. **Generalized cycles.** Let us now recall the concept of generalized cycles on a Riemann surface as in [12], which will help us introduce suitable local coordinates in the space of spectral curves. These local coordinates can be seen as deformation parameters giving rise to families of spectral curves and will play a key role when producing our system of PDEs for a large class of spectral curves.

The so-called times t_i, introduced in [12], can be viewed as local coordinates in the space of spectral curves. Time deformations ∂_{t_i} belong to the tangent space, which is isomorphic to the space of meromorphic differential forms on the spectral curve, and via form-cycle duality it can be identified with the space of generalized cycles on the spectral curve. In [12], generalized cycles are defined as elements of the dual of the space of meromorphic forms on Σ such that integrating $\omega_{0,2}=B$ on them gives meromorphic 1-forms.

In practice a generating family is given by 3 kinds of cycles, dual to 3 kinds of forms:

- **1st kind cycles:** This type of cycles are usual non-contractible cycles, i.e. elements $[\gamma]\in\mathrm{H}_1(\Sigma,\mathbb{C})$. If Σ is compact of genus g, then $\dim\mathrm{H}_1(\Sigma,\mathbb{C})=2g$.
- **2nd kind cycles:** A cycle of second type $\gamma=\gamma_p.f$ consists of a small circle γ_p around a point $p\in\Sigma$ (small in the sense of projective limit) weighted by a function f holomorphic in a neighborhood of γ_p and meromorphic in a neighborhood of p, with a possible pole at p (of any degree), i.e. by definition $\int_\gamma\omega\coloneqq 2\pi i\,\mathrm{Res}_p\,f\omega$.
- **3rd kind cycles:** They are open chains $\gamma=\gamma_{q\to p}$ (paths up to homotopic deformation with fixed endpoints), whose boundaries $\partial\gamma=[p]-[q]$ are degree zero divisors.

Let $x:\Sigma\to\mathbb{C}$ be the meromorphic function that makes the spectral curve a branched cover of the Riemann sphere. A basis of functions which are meromorphic in a neighborhood of $p\in\Sigma$ is given by

$$\{\xi_p^k\}_{k\in\mathbb{Z}},\ \text{with}\ \xi_p=(x-x(p))^{1/\mathrm{ord}_p(x)}.$$

If $x(p)=\infty$ we set $\xi_p=x^{1/\mathrm{ord}_p(x)}$, with $\mathrm{ord}_p(x)<0$. The following set of cycles generates an integer lattice in the space of second kind cycles:

$$\mathcal{A}_{p,k}=\gamma_p.\xi_p^k,\ \ p\in\Sigma,k\geqslant 0, \tag{2}$$

$$\mathcal{B}_{p,k}=\frac{1}{2\pi i}\gamma_p.\frac{\xi_p^{-k}}{k},\ \ p\in\Sigma,k\geqslant 1. \tag{3}$$

Given a meromorphic 1-form ω on Σ, for every pole p of ω, we define for every $j \geqslant 0$ the *KP times*:

$$t_{p,j} = \operatorname*{Res}_{p} (\xi_p)^j \omega = \frac{1}{2\pi i} \int_{\mathcal{A}_{p,j}} \omega := \frac{1}{2\pi i} \int_{\gamma_p} \xi_p^j \, \omega, \tag{4}$$

so that

$$\omega \sim \sum_{j=0}^{\deg_p(\omega)} t_{p,j} \xi_p^{-j-1} d\xi_p + \text{analytic at } p. \tag{5}$$

Since we assumed Σ to be compact, the number of poles is finite. Moreover, all the times with $j \geqslant \deg_p \omega$ are vanishing. Therefore, only a finite number of times are non-zero.

1.3. **Context and outline.** One of our motivations to study the problem of quantum curves for any spectral curve with a global involution was two be able to recover the whole isomonodromic system associated to Painlevé I just from loop equations. We also aimed to give the first quantum curves for spectral curves of genus $g > 1$ and we were especially interested to see how introducing deformations with respect to the times could give rise to systems of PDEs that we consider more natural in general.

1.3.1. *Comparison to the literature.* In [23], they generalize the techniques employed in [22] to find the quantum curves for admissible curves to apply them to the family of genus one spectral curves given by the Weierstrass equation. They find an order two differential operator that annihilates the perturbative wave-function ψ. However, it is not a quantum curve, since it contains infinitely many $\hbar$ corrections which are not meromorphic functions of x. They also check the first orders of the conjectural quantum curve [11, 14, 3] for the non-perturbative wave-function.

In [19], they focus on the Painlevé I spectral curve, which is a degenerate torus, and they get from topological recursion a PDE that annihilates the wave function, which is compatible with the isomonodromic system and, together with another identity coming from integrable systems, provides a quantum curve that annihilates the wave function. In [17], the first author slightly generalizes the same results to the case of any elliptic curve, that is he considers not only the degenerate case of Painlevé I, but tori where none of the two cycles are pinched. In both papers, they show that the $\hbar$ corrections from [23] can be controlled by a derivative with respect to a deformation parameter. The quantum curves still contain inifinitely many $\hbar$-correction terms, but in this case, these corrections are given by the asymptotic expansion of the solution of Painlevé I around $\hbar \to 0$ [16].

In [18], the approach is reversed: they prove that Lax pairs associated with $\hbar$-dependent Painlevé equations satisfy the topological type property of [1], which implies that one can reconstruct the $\hbar$-expansion of the isomonodromic τ-function from to topological recursion. Finally, in [21], they generalize this result showing that it is always possible to deform a differential equation $\partial_x \Psi(x) = \mathcal{L}(x)\Psi(x)$, with $\mathcal{L}(x) \in \mathfrak{sl}_2(\mathbb{C})$ by introducing a formal parameter $\hbar$ in such a way that it satisfies the topological type property.

In the present work, we recover the PDE from [19, 17] from loop equations (which are necessary for topological recursion, but not sufficient) and as part of a system that we obtain because we consider a wave-function where the integrals are over any divisor of degree zero. With our system, we are able to recover the whole isomonodromic system associated to Painlevé I just from loop equations. We also give an additional meaning to the deformation parameter that appears naturally in [19, 17] for the case of elliptic curves, making use of the powerful idea of deforming with respect to the generalized cycles introduced in the previous section. The elliptic curve case is a very concrete case in which there is only one such deformation parameter, but we see that we need to consider several in the higher genus cases.

1.3.2. *Outline.* In Section 2 we introduce the type of curves we consider in this work and relate them to the concept of spectral curves as input of the topological recursion. We also give the link to the spectral curves in the setting of isomonodromy systems, which serves as a motivation to us. Moreover, we compute the deformation parameters of the family of elliptic curves, which recovers the Painlevé isomonodromy system setting in the degenerate case; in particular, the so-called KP times.

In Section 3 we recall the loop equations for our specific setting and deduce some interesting consequences relating them to time deformations, which appear when considering spectral curves of genus $g > 0$.

In Section 4 we prove our main result. We obtain from the loop equations a system of partial differential equations that annihilates our wave-function defined from topological recursion integrating over a general divisor. We also give the shape of this system in the particular cases of genus zero and elliptic curves. Finally, we consider our system for the particular case of a two-point divisor, which then consists of only two PDEs, with derivations with respect to two spectral variables and the deformation parameters, called times. We are able to combine the two PDEs in such a way that we eliminate one of the spectral variables.

In Section 5, we argue that if the spectral curve comes from an isomonodromic system, then the topological recursion non-perturbative wave function has to coincide with the solution of the isomonodromic system, which implies an ODE, which is the quantum curve we were looking for. As particular interesting cases, we recover the Painlevé system and equation, and its higher analogues defined in terms of Gelfand-Dikii polynomials.

Acknowledgements. B.E.'s work is supported by the ERC Synergie Grant ERC-2018-SyG 810573 "ReNewQuantum". It is also partly supported by the ANR grant Quantact : ANR-16-CE40-0017. E.G.-F. was supported by the public grant "Jacques Hadamard" as part of the Investissement d'avenir project, reference ANR-11-LABX-0056-LMH, LabEx LMH.

While we were finishing this manuscript, we learnt that Orantin and Marchal were working on exactly the same topic, and we are grateful to them to have informed us, and for valuable discussions about this. We plan to work together on the upcoming generalization to plane curves of arbitrary degrees.

We would also like to thank R. Belliard, M. Bergère, G. Borot, V. Bouchard, J. Zinn-Justin and A. Voros for valuable discussions on the subject.

2. Spectral curves with a global involution

In this article we focus on algebraic plane curves of the form

$$y^2 = P(x), \tag{6}$$

with $P(x) \in \mathbb{C}[x]$ an arbitrary polynomial of x, and we will generalize to $y^2 = R(x)$ with $R(x) \in \mathbb{C}(x)$ an arbitrary rational function of x.

The degree of the polynomial $y^2 - R(x)$ is related to the genus of the curve. For example, in the case in which R is a polynomial of degree $2m+1$ or $2m+2$ the curve has genus $\hat{g} \leqslant m$, with equality if the plane curve is smooth. If the degree is odd, the curve has one point at infinity and if the degree is even, the curve has two points at infinity. If $\hat{g} > 1$, the curve is called hyperelliptic; if $\hat{g} = 1$ (with a distinguished point), it is called elliptic, and if $\hat{g} = 0$, it is called rational.

2.1. **Spectral curves as input of the topological recursion.**

The method of topological recursion associates to a spectral curve $\mathcal{S}$ a doubly indexed family of meromorphic multi-differentials $\omega_{g,n}$ on Σ^n:

$$\text{TR:} \quad \text{Spectral curve } \mathcal{S} = (\Sigma, x, y\mathrm{d}x, B) \rightsquigarrow \text{Invariants } \omega_{g,n} \ (F_g = \omega_{g,0}).$$

A spectral curve is the data of Σ a Riemann surface, $x : \Sigma \to \mathbb{CP}^1$ a holomorphic projection to the base $\mathbb{CP}^1$, making Σ a ramified cover of the sphere, $y\mathrm{d}x$ a meromorphic 1-form on Σ, and B a 2nd kind fundamental differential, i.e. a symmetric $1 \otimes 1$ form on $\Sigma \times \Sigma$ with normalized double pole on the diagonal and no other pole, behaving near the diagonal as:

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2} + \text{holomorphic at } z_1 = z_2.$$

In case the spectral curve is of genus 0, i.e. $\Sigma = \mathbb{CP}^1$, it is known that such a B is unique and is worth

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

If the genus $\hat{g}$ of Σ is $\geqslant 1$, B is not unique since one can add any symmetric bilinear tensor product of holomorphic 1-forms. A way to find a unique one is to choose a Torelli marking, which is a choice of a symplectic basis $\{\{\mathcal{A}_i\}_{i=1}^{\hat{g}}, \{\mathcal{B}_i\}_{i=1}^{\hat{g}}\}$ of $H_1(\Sigma, \mathbb{Z})$. There exists a unique B normalized on the $\mathcal{A}$-cycles of $H_1(\Sigma, \mathbb{Z})$

$$\oint_{\mathcal{A}_i} B(z_1, \cdot) = 0$$

Such a bi-differential has a natural construction in algebraic geometry and is called the normalized *fundamental differential of the second kind* on Σ. See [13] for constructing B for general algebraic plane curves.

Remark 2.1. The coordinate $x : \Sigma \to \mathbb{CP}^1$ in the definition of spectral curve can be thought as a ramified covering of the sphere. We call *degree* of the spectral curve the number of sheets of the covering, i.e. the number of preimages of a generic point. In this article, we focus on spectral curves of degree 2 with a global involution $(x, y) \mapsto (x, -y)$.

2.2. **Spectral curves from isomonodromic systems.** Painlevé transcendents have their origin in the study of special functions and of isomonodromic deformations of linear differential equations. They are solutions to certain nonlinear second-order ordinary differential equations in the complex plane with the Painlevé property, i.e. the only movable singularities are poles.

An $\hbar$-dependent *Lax pair* is a pair $(\mathcal{L}(x, t; \hbar), \mathcal{R}(x, t; \hbar))$ of 2×2 matrices, whose entries are rational functions of x and holomorphic in t such that the system of partial differential equations

$$\begin{cases} \hbar \dfrac{\partial}{\partial x} \Psi(x, t) = \mathcal{L}(x, t; \hbar) \Psi(x, t), \\ \hbar \dfrac{\partial}{\partial t} \Psi(x, t) = \mathcal{R}(x, t; \hbar) \Psi(x, t) \end{cases}$$

is compatible. We call such a system an *isomonodromy system*.

The compatibility condition, i.e. $\frac{\partial}{\partial t \partial x} \Psi = \frac{\partial}{\partial x \partial t} \Psi$, is equivalent to the so-called zero-curvature equation:

$$\hbar \frac{\partial \mathcal{L}}{\partial t} - \hbar \frac{\partial \mathcal{R}}{\partial x} + [\mathcal{L}, \mathcal{R}] = 0.$$

In [20], Jimbo and Miwa gave a list of the Lax pairs whose compatibility conditions are equivalent to the six Painlevé equations.

Let us consider the expansion around $\hbar = 0$ of the first equation of the system: $\mathcal{L}(x, t; \hbar) = \sum_{k \geqslant 0} \hbar^k \mathcal{L}_k(x, t)$. The associated *spectral curve* is given by

$$\det(y \, \mathrm{Id} - \mathcal{L}_0(x, t)) = 0, \tag{7}$$

which is actually a family of algebraic curves parametrized by t.

2.2.1. *Motivational example: The first Painlevé equation.* Let us consider the first Painlevé equation with a formal small parameter $\hbar$:

$$P_I:\ \frac{\hbar^2}{2}\frac{\partial^2}{\partial t^2}U - 3U^2 = t.$$

The leading term $u = u(t)$ of a formal power series solution $U(t,\hbar) = u(t) + \sum_{k\geqslant 1}\hbar^{2k}u_k(t)$ satisfies $t = -3u^2$ and determines the subleading terms recursively:

$$u_k = c_k u^{1-5k}, c_k \in \mathbb{Q}$$

by the recursion

$$c_0 = 1 \quad , \quad 2c_{k+1} = \frac{25k^2-1}{6^3}c_k - \sum_{j=1}^{k} c_j c_{k+1-j}.$$

The coefficient $u_k(t)$ has a singularity at $u = 0$, i.e. $t = 0$. This special point is called a turning point of P_I. We shall assume that $t \neq 0$. We denote by $\dot{U}$ the derivative with respect to t of $U(t,\hbar)$.

The Tau-function $\mathcal{T}(t)$ is defined in such a way that

$$U(t) = -\hbar^2\, \frac{\partial^2}{\partial t^2}\log\mathcal{T}.$$

The Painlevé equation ensures that $\mathcal{T}$ is an entire function with simple zeros at the movable poles of U.

The Lax pair associated to the first Painlevé equation is given by

$$\mathcal{L}(x,t;\hbar) := \begin{pmatrix} \frac{\hbar}{2}\dot{U} & x-U \\ (x-U)(x+2U)+\frac{\hbar^2}{2}\ddot{U} & -\frac{\hbar}{2}\dot{U}\end{pmatrix} \text{ and } \mathcal{R}(x,t;\hbar) := \begin{pmatrix} 0 & 1 \\ x+2U & 0\end{pmatrix}. \tag{8}$$

The leading term of $\mathcal{L}$ in its expansion around $\hbar = 0$ is given by

$$\mathcal{L}_0(x,t) = \begin{pmatrix} 0 & x-u \\ x^2+ux-2u^2 & 0\end{pmatrix}.$$

The spectral curve reads

$$\det(y\,\mathrm{Id} - \mathcal{L}_0(x,t)) = y^2 - (x-u)^2(x+2u) = 0, \tag{9}$$

which is actually a family of algebraic curves parametrized by t. Since we have assumed $t \neq 0$, the two roots of $R(x) = (x-u)^2(x+2u)$ $x = u$ and $x = -2u$ are distinct, but the root $x = u$ has multiplicity 2. These curves have genus 0 or, more precisely, constitute a family of tori with one of the cycles pinched.

In general, we want to study the family of tori given by

$$y^2 = x^3 + tx + V, \tag{10}$$

where $R(x) = x^3 + tx + V$ has three different roots. The case $t = -3u^2$, $V = 2u^3$ recovers the particular degenerate case (9).

2.3. **Parametrizations and deformation parameters of the elliptic case.** In the elliptic case, we give the parametrizations and compute the coordinates because it has more structure than the genus 0 case, since we need to introduce one deformation parameter, and it also illustrates how the general case works. The degenerate case corresponds to Painlevé I, which is our prototypical example, when making the connection to isomonodromy systems.

2.3.1. *Degenerate case.* When $V = 2u^3$, consider the parametrization of the spectral curve given by

$$\begin{cases} x(z) = z^2 - 2u, \\ y(z) = z^3 - 3uz, \end{cases}$$

with $z \in \Sigma = \mathbb{C}\mathrm{P}^1$ and the fundamental form of the second kind on Σ:

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

It satisfies

$$y^2 = x^3 + tx + V,$$

with $t = -3u^2$, $V = 2u^3$.

This curve has one ramification point at $z = 0$ and one pole at $z = \infty$.

Near $z = \infty$, we have

$$y \sim x^{\frac{3}{2}} + \frac{t}{2x^{\frac{1}{2}}} + \frac{V}{2x^{\frac{3}{2}}} - \frac{t^2}{8x^{\frac{5}{2}}} - \frac{tV}{4x^{\frac{7}{2}}} + O(x^{-\frac{9}{2}}).$$

This implies that

$$t_{\infty,1} = \frac{1}{2\pi i} \int_{\mathcal{A}_{\infty,1}} y\mathrm{d}x = \operatorname*{Res}_{z\to\infty} x^{-\frac{1}{2}}\, y\mathrm{d}x = -t,$$

$$\int_{\mathcal{B}_{\infty,1}} y\mathrm{d}x = \operatorname*{Res}_{z\to\infty} x^{\frac{1}{2}}\, y\mathrm{d}x = -V,$$

$$t_{\infty,5} = \frac{1}{2\pi i} \int_{\mathcal{A}_{\infty,5}} y\mathrm{d}x = \operatorname*{Res}_{z\to\infty} x^{-\frac{5}{2}}\, y\mathrm{d}x = -2,$$

$$\int_{\mathcal{B}_{\infty,5}} y\mathrm{d}x = \frac{1}{5}\operatorname*{Res}_{z\to\infty} x^{\frac{5}{2}}\, y\mathrm{d}x = \frac{tV}{10},$$

where we use the generalized cycles $\mathcal{A}_{p,k}$ and $\mathcal{B}_{p,k}$ defined in (2) and we have considered only the values $k = 1, 5$ for which $t_{\infty,k} \neq 0$.

The degenerate cycle $\mathcal{A}$ corresponds to a small simple closed curve encircling $z_0 = \sqrt{3u}$ and the cycle $\mathcal{B}$ corresponds to the chain $(-\sqrt{3u} \to \sqrt{3u})$. Therefore, we have

$$\epsilon = \frac{1}{2\pi i} \oint_{\mathcal{A}} y\mathrm{d}x = 0 \quad , \quad I = \oint_{\mathcal{B}} y\mathrm{d}x = \frac{-8}{15}\, (3u)^{\frac{5}{2}}.$$

The prepotential F_0 [12, 15] is worth

$$F_0 = \frac{1}{2}\left(\epsilon I + \sum_k t_{\infty,k} \int_{\mathcal{B}_{\infty,k}} y\mathrm{d}x\right) = \frac{1}{2}\left(tV - 2\frac{tV}{10}\right) = \frac{2}{5}tV = -\frac{12}{5}u^5$$

and satisfies

$$\frac{\partial F_0}{\partial t} = 2u^3 = V \quad , \quad \frac{\partial^2 F_0}{\partial t^2} = \frac{\partial V}{\partial t} = -u.$$

2.3.2. *Non-degenerate case: elliptic curves.* When $V \neq 2u^3$, we shall consider the Weierstrass parametrization of the torus of modulus τ, and with a scaling ν:

$$\begin{cases} x(z) = \nu^2 \wp(z), \\ y(z) = \frac{\nu^3}{2} \wp'(z), \end{cases}$$

with $z \in \Sigma = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ the torus of modulus τ. The fundamental form of the second kind on Σ, normalized on the $\mathcal{A}$ cycle is:

$$B(z_1, z_2) = (\wp(z_1 - z_2) + G_2(\tau)) \mathrm{d}z_1 \mathrm{d}z_2.$$

with $G_k(\tau)$ the k^{th} Eisenstein series. It satisfies

$$y^2 = x^3 + tx + V,$$

with

$$t = -15\nu^4 G_4(\tau), \quad , \quad V = -35\nu^6 G_6(\tau).$$

Instead of parametrizing the spectral curve with t and V, we shall parametrize it with t and ϵ where

$$\epsilon = \frac{1}{2\pi i} \oint_{\mathcal{A}} y \mathrm{d}x = 3\nu^5 G_4'(\tau).$$

We shall now write

$$V = V(t, \epsilon).$$

We have:

$$\mathrm{d}V = 2\pi i \nu \mathrm{d}\epsilon - \nu^2 G_2(\tau) \mathrm{d}t.$$

This curve has 3 ramification points at $z = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$, and one pole at $z = 0$.

Near $z = 0$, we have

$$y \sim x^{\frac{3}{2}} + \frac{t}{2x^{\frac{1}{2}}} + \frac{V}{2x^{\frac{3}{2}}} - \frac{t^2}{8x^{\frac{5}{2}}} - \frac{tV}{4x^{\frac{7}{2}}} + O(x^{-\frac{9}{2}}).$$

This implies that

$$t_{\infty,1} = \frac{1}{2\pi i} \int_{\mathcal{A}_{\infty,1}} y \mathrm{d}x = \operatorname*{Res}_{z \to 0} x^{-\frac{1}{2}} \, y \mathrm{d}x = -t,$$

$$\int_{\mathcal{B}_{\infty,1}} y \mathrm{d}x = \operatorname*{Res}_{z \to 0} x^{\frac{1}{2}} \, y \mathrm{d}x = -V.$$

$$t_{\infty,5} = \frac{1}{2\pi i} \int_{\mathcal{A}_{\infty,5}} y \mathrm{d}x = \operatorname*{Res}_{z \to \infty} x^{-\frac{5}{2}} \, y \mathrm{d}x = -2,$$

$$\int_{\mathcal{B}_{\infty,5}} y \mathrm{d}x = \frac{1}{5} \operatorname*{Res}_{z \to \infty} x^{\frac{5}{2}} \, y \mathrm{d}x = \frac{tV}{10}.$$

We also define

$$\epsilon = \frac{1}{2\pi i} \oint_{\mathcal{A}} y \mathrm{d}x,$$

$$I = \oint_{\mathcal{B}} y \mathrm{d}x.$$

The prepotential $F_0 = \omega_{0,0}$ is worth

$$F_0 = \frac{1}{2} \left(tV - 2\frac{tV}{10} + I\epsilon \right) = \frac{2}{5} tV + \frac{1}{2} I\epsilon,$$

and satisfies

$$\mathrm{d}F_0 = V \mathrm{d}t + I \mathrm{d}\epsilon.$$

In terms of the torus modulus τ and scaling ν, we have

$$t = -15\nu^4 G_4(\tau), \qquad V = -35\nu^6 G_6(\tau), \qquad \epsilon = 3\nu^5 G_4'(\tau), \qquad I = 2\pi i\tau\epsilon + \frac{4}{5}\nu t.$$

We also have

$$F_1 = \frac{1}{48}\log\left(4t^3 + 27V^2\right) + \frac{1}{4}\log\frac{2}{\nu}.$$

3. Loop equations and deformation parameters

We start by recalling the loop equations for the topological recursion applied to any spectral curve of degree 2 with a global involution. Let $y^2 = R(x)$, with $R \in \mathbb{C}(x)$. The family of curves that we consider has the global involution $z \mapsto -z$, i.e. $x(z) = x(-z)$.

Let $\omega_{0,1}(z) := y(z)\mathrm{d}x(z)$, $\omega_{0,2}(z_1, z_2) := B(z_1, z_2)$ and $\omega_{g,n}$ for $2g - 2 + n > 0$ be defined as the topological recursion amplitudes for this initial data [15].

The loop equations for this particular case read:

Theorem 3.1. [15] *The linear loop equations read:*

$$\omega_{g,n+1}(z, z_1, \dots, z_n) + \omega_{g,n+1}(-z, z_1, \dots, z_n) = \delta_{g,0}\delta_{n,1}\,\frac{\mathrm{d}x(z)\mathrm{d}x(z_1)}{(x(z) - x(z_1))^2}. \tag{11}$$

The quadratic loop equations claim that the following expression

$$\frac{1}{\mathrm{d}x(z)^2}\left(\omega_{g-1,n+2}(z, -z, z_1, \dots, z_n) + \sum_{\substack{g_1+g_2=g,\\ I_1\sqcup I_2=\{z_1,\dots,z_n\}}} \omega_{g_1,1+|I_1|}(z, I_1)\omega_{g_2,1+|I_2|}(-z, I_2)\right) \tag{12}$$

is a rational function of $x(z)$ with no poles at the branch-points.

We will make use of an immediate consequence of the loop equations:

Corollary 3.2. *For all $g, n \geqslant 0$,*

$$\begin{aligned} P_{g,n}(x(z); z_1, \dots, z_n) \;:=\; & \frac{-1}{\mathrm{d}x(z)^2}\Bigg(\omega_{g-1,n+2}(z, -z, z_1, \dots, z_n) \\ & + \sum_{\substack{g_1+g_2=g,\\ I_1\sqcup I_2=\{z_1,\dots,z_n\}}} \omega_{g_1,1+|I_1|}(z, I_1)\omega_{g_2,1+|I_2|}(-z, I_2)\Bigg) \\ & + \sum_{i=1}^{n} \mathrm{d}_i\left(\frac{1}{x(z) - x(z_i)}\frac{\omega_{g,n}(z_1, \dots, -z_i, \dots, z_n)}{\mathrm{d}x(z_i)}\right) \end{aligned} \tag{13}$$

is a rational function of $x(z)$ that has no poles at the branch-points and no poles when $x(z) = x(z_i)$.

Proof. First, the expression (13) is even under $z \to -z$ and meromorphic on Σ, hence a meromorphic function of $x(z) \in \mathbb{CP}^1$, i.e. a rational function of $x(z)$. From the loop equations, it has no pole at branchpoints. Let us study the behavior at $z = z_i$. The only term in (12) that contains a pole at $z = z_i$ is

$$\frac{1}{\mathrm{d}x(z)^2}B(z, z_i)\,\omega_{g,n}(-z, z_1, \dots, \hat{z}_i, \dots, z_n).$$

Remark that $B(-z, z_i)$ has no pole at $z = z_i$ and

$$B(z, z_i) + B(-z, z_i) = \frac{\mathrm{d}x(z)\mathrm{d}x(z_i)}{(x(z) - x(z_i))^2} = \mathrm{d}_i\left(\frac{\mathrm{d}x(z)}{x(z) - x(z_i)}\right).$$

Therefore we add a term without any poles to the previous one and consider the term with a pole at $z = z_i$ to be

$$\frac{1}{\mathrm{d}x(z)^2}\frac{\mathrm{d}x(z)\mathrm{d}x(z_i)}{(x(z)-x(z_i))^2}\,\omega_{g,n}(-z,z_1,\dots,\hat{z_i},\dots,z_n).$$

We can write it as

$$\mathrm{d}_i\Bigg(\left(\frac{1}{x(z)-x(z_i)}\right)\Bigg(\frac{\omega_{g,n}(-z,z_1,\dots,\hat{z_i},\dots,z_n)}{\mathrm{d}x(z)} - \frac{\omega_{g,n}(-z_i,z_1,\dots,\hat{z_i},\dots,z_n)}{\mathrm{d}x(z_i)} + \frac{\omega_{g,n}(-z_i,z_1,\dots,\hat{z_i},\dots,z_n)}{\mathrm{d}x(z_i)}\Bigg)\Bigg). \tag{14}$$

The sum of the first two terms does not have a pole at $z = z_i$. Therefore subtracting the last term for all $i = 1, \dots, n$, we obtain an expression with no poles at $z = z_i$. Since this expression is an even function of z, there is no pole at $z = -z_i$ either. □

Recall that in general $\omega_{0,2} = B$ can have poles only at coinciding points and the $\omega_{g,n}$'s with $2g-2+n > 0$ can have poles only at ramification points. Therefore, from the corollary we see that $P_{g,n}(x(z), z_1, \dots, z_n)$ as a function of z can only have poles at the poles of $\omega_{0,1} = y\mathrm{d}x$.

3.1. **Relation to time deformation for Painlevé I.** Now we restrict ourselves to curves described by polynomials of the form

$$y^2 = x^3 + tx + V,$$

with $t = -3u^2$ and $V = 2u^3$. In this case, $\omega_{0,1} = y\mathrm{d}x$ can only have poles at $x(z) = \infty$.

The topological recursion amplitudes for $2g-2+n \geqslant 0$ are analytic away from branchpoints; in particular they are analytic at ∞, with the following behavior near $z = \infty$:

$$\omega_{g,n}(z, z_1, \dots, z_n) = O(z^{-2}).$$

In our case, this implies the following behavior at $x(z) = \infty$:

$$\frac{\omega_{g,n+1}(z,z_1,\dots,z_n)}{\mathrm{d}x(z)} = O(x(z)^{\frac{-3}{2}}), \text{ for } 2g-2+n \geqslant 0. \tag{15}$$

Since the only pole can come from terms that contain $\omega_{0,1} = y\mathrm{d}x$, $P_{g,n}$ has the following behavior at $x(z) \to \infty$:

$$P_{g,n}(x(z), z_1, \dots, z_n) = 2\frac{y(z)\mathrm{d}x(z)\,\omega_{g,n+1}(z,z_1,\dots,z_n)}{\mathrm{d}x(z)^2} + O(x(z)^{-3}),$$

i.e.

$$P_{g,n}(x(z), z_1, \dots, z_n) = 2y(z)O(x(z)^{\frac{-3}{2}}) + O(x(z)^{-3}) = O(1), \text{ for } 2g-2+n \geqslant 0, \tag{16}$$

where the last behavior comes from the fact that in the elliptic curve case, we have $y \sim x^{\frac{3}{2}}$.

We have seen that $P_{g,n}$ is a polynomial of degree 0, that is independent of z, and can be written:

$$P_{g,n}(x(z), z_1, \dots, z_n) = 2\lim_{z\to\infty} x(z)^{\frac{3}{2}}\,\frac{\omega_{g,n+1}(z,z_1,\dots,z_n)}{\mathrm{d}x(z)}.$$

Corollary 3.3. *For* $(g,n) \neq (0,0),(0,1)$*:*

$$P_{g,n}(x(z), z_1, \dots, z_n) = -\oint_{\mathcal{B}_{\infty,1}} \omega_{g,n+1}(z, z_1, \dots, z_n) = \frac{\partial}{\partial t}\omega_{g,n}(z_1, \dots, z_n), \tag{17}$$

with $\mathcal{B}_{\infty,1}$ *the second kind cycle given by* $\frac{1}{2\pi i}\mathcal{C}_\infty\sqrt{x(z)}$*, where* $\mathcal{C}_\infty$ *denotes a small contour around* ∞*.*

Moreover

$$P_{0,0}(x(z)) = y(z)^2 = x^3 + tx + V = x^3 + tx + \frac{\partial}{\partial t}\omega_{0,0}, \tag{18}$$

$$P_{0,1}(x(z), z_1) = 2\frac{y(z)}{\mathrm{d}x(z)}B(z, z_1) - \mathrm{d}_1\left(\frac{y(z) + y(z_1)}{x(z) - x(z_1)}\right) = \frac{\partial}{\partial t}\omega_{0,1}(z_1). \tag{19}$$

Proof. The expressions for $(g, n) = (0, 0), (0, 1)$ are direct computations using (13).

$$P_{0,1}(x(z), z_1) = \frac{y(z)}{\mathrm{d}x(z)}(B(z, z_1) - B(-z, z_1)) - \mathrm{d}_1\left(\frac{y(z_1)}{x(z) - x(z_1)}\right).$$

In order to get the second equality in (19), observe from the first equality that also $P_{0,1}$ has the following behavior at $x(z) \to \infty$:

$$P_{0,1}(x(z), z_1) = 2\frac{y(z)\, B(z, z_1)}{\mathrm{d}x(z)} + O(x(z)^{-1}) = 2y(z)O(x(z)^{\frac{-3}{2}}) + O(x(z)^{-1}) = O(1).$$

For $(g, n) \neq (0, 0)$, since $P_{g,n}$ is constant with respect to z, we can write

$$\begin{aligned} P_{g,n} &= 2\lim_{x(z)\to\infty} x(z)^{\frac{3}{2}}\ \frac{\omega_{g,n+1}(z, z_1, \ldots, z_n)}{\mathrm{d}x(z)} \\ &= -\operatorname*{Res}_{z\to\infty}\ \sqrt{x(z)}\ \omega_{g,n+1}(z, z_1, \ldots, z_n) = \\ &= -\oint_{\mathcal{B}_{\infty,1}} \omega_{g,n+1}(z, z_1, \ldots, z_n). \end{aligned} \tag{20}$$

Moreover, since $t = -t_{\infty,1}$, we have $\frac{\partial}{\partial t}\omega_{0,1} = -\frac{\partial}{\partial t_{\infty_1}}\omega_{0,1} = -\int_{\mathcal{B}_{\infty,1}}\omega_{0,2}$, and since B is the Bergman kernel normalized on the $\mathcal{A}$-cycles, we also have $\frac{\partial}{\partial t_{\infty,1}}\omega_{0,2} = \int_{\mathcal{B}_{\infty,1}}\omega_{0,3}$. Knowing that, it is proved in [15, 12] that

$$\oint_{\mathcal{B}_{\infty,1}} \omega_{g,n+1}(z, z_1, \ldots, z_n) = \frac{\partial}{\partial t_{\infty,1}}\omega_{g,n}(z_1, \ldots, z_n) = -\frac{\partial}{\partial t}\omega_{g,n}(z_1, \ldots, z_n).$$

□

3.2. **Generalization to any plane curve with a global involution.** Our goal is to find the relation between $P_{g,n}$ from loop equations and an operator depending on time deformations acting on the topological recursion amplitudes. In this section, we generalize the relation that we have just found in the Painlevé I case to all plane curves with a global involution.

Consider algebraic curves of the form

$$y^2 = R(x), \tag{21}$$

with $R(x) \in \mathbb{C}(x)$ an arbitrary rational function.

This is parametrized by a pair of meromorphic functions x, y on a Riemann surface Σ. The ramified covering given by $x : \Sigma \to \mathbb{CP}^1$ is a double cover. Depending on the parity of the behavior of y at $x \to \infty$, x has either one double pole (order $d = -2$) or 2 simple poles (order $d = -1$). Let us denote σ the global involution which sends $(x, y) \mapsto (x, -y)$.

The 1-form $\omega_{0,1} = y\mathrm{d}x$ can have a pole over $x = \infty$ and poles over the zeros of the denominator of $R(x)$. We call ζ_i the poles of $\omega_{0,1}$ of respective degrees given by $m_i + 1$.

Let us define $d_i := \mathrm{ord}_{\zeta_i}(x)$. If ζ_i is not a pole of x, we assume that it is not a ramification point, hence $d_i = 1$. If ζ_i is a pole of x, then d_i can be either -2 or -1 as we commented, depending on ζ_i being a ramification point or not. Notice that if ζ_i is a pole, so is $\sigma(\zeta_i)$, and thus poles come in pairs. Moreover, we can only have $\sigma(\zeta_i) = \zeta_i$, if ζ_i is a double pole of x, which corresponds to the case $d_i = -2$. Near ζ_i we use the local variable

$$\xi_i = (x - x(\zeta_i))^{\frac{1}{d_i}}, \tag{22}$$

where we define $x(\zeta_i) = 0$, if ζ_i is a pole of x.

We write the Laurent expansion:

$$y\mathrm{d}x \sim \sum_{j=0}^{m_i} t_{\zeta_i,j}\ \xi_i^{-1-j}\mathrm{d}\xi_i + \text{analytic at } \zeta_i. \tag{23}$$

This defines the local KP times [12]

$$t_{\zeta_i,j} = \operatorname*{Res}_{\zeta_i} \xi_i^j y\mathrm{d}x = \frac{1}{2\pi i}\oint_{\mathcal{A}_{\zeta_i,j}} y\mathrm{d}x. \tag{24}$$

Notice that for poles for which $\sigma(\zeta_i) \neq \zeta_i$, we have

$$t_{\sigma(\zeta_i),j} = -t_{\zeta_i,j}. \tag{25}$$

Again, loop equations imply that the $P_{g,n}$ defined in (13)

$$P_{g,n}(x; z_1, \ldots, z_n) \tag{26}$$

has no pole at coinciding points or at branchpoints. It must be a rational function of x, whose poles can be only at the poles of $y\mathrm{d}x$, i.e. at the poles of $R(x)$ and possibly at $x = \infty$.

Proposition 3.4. *Defining the operator*

$$\begin{aligned} L(x) \quad := \quad & \sum_{i, x(\zeta_i)=\infty} \sum_{j=1-2d_i}^{m_i} t_{\zeta_i,j} \sum_{0\leqslant k\leqslant \frac{1-j}{d_i}-2} x^k\left(-\frac{j}{d_i} - k - 2\right)\frac{\partial}{\partial t_{\zeta_i,j+d_i(k+2)}} \\ & + \sum_{i, x(\zeta_i)\neq\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k=0}^{j} (x - x(\zeta_i))^{-(k+1)}(j+1-k)\frac{\partial}{\partial t_{\zeta_i,j+1-k}}, \end{aligned}$$

then,

$$P_{g,n}(x; z_1, \ldots, z_n) = L(x).\omega_{g,n}(z_1, \ldots, z_n). \tag{28}$$

Proof. Let us write the Cauchy formula for $x = x(z)$

$$\begin{aligned} P_{g,n}(x; z_1, \ldots, z_n) &= \operatorname*{Res}_{x'\to x} \frac{\mathrm{d}x'}{x'-x}\ P_{g,n}(x'; z_1, \ldots, z_n) \\ &= \frac{1}{2}\operatorname*{Res}_{z'\to z} \frac{\mathrm{d}x(z')}{x(z')-x(z)}\ P_{g,n}(x(z'); z_1, \ldots, z_n) \\ &\quad +\frac{1}{2}\operatorname*{Res}_{z'\to\sigma(z)} \frac{\mathrm{d}x(z')}{x(z')-x(z)}\ P_{g,n}(x(z'); z_1, \ldots, z_n) \\ &= \frac{1}{2}\sum_i \operatorname*{Res}_{z'\to\zeta_i} \frac{\mathrm{d}x(z')}{x(z)-x(z')}\ P_{g,n}(x(z'); z_1, \ldots, z_n) \\ &= -\frac{1}{2}\sum_{i, x(\zeta_i)=\infty}\sum_{k\geqslant 0} x(z)^k \operatorname*{Res}_{z'\to\zeta_i} x(z')^{-(k+1)}\mathrm{d}x(z')\ P_{g,n}(x'; z_1, \ldots, z_n) \\ &\quad +\frac{1}{2}\sum_{i, x(\zeta_i)\neq\infty}\sum_{k\geqslant 0} \xi_i(z)^{-(k+1)} \operatorname*{Res}_{z'\to\zeta_i} \xi_i(z')^k\mathrm{d}x(z')\ P_{g,n}(x'; z_1, \ldots, z_n). \end{aligned} \tag{29}$$

Since the behavior at the poles is given by the terms containing $\omega_{0,1} = y\mathrm{d}x$ in (13), near any of the poles ζ_i we have

$$P_{g,n}(x(z); z_1, \ldots, z_n) \sim \frac{2y(z)}{\mathrm{d}x(z)}\omega_{g,n+1}(z, z_1, \ldots, z_n) + O(\xi_i^{-2(d_i-1)}). \tag{30}$$

First consider the poles over $x = \infty$, with $d_i = -1$ or $d_i = -2$, which contribute to $P_{g,n}$ as

$$\begin{aligned}
& - \sum_{i,x(\zeta_i)=\infty} \sum_{k\geqslant 0} x(z)^k \operatorname*{Res}_{z'\to\zeta_i} x(z')^{-(k+1)} y(z')\omega_{g,n+1}(z', z_1, \ldots, z_n) \\
= & - \sum_{i,x(\zeta_i)=\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k\geqslant 0} x(z)^k \operatorname*{Res}_{z'\to\zeta_i} \xi_i(z')^{-d_i(k+1)}\xi_i(z')^{-j-1}\frac{1}{d_i\,\xi_i(z')^{d_i-1}}\omega_{g,n+1}(z', z_1, \ldots, z_n) \\
= & - \sum_{i,x(\zeta_i)=\infty} \frac{1}{d_i}\sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k\geqslant 0} x(z)^k \operatorname*{Res}_{z'\to\zeta_i} \xi_i(z')^{-d_i(k+2)-j}\omega_{g,n+1}(z', z_1, \ldots, z_n) \\
= & - \sum_{i,x(\zeta_i)=\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k\geqslant 0} x(z)^k \Big(k+2+\frac{j}{d_i}\Big)\int_{\mathcal{B}_{\zeta_i,j+d_i(k+2)}} \omega_{g,n+1}(z', z_1, \ldots, z_n) \\
= & - \sum_{i,x(\zeta_i)=\infty} \sum_{j=1-2d_i}^{m_i} t_{\zeta_i,j} \sum_{0\leqslant k\leqslant \frac{1-j}{d_i}-2} x(z)^k \Big(k+2+\frac{j}{d_i}\Big)\frac{\partial}{\partial t_{\zeta_i,j+d_i(k+2)}}\omega_{g,n}(z_1, \ldots, z_n).
\end{aligned}$$

The finite poles contribute as

$$\begin{aligned}
& \frac{1}{2}\sum_{i,x(\zeta_i)\neq\infty} \sum_{k\geqslant 0} \xi_i(z)^{-(k+1)} \operatorname*{Res}_{z'\to\zeta_i} \xi_i(z')^k dx(z')\ P_{g,n}(x'; z_1, \ldots, z_n) \\
= & \sum_{i,x(\zeta_i)\neq\infty} \sum_{k\geqslant 0} \xi_i(z)^{-(k+1)} \operatorname*{Res}_{z'\to\zeta_i} \xi_i(z')^k y(z')\omega_{g,n+1}(z', z_1, \ldots, z_n) \\
= & \sum_{i,x(\zeta_i)\neq\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k\geqslant 0} \xi_i(z)^{-(k+1)} \operatorname*{Res}_{z'\to\zeta_i} \xi_i(z')^{k-j-1}\omega_{g,n+1}(z', z_1, \ldots, z_n) \\
= & \sum_{i,x(\zeta_i)\neq\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k=0}^{j} \xi_i(z)^{-(k+1)}(j+1-k)\int_{\mathcal{B}_{\zeta_i,j+1-k}} \omega_{g,n+1}(z', z_1, \ldots, z_n) \\
= & \sum_{i,x(\zeta_i)\neq\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k=0}^{j} \xi_i(z)^{-(k+1)}(j+1-k)\frac{\partial}{\partial t_{\zeta_i,j+1-k}}\omega_{g,n}(z_1, \ldots, z_n).
\end{aligned}$$

□

We have just found a differential operator $L(x)$ in the times, whose coefficients are rational functions of x, with poles at $x = \infty$ or $x = x(\zeta_i)$, i.e. the same poles as $R(x)$, with at most the same degrees.

Example 3.1. In the elliptic case of curves of the form $y^2 = x^3 + tx + V$ we have only one pole, at $\zeta_i = \infty$, of degree $m_i = 5$, with $d_i = -2$. The only non-vanishing times are $t_{\infty,5} = -2$ and $t_{\infty,1} = -t$, and thus only the terms with $j = 5$ and $k = 0$ contribute:

$$L(x) = \sum_{j=1,5} t_{\infty,j} \sum_{0\leqslant k\leqslant -2+(j-1)/2} x^k (j/2-k-2)\frac{\partial}{\partial t_{\infty,j-2(k+2)}}$$

$$
\begin{aligned}
&= \sum_{j=5} \sum_{k=0} t_{\infty,5}(j/2-k-2)\frac{\partial}{\partial t_{\infty,j-4}} \\
&= -2\,(\frac{5}{2}-2)\frac{\partial}{\partial t_{\infty,1}} \\
&= -\frac{\partial}{\partial t_{\infty,1}} \\
&= \frac{\partial}{\partial t}.
\end{aligned}
$$

Example 3.2. In the Airy case, $y^2 = x$, we have only one pole, at $\zeta_i = \infty$, of degree $m_i = 3$, with $d_i = -2$. The sum is empty and

$$L(x) = 0.$$

Remark 3.3. More generally, the admissible curves considered in [22], are those for which

$$L(x) = 0.$$

4. PDE for any degree 2 curve

For $r \geqslant 1$, let $D = \sum_{i=1}^{r} \alpha_i [p_i]$ be a divisor on Σ, with $p_i \in \Sigma$. We call $\sum_i \alpha_i$ the *degree* of the divisor and denote $\mathrm{Div}_0(\Sigma)$ the set of divisors of degree 0. For $D \in \mathrm{Div}_0(\Sigma)$, we define the integration of a 1-form $\rho(z)$ on Σ as

$$\int_D \rho(z) := \sum_i \alpha_i \int_o^{p_i} \rho(z), \tag{34}$$

where $o \in \Sigma$ is an arbitrary base point. This integral is well defined locally, meaning that it is independent of the base point o because the degree of the divisor is zero, however it depends on a choice of homotopy class from o to p_i.

For $(g,n) \neq (0,2)$ consider the functions of D, defined locally:

$$F_{g,0}(D) := F_g, \tag{35}$$

$$F_{g,n}(D) := \overbrace{\int_D \cdots \int_D}^{n} \omega_{g,n}(z_1,\ldots,z_n), \tag{36}$$

$$F'_{g,n}(z,D) := \frac{1}{\mathrm{d}x(z)} \overbrace{\int_D \cdots \int_D}^{n-1} \omega_{g,n}(z,z_2\ldots,z_n), \tag{37}$$

$$F''_{g,n}(z,\tilde{z},D) := \frac{1}{\mathrm{d}x(z)\mathrm{d}x(\tilde{z})} \overbrace{\int_D \cdots \int_D}^{n-2} \omega_{g,n}(z,\tilde{z},z_3,\ldots,z_n). \tag{38}$$

Recall that

$$B(z_1,z_2) := \mathrm{d}_1\mathrm{d}_2 \log\left(E(z_1,z_2)\sqrt{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}\right),$$

with $E(z_1,z_2)$ being the prime form, which is defined in [15], and satisfies that it vanishes only if $z_1 = z_2$ with a simple zero and has no pole.

For $(g,n)=(0,2)$ define:

$$F_{0,2}(D) := 2\sum_{i<j}\alpha_i\alpha_j \log\left(E(p_i,p_j)\sqrt{\mathrm{d}x(p_i)\mathrm{d}x(p_j)}\right), \tag{39}$$

$$F'_{0,2}(z,D) := \frac{1}{\mathrm{d}x(z)}\mathrm{d}_z\left(\sum_{i=1}^{r}\alpha_i \log\left(E(z,p_i)\sqrt{\mathrm{d}x(z)\mathrm{d}x(p_i)}\right)\right), \tag{40}$$

$$F''_{0,2}(z,\tilde z,D) := \frac{B(z,\tilde z)}{\mathrm{d}x(z)\mathrm{d}x(\tilde z)}. \tag{41}$$

Since the $\omega_{g,n}$ are symmetric, we have the following relations for $(g,n)\neq(0,2)$:

$$\frac{\mathrm{d}}{\mathrm{d}x_i}F_{g,n}(D) = n\alpha_i F'_{g,n}(p_i,D), \tag{42}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x_i}\right)^2 F_{g,n}(D) = n(n-1)\alpha_i^2 F''_{g,n}(p_i,p_i,D) + n\alpha_i\left(\frac{\mathrm{d}}{\mathrm{d}\tilde x}F'_{g,n}(\tilde p,D)\right)_{\tilde p=p_i}, \tag{43}$$

where $x_i := x(p_i)$, $\tilde x := x(\tilde p)$, and $\frac{\mathrm{d}}{\mathrm{d}x}$ acts on meromorphic functions by taking exterior derivative and dividing by $\mathrm{d}x$, which amounts to derivate an analytic expansion of the meromorphic function with respect to a local variable x.

For $(g,n)=(0,2)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x_i}F_{0,2}(D) = 2\alpha_i\lim_{z\to p_i}\left(F'_{0,2}(z,D) - \alpha_i\frac{\mathrm{d}}{\mathrm{d}x(z)}\log\left(E(z,p_i)\sqrt{\mathrm{d}x(z)\mathrm{d}x(p_i)}\right)\right) \tag{44}$$

$$= 2\alpha_i\sum_{j\neq i}\alpha_j\frac{\mathrm{d}}{\mathrm{d}x(p_i)}\log\left(E(p_i,p_j)\sqrt{\mathrm{d}x(p_i)\mathrm{d}x(p_j)}\right). \tag{45}$$

Integrating the first part of Theorem 3.1 over a divisor D of degree 0, we obtain:

Lemma 4.1.

$$F'_{0,2}(z,D) + F'_{0,2}(-z,D) = \sum_{i=1}^{r}\frac{\alpha_i}{x(z)-x(p_i)}. \tag{46}$$

Lemma 4.2. *For t any KP time, we obtain*

$$\int_D\int_D\int_{\mathcal{B}_{\infty,1}}\omega_{0,3}(z,z_1,z_2) = \int_D\int_D\frac{\partial}{\partial t}\omega_{0,2}(z_1,z_2) = \frac{\partial}{\partial t}F_{0,2}(D). \tag{47}$$

Proof.

$$\int_D\int_D\left(B(z_1,z_2) - \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\right) + 2\sum_{i<j}\alpha_i\alpha_j\log(x_i-x_j) = \tag{48}$$

$$2\sum_{i<j}\alpha_i\alpha_j\log\left(\frac{E(p_i,p_j)\sqrt{\mathrm{d}x_i\mathrm{d}x_j}}{x_i-x_j}\right) + 2\sum_{i<j}\alpha_i\alpha_j\log(x_i-x_j) + \sum_i\alpha_i^2\log\frac{\mathrm{d}x_i}{\mathrm{d}x_i} = F_{0,2}(D).$$

Taking the derivative with respect to t of the first line gives the left hand side of (47) because we are taking this derivative at fixed x. ☐

Consider $\mathrm{d}E(z,p_1) := \mathrm{d}_z\log\left(E(z,p_1)\sqrt{\mathrm{d}x(z)\mathrm{d}x(p_1)}\right)$ and observe that

$$\lim_{z\to p_1}\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x(z)} - \frac{1}{x(z)-x(p_1)} = 0. \tag{49}$$

With this notation one can rewrite (45) as:

$$F'_{0,2}(z,D) = \sum_{i=1}^{r} \alpha_i \frac{\mathrm{d}E(z,p_i)}{\mathrm{d}x(z)}$$

and $\frac{\mathrm{d}}{\mathrm{d}x_i} F_{0,2}(D) = 2\alpha_i \sum_{j\neq i} \alpha_j \frac{\mathrm{d}E(p_i,p_j)}{\mathrm{d}x(p_i)}$.

We define

$$S_0(D,t) := \int_D y\mathrm{d}x = F_{0,1}(D), \quad S_1(D,t) := \log \prod_{i<j} \left(E(p_i,p_j)\sqrt{\mathrm{d}x(p_i)\mathrm{d}x(p_j)} \right)^{\alpha_i\alpha_j} = \frac{F_{0,2}(D)}{2},$$

$$S_m(D,t) := \sum_{\substack{2g-2+n=m-1 \\ g\geqslant 0, n\geqslant 1}} \frac{F_{g,n}(D)}{n!},$$

and

$$\psi(D,t,\hbar) := \exp(S(D,t,\hbar)), \text{ with } S(D,t,\hbar) := \sum_{m=0}^{\infty} \hbar^{m-1} S_m(D,t). \tag{50}$$

Theorem 4.3. *Let $F := \sum_{g>0} \hbar^{2g} F_g$. For every $k = 1,\dots,r$, we obtain*

$$\hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_k^2} - \sum_{i\neq k} \frac{\frac{\mathrm{d}}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_k}\frac{\mathrm{d}}{\mathrm{d}x_k}}{x_k - x_i} - L(x_k) + \sum_{\substack{i\neq j \\ i\neq k, j\neq k}} \frac{\alpha_i\alpha_j}{(x_k-x_i)(x_i-x_j)} \right)\psi = \left(R(x_k) + L(x_k).F\right)\psi. \tag{51}$$

Proof. We will give the proof of the claim for the case $k=1$, but it works exactly the same for every k.

Let us first consider the generic situation with $(g,n) \neq (0,0),(0,1),(0,2),(1,0)$. For $n=0$, we can write $P_{g,0}(x(z))$ as

$$F''_{g-1,2}(z,z,D) + \sum_{g_1+g_2=g} F'_{g_1,1}(z,D)F'_{g_2,1}(z,D) = L(x).F_{g,0}(D) = L(x).F_g(D). \tag{52}$$

Setting $z = p_1$, $x_1 = x(p_1)$ and using (42) and (43), we obtain

$$\frac{1}{2\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{g-1,2}(D) - \frac{1}{\alpha_1}\left(\frac{\mathrm{d}}{\mathrm{d}x(\tilde{p})}F'_{g-1,2}(\tilde{p},D)\right)_{\tilde{p}=p_1} + \sum_{g_1+g_2=g} \frac{1}{\alpha_1^2}\frac{\mathrm{d}F_{g_1,1}(D)}{\mathrm{d}x_1}\frac{\mathrm{d}F_{g_2,1}(D)}{\mathrm{d}x_1}.$$

Making use of (42) again, we get

$$P_{g,0}(x(z)) = \sum_{g_1+g_2=g} \frac{1}{\alpha_1^2}\frac{\mathrm{d}F_{g_1,1}(D)}{\mathrm{d}x_1}\frac{\mathrm{d}F_{g_2,1}(D)}{\mathrm{d}x_1} = L(x_1).F_g(D). \tag{53}$$

For $n>0$, we integrate (13) n times over D and use (17) to get

$$F''_{g-1,n+2}(z,z,D) + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}}^{\text{no }(0,2)} \binom{n}{n_1} F'_{g_1,n_1+1}(z,D)F'_{g_2,n_2+1}(z,D) +$$
$$- 2nF'_{0,2}(-z,D)F'_{g,n}(z,D) - n\sum_{i=1}^{r}\alpha_i \frac{F'_{g,n}(p_i,D) - F'_{g,n}(z,D)}{x(z)-x(p_i)} = L(x).F_{g,n}(D),$$

where the sum of the second term is taken over $g_i, n_i \geqslant 0$ and "no $(0,2)$" means we exclude the case $(g_i, n_i) = (0,1)$ for $i = 1, 2$, and we have used that

$$(F'_{0,2}(z,D) - F'_{0,2}(-z,D))F'_{g,n}(z,D) = -2F'_{0,2}(-z,D)F'_{g,n}(z,D) + \sum_{i=1}^{r} \alpha_i \frac{F'_{g,n}(z,D)}{x(z) - x(p_i)},$$

which follows from (46).

Letting $z = p_1$, $x_i = x(p_i)$, dividing by $n!$ and using (42) and (43), we obtain

$$\frac{1}{(n+2)!\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{g-1,n+2}(D) - \frac{1}{(n+1)!\alpha_1}\left(\frac{\mathrm{d}}{\mathrm{d}x(\tilde{p})} F'_{g-1,n+2}(\tilde{p},D)\right)_{\tilde{p}=p_1} + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}}^{\text{no }(0,2)} \frac{1}{(n_1+1)!(n_2+1)!\alpha_1^2} \frac{\mathrm{d}F_{g_1,n_1+1}(D)}{\mathrm{d}x_1} \frac{\mathrm{d}F_{g_2,n_2+1}(D)}{\mathrm{d}x_1}$$
$$- \frac{1}{n!}\sum_{i=2}^{r} \frac{1}{x(p_1) - x(p_i)}\left(\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_i} - \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1}\right) + \frac{\alpha_1}{(n-1)!}\left(\frac{\mathrm{d}F'_{g,n}(\tilde{p},D)}{\mathrm{d}x(\tilde{p})}\right)_{\tilde{p}=p_1}$$
$$- \frac{2}{n!\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1}\; F'_{0,2}(-p_1,D) = L(x_1).\frac{F_{g,n}(D)}{n!}.$$

Using (42) again, we obtain the following expression for the left hand side:

$$\frac{1}{(n+2)!\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{g-1,n+2}(D) - \frac{1}{(n+2)!\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{g-1,n+2}(D)$$
$$+ \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{g,n}(D) + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}}^{\text{no }(0,2)} \frac{1}{(n_1+1)!(n_2+1)!\alpha_1^2} \frac{\mathrm{d}F_{g_1,n_1+1}(D)}{\mathrm{d}x_1} \frac{\mathrm{d}F_{g_2,n_2+1}(D)}{\mathrm{d}x_1}$$
$$- \frac{1}{n!}\sum_{i=2}^{r} \frac{1}{x(p_1) - x(p_i)}\left(\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1}\right)$$
$$+ \frac{2}{n!\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1}\left(\sum_{i=2}^{r} \frac{\alpha_i}{x(p_1) - x(p_i)} - \; F'_{0,2}(-p_1,D)\right) =$$
$$\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{g,n}(D) + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}}^{\text{no }(0,2)} \frac{1}{(n_1+1)!(n_2+1)!\alpha_1^2} \frac{\mathrm{d}F_{g_1,n_1+1}(D)}{\mathrm{d}x_1} \frac{\mathrm{d}F_{g_2,n_2+1}(D)}{\mathrm{d}x_1}$$
$$- \frac{1}{n!}\sum_{i=2}^{r} \frac{1}{x(p_1) - x(p_i)}\left(\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1}\right)$$
$$+ \frac{2}{n!\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1}\left(F'_{0,2}(\tilde{p},D) - \frac{\alpha_1}{x(\tilde{p}) - x(p_1)}\right)_{\tilde{p}=p_1}.$$

For all $\ell \geqslant 3$, we sum this expression for all $g \geqslant 0, n \geqslant 1$ such that $2g-2+n = \ell-2$ and $P_{g,0}(x(z_1))$ from (53) for all $g \geqslant 0$ such that $2g-2 = \ell-2$, and we use (44) to obtain:

$$\sum_{\substack{2g+n=\ell\\ g\geqslant 0,n\geqslant 1}} \left(\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{g,n}(D) - \frac{1}{n!}\sum_{i=2}^{r} \frac{1}{x(p_1)-x(p_i)} \left(\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1} \right) \right)$$
$$+ \frac{1}{\alpha_1^2}\sum_{\ell_1+\ell_2=\ell} \left(\sum_{\substack{2g_1-2+n_1=\ell_1-1\\ g_1\geqslant 0,n_1\geqslant 1}} \frac{1}{n_1!}\frac{\mathrm{d}F_{g_1,n_1}(D)}{\mathrm{d}x_1} \sum_{\substack{2g_2-2+n_2=\ell_1-1\\ g_1\geqslant 0,n_2\geqslant 1}} \frac{1}{n_2!}\frac{\mathrm{d}F_{g_2,n_2}(D)}{\mathrm{d}x_1} \right) = \sum_{\substack{2g+n=\ell\\ g\geqslant 0,n\geqslant 0}} L(x_1).F_{g,n}.$$

Therefore, for $\ell \geqslant 3$, we have proved

$$\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S_{\ell-1} + \frac{1}{\alpha_1^2}\sum_{\ell_1+\ell_2=\ell} \frac{\mathrm{d}}{\mathrm{d}x_1}S_{\ell_1}\frac{\mathrm{d}}{\mathrm{d}x_1}S_{\ell_2} - \sum_{i=2}^{r} \frac{\frac{\mathrm{d}S_{\ell-1}}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S_{\ell-1}}{\mathrm{d}x_1}}{x(p_1)-x(p_i)}$$
$$= L(x_1).S_{\ell-1} + \begin{cases} L(x_1).F_{\ell/2}, & \ell \text{ even,}\\ 0, & \text{odd,}\end{cases}$$

that is

$$(54)\quad [\hbar^\ell]\left[\hbar^2\left(\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S + \frac{1}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S\frac{\mathrm{d}}{\mathrm{d}x_1}S - \sum_{i=2}^{r}\frac{\frac{\mathrm{d}S}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S}{\mathrm{d}x_1}}{x(p_1)-x(p_i)}\right)\right] =$$
$$[\hbar^\ell]\left[\hbar^2 L(x_1).S + (R(x_1) + L(x_1).F)\right].$$

Let us finally consider the special cases:

- For $(g,n) = (0,0)$ we get

$$(55)\qquad \frac{1}{\alpha_1^2}\left(\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_1}\right)^2 = R(x_1).$$

$$\frac{1}{\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}S_0\right)^2 = R(x_1).$$

$$[\hbar^0]\left(\hbar^2\left(\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S + \frac{1}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S\frac{\mathrm{d}}{\mathrm{d}x_1}S - \sum_{i=2}^{r}\frac{\frac{\mathrm{d}S}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S}{\mathrm{d}x_1}}{x(p_1)-x(p_i)} - L(x_1).S\right) - (R(x_1) + L(x_1).F)\right) = 0.$$

- For $(g,n) = (0,1)$ we get

$$2F'_{0,1}(z,D)F'_{0,2}(z,D) - \sum_{i=1}^{r}\alpha_i\frac{F'_{0,1}(p_i,D) + F'_{0,1}(z,D)}{x(z)-x(p_i)} = L(x).F_{0,1}(D).$$

thus

$$2F'_{0,1}(z,D)F'_{0,2}(z,D) - 2\alpha_1\frac{F'_{0,1}(z,D)}{x(z)-x(p_1)} - \alpha_1\frac{F'_{0,1}(p_1,D) - F'_{0,1}(z,D)}{x(z)-x(p_1)}$$
$$- \sum_{i=2}^{r}\alpha_i\frac{F'_{0,1}(p_i,D) + F'_{0,1}(z,D)}{x(z)-x(p_i)} = L(x).F_{0,1}(D).$$

At $z = p_1$ this gives

$$2F'_{0,1}(p_1,D)\left(F'_{0,2}(z,D)-\alpha_1\frac{1}{x(z)-x(p_1)}\right)_{z=p_1}+\alpha_1\left(\frac{\mathrm{d}F'_{0,1}(z,D)}{\mathrm{d}x}\right)_{z=p_1}$$
$$-\sum_{i=2}^{r}\frac{1}{x(p_1)-x(p_i)}\left(\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_1}\right) = L(x_1).F_{0,1}(D).$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{0,1}(D)+\frac{1}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}F_{0,1}(D)\frac{\mathrm{d}}{\mathrm{d}x_1}F_{0,2}(D)-\sum_{i=2}^{r}\frac{\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_1}}{x(p_1)-x(p_i)} = L(x_1).F_{0,1}(D).$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S_0+\frac{2}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S_0\frac{\mathrm{d}}{\mathrm{d}x_1}S_1-\sum_{i=2}^{r}\frac{\frac{\mathrm{d}S_0}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S_0}{\mathrm{d}x_1}}{x(p_1)-x(p_i)} = L(x_1).S_0.$$

$$[\hbar]\left(\hbar^2\left(\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S+\frac{1}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S\frac{\mathrm{d}}{\mathrm{d}x_1}S-\sum_{i=2}^{r}\frac{\frac{\mathrm{d}S}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S}{\mathrm{d}x_1}}{x(p_1)-x(p_i)}-L(x_1).S\right)-(R(x_1)+L(x_1).F)\right)=0.$$

- For $(g,n)=(0,2)$ we first rewrite (13):

$$(56)\quad P_{0,2}(x(z),z_1,z_2)-2y(z)\frac{\omega_{0,3}(z,z_1,z_2)}{\mathrm{d}x} =$$
$$-\frac{B(z,z_1)B(-z,z_2)}{(\mathrm{d}x)^2}-\frac{B(-z,z_1)B(z,z_2)}{(\mathrm{d}x)^2}+\mathrm{d}_1\frac{B(z_2,-z_1)}{(x-x_1)\mathrm{d}x_1}+\mathrm{d}_2\frac{B(z_1,-z_2)}{(x-x_2)\mathrm{d}x_2} =$$
$$2\frac{B(z,z_1)B(z,z_2)}{(\mathrm{d}x)^2}+\mathrm{d}_1\frac{1}{x-x_1}\left(\frac{B(z_2,-z_1)}{\mathrm{d}x_1}-\frac{B(z,z_2)}{\mathrm{d}x}\right)+\mathrm{d}_2\frac{1}{x-x_2}\left(\frac{B(z_1,-z_2)}{\mathrm{d}x_2}-\frac{B(z,z_1)}{\mathrm{d}x}\right) =$$
$$2\frac{B(z,z_1)B(z,z_2)}{(\mathrm{d}x)^2}-\mathrm{d}_1\frac{1}{x-x_1}\left(\frac{B(z_2,z_1)}{\mathrm{d}x_1}+\frac{B(z,z_2)}{\mathrm{d}x}\right)-\mathrm{d}_2\frac{1}{x-x_2}\left(\frac{B(z_1,z_2)}{\mathrm{d}x_2}+\frac{B(z,z_1)}{\mathrm{d}x}\right)$$
$$+\mathrm{d}_1\mathrm{d}_2\frac{1}{x-x_1}\frac{1}{x-x_2}.$$

Now we integrate twice over D:

$$(57)\quad \int_D\int_D P_{0,2}(x(z),z_1,z_2)-2F'_{0,1}(z,D)F'_{0,3}(z,D) =$$
$$2(F'_{0,2}(z,D))^2-2\sum_{i=1}^{r}\alpha_i\frac{1}{x-x_i}F'_{0,2}(z,D)-\sum_{i=1}^{r}\frac{2\alpha_i}{x-x_i}\sum_{j\neq i}\alpha_j\left(\frac{\mathrm{d}E(p_i,p_j)}{\mathrm{d}x_i}-\frac{1}{x_i-x_j}\right).$$

We introduce $\hat{F}'_{0,2}(z,D)=F'_{0,2}(z,D)-\alpha_1\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x}$ and obtain:

$$(58)\quad 2(\hat{F}'_{0,2}(z,D))^2+4\alpha_1\hat{F}'_{0,2}(z,D)\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x}+2\alpha_1^2\frac{(\mathrm{d}E(z,p_1))^2}{(\mathrm{d}x)^2}-\frac{2\alpha_1}{x-x_1}\hat{F}'_{0,2}(z,D)-$$
$$\frac{2\alpha_1^2}{x-x_1}\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x}-2\sum_{i\neq 1}\frac{\alpha_i}{x-x_i}\hat{F}'_{0,2}(z,D)-2\sum_{i\neq 1}\frac{\alpha_1\alpha_i}{x-x_i}\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x}-$$
$$\frac{1}{x-x_1}\frac{\mathrm{d}}{\mathrm{d}x_1}F_{0,2}(D)-\sum_{i\neq 1}\frac{1}{x-x_i}\frac{\mathrm{d}}{\mathrm{d}x_i}F_{0,2}(D)+2\sum_{\substack{i\neq j\\ i\neq 1}}\frac{\alpha_i\alpha_j}{(x-x_i)(x_i-x_j)}+2\sum_{j\neq 1}\frac{\alpha_1\alpha_j}{(x-x_1)(x_1-x_j)}.$$

Observe that

$$\lim_{z\to p_1}\left(-\sum_{i\neq 1}\frac{\alpha_1\alpha_i}{x-x_i}\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x}+\sum_{j\neq 1}\frac{\alpha_1\alpha_j}{(x-x_1)(x_1-x_j)}\right)=\sum_{i\neq 1}\frac{\alpha_1\alpha_i}{(x_1-x_i)^2}. \tag{59}$$

Using this, at $z=p_1$ we obtain

$$\frac{1}{2\alpha_1^2}\left(\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_1}\right)^2+\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2F_{0,2}(D)-\sum_{i\neq 1}\frac{1}{x_1-x_i}\left(\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_1}\right)+ \\ 2\alpha_1^2\mathcal{S}(p_1)+2\sum_{\substack{i\neq j\\ i\neq 1,j\neq 1}}\frac{\alpha_i\alpha_j}{(x_1-x_i)(x_i-x_j)}, \tag{60}$$

where we have called $\mathcal{S}(p_1)$ the limit

$$\lim_{z\to p_1}\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x(z)}\left(\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x(z)}-\frac{1}{x(z)-x(p_1)}\right).$$

Now observe that, using the Lemma 4.2, we obtain

$$\begin{aligned} L(x(z)).\omega_{0,2}(z_{1,2}) &= L(x(z)).\left(\omega_{0,2}(z_{1,2})-\frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\right) \\ &= -\sum_{i,x(\zeta_i)=\infty}\sum_{j=0}^{m_i}t_{\zeta_i,j}\sum_{0\leqslant k\leqslant -2-j/d_i}x(z)^k(k+2+\frac{j}{d_i})\int_{\mathcal{B}_{\zeta_i,j+d_i(k+2)}}\omega_{0,3}(z',z_1,z_2) \\ &\quad +\sum_{i,x(\zeta_i)\neq\infty}\sum_{j=0}^{m_i}t_{\zeta_i,j}\sum_{k=0}^{j}\xi_i(z)^{-(k+1)}(k-j-1)\int_{\mathcal{B}_{\zeta_i,j+1-k}}\omega_{0,3}(z',z_1,z_2), \end{aligned}$$

which implies that

$$L(x).F_{0,2}(D)=\int_D\int_D P_{0,2}(x;z_1,z_2). \tag{62}$$

For the first term of (57), we thus obtain:

$$L(x_1).\frac{F_{0,2}(D)}{2}=\frac{1}{3\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}F_{0,1}(D)\frac{\mathrm{d}}{\mathrm{d}x_1}F_{0,3}(D)+\frac{1}{4\alpha_1^2}\left(\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_1}\right)^2+\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2F_{0,2}(D)+ \\ \alpha_1^2\mathcal{S}(p_1)-\frac{1}{2}\sum_{i\neq 1}\frac{1}{x_1-x_i}\left(\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_1}\right)+\sum_{\substack{i\neq j\\ i\neq 1,j\neq 1}}\frac{\alpha_i\alpha_j}{(x_1-x_i)(x_i-x_j)}. \tag{63}$$

- For $(g,n)=(1,0)$ we get

$$\frac{-B(z,-z)}{\mathrm{d}x(z)^2}+2F'_{0,1}(z,D)F'_{1,1}(z,D)\ =L(x_1).F_{1,0}(D).$$

At $z=p_1$ this gives

$$\frac{-B(p_1,-p_1)}{\mathrm{d}x_1^2}+2\frac{1}{\alpha_1^2}\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_1}\frac{\mathrm{d}F_{1,1}(D)}{\mathrm{d}x_1}=L(x_1).F_{1,0}(D).$$

Using that $\frac{B(p_1,-p_1)}{\mathrm{d}x_1^2} = \mathcal{S}(p_1)$ and summing the expressions for $(0,2)$ and $(1,0)$, we obtain:

$$\begin{aligned}(64)\quad \left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S_1 + \frac{1}{\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}S_1\right)^2 + \frac{2}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S_0\frac{\mathrm{d}}{\mathrm{d}x_1}S_2 - \sum_{i=2}^{r}\frac{\frac{\mathrm{d}S_1}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S_1}{\mathrm{d}x_1}}{x(p_1)-x(p_i)} \\ + (\alpha_1^2-1)\mathcal{S}(p_1) + \sum_{\substack{i<j\\ i\neq 1}}\frac{\alpha_i\alpha_j}{(x_1-x_i)(x_1-x_j)} = L(x_1).S_1 + L(x_1).F_1.\end{aligned}$$

that is

$$\begin{aligned}(65)\quad [\hbar^2]\left[\hbar^2\left(\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S + \frac{1}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S\frac{\mathrm{d}}{\mathrm{d}x_1}S - \sum_{i=2}^{r}\frac{\frac{\mathrm{d}S}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S}{\mathrm{d}x_1}}{x(p_1)-x(p_i)} + (\alpha_1^2-1)\mathcal{S}(p_1) + (\star)\right)\right] = \\ [\hbar^2]\left[\hbar^2 L(x_1).S + (R(x_1)+L(x_1).F)\right],\end{aligned}$$

with

$$(66)\qquad (\star) = \sum_{\substack{i\neq j\\ i\neq 1, j\neq 1}}\frac{\alpha_i\alpha_j}{(x_1-x_i)(x_i-x_j)}.$$

Assuming $\alpha_1 = \frac{1}{\alpha_1}$, and summing over all topologies, we get the claim. □

Remark 4.1. Very often in the literature a different convention is used to regularize the $(0,2)$ term of the wave function:

$$\tilde{\psi}(D,t,\hbar) := \exp\left(\tilde{S}_1(D,t) + \sum_{m\geqslant 0, m\neq 1}\hbar^{m-1}S_m(D,t)\right),$$

where $\tilde{S}_1(D,t) := \frac{1}{2}\int_D\int_D\left(B(z_1,z_2) - \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\right)$. Using (48), we obtain that the relation to our wave function is the following

$$\psi(D,t,\hbar) = \tilde{\psi}(D,t,\hbar)\cdot\prod_{i<j}(x_i-x_j)^{\alpha_i\alpha_j}.$$

4.1. **PDE for Airy curve.** In this particular case, we had $P_{g,n} = 0$.

Therefore, in this case we obtain the following system of PDEs:

$$(67)\qquad \hbar^2\left(\frac{\mathrm{d}^2}{\mathrm{d}x_k^2} - \sum_{i\neq k}\frac{\frac{\mathrm{d}}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_k}\frac{\mathrm{d}}{\mathrm{d}x_k}}{x_k-x_i} + \sum_{\substack{i\neq j\\ i\neq k, j\neq k}}\frac{\alpha_i\alpha_j}{(x_k-x_i)(x_i-x_j)}\right)\psi = x\psi,$$

for every $k = 1,\ldots,r$.

Example 4.2. Considering the divisor $D = [z_1]-[z_2]$, sending $z_2\to\infty$ and regularizing the $(0,1)$ factor of the wave function, we recover the Airy quantum curve from our PDE for $k=1$, with $x = x_1$:

$$\left(\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\right)\psi = 0.$$

4.2. **PDE for Painlevé case.** In this case we have $P_{g,n} = \frac{\partial}{\partial t}\omega_{g,n}(z_1,\ldots,z_n)$.

Therefore, we obtain the following PDE:

$$(68)\qquad \left(\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x_k^2} - \hbar^2\sum_{i\neq k}\frac{\frac{\mathrm{d}}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_k}\frac{\mathrm{d}}{\mathrm{d}x_k}}{x_k-x_i} - \frac{\partial}{\partial t} + (\star)\right)\psi(D) = (P(x) + \frac{\partial}{\partial t}F)\psi,$$

for every $k = 1,\ldots,r$, where $(\star)$ is given by (66).

4.3. **Reduced equation.** Consider a divisor $D=[z]-[z']$ with 2 points, and call $x=x(z), x'=x(z')$. The equation we have obtained is a PDE: it involves both $\mathrm{d}/\mathrm{d}x$ and $\mathrm{d}/\mathrm{d}x'$, as well as partial derivatives with respect to times when $L(x)\neq 0$. Let us show here that it is possible to eliminate $\mathrm{d}/\mathrm{d}x'$ and arrive to an equation involving only $\mathrm{d}/\mathrm{d}x$, as well as possibly times derivatives.

Define

$$\tilde\psi(z,z') := (x-x')\psi([z]-[z'],t,\hbar)e^{F}. \tag{69}$$

Define the differential operators

$$\mathcal{D} := \hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - L(x) - R(x), \tag{70}$$

$$\mathcal{D}' := \hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x'^2} - L(x') - R(x'). \tag{71}$$

Equation (51) is equivalent to

$$\mathcal{D}\tilde\psi = \frac{\hbar^2}{x-x'}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\mathrm{d}}{\mathrm{d}x'}\right)\tilde\psi = -\mathcal{D}'\tilde\psi. \tag{72}$$

In particular this implies

$$\hbar^2\frac{\mathrm{d}}{\mathrm{d}x'}\tilde\psi = -\hbar^2\frac{\mathrm{d}}{\mathrm{d}x}\tilde\psi + (x-x')\mathcal{D}\tilde\psi, \tag{73}$$

and applying $\mathrm{d}/\mathrm{d}x'$ again we find

$$\begin{aligned}
\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x'^2}\tilde\psi &= -\mathcal{D}\tilde\psi - \hbar^2\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x'}\tilde\psi + (x-x')\mathcal{D}\frac{\mathrm{d}}{\mathrm{d}x'}\tilde\psi \\
&= \hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde\psi - 2\mathcal{D}\tilde\psi - (x-x')\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{D}\tilde\psi + \mathcal{D}\frac{\mathrm{d}}{\mathrm{d}x}\tilde\psi - \hbar^{-2}\mathcal{D}(x-x')\mathcal{D}\right)\tilde\psi.
\end{aligned}$$

Therefore

$$\begin{aligned}
0 &= (\mathcal{D}+\mathcal{D}')\tilde\psi \\
&= \mathcal{D}\tilde\psi + (\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x'^2} - R(x') - L(x'))\tilde\psi \\
&= \left(\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \mathcal{D} - R(x') - L(x') - (x-x')\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{D}\tilde\psi + \mathcal{D}\frac{\mathrm{d}}{\mathrm{d}x}\tilde\psi - \hbar^{-2}\mathcal{D}(x-x')\mathcal{D}\right)\right)\tilde\psi \\
&= \left(R(x) - R(x') + (L(x)-L(x')) - (x-x')\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{D}\tilde\psi + \mathcal{D}\frac{\mathrm{d}}{\mathrm{d}x}\tilde\psi - \hbar^{-2}\mathcal{D}(x-x')\mathcal{D}\right)\right)\tilde\psi \\
&= \left(R(x) - R(x') + L(x) - L(x') - (x-x')\left(-\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{D}\tilde\psi + \mathcal{D}\frac{\mathrm{d}}{\mathrm{d}x}\tilde\psi - \hbar^{-2}(x-x')D^2\right)\right)\tilde\psi \\
&= \left(R(x) - R(x') + L(x) - L(x') - (x-x')\left(\frac{\mathrm{d}R(x)}{\mathrm{d}x} + \hbar\frac{\mathrm{d}L(x)}{\mathrm{d}x} - \hbar^{-2}(x-x')\mathcal{D}^2\right)\right)\tilde\psi
\end{aligned}$$

and thus

$$\frac{R(x)-R(x')}{x-x'}\tilde\psi + \frac{L(x)-L(x')}{x-x'}\tilde\psi = \left(\frac{\mathrm{d}R(x)}{\mathrm{d}x} + \frac{\mathrm{d}L(x)}{\mathrm{d}x} - \hbar^{-2}(x-x')\mathcal{D}^2\right)\tilde\psi. \tag{76}$$

Finally,

$$\mathcal{D}^2\tilde\psi = \frac{\hbar^2}{x-x'}\left(\frac{R(x)-R(x')}{x-x'} + \frac{L(x)-L(x')}{x-x'} - \frac{\mathrm{d}R(x)}{\mathrm{d}x} - \frac{\mathrm{d}L(x)}{\mathrm{d}x}\right)\tilde\psi. \tag{77}$$

This equation is a PDE, with rational coefficients $\in \mathbb{C}(x)$, involving $\mathrm{d}/\mathrm{d}x$ and $\partial/\partial t_k$s but no $\mathrm{d}/\mathrm{d}x'$ anymore.

Notice that the right hand side is of order $O(\hbar^2)$ in the limit $\hbar \to 0$, and $\mathcal{D} \to \hat{y}^2 - R(x)$, where $\hat{y} = \hbar \mathrm{d}/\mathrm{d}x$.

5. Quantum curves

The goal now is to prove that $\psi(D)$ obeys an isomonodromic system type of equation, and in particular this implies the existence of a quantum curve $\hat{P}(x, \hat{y}, \hbar)$ that annihilates ψ. To this purpose, we first prove that $\psi([z] - [z'])$ coincides with the integrable kernel of an isomonodromic system. The way to prove it generalizes the method of [2], i.e. first proving that the ratio of ψ and the integrable kernel has to be a formal series of the form $1 + O(z'^{-1})$ and then showing that the only solution of equations (51) which has that behavior implies that the ratio must be 1.

5.1. **Painlevé I (genus 0 case).**

We shall prove that $\psi([z] - [z'])$ coincides with the integrable kernel associated to the Painlevé I kernel.

Consider a solution of the Painlevé system (8), written as

$$\Psi(x) = \begin{pmatrix} A(x) & B(x) \\ \tilde{A}(x) & \tilde{B}(x) \end{pmatrix}, \quad \det \Psi(x) = 1, \tag{78}$$

i.e. $\Psi(x)$ satisfying (8):

$$\left(\hbar \frac{\partial}{\partial x} - \mathcal{L}(x, t; \hbar)\right) \Psi = 0 \quad , \quad \left(\hbar \frac{\partial}{\partial t} - \mathcal{R}(x, t; \hbar)\right) \Psi = 0.$$

Define $A(x), \tilde{A}(x), B(x), \tilde{B}(x)$ as WKB $\hbar$-formal series solutions, with leading orders

$$\begin{aligned}
A(x) &\sim \frac{i}{2\sqrt{z}}\, e^{\hbar^{-1}\int_0^x y\mathrm{d}x}(1 + O(\hbar)), \\
B(x) &\sim \frac{i}{2\sqrt{z}}\, e^{-\hbar^{-1}\int_0^x y\mathrm{d}x}(1 + O(\hbar)), \\
\tilde{A}(x) &\sim i\sqrt{z}\, e^{\hbar^{-1}\int_0^x y\mathrm{d}x}(1 + O(\hbar)), \\
\tilde{B}(x) &\sim -i\sqrt{z}\, e^{-\hbar^{-1}\int_0^x y\mathrm{d}x}(1 + O(\hbar)),
\end{aligned}$$

and with each coefficient of higher powers of $\hbar$ in $(1 + O(\hbar))$ being a polynomial of $1/z$ that tends to 0 as $z \to \infty$. The integrable kernel is defined as (a WKB formal series of $\hbar$):

$$K(x, x') := \frac{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')}{x - x'}. \tag{80}$$

From the isomonodromic system (8), one can verify that this kernel obeys the same equation (51) as $\psi([z] - [z'])$. In fact, they are equal:

Theorem 5.1. *We have, as formal WKB power series of* $\hbar$

$$\psi([z] - [z'], t, \hbar) = \frac{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')}{x - x'}, \tag{81}$$

where $x = x(z)$ *and* $x' = x(z')$.

Proof. Define the ratio

$$H(z, z') := \frac{(x - x')\psi([z] - [z'], t, \hbar)}{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')}. \tag{82}$$

It is a formal series of $\hbar$ whose coefficients are rational functions of z and z'. The leading orders show that

$$H(z,z') = 1 + O(\hbar). \tag{83}$$

Moreover, at each order of $\hbar$, the coefficient is a polynomial of $1/z, 1/z'$, which tends to 0 at $z, z' \to \infty$:

$$H(z,z') - 1 \in \frac{1}{zz'}\mathbb{C}[z^{-1}, z'^{-1}][[\hbar]]. \tag{84}$$

We shall prove that $H = 1$ by following the method of [2]. Let us assume that $H \neq 1$, and write

$$H(z,z') = 1 + H_M(z)z'^{-M} + O(z'^{-M-1}), \tag{85}$$

where $M \geqslant 1$ is the smallest possible power of z' whose coefficient H_M would be $\neq 0$ as a formal series of $\hbar$. Equation (72) implies

$$\frac{\mathrm{d}^2 H}{\mathrm{d}x'^2} + 2\frac{\mathrm{d}\ln B(x')}{\mathrm{d}x'}\frac{\mathrm{d}H}{\mathrm{d}x'} + 2\frac{\mathrm{d}\ln(\tilde{A}(x)/A(x) - \tilde{B}(x')/B(x'))}{\mathrm{d}x'}\frac{\mathrm{d}H}{\mathrm{d}x'} = \frac{1}{x'-x}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x'}\right)H, \tag{86}$$

whose leading power of z' comes only from the second term and is

$$2y(z')H_M(z)z'^{-M} + O(z'^{-M-1}) = 0, \tag{87}$$

implying that $H_M = 0$, and thus contradicting the hypothesis that $H \neq 1$. This proves the theorem. □

As a corollary this implies that

$$\lim_{z'\to\infty} \frac{(x(z) - x(z'))\psi([z]-[z'],t,\hbar)}{\tilde{B}(x(z'))} = A(x(z)). \tag{88}$$

In other words

$$\frac{i}{2\sqrt{z}} e^{\hbar^{-1}\int_0^z \omega_{0,1}}\ e^{\sum_{(g,n)\neq(0,1),(0,2)} \frac{\hbar^{2g-2+n}}{n!}\int_\infty^z \cdots \int_\infty^z \omega_{g,n}} = A(x(z)). \tag{89}$$

In the Painlevé system, the function $A(x)$ is annihilated by the quantum curve

$$\hat{y}^2 - \left((x-U)^2(x+2U) + \frac{\hbar^2}{2}\ddot{U}(x-U) + \frac{\hbar^2}{4}\dot{U}^2\right) - \frac{\hbar^2}{2(x-U)}\dot{U} - \frac{\hbar}{x-U}\hat{y}. \tag{90}$$

5.2. **General genus 0 case.** The same argument applies to any isomonodromic system. Assume that we have a Lax pair of type (8), whose spectral curve in the limit $\hbar \to 0$ is a genus zero curve of the form $\det(y - \mathcal{L}_0(x)) = y^2 - R(x) = 0$, and which has a WKB formal power series solution in the form

$$\begin{aligned}
A(x) &\sim \frac{i}{\sqrt{2\mathrm{d}x/\mathrm{d}z}}\, e^{\hbar^{-1}\int_0^x y\mathrm{d}x}(1+O(\hbar)),\\
B(x) &\sim \frac{i}{\sqrt{2\mathrm{d}x/\mathrm{d}z}}\, e^{-\hbar^{-1}\int_0^x y\mathrm{d}x}(1+O(\hbar)),\\
\tilde{A}(x) &\sim i\sqrt{\frac{1}{2}\mathrm{d}x/\mathrm{d}z}\, e^{\hbar^{-1}\int_0^x y\mathrm{d}x}(1+O(\hbar)),\\
\tilde{B}(x) &\sim -i\sqrt{\frac{1}{2}\mathrm{d}x/\mathrm{d}z}\, e^{-\hbar^{-1}\int_0^x y\mathrm{d}x}(1+O(\hbar)),
\end{aligned}$$

where each coefficient of higher powers of $\hbar$ in $(1+O(\hbar))$ is a rational function of z that tends to 0 at poles $z \to \zeta$, where we have chosen a pole $x(\zeta) = \infty$. This pole has degree $-d = 1$ or $-d = 2$, and $\xi = x^{1/d}$ is a local coordinate near the pole. We emphasize that for all genus 0 spectral curves where $y^2 = P_{\mathrm{odd}}(x)$ with $P_{\mathrm{odd}}(x)$ an odd polynomial of x, such systems are explicitely known as Gelfand-Dikii systems [16] described in section 5.4 below, and for more general spectral curves, it was proved in [21] that a 2×2 autonomous system $\mathcal{L}_0$ always admits an $\hbar$-deformation $\mathcal{L}$ with this property.

We define the following formal series of $\hbar$:

$$H(z, z') \coloneqq \frac{(x(z) - x(z'))\psi([z] - [z'])}{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')} = 1 + O(\hbar). \tag{92}$$

It is easy to see that the integrable kernel satisfies equation (72), and thus H satisfies the PDE (86).

Moreover the coefficients of H are analytic functions of z', which tend to 0 at $z' \to \zeta$. Let us write $H(z, z') = 1 + O(x'^{1/d})$. The leading order $H = 1 + H_M(z)x'^{M/d} + O(x'^{(M+1)/d})$ must satisfy $y(z')H_M(z)x'^{M/d} = O(x'^{(M+1)/d})$, and therefore $H = 1$.

This implies that $\psi([z] - [z'])$ coincides with the integrable kernel

$$\psi([z] - [z']) = \frac{A(x(z))\tilde{B}(x(z')) - \tilde{A}(x(z))B(x(z'))}{x(z) - x(z')}. \tag{93}$$

Then taking the limit $z' \to \zeta$ this implies that

$$\lim_{z' \to \zeta} \frac{(x(z) - x(z'))\psi([z] - [z'])}{\tilde{B}(x(z'))} = A(x(z)). \tag{94}$$

Knowing that $A(x), \tilde{A}(x)$ satisfy an isomonodromic system with first equation

$$\hbar \frac{\partial}{\partial dx} \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix} = \mathcal{L}(x, t, \hbar) \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix}, \tag{95}$$

with

$$\mathcal{L}(x, t, \hbar) = \begin{pmatrix} \alpha(x, t, \hbar) & \beta(x, t, \hbar) \\ \gamma(x, t, \hbar) & \delta(x, t, \hbar) \end{pmatrix}, \tag{96}$$

where $\alpha, \beta, \gamma, \delta$ rational functions of x, with coefficients being formal power series of $\hbar$, we get the quantum curve annihiliating $A(x)$:

$$\hat{y}^2 - (\alpha + \delta)\hat{y} + (\alpha\delta - \beta\gamma) + \hbar \left(\mathrm{d}\alpha/\mathrm{d}x - \alpha \frac{\mathrm{d}\beta/\mathrm{d}x}{\beta} - \frac{\mathrm{d}\beta/\mathrm{d}x}{\beta} \hat{y} \right). \tag{97}$$

Its classical part $\hbar \to 0$ is indeed the spectral curve

$$y^2 - (\alpha(x, t, 0) + \delta(x, t, 0))y(\alpha(x, t, 0)\delta(x, t, 0) - \beta(x, t, 0)\gamma(x, t, 0)) = \det(y - \mathcal{L}_0(x)). \tag{98}$$

5.3. **Higher genus case.** If the curve $y^2 = R(x)$ has genus $\hat{g} > 0$, it was verified in [23] (for $\hat{g} = 1$), and argued in [3], that the perturbative wave function cannot satisfy the quantum curve, and in fact just because it is not a function (order by order in $\hbar$) on the spectral curve. Indeed, multiple integrals of type $\int_o^z \dots \int_o^z \omega_{g,n}$ are not invariant after z goes around a cycle, and don't transform as Abelian differentials. It was argued in [15, 3, 12] that only the non-perturbative wave function of [11, 14] can be a wave function and can obey a quantum curve, and this was proved up to the 3rd non trivial powers of $\hbar$ for arbitrary curves in [3], and verified to many orders for elliptic curves in [23].

Consider a curve $y^2 = R(x)$ with genus $\hat{g} > 0$, let $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$ be a symplectic basis of cycles of $H_1(\Sigma, \mathbb{Z})$ (i.e. integer cycles). We choose the Bergman kernel normalized on $\mathcal{A}$-cycles.

Define the 1st kind times

$$\epsilon_i = \frac{1}{2i\pi} \oint_{\mathcal{A}_i} y \mathrm{d}x. \tag{99}$$

Since (51) is a linear PDE, any linear combination of solutions is solution. Moreover, since the coefficients of the PDE do not involve the times ϵ_i, we remark that shifting $\epsilon_i \to \epsilon_i + n_i$ is another solution, which we denote as follows

$$\psi(\{\epsilon_i \to \epsilon_i + n_i\}; [z] - [z']). \tag{100}$$

The transseries linear combination introduced in [11, 14, 3, 12]

$$\hat{\psi}([z] - [z']) = \frac{1}{\hat{\mathcal{T}}} \sum_{n_1,\dots,n_{\hat{g}} \in \mathbb{Z}^{\hat{g}}} \psi(\{\epsilon_i \to \epsilon_i + n_i\}; [z] - [z']) Z(\{\epsilon_i \to \epsilon_i + n_i\}), \tag{101}$$

where

$$\hat{\mathcal{T}} = \sum_{n_1,\dots,n_{\hat{g}} \in \mathbb{Z}^{\hat{g}}} Z(\{\epsilon_i \to \epsilon_i + n_i\}) \tag{102}$$

is thus also a solution of the same PDE. Moreover, this combination is obviously independent [12] of the integration homotopy class from z' to z, and it is, order by order as a transseries of $\hbar$ a function of z and z' on the spectral curve.

From there, the same argument as the genus zero case applies. Assume that there is an isomonodromic system $(\mathcal{L}, \mathcal{R})$ whose associated spectral curve is our spectral curve, with

$$\Psi(x) = \begin{pmatrix} A(x) & B(x) \\ \tilde{A}(x) & \tilde{B}(x) \end{pmatrix}, \tag{103}$$

a formal transseries solution. Then the formal transseries

$$H(z, z') = \frac{(x(z) - x(z'))\psi([z] - [z'])}{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')} = 1 + O(\hbar) \tag{104}$$

satisfies the PDE (86), and is such that $H(z, z') = 1 + O(x'^{1/d})$. The leading order $H = 1 + H_M(z)x'^{M/d} + O(x'^{(M+1)/d})$ must satisfy $y(z')H_M(z)x'^{M/d} = O(x'^{(M+1)/d})$, and therefore $H = 1$. This implies that

$$\psi([z] - [z']) = \frac{A(x(z))\tilde{B}(x(z')) - \tilde{A}(x(z))B(x(z'))}{x(z) - x(z')}. \tag{105}$$

This also implies that

$$\lim_{z' \to \zeta} \frac{(x(z) - x(z'))\psi([z] - [z'])}{\tilde{B}(x(z'))} = A(x(z)). \tag{106}$$

Since $A(x), \tilde{A}(x)$ satisfy the isomonodromic system

$$\hbar \frac{\partial}{\partial x} \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix} = \mathcal{L}(x) \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix}, \tag{107}$$

where

$$\mathcal{L}(x, t, \hbar) = \begin{pmatrix} \alpha(x, t, \hbar) & \beta(x, t, \hbar) \\ \gamma(x, t, \hbar) & \delta(x, t, \hbar) \end{pmatrix}$$

we find the quantum curve annihiliating $A(x)$:

$$\hat{y}^2 - (\alpha(x) + \delta(x))\hat{y} + (\alpha(x)\delta(x) - \beta(x)\gamma(x)) + \hbar \left(\alpha'(x) - \alpha(x)\frac{\beta'(x)}{\beta(x)} - \frac{\beta'(x)}{\beta(x)}\hat{y} \right). \tag{108}$$

Its classical part $\hbar \to 0$ is indeed the equation

$$\det(y - \mathcal{L}(x, t, 0)) = 0. \tag{109}$$

5.4. **Examples: Gelfand-Dikii systems.** These systems generalize the Painlevé I equation; they appear in the enumeration of maps in the large size limit [16]. For these Gelfand-Dikii systems, the proof that $\psi([z]-[z'])$ coincides with the integrable kernel (which then implies the quantum curve) can be found in [16] chapter 5, by another method. Here let us provide another proof with our current method.

The Gelfand-Dikii polynomials are defined as differential polynomials of a function $U(t)$, by the recursion

$$R_0(U) = 2 \quad , \quad \frac{\partial}{\partial t}R_{k+1}(U) = -2U\frac{\partial R_k(U)}{\partial t} - R_k(U)\frac{\partial U}{\partial t} + \frac{\hbar^2}{4}\frac{\partial^2 R_k(U)}{\partial t^2}. \tag{110}$$

At each step the integration constant is chosen so that $R_k(U)$ is homogeneous in powers of U and $\partial^2/\partial t^2$. The first fews are given by

$$\begin{aligned} R_0 &= 2, \\ R_1 &= -2U, \\ R_2 &= 3U^2 - \frac{\hbar^2}{2}\ddot{U}, \\ R_3 &= -5U^3 + \frac{5\hbar^2}{2}U\ddot{U} + \frac{5\hbar^2}{4}\dot{U}^2 - \frac{\hbar^4}{8}\frac{\partial^4 U}{\partial t^4} \\ &\dots \end{aligned}$$

Let an integer $m \geqslant 1$, and let $\tilde{t}_0, \tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_m$ be a set of "times". Let $U(t; \tilde{t}_0, \tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_m)$ be a solution of the following non-linear ODE:

$$\sum_{j=0}^{m} \tilde{t}_j R_{j+1}(U) = t. \tag{112}$$

Notice that, formally, $t = -2\tilde{t}_{-1}$. The case $m = 1$, $R_2(U) = t$ is the Painlevé I equation. The case $m = 2$, $R_3(U) = t$ is called the Lee-Yang equation. The case $m = 0$ is simply $U(t) = -\frac{t}{2\tilde{t}_0}$.

Consider the Lax pair (adopting the normalizations of [16]) given by

$$\mathcal{R}(x,t,\hbar) = \begin{pmatrix} 0 & 1 \\ x + 2U(t) & 0 \end{pmatrix} \tag{113}$$

and

$$\mathcal{L}(x,t,\hbar) = \sum_{j=0}^{m} \tilde{t}_j \mathcal{L}_j(x,t,\hbar), \tag{114}$$

where

$$\mathcal{L}_j(x,t,\hbar) = \begin{pmatrix} \alpha_j(x,t) & \beta_j(x,t) \\ \gamma_j(x,t) & -\alpha_j(x,t) \end{pmatrix} \text{ with,} \tag{115}$$

(116)

$$\beta_j(x,t) = \frac{1}{2}\sum_{k=0}^{j} x^{j-k} R_k(U), \quad \alpha_j(x,t) = -\frac{\hbar}{2}\frac{\partial}{\partial t}\beta_j(x,t), \quad \gamma_j(x,t) = (x+2U)\beta_j(x,t) + \hbar\frac{\partial}{\partial t}\alpha_j(x,t).$$

One can easily verify that the Gelfand-Dikii polynomials are such that the zero curvature equation is satisfied

$$\hbar\frac{\partial}{\partial t}\mathcal{L}(x,t,\hbar) + \hbar\frac{\partial}{\partial x}\mathcal{R}(x,t,\hbar) = [\mathcal{R}(x,t,\hbar), \mathcal{L}(x,t,\hbar)]. \tag{117}$$

The differential equation (112) admits a formal power series solution $U(t,\hbar)$ with only even powers of $\hbar$:

$$U(t,\hbar)=\sum_k \hbar^{2k}u_k(t), \tag{118}$$

whose first term $u(t)=u_0(t)$ satisfies an algebraic equation

$$\sum_{j=0}^{m}\frac{(2j+1)!}{j!(j+1)!}\tilde{t}_j(-u/2)^{j+1}=-\frac{1}{4}t. \tag{119}$$

In the Painlevé I case, $m=1$, $\tilde{t}_k=\delta_{k,1}$, we recover $t=-3u^2$.

The spectral curve, in the limit $\hbar\to 0$:

$$\det(y-\mathcal{L}(x,t,0))=0 \tag{120}$$

is always a genus 0 curve. It admits the rational parametrization

$$\left\{\begin{array}{l} x(z)=z^2-2u(t)\\ y(z)=\sum_{j=0}^{m}\tilde{t}_j\left(z^{2j+1}(1-2u(t)/z^2)^{j+\frac{1}{2}}\right)_+ \end{array}\right. , \tag{121}$$

where $()_+$ means the positive part in the Laurent series expansion near $z=\infty$, i.e.

$$y(z)=\sum_{j=0}^{m}\tilde{t}_j\sum_{k=0}^{j}(-u)^k\frac{(2j+1)!!}{(2j-2k+1)!!}\ z^{2j-2k+1}.$$

In the Painlevé I case, $\tilde{t}_j=\delta_{j,1}$, we recover $y(z)=z^3-3uz$.
In the case $m=0$, with $\tilde{t}_0=1$, we recover the Airy system

$$\mathcal{L}(x,t,\hbar)=\begin{pmatrix}0 & 1\\ x-t & 0\end{pmatrix} \tag{122}$$

with spectral curve $y^2=x-t$.

Let $\Psi(x,t,\hbar)$ as follows

$$\Psi(x,t,\hbar)=\begin{pmatrix}A(x) & B(x)\\ \tilde{A}(x) & \tilde{B}(x)\end{pmatrix} \tag{123}$$

be a WKB $\hbar$ formal series solution of

$$\hbar\frac{\partial}{\partial x}\Psi(x,t,\hbar)=-\mathcal{L}(x,t,\hbar)\Psi(x,t,\hbar)\quad,\quad \hbar\frac{\partial}{\partial t}\Psi(x,t,\hbar)=\mathcal{R}(x,t,\hbar)\Psi(x,t,\hbar). \tag{124}$$

Our previous results show that the formal series $\psi([z]-[\infty],\tilde{t},\hbar)$ coincide with

$$A(x,t,\hbar)=\frac{1}{\sqrt{2z}}e^{\hbar^{-1}\int_0^z ydx}e^{\sum_{(g,n)\neq(0,1),(0,2)}\frac{\hbar^{2g-2+n}}{n!}\int_\infty^z\cdots\int_\infty^z\omega_{g,n}}, \tag{125}$$

and is annihiliated by the quantum curve

$$\hat{y}^2-(\alpha(x)+\delta(x))\hat{y}+(\alpha(x)\delta(x)-\beta(x)\gamma(x))+\hbar\left(\alpha'(x)-\alpha(x)\frac{\beta'(x)}{\beta(x)}-\frac{\beta'(x)}{\beta(x)}\hat{y}\right). \tag{126}$$

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