# Two-loop $n$-point anomalous amplitudes in $\mathcal{N}=4$ supergravity 

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#### Abstract

We compute the anomalous two-loop four-point amplitudes in $\mathcal{N}=4$ pure supergravity, using unitarity and the double-copy construction. We also present all terms determined by four-dimensional cuts in two all-multiplicity two-loop anomalous superamplitudes. This result provides the first twoloop $n$-point gravity amplitude, up to a class of undetermined rational terms, which are absent at four points. We show that a recently proposed finite counterterm cancels these amplitudes to this order. We argue that the counterterm does not spoil the three-loop finiteness of anomalous amplitudes in the $\mathcal{N}=4$ theory.


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Introduction: All but the simplest perturbative calculations in quantum gravity using standard Lagrangian methods quickly lead into a swamp of intractable intermediate expressions. Modern approaches, in particular generalized unitarity [1, 2] and double-copy relations between gravity and gauge theories [3 5], have made possible multi-loop calculations [6] , as well as loop-level results with large numbers of external legs [7].

Multiloop amplitudes in supergravity theories offer a window on ultraviolet properties [8 11]. Supersymmetry constraints push possible divergences to higher-loop order; but are they the whole story? Extended supergravity theories display nontrivial enhanced cancellations of ultraviolet singularities [9, 10]. These have yet to be understood from standard symmetry arguments, despite valiant attempts [12]. (See Ref. 13] for recent progress.) The only explicit divergence found in $D=4$ pure supergravity is in the $\mathcal{N}=4$ theory [14]. This theory possesses another important feature, not found in supergravity theories with more supersymmetries: an anomaly [15] in a $U(1)$ subgroup of its $S U(1,1)$ duality group. As a consequence, certain classes of amplitudes that vanish at tree level, thanks to this symmetry, fail to do so at one loop (16].

As shown in Ref. [17], an appropriate finite local counterterm can cancel the one-loop anomalous amplitudes. This was shown explicitly for all five-point amplitudes and certain infinite classes of anomalous amplitudes. The local counterterm appears to effectively adjust the regularization scheme so as to preserve the $U(1)$ subgroup responsible for the vanishing of amplitudes. We may ask: do the cancellations continue to higher loops?

In this Letter, we evaluate the four-point two-loop anomalous amplitudes in $\mathcal{N}=4$ supergravity, and all but rational contributions to two all-multiplicity two-
loop anomalous amplitudes. We use the loop version of the Bern-Carrasco-Johansson (BCJ) double copy [5] to compute the complete four-point amplitudes, and fourdimensional cuts to compute the polylogarithmic terms of the $n$-point amplitudes. These are the first-ever allmultiplicity results for any gravity amplitudes at two loops. We show that the same finite counterterm that removes the anomalous one-loop amplitudes also removes the two-loop ones. These results make it plausible that all anomalous amplitudes are completely removed by the counterterm of Ref. 17]. If this is indeed true, it would have very interesting consequences for the ultraviolet properties of the $\mathcal{N}=4$ theory, a point to which we return below.

Review: The spectrum of pure $\mathcal{N}=4$ supergravity 18] has two supermultiplets:

$$
\begin{align*}
& \Phi^{+}=\left\{h^{++}, \psi_{a}^{+}, A_{a b}^{+}, \chi_{a b c}^{+}, \bar{t}\right\} \\
& \Phi^{-}=\left\{h^{--}, \psi_{a}^{-}, A_{a b}^{-}, \chi_{a b c}^{-}, t\right\} \tag{1}
\end{align*}
$$

where $\pm$ indicates the helicity; $a, b, c$ are $S U(4)_{R}$ symmetry indices; $h$ is the graviton, $\psi$ the gravitino, $A$ the vector, $\chi$ the spin- $\frac{1}{2}$ fermion and $(t, \bar{t})$ the complex scalar. Using the BCJ loop-level construction [5], we can build the $\mathcal{N}=4$ theory as a double copy of $\mathcal{N}=4$ super-YangMills and pure Yang-Mills theories [9, 14, 19, 20]. The two multiplets in Eq. (1) correspond respectively to the $\mathcal{N}=4$ super-Yang-Mills multiplet tensored with either positive- or negative-helicity gluons.

We may classify amplitude multiplets according to their helicity-violating ( $\mathrm{N}^{k} \mathrm{MHV}$ ) degree, $k=0, \ldots, n-4$ along with the numbers $n_{+}$and $n_{-}$of particles in the $\Phi^{+}$ and $\Phi^{-}$multiplets [16]. The former corresponds to the supersymmetric side of the double copy, while $n_{ \pm}$corre-


FIG. 1. Spanning sets of unitarity cuts at two loops: (a) three-particle cuts (b) two-particle cuts. The shaded blobs denote tree-level amplitudes, and the annulus denotes a oneloop amplitude.
spond to the nonsupersymmetric side:

$$
\begin{equation*}
M_{n, k}^{\left(n_{+}, n_{-} ; L\right)} \equiv M_{n, k}^{(L)}\left(1^{-}, \ldots, n_{-}^{-},\left(n_{-}+1\right)^{+}, \ldots, n^{+}\right) \tag{2}
\end{equation*}
$$

Superscripts denote the type of multiplet, $n=n_{+}+n_{-}$ and $L$ is the loop order. In all amplitudes, we omit a factor of $(\kappa / 2)^{n+2 L-2} c_{\Gamma}^{L} /(4 \pi)^{L(2-\epsilon)}$, where $\epsilon=(4-D) / 2$ is the dimensional regulator and where $c_{\Gamma}$ is given Eq. (3.2) of Ref. [21]. In this Letter, we will consider only MHV amplitudes where $k=0$. Vanishing amplitudes excluded by supersymmetry Ward identities are already eliminated by the use of on-shell superspace. At tree level, the $U(1)$ selection rule [16] leaves only the $M_{n, k}^{(n-k-2, k+2 ; 0)}$ amplitudes nonvanishing. The four-point superamplitude at tree level has $k=0$,

$$
\begin{equation*}
M_{n, 0}^{(2,2 ; 0)}=i \frac{\langle 12\rangle^{4}[12]}{\langle 34\rangle} \frac{\delta^{(8)}(Q)}{\prod_{i<j}\langle i j\rangle} \tag{3}
\end{equation*}
$$

The spinor inner products $\langle a b\rangle$ and $[a b]$ follow the conventions in Ref. [22] and $Q$ is the supermomentum in the usual on-shell superspace [23]. Beyond tree level, the anomaly gives nonvanishing values to other amplitudes. The tree-level vanishing of anomalous amplitudes does eliminate all divergences from the corresponding one-loop amplitudes, which are also purely rational functions of the external spinors.

Formulas for all but two classes of $k=0$ one-loop superamplitudes were conjectured based on soft limits in Ref. 17]. We will be interested in higher-loop corrections to the following superamplitudes,

$$
\begin{align*}
M_{n, 0}^{(0, n ; 1)} & =i(n-3)!\delta^{(8)}(Q), \\
M_{n, 0}^{(1, n-1 ; 1)} & =-i(n-4)!\delta^{(8)}(Q) \sum_{r=1}^{n-1} \frac{[n r]\langle r x\rangle\langle r y\rangle}{\langle n r\rangle\langle n x\rangle\langle n y\rangle} . \tag{4}
\end{align*}
$$

The latter expression is independent of the (arbitrary) reference spinors $x, y$.

Two-loop amplitudes: We compute the two-loop fourpoint anomalous amplitude of $\mathcal{N}=4$ supergravity using the double-copy construction applied to the amplitudes of Refs. [24, 25], as described in Ref. [20]. The general form
for the two-loop superamplitudes was presented there,

$$
\begin{align*}
& M^{\left(n_{+}, n_{-} ; 2\right)}(1,2,3,4)=-i s_{12} s_{23} A_{\mathcal{N}=4}^{\text {tree }}(1,2,3,4) \\
& \times[ \tag{5}
\end{align*}
$$

where $s_{i j}=\left(k_{i}+k_{j}\right)^{2}, A_{\mathcal{N}=4}^{\text {tree }}(1,2,3,4)$ is the fourpoint tree-level superamplitude of $\mathcal{N}=4$ super-YangMills theory and $A_{1234, ~ \mathrm{YM}}^{\mathrm{P}}$ and $A_{1234, \mathrm{YM}}^{\mathrm{NP}}$ are respectively the color-ordered planar and nonplanar subamplitudes of pure Yang-Mills theory with $n_{+}$positive helicities and $n_{-}$negative helicities, extracted from the QCD computation in Ref. [25]. This form holds for both anomalous and nonanomalous superamplitudes. The latter were presented explicitly in Ref. [20]. (See Section 4 of that reference for further details.)

This simple connection between the integrated supergravity amplitudes and those of pure Yang-Mills theory is special to the four-point amplitude. It follows from the $\mathcal{N}=4$ super-Yang-Mills diagram numerators' independence of loop momenta. At higher points, we can nonetheless compute the polylogarithmic parts of the two-loop amplitudes. A spanning set of unitarity cuts, from which they can be determined completely, is shown in Fig. 1. We can determine these terms using four-dimensional momenta and helicities for the cut lines. (However, one-loop amplitudes sitting on either side of the cut must be evaluated in $D$ dimensions.) This is similar to the computation of the all-multiplicity all-plus QCD amplitude in Ref. [26].

The following observations greatly simplify this calculation. First, cuts that decompose an anomalous amplitude into a product of trees automatically vanish when the cut lines are placed in four dimensions. This happens because four-dimensional cuts of anomalous amplitudes at any loop order must have an anomalous amplitude somewhere, otherwise the original amplitude will not carry a nonzero $U(1)$ charge. Thus we need only consider those cuts involving a one-loop anomalous amplitude, as shown in Fig. 1b.

Furthermore, for cuts where the configuration of external momenta is MHV, the amplitudes on both sides of the cut must also be MHV, else the supersums will vanish. This limits the number of particles that can enter the tree-level amplitude, by convention to the left of the cut as in Fig. 10. For instance, for the $M_{n, 0}^{(0, n ; 2)}$ superamplitudes, the only suitable tree-level MHV amplitude is the four-point one, as shown in the example of Fig. 2, The other side of the cut will be an anomalous one-loop amplitude. It is straightforward to compute them using


FIG. 2. The only nonzero class of four-dimensional cuts of the two-loop all minus amplitude.

Eqs. (3) and (4),

$$
\begin{align*}
\int d^{4} \eta_{\ell_{1}} d^{4} \eta_{\ell_{2}} & M_{n, 0}^{(2,2 ; 0)}\left(1^{-}, 2^{-}, \ell_{2}^{+}, \ell_{1}^{+}\right) \\
& \times M_{n, 0}^{(0, n ; 1)}\left(-\ell_{1}^{-},-\ell_{2}^{-}, 3^{-}, \ldots, n^{-}\right)  \tag{6}\\
= & \frac{1}{2} M_{n, 0}^{(0, n ; 1)} \frac{s_{12}^{2}}{\left(\ell_{1}+k_{1}\right)^{2}}+\left(\ell_{1} \leftrightarrow \ell_{2}\right)
\end{align*}
$$

For the $M_{n, 0}^{(1, n-1 ; 2)}$ superamplitudes, there are two additional classes of cuts: one with a four-point amplitude to the left of the cut, but where the two external legs on the left-hand side in Fig. 2 are taken from different multiplet types instead of both from $\Phi^{-}$; the other class, with a five-point amplitude to the left of the cut, with two external $\Phi^{-}$and one $\Phi^{+}$. More generally the cuts of $M_{n, 0}^{\left(n_{+}, n_{-} ; 2\right)}$ can contain tree-level amplitudes with at most $n_{+}+4$ particles.

We obtain,

$$
\left.\begin{array}{rl}
M_{n, 0}^{(0, n ; 2)} & =-M_{n, 0}^{(0, n ; 1)}(\epsilon) \sum_{i<j}^{n} s_{i j} \\
M_{n, 0}^{(1, n-1 ; 2)} & =-M_{n, 0}^{(1, n-1 ; 1)}(\epsilon) \sum_{i<j}^{n} s_{i j}  \tag{7}\\
- & \sum_{i<j}^{n-1} c_{i, n ; j}(s_{i j}^{n} \underbrace{n}_{i}+s_{j n}^{2} \underbrace{2}_{i}
\end{array}\right),
$$

up to possible additional rational terms. Here,

$$
\begin{equation*}
c_{i, n ; j}=-i \frac{[i n]\langle i j\rangle^{2}}{\langle i n\rangle\langle n j\rangle^{2}} \delta^{(8)}(Q) \tag{8}
\end{equation*}
$$

The would-be spurious singularities in these coefficients are in fact absent, as they are canceled by the polylogarithmic content of the accompanying integrals. The integrals appearing in these expressions are the box with a chiral numerator 27],
${ }_{i}^{n}=\int \frac{d^{D} \ell}{i \pi^{D / 2} c_{\Gamma}} \frac{\langle n| j \ell(\ell+i) \mid n]}{\ell^{2}\left(\ell+k_{i}\right)^{2}\left(\ell+k_{i}+k_{n}\right)^{2}\left(\ell-k_{j}\right)^{2}}$
$=\operatorname{Li}_{2}\left(1-\frac{K^{2}}{s_{i j}}\right)+\operatorname{Li}_{2}\left(1-\frac{K^{2}}{s_{i n}}\right)+\frac{1}{2} \ln ^{2}\left(\frac{s_{i j}}{s_{i n}}\right)+\zeta_{2} ;(9)$
the normalized scalar triangle,

$$
\begin{align*}
j & =\int \frac{d^{D} \ell}{i \pi^{D / 2} c_{\Gamma}} \frac{s_{i j}}{\ell^{2}\left(\ell+k_{i}\right)^{2}\left(\ell+k_{i}+k_{j}\right)^{2}}  \tag{10}\\
& =\frac{1}{\epsilon^{2}}-\frac{1}{\epsilon} \log \left(s_{i j}\right)+\log ^{2}\left(s_{i j}\right)
\end{align*}
$$

and a chiral triangle that integrates to zero,

$$
{ }_{i}^{n}:=\int \frac{d^{D} \ell}{i \pi^{D / 2} c_{\Gamma}} \frac{\langle n| j i \ell \mid n]^{2}}{\ell^{2}\left(\ell+k_{i}\right)^{2}\left(\ell+k_{i}+k_{n}\right)^{2}}=0
$$

The latter triangle does not contribute to the amplitude but is needed to match the cuts at the integrand level.

The results in Eq. (7) require the one-loop amplitude to higher order in $\epsilon$. The cross terms between the $\mathcal{O}(\epsilon)$ contributions and the leading $1 / \epsilon$ pole in the one-loop triangles in Eq. (77) give rise to finite contributions. We will call these 'IR-O $(\epsilon)$ ' cross terms. Weinzierl has argued [28] that such contributions ultimately cancel against corresponding ones in real-emission corrections when computing physical cross sections. Ref. [25] provides an explicit example of such a cancellation in QCD at two loops.

We have computed the $\mathcal{O}(\epsilon)$ terms for $n=4$ using the double copy as explained in Refs. [16, 21], with the result,

$$
\begin{equation*}
M_{4,0}^{\left(n_{+}, n_{-} ; 1\right)}(\epsilon)=M_{4,0}^{\left(n_{+}, n_{-} ; 1\right)}\left[1+\epsilon g^{\left(n_{+}, n_{-}\right)}+\mathcal{O}\left(\epsilon^{2}\right)\right] \tag{12}
\end{equation*}
$$

where,

$$
\begin{align*}
g^{(0,4)} & =\left[-g_{0}+\frac{8}{6}\right]+\operatorname{cyclic}(s, t, u) \\
g^{(1,3)} & =\left[2 g_{0}+\frac{1}{2} L\left(-s, \mu^{2}\right)+\frac{8}{6}\right]+\operatorname{cyclic}(s, t, u),  \tag{13}\\
g_{0} & =\frac{t u}{6 s^{2}}\left(L^{2}(t, u)+\pi^{2}\right)+\frac{s^{2}}{3 t u} L\left(-s, \mu^{2}\right)
\end{align*}
$$

with $L(v, w)=\log (v / w), s=s_{12}, t=s_{23}$ and $u=s_{13}$. The terms higher order in $\epsilon$ are not known explicitly beyond $n=4$. Integrands for both the super-YangMills side and a BCJ form of the pure Yang-Mills side of the double copy for arbitrary multiplicity were given in Ref. 1, 29]. Four-dimensional cuts do not determine these higher-order terms. They are nonetheless required by soft limits in order to match the explicit four-point two-loop calculation described above Eq. (5). There are, however, terms in Eq. (7) for $n \geq 5$ that remain incompletely determined: we cannot rule out additional rational terms that vanish in all soft and complexified collinear limits [7, 17] and have no overall spurious singularities.
One-loop counterterm amplitudes: Ref. 17] showed that one-loop anomalous amplitudes can be canceled by corresponding tree-level amplitudes with a single insertion
of a finite counterterm (see also Ref. 30]),

$$
\begin{align*}
& \Delta S_{\mathrm{ct}}=-\frac{1}{2(4 \pi)^{2}} \int d^{4} x\left((1-\log (1-\bar{t}))\left(R^{+}\right)^{2}\right.  \tag{14}\\
&\left.+(1-\log (1-t))\left(R^{-}\right)^{2}\right)+ \text { SUSY }
\end{align*}
$$

In this equation, $R^{ \pm}$are the self-dual and anti-self-dual components of the Riemann tensor, and $t$ and $\bar{t}$ are the scalars from the two supermultiplets in Eq. (1). We wish to explore this cancellation to one additional order in perturbation theory. Amplitudes with an insertion of the counterterm can be obtained using the double copy for higher-dimensional operators 31]. In particular we use the double copy of $\mathcal{N}=4$ SYM with matrix elements of the $F^{3}$ operator added to pure Yang-Mills theory 32]. We can then write one-loop four-point counterterm superamplitudes as linear combinations of products of the former with the latter,

$$
\begin{align*}
& M_{\mathrm{ct}}^{(1)}(1,2,3,4)=-i c_{H} s_{12} s_{23} A_{\mathcal{N}=4}^{\text {tree }}(1,2,3,4) \times \\
& \left(A_{F^{3}}^{(1)}(1,2,3,4)+A_{F^{3}}^{(1)}(1,3,4,2)+A_{F^{3}}^{(1)}(1,4,2,3)\right), \tag{15}
\end{align*}
$$

where $c_{H}$ is an integer factor dependent on the choice of external states (essentially on the $U(1)$ charge violation). The color-ordered $F^{3}$ amplitudes were constructed using $D$-dimensional unitarity cuts and will be presented elsewhere [33]. The sum of $M_{4,0}^{(0,4 ; 1)}$ and the corresponding $M_{\mathrm{ct}}$, or of $M_{4,0}^{(1,3 ; 1)}$ and its corresponding $M_{\mathrm{ct}}$, is zero up to IR- $\mathcal{O}(\epsilon)$ cross terms. From the viewpoint of unitarity, adding the finite counterterm cancels the one-loop amplitude that appears on the right-hand side of the cut in Fig. 10. The same cancellation continues at higher points, again up to IR- $\mathcal{O}(\epsilon)$ cross terms. Similar cross terms were first noted in Refs. 34, 35] in the context of the usual axial anomaly in dimensional regularization. Here too they were shown to cancel against real-emission terms in a physical quantity. The surviving terms in our amplitudes are exactly of this type, arising exclusively because we use a dimensional regulator. In particular, their presence will have no effect on ultraviolet divergences at higher loops.

Implications at higher loops: Setting aside possible rational terms in the two-loop anomalous amplitudes, all terms in the three-loop anomalous amplitudes detectable in the four-dimensional cuts shown in Fig. 3 will vanish in the presence of the counterterm. One may further conjecture that this pattern continues to higher loops as well, making all anomalous amplitudes vanish to all orders.

What about ultraviolet divergences? Supersymmetry and power-counting rule out ultraviolet divergences in $\mathcal{N}=4$ supergravity at one and two loops. Symmetry considerations as presently understood admit a counterterm allowing a divergence to appear at three loops [12, 36]; explicit calculation surprisingly shows that its coefficient vanishes (9], and the theory remains finite.


FIG. 3. Spanning sets of unitarity cuts at three loops: (a) four-particle cuts (b) three-particle cuts (c) two-particle cuts and (d) additional two-particle cuts. Here, the annulus denotes a one-loop amplitude or a counterterm insertion, and the double annulus denotes a two-loop amplitude. The loop amplitudes can be anomalous or not.

As leading ultraviolet divergences are detectable in fourdimensional cuts, the absence of rational terms in the two-loop amplitudes would imply that the addition of the counterterm does not spoil the ultraviolet finiteness of three-loop anomalous amplitudes. A spanning set of cuts for the three-loop four-point amplitude is shown in Fig. 3, Only the four-dimensional cuts in Fig. 3b-d contribute to anomalous amplitudes. The calculation of Ref. 9] shows that their sum gives no ultraviolet divergence; the addition of the counterterm to the theory simply adds an equal but opposite contribution, which again vanishes. The nonanomalous amplitudes receive new contributions when both sides of the cuts have equal and opposite $U(1)$ charges: one counterterm insertion and one one-loop amplitude, or two counterterm insertions. These require further study.

At four loops, an ultraviolet divergence does appear in three distinct superamplitudes [14],

$$
\begin{align*}
\left.M_{\mathcal{N}=4}^{(4)}\right|_{\text {div }} & =\frac{1}{\epsilon} \frac{\left(264 \zeta_{3}-1\right)}{288} \\
& \times\left\{M_{D^{2} R^{4}}^{(2,2)}, \quad-3 M_{D^{4} t R^{3}}^{(1,3)}, \quad 60 M_{D^{6} t^{2} R^{2}}^{(0,4)}\right\} \tag{16}
\end{align*}
$$

where $M_{\mathcal{O}}^{\left(n_{+}, n_{-}\right)}$denotes the tree-level superamplitude with given numbers of $\Phi^{ \pm}$multiplets and one insertion of the supersymmetrization of the operator $\mathcal{O}$. Its structure is quite unusual, appearing at the same order in both nonanomalous and anomalous four-point superamplitudes. As the anomalous superamplitudes have structure similar the nonanomalous amplitudes at one lower loop, one might have expected the ultraviolet divergence to have appeared at a higher loop order than the nonanomalous one.

The all-loop conjecture above implies that adding the counterterm would eliminate the anomalous amplitudes from this ultraviolet divergence. The fate of the nonanomalous superamplitude on the right-hand side of Eq. (16) remains to be computed explicitly.

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