Real-Space Renormalization for disordered systems at the level of Large Deviations

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The real-space renormalization procedures on hierarchical lattices have been much studied for many disordered systems in the past at the level of their typical fluctuations. In the present paper, the goal is to analyze instead the renormalization flows for the tails of probability distributions in order to extract the scalings of their large deviations and the tails behaviors of the corresponding rate functions. We focus on the renormalization rule for the ground-state energy of the Directed Polymer model in a random medium, and study the various renormalization flows that can emerge for the tails as a function of the tails of the initial condition.

I. INTRODUCTION

The theory of large deviation has a long history in mathematics (see the books [1–6] and references therein), in particular in the area of disordered systems (see the the books [7–9], the review [10] and references therein). In physics, the explicit use of the large deviations framework is more recent but is nowadays recognized as the unifying language for equilibrium, non-equilibrium and dynamical systems (see the reviews [11–13] and references therein). In particular, this point of view has turned out to be essential to formulate the statistical physics approach of non-equilibrium dynamics (see the reviews [14–20] and the PhD Theses [21–24] and the HDR Thesis [25]).

It is thus natural to revisit also classical and quantum disordered systems from the perspective of large deviations [26]. In particular, in the field of real-space renormalization procedures for classical statistical physics models, the focus of previous studies has been mostly the region of typical fluctuations around typical values, but it is interesting to study now how their large deviations properties emerge from the renormalization flows. In the present paper, we have chosen to focus on the renormalization rule for the intensive energy of the ground state of the Directed Polymer on a hierarchical lattice of parameters (A, B) (see section II for more details)

$$x_{n+1} = \max_{1\leq b\leq B} \left(\frac{1}{A} \sum_{a=1}^{A} x_n^{(a,b)} \right) \tag{1}$$

in order to study the renormalization flows for tails $x \to \pm\infty$ of the corresponding probability distributions $\mathcal{P}_n(x)$ as a function of the exponents $\alpha^{\pm}$ characterizing the exponential decays of the initial condition at generation $n = 0$

$$\mathcal{P}_0(x) \underset{x\to\pm\infty}{\propto} e^{-\lambda_0^{\pm} |x|^{\alpha^{\pm}}} \tag{2}$$

Besides its physical interpretation for the Directed Polymer model, the RG rule of Eq. 1 is also interesting on its own from the general point of view of probabilities, because it mixes the basic operations 'sum over A variables' and 'max over B variables'. Of course, two special limiting cases are very well-known :

(i) in the special case $B = 1$, x_n reduces to the empirical average of A^n independent variables, which is the most studied problem in the whole history of probability. The possible typical fluctuations are classified in terms of the Gaussian distribution of the Central Limit Theorem (see [27–29] for the renormalization point of view) and in terms of the Lévy stable laws (when the variance does not exist). While the standard theory for the large deviations of the empirical average focuses on the case of symmetric large deviations [11, 13], the case of asymmetric large deviations (with different scalings for rare values bigger or smaller than the typical value) have also attracted a lot of attention recently [30–35].

(ii) in the special case $A = 1$, x_n reduces to the empirical maximum of B^n independent variables. The possible typical fluctuations are classified in terms of the three universality classes Gumbel-Fréchet-Weibull of Extreme Value Statistics [36, 37], with many applications in various physics domains (see the reviews [38–40] and references therein) and have been much studied from the renormalization perspective [41–45]. The large deviations properties of the empirical maximum have been found to be asymmetric [35, 46, 47], as a consequence of the following obvious asymmetry : an 'anomalously good' maximum requires only one anomalously good variable, while an 'anomalously bad' maximum requires that all variables are anomalously bad.

The paper is organized as follows. In section II, we recall the origin of the RG rule of Eq. 1 for the Directed Polymer model on the hierarchical lattice of parameters (A, B). The various renormalization flows that can emerge for the tails $x \to \pm\infty$ of the probability distributions as a function of the exponents $\alpha^{\pm}$ of the exponential decay of the initial condition (Eq 2) are then discussed in the following sections with their consequences for the large deviations

properties : the generic large deviation form with respect to the length L_n for the tail $x \to +\infty$ that emerges for $\alpha^+ > 1$ is studied in section III; the generic large deviation form with respect to the volume L_n^d for the tail $x \to -\infty$ that emerges for $\alpha^- > 1$ is analyzed in section IV; the anomalous large deviations properties that emerge when the initial condition decays only as a stretched exponential $0 < \alpha^\pm < 1$ are then discussed for the tails $x \to +\infty$ and $x \to -\infty$ in sections V and VI respectively; the intermediate cases $\alpha^+ = 1$ and $\alpha^- = 1$ are considered in sections VII and VIII respectively. Our conclusions are summarized in Section IX. The Appendix A contains a reminder on the tails properties of the empirical average of independent variables.

II. REAL SPACE RENORMALIZATION AT THE LEVEL OF LARGE DEVIATIONS

A. Real Space Renormalization on the hierarchical diamond lattice with two parameters (A, B)

Among real-space renormalization procedures for classical statistical physics models (see the reviews [48–50] and references therein), Migdal-Kadanoff block renormalizations [51, 52] play a special role because they can be considered in two ways, either as approximate renormalization procedures on hypercubic lattices, or as exact renormalization procedures on certain hierarchical lattices [53–55]. One of the most studied hierarchical lattice is the diamond lattice which is constructed recursively from a single link called here generation $n = 0$: generation $n = 1$ consists of B branches, each branch containing A bonds in series (the standard choice is $A = 2$ but here it will be more convenient to keep the generic notation A in order to see more clearly its role in the various computations); generation $n = 2$ is obtained by applying the same transformation to each bond of the generation $n = 1$. At generation n, the length L_n between the two extreme sites is

$$L_n \quad = AL_{n-1} = A^2 L_{n-2} = .. = A^n L_0 = A^n \tag{3}$$

while the volume V_n (defined as the number of bonds) evolve as

$$V_n \quad = (AB)V_{n-1} = (AB)^2 V_{n-2} ... = (AB)^n V_0 = (AB)^n \tag{4}$$

The effective fractal dimension d defined by $V_n = L_n^d$ thus reads

$$d = \frac{\ln(AB)}{\ln A} = 1 + \frac{\ln B}{\ln A} \tag{5}$$

On this diamond lattice, many disordered models have been studied, including the diluted Ising model [56], the ferromagnetic random Potts model [57–60], spin-glasses [61–71] and the directed polymer model in a random medium [72–83]. In this paper, we will focus only on the ground-state energy of this directed polymer model.

B. RG rules for the intensive ground state energy x_n of the Directed Polymer Model

The extensive ground state energy of the Directed Polymer Model follows the renormalization rule [72]

$$E_{n+1} = \min_{1 \leq b \leq B} \left(\sum_{a=1}^{A} E_n^{(a,b)} \right) \tag{6}$$

where $E_n^{(a,b)}$ are (AB) independent energies of generation n. In order to analyze large deviations, it is convenient to focus on the intensive variable

$$x_n \equiv -\frac{E_n}{L_n} = -\frac{E_n}{A^n} \tag{7}$$

that evolves with the RG rule of Eq. 1 mentioned in the Introduction.

C. RG rules for the probability distributions $\mathcal{P}_n(x)$

The RG rule of Eq. 1 concerning random variables can be translated as follows for their probability distributions. If $\mathcal{P}_n(x)$ denotes the probability distribution of the independent intensive variables $x_n^{(a,b)}$ at generation n, the probability distribution $\mathcal{P}_{n+1}(x)$ at the next generation $(n+1)$ is then obtained via the two following steps [72] :

(1) the probability distribution $\mathcal{A}_n(x)$ of the B independent empirical averages

$$x_n^{(b)} \equiv \frac{1}{A}\sum_{a=1}^{A} x_n^{(a,b)} \tag{8}$$

is given by the convolution

$$\mathcal{A}_n(x) = \int_{-\infty}^{+\infty} dx_1 ... \int_{-\infty}^{+\infty} dx_A \mathcal{P}_n(x_1)...\mathcal{P}_n(x_A)\delta\left(x - \frac{1}{A}\sum_{a=1}^{A} x_a\right) \tag{9}$$

The tails properties of this convolution $\mathcal{A}_n(x)$ depend on the tails properties of $\mathcal{P}_n(x)$, as recalled in Appendix A.

(2) the probability distribution $\mathcal{P}_{n+1}(x)$ corresponds to the distribution of the maximum of B independent variables $x_n^{(b)}$ of Eq. 8

$$\mathcal{P}_{n+1}(x) = B\mathcal{A}_n(x)\left[\int_{-\infty}^{x} dx' \mathcal{A}_n(x')\right]^{B-1} = B\mathcal{A}_n(x)\left[1 - \int_{x}^{+\infty} dx' \mathcal{A}_n(x')\right]^{B-1} \tag{10}$$

The tails of $\mathcal{P}_{n+1}(x)$ for $x \to \pm\infty$ are thus related to the tails of $\mathcal{A}_n(x)$ as follows

$$\begin{aligned} \mathcal{P}_{n+1}(x) &\underset{x\to+\infty}{\simeq} B\mathcal{A}_n(x) \\ \mathcal{P}_{n+1}(x) &\underset{x\to-\infty}{\simeq} B\mathcal{A}_n(x)\left[\int_{-\infty}^{x} dx' \mathcal{A}_n(x')\right]^{B-1} \end{aligned} \tag{11}$$

D. Renormalization for the tails $x \to \pm\infty$ of the probability distributions $\mathcal{P}_n(x)$

The goal of the following sections is to analyze the renormalization rules for the functions $f_n^{\pm}(x)$ that characterize the two tails $x \to \pm\infty$

$$\mathcal{P}_n(x) \underset{x\to\pm\infty}{\simeq} e^{-f_n^{\pm}(x)} \tag{12}$$

in order to extract the large deviation properties for large n.

The general expectation for the Directed Polymer model in a random medium of dimension d is that the region of values bigger than the typical value ($x \geq x_{typ}$) should display a large deviation form with respect to the length L_n (because an anomalously good ground state energy requires only L anomalously good bond energies along the polymer)

$$\mathcal{P}_n(x) \underset{L_n\to+\infty}{\propto} e^{-L_n I^{+}(x)} \quad \text{for} \quad x \geq x^{typ} \tag{13}$$

while the region of values smaller than the typical value ($x \leq x_{typ}$) should display instead a large deviation form with respect to the volume $V_n = L_n^d$ (because an 'anomalously bad' ground state energy requires L^d bad bond energies in the sample)

$$\mathcal{P}_n(x) \underset{L_n\to+\infty}{\propto} e^{-L_n^d I^{-}(x)} \quad \text{for} \quad x \leq x^{typ} \tag{14}$$

This asymmetric large deviation form has been computed exactly for the Directed Polymer in dimension $d = 1 + 1$ [84, 85] that belongs to the Kardar-Parisi-Zhang universality class (see the various models and interpretations in the review [86]). Here our goal will be thus to derive this asymmetric large deviation form and to compute explicitly the tails $x \to \pm\infty$ of the corresponding rate functions $I^{\pm}(x)$ for the hierarchical lattices of parameters (A, B).

To be more concrete, we will focus on the initial conditions of the form

$$\mathcal{P}_0(x) \underset{x\to\pm\infty}{\simeq} K_0^{\pm}|x|^{\nu_0^{\pm}-1} e^{-\lambda_0^{\pm}|x|^{\alpha^{\pm}}} \tag{15}$$

where the most important parameters are the exponents $\alpha^{\pm} > 0$ characterizing the leading exponential decays. These parameters $\alpha^{\pm}$ will be conserved by the flow, while the other parameters may be renormalized, i.e. the tails at generation n will be of the form

$$\mathcal{P}_n(x) \underset{x\to\pm\infty}{\simeq} K_n^{\pm}|x|^{\nu_n^{\pm}-1} e^{-\lambda_n^{\pm}|x|^{\alpha^{\pm}}} \tag{16}$$

and thus correspond to the following special form of the tail functions $f_n^{\pm}(x)$

$$f_n^{\pm}(x) \underset{x\to\pm\infty}{\simeq} \lambda_n^{\pm}|x|^{\alpha^{\pm}} + (1-\nu_n^{\pm})\ln|x| - \ln\left(K_n^{\pm}\right) \tag{17}$$

In the following sections, we discuss the various renormalization flows of these tails $x \to \pm\infty$ that can emerge as a function of the exponents $\alpha^{\pm}$ of the initial condition of Eq. 15 : we will first consider the cases $\alpha^{\pm} > 1$ that indeed lead to the expected large deviations forms of Eqs 13 and 14, then turn to the cases $0 < \alpha^{\pm} < 1$ that lead to anomalous large deviations forms with respect to Eqs 13 and 14 and finish with the intermediate cases $\alpha^{\pm} = 1$ that require a special analysis.

III. RENORMALIZATION FLOW FOR THE TAIL $x \to +\infty$ WHEN $\alpha^+ > 1$

A. Functional renormalization for the tail function $f_n^+(x)$

As recalled in Appendix A, when the tail function $f_n^+(x)$ of Eq. 12 satisfies the conditions of Eq. A7, the tail of the distribution of the convolution $\mathcal{A}_n(x)$ of Eq. 9 has been studied in detail in Ref [87] and the output is the 'democratic' formula of Eq. A6

$$\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} e^{-Af_n^+(x)}\sqrt{A}\left(\frac{2\pi}{(f_n^+)''(x)}\right)^{\frac{A-1}{2}} \tag{18}$$

The tail $x \to +\infty$ of Eq. 11 is simply given by

$$\mathcal{P}_{n+1}(x) \underset{x\to+\infty}{\simeq} B\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} Be^{-Af_n^+(x)}\sqrt{A}\left(\frac{2\pi}{(f_n^+)''(x)}\right)^{\frac{A-1}{2}} \tag{19}$$

The identification with $\mathcal{P}_{n+1}(x \to +\infty) \simeq e^{-f_{n+1}^+(x)}$ of Eq. 12 yields the functional RG rule for the tail function $f_n^+(x)$

$$f_{n+1}^+(x) = Af_n^+(x) + (A-1)\ln\left(\sqrt{\frac{(f_n^+)''(x)}{2\pi}}\right) - \ln\left(B\sqrt{A}\right) \tag{20}$$

B. Explicit solution of the RG flow for the special form of Eq. 17 when $\alpha^+ > 1$

The special form of Eq. 17

$$\begin{aligned} f_n^+(x) &\underset{x\to+\infty}{\simeq} \lambda_n^+ x^{\alpha^+} + (1-\nu_n^+)\ln x - \ln\left(K_n^+\right) \\ (f_n^+)''(x) &\underset{x\to+\infty}{\simeq} \lambda_n^+\alpha^+(\alpha^+-1)x^{\alpha^+-2} + \frac{(\nu_n^+-1)}{x^2} \underset{x\to+\infty}{\simeq} \lambda_n^+\alpha^+(\alpha^+-1)x^{\alpha^+-2} \end{aligned} \tag{21}$$

satisfies the conditions of Eq. A7 in the region $\alpha^+ > 1$, and remains closed under the functional RG flow of Eq. 20 with the following RG rules for the parameters

$$\begin{aligned} \lambda_{n+1}^+ &= A\lambda_n^+ \\ \nu_{n+1}^+ &= A\left(\nu_n^+ - \frac{\alpha^+}{2}\right) + \frac{\alpha^+}{2} \\ \ln(K_{n+1}^+) &= A\ln(K_n^+) + (A-1)\ln\left(\sqrt{\frac{2\pi}{\lambda_n^+\alpha^+(\alpha^+-1)}}\right) + \ln(B\sqrt{A}) \end{aligned} \tag{22}$$

In terms of the initial condition at generation $n = 0$, the solution reads

$$\begin{aligned} \lambda_n^+ &= A^n\lambda_0^+ \\ \nu_n^+ &= A^n\left(\nu_0^+ - \frac{\alpha^+}{2}\right) + \frac{\alpha^+}{2} \\ \ln(K_n^+) &= A^n\left[\frac{\ln B}{A-1} + \ln\left(K_0^+\sqrt{\frac{2\pi}{\lambda_0^+\alpha^+(\alpha^+-1)}}\right)\right] + \frac{n}{2}\ln A - \frac{\ln B}{A-1} - \ln\left(\sqrt{\frac{2\pi}{\lambda_0^+\alpha^+(\alpha^+-1)}}\right) \end{aligned} \tag{23}$$

Putting everything together, it is convenient to gather all the terms involving the length $L_n = A^n$ to obtain the final result for the tail function of Eq. 21

$$\begin{aligned} f_n^+(x) \quad & \underset{x\to+\infty}{\simeq} A^n \left[\lambda_0^+ x^{\alpha^+} - \frac{\ln B}{A-1} - \ln\left(K_0^+\sqrt{\frac{2\pi}{\lambda_0^+\alpha^+(\alpha^+-1)}}\right) + \left(\frac{\alpha^+}{2} - \nu_0^+\right)\ln x\right] \\ & -\frac{n}{2}\ln A + \frac{\ln B}{A-1} + \ln\left(\sqrt{\frac{2\pi}{\lambda_0^+\alpha^+(\alpha^+-1)}}\right) + \left(1-\frac{\alpha^+}{2}\right)\ln x \end{aligned} \tag{24}$$

C. Conclusion for the large deviations in the tail $x \to +\infty$ when $\alpha^+ > 1$

The RG solution of Eq. 24 thus corresponds to the expected large deviation form with respect to the length $L_n = A^n$ of Eq. 13. In addition, the corresponding rate function $I^+(x)$ of Eq. 13 displays the tail behavior

$$I^+(x) \quad \underset{x\to+\infty}{\simeq} \lambda_0^+ x^{\alpha^+} - \frac{\ln B}{A-1} - \ln\left(K_0^+\sqrt{\frac{2\pi}{\lambda_0^+\alpha^+(\alpha^+-1)}}\right) + \left(\frac{\alpha^+}{2} - \nu_0^+\right)\ln x \tag{25}$$

D. Example with the simplest special solution

The special solution of Eq. 24 does not contain terms in $(\ln x)$ for the initial conditions satisfying

$$\begin{aligned} \alpha^+ &= 2 \\ \nu_0^+ &= 1 \end{aligned} \tag{26}$$

It is thus interesting to consider the normalized Gaussian initial distribution at generation $n = 0$

$$\begin{aligned} \mathcal{P}_0(x) &= K_0^+ e^{-\lambda_0^+ x^2} \\ K_0^+ &= \sqrt{\frac{\lambda_0^+}{\pi}} \end{aligned} \tag{27}$$

The special solution of Eq. 24 then simplifies into

$$f_n^+(x) \quad \underset{x\to+\infty}{\simeq} A^n\left[\lambda_0^+ x^2 - \frac{\ln B}{A-1}\right] - \frac{n}{2}\ln A + \frac{\ln B}{A-1} - \ln\left(\sqrt{\frac{\lambda_0^+}{\pi}}\right) \tag{28}$$

and corresponds for the probability distribution to the tail

$$\mathcal{P}_n(x) \underset{x\to+\infty}{\simeq} e^{-f_n^+(x)} \underset{x\to+\infty}{\simeq} \sqrt{\frac{\lambda_0^+ A^n}{\pi}} e^{-A^n\left[\lambda_0^+ x^2 - \frac{\ln B}{A-1}\right] - \frac{\ln B}{A-1}} = \sqrt{\frac{\lambda_0^+ A^n}{\pi}} e^{-A^n\left[x^2 - (x_n^+)^2\right]} \tag{29}$$

where

$$(x_n^+)^2 = \frac{\ln B}{A-1}\left(1 - \frac{1}{A^n}\right) \tag{30}$$

IV. RENORMALIZATION FLOW FOR THE TAIL $x \to -\infty$ WHEN $\alpha^- > 1$

A. Functional renormalization for the tail function $f_n^-(x)$

As recalled in Appendix A, when the tail function $f_n^-(x)$ of Eq. 12 satisfies the conditions of Eq. A7, the tail of the distribution of the convolution $\mathcal{A}_n(x)$ of Eq. 9 is given by the 'democratic' formula of Eq. A6

$$\mathcal{A}_n(x) \underset{x\to-\infty}{\simeq} e^{-Af_n^-(x)}\sqrt{A}\left(\frac{2\pi}{(f_n^-)''(x)}\right)^{\frac{A-1}{2}} \tag{31}$$

As a consequence, the corresponding cumulative distribution displays the tail

$$\int_{-\infty}^{x} dx' \mathcal{A}_n(x') \underset{x\to-\infty}{\simeq} \int_{-\infty}^{x} dx' e^{-Af_n^-(x')} \sqrt{A} \left(\frac{2\pi}{(f_n^-)''(x')}\right)^{\frac{A-1}{2}} \underset{x\to-\infty}{\simeq} e^{-Af_n^-(x)} \frac{\sqrt{A}}{A[-(f_n^-)'(x)]} \left(\frac{2\pi}{(f_n^-)''(x)}\right)^{\frac{A-1}{2}} \tag{32}$$

So the tail $x \to -\infty$ of Eq. 11 is given by

$$\begin{aligned} \mathcal{P}_{n+1}(x) &\underset{x\to-\infty}{\simeq} B\mathcal{A}_n(x) \left[\int_{-\infty}^{x} dx' \mathcal{A}_n(x')\right]^{B-1} \\ &\underset{x\to-\infty}{\simeq} BA[-(f_n^-)'(x)] \left[e^{-Af_n^-(x)} \frac{\sqrt{A}}{A[-(f_n^-)'(x)]} \left(\frac{2\pi}{(f_n^-)''(x)}\right)^{\frac{A-1}{2}}\right]^{B} \end{aligned} \tag{33}$$

The identification of the tail $\mathcal{P}_{n+1}(x \to -\infty) \simeq e^{-f_{n+1}^-(x)}$ of Eq. 12 yields the functional RG rule for the tail function $f_n^-(x)$

$$f_{n+1}^-(x) = ABf_n^-(x) + (A-1)B\ln\left(\sqrt{\frac{(f_n^-)''(x)}{2\pi}}\right) + (B-1)\ln[-(f_n^-)'(x)] + \left(\frac{B}{2}-1\right)\ln A - \ln B \tag{34}$$

B. Explicit solution of the RG flow for the special form of Eq. 17

The special form of Eq. 17

$$\begin{aligned} f_n^-(x) &\underset{x\to-\infty}{\simeq} \lambda_n^-(-x)^{\alpha^-} + (1-\nu_n^-)\ln(-x) - \ln\left(K_n^-\right) \\ (f_n^-)'(x) &\underset{x\to-\infty}{\simeq} -\lambda_n^-\alpha^-(-x)^{\alpha^--1} + \frac{1-\nu_n^-}{x} \\ (f_n^-)''(x) &\underset{x\to-\infty}{\simeq} \lambda_n^-\alpha^-(\alpha^--1)(-x)^{\alpha^--2} + \frac{(\nu_n^--1)}{x^2} \underset{x\to-\infty}{\simeq} \lambda_n^-\alpha^-(\alpha^--1)(-x)^{\alpha^--2} \end{aligned} \tag{35}$$

satisfies the conditions of Eq. A7 in the region $\alpha^- > 1$, and remains closed under the functional RG flow of Eq. 34 with the following RG rules for the parameters

$$\begin{aligned} \lambda_{n+1}^- &= AB\lambda_n^- \\ \nu_{n+1}^- &= AB\nu_n^- + \frac{\alpha^-}{2}(2-B-AB) \\ \ln(K_{n+1}^-) &= AB\ln(K_n^-) - (AB-1)\ln(\lambda_n^-) + (A-1)B\ln\left(\sqrt{\frac{2\pi}{\alpha^-(\alpha^--1)}}\right) - (B-1)\ln(\alpha^-) - (B-2)\frac{\ln A}{2} + \ln(B) \end{aligned} \tag{36}$$

In terms of the initial condition at generation $n = 0$, the solution reads

$$\begin{aligned} \lambda_n^- &= (AB)^n\lambda_0^- \\ \nu_n^- &= (AB)^n\left(\nu_0^- - \frac{\alpha^-}{2}\omega\right) + \frac{\alpha^-}{2}\omega \\ \ln(K_n^-) &= (AB)^n\left[\ln(K_0^-) + v\right] + \frac{n}{2}\omega\ln(AB) - v \end{aligned} \tag{37}$$

where we have introduced the notation

$$\begin{aligned} \omega &\equiv 1 + \frac{B-1}{AB-1} \\ v &\equiv -\ln(\lambda_0^-) + \frac{(A-1)B}{AB-1}\left[\frac{\ln\sqrt{B}}{AB-1} + \ln\left(\sqrt{\frac{2\pi}{\alpha^-(\alpha^--1)}}\right)\right] - \frac{B-1}{AB-1}\left[\frac{AB}{AB-1}\ln\sqrt{A} + \ln(\alpha^-)\right] \end{aligned} \tag{38}$$

Putting everything together, the tail function of Eq. 35 reads

$$\begin{aligned} f_n^-(x) \underset{x\to-\infty}{\simeq} & (AB)^n\left[\lambda_0^-|x|^{\alpha^-} - \ln(K_0^-) - v + \left(\frac{\alpha^-}{2}\omega - \nu_0^-\right)\ln|x|\right] \\ & -\frac{n}{2}\omega\ln(AB) + v + \left(1 - \frac{\alpha^-}{2}\omega\right)\ln|x| \end{aligned} \tag{39}$$

C. Conclusion for the large deviations in the tail $x \to -\infty$ when $\alpha^- > 1$

The RG solution of Eq. 39 thus corresponds to the expected large deviation form with respect to the volume $V_n = L_n^d = (AB)^n$ of Eq. 14. The corresponding rate function $I^-(x)$ of Eq. 14 displays the tail behavior

$$\begin{aligned} I^-(x) \quad & \underset{x\to-\infty}{\simeq} \lambda_0^- |x|^{\alpha^-} + \left(\frac{\alpha^-}{2}\omega - \nu_0^-\right) \ln|x| - \ln(K_0^-) - v \\ & = \lambda_0^- |x|^{\alpha^-} + \left[\frac{\alpha^-}{2}\left(1 + \frac{B-1}{AB-1}\right) - \nu_0^-\right] \ln|x| \\ & + \ln(\lambda_0^-) - \ln(K_0^-) - \frac{(A-1)B}{AB-1}\left[\frac{\ln\sqrt{B}}{AB-1} + \ln\left(\sqrt{\frac{2\pi}{\alpha^-(\alpha^- - 1)}}\right)\right] + \frac{B-1}{AB-1}\left[\frac{AB}{AB-1}\ln\sqrt{A} + \ln(\alpha^-)\right] \end{aligned} \tag{40}$$

V. RENORMALIZATION FLOW FOR THE TAIL $x \to +\infty$ WHEN $0 < \alpha^+ < 1$

A. Functional renormalization for the tail function $f_n^+(x)$

As recalled in Appendix A, the tail of the distribution of the convolution $\mathcal{A}_n(x)$ of Eq. 9 is then given by the 'monocratic formula' of Eq. A14

$$\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} A^2 e^{-f_n^+(Ax)} \tag{41}$$

so the tail $x \to +\infty$ of Eq. 11 becomes

$$\mathcal{P}_{n+1}(x) \quad \underset{x\to+\infty}{\simeq} B\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} BA^2 e^{-f_n^+(Ax)} \tag{42}$$

The identification with $\mathcal{P}_{n+1}(x \to +\infty) \simeq e^{-f_{n+1}^+(x)}$ of Eq. 12 yields the functional RG rule for the tail function $f_n^+(x)$

$$f_{n+1}^+(x) = f_n^+(Ax) - \ln\left(BA^2\right) \tag{43}$$

instead of the functional RG rule of Eq. 20.

B. Explicit solution of the RG flow for the special form of Eq. 17 for $0 < \alpha^+ < 1$

The special form of Eq. 17

$$f_n^+(x) \quad \underset{x\to+\infty}{\simeq} \lambda_n^+ x^{\alpha^+} + (1 - \nu_n^+)\ln x - \ln\left(K_n^+\right) \tag{44}$$

remains closed for the functional RG rule of Eq. 43 with the following RG rules for the parameters

$$\begin{aligned} \lambda_{n+1}^+ &= A^{\alpha^+}\lambda_n^+ \\ \nu_{n+1}^+ &= \nu_n^+ \\ \ln(K_{n+1}^+) &= \ln(K_n^+) + (\nu_n^+ + 1)\ln A + \ln B \end{aligned} \tag{45}$$

In terms of the initial condition at generation $n = 0$, the solution reads

$$\begin{aligned} \lambda_n^+ &= A^{n\alpha^+}\lambda_0^+ \\ \nu_n^+ &= \nu_0^+ \\ \ln(K_n^+) &= \ln(K_0^+) + n\left[(\nu_0^+ + 1)\ln A + \ln B\right] \end{aligned} \tag{46}$$

Putting everything together, the tail function of Eq. 44 reads

$$f_n^+(x) \quad \underset{x\to+\infty}{\simeq} A^{n\alpha^+}\lambda_0^+ x^{\alpha^+} + (1 - \nu_0^+)\ln x - \ln(K_0^+) - n\left[(\nu_0^+ + 1)\ln A + \ln B\right] \tag{47}$$

C. Conclusion for the anomalous large deviations in the tail $x \to +\infty$ when $0 < \alpha^+ < 1$

The solution of Eq. 47 thus corresponds to the following anomalous large deviation form with respect to the length $L_n = A^n$

$$\mathcal{P}_n(x) \underset{L_n \to +\infty}{\propto} e^{-L_n^{\alpha^+} J^+(x)} \quad \text{for} \quad x \geq x^{typ} \tag{48}$$

instead of the standard form of Eq. 13. The corresponding rate function $J^+(x)$ displays the tail behavior

$$J^+(x) \underset{x \to +\infty}{\simeq} \lambda_0^+ x^{\alpha^+} \tag{49}$$

VI. RENORMALIZATION FLOW FOR THE TAIL $x \to -\infty$ WHEN $0 < \alpha^- < 1$

A. Functional renormalization for the tail function $f_n^-(x)$

As recalled in Appendix A, the tail of the distribution of the convolution $\mathcal{A}_n(x)$ of Eq. 9 is then given by the 'monocratic formula of Eq. A14

$$\mathcal{A}_n(x) \underset{x \to -\infty}{\simeq} A^2 e^{-f_n^-(Ax)} \tag{50}$$

The corresponding cumulative distribution displays the tail

$$\int_{-\infty}^{x} dx' \mathcal{A}_n(x') \underset{x \to -\infty}{\simeq} \int_{-\infty}^{x} dx' A^2 e^{-f_n^-(Ax)} \underset{x \to -\infty}{\simeq} \frac{A}{[-(f_n^-)'(Ax)]} e^{-f_n^-(Ax)} \tag{51}$$

and leads to the following result for the tail $x \to -\infty$ of Eq. 11 i

$$\mathcal{P}_{n+1}(x) \underset{x \to -\infty}{\simeq} B\mathcal{A}_n(x) \left[\int_{-\infty}^{x} dx' \mathcal{A}_n(x') \right]^{B-1} \underset{x \to -\infty}{\simeq} \frac{BA^{B+1}}{[-(f_n^-)'(Ax)]^{B-1}} e^{-Bf_n^-(Ax)} \tag{52}$$

The identification with $\mathcal{P}_{n+1}(x \to -\infty) \simeq e^{-f_{n+1}^-(x)}$ of Eq. 12 yields the functional RG rule for the tail function $f_n^-(x)$

$$f_{n+1}^-(x) = Bf_n^-(Ax) + (B-1)\ln[-(f_n^-)'(Ax)] - (B+1)\ln A - \ln B \tag{53}$$

instead of the functional RG rule of Eq. 34.

B. Explicit solution of the RG flow for the special form of Eq. 17 for $0 < \alpha^- < 1$

The special form of Eq. 17

$$\begin{aligned} f_n^-(x) &\underset{x \to -\infty}{\simeq} \lambda_n^-(-x)^{\alpha^-} + (1-\nu_n^-)\ln(-x) - \ln\left(K_n^-\right) \\ (f_n^-)'(x) &\underset{x \to -\infty}{\simeq} -\lambda_n^-\alpha^-(-x)^{\alpha^- - 1} + \frac{1-\nu_n^-}{x} \underset{x \to -\infty}{\simeq} -\lambda_n^-\alpha^-(-x)^{\alpha^- - 1} \end{aligned} \tag{54}$$

remains closed for the functional RG rule of Eq. 53 with the following RG rules for the parameters

$$\begin{aligned} \lambda_{n+1}^- &= BA^{\alpha^-}\lambda_n^- \\ \nu_{n+1}^- &= B\nu_n^- - (B-1)\alpha^- \\ \ln(K_{n+1}^-) &= B\ln(K_n^-) - (B-1)\ln(\alpha^-\lambda_n^-) + \left[B(\nu_n^- + 1) - (B-1)\alpha^-\right]\ln A + \ln B \end{aligned} \tag{55}$$

In terms of the initial condition at generation $n = 0$, the solution reads

$$\begin{aligned} \lambda_n^- &= \left(BA^{\alpha^-}\right)^n \lambda_0^- \\ \nu_n^- &= B^n(\nu_0^- - \alpha^-) + \alpha^- \\ \ln(K_n^-) &= B^n\left[\ln(K_0^-) + n(\nu_0^- - \alpha^-)\ln A - \ln(\lambda_0^-\alpha^-) + \frac{B}{B-1}\ln A\right] + n\ln(BA^{\alpha^-}) + \ln(\lambda_0^-\alpha^-) - \frac{B}{B-1}\ln A \end{aligned} \tag{56}$$

Putting everything together, the tail function of Eq. 54 reads

$$f_n^-(x) \underset{x\to-\infty}{\simeq} \left(BA^{\alpha^-}\right)^n \lambda_0^- |x|^{\alpha^-} - B^n \left[(\nu_0^- - \alpha^-)|x| + \ln(K_0^-) + n(\nu_0^- - \alpha^-)\ln A - \ln(\lambda_0^-\alpha^-) + \frac{B}{B-1}\ln A\right]$$
$$-n\ln(BA^{\alpha^-}) + (1-\alpha^-)|x| - \ln(\lambda_0^-\alpha^-) + \frac{B}{B-1}\ln A \tag{57}$$

C. Conclusion for the anomalous large deviations in the tail $x \to -\infty$ when $0 < \alpha^- < 1$

The solution of Eq. 57 thus corresponds to the following anomalous large deviation form in $\left(BA^{\alpha^-}\right)^n = L_n^{d-1+\alpha^-}$

$$\mathcal{P}_n(x) \underset{L_n\to+\infty}{\propto} e^{-L_n^{d-1+\alpha^-} J^-(x)} \quad \text{for} \quad x \le x^{typ} \tag{58}$$

instead of the standard form of Eq. 14. The corresponding rate function $J^-(x)$ displays the tail behavior

$$J^-(x) \underset{x\to-\infty}{\simeq} \lambda_0^- |x|^{\alpha^-} \tag{59}$$

VII. RENORMALIZATION FLOW FOR THE TAIL $x \to +\infty$ FOR THE INTERMEDIATE CASE $\alpha^+ = 1$

A. Explicit solution of the RG flow for the special form of Eq. 17 for $\alpha^+ = 1$ and $\nu_0^+ > 0$

In this section, we wish to analyze the closed RG flow for the special form of Eq. 16 when $\alpha^+ = 1$

$$\mathcal{P}_n(x) \underset{x\to+\infty}{\simeq} K_n^+ x^{\nu_n^+ - 1} e^{-\lambda_n^+ x} \tag{60}$$

As explained in the Appendix A, the tail of the convolution of Eq. 9 is then given by Eq. A18 if $\nu_n^+ > 0$

$$\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} \frac{A^{A\nu_n^+}\left[K_n^+\Gamma(\nu_n^+)\right]^A}{\Gamma(A\nu_n^+)} x^{A\nu_n^+ - 1} e^{-A\lambda_n^+ x} \tag{61}$$

Then Eq. 11 yields that the tail at generation $(n+1)$ reads

$$\mathcal{P}_{n+1}(x) \underset{x\to+\infty}{\simeq} B\mathcal{A}_n(x) \underset{x\to+\infty}{\simeq} B\frac{A^{A\nu_n^+}\left[K_n^+\Gamma(\nu_n^+)\right]^A}{\Gamma(A\nu_n^+)} x^{A\nu_n^+ - 1} e^{-A\lambda_n^+ x} \tag{62}$$

The identification with the notations of Eq. 60 at generation $(n+1)$ leads to the following RG rules for the parameters

$$\begin{aligned} \lambda_{n+1}^+ &= A\lambda_n^+ \\ \nu_{n+1}^+ &= A\nu_n^+ \\ \ln(K_{n+1}^+) &= A\left[\ln(K_n^+) + \ln\left(\Gamma(\nu_n^+)\right) + \nu_n^+ \ln A\right] - \ln\left(\Gamma(A\nu_n^+)\right) + \ln(B) \end{aligned} \tag{63}$$

Taking into account the initial condition at generation $n = 0$, the solution reads

$$\begin{aligned} \lambda_n^+ &= A^n \lambda_0^+ \\ \nu_n^+ &= A^n \nu_0^+ \\ \ln(K_n^+) &= A^n\left[\frac{\ln B}{A-1} + n\nu^+ \ln A + \ln(K_0^+) + \ln(\Gamma(\nu^+))\right] - \ln\left(\Gamma(A^n\nu^+)\right) - \frac{1}{A-1}\ln B \end{aligned} \tag{64}$$

so this solution satisfies the validity condition $\nu_n^+ > 0$ for any n if the initial condition does $\nu_0^+ > 0$.

Putting everything together, the tail function $f_n(x)$ of Eq. 17 reads

$$f_n^+(x) \underset{x\to+\infty}{\simeq} \lambda_n^+ x + (1-\nu_n^+)\ln x - \ln\left(K_n^+\right) \tag{65}$$
$$\underset{x\to+\infty}{\simeq} A^n\left[\lambda_0^+ x - \nu_0^+ \ln x - \frac{\ln B}{A-1} - n\nu_0^+ \ln A - \ln(K_0^+) - \ln(\Gamma(\nu_0^+))\right] + \ln x + \ln\left(\Gamma(A^n\nu_0^+)\right) + \frac{1}{A-1}\ln B$$

B. Conclusion for the large deviations in the tail $x \to +\infty$ for $\alpha^+ = 1$ and $\nu_0^+ > 0$

To extract the large deviation form from the solution of Eq. 65, one needs to use the Stirling formula for the Gamma function of $z = A^n \nu_0^+ \gg 1$

$$\Gamma(A^n \nu_0^+) \underset{n \gg 1}{\simeq} \sqrt{2\pi} (A^n \nu_0^+)^{A^n \nu_0^+ - \frac{1}{2}} e^{-A^n \nu_0^+} \tag{66}$$

Plugging its logarithm

$$\ln\left(\Gamma(A^n \nu_0^+)\right) \underset{n \gg 1}{\simeq} A^n \left[n \nu_0^+ \ln A + \nu_0^+ \ln(\nu_0^+) - \nu_0^+ \right] + \ln(\sqrt{2\pi}) - \frac{1}{2}\left(n \ln A + \ln(\nu_0^+)\right) \tag{67}$$

into Eq. 65 yields to the standard large deviation form with respect to the length $L_n = A^n$ of Eq. 13 and the corresponding rate function $I^+(x)$ displays the tail behavior

$$I^+(x) \underset{x \to +\infty}{\simeq} \lambda_0^+ x - \frac{\ln B}{A-1} - \ln(K_0^+) - \ln(\Gamma(\nu_0^+)) + \nu_0^+ \ln(\nu_0^+) - \nu_0^+ - \nu_0^+ \ln x \tag{68}$$

instead of Eq. 25.

VIII. RENORMALIZATION FLOW FOR THE TAIL $x \to -\infty$ FOR THE INTERMEDIATE CASE $\alpha^- = 1$

A. Explicit solution of the RG flow for the special form of Eq. 17 for $\alpha^- = 1$ and $\nu_0^- \geq \frac{B-1}{AB-1}$

In this section, we wish to analyze the closed RG flow for the special form of Eq. 16 when $\alpha^- = 1$

$$\mathcal{P}_n(x) \underset{x \to -\infty}{\simeq} K_n^- |x|^{\nu_n^- - 1} e^{-\lambda_n^- |x|} \tag{69}$$

As explained in the Appendix A, the tail of the convolution of Eq. 9 is then given by the analog of Eq. A18 if $\nu_n^- > 0$

$$\mathcal{A}_n(x) \underset{x \to -\infty}{\simeq} \frac{A^{A\nu_n^-} \left[K_n^- \Gamma(\nu_n^-)\right]^A}{\Gamma(A\nu_n^-)} |x|^{A\nu_n^- - 1} e^{-A\lambda_n^- |x|} \tag{70}$$

Then the corresponding cumulative distribution displays the tail

$$\int_{-\infty}^{x} dx' \mathcal{A}_n(x') \underset{x \to -\infty}{\simeq} \frac{A^{A\nu_n^-} \left[K_n^- \Gamma(\nu_n^-)\right]^A}{A\lambda_n^- \Gamma(A\nu_n^-)} |x|^{A\nu_n^- - 1} e^{-A\lambda_n^- |x|} \tag{71}$$

As a consequence, the tail at generation $(n+1)$ of Eq. 11 reads

$$\mathcal{P}_{n+1}(x) = B \mathcal{A}_n(x) \left[\int_{-\infty}^{x} dx' \mathcal{A}_n(x') \right]^{B-1} \underset{x \to -\infty}{\simeq} \frac{B}{[A\lambda_n^-]^{B-1}} \left[\frac{A^{A\nu_n^-} \left[K_n^- \Gamma(\nu_n^-)\right]^A}{\Gamma(A\nu_n^-)} |x|^{A\nu_n^- - 1} e^{-A\lambda_n^- |x|} \right]^B \tag{72}$$

The identification with the notations of Eq. 69 at generation $(n+1)$ leads to the following RG rules for the parameters

$$\begin{aligned}
\lambda_{n+1}^- &= AB\lambda_n^- \\
\nu_{n+1}^- &= AB\nu_n^- - (B-1) \\
\ln(K_{n+1}^-) &= AB \ln(K_n^-) + B\left[A \ln\left(\Gamma(\nu_n^-)\right) - \ln\left(\Gamma(A\nu_n^-)\right)\right] + \nu_n^- AB \ln A - (B-1)\left[\ln(\lambda_n^-) + \ln A\right] + \ln B
\end{aligned} \tag{73}$$

Taking into account the initial condition at generation $n = 0$, the solution reads

$$\begin{aligned}
\lambda_n^- &= (AB)^n \lambda_0^- \\
\nu_n^- &= (AB)^n \left[\nu_0^- - \frac{B-1}{AB-1}\right] + \frac{B-1}{AB-1} \\
\ln(K_n^-) &= (AB)^n \left[\ln(K_0^-) + n\left(\nu_0^- - \frac{B-1}{AB-1}\right)\ln A + \frac{B(A-1)}{(AB-1)^2}\ln B - \frac{B-1}{AB-1}\ln(\lambda_0^-)\right] \\
&\quad + n\frac{B-1}{(AB-1)}\ln(AB) - \frac{B(A-1)}{(AB-1)^2}\ln B + \frac{B-1}{AB-1}\ln(\lambda_0^-) \\
&\quad + \sum_{k=0}^{n-1} (AB)^{n-1-k} B\left[A \ln\left(\Gamma(\nu_k^-)\right) - \ln\left(\Gamma(A\nu_k^-)\right)\right]
\end{aligned} \tag{74}$$

so this solution satisfies the validity condition $\nu_n^- > 0$ for any n if the initial condition satisfies $\nu_0^- \geq \frac{B-1}{AB-1}$.

B. Conclusion for the large deviations in the tail $x \to -\infty$ for $\alpha^- = 1$ and $\nu_0^- > \frac{B-1}{AB-1}$

To extract the large deviation form from the solution of Eq. 74, one needs to use the Stirling formula for $\Gamma(\nu_k^-)$ and $\Gamma(A\nu_k^-)$ to obtain the asymptotic behavior of the difference

$$\left[A \ln\left(\Gamma(\nu_k^-)\right) - \ln\left(\Gamma(A\nu_k^-)\right)\right] \underset{k \gg 1}{\simeq} -(AB)^k A \left(\nu_0^- - \frac{B-1}{AB-1}\right) \ln A - k\frac{A-1}{2}\ln(AB) + \left[(A-1)\ln(\sqrt{2\pi}) - \frac{A-1}{2}\ln\left(\nu_0^- - \frac{B-1}{AB-1}\right) + \left(\frac{A-1}{AB-1} - \frac{1}{2}\right)\ln A\right] \quad (75)$$

As a consequence, the leading terms of order $(AB)^n$ in the solution $\ln(K_n^-)$ of Eq 74 is given by

$$\ln(K_n^-) \underset{n \gg 1}{\simeq} (AB)^n \left[\ln(K_0^-) - \frac{B}{2(AB-1)}\ln A - \frac{B-1}{AB-1}\ln(\lambda_0^-) + \frac{B(A-1)}{AB-1}\left(\frac{\ln(AB)}{2(AB-1)} + \ln\left(\sqrt{\frac{2\pi}{\nu_0^- - \frac{B-1}{AB-1}}}\right)\right)\right] + ... \quad (76)$$

One thus obtains the standard large deviation form with respect to the volume $L_n^d = (AB)^n$ of Eq. 14 and the corresponding rate function $I^-(x)$ displays the tail behavior

$$I^-(x) \underset{x \to -\infty}{\simeq} \lambda_0^- |x| - \left(\nu_0^- - \frac{B-1}{AB-1}\right)\ln|x| - \ln(K_0^-) + \frac{B-1}{AB-1}\ln(\lambda_0^-) + \frac{B}{2(AB-1)}\ln A - \frac{B(A-1)}{AB-1}\left(\frac{\ln(AB)}{2(AB-1)} + \ln\left(\sqrt{\frac{2\pi}{\nu_0^- - \frac{B-1}{AB-1}}}\right)\right) \quad (77)$$

instead of Eq. 40.

IX. CONCLUSIONS

In this paper, we have revisited the renormalization rule for the ground-state energy of the Directed Polymer model on a hierarchical lattice of parameters (A, B) in order to analyze the renormalization flows for the tails of probability distributions as a function of the exponents $\alpha^\pm$ characterizing the exponential decays of the initial condition at generation $n = 0$. In each case, the explicit solution has allowed to extract the scalings involved in the large deviations properties and the tail behaviors of the corresponding rate functions. Our main conclusions can be summarized as follows :

(i) the generic large deviation form with respect to the length L_n for the tail $x \to +\infty$ emerges only for $\alpha^+ \geq 1$, while the stretched exponential $0 < \alpha^+ < 1$ initial conditions lead to anomalous large deviations in $L_n^{\alpha^+}$.

(ii) the generic large deviation form with respect to the volume L_n^d for the tail $x \to -\infty$ emerges only for $\alpha^- \geq 1$, while the stretched exponential $0 < \alpha^- < 1$ initial conditions lead to anomalous large deviations in $L_n^{d-1+\alpha^-}$.

This example shows that it is interesting to analyze the renormalization flows of disordered systems at the level of large deviations, in order to go beyond the region of typical fluctuations that have been much studied in the past.

Appendix A: Tail analysis for the empirical average of a finite number A of random variables

In this Appendix, we consider a finite number A of independent random variables x_a distributed with some probability distribution $\mathcal{P}(x)$ whose tail for $x \to +\infty$ is characterized by the function $f(x)$

$$\mathcal{P}(x) \underset{x \to +\infty}{\simeq} e^{-f(x)} \quad (A1)$$

The empirical average

$$x \equiv \frac{1}{A}\sum_{a=1}^{A} x_a \quad (A2)$$

is distributed with the convolution

$$\mathcal{A}(x) = \int_{-\infty}^{+\infty} dx_1 ... \int_{-\infty}^{+\infty} dx_A \mathcal{P}(x_1)...\mathcal{P}(x_A)\delta\left(x - \frac{1}{A}\sum_{a=1}^{A} x_a\right) \tag{A3}$$

The tail behavior as $x \to +\infty$ of this convolution depends on the tail of Eq. A1. For concreteness, it will be convenient to consider the family

$$\mathcal{P}(x) \underset{x\to+\infty}{\simeq} K x^{\nu-1} e^{-\lambda x^\alpha} \tag{A4}$$

so that the corresponding tail function $f(x)$ of Eq. A1 and its second derivative read

$$\begin{aligned} f(x) &= \lambda x^\alpha + (1-\nu)\ln x - \ln K \\ f''(x) &= \lambda\alpha(\alpha-1)x^{\alpha-2} + \frac{\nu-1}{x^2} \end{aligned} \tag{A5}$$

1. The 'democratic' formula for $\alpha > 1$

The 'democratic' formula obtained in Ref [87]

$$\mathcal{A}^{democratic}(x) \underset{x\to+\infty}{\simeq} e^{-Af(x)}\sqrt{A}\left(\frac{2\pi}{f''(x)}\right)^{\frac{A-1}{2}} \tag{A6}$$

can be understood from two points of view.

a. *'Democratic' saddle-point analysis of Ref [87]*

The formula of Eq. A6 has been derived in Ref [87] from the saddle-point evaluation of the convolution of Eq. A3 around the symmetric solution $x_a = x$ for $a = 1, 2.., A$ with the two validity conditions (see [87] for very detailed discussions and various formulations of the validity conditions)

$$\begin{aligned} & f''(x) > 0 \\ & x^2 f''(x) \underset{x\to+\infty}{\simeq} +\infty \end{aligned} \tag{A7}$$

For the special family of Eq. A4, these conditions are satisfied only in the region

$$\alpha > 1 \tag{A8}$$

while they are not satisfied for $0 < \alpha \leq 1$.

b. *Alternative derivation via the tail $k \to +\infty$ of the cumulant generating function*

Another way to understand Eq. A6 involves the cumulant generating function $\phi(k)$

$$e^{\phi(k)} \equiv \int_{-\infty}^{+\infty} dx e^{kx}\mathcal{P}(x) = \int_{-\infty}^{+\infty} dx e^{kx - f(x)} \tag{A9}$$

For $\alpha > 1$, this cumulant generating function exists even for large k, and the tail for $k \to +\infty$ is determined by the tail for $x \to +\infty$ of Eq. A1 via the saddle-point evaluation of Eq. A9 around the large saddle-point value x_k satisfying

$$f'(x_k) = k \tag{A10}$$

that leads to the asymptotic result

$$e^{\phi(k)} \underset{k\to+\infty}{\simeq} \int_{-\infty}^{+\infty} dx e^{kx_k - f(x_k) - \frac{(x-x_k)^2}{2} f''(x_k)} = e^{kx_k - f(x_k)}\sqrt{\frac{2\pi}{f''(x_k)}} \tag{A11}$$

The scaled cumulant generating function associated to the empirical average of Eq. A3 is simply given by the power A of Eq. A9

$$\int_{-\infty}^{+\infty} dx e^{Akx} \mathcal{A}(x) = \left(\int_{-\infty}^{+\infty} dx e^{kx} \mathcal{P}(x) \right)^A = e^{A\phi(k)} \tag{A12}$$

Eq A11 then yields that the tail for $k \to +\infty$ is given by

$$\int_{-\infty}^{+\infty} dx e^{kAx} \mathcal{A}_n(x) \underset{k \to +\infty}{\simeq} \left(e^{kx_k - f(x_k)} \sqrt{\frac{2\pi}{f''(x_k)}} \right)^A = e^{Akx_k - Af(x_k)} \left(\frac{2\pi}{f''(x_k)} \right)^{\frac{A-1}{2}} \sqrt{\frac{2\pi}{f''(x_k)}} \tag{A13}$$

that corresponds indeed to the saddle-point evaluation of the tail of Eq. A6.

2. The 'monocratic' formula for $0 < \alpha < 1$

The 'monocratic' formula corresponds to the cases where the tail $x \to +\infty$ of the convolution of Eq. A3 is dominated by the drawing of the anomalously large value $y \simeq Ax$ for the maximum of the A variables $(x_1, .., x_A)$, while the other $(A-1)$ values remain typical, so that one obtains the tail behavior

$$\mathcal{A}^{monocratic}(x) \underset{x \to +\infty}{\simeq} A \int dy \mathcal{P}(y) \delta\left(x - \frac{y}{A}\right) = A^2 \mathcal{P}(Ax) = A^2 e^{-f(Ax)} \tag{A14}$$

that indeed gives a bigger result than the 'democratic' formula of Eq. A6 for $0 < \alpha < 1$.

3. The intermediate case $\alpha = 1$ when $\nu > 0$

For the intermediate case $\alpha = 1$ of Eq. A4

$$\mathcal{P}(x) \underset{x \to +\infty}{\simeq} K x^{\nu - 1} e^{-\lambda x} \tag{A15}$$

one can use neither the 'democratic' formula nor the 'monocratic' formula described above. For $\nu > 0$, the cumulant generating function $\phi(k)$ of Eq. A9 exists only for $k < \lambda$ and diverges as $k \to \lambda$. This singularity as $k \to \lambda$ is then governed by the tail $x \to +\infty$ of Eq. 60 that one assumes to be valid in the region $x > C$ (where C is some fixed large constant $C > 0$)

$$e^{\phi(k)} \underset{k \to \lambda}{\simeq} \int_C^{+\infty} dx K x^{\nu-1} e^{-(\lambda - k)x} = \frac{K}{(\lambda - k)^\nu} \int_{C(\lambda - k)}^{+\infty} dt t^{\nu - 1} e^{-t} \underset{k \to \lambda}{\simeq} \frac{K\ \Gamma(\nu)}{(\lambda - k)^\nu} \tag{A16}$$

The scaled cumulant generating function of Eq. A12 associated to the empirical average of Eq. A3 then displays the singularity

$$\int_{-\infty}^{+\infty} dx e^{Akx} \mathcal{A}(x) = e^{A\phi(k)} \underset{k \to \lambda}{\simeq} \frac{[K\ \Gamma(\nu)]^A}{(\lambda - k)^{A\nu}} \tag{A17}$$

that corresponds to the following tail as $x \to +\infty$

$$\mathcal{A}(x) \underset{x \to +\infty}{\simeq} \frac{A^{A\nu} [K\Gamma(\nu)]^A}{\Gamma(A\nu)} x^{A\nu - 1} e^{-A\lambda x} \tag{A18}$$

4. Final remark on the similarities and differences with the large deviations of the empirical average

In this Appendix, we have considered as in Ref [87] the problem of the tail $x \to +\infty$ of the empirical average of a finite number A of independent variables, while the standard large deviations problem for the empirical average focuses instead on a large number $A \to +\infty$ of independent variables, while x remains finite. The two problems are

thus clearly different, but they nevertheless display some similarities as discussed in detail in Ref [87], and the two democratic/monocratic behaviors have also been much studied in the large deviation regime [30–35].

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[1] R.S. Ellis, "Entropy, Large Deviations, and Statistical Mechanics", Springer Verlag New-York (1985).
[2] J.D. Deuschel and D.W. Stroock, "Large Deviations", Academic Press Boston (1989).
[3] A. Dembo and O. Zeitouni, "Large Deviations Techniques and Applications", Springer Verlag Berlin (1998).
[4] F. den Hollander, "Large Deviations", Fields Institute Monographs, American Mathematical Society, Providence (2008).
[5] R. Azencott, M.I. Freidlin and S.R.S. Varadhan, "Large Deviations at Saint-Flour ", Probability at Saint-Flour, Springer Heidelberg (2012)
[6] F. Rassoul-Agha and T. Seppalainen, "A Course on Large Deviations with an Introduction to Gibbs Measures", Graduate Studies in Mathematics volume 162, American Mathematical Society (2015)
[7] M. Talagrand, "Spin-glasses : a challenge for mathematicians", springer New-York (2003)
[8] A. Bovier, "Statistical mechanics of disordered systems : a mathematical perspective", Cambridge University Press, Cambridge (2006).
[9] F. Comets, "Directed Polymers in Random Environments", Probability in Saint-Flour, Lecture Notes in Mathematics 2175, Springer International Publishing (2017).
[10] O. Zeitouni, J. Phys. A Math. Gen. 39, R433 (2006).
[11] Y. Oono, Progress of Theoretical Physics Supplement 99, 165 (1989).
[12] R.S. Ellis, Physica D 133, 106 (1999).
[13] H. Touchette, Phys. Rep. 478, 1 (2009).
[14] B. Derrida, J. Stat. Mech. P07023 (2007).
[15] R J Harris and G M Schütz, J. Stat. Mech. P07020 (2007).
[16] E.M. Sevick, R. Prabhakar, S. R. Williams, D. J. Searles, Ann. Rev. of Phys. Chem. Vol 59, 603 (2008).
[17] H. Touchette and R.J. Harris, chapter "Large deviation approach to nonequilibrium systems" of the book "Nonequilibrium Statistical Physics of Small Systems: Fluctuation Relations and Beyond", Wiley 2013.
[18] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim Rev. Mod. Phys. 87, 593 (2015).
[19] A. Lazarescu, J. Phys. A: Math. Theor. 48 503001 (2015).
[20] A. Lazarescu, J. Phys. A: Math. Theor. 50 254004 (2017).
[21] A. de La Fortelle, PhD (2000).
[22] V. Lecomte, PhD Thesis (2007).
[23] R. Chétrite, PhD Thesis (2008).
[24] B. Wynants, PhD Thesis (2010), arXiv:1011.4210.
[25] R. Chétrite, HDR Thesis (2018).
[26] C. Monthus, arXiv:1903.05899.
[27] G. JonaLasinio, Nuovo Cimento 26, 98 (1975).
[28] G. JonaLasinio, Phys. Rep. 352, 439 (2001).
[29] I. Calvo, J. C. Cuchi, J. G. Esteve and F. Falceto, J. Stat. Phys. 141, 409 (2010).
[30] S.V. Nagaev, Theory of Probability and its Applications, 14(1):5164. (1969)
[31] M. R. Evans and S. N. Majumdar, J. Stat. Mech. P05004 (2008).
[32] N. Gantert, K. Ramanan and F. Rembart, Electronic Communications in Probability 19 , 41, 1 (2014).
[33] C. Godrèche, arXiv:1812.02513.
[34] J. Szavits-Nossan, M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. 112, 020602 (2014) and J. Phys. A: Math. Theor. 47 (2014) 455004 (2014).
[35] C. Monthus, arXiv:1904.02448
[36] E.J. Gumbel, " Statistics of extreme" (Columbia University Press, NY 1958).
[37] J. Galambos, " The asymptotic theory of extreme order statistics" (Krieger , Malabar, FL 1987).
[38] J. P. Bouchaud and M. Mézard, J. Phys. A: Math. Gen. 30, 7997 (1997)
[39] M. Clusel and E. Bertin, Int. J. Mod. Phys. B 22, 3311 (2008)
[40] J.Y. Fortin and M. Clusel, J. Phys. A: Math. Theor. 48 183001 (2015).
[41] G. Gyorgyi, N. R. Moloney, K. Ozogany, and Z. Racz Phys. Rev. Lett. 100, 210601 (2008).
[42] G. Györgyi, N. R. Moloney, K. Ozogány, Z. Rácz, and M. Droz, Phys. Rev. E 81, 041135 (2010)
[43] E. Bertin and G. Györgyi, J. Stat. Mech. P08022 (2010).
[44] I. Calvo, J. C. Cuchi, J. G. Esteve, and F. Falceto, Phys. Rev. E 86 041109 (2012)
[45] F. Angeletti, E. Bertin, and P. Abry, J. Phys. A: Math. Theor. 45 115004 (2012).
[46] R. Giuliano and C. Macci, Communications in Statistics - Theory and Methods, 43:6, 1077-1098 (2014)
[47] P. Vivo, Eur. J. Phys. 36 055037 (2015).
[48] Th. Niemeijer, J.M.J. van Leeuwen, "Renormalization theories for Ising spin systems" in Domb and Green Eds, "Phase Transitions and Critical Phenomena" (1976).
[49] T.W. Burkhardt and J.M.J. van Leeuwen, "Real-space renormalizations", Topics in current Physics, Vol. 30, Spinger, Berlin (1982).

[50] B. Hu, Phys. Rep. 91, 233 (1982).
[51] A.A. Migdal, Sov. Phys. JETP 42, 743 (1976).
[52] L.P. Kadanoff, Ann. Phys. 100, 359 (1976).
[53] A.N. Berker and S. Ostlund, J. Phys. C 12, 4961 (1979).
[54] M. Kaufman and R. B. Griffiths, Phys. Rev. B 24, 496 - 498 (1981).
[55] R. B. Griffiths and M. Kaufman, Phys. Rev. B 26, 5022 (1982).
[56] C. Jayaprakash, E. K. Riedel and M. Wortis, Phys. Rev. B 18, 2244 (1978)
[57] W. Kinzel and E. Domany, Phys. Rev. B 23, 3421 (1981).
[58] B. Derrida and E. Gardner, J. Phys. A 17, 3223 (1984); B. Derrida, Les Houches (1984).
[59] D. Andelman and A.N. Berker, Phys. Rev. B 29, 2630 (1984).
[60] C. Monthus and T. Garel, Phys. Rev. B 77, 134416 (2008).
[61] A. P. Young and R. B. Stinchcombe, J. Phys. C 9 (1976) 4419.
[62] B. W. Southern and A. P. Young J. Phys. C 10 (1977) 2179.
[63] S.R. McKay, A.N. Berker and S. Kirkpatrick, Phys. Rev. Lett. 48 (1982) 767.
[64] E. J. Hartford, J. Appl. Phys. 70, 6068 (1991).
[65] E. Gardner, J. Physique 45, 115 (1984).
[66] A.J. Bray and M. A. Moore, J. Phys. C 17 (1984) L463.
[67] J.R. Banavar and A.J. Bray, Phys. Rev. B 35, 8888 (1987).
[68] M. A. Moore, H. Bokil, B. Drossel, Phys. Rev. Lett. 81 (1998) 4252.
[69] M. Nifle and H.J. Hilhorst, Phys. Rev. Lett. 68 (1992) 2992.
[70] M. Ney-Nifle and H.J. Hilhorst, Physica A 193 (1993) 48.
[71] M.J. Thill and H.J. Hilhorst, J. Phys. I France 6, 67 (1996)
[72] B. Derrida and R.B. Griffiths, Eur.Phys. Lett. 8 , 111 (1989).
[73] J. Cook and B. Derrida, J. Stat. Phys. 57, 89 (1989).
[74] T. Halpin-Healy, Phys. Rev. Lett. 63, 917 (1989); Phys. Rev. A , 42 , 711 (1990).
[75] S. Roux, A. Hansen, L R da Silva, LS Lucena and RB Pandey, J. Stat. Phys. 65, 183 (1991).
[76] L. Balents and M. Kardar, J. Stat. Phys. 67, 1 (1992).
[77] E. Medina and M. Kardar, J. Stat. Phys. 71, 967 (1993).
[78] M.S. Cao, J. Stat. Phys. 71, 51 (1993).
[79] LH Tang J Stat Phys 77, 581 (1994).
[80] S. Mukherji and S. M. Bhattacharjee, Phys. Rev. E 52, 1930 (1995).
[81] R. A. da Silveira and J. P. Bouchaud, Phys. Rev. Lett. 93, 015901 (2004)
[82] C. Monthus and T. Garel, Phys. Rev. E 77, 021132 (2008)
[83] C. Monthus and T. Garel, J. Stat. Mech. P01008 (2008).
[84] B. Derrida and J.L. Lebowitz, Phys. Rev. Lett. 80, 209 (1998)
[85] D.S. Dean and S.N. Majumdar, Phys. Rev. Lett. 97, 160201 (2006) and Phys. Rev. E 77, 041108 (2008).
[86] S. N. Majumdar and G. Schehr, J. Stat. Mech. P01012 (2014)
[87] U. Frisch and D. Sornette, J. Phys. I France 7, 1155 (1997).