IPhT-T19/052, Hal cea-02126908, CRM-3374, IHES-???

Large genus behavior of topological recursion

B. Eynard[123]

[1] Institut de Physique Théorique de Saclay, UMR 2681,
F-91191 Gif-sur-Yvette Cedex, France.
[2] CRM, Centre de recherches mathématiques de Montréal,
Université de Montréal, QC, Canada,
[3] IHES Bures sur Yvette.

Abstract

We show that for a rather generic set of regular spectral curves, the *Topological–Recursion* invariants F_g grow at most like $O((\beta g)! r^{-g})$ with some $r > 0$ and $\beta \leq 5$.

1 Introduction

Topological–Recursion [8, 4, 2, 7, 9, 1] associates to an object called a "spectral curve" $\mathcal{S}$, a double sequence (indexed by two non-negative integers g, n) of differential forms, that we shall call its "TR-invariants":

$$\begin{array}{rrcl} \mathrm{TR}: & \text{Spectral curves} & \to & \text{invariants} \\ & \mathcal{S} & \mapsto & \{\omega_{g,n}(\mathcal{S})\}_{g,n} \end{array} \tag{1-1}$$

where $\omega_{g,n}(\mathcal{S})$ is a symmetric multidifferential n-form, and for $n = 0$, $\omega_{g,0}(\mathcal{S})$ is denoted $F_g(\mathcal{S}) \in \mathbb{C}$ is a complex number (a 0-form).

These invariants play an importamt role in enumerative geometry, in integrable systems, in string theory, in WKB approximation, in random matrices, ... etc, see reviews [7, 9].

The main question of this article is: **how $F_g(\mathcal{S})$ behaves at large g, and more generally how $\omega_{g,n}(\mathcal{S})$ behaves at large g ? Is the series $\sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\mathcal{S})$ summable ?**

We shall establish some bounds, under reasonable smoothness assumptions on the spectral curve $\mathcal{S}$. We shall find that the series

$$\sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\mathcal{S}) \tag{1-2}$$

is an asymptotic series with factorially bounded coefficients, thus having a Borel transform converging in a disc. We postpone to a following article the issue of whether this is a resurgent series and whether it can be Borel-ressumed.

2 Bound on the growth

2.1 Hypothesis

We consider a spectral curve

$$\mathcal{S} = (\Sigma, \mathrm{x}, \mathrm{y}, B), \tag{2-1}$$

where:

- Σ is a Riemann surface (it needs not be compact neither connected, for example it could be a union of disjoint discs, = a "local curve"),
- $\mathrm{x} : \Sigma \to \mathbb{C}P^1$ is a holomorphic function, it makes Σ a ramified cover of (an open domain of) the Riemann sphere $\mathbb{C}P^1$, and in particular it can have ramification points.

  We shall moreover assume that x has only simple ramification points, at which the 1-form $d\mathrm{x}$ has only **simple zeros**, and only a finite number of them, we denote the set of ramification points:

  $$\mathfrak{R} = \{a \mid d\mathrm{x}(a) = 0\}. \tag{2-2}$$

- y is a meromorphic 1-form on Σ, that is holomorphic in a neighborhood of ramification points. We shall denote $\mathrm{y} = ydx$ where y is thus a holomorphic function in a neighborhood of ramification points. Remark: In the "local curve" definition of topological recursion, all what is needed is y to be a formal series, with possibly a zero radius of convergence, here we assume something much stronger: that y is analytic in a neighborhood of every a. However we don't care about how y could have poles or singularities outside of these neighborhoods of $\mathfrak{R}$.

  We shall furthermore assume that at any ramification point a, we have

  $$dy \neq 0 \quad \text{at} \quad a. \tag{2-3}$$

These assumptions are generic, they indicate that near a branch point a, y behaves like a square-root:

$$y(p) \sim y(a) + y'(a)\sqrt{\mathrm{x}(p) - \mathrm{x}(a)} + O(\mathrm{x}(p) - \mathrm{x}(a)) \qquad , \quad \mathrm{y} = y dx. \tag{2-4}$$

- B is a meromorphic bidifferential on $\Sigma \times \Sigma$, with double pole at coinciding points, and no other poles, normalized, in any local coordinate ζ as

$$B(p_1, p_2) \underset{p_1 \to p_2}{\sim} \frac{d\zeta(p_1) \otimes d\zeta(p_2)}{(\zeta(p_1) - \zeta(p_2))^2} + \text{analytic}. \tag{2-5}$$

- Let us define for $p \in \Sigma$:

$$\rho(p) = \sqrt{\prod_{a \in \mathfrak{R}} (\mathrm{x}(p) - a)}. \tag{2-6}$$

For some $0 < R < 1$ we are going to consider the domain of Σ

$$\Sigma_R = \{p \in \Sigma \mid |\rho(p)| \leq R\}. \tag{2-7}$$

We assume that the radius R is small enough so that Σ_R is a union of disjoint discs, whose centers are the ramification points. We make once for all a choice of squareroot in the definition of ρ, so that ρ is analytic in each disc, and is thus a local coordinate in each disc.

Definition 2.1 *Let*

$$C = |\mathfrak{R}| \sup_{p, p_1 \in \Sigma_R} \left| K(p_1, p) \frac{d\rho(p)}{d\rho(p_1)} \right| \ |\rho(p)^2 - \rho(p_1)^2| \ |\rho(p)| \tag{2-8}$$

$$B = \sup_{p, p_1 \in \Sigma_R} \left| \frac{B(p_1, p)}{d\rho(p) d\rho(p_1)} \right| \ |\rho(p) - \rho(p_1)|^2 \ . \tag{2-9}$$

Here K is the *Topological–Recursion* kernel (see [8]), worth

$$K(p_1, p) = \frac{1}{2} \frac{\int_{p'=\sigma_a(p)}^{p} B(p_1, p')}{(\mathrm{y}(p) - \mathrm{y}(\sigma_a(p)))}, \tag{2-10}$$

where $\sigma_a(p)$ denotes the unique point such that $\rho(\sigma_a(p)) = -\rho(p)$ in the disc around a.

Our hypothesis imply that B and C are $< \infty$.

2.2 The bounds

The following theorem is the main result in this paper

Theorem 2.1 (Bound) *If $2g-2+n>0$, $n\geq 1$ and $p_1,\dots,p_n\in\Sigma_R$, we have the bound*

$$\left|\frac{\omega_{g,n}(p_1,\dots,p_n)}{d\rho(p_1)\dots d\rho(p_n)}\right|\leq (n-1)!\ C_{g,n}\ \frac{C^{2g-2+n}\ B^{g-1+n}}{\left(\inf_{i\in\{1,\dots,n\}}|\rho(p_i)|\right)^{2d_{g,n}+2n}} \tag{2-11}$$

where

$$d_{g,n}=3g-3+n\quad,\quad D_{g,n}=d_{g,n}+n \tag{2-12}$$

and $C_{g,n}$ is the sequence defined by $C_{0,3}=1$, $C_{1,1}=1$, $C_{g,0}=0$, and by recursion

$$\begin{aligned} C_{g,n+1} \quad = \quad & \left((n+1)C_{g-1,n+2}+\sum_{g_1+g_2=g,\ n_1+n_2=n}^{stable} C_{g_1,n_1+1}\ C_{g_2,n_2+1}\right)\ \frac{(D_{g,n+1}+1)^{D_{g,n+1}+1}}{(D_{g,n+1})^{D_{g,n+1}}} \\ & +2C_{g,n}\ \frac{(2D_{g,n+1}+1)^{2D_{g,n+1}+1}}{3^3(2D_{g,n+1}-2)^{2D_{g,n+1}-2}} \end{aligned} \tag{2-13}$$

where "stable" means $(g_i,n_i+1)\neq(0,1),(0,2)$.

We shall use the following lemma, that we admit (proof straightforward)

Lemma 2.1 *If $k>0$ and $d>0$*

$$\inf_{\eta\in]0,1[}\frac{1}{(1-\eta)^k\eta^d}=\frac{(d+k)^{d+k}}{k^k\ d^d}\leq\frac{e^k}{k^k}\ (d+k)^k. \tag{2-14}$$

Proof of theorem 2.1.

Since this is the main result of this paper, we do the proof here in full detail.

First we write

$$W_{g,n}(p_1,\dots,p_n)=\frac{\omega_{g,n}(p_1,\dots,p_n)}{d\rho(p_1)\dots d\rho(p_n)}, \tag{2-15}$$

which is now a meromorphic function on $(\Sigma_R)^n$, with poles only at $\rho(p_i)=0$.

In all what follows we shall write

$$r_i=|\rho(p_i)|, \tag{2-16}$$

$$r_{\min}=\min_i r_i, \tag{2-17}$$

$$\eta_i=\frac{r_i}{r_{\min}}\geq 1. \tag{2-18}$$

By definition of topological recursion [8] we have

$$\omega_{g,n+1}(p_1,\dots,p_{n+1})\quad=\quad\sum_{a\in\mathfrak{R}}\frac{1}{2\pi \mathrm{i}}\oint_{p\in\mathcal{C}_a}K(p_1,p)\Big[$$

$$\sum_{g_1+g_2=g, I_1 \sqcup I_2=\{p_2,\dots,p_{n+1}\}}^{\text{stable}} \omega_{g_1,1+|I_1|}(p, I_1)\omega_{g_2,1+|I_2|}(\sigma_a(p), I_2)\Big]$$
$$+\sum_{a\in\mathfrak{R}} \frac{1}{2\pi \mathrm{i}} \oint_{p\in\mathcal{C}_a} K(p_1,p)\omega_{g-1,n+1}(p,\sigma_a(p),p_2,\dots,p_{n+1})$$
$$+\sum_{j=2}^{n+1}\sum_{a\in\mathfrak{R}} \frac{2}{2\pi \mathrm{i}} \oint_{p\in\mathcal{C}_a} K(p_1,p)B(\sigma_a(p),p_j)\ \omega_{g,n}(p,p_2,\dots,\widehat{p_j},\dots,p_{n+1}) \tag{2-19}$$

where, for each term, $\mathcal{C}_a$ is any small–enough circle around a, that we can choose to write as a circle in the coordinate $\rho(p)$ as:

$$\rho(p) = r\ e^{\mathrm{i}\,\theta} \qquad , \qquad \theta\in[0,2\pi]. \tag{2-20}$$

"Small-enough" means that the value of the radius $r>0$ has to be chosen so that the circle doesn't enclose any point other than a, at which the integrand could have poles, in particular, since $K(p,p_1)$ has a pole at $\rho(p)=\pm\rho(p_1)$, so we must have

$$r<r_1, \tag{2-21}$$

and for the last line of (2-19), for each value of j, since $B(\sigma_a(p),p_j)$ has a pole at $\rho(p)=-\rho(p_j)$, we must have

$$r<r_j. \tag{2-22}$$

We shall thus choose

$$r=\eta\ r_{\min} \quad , \quad \eta\in]0,1[. \tag{2-23}$$

The residue is independent of the value of $\eta\in]0,1[$, and therefore we shall eventually choose the value of η that will minimize the bound.

- We start with $(g,n)=(1,1)$:

$$\omega_{1,1}(p_1) \ = \ \sum_{a\in\mathfrak{R}} \frac{1}{2\pi \mathrm{i}} \oint_{p\in\mathcal{C}_a} K(p_1,p)\Big[B(p,\sigma_a(p))\Big] \tag{2-24}$$

From (2-8), (2-9) we have for any $\eta\in]0,1[$

$$\begin{aligned} |W_{1,1}(p_1)| &\leq CB\frac{1}{2\pi}\oint_{|\rho(p)|=r=\eta|\rho(p_1)|} \frac{|d\rho(p)/\rho(p)|}{|\rho(p_1)^2-\rho(p)^2|\ 4\ |\rho(p)|^2} \\ &\leq CB\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{4\ (r_1^2-r^2)\ r^2} \\ &\leq \frac{CB}{4\ r_1^4}\ \frac{1}{(1-\eta^2)\eta^2} \\ &\leq \frac{CB}{r_1^4} \qquad \leftarrow \quad \text{with } \eta=\frac{1}{\sqrt{2}}, \end{aligned} \tag{2-25}$$

so that the theorem holds with

$$C_{1,1} = 1. \tag{2-26}$$

• Then for $(g,n)=(0,3)$, topological recursion gives:

$$\omega_{0,3}(p_1,p_2,p_3) \;=\; 2\sum_{a\in\mathfrak{R}} \frac{1}{2\pi \mathrm{i}} \oint_{p\in\mathcal{C}_a} K(p_1,p)\Big[B(p,p_2)\ B(\sigma_a(p),p_3)\Big] \tag{2-27}$$

$$\begin{aligned}
|W_{0,3}(p_1,p_2,p_3)| &\leq 2CB^2 \frac{1}{2\pi}\oint_{|\rho(p)|=r} \frac{|d\rho(p)/\rho(p)|}{|\rho(p_1)^2-\rho(p)^2|}\ \frac{1}{|\rho(p)-\rho(p_2)|^2}\ \frac{1}{|\rho(p)+\rho(p_3)|^2} \\
&\leq 2CB^2\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(r_1^2-r^2)}\ \frac{1}{(r_2-r)^2\ (r_3-r)^2} \\
&\leq \frac{2CB^2}{r_{\min}^6}\ \frac{1}{(\eta_1^2-\eta^2)\ (\eta_2-\eta)^2\ (\eta_3-\eta)^2} \\
&\leq \frac{2CB^2}{r_{\min}^6}\ \frac{1}{(1-\eta)^5} \\
&\leq \frac{2CB^2}{r_{\min}^6} \qquad \leftarrow \quad \text{with } \eta\to 0,
\end{aligned} \tag{2-28}$$

so that the theorem holds with

$$C_{0,3} = 1. \tag{2-29}$$

• The bound shall then be proved by recursion. Let (g,n) such that $2g+n>2$. Assume that the bounds are already proved for all $W_{g',n'}$ such that $2\leq 2g'+n' < 2g+n+1$, we shall now prove it for $W_{g,n+1}$.

From the recursion hypotyhesis, and assuming that we choose the circle $\mathcal{C}_a$ of radius $r=\eta r_{\min}$, we have (we write $|I_1|=n_1$, $|I_2|=n_2$, so that $n_1+n_2=n$)

$$\begin{aligned}
&\left|\sum_{a\in\mathfrak{R}} \frac{1}{2\pi\mathrm{i}}\oint_{p\in\mathcal{C}_a} d\rho(p)^2 K(p_1,p) W_{g_1,1+n_1}(p,I_1)\ W_{g_2,1+n_2}(\sigma_a(p),I_2)\right| \\
\leq\ & \frac{C}{2\pi}\oint_{p\in\mathcal{C}_a}\frac{|d\rho(p)/\rho(p)|}{|\rho(p_1)^2-\rho(p)^2|}\frac{n_1!\ C_{g_1,1+n_1}\ C^{2g_1-2+1+n_1}\ B^{g_1+n_1}}{|\rho(p)|^{2d_{g_1,1+n_1}+2n_1+2}} \\
& \frac{n_2!\ C_{g_2,1+n_2}\ C^{2g_2-2+1+n_2}\ B^{g_2+n_2}}{|\rho(p)|^{2d_{g_2,1+n_2}+2n_2+2}} \\
\leq\ & n_1!\ n_2!\ C_{g_1,1+n_1}\ C_{g_2,1+n_2}\ C^{2g-2+n+1}\ B^{g-1+n+1} \\
& \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(r_1^2-r^2)}\frac{1}{r^{2d_{g_1,1+n_1}+2d_{g_2,1+n_2}+2n+4}} \\
\leq\ & n_1!\ n_2!\ C_{g_1,1+n_1}\ C_{g_2,1+n_2}\ C^{2g-2+n+1}\ B^{g-1+n+1}\ \frac{1}{(r_1^2-r^2)}\frac{1}{r^{2d_{g,n+1}+2n}} \\
\leq\ & \frac{n_1!\ n_2!\ C_{g_1,1+n_1}\ C_{g_2,1+n_2}\ C^{2g-2+n+1}\ B^{g-1+n+1}}{r_{\min}^{2d_{g,n+1}+2(n+1)}}\ \frac{1}{(\eta_1^2-\eta^2)\ \eta^{2d_{g,n+1}+2n}} \\
\leq\ & \frac{n_1!\ n_2!\ C_{g_1,1+n_1}\ C_{g_2,1+n_2}\ C^{2g-2+n+1}\ B^{g-1+n+1}}{r_{\min}^{2d_{g,n+1}+2(n+1)}}\ \frac{1}{(1-\eta^2)\ \eta^{2d_{g,n+1}+2n}}.
\end{aligned} \tag{2-30}$$

By a similar reasoning we get when $g > 0$

$$\left|\sum_{a\in\mathfrak{R}} \frac{1}{2\pi \mathrm{i}} \oint_{p\in\mathcal{C}_a} d\rho(p)^2 K(p_1,p) W_{g-1,n+2}(p,\sigma_a(p),p_2,\dots,p_{n+1})\right|$$
$$\leq \frac{C}{2\pi} \oint_{p\in\mathcal{C}_a} \frac{|d\rho(p)/\rho(p)|}{|\rho(p_1)^2-\rho(p)^2|} \frac{(n+1)!\ C_{g-1,n+2}\ C^{2g-4+2+n}\ B^{g-1+n+1}}{|\rho(p)|^{2d_{g-1,n+2}+2(n+2)}}$$
$$\leq \frac{(n+1)! C_{g-1,n+2}\ C^{2g-2+n+1}\ B^{g-1+n+1}}{r_{\min}^{2d_{g,n+1}+2(n+1)}} \frac{1}{(1-\eta^2)\ \eta^{2d_{g,n+1}+2n}}. \tag{2-31}$$

By a similar reasoning we get when $n > 0$, and $j = 2,\dots,n+1$:

$$\left|\sum_{a\in\mathfrak{R}} \frac{1}{2\pi \mathrm{i}} \oint_{p\in\mathcal{C}_a} d\rho(p) K(p_1,p) B(\sigma_a(p),p_j)\ W_{g,n}(p,p_2,\dots,\widehat{p_j},\dots,p_{n+1})\right|$$
$$\leq \frac{(n-1)! C_{g,n}\ C^{2g-2+n+1}\ B^{g-1+n+1}}{r_{\min}^{2d_{g,n+1}+2(n+1)}} \frac{1}{(\eta_1^2-\eta^2)\ (\eta_j-\eta)^2\ \eta^{2d_{g,n+1}+2n-2}}$$
$$\leq \frac{(n-1)! C_{g,n}\ C^{2g-2+n+1}\ B^{g-1+n+1}}{r_{\min}^{2d_{g,n+1}+2(n+1)}} \frac{1}{(1-\eta)^3\ \eta^{2d_{g,n+1}+2n-2}}. \tag{2-32}$$

Using lemma 2.1, the recursion hypothesis will be satisfied with

$$C_{g,n+1} = \left((n+1) C_{g-1,n+2} + \sum_{g_1+g_2=g,\ n_1+n_2=n}^{\text{stable}} C_{g_1,n_1+1}\ C_{g_2,n_2+1} \right) \frac{(D_{g,n+1}+1)^{D_{g,n+1}+1}}{(D_{g,n+1})^{D_{g,n+1}}}$$
$$+2C_{g,n}\ \frac{(2D_{g,n+1}+1)^{2D_{g,n+1}+1}}{3^3(2D_{g,n+1}-2)^{2D_{g,n+1}-2}}. \tag{2-33}$$

□

Remark 2.1 The exponent of $1/r_{\min}$ i.e. $2d_{g,n}+2n$ is optimal, indeed it is reached for the Airy spectral curve, and is in agreement with [5, 6].

Remark 2.2 But the coefficient $C_{g,n}$ is probably far from being optimal, it was obtained by bounding the integral by the integral of the absolute value, ignoring the phase oscillations, which could produce large cancellations. We are clearly overestimating here.

2.2.1 Factorial Bound

Theorem 2.2 *We have the bounds:*

$$C_{g,n} \leq t\ r^{-g} s^{-n}\ (5g-5+3n)! \tag{2-34}$$

$$C_{g,n} \leq 9\ (5g-5+3n)!\ e^{4g-4+3n} 80^{2g-2+n} 3^{3-3g-3n} 14^{-g} \tag{2-35}$$

where

$$s = \frac{27}{80}\ e^{-3}, \tag{2-36}$$

$$r = \frac{14\times 27}{80^2}\ e^{-4}, \tag{2-37}$$

$$t = \frac{3^5}{80^2}\ e^{-4}. \tag{2-38}$$

The bound can also be written

$$C_{g,n} \leq 9\ (5g-5+3n)!\ e^{4g-4+3n} 3^{5g-5+n} 14^{-g}. \tag{2-39}$$

proof:

We shall prove the theorem by recursion. First observe that it is satisfied for $C_{0,3}=1$, $C_{1,1}=1$ and $C_{g,0}=0$. Assume that it is satisfied for all $C_{g',n'}$ such that $2g'+n'<2g+n+1$. We shall now prove it for $C_{g,n+1}$.

Define

$$A_{g,n}=5g-5+3n \quad , \quad \kappa_{g,n}=2g-2+n \quad , \quad D_{g,n}=3g-3+2n. \tag{2-40}$$

For stable (g,n) (i.e. $(g,n)\neq(0,1),(0,2)$) and with $n\geq 1$ we have

$$\kappa_{g,n}\geq 1 \quad , \quad A_{g,n}\geq 3 \quad , \quad D_{g,n}\geq 2. \tag{2-41}$$

We shall need the following inequalities:

-
$$D_{g,n}+1 \leq D_{g,n}+\kappa_{g,n} = A_{g,n}. \tag{2-42}$$

-
$$n = 2A_{g,n}-5\kappa_{g,n} \leq 2A_{g,n}-5 = 2(A_{g,n}-1)-1 \tag{2-43}$$

- for all $u\in]0,\frac{5}{2}[$ we have

$$g-1 = \frac{1}{5-2u}(A_{g,n}-u\kappa_{g,n}-3n+un) \leq \frac{A_{g,n}-3}{5-2u} \quad \Longrightarrow \quad g+1 \leq \frac{A_{g,n}+7-4u}{5-2u}. \tag{2-44}$$

  The case $u=\frac{9}{4}$ gives

$$g+1 \leq 2(A_{g,n}-2). \tag{2-45}$$

- The number of stable pairs $(g_1,1+n_1),(g_2,1+n_2)$ such that $g_1+g_2=g$ and $n_1+n_2=n$, is:

$$(g+1)(n+1)-4 \leq 4(A_{g,n+1}-2)(A_{g,n+1}-1-\frac{1}{2}). \tag{2-46}$$

- We have

$$A_{g-1,n+2} = A_{g,n+1}-2 \tag{2-47}$$

$$A_{g_1,n_1+1}+A_{g_2,n_2+1}-1 = A_{g,n+1}-3 \tag{2-48}$$

$$A_{g,n} = A_{g,n+1}-3. \tag{2-49}$$

We shall use the property that for any a, b strictly positive integers, we have $a!b! \leq (a+b-1)!$. This implies that

$$A_{g_1,n_1+1}!A_{g_2,n_2+1}! \leq (A_{g_1,n_1+1} + A_{g_2,n_2+1} - 1)! = (A_{g,n+1} - 3)! \tag{2-50}$$

$$A_{g-1,n+2}! = (A_{g,n+1} - 2)! = (A_{g,n+1} - 3)!(A_{g,n+1} - 2) \tag{2-51}$$

$$A_{g,n}! = (A_{g,n+1} - 3)! \tag{2-52}$$

From lemma 2.1, we have:

$$\begin{aligned} C_{g,n+1} \;\leq\; & \left((n+1)C_{g-1,n+2} + \sum_{g_1+g_2=g,\ n_1+n_2=n}^{\text{stable}} C_{g_1,n_1+1}\ C_{g_2,n_2+1}\right)\ e(D_{g,n+1}+1) \\ & +2C_{g,n}\ \frac{e^3}{3^3}(2D_{g,n+1}+1)^3, \end{aligned} \tag{2-53}$$

now using the recursion hypothesis we have

$$\begin{aligned} C_{g,n+1} \;\leq\; & t\ r^{-g}s^{-n-1}\Big(\frac{r}{s}(n+1)A_{g-1,n+2}! \\ & +ts^{-1}\sum_{g_1+g_2=g,\ n_1+n_2=n}^{\text{stable}} A_{g_1,n_1+1}!\ A_{g_2,n_2+1}!\Big)\ e(D_{g,n+1}+1) \\ & +t\ r^{-g}s^{-n-1}A_{g,n}!\ \frac{2^4e^3s}{3^3}(D_{g,n+1}+\frac{1}{2})^3, \end{aligned} \tag{2-54}$$

and thus

$$\begin{aligned} \frac{C_{g,n+1}}{t\ r^{-g}s^{-n-1}A_{g,n+1}!} \;\leq\; & \frac{1}{A_{g,n+1}(A_{g,n+1}-1)(A_{g,n+1}-2)}\Big(\Big(\frac{r}{s}(n+1)(A_{g,n+1}-2) \\ & +\frac{t}{s}((g+1)(n+1)-4)\Big)\ e(D_{g,n+1}+1) \\ & +\frac{2^4e^3s}{3^3}(D_{g,n+1}+\frac{1}{2})^3\Big), \end{aligned} \tag{2-55}$$

Remark that $D_{g,n+1}+1 \leq A_{g,n+1}$ and $D_{g,n+1}+\frac{1}{2} \leq A_{g,n+1}$, therefore

$$\begin{aligned} \frac{C_{g,n+1}}{t\ r^{-g}s^{-n-1}A_{g,n+1}!} \;\leq\; & \frac{e}{(A_{g,n+1}-1)(A_{g,n+1}-2)}\Big(\Big(\frac{r}{s}(n+1)(A_{g,n+1}-2) \\ & +\frac{t}{s}((g+1)(n+1)-4)\Big) \\ & +\frac{2^4e^2s}{3^3}(D_{g,n+1}+\frac{1}{2})^2\Big), \end{aligned} \tag{2-56}$$

We define

$$er/s = c'' = 14/80 \tag{2-57}$$

$$et/s = c = 9/80 \tag{2-58}$$

$$2^4e^3s/3^3 = c' = 16/80. \tag{2-59}$$

writing $A = A_{g,n+1}$, we have

$$\begin{aligned}
& c''(n+1)(A_{g,n+1}-2) + c((g+1)(n+1)-4) + c'(D_{g,n+1}+\frac{1}{2})^2 \\
\leq\ & c''(2(A-1)-1)(A-2) + c(2(A-2)(2(A-1)-1)-4) + c'(A-\frac{1}{2})^2 \\
\leq\ & (2c''+4c)(A-1)(A-2) - (c''+2c)(A-2) - 4c + c'(A^2-A+\frac{1}{4}) \\
\leq\ & (2c''+4c)(A-1)(A-2) - (c''+2c)(A-2) - 4c + c'((A-1)(A-2)+2A-2+\frac{1}{4}) \\
\leq\ & (2c''+4c+c')(A-1)(A-2) - (c''+2c)(A-2) - 4c + c'(2A-2+\frac{1}{4}) \\
\leq\ & (2c''+4c+c')(A-1)(A-2) - (c''+2c-2c')(A-2) - 4c + c'(2+\frac{1}{4})
\end{aligned} \tag{2-60}$$

We have

$$c'' + 2c - 2c' = 0 \tag{2-61}$$

$$4c - \frac{9}{4}c' = 0 \tag{2-62}$$

and

$$2c'' + 4c + c' = 1. \tag{2-63}$$

This implies

$$c''(n+1)(A_{g,n+1}-2) + c((g+1)(n+1)-4) + c'(D_{g,n+1}+\frac{1}{2})^2 \leq (A_{g,n+1}-1)(A_{g,n+1}-2) \tag{2-64}$$

which implies the bound for $C_{g,n+1}$. $\square$

2.3 Bounds for F_g

For $g \geq 2$ we have [8]

$$F_g = \frac{1}{2g-2} \sum_{a\in\mathfrak{R}} \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_a} \omega_{g,1}(p)\Phi(p) \tag{2-65}$$

where $d\Phi = (y - y(a))dx$. Our assumption that y behaves like a square-root implies that $\Phi(p) - \Phi(a)$ behaves like $O(\rho(p)^3)$. Let us define

$$\tilde{C} = \frac{1}{\#\mathfrak{R}} BC \sup_{p\in\Sigma_R} |\Phi(p) - \Phi(a)|\ |\rho(p)|^{-3}. \tag{2-66}$$

Theorem 2.3 *For $g \geq 2$ we have*

$$|F_g| \leq \tilde{C} C^{2g-2} B^{g-1}\ \frac{1}{R^{6g-6}}\ \frac{C_{g,1}}{2g-2}. \tag{2-67}$$

$$|F_g| \leq \tilde{C} \frac{9}{80e} C^{2g-2} B^{g-1}\ \frac{1}{R^{6g-6}}\ r^{-g} \frac{(5g-2)!}{2g-2}. \tag{2-68}$$

proof: Choosing the circle of radius $|\rho(p)| = R$, one has

$$\begin{aligned} (2g-2)|F_g| &\leq \frac{1}{2\pi}\frac{\tilde{C}}{BC}\int_0^{2\pi} C^{2g-2+1}B^{g-1+1}C_{g,1}\frac{R^3}{R^{2d_{g,1}+2}}Rd\theta \\ &\leq \tilde{C}C^{2g-2}B^{g-1}C_{g,1}\frac{1}{R^{2d_{g,1}-2}} \\ &\leq \tilde{C}C^{2g-2}B^{g-1}C_{g,1}\frac{1}{R^{6g-6}}. \end{aligned} \quad (2\text{-}69)$$

□

Remark that R was constrained by the condition that discs $|\rho(p)| < R$ are disjoints, in other words R somehow measures the "distance between ramification points", and thus we recover the well known fact that F_g diverges when ramification points meet.

Conclusion: Borel transform and resurgence

In this article, we have showed that, under reasonable generic asumptions $F_g(\mathcal{S})$ has a factorial growth at large g, of speed at most $(5g)!$. We already pointed out that this is an upper bound, probably overestimated, and indeed for most known examples, F_g has actually a factorial growth of order $(2g)!$.

Let us assume that F_g has a factorial growth of order $(\beta g)!$ with $\beta \leq 5$.

We may define

$$\hat{F}(\mathcal{S}, s) = \sum_{g=0}^{\infty} \frac{s^{\beta g}}{(\beta g)!}\, F_g(\mathcal{S}) \quad (2\text{-}70)$$

which is absolutely convergent in a disc.

It may happen that it is an entire function convergent in the whole complex plane $\mathbb{C}$ (this is the case where the growth of F_g was actually slower than $(\beta g)!$, and one could choose a smaller value of β).

If $\hat{F}(\mathcal{S}, s)$ would be analytically continuable beyond its convergence disc, up to ∞, we would recover F by the Laplace transform

$$F(\mathcal{S}, \hbar) = \hbar^{-2-\frac{2}{\beta}} \int_0^{\infty} ds e^{-s\hbar^{-\frac{2}{\beta}}}\hat{F}(\mathcal{S}, s). \quad (2\text{-}71)$$

This requires to know if $\hat{F}(\mathcal{S}, s)$ can be analytically continued beyond its convergence disc, up to ∞, in other words this requires to know if F_g is a resurgent series [3].

Equivalently this needs to know where the singularities of $\hat{F}(\mathcal{S}, s)$ can be, or what are the possible divergences at ∞.

If $\hat{F}$ has singularities at finite distance, we may get contributions to F of the type

$$e^{-s_{\text{sing}}\hbar^{-\frac{2}{\beta}}}. \quad (2\text{-}72)$$

If $\beta = 2$ we would get corrections in $e^{-\hbar^{-1}}$.

If $\beta > 2$ and $\hat{F}$ is an entire function and behaves at ∞ as

$$\hat{F} \sim e^{s^\alpha} \tag{2-73}$$

We may get contributions to F of the type

$$e^{-\hbar^{\frac{-2\alpha}{\beta(\alpha-1)}}}. \tag{2-74}$$

For instance if $\alpha = \frac{\beta}{\beta-2}$ we would get corrections in $e^{-\hbar^{-1}}$.

We shall study the resurgence properties in a forthcoming work...

Acknowledgments

This work was supported by the ERC Starting Grant no. 335739 “Quantum fields and knot homologies” funded by the European Research Council under the European Union’s Seventh Framework Programme. It is also partly supported by the ANR grant Quantact : ANR-16-CE40-0017. I wish to thank IHES and M. Kontsevich.

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