Topological phase transitions in random Kitaev α-chains

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The topological phases of random Kitaev α-chains are labelled by the number of localized edge Majorana Zero Modes. The critical lines between these phases thus correspond to delocalization transitions for these localized edge Majorana Zero Modes. For the random Kitaev chain with next-nearest couplings, where there are three possible topological phases $n = 0, 1, 2$, the two Lyapunov exponents of Majorana Zero Modes are computed for a specific solvable case of Cauchy disorder, in order to analyze how the phase diagram evolves as a function of the disorder strength. In particular, the direct phase transition between the phases $n = 0$ and $n = 2$ is possible only in the absence of disorder, while the presence of disorder always induces an intermediate phase $n = 1$, as found previously via numerics for other distributions of disorder.

I. INTRODUCTION

Many one-dimensional quantum models involving N quantum spins $S = 1/2$ (see the Appendix) or N spinless Dirac fermions can be reformulated in terms of $2N$ of Majorana operators γ_j that are hermitian $\gamma_j^\dagger = \gamma_j$, square to the Identity $\gamma_j^2 = \mathbb{1}$ and anti-commute with each other. For models respecting the Time-Reversal-Symmetry T defined by its action [1, 2]

$$\begin{aligned} TiT^{-1} &= -i \\ T\gamma_{2j-1}T^{-1} &= \gamma_{2j-1} \\ T\gamma_{2j}T^{-1} &= -\gamma_{2j} \end{aligned} \tag{1}$$

it is actually useful to relabel the Majorana operators with the flavors a and b to stress their different behaviors with respect to T

$$\begin{aligned} \gamma_{2j-1} &= a_j \\ \gamma_{2j} &= b_j \end{aligned} \tag{2}$$

Among the Hamiltonians respecting also the the total parity

$$P = i^N \gamma_1\gamma_2\gamma_3\gamma_4...\gamma_{2N-1}\gamma_{2N} \tag{3}$$

the simplest ones are the free-fermionic Kitaev α-chains [1–5]

$$H_\alpha = i\sum_m b_m K_{m,m+\alpha} a_{m+\alpha} \tag{4}$$

where the diagonalization simply corresponds to the pairing of $(b_n, a_{m+\alpha})$, even for random couplings $K_{m,m+\alpha}$. Then the possible edge Majorana zero modes are localized on single sites and are thus obvious. In particular, $n = |\alpha|$ counts the number of Majorana Zero Modes of type a or b located near the two edges. For instance, $\alpha = 0$ corresponds to $n = 0$ zero modes, $\alpha = 1$ corresponds to the single zero mode a_1 on the left and the single zero mode b_N on the right, $\alpha = 2$ corresponds to two zero modes (a_1, a_2) on the left and two zero modes (b_{N-1}, b_N) on the right, etc.

Many interesting random free Majorana models respecting the (P, T) symmetries, corresponding either to standard quantum spin models (see the Appendix) or to spinless Dirac fermions superconducting models [1, 2, 6], can be rewritten as a linear combinations with a finite number of values of α [3–5]

$$H = \sum_\alpha H_\alpha = i\sum_m b_m\left(\sum_\alpha K_{m,m+\alpha}a_{m+\alpha}\right) \tag{5}$$

The topological phases are characterized by the number of edge localized Majorana Zero Modes, while the phase transitions between them correspond to delocalization transitions for these edge Majorana Zero Modes. For instance, the linear combination of the three values $\alpha = 0, 1, 2$

$$H = H_0 + H_1 + H_2 = i\sum_n b_n\left(\sum_{\alpha=0}^{2} K_{n,n+\alpha}a_{n+\alpha}\right) \tag{6}$$

has been much considered recently, mostly with homogeneous couplings [3, 4, 7–9] but also with random couplings [9], in order to analyze the phase diagram of the three possible phases $n = 0, 1, 2$ and the phase transitions between them. The standard method to analyze the phase diagram of the model of Eq 6 [3, 4, 7–9], or more generally of the free-Majorana-models of Eq. 5, is the transfer matrix computation of the Majorana Zero Modes to characterize their localization properties near the edges [6, 10–19]. In the random case, the analysis of the Lyapunov exponents of the product of random transfer matrices [6, 9, 10, 12–16] is thus based on the methods that have been developed in the field of Anderson localization and other one-dimensional disordered models (see the books [20, 21] and the more recent Lectures Notes [22] as well as references therein).

The goal of the present paper is to compute the exact phase diagram between the topological phases $n = 0, 1, 2$ for the Hamiltonian of Eq. 6 for a special case of disorder. The paper is organized as follows. In section II, we recall the general method to compute zero modes in random Kitaev α-chains, and the explicit application for arbitrary disorder to the three cases $(H_0 + H_1)$, $(H_0 + H_2)$ and $(H_1 + H_2)$ containing only two values of α among the three values of Eq. 6. In section III, we focus on the topological phases of the random Hamiltonian $H_0 + H_1 + H_2$ of Eq. 6, where the localizations properties of the Majorana zero modes can be then obtained via the product of 2×2 random transfer matrix and via the Riccati recurrence method. The explicit solution for a special type of Cauchy disorder is given in section IV, in order to analyze the changes of the phase diagram as a function of the disorder strength. Our conclusions are summarized in V. Appendix A contains a short dictionary between Majorana and quantum spin chains.

II. REMINDER ON EDGE ZERO MODES IN RANDOM KITAEV α-CHAINS

A. General method to compute edge Majorana Zero Mode

For the general PT-symmetric quadratic models of Eq. 5, where each Majorana fermion of a given flavor, respectively a or b, interacts only with Majorana fermions of the other flavor, respectively b or a, the zero modes can be also separated into the two flavors, and can be constructed from some linear combination of the Majorana fermions of flavor a only

$$A = \sum_j u_j a_j \tag{7}$$

or from some linear combination of the Majorana fermions of flavor b only

$$B = \sum_j v_j b_j \tag{8}$$

The linear combination A of Eq. 7 will be a Majorana fermion if the coefficients u_j are real $u_j^* = u_j$ and if they satisfy the normalization condition

$$\sum_j u_j^2 = 1 \tag{9}$$

This Majorana fermion A will be a Zero Mode if the commutator with the Hamiltonian vanish

$$0 = [H, A] = 2i \sum_j u_j \sum_\alpha b_{j-\alpha} K_{j-\alpha,n} = 2i \sum_m b_m \left(\sum_\alpha K_{m,m+\alpha} u_{m+\alpha} \right) \tag{10}$$

This condition yields the following recursion for the coefficients u_j that should be satisfied for any m

$$0 = \sum_\alpha K_{m,m+\alpha} u_{m+\alpha} \tag{11}$$

i.e. the coefficients u_j correspond to a right eigenvector of the coupling matrix $K_{m,m+\alpha}$ associated to the zero eigenvalue.

Of course, on can apply the same analysis for the Majorana Zero Mode B of flavor b of Eq. 8, and one obtains that the real normalized coefficients v_j correspond to a left eigenvector of the coupling matrix $K_{m,m+\alpha}$ associated to the zero eigenvalue, but this will not be discussed further here in order to avoid repetitions.

B. Example $H = H_0 + H_1$

For the case

$$H = H_0 + H_1 = i\sum_{n=1}^{N} b_n K_{n,n} a_n + i\sum_{n=1}^{N-1} b_n K_{n,n+1} a_{n+1} \tag{12}$$

corresponding to the random quantum spin Ising chain (see Eqs A3 and A4 in Appendix A), the competition is between the phase $n = 0$ with no zero mode (when H_0 dominates over H_1) and the phase $n = 1$ with one zero mode of flavor a localized on the left edge and one zero mode of flavor b localized on the right edge (when H_1 dominates over H_0). To analyze the phase diagram, one thus needs to study the existence of a zero mode of flavor a localized on the left edge via the recursion of Eq. 11

$$0 = K_{m,m} u_m + K_{m,m+1} u_{m+1} \tag{13}$$

This simple recursion between two consecutive coefficients

$$u_{m+1} = -\frac{K_{m,m}}{K_{m,m+1}} u_m \tag{14}$$

can be trivially solved for any realization of the random couplings

$$u_{m+1} = \left(-\frac{K_{m,m}}{K_{m,m+1}}\right) ... \left(-\frac{K_{11}}{K_{12}}\right) u_1 = u_1 \prod_{k=1}^{m} \left(-\frac{K_{k,k}}{K_{k,k+1}}\right) \tag{15}$$

The normalization condition of Eq. 9

$$1 = \sum_{n=1}^{N} u_n^2 = u_1^2 \sum_{n=1}^{N} \prod_{k=1}^{n-1} \left(\frac{K_{k,k}^2}{K_{k,k+1}^2}\right) \tag{16}$$

corresponds to the well-known structure of Kesten random variables [23–27] and has been much discussed in relation with the surface magnetization in the ground-state of the one-dimensional transverse field Ising chain [28–30].

The Lyapunov exponent of this zero mode is then determined by the disorder-average of the logarithms of the couplings

$$\gamma^{\{H_0+H_1\}} \equiv \lim_{N\to+\infty} \frac{\ln |\frac{u_{N+1}}{u_1}|}{N} = \lim_{N\to+\infty} \frac{\sum_{k=1}^{N} \ln |\frac{K_{k,k}}{K_{k,k+1}}|}{N} = \overline{\ln |K_{k,k}| - \ln |K_{k,k+1}|} \tag{17}$$

and allows to determine if one can construct a localized zero mode near the left edge satisfying the normalization of Eq. 16.

One obtains the following topological phase diagram :

The phase $n = 1$ corresponds to the region of negative Lyapunov exponent $\gamma^{\{H_0+H_1\}} < 0$, where the normalized zero mode of flavor a is localized near the left edge and typically decays exponentially.

The phase $n = 0$ corresponds to the region of positive Lyapunov exponent $\gamma^{\{H_0+H_1\}} > 0$, where there is no normalizable zero mode of flavor a localized near the left edge.

The phase transition between the two phases $n = 0, 1$ corresponds to the vanishing of the Lyapunov exponent $\gamma^{\{H_0+H_1\}} = 0$, i.e. to the well-known criterion in the language of the Random Transverse Field Ising Chain [31, 32]. The corresponding Infinite Disorder character of the transition is reviewed in [33, 34].

C. Example $H = H_0 + H_2$

For the case

$$H = H_0 + H_2 = i\sum_{n=1}^{N} b_n K_{n,n} a_n + i\sum_{n=1}^{N-2} b_n K_{n,n+2} a_{n+2} \tag{18}$$

(see Eqs A3 and A5 in Appendix A for the translation in the spin language), Eq. 11 yields one solvable recursion for the odd coefficients

$$u_{2N+1} = \left(-\frac{K_{2N-1,2N-1}}{K_{2N-1,2N+1}}\right) u_{2N-1} = u_1 \prod_{k=1}^{m} \left(-\frac{K_{2m-1,2m-1}}{K_{2m-1,2m+1}}\right) \tag{19}$$

and one solvable recursion for the even coefficients

$$u_{2N+2} = \left(-\frac{K_{2N,2N}}{K_{2N,2N+2}}\right) u_{2N} = u_2 \prod_{k=1}^{m} \left(-\frac{K_{2m,2m}}{K_{2m,2m+2}}\right) \tag{20}$$

The corresponding Lyapunov exponents

$$\gamma_{odd}^{\{H_0+H_2\}} = \lim_{N\to+\infty} \frac{\ln |\frac{u_{2N+1}}{u_1}|}{2N} = \lim_{N\to+\infty} \frac{\sum_{m=1}^{N} \ln |\frac{K_{2m-1,2m-1}}{K_{2m-1,2m+1}}|}{2N} = \frac{1}{2}\overline{\ln |K_{k,k}| - \ln |K_{k,k+2}|} \tag{21}$$

and

$$\gamma_{even}^{\{H_0+H_2\}} = \lim_{N\to+\infty} \frac{\ln |\frac{u_{2N+2}}{u_2}|}{2N} = \lim_{N\to+\infty} \frac{\sum_{m=1}^{N} \ln |\frac{K_{2m,2m}}{K_{2m,2m+2}}|}{2N} = \frac{1}{2}\overline{\ln |K_{k,k}| - \ln |K_{k,k+2}|} \tag{22}$$

are thus equal to the value

$$\gamma_{odd}^{\{H_0+H_2\}} = \gamma_{even}^{\{H_0+H_2\}} = \frac{1}{2}\overline{\ln |K_{k,k}| - \ln |K_{k,k+2}|} \equiv \gamma^{\{H_0+H_2\}} \tag{23}$$

The phase $n = 2$ corresponds to to the region $\gamma^{\{H_0+H_2\}} < 0$, where the two zero modes are localized near the left edge and display the same typical exponential decay.

The phase $n = 0$ corresponds to to the region $\gamma^{\{H_0+H_2\}} > 0$, where there is no localized edge zero mode.

The phase transition between the two phases $n = 0, 2$ corresponds to the simultaneous delocalization transition $\gamma^{\{H_0+H_2\}} = 0$ of the two zero modes.

D. Example $H = H_1 + H_2$

For the case

$$H = H_1 + H_2 = i \sum_{n=1}^{N-1} b_n K_{n,n+1} a_{n+1} + i \sum_{n=1}^{N-2} b_n K_{n,n+2} a_{n+2} \tag{24}$$

(see Eqs A4 and A5 in Appendix A for the translation in the spin language), Eq. 11 reads

$$0 = K_{m,m+1} u_{m+1} + K_{m,m+2} u_{m+2} \tag{25}$$

So there always exists the trivial zero mode a_1 localized on the single site $m = 1$, corresponding to the singular Lyapunov exponent

$$\gamma_{-}^{\{H_0+H_2\}} = -\infty \tag{26}$$

while the possible second zero mode has for coefficients

$$u_{N+2} = \left(-\frac{K_{N,N+1}}{K_{N,N+2}}\right) u_{N+1} = u_2 \prod_{m=1}^{N} \left(-\frac{K_{m,m+1}}{K_{m,m+2}}\right) \tag{27}$$

and is thus characterized by the Lyapunov exponent

$$\gamma_{+}^{\{H_0+H_2\}} = \lim_{N\to+\infty} \frac{\ln |\frac{u_{N+2}}{u_2}|}{N} = \lim_{N\to+\infty} \frac{\sum_{m=1}^{N} \ln |\frac{K_{m,m+1}}{K_{m,m+2}}|}{N} = \overline{\ln |K_{k,k+1}| - \ln |K_{k,k+2}|} \tag{28}$$

The phase $n = 2$ corresponds to $\gamma_{+}^{\{H_1+H_2\}} < 0$, where the second zero mode is localized near the left edge.

The phase $n = 1$ corresponds to $\gamma_{+}^{\{H_1+H_2\}} > 0$, where one cannot construct a second normalized zero mode localized near the left edge.

The phase transition between the two phases $n = 1, 2$ corresponds to the delocalization transition $\gamma_{+}^{\{H_1+H_2\}} = 0$ of the second zero mode.

E. Discussion

As seen on the three examples above, the linear combination $(H_{\alpha_1} + H_{\alpha_2})$ involving only two values of α can be studied via the explicit computation of the possible zero modes for arbitrary couplings, so that the location of the phase transition between the two possible topological phases are exactly known in the presence of arbitrary disorder. In the following section, we focus on the case $H = H_0 + H_1 + H_2$ involving three values of α.

III. STUDY OF THE TOPOLOGICAL PHASES OF RANDOM HAMILTONIAN $H = H_0 + H_1 + H_2$

For the Hamiltonian $H = H_0 + H_1 + H_2$ of Eq. 6 (see Eqs A3, A4 and A5 in Appendix A for the translation in the spin language), the recursion equation 11 for the coefficients of the zero mode

$$0 = K_{m,m}u_m + K_{m,m+1}u_{m+1} + K_{m,m+2}u_{m+2} \tag{29}$$

corresponds to a linear recurrence involving three consecutive terms

$$u_{m+2} = -\frac{K_{m,m+1}}{K_{m,m+2}}u_{m+1} - \frac{K_{m,m}}{K_{m,m+2}}u_m \tag{30}$$

A. Product of random 2×2 matrices

It is standard to rewrite the recurrence of Eq. 30 as

$$\begin{pmatrix} u_{m+2} \\ u_{m+1} \end{pmatrix} = T_m \begin{pmatrix} u_{m+1} \\ u_m \end{pmatrix} \tag{31}$$

in terms of the 2×2 transfer matrix

$$T_m = \begin{pmatrix} -\frac{K_{m,m+1}}{K_{m,m+2}} & -\frac{K_{m,m}}{K_{m,m+2}} \\ 1 & 0 \end{pmatrix} \tag{32}$$

so that the solution can be obtained from the product of the random transfer matrices

$$\begin{pmatrix} u_{N+2} \\ u_{N+1} \end{pmatrix} = T_N ... T_1 \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} \tag{33}$$

Since the product

$$\mathcal{T}_N \equiv T_N ... T_1 \tag{34}$$

is a 2×2 matrix, the product of its two eigenvalues $\tau_\pm^{(N)}$ can be computed from its determinant as

$$\tau_+^{(N)}\tau_-^{(N)} = det(\mathcal{T}_N) = \prod_{m=1}^{N} det(T_m) = \prod_{m=1}^{N} \left(\frac{K_{m,m}}{K_{m,m+2}} \right) \tag{35}$$

As a consequence, the two corresponding Lyapunov exponents $\gamma^\pm$

$$\gamma_\pm \equiv \lim_{N \to +\infty} \frac{\ln |\tau_\pm^{(N)}|}{N} \tag{36}$$

satisfy the simple sum rule [9]

$$\gamma_+ + \gamma_- = \lim_{N \to +\infty} \frac{\ln |det(\mathcal{T}_N)|}{N} = \lim_{N \to +\infty} \frac{\sum_{m=1}^{N} \ln |\frac{K_{m,m}}{K_{m,m+2}}|}{N} = \overline{\ln |K_{m,m}| - \ln |K_{m,m+2}|} \tag{37}$$

with the following consequences [9] with the ordering $\gamma_- \leq \gamma_+$:

(i) if $\overline{\ln |K_{m,m}| - \ln |K_{m,m+2}|} < 0$, then the smallest Lyapunov exponent has to be negative $\gamma_- < 0$ so there is at least one zero mode, and the only possibles phases are $n = 1, 2$ (the phase $n = 0$ is excluded), while the transition between the two corresponds to the vanishing of the biggest Lyapunov exponent

$$\gamma_+^{criti(n=1,2)} = 0 \tag{38}$$

(ii) if $\overline{\ln |K_{m,m}| - \ln |K_{m,m+2}|} > 0$, then the biggest Lyapunov exponent has to be positive $\gamma_+ > 0$, so there is at most one zero mode, and the only possibles phases are $n = 0, 1$ (the phase $n = 2$ is excluded), while the transition between the two corresponds to the vanishing of the smallest Lyapunov exponent

$$\gamma_-^{criti(n=0,1)} = 0 \tag{39}$$

(iii) if $\overline{\ln |K_{m,m}| - \ln |K_{m,m+2}|} = 0$, then either both Lyapunov exponent vanish $\gamma_- = 0 = \gamma_+$ corresponding to the simultaneous delocalization transition of the two zero modes (see the example previously discussed at the end of the subsection II C), or the two Lyapunov exponents are opposite $\gamma_- < 0 < \gamma_+ = -\gamma_-$ corresponding to the phase $n = 1$.

B. Non-linear recurrence for the Riccati ratio

Another standard approach [20–22] involves the introduction of the Riccati ratios

$$R_m \equiv \frac{u_{m+1}}{u_m} \tag{40}$$

in order to transform the second-order linear recurrence of Eq. 30 into the first-order non-linear recurrence

$$R_{m+1} = -\frac{K_{m,m+1}}{K_{m,m+2}} - \frac{K_{m,m}}{K_{m,m+2}} \frac{1}{R_m} \tag{41}$$

In terms of these Riccati ratios, the biggest Lyapunov exponent reads

$$\gamma_+ = \lim_{N \to +\infty} \frac{\ln |\frac{u_{N+1}}{u_1}|}{N} = \lim_{N \to +\infty} \frac{\sum_{m=1}^{N} \ln |R_m|}{N} = \int_{-\infty}^{+\infty} dR \, \mathcal{P}_{st}(R) \ln |R| \tag{42}$$

where $\mathcal{P}_{st}(R)$ denotes the attractive stationary distribution for the recurrence of Eq. 41.

This formulation in terms of the Riccati ratios is also useful to characterize the finite-size fluctuations via the Central-Limit-Theorem

$$\ln \left| \frac{u_{N+1}}{u_1} \right| = \sum_{m=1}^{N} \ln |R_m| \underset{N \to +\infty}{\simeq} \gamma_+ N + \sqrt{N} u \tag{43}$$

where u is a Gaussian variable of zero mean and of variance given by the variance of $(\ln |R|)$ computed with the stationary distribution $\mathcal{P}_{st}(R)$. As a consequence, the phase transition corresponds to an Infinite Disorder Fixed point, where the typical correlation exponent $\nu_{typ} = 1$ characterizes the vanishing of γ_+ as a function of the control parameter, while the average correlation exponent $\nu_{av} = 2$ characterizes the sample-to-sample fluctuations, as in the much studied Random Transverse Field Ising Chain (see the review [33]). Hence Strong Disorder Renormalization has been used to analyze the critical points and the Griffiths effects for this type of models in the langage of dirty superconductors [6].

C. Reminder on the numerical results for the case of random couplings K_{mm} [9]

In the presence of arbitrary disorder, recurrences like Eq. 41 are not exactly soluble and are usually studied numerically, for instance the results for the case with random $K_{m,m}$ and non-random $K_{m,m+1}$ and $K_{m,m+2}$ can be found in Ref. [9], with the following conclusions for the phase diagram :

(a) The direct phase transition between the phases $n = 0$ and $n = 2$ that requires the simultaneous delocalization of the two zero modes $\gamma_- = 0 = \gamma_+$ is possible only in the absence of disorder, while the presence of disorder induces a splitting between the two Lyapunov exponents and thus introduces an intermediate phase $n = 1$ even for arbitrary weak disorder. More generally, the Lyapunov spectrum is expected to be non-degenerate in random systems so that the phase transitions only change the topological index by one [6].

(b) The presence of disorder may favor the existence of Majorana Zero Mode, i.e. in the phase diagram, a point corresponding to the phase $n = 0$ for the pure model may belong to the phase $n = 1$ in the presence of sufficient disorder, or similarly a point corresponding to the phase $n = 1$ for the pure model may belong to the phase $n = 2$ in the presence of sufficient disorder.

(c) As the disorder becomes very strong for the couplings $K_{m,m}$, one expects transition $n = 2 \to n = 1 \to n = 0$ towards the trivial phase $n = 0$.

In the following, it is thus interesting to analyze the same questions for the phase diagram corresponding to another type of disorder, namely in the coupling $K_{m,m+1}$.

IV. EXPLICIT SOLUTION FOR $H = H_0 + H_1 + H_2$ WITH CAUCHY DISORDER IN $K_{m,m+1}$

Among the exactly soluble cases of first-order non-linear recurrence for Riccati ratios [20–22], the simplest explicit case is the Lloyd model [35, 36], as already used in the context of random Majorana models in [12]. For our present model, this soluble case for the Riccati recurrence corresponds the case where the couplings $K_{m,m}$ and $K_{m,m+2}$ are non-random

$$\begin{aligned} K_{m,m} &= k_0 \\ K_{m,m+2} &= k_2 \end{aligned} \tag{44}$$

while the couplings $K_{m,m+1}$ are distributed with the Cauchy distribution $C_{k_1,W}$ of average k_1 and broadness W

$$C_{k_1,W}(K_{m,m+1}) = \frac{1}{\pi} \frac{W}{(K_{m,m+1} - k_1)^2 + W^2} \tag{45}$$

A. Recurrence for the parameters of the Cauchy distribution

Then the recurrence of Eq. 41 simplifies into

$$R_{m+1} = -\frac{K_{m,m+1}}{k_2} - \frac{k_0}{k_2} \frac{1}{R_m} \tag{46}$$

Since the Cauchy distribution is stable with respect to both addition and inversion, the Riccati ratios R_m are then also distributed with the Cauchy distribution C_{x_m,y_m} with some average x_m and some broadness $y_m > 0$

$$C_{x_m,y_m}(R_m) = \frac{1}{\pi} \frac{y_m}{(R_m - x_m)^2 + y_m^2} \tag{47}$$

with the Fourier transform

$$\int_{-\infty}^{+\infty} dR_m e^{iqR_m} C_{x_m,y_m}(R_m) = e^{iqx_m - |q| y_m} \tag{48}$$

The recurrence of Eq. 46 yields the following recurrence for the Cauchy distributions

$$C_{x_{m+1},y_{m+1}}(R_{m+1}) = \int dK_{m,m+1} C_{k_1,W}(K_{m,m+1}) \int dR_m C_{x_m,y_m}(R_m) \delta\left(R_{m+1} + \frac{K_{m,m+1}}{k_2} + \frac{k_0}{k_2}\frac{1}{R_m}\right) \tag{49}$$

or equivalently in Fourier transform

$$\begin{aligned} e^{iqx_{m+1} - |q| y_{m+1}} &= \int dR_{m+1} e^{i\omega R_{m+1}} C_{x_{m+1},y_{m+1}}(R_{m+1}) \\ &= \int dK_{m,m+1} C_{k_1,W}(K_{m,m+1}) e^{-i\frac{q}{k_2} K_{m,m+1}} \int dR_m C_{x_m,y_m}(R_m) e^{-iq\frac{k_0}{k_2}\frac{1}{R_m}} \\ &= e^{-i\frac{q}{k_2}k_1 - |\frac{q}{k_2}| W} e^{-iq\frac{k_0}{k_2}\frac{x_m}{x_m^2+y_m^2} - |\frac{qk_0}{k_2}| \frac{y_m}{x_m^2+y_m^2}} \end{aligned} \tag{50}$$

The recurrence for the average x_m and the broadness y_m thus reads

$$\begin{aligned} x_{m+1} &= -\frac{k_1}{k_2} - \frac{k_0}{k_2}\frac{x_m}{x_m^2+y_m^2} \\ y_{m+1} &= \frac{W}{|k_2|} + \left|\frac{k_0}{k_2}\right| \frac{y_m}{x_m^2+y_m^2} \end{aligned} \tag{51}$$

and the corresponding attractive fixed point (x_f, y_f) satisfies

$$\begin{aligned} x_f &= -\frac{k_1}{k_2} - \frac{k_0}{k_2}\frac{x_f}{x_f^2+y_f^2} \\ y_f &= \frac{W}{|k_2|} + \left|\frac{k_0}{k_2}\right| \frac{y_f}{x_f^2+y_f^2} \end{aligned} \tag{52}$$

From the stationary distribution $\mathcal{P}_{st}(R) = C_{x_f,y_f}(R)$ of the Riccati ratio R, Eq. 42 yields the biggest Lyapunov exponent

$$\gamma_+ = \int_{-\infty}^{+\infty} dR \mathcal{P}_{st}(R) \ln|R| = \int dR \frac{1}{\pi} \frac{y_f}{(R-x_f)^2+y_f^2} \ln|R| = \frac{\ln(x_f^2+y_f^2)}{2} \tag{53}$$

while the sum rule of Eq. 37 yields the other Lyapunov exponent

$$\gamma_- = \ln\left|\frac{k_0}{k_2}\right| - \gamma_+ \tag{54}$$

The solution of the fixed-point system of Eq. 52 depends on the sign of the ratio $\frac{k_0}{k_2}$, i.e. whether the two couplings k_0 and k_2 have the same sign or not. These two cases are thus analyzed in the two following subsections respectively.

B. Lyapunov exponents for the case $\frac{k_0}{k_2} = -\left|\frac{k_0}{k_2}\right| < 0$

When the two couplings k_0 and k_2 have opposite signs, Eq 52 can be rewritten as

$$\begin{aligned} x_f &= -\frac{\frac{k_1}{k_2}}{1 - \left|\frac{k_0}{k_2}\right| \frac{1}{x_f^2+y_f^2}} \\ y_f &= \frac{\frac{W}{|k_2|}}{1 - \left|\frac{k_0}{k_2}\right| \frac{1}{x_f^2+y_f^2}} \end{aligned} \tag{55}$$

It is then convenient to introduce the polar coordinates

$$\begin{aligned} x_f &= r_f \cos\theta_f \\ y_f &= r_f \sin\theta_f \end{aligned} \tag{56}$$

Since the broadness is positive $y_f > 0$, the angle belongs to $\theta \in [0, \pi[$ and is determined by its tangent

$$\tan\theta_f = \frac{y_f}{x_f} \quad = -\frac{W k_2}{k_1 |k_2|} \tag{57}$$

while the modulus r_f satisfies the bound $\left(1 - \left|\frac{k_0}{k_2}\right| \frac{1}{r_f^2}\right) > 0$ and thus the equation

$$r_f = \sqrt{x_f^2+y_f^2} = \frac{\frac{\sqrt{k_1^2+W^2}}{|k_2|}}{1 - \left|\frac{k_0}{k_2}\right| \frac{1}{r_f^2}} \tag{58}$$

The rewriting as a second order equation

$$r_f^2 - \frac{\sqrt{k_1^2+W^2}}{|k_2|} r_f - \left|\frac{k_0}{k_2}\right| = 0 \tag{59}$$

yields that the only positive solution reads

$$r_f = \frac{\sqrt{k_1^2+W^2} + \sqrt{k_1^2+W^2+4|k_0k_2|}}{2|k_2|} \tag{60}$$

The biggest Lyapunov exponent of Eq 53 becomes

$$\gamma_+ = \frac{\ln(x_f^2+y_f^2)}{2} = \ln r_f = \ln\left(\frac{\sqrt{k_1^2+W^2} + \sqrt{k_1^2+W^2+4|k_0k_2|}}{2|k_2|}\right) \tag{61}$$

while the other Lyapunov exponent $\gamma_- \leq \gamma_+$ can be obtained from the sum of Eq. 37

$$\gamma_- = \ln|k_0| - \ln|k_2| - \gamma_+ = \ln\left(\frac{2|k_0|}{\sqrt{k_1^2+W^2} + \sqrt{k_1^2+W^2+4|k_0k_2|}}\right) \tag{62}$$

1. *Phase $n = 0$*

The phase $n = 0$ corresponds to two positive Lyapunov exponents $0 < \gamma^-(\leq \gamma^+)$, i.e. to the region

$$\sqrt{k_1^2+W^2} + \sqrt{k_1^2+W^2+4|k_0k_2|} < 2|k_0| \tag{63}$$

which can be simplified into

$$\sqrt{k_1^2+W^2} + |k_2| < |k_0| \tag{64}$$

2. *Phase $n = 2$*

The phase $n = 2$ corresponds to two negative Lyapunov exponents $(\gamma^- \leq)\gamma^+ < 0$, i.e. to the region

$$\sqrt{k_1^2+W^2} + \sqrt{k_1^2+W^2+4|k_0k_2|} < 2|k_2| \tag{65}$$

which can be simplified into

$$\sqrt{k_1^2+W^2} + |k_0| < |k_2| \tag{66}$$

3. *Phase $n = 1$*

Finally, the phase $n = 1$ corresponds to the case $\gamma^- < 0 < \gamma^+$, i.e. to the region

$$(|k_2| - |k_0|)^2 < k_1^2 + W^2 \tag{67}$$

4. *Conclusion : Locations of the phase transitions between the three phases in the region $\frac{k_0}{k_2} = -\left|\frac{k_0}{k_2}\right| < 0$*

The critical line between the phases $n = 0$ and $n = 1$ corresponds to

$$\sqrt{k_1^2+W^2} + |k_2| = |k_0| \tag{68}$$

The critical line between the phases $n = 1$ and $n = 2$ corresponds to

$$\sqrt{k_1^2 + W^2} + |k_0| = |k_2| \tag{69}$$

A direct transition between the phases $n = 0$ and $n = 2$ requires the condition

$$\sqrt{k_1^2 + W^2} = |k_0| - |k_2| = |k_2| - |k_0| \tag{70}$$

which can be fulfilled only for the case $|k_0| = |k_2|$ and $(k_1 = 0, W = 0)$ where the couplings $K_{m,m+1}$ all vanish (this case has been previously discussed in the subsection II C).

C. Lyapunov exponents for the case $\frac{k_0}{k_2} = +\left|\frac{k_0}{k_2}\right| > 0$

When the two couplings k_0 and k_2 have the same sign, it is convenient to recast the system of Eq 52

$$\begin{aligned} x_f &= -\frac{k_1}{k_2} - \frac{k_0}{k_2}\frac{x_f}{x_f^2 + y_f^2} \\ y_f &= \frac{W}{|k_2|} + \frac{k_0}{k_2}\frac{y_f}{x_f^2 + y_f^2} \end{aligned} \tag{71}$$

into the following single equation for the complex variable $z_f = x_f + iy_f$

$$z_f = -\omega - \frac{k_0}{k_2}\frac{1}{z_f} \tag{72}$$

where we have introduced the notation

$$\omega \equiv \frac{k_1}{k_2} - i\frac{W}{|k_2|} \tag{73}$$

Rewriting Eq 71 as a second degree equation

$$z_f^2 + \omega z_f + \frac{k_0}{k_2} = 0 \tag{74}$$

one obtains the two solutions

$$z_{f\pm} = \frac{-\omega \pm \sqrt{\Delta}}{2} \tag{75}$$

in terms of the discriminant

$$\Delta = \omega^2 - 4\frac{k_0}{k_2} \tag{76}$$

Since we are interested into the squares of the modulus of the solutions, it is actually convenient to use that their product reads

$$|z_{f+}|^2 |z_{f-}|^2 = \frac{k_0^2}{k_2^2} \tag{77}$$

while their sum is given by

$$S \equiv |z_{f+}|^2 + |z_{f-}|^2 = \frac{|\omega^2| + |\Delta|}{2} = \frac{|\omega^2| + \left|\omega^2 - 4\frac{k_0}{k_2}\right|}{2} \tag{78}$$

So $r_f^2 = |z_f|^2$ satisfies the second order equation

$$r_f^4 - S r_f^2 + \frac{k_0^2}{k_2^2} = 0 \tag{79}$$

of discriminant

$$D = S^2 - 4\frac{k_0^2}{k_2^2} = \frac{\left(|\omega^2| + \left|\omega^2 - 4\frac{k_0}{k_2}\right|\right)^2 - 16\frac{k_0^2}{k_2^2}}{4} = \frac{\left(|\omega^2| + \left|\omega^2 - 4\frac{k_0}{k_2}\right| + 4\frac{k_0}{k_2}\right)\left(|\omega^2| + \left|\omega^2 - 4\frac{k_0}{k_2}\right| - 4\frac{k_0}{k_2}\right)}{4} \tag{80}$$

which is positive (as it should to have real roots) as a consequence of the triangular inequality. From the two solutions

$$r^2_{f\pm} = \frac{S \pm \sqrt{D}}{2} \tag{81}$$

one obtains that the biggest Lyapunov exponent reads

$$\gamma_+ = \frac{\ln(r^2_{f+})}{2} = \frac{\ln\left(\frac{S+\sqrt{D}}{2}\right)}{2} \tag{82}$$

while the smallest reads (Eq. 37)

$$\gamma_- = \ln|k_0| - \ln|k_2| - \gamma_+ = \frac{\ln\frac{\frac{k_0^2}{k_2^2}}{(r^2_{f+})}}{2} = \frac{\ln(r^2_{f-})}{2} = \frac{\ln\left(\frac{S-\sqrt{D}}{2}\right)}{2} \tag{83}$$

In terms of the initial parameters, Eq 78 reads

$$S = \frac{k_1^2 + W^2 + \sqrt{[(k_1 + 2\sqrt{k_0 k_2})^2 + W^2][(k_1 - 2\sqrt{k_0 k_2})^2 + W^2]}}{2k_2^2} = \frac{k_1^2 + W^2 + \sqrt{(k_1^2 - 4k_0k_2)^2 + 2W^2(k_1^2 + 4k_0k_2) + W^4}}{2k_2^2} \tag{84}$$

1. Phase $n = 2$

The phase $n = 2$ corresponds to two negative Lyapunov exponents $(\gamma^- \leq)\gamma^+ < 0$, i.e. to the region

$$S < 1 + \frac{k_0^2}{k_2^2} < 2 \tag{85}$$

leading finally to the two conditions

$$\begin{aligned} &|k_0| < |k_2| \\ &k_1^2(k_2 - k_0)^2 + W^2(k_2 + k_0)^2 < (k_2 - k_0)^2(k_2 + k_0)^2 \end{aligned} \tag{86}$$

For the non-random case $W = 0$, these two conditions reduce to

$$\begin{aligned} &|k_0| < |k_2| \\ &|k_1| < |k_2 + k_0| \end{aligned} \tag{87}$$

2. Phase $n = 1$

The phase $n = 1$ corresponds to the case $\gamma^- < 0 < \gamma^+$, i.e. to the region

$$S > 1 + \frac{k_0^2}{k_2^2} \tag{88}$$

that finally leads to the condition

$$(k_2 - k_0)^2(k_2 + k_0)^2 < k_1^2(k_2 - k_0)^2 + W^2(k_2 + k_0)^2 \tag{89}$$

For the non-random case $W = 0$, Eq 89 reduces to

$$|k_2 + k_0| < |k_1| \tag{90}$$

3. Phase $n = 0$

The phase $n = 0$ corresponds to two positive Lyapunov exponents $0 < \gamma^-(\leq \gamma^+)$, i.e. to the region

$$2 < S < 1 + \frac{k_0^2}{k_2^2} \tag{91}$$

leading finally to the two conditions

$$\begin{aligned} &|k_2| < |k_0| \\ &k_1^2(k_2 - k_0)^2 + W^2(k_2 + k_0)^2 < (k_2 - k_0)^2(k_2 + k_0)^2 \end{aligned} \tag{92}$$

For the non-random case $W = 0$, these two conditions reduce to

$$\begin{aligned} &|k_2| < |k_0| \\ &|k_1| < |k_2 + k_0| \end{aligned} \tag{93}$$

4. Conclusion : Location of the phase transition between the three phases in the region $\frac{k_0}{k_2} = + \left| \frac{k_0}{k_2} \right| > 0$

The critical line between the phases $n = 1$ and $n = 2$ corresponds to to the two conditions

$$\begin{aligned} &|k_0| < |k_2| \\ &k_1^2(k_2 - k_0)^2 + W^2(k_2 + k_0)^2 = (k_2 - k_0)^2(k_2 + k_0)^2 \end{aligned} \tag{94}$$

For the non-random case $W = 0$, this reduces to

$$\begin{aligned} &|k_0| < |k_2| \\ &|k_1| = |k_2 + k_0| \end{aligned} \tag{95}$$

The critical line between the phases $n = 0$ and $n = 1$ corresponds to to the two conditions

$$\begin{aligned} &|k_2| < |k_0| \\ &k_1^2(k_2 - k_0)^2 + W^2(k_2 + k_0)^2 = (k_2 - k_0)^2(k_2 + k_0)^2 \end{aligned} \tag{96}$$

For the non-random case $W = 0$, this reduces to

$$\begin{aligned} &|k_2| < |k_0| \\ &|k_1| = |k_2 + k_0| \end{aligned} \tag{97}$$

A direct transition between the phases $n = 0$ and $n = 2$ requires the condition

$$S = 1 + \frac{k_0^2}{k_2^2} = 2 \tag{98}$$

Since in the present section the two couplings k_0 and k_2 have the same sign, one obtains the conditions

$$\begin{aligned} &k_0 = k_2 \\ &W = 0 \\ &k_1^2 < 2(k_0^2 + k_2^2) = 4k_0^2 \end{aligned} \tag{99}$$

i.e. it is only possible in the absence of disorder $W = 0$.

D. Figures of the phase diagram in the plane $(\frac{k_0}{k_2}, \frac{k_1}{k_2})$ for various disorder strengths $\frac{W}{|k_2|}$

It is now interesting to draw the phase diagram in the plane of the reduced variables

$$x = \frac{k_0}{k_2}$$
$$y = \frac{k_1}{k_2} \tag{100}$$

in order to see how it evolves as a function of the reduced disorder strength

$$w = \frac{W}{|k_2|} \tag{101}$$

For the non-random case $w = 0$, the phase diagram has been already discussed in previous works [3, 4, 7–9] and is drawn as the first picture on Figure 1 as a comparison with the random cases $w > 0$.

1. *Locations of the phase transitions between the three phases in the region $x = \frac{k_0}{k_2} < 0$*

In the half-plane $x < 0$, the critical line between the phases $n = 0$ and $n = 1$ corresponds to the full hyperbola branch (Eq 68)

$$y = \pm\sqrt{(x+1)^2 - w^2}$$
$$x \leq -1 - w \tag{102}$$

The critical line between the phases $n = 1$ and $n = 2$ corresponds to the truncated other branch of the same hyperbola

$$y = \pm\sqrt{(x+1)^2 - w^2}$$
$$-1 + w \leq x \leq 0 \tag{103}$$

that exists only for sufficiently small reduced disorder $w < 1$ (see Fig 1).

In the non-random case $w = 0$, this hyperbola degenerate into the two straight lines $y = \pm(x+1)$ (see Fig 1) : the direct transition between the phases $n = 0$ and $n = 2$ is then reduced to their intersection point $(x = -1, y = 0)$.

2. *Location of the phase transitions between the three phases in the region $x = \frac{k_0}{k_2} > 0$*

In the half-plane $x > 0$, the critical line between the phases $n = 1$ and $n = 2$ corresponds to the truncated branch

$$y = \pm(1+x)\sqrt{1 - \frac{w^2}{(1-x)^2}}$$
$$0 \leq x \leq 1 - w \tag{104}$$

that exists only for sufficiently small reduced disorder $w < 1$ (see Fig 1). The critical line between the phases $n = 0$ and $n = 1$ corresponds to to the full other branch of the same curve

$$y = \pm(1+x)\sqrt{1 - \frac{w^2}{(1-x)^2}}$$
$$1 + w \leq x \tag{105}$$

For the non-random case $w = 0$, the curve above degenerates into the straight lines $y = \pm(1+x)$ and the direct transition between the phases $n = 0$ and $n = 2$ becomes possible along the vertical segment (see Fig 1)

$$-2 \leq y \leq +2$$
$$x = 1 \tag{106}$$

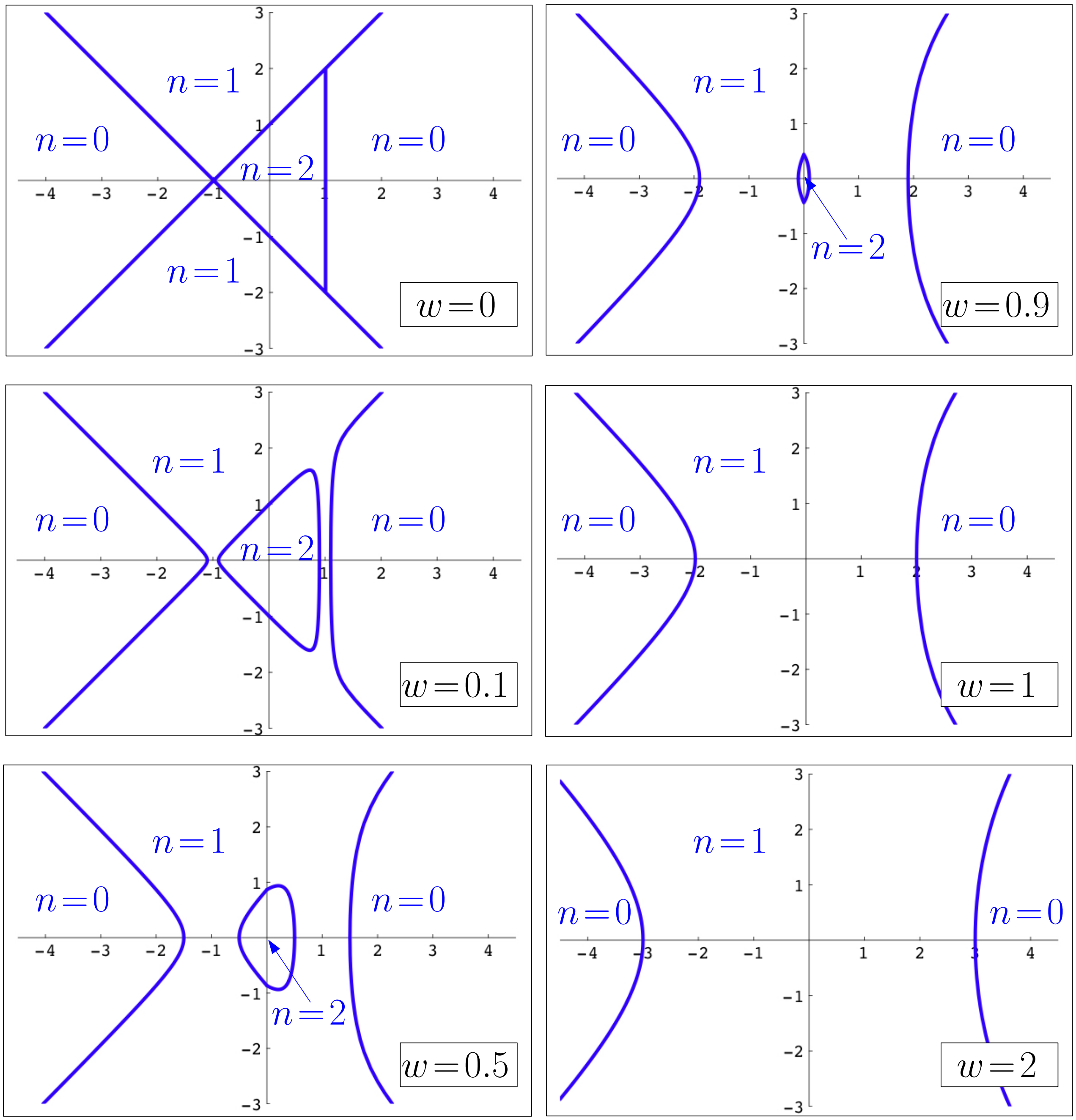

Figure 1: Phase diagram of the three topological phases $n = 0, 1, 2$ in the plane $(x = \frac{k_0}{k_2}, y = \frac{k_1}{k_2})$ for various disorder strengths $w = \frac{W}{|k_2|}$. Note that the direct transition between $n = 0$ and $n = 2$ occurs only in the non-random case $w = 0$ and disappears for any arbitrary disorder via an intermediate phase $n = 1$, as shown with the examples $w = 0.1$, $w = 0.5$ and $w = 0.9$. For sufficiently strong disorder $w \geq 1$, the phase $n = 2$ cannot exist anymore, as shown with the examples $w = 1$ and $w = 2$.

V. CONCLUSIONS

In this paper, we have considered the topological phase transitions in random Kitaev α-chains. We have first recalled how the edge Majorana Zero Modes could be computed for any realization of disorder for Hamiltonians involving only two values of α. We have then focused on the random Hamiltonian $(H_0 + H_1 + H_2)$ containing three values of α, where the localization properties of the edge Majorana Zero Modes can be analyzed via the product of 2×2 random matrices and via the Riccati non-linear recurrence. For the special case of Cauchy disorder in the couplings $K_{m,m+1}$, we have computed explicitly the two Lyapunov exponents in order to analyze how the phase diagram of the three topological phases $n = 0, 1, 2$ evolves as a function of the disorder strength. In particular, we have obtained that the direct phase transition between the phases $n = 0$ and $n = 2$ becomes impossible in the presence of disorder that always induces an intermediate phase $n = 1$, as found previously via numerics for other distributions of disorder [9], and in agreement with the more general expectation that topological phase transitions in random systems only change the topological index by one as a consequence of the non-degeneracy of the Lyapunov spectrum [6]. We have also obtained that the phase $n = 2$ completely disappears for strong enough disorder ($w \geq 1$ in Figure 1).

Appendix A: Dictionary between Majorana fermions and quantum spin chains

For a chain of N quantum spins described by Pauli matrices, the $(2N)$ string operators

$$\begin{aligned} a_j = \gamma_{2j-1} &\equiv \left(\prod_{k=1}^{j-1} \sigma_k^z \right) \sigma_j^x \\ b_j = \gamma_{2j} &\equiv \left(\prod_{k=1}^{j-1} \sigma_k^z \right) \sigma_j^y \end{aligned} \tag{A1}$$

satisfy the Majorana anticommutation relations

$$\gamma_k \gamma_l + \gamma_l \gamma_k = 2\delta_{kl} \tag{A2}$$

The first examples of Kitaev α-chains of Eq. 4 reads in the spin language

$$H_{(\alpha=0)} = i \sum_m b_m K_{m,m} a_m = \sum_m K_{m,m} \sigma_m^z \tag{A3}$$

$$H_{(\alpha=1)} = i \sum_m b_m K_{m,m+1} a_{m+1} = - \sum_m K_{m,m+1} \sigma_m^x \sigma_{m+1}^x \tag{A4}$$

and

$$H_{(\alpha=2)} = i \sum_m b_m K_{m,m+2} a_{m+2} = - \sum_m K_{m,m+2} \sigma_m^x \sigma_{m+1}^z \sigma_{m+2}^x \tag{A5}$$

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