Multifractality in the generalized Aubry-André quasiperiodic localization model with power-law hoppings or power-law Fourier coefficients

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The nearest-neighbor Aubry-André quasiperiodic localization model is generalized to include power-law translation-invariant hoppings $T_l \propto t/l^a$ or power-law Fourier coefficients $W_m \propto w/m^b$ in the quasi-periodic potential. The Aubry-André duality between T_l and W_m is manifest when the Hamiltonian is written in the real-space basis and in the Fourier basis on a finite ring. The perturbative analysis in the amplitude t of the hoppings yields that the eigenstates remain power-law localized in real space for a > 1 and are critical for $a_c = 1$ where they follow the Strong Multifractality linear spectrum, as in the equivalent model with random disorder. The perturbative analysis in the amplitude w of the quasi-periodic potential yields that the eigenstates remain delocalized in real space (power-law localized in Fourier space) for b > 1 and are critical for $b_c = 1$ where they follow the Weak Multifractality gaussian spectrum in real space (or Strong Multifractality linear spectrum in the Fourier basis). This critical case $b_c = 1$ for the Fourier coefficients W_m corresponds to a periodic function with discontinuities, instead of the cosinus of the standard self-dual Aubry-André model.

I. INTRODUCTION

Although the phenomenon of Anderson Localization is mostly studied in the presence of random potentials (see the reviews [1–4]), the case of non-random quasi-periodic potentials has also attracted a lot of interest over the years [5–22]. More recently in the presence of interactions, the Many-Body-Localization (see the reviews [23, 24] and references therein) has been also considered for the quasi-periodic case [25–29] in order to understand the similarities and the differences with the random case.

In the simplest case of the nearest-neighbor one-dimensional Anderson Localization model, it is well known that any random potential (even extremely weak) leads to exponentially localized eigenfunctions, while the corresponding nearest-neighbor self-dual Aubry-André quasiperiodic model [8] displays a phase transition between a localized phase and a delocalized ballistic phase. In the presence of power-law long-ranged hoppings $T(l) \propto l^{-a}$, one-dimensional Anderson localization models with disorder are know to become critical for the value $a_c = 1$ where the multifractality of eigenfunctions changes continuously as a function of the amplitude between strong and weak multifractality (see the review [4] and references therein). In the present paper, we thus wish to consider similarly the long-ranged version of the Aubry-André quasiperiodic model, where the hoppings decay only as a power-law $T(l) \propto l^{-a}$. As a consequence of the Aubry-André duality [8], it will be also interesting to focus on the case where the Fourier coefficients of the quasi-periodic potential decay only as a power-law $W_m \propto m^{-b}$. The main goal is to analyze the multifractal properties of eigenstates in these two cases.

The paper is organized as follows. In section II, the generalized version of the Aubry-André model with arbitrary translation-invariant hoppings and arbitrary Fourier coefficients is defined on the infinite lattice. In section III, we describe the properties of the model for a finite ring of N sites, where the Aubry-André duality between the N sites and the N Fourier modes is obvious. In section IV, the case of power-law hoppings with a small amplitude is studied perturbatively and leads to the strong multifractality spectrum of eigenstates. In section V, the case of power-law Fourier coefficients with a small amplitude is studied perturbatively and leads to the weak multifractality spectrum. Our conclusions are summarized in VI. In Appendix A, some differences with the random case are stressed.

II. GENERALIZED AUBRY-ANDRÉ ON THE INFINITE LATTICE

In this paper, we consider the generalized version of the one-dimensional Aubry-André Hamiltonian [8] containing two competing contributions

$$H = H^{deloc} + H^{loc} \tag{1}$$

where H^{deloc} contains translation-invariant hoppings that may be long-ranged, and where H^{loc} contains on-sites energies following a quasi-periodic function containing arbitrary Fourier coefficients. Let us now discuss separately their properties.

A. H^{deloc} containing translation-invariant hoppings

The contribution H^{deloc} contains hoppings $H^{deloc}_{nn'}$ between different sites $n \neq n'$, with the translation-invariance property

$$H_{nn'}^{deloc} = T_{n-n'} \tag{2}$$

and the hermitian property $H^{deloc}_{nn^\prime}=(H^{deloc}_{n^\prime n})^*$ corresponding to

$$T_l = T_l^* \tag{3}$$

On the infinite lattice, H^{deloc} can be then rewritten with the various forms

$$H^{deloc} = \sum_{n \neq n'} H_{nn'} |n \rangle \langle n'| = \sum_{n \neq n'} T_{n-n'} |n \rangle \langle n'|$$

$$= \sum_{n=-\infty}^{+\infty} \sum_{l \neq 0} T_l |n \rangle \langle n-l| = \sum_{n=-\infty}^{+\infty} \sum_{l=1}^{+\infty} (T_l |n \rangle \langle n-l| + T_l^* |n-l| \rangle \langle n|)$$
(4)

The translation invariance yields that its eigenstates are delocalized Fourier modes.

B. *H^{loc}* containing quasi-periodic on-site energies

The contribution

$$H^{loc} = \sum_{n=-\infty}^{+\infty} H_{nn} |n\rangle \langle n|$$
(5)

contains on-site energies H_{nn} following the quasi-periodic form

$$H_{nn} = W(gn) \tag{6}$$

where g is usually chosen in terms of the inverse Golden mean

$$\frac{g}{2\pi} = \frac{\sqrt{5} - 1}{2} \tag{7}$$

while W(x) is a real 2π -periodic function of zero-mean that can be defined by its Fourier expansion

$$W(x) = \sum_{m=-\infty}^{+\infty} W_m e^{imx} = \sum_{m=1}^{+\infty} (W_m e^{imx} + W_m^* e^{-imx})$$
(8)

where the Fourier coefficients satisfy $W_{-m} = W_m^*$ and $W_{m=0} = 0$.

C. Standard nearest-neighbor self-dual Aubry-André model

The standard self-dual Aubry-André model [8] corresponds to hoppings limited to nearest-neighbors $l = \pm 1$

$$T_{l}^{(AA)} = \delta_{l,1} + \delta_{l,-1} \tag{9}$$

and to the function $W(x) = \lambda 2 \cos(x+h)$ involving only the first Fourier coefficients $m = \pm 1$

$$W_m^{(AA)} = \lambda e^{imh} (\delta_{m,1} + \delta_{m,-1}) \tag{10}$$

Aubry and André [8] have shown that this model satisfy a remarkable self-duality. The main consequence is that the exact critical point $\lambda_c = 1$ separates the delocalized ballistic phase $\lambda < 1$ from the localized phase $\lambda > 1$ characterized by the exact localization length $\xi = \frac{1}{\ln \lambda}$ [8].

D. Long-Ranged power-law hoppings

In the present paper, we wish to analyze the cases where the hoppings decay as a power-law with some exponent $a \ge 1$

$$T_l^{(a)} = it \frac{(1 - \delta_{l,0}) \operatorname{sgn}(l)}{|l|^a}$$
(11)

including the critical case $a_c = 1$

$$T_l^{(a_c=1)} = it \frac{(1-\delta_{l,0})}{l}$$
(12)

known as the Calogero-Moser matrix model (see [30] and references therein). In particular we will focus in section IV on the strong multifractality regime occuring for the critical case $a_c = 1$ with a small amplitude t.

Note that the presence of the imaginary factor i in the critical case of Eq 12 is essential to obtain a well defined model for $a_c = 1$: the real symmetric power-law case $T(l) = \frac{t}{|l|^a}$ which has been studied in various contexts [31–34] is well defined only for a > 1, because the Fourier mode K = 0 would have an infinite energy for $a_c = 1$. For a > 1, there is no Anderson transition in the middle of the spectrum, but only near the edge of the spectrum where the level spacing is anomalous (see section A 2 in Appendix).

E. Long-Ranged Fourier coefficients

Since there exists some duality between the hoppings T_l and the Fourier coefficients W_m (see more details in section IIID), we will also focus on the case where the Fourier coefficients W_m decay as a power-law with some exponent exponent $b \ge 1$

$$W_m^{(b)} = -iw \frac{(1 - \delta_{m,0}) \operatorname{sgn}(m)}{|m|^b}$$
(13)

including the critical case $b_c = 1$

$$W_m^{(b_c=1)} = -iw \frac{(1-\delta_{m,0})}{m}$$
(14)

where the weak multifractality regime occuring for small amplitude w will be analyzed in section V.

From the general theory of Fourier series, it is well-known that the decay of the Fourier coefficients W_m for large m directly reflects the regularity properties of the function W(x) of Eq. 8 : the critical decay as 1/m corresponds to a function W(x) presenting discontinuities, the decay as $1/m^2$ corresponds to a continuous function W(x) whose first derivative presents discontinuities, and so on.

For instance, the critical case $b_c = 1$ of Eq. 14 corresponds to the 2π -periodic linear function

$$W^{(b_c=1)(N=\infty)}(0 < x < 2\pi) = 2w \sum_{m=1}^{+\infty} \frac{1}{m} \sin(mx) = w(\pi - x) \quad \text{for } 0 < x < 2\pi$$
(15)

with the following discontinuity at $x = 0 [\text{modulo} 2\pi]$

$$W^{(b_c=1)(N=\infty)}(x=0^+) = w\pi$$

$$W^{(b_c=1)(N=\infty)}(x=0) = 0$$

$$W^{(b_c=1)(N=\infty)}(x=0^-) = -w\pi$$
(16)

III. GENERALIZED VERSION OF THE AUBRY-ANDRÉ MODEL ON A FINITE RING OF N SITES

Since our goal is to analyze the multifractal properties of critical eigenstates via perturbation theory, it is convenient to consider the finite-size version of the above model in order to have discrete energy levels. Another advantage is that the Aubry-André duality is much clearer when the N sites and the N Fourier modes play exactly similar roles,

even if the duality has been first formulated for the infinite lattice [8]. In this section, we thus describe how the infinite-lattice model described in the previous section can be defined on a finite ring containing N sites with periodic boundary conditions $|n + N \rangle = |n \rangle$. To avoid discussions on the differences between even and odd sizes N, it will be convenient to focus only on the odd case with the notation

$$N = 2P + 1 \tag{17}$$

A. Fourier diagonalization of H^{deloc}

It is convenient to define the finite-size version of Eq. 4 as

$$H^{deloc(N)} = \sum_{n=0}^{N-1} \sum_{l=1}^{P} (T_l | n > < n - l | + T_l^* | n - l > < n |)$$
(18)

In the basis K = 0, ..., N - 1 of the N Fourier modes

$$|K\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{i2\pi K \frac{n}{N}} |n\rangle$$
(19)

with the orthonormalization

$$\langle K'|K \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{i2\pi(K-K')\frac{n}{N}} = \delta_{K',K}$$
 (20)

one obtains the diagonal form

$$H^{deloc(N)} = \sum_{K=0}^{N-1} E_K^{deloc} |K| > < K|$$
(21)

with the Fourier eigenvalues

$$E_K^{deloc} = \sum_{l=1}^P (T_l e^{-i2\pi K \frac{l}{N}} + T_l^* e^{i2\pi K \frac{l}{N}})$$
(22)

For instance the nearest-neighbor hopping of Eq. 9 corresponds to the usual result

$$E_K^{deloc(AA)} = 2\cos\left(2\pi K \frac{l}{N}\right) \tag{23}$$

while the power-law case of Eq. 11 corresponds to

$$E_K^{deloc(a)} = 2t \sum_{l=1}^P \frac{1}{l^a} \sin\left(2\pi K \frac{l}{N}\right)$$
(24)

In the thermodynamic limit $N \to +\infty$ where the momentum $k = 2\pi \frac{K}{N}$ becomes continuous in the Brillouin zone $[0, 2\pi]$, the energy

$$E^{deloc(a)(N=\infty)}(k) = 2t \sum_{l=1}^{+\infty} \frac{1}{l^a} \sin(kl)$$
 (25)

remains finite for a > 1 where the Fourier series converges absolutely. For the critical case $a_c = 1$ of Eq. 12, the absolute convergence is lost, but one recognizes the sine-Fourier decomposition of the odd 2π -periodic function of Eq. 15 following the linear form on $]0, 2\pi[$

$$E^{deloc(a_c=1)(N=\infty)}(0 < k < 2\pi) = 2t \sum_{l=1}^{+\infty} \frac{1}{l} \sin(kl) = t(\pi - k) \quad \text{for } 0 < k < 2\pi$$
(26)

$$E^{deloc(a_c=1)(N=\infty)}(k=0^+) = t\pi$$

$$E^{deloc(a_c=1)(N=\infty)}(k=0) = 0$$

$$E^{deloc(a_c=1)(N=\infty)}(k=0^-) = -t\pi$$
(27)

For large N, the Fourier modes K = 1, ..., N - 1 thus follow the linear ramp of Eq. 25

$$E_K^{deloc(a_c=1)} \simeq t\left(\pi - 2\pi \frac{K}{N}\right) \quad \text{for } K = 1, .., N-1$$
 (28)

while the mode K = 0 is exactly in the middle of the spectrum

$$E_{K=0}^{deloc(a_c=1)} = 0 (29)$$

and has for neighboring energy levels the Fourier modes corresponding to K = P et K = P + 1

$$E_{K=P}^{deloc(a_c=1)} = \frac{\pi t}{N}$$

$$E_{K=P+1}^{deloc(a_c=1)} = -\frac{\pi t}{N}$$
(30)

B. Properties of the quasiperiodic on-site energies H_{nn} on a finite ring

On a finite ring of N sites, it is convenient to keep only the Fourier modes $-P \le m \le P$ of the 2π periodic function of Eq. 8

$$W_N(x) = \sum_{m=-P}^{+P} W_m e^{imx} = \sum_{m=1}^{P} (W_m e^{imx} + W_m^* e^{-imx})$$
(31)

and to ask that the quasi-periodic form of Eq. 6

$$H_{nn} = W_N(gn) = \sum_{m=-P}^{+P} W_m e^{imgn}$$
(32)

is compatible with the periodic boundary condition $n \rightarrow n + N$ of the ring. This condition $e^{igN} = 1$ yields the choice

$$\frac{g}{2\pi} = \frac{G}{N} \tag{33}$$

where the integer $G = G_N$ is chosen to ensure that the successive fractions $\frac{G_N}{N}$ converge towards the inverse Golden mean Eq. 7: the standard solution consists in choosing $N = F_i$ and $G_N = F_{i-1}$ in terms of the Fibonacci numbers satisfying the recurrence $F_i = F_{i-1} + F_{i-2}$.

In summary, the on-site energies on the finite ring read

$$H_{nn} = \sum_{m=-P}^{+P} W_m e^{im2\pi\frac{G}{N}n} = \sum_{m=1}^{P} \left(W_m e^{im2\pi\frac{G}{N}n} + W_m^* e^{-im2\pi\frac{G}{N}n} \right)$$
(34)

In terms of the Fourier modes of Eq. 19, the localized contribution of the Hamiltonian reads

$$H^{loc} = \sum_{n=0}^{N-1} H_{nn} |n\rangle \langle n| = \sum_{K=0}^{N-1} \sum_{K'=0}^{N-1} H^{loc}_{KK'} |K\rangle \langle K'|$$
(35)

with

$$H_{KK'}^{loc} = \frac{1}{N} \sum_{n=0}^{N-1} e^{i2\pi (K'-K)\frac{n}{N}} H_{nn} = \sum_{m=-P}^{+P} W_m \frac{1}{N} \sum_{n=0}^{N-1} e^{i2\pi (K'-K+mG)\frac{n}{N}}$$
$$= \sum_{m=-P}^{+P} W_m \delta_{K'-K+mG[N]}$$
(36)

where the notation [N] means (modulo N). So the interaction between two Fourier modes K and K' depend on the integer m satisfying K - K' = mG[N]. For instance, in the Aubry-André case of Eq. 10, Eq. 36 reduces to

$$H_{KK'}^{loc} = \lambda e^{ih} \delta_{K'-K+G[N]} + \lambda e^{-ih} \delta_{K'-K-G[N]}$$

$$\tag{37}$$

so that each Fourier mode K interacts only with two other Fourier modes $K' = K \pm G[N]$.

C. Quasiperiodic Fourier basis

The form of Eq. 36 for the interaction between two Fourier modes (K, K') suggests that it is useful to reparametrize the Fourier modes K = 0, ..., N - 1 in terms of the new integer Q = 0, ..., N - 1 satisfying [16, 18]

$$K = QG[N] \tag{38}$$

This amounts to introduce the alternative Fourier basis adapted to the quasi-periodicity

$$|Q> = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{i2\pi Q \frac{Gn}{N}} |n>$$
(39)

The orthonormalization

$$\langle Q'|Q \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{i2\pi (Q-Q')\frac{Gn}{N}} = \delta_{(Q-Q')G=0[N]} = \delta_{Q',Q}$$
(40)

can be understood as follows : besides the trivial solution Q = Q', one cannot find another solution (Q - Q')G = jNcorresponding to some integer $j \neq 0$: indeed the equation $\frac{j}{Q-Q'} = \frac{G}{N}$ has no solution for Q - Q' = 1, ..., N - 1 since the irreducible fraction $\frac{G}{N}$ cannot be written as a fraction with a smaller denominator.

The advantage of this new basis is that the interaction of Eq. 34 becomes much simpler

$$H_{QQ'}^{loc} = \sum_{m=-P}^{+P} W_m \delta_{(Q'-Q+m)G[N]} = W_{m=Q-Q'[N]}$$
(41)

as it is directly given by the Fourier coefficient W_m corresponding to m = Q - Q'[N]. For instance, in the Aubry-André case of Eq. 10, the interaction is only between nearest-neighbors for the new indices (Q, Q')

$$H_{QQ'}^{loc} = W_1(e^{ih}\delta_{Q'-Q+1} + e^{-ih}\delta_{Q'-Q-1})$$
(42)

The other contribution H^{deloc} that was diagonal in K remains of course diagonal in Q, but the corresponding eigenvalues E_K^{deloc} of Eq. 22 that were well ordered in K become

$$E_Q^{deloc} = \sum_{l=1}^P (T(l)e^{-i2\pi QG\frac{l}{N}} + T^*(l)e^{i2\pi QG\frac{l}{N}})$$
(43)

This is thus similar to the quasi-periodic form of the on-site energies of Eq. 34 where W_m has been replaced by $T^*(l)$.

D. Duality between the real-space basis and the quasiperiodic Fourier basis

The discussion of the quasiperiodic Fourier basis of the previous section shows the following Aubry-André duality for the finite-size model that we consider :

(i) In the real space basis $|n\rangle$, the delocalized Hamiltonian H^{deloc} is a circulant matrix defined by the coefficients T_l

$$H_{nn'}^{deloc} = T_{n-n'} \tag{44}$$

while the localized Hamiltonian H^{loc} is diagonal and follows the quasiperiodic form defined by Fourier coefficients W_m

$$H_{nn}^{loc} = \sum_{m=-P}^{+P} W_m e^{im2\pi\frac{G}{N}n} = \sum_{m=1}^{P} \left(W_m e^{im2\pi\frac{G}{N}n} + W_m^* e^{-im2\pi\frac{G}{N}n} \right)$$
(45)

(ii) In the quasiperiodic Fourier basis $|Q\rangle$, the delocalized Hamiltonian H^{deloc} is diagonal and follows the quasiperiodic form of Fourier coefficients T_l^*

$$E_{QQ}^{deloc} = \sum_{l=1}^{P} (T(l)e^{-i2\pi QG\frac{l}{N}} + T^*(l)e^{i2\pi QG\frac{l}{N}})$$
(46)

while the localized Hamiltonian H^{loc} is a circulant matrix defined by the coefficients W_m

$$H_{QQ'}^{loc} = W_{Q-Q'} \tag{47}$$

For instance, the standard Aubry-André duality corresponds to the case where both T_l and W_m are limited to nearest neighbors

$$T_{l}^{(AA)} = \delta_{l,1} + \delta_{l,-1} W_{m}^{(AA)} = \lambda(\delta_{m,1} + \delta_{m,-1})$$

so that the self-dual point $\lambda_c = 1$ correspond to the critical point between the localized and the delocalized phases.

For the present power-law models that we consider, this Aubry-André duality relates the power-law hopping case of Eq. 11 and the power-law Fourier coefficients of Eq. 13.

IV. PERTURBATION THEORY IN THE AMPLITUDE t OF THE POWER-LAW HOPPINGS

In this section, we focus on the case of Eq 11 where the hoppings T_l decay as a power-law of exponent $a \ge 1$ with some small amplitude t, while the Fourier coefficients W_m of finite amplitude can be either short-ranged or long-ranged.

A. Perturbation theory for the eigenstates

If $H^{deloc} = 0$, the N eigenstates are completely localized on the sites n

$$|\psi_n^{loc(0)}\rangle = |n\rangle \tag{48}$$

and the corresponding eigenvalues given by the on-site energies

$$E_n^{loc(0)} = H_{nn} \tag{49}$$

are non-degenerate. The first order perturbation theory in H^{deloc} yields the eigenstates

$$\begin{aligned} |\psi_n^{loc(0+1)} > &= |n > + \sum_{n'} |n' > \frac{H_{n'n}^{deloc}}{H_{nn} - H_{n'n'}} \\ &= |n > + \sum_{l=-P,+P} |n+l > \frac{T_l}{H_{nn} - H_{n+l,n+l}} \end{aligned}$$
(50)

For a given eigenstate indexed by the site n of the zero-order, the density on the other sites l = 1, ..., N - 1 is thus given at lowest order by

$$\rho_l \equiv \left| < n + l |\psi_n^{loc(0+1)} > \right|^2 = \left| \frac{T_l}{H_{nn} - H_{n+l,n+l}} \right|^2 \tag{51}$$

In this perturbation theory, one expects that the most dangerous term corresponds to the smallest denominator associated to the closest energy level : the difference $(H_{nn} - H_{n+l,n+l})$ between these two neighboring energy levels scales as the level spacing

$$\Delta_N \propto \frac{1}{N} \tag{52}$$

This has to be compared with the scaling of the hopping on the scale of the system-size $l \propto N$

$$T_l \underset{l \propto N}{\propto} \frac{1}{N^a} \tag{53}$$

This scaling argument yields that the perturbation theory remains consistent at large size N only for a > 1, while $a_c = 1$ corresponds to the critical case.

B. Multifractal analysis

In the multifractal formalism, one is interested into the singularity spectrum $f(\alpha)$ governing the leading exponential behavior of the probability that the density of Eq. 51 scales as $\rho_l \propto N^{-\alpha}$ [4]

$$\mathcal{P}(\alpha) \propto N^{f(\alpha)-1} \tag{54}$$

For the energy difference $|H_{nn} - H_{n+l,n+l}|$ appearing in the denominator of Eq. 51, the important property is the level spacing of Eq. 52. For instance, there are a finite number O(1) of states that have an energy difference scaling as the level spacing $\frac{1}{N}$, while there is an extensive number O(N) of states that have a finite energy difference O(1). More generally, if the states are re-labelled in the order of the energy difference with some index p = 1, ..., N, the change of variables $p = N^x$ with $0 \le x \le 1$ and $dp = N^x(\ln N)dx$ yields that the number of states having an energy difference of order $N^x(\Delta_N) = N^{x-1}$ scales as $N^x(\ln N)$. So the probability to have an energy difference of order N^{x-1} scales as

$$\operatorname{Prob}(|H_{nn} - H_{n+l,n+l}| \propto N^{x-1}) \propto N^{x-1}(\ln N)\theta(0 \le x \le 1)$$
(55)

For the power-law hopping T_l of Eq. 11 appearing in the numerator of Eq. 51, the change of variable $l = N^y$ with $0 \le y \le 1$ and $dl = N^y (\ln N) dy$ yields that the probability to have a hopping scaling as $|T_l| \propto N^{-ay}$ scales as

$$\operatorname{Prob}(|T_l| \propto N^{-ay}) \propto N^{y-1} (\ln N) \theta (0 \le y \le 1)$$
(56)

So the probability of Eq. 54 to have a density ρ_l of Eq. 51 scaling as $N^{-\alpha}$ can be evaluated from Eqs 55 and 56 with the correspondence $N^{-\alpha} = \left(\frac{N^{-ay}}{N^{x-1}}\right)^2$ as

$$\mathcal{P}_{a}(\alpha) \propto \int_{0}^{1} dx (\ln N) N^{x-1} \int_{0}^{1} dy (\ln N) N^{y-1} \delta(\alpha + 2 - 2x - 2ay) = (\ln N)^{2} N^{\frac{\alpha}{2} - 1} \int_{0}^{1} dy N^{(1-a)y} \quad \theta\left(\frac{\alpha}{2a} \le y \le \frac{\alpha}{2a} + \frac{1}{a}\right)$$
(57)

Since the exponent α is positive $\alpha \ge 0$, the lower bound of the integral is $y_{min} = \frac{\alpha}{2a} \ge 0$.

For the critical case $a_c = 1$, Eq. 57 becomes

$$\mathcal{P}_{a_c=1}(\alpha) = (\ln N)^2 N^{\frac{\alpha}{2}-1} \int dy \ \theta\left(\frac{\alpha}{2} \le y \le 1\right)$$
$$= \left(1 - \frac{\alpha}{2}\right) (\ln N)^2 N^{\frac{\alpha}{2}-1} \theta(0 \le \alpha \le 2)$$
(58)

so that the corresponding multifractal spectrum of Eq. 54

$$f_{a_c=1}(\alpha) = \frac{\alpha}{2}\theta(0 \le \alpha \le 2) \tag{59}$$

is the well-known 'Strong Multifractality' critical spectrum [35, 36] found at Anderson Localization Transition in the limit of infinite dimension $d \to +\infty$ [4]. It appears similarly in the long-ranged power-law hopping model in the presence of random (instead of quasi-periodic) on-site energies [30, 37–51]. It has been also found in various matrix

models, in particular in the generalized Rosenzweig-Potter matrix model of [52], and in the Lévy Matrix Model [53], as well as in Many-Body-Localization models [54, 55].

For a > 1, the integral over y in Eq. 57 is dominated by the lower value $y_{min} = \frac{\alpha}{2a}$, so that it is convenient to make the change of variables $y = \frac{\alpha}{2a} + \frac{u}{\ln N}$ to obtain

$$\mathcal{P}_{a>1}(\alpha) = (\ln N)^2 N^{\frac{\alpha}{2}-1} \int_0^{(1-\frac{\alpha}{2a})\ln N} \frac{du}{\ln N} N^{(1-a)\frac{\alpha}{2a}+(1-a)\frac{u}{\ln N}} \quad \theta \ (0 \le \alpha \le 2a)$$
$$= (\ln N) N^{\frac{\alpha}{2a}-1} \theta \ (0 \le \alpha \le 2a) \int_0^{(1-\frac{\alpha}{2a})\ln N} du e^{-(a-1)u}$$
$$= (\ln N) N^{\frac{\alpha}{2a}-1} \theta \ (0 \le \alpha \le 2a) \frac{1-N^{-(a-1)(1-\frac{\alpha}{2a})}}{a-1}$$
(60)

so that the corresponding multifractal spectrum of Eq. 54

$$f_{a>1}(\alpha) = \frac{\alpha}{2a}\theta(0 \le \alpha \le 2a) \tag{61}$$

is also linear as Eq. 59 and has been been found in other models in the localized phase close to the critical point describe by the strong multifractality spectrum [52–55].

C. Inverse Participation Ratios Y_q of arbitrary index q

The corresponding leading behavior of Inverse Participation Ratios of arbitrary index q [4]

$$Y_q^{(a)} \equiv \sum_{l=1}^{N-1} p_l^q \simeq N \int d\alpha \mathcal{P}_a(\alpha) N^{-q\alpha} \propto N^{(1-q)D(q)}$$
(62)

involving the generalized dimensions D(q) is governed by the Legendre transform of the singularity spectrum $f(\alpha)$ as a consequence of the saddle-point evaluation of the integral. However here we wish to include also the logarithmic prefactors.

For $a_c = 1$, Eq. 58 yields

$$Y_{q}^{(a_{c}=1)} \simeq (\ln N)^{2} \int_{0}^{2} d\alpha \left(1 - \frac{\alpha}{2}\right) N^{\left(\frac{1}{2} - q\right)\alpha}$$

$$= (\ln N)^{2} \left[\frac{\left(1 - \frac{\alpha}{2} + \frac{1}{2\left(\frac{1}{2} - q\right)\ln N}\right)}{\left(\frac{1}{2} - q\right)\ln N} N^{\left(\frac{1}{2} - q\right)\alpha} \right]_{\alpha=0}^{\alpha=2}$$

$$= \frac{N^{1-2q} - 1 - (1 - 2q)\ln N}{2\left(\frac{1}{2} - q\right)^{2}}$$
(63)

so that the leading behavior for large N changes at $q_c = \frac{1}{2}$

$$Y_{q<\frac{1}{2}}^{(a_c=1)} \propto \frac{N^{1-2q}}{2\left(\frac{1}{2}-q\right)^2}$$

$$Y_{q=\frac{1}{2}}^{(a_c=1)} \propto (\ln N)^2$$

$$Y_{q>\frac{1}{2}}^{(a_c=1)} \propto \frac{\ln N}{N \to +\infty} (q-\frac{1}{2})$$
(64)

For a > 1, Eq. 60 yields

$$Y_q^{(a>1)} \simeq \frac{\ln N}{a-1} \int_0^{2a} d\alpha N^{\left(\frac{1}{2a}-q\right)\alpha} = \frac{N^{1-2aq}-1}{(1-a)\left(\frac{1}{2a}-q\right)}$$
(65)

so that the leading behavior for large N changes at $q_c = \frac{1}{2a}$

$$Y_{q<\frac{1}{2a}}^{(a>1)} \qquad \underset{N \to +\infty}{\propto} \frac{N^{1-2aq}}{(1-a)\left(\frac{1}{2a}-q\right)}$$

$$Y_{q=\frac{1}{2a}}^{(a>1)} \qquad \underset{N \to +\infty}{\propto} \frac{2a}{a-1}\ln N$$

$$Y_{q>\frac{1}{2a}}^{(a>1)} \qquad \underset{N \to +\infty}{\propto} \frac{1}{(1-a)\left(q-\frac{1}{2a}\right)}$$
(66)

D. Discussion

In summary, the multifractality of eigenvectors for power-law hopping of small amplitude t is the same for quasiperiodic or random energies of finite amplitude :

(i) for a > 1, the eigenstates remain localized, in the sense that the Inverse Participation Ratios Y_q remain finite above some threshold $q > \frac{1}{2a}$ including q = 1 corresponding to the normalizability of eigenstates, but they are nevertheless described by the multifractal spectrum of Eq. 61 as a consequence of their power-law localization.

(ii) at the critical point $a_c = 1$, the Inverse Participation Ratios Y_q diverge logarithmically above the threshold $q > \frac{1}{2}$, and the critical eigenstates follow the Strong Multifractality spectrum of Eq. 59.

V. PERTURBATION THEORY IN THE AMPLITUDE w OF THE FOURIER COEFFICIENTS

In this section, we focus on the case of Eq 13 where the Fourier coefficients W_m of the quasiperiodic potential decay as a power-law of exponent $b \ge 1$ with some small amplitude w, while the hoppings T_l of finite amplitude can be either short-ranged or long-ranged. In particular, the critical case $b_c = 1$ of Eq. 14 corresponds in the thermodynamic limit to the 2π -periodic ramp function of Eq. 15 with the discontinuity of Eq. 16.

A. First order perturbation theory for the eigenstates

If $H^{loc} = 0$, the N eigenstates are the Fourier modes, written here with the quasi-periodic label Q of Eq. 38

$$|\psi_Q^{deloc(0)}\rangle = |Q\rangle \tag{67}$$

and the corresponding eigenvalues

$$E_Q^{deloc(0)} = E_{QQ}^{deloc} \tag{68}$$

are non-degenerate for the critical case $a_c = 1$ of Eq. 14 we focus on, as discussed in more details for the dual case concerning the hoppings (Eqs 28 29 30).

The first order perturbation theory in perturbation theory yields the eigenstates

$$\begin{aligned} |\psi_{Q}^{deloc(0+1)} \rangle &= |Q\rangle + \sum_{Q'} |Q'\rangle \frac{H_{Q'Q}^{loc}}{E_{QQ}^{deloc} - E_{Q'Q'}^{deloc}} \\ &= |Q\rangle + \sum_{m=-P,+P} |Q+m\rangle \frac{W_{m}}{E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc}} \end{aligned}$$
(69)

B. Inverse participation Ratios I_q in the Fourier basis

As a consequence, the weights of this eigenstate in this Fourier basis

$$\tilde{\rho}_m \equiv \left| \frac{W_m}{E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc}} \right|^2 \tag{70}$$

in this Fourier basis have exactly the same properties as the densities in the real space basis of Eq 51 for the power-law hoppings discussed in the previous section : in particular, the Inverse participation ratios in this Fourier basis

$$I_q \equiv \sum_m \left| \frac{W_m}{E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc}} \right|^{2q}$$
(71)

follow the equivalent of Eq 64 for the critical base $b_c = 1$

$$I_{q<\frac{1}{2}}^{(b_c=1)} \propto \frac{N^{1-2q}}{2\left(\frac{1}{2}-q\right)^2}$$

$$I_{q=\frac{1}{2}}^{(b_c=1)} \propto (\ln N)^2$$

$$I_{q>\frac{1}{2}}^{(b_c=1)} \propto \frac{\ln N}{N\to +\infty} \left(\frac{\ln N}{\left(q-\frac{1}{2}\right)}\right)$$
(72)

and the equivalent of Eq 64 for b > 1

$$I_{q<\frac{1}{2b}}^{(b>1)} \propto \frac{N^{1-2bq}}{(1-b)\left(\frac{1}{2b}-q\right)}$$

$$I_{q=\frac{1}{2b}}^{(b>1)} \propto \frac{2b}{b-1}\ln N$$

$$I_{q>\frac{1}{2b}}^{(b>1)} \propto \frac{1}{(1-b)\left(q-\frac{1}{2b}\right)}$$
(73)

However to analyze the localization properties in real space, one needs instead to study the Inverse participation Ratios Y_q in the real-space basis.

C. Inverse participation Ratios Y_q in the real-space basis

We need to consider the expansion up to second order in the amplitude w to obtain the Inverse participation Ratios Y_q in the real-space basis. Let us assume that the eigenstates are expanded up to second order in the perturbation H^{loc}

$$|\psi_Q^{(0+1+2)}\rangle = |\psi_Q^{(0)}\rangle + |\psi_Q^{(1)}\rangle + |\psi_Q^{(2)}\rangle$$
(74)

The normalization

$$1 = \langle \psi_Q^{(0+1+2)} | \psi_Q^{(0+1+2)} \rangle = \left(\langle \psi_Q^{(0)} \rangle | + \langle \psi_Q^{(1)} | + \langle \psi_Q^{(2)} | \right) \left(| \psi_Q^{(0)} \rangle + | \psi_Q^{(1)} \rangle \right) + | \psi_Q^{(2)} \rangle$$
(75)

yields the following conditions for the first and second orders

$$\begin{array}{ll}
0 & = <\psi_Q^{(1)}|\psi_Q^{(0)} > + <\psi_Q^{(0)}|\psi_Q^{(1)} > \\
0 & = <\psi_Q^{(1)}|\psi_Q^{(1)} > + <\psi_Q^{(2)}|\psi_Q^{(0)} > + <\psi_Q^{(0)}|\psi_Q^{(2)} > \\
\end{array} \tag{76}$$

Let us now consider the perturbation up to second order of the Inverse participation Ratios Y_q

$$Y_{q}^{(0+1+2)} = \sum_{n=1}^{N} \left(| < n | \psi_{Q}^{(0+1+2)} > |^{2} \right)^{q} = \sum_{n=1}^{N} \left(< \psi_{Q}^{(0+1+2)} | n > < n | \psi_{Q}^{(0+1+2)} > \right)^{q}$$

$$= \sum_{n=1}^{N} \left[< \psi_{Q}^{(0)} | n > < n | \psi_{Q}^{(0)} > + (< \psi_{Q}^{(0)} | n > < n | \psi_{Q}^{(1)} > + < \psi_{Q}^{(1)} | n > < n | \psi_{Q}^{(0)} > \right)$$

$$+ \left(< \psi_{Q}^{(0)} | n > < n | \psi_{Q}^{(2)} > + < \psi_{Q}^{(2)} | n > < n | \psi_{Q}^{(0)} > + < \psi_{Q}^{(1)} | n > < n | \psi_{Q}^{(1)} > \right]^{q}$$

$$(77)$$

Using the delocalized value of the leading contribution $\langle \psi_Q^{(0)} | n \rangle \langle n | \psi_Q^{(0)} \rangle = \frac{1}{N}$, the expansion of Eq. 77 up to second order becomes

$$\begin{split} Y_q^{(0+1+2)} &= \sum_{n=1}^N N^{-q} [1 + N(\langle \psi_Q^{(0)} | n \rangle \langle n | \psi_Q^{(1)} \rangle + \langle \psi_Q^{(1)} | n \rangle \langle n | \psi_Q^{(0)} \rangle) \\ &+ N(\langle \psi_Q^{(0)} | n \rangle \langle n | \psi_Q^{(2)} \rangle + \langle \psi_Q^{(2)} | n \rangle \langle n | \psi_Q^{(0)} \rangle + \langle \psi_Q^{(1)} | n \rangle \langle n | \psi_Q^{(1)} \rangle)]^q \\ &= \sum_{n=1}^N N^{-q} [1 + qN(\langle \psi_Q^{(0)} | n \rangle \langle n | \psi_Q^{(1)} \rangle + \langle \psi_Q^{(1)} | n \rangle \langle n | \psi_Q^{(0)} \rangle) \\ &+ \frac{q(q-1)}{2} N^2 (\langle \psi_Q^{(0)} | n \rangle \langle n | \psi_Q^{(1)} \rangle + \langle \psi_Q^{(1)} | n \rangle \langle n | \psi_Q^{(0)} \rangle)^2 \\ &+ qN(\langle \psi_Q^{(0)} | n \rangle \langle n | \psi_Q^{(2)} \rangle + \langle \psi_Q^{(2)} | n \rangle \langle n | \psi_Q^{(0)} \rangle + \langle \psi_Q^{(1)} | n \rangle \langle n | \psi_Q^{(1)} \rangle)] \\ &= N^{1-q} + qN^{1-q} (\langle \psi_Q^{(0)} | \psi_Q^{(1)} \rangle + \langle \psi_Q^{(1)} | \psi_Q^{(0)} \rangle) \\ &+ N^{2-q} \frac{q(q-1)}{2} \sum_{n=1}^N (\langle \psi_Q^{(0)} | n \rangle \langle n | \psi_Q^{(1)} \rangle + \langle \psi_Q^{(1)} | n \rangle \langle n | \psi_Q^{(0)} \rangle)^2 \\ &+ N^{1-q} q(\langle \psi_Q^{(0)} | \psi_Q^{(2)} \rangle + \langle \psi_Q^{(2)} | \psi_Q^{(0)} \rangle + \langle \psi_Q^{(1)} | \psi_Q^{(1)} \rangle) \end{split}$$
(78)

Using Eqs 76, this simplifies into

$$Y_q^{(0+1+2)} = N^{1-q} \left[1 + \frac{q(q-1)}{2} \mathcal{S}_N \right]$$
(79)

where

$$S_N \equiv N \sum_{n=1}^N (\langle \psi_Q^{(0)} | n \rangle \langle n | \psi_Q^{(1)} \rangle + \langle \psi_Q^{(1)} | n \rangle \langle n | \psi_Q^{(0)} \rangle)^2$$
(80)

only depends on the first correction $|\psi_Q^{(1)}>$ of the eigenstate of Eq. 69

$$|\psi_Q^{(1)}\rangle = \sum_{m=-P,+P} |Q+m\rangle \frac{W_m}{E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc}}$$
 (81)

Using the explicit expression of Eq 39 for the Fourier modes, one obtains

$$<\psi_Q^{(0)}|n>< n|\psi_Q^{(1)}> = \frac{1}{N}\sum_{m=-P,+P}e^{i2\pi m\frac{Gn}{N}}\frac{W_m}{E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc}}$$
(82)

so that

$$<\psi_Q^{(0)}|n>< n|\psi_Q^{(1)}> + <\psi_Q^{(1)}|n>< n|\psi_Q^{(0)}> = \frac{1}{N}\sum_{m=-P,+P}\frac{W_m e^{i2\pi m\frac{Gn}{N}} + W_m^* e^{-i2\pi m\frac{Gn}{N}}}{E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc}}$$
(83)

The sum of Eq. 80 thus reads $% \left({{{\rm{E}}_{\rm{B}}}} \right)$

$$S_{N} = N \sum_{n=1}^{N} (\langle \psi_{Q}^{(0)} | n \rangle \langle n | \psi_{Q}^{(1)} \rangle + \langle \psi_{Q}^{(1)} | n \rangle \langle n | \psi_{Q}^{(0)} \rangle)^{2}$$

$$= \sum_{m=-P,+P} \sum_{m'=-P,+P} \frac{\frac{1}{N} \sum_{n=1}^{N} (W_{m} e^{i2\pi m \frac{G_{n}}{N}} + W_{m}^{*} e^{-i2\pi m \frac{G_{n}}{N}}) (W_{m'} e^{i2\pi m' \frac{G_{n}}{N}} + W_{m'}^{*} e^{-i2\pi m' \frac{G_{n}}{N}})}{(E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc}) (E_{QQ}^{deloc} - E_{Q+m',Q+m'}^{deloc})}$$

$$= \sum_{m=-P,+P} \sum_{m'=-P,+P} \frac{(W_{m} W_{m'} + W_{m}^{*} W_{m'}^{*}) \delta_{m+m'} + (W_{m} W_{m'}^{*} + W_{m}^{*} W_{m'}) \delta_{m-m'}}{(E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc}) (E_{QQ}^{deloc} - E_{Q+m',Q+m'}^{deloc})}$$

$$= \sum_{m=-P,+P} \frac{(W_{m} W_{-m} + W_{m}^{*} W_{-m}^{*})}{(E_{QQ}^{deloc} - E_{Q-m,Q-m}^{deloc})} + \sum_{m=-P,+P} \frac{(W_{m} W_{m}^{*} + W_{m}^{*} W_{m})}{(E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc})}$$

$$(84)$$

Since $W_{-m} = W_m^*$, one finally obtains

$$S_N = \sum_{m=-P,+P} \frac{2|W_m|^2}{(E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc})(E_{QQ}^{deloc} - E_{Q-m,Q-m}^{deloc})} + \sum_{m=-P,+P} \frac{2|W_m|^2}{(E_{QQ}^{deloc} - E_{Q+m,Q+m}^{deloc})^2}$$
(85)

The second contribution is simply $(2I_1)$ where $I_{q=1}$ of Eq. 71 is the IPR in the Fourier basis for q = 1 The first contribution is less singular, since the numerator is the same, but the two denominators are distinct instead of coinciding. So one obtains the following conclusions

(i) for b > 1, $I_{q=1}$ of Eq. 73 remains finite, so S_N remains also finite O(1) in the thermodynamic limit $N \to +\infty$, and the Inverse Participation ratios keep their delocalized scaling

$$Y_q^{(0+1+2)} = N^{1-q} \left[1 + \frac{q(q-1)}{2} \mathcal{S}_{N=\infty} \right] \propto N^{(1-q)D^{deloc}(q)}$$
(86)

where the generalized dimensions coinciding with unity

$$D^{deloc}(q) = 1 \tag{87}$$

(ii) for $b_c = 1$, the logarithmic divergence of $I_{q=1}$ of Eq. 72 induces the logarithmic divergence of the sum with a small amplitude (δw^2)

$$S_N \simeq \delta w^2 \ln N \tag{88}$$

As a consequence, the Inverse Participation Ratio

$$Y_q^{(0+1+2)} = N^{1-q} \left[1 + \frac{q(q-1)}{2} \delta \ln N \right] \simeq N^{1-q} N^{\frac{q(q-1)}{2}\delta} = N^{(1-q)D^{criti}(q)}$$
(89)

where the generalized dimensions

$$D^{criti}(q) = 1 - \frac{q}{2}\delta w^2 \tag{90}$$

are only slightly different from their delocalized value (Eq 87). These properties are well-known as the weakmultifractality regime (see [4] and references therein).

VI. CONCLUSION

In this paper, we have considered the generalized version of the nearest-neighbor Aubry-André quasiperiodic localization model in order to include power-law translation-invariant hoppings $T_l \propto t/l^a$ or power-law Fourier coefficients $W_m \propto w/m^b$ in the quasi-periodic potential. We have first recalled the Aubry-André duality existing between T_l and W_m when the model is written in the real-space basis and in the Fourier basis on a finite ring. Via the perturbative analysis in the amplitude t of the hoppings, we have obtained that the eigenstates remain power-law localized for a > 1 and become critical at $a_c = 1$ where they follow the Strong Multifractality linear spectrum, as in the equivalent model with random disorder. Via the perturbative analysis in the amplitude w of the quasi-periodic potential, we have obtained that eigenstates remain delocalized in real space (power-law localized in Fourier space) for b > 1 and become critical at $b_c = 1$ where they follow the Weak Multifractality gaussian spectrum in real space (or Strong Multifractality linear spectrum in the Fourier basis). This critical case $b_c = 1$ for the Fourier coefficients W_m that we have studied corresponds to a periodic linear function with jumps, instead of the cosinus function of the self-dual Aubry-André. More generally, our conclusion is that any periodic function W(x) of weak amplitude w displaying discontinuities, i.e. characterized by Fourier coefficients decaying only as 1/m, makes the nearest-neighbor Aubry-André model critical.

To go beyond the perturbative analysis in the amplitudes t and w described in the present paper, it would be interesting to study numerically how the multifractal properties of eigenstates evolve as a function of these amplitudes.

Appendix A: Differences between a weak random potential and a weak quasiperiodic potential

While the cases of strong random potential and strong quasi-periodic potential are very similar, the cases of weak random potential and weak quasi-periodic potentials are completely different as a consequence of the different couplings existing between the Fourier modes, as discussed in this Appendix.

1. Generic instability of the weak-disorder expansion around Fourier modes

For the quasi-periodic case considered in the main text, the couplings between two Fourier modes (K, K') is governed by Eq. 36, or equivalently by Eq. 41 after the relabelling (Q, Q'): the interaction is thus directly determined by the Fourier coefficients W_m .

In the random case where the on-sites energies H_{nn} are random variables of zero-average and variance w^2 , the coupling between two Fourier modes (K, K') given by

$$H_{KK'}^{loc} = \frac{1}{N} \sum_{n=0}^{N-1} e^{i2\pi (K'-K)\frac{n}{N}} H_{nn}$$
(A1)

is a random variable of zero-average and of variance

$$\frac{|H^{loc}(K - K')|^2}{N} = \frac{w^2}{N}$$
(A2)

for any difference (K' - K): effectively, it is thus some 'mean-field' model where all the N Fourier modes are coupled via some interaction decaying with the system-size N: this is completely different from the quasi-periodic case considered in the text where the interaction between Fourier modes is governed by the Fourier coefficients W_m . The typical order of magnitude

$$H_N^{loc}(K - K') \underset{typ}{\propto} \frac{w}{N^{\frac{1}{2}}}$$
(A3)

is much bigger in scaling than the level spacing of the diagonal elements E_K^{deloc} in the Fourier basis

$$\Delta_N \propto \frac{t}{N} \tag{A4}$$

As a consequence, the perturbative expansion of eigenfunctions around Fourier modes that was described in section V for the quasiperiodic case looses its meaning in the random case as soon as the ratio

$$\frac{H_N^{loc}(K-K')}{\Delta_N} \mathop{\propto}\limits_{typ} \frac{wN^{\frac{1}{2}}}{t}$$
(A5)

becomes of order unity. The corresponding maximal size

$$N \le N_{max} = \left(\frac{t}{w}\right)^2 \tag{A6}$$

has the same scaling as the localization length $\xi \propto \left(\frac{t}{W}\right)^2$ that has been computed in the nearest-neighbor Anderson model [56, 57].

Another way to understand this instability of the weak-disorder expansion is that in the Fourier basis, the matrix actually corresponds to the Generalized-Rosenzweig-Porter model that has been much studied recently [52, 58–61] : the diagonal elements are finite O(1), while the off-diagonal elements are random of order N^{-b} : here the value is $b = \frac{1}{2}$ as in the Gaussian Orthogonal Ensemble and is thus well beyong the critical point $b_c = 1$.

2. Special stability of the weak-disorder expansion for anomalous level spacing

The argument above based of the usual level spacing of Eq. A4 has to be modified if the level spacing behaves differently in some region of the spectrum, as a consequence of some singularity in the density of states. For instance for the case where the hoppings are real symmetric decaying as the power-law $T(l) = \frac{t}{|l|^a}$ with a > 1 [31–34], the anomalous level spacing near zero momentum k = 0

$$\Delta_N(k\simeq 0) \propto \frac{t}{N^{a-1}} \tag{A7}$$

changes the ratio of Eq. A5 into

$$\frac{H_N^{loc}(K-K')}{\Delta_N} \mathop{\propto}\limits_{typ} \frac{WN^{a-\frac{3}{2}}}{t}$$
(A8)

so that the weak disorder expansion around Fourier modes is stable for $1 < a < \frac{3}{2}$ [31–34].

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