Return probability of N fermions released from a 1D confining potential

P L Krapivsky[1,2], J M Luck[2] and K Mallick[2]

[1] Department of Physics, Boston University, Boston, MA 02215, USA
[2] Institut de Physique Théorique, Université Paris-Saclay, CEA and CNRS, 91191 Gif-sur-Yvette, France

Abstract. We consider N non-interacting fermions prepared in the ground state of a 1D confining potential and submitted to an instantaneous quench consisting in releasing the trapping potential. We show that the quantum return probability of finding the fermions in their initial state at a later time falls off as a power law in the long-time regime, with a universal exponent depending only on N and on whether the free fermions expand over the full line or over a half-line. In both geometries the amplitudes of this power-law decay are expressed in terms of finite determinants of moments of the one-body bound-state wavefunctions in the potential. These amplitudes are worked out explicitly for the harmonic and square-well potentials. At large fermion numbers they obey scaling laws involving the Fermi energy of the initial state. The use of the Selberg-Mehta integrals stemming from random matrix theory has been instrumental in the derivation of these results.

E-mail:
pkrapivsky@gmail.com,jean-marc.luck@ipht.fr,kirone.mallick@ipht.fr

1. Introduction

Dynamical properties of a quantum system subject to a quench, i.e., a sudden modification of its Hamiltonian, have become an active field of research (see [1, 2, 3], and [4, 5, 6, 7, 8] for reviews). This renewed interest has been largely motivated by experimental progress in cold atom physics, where the evolution of a condensate after removing a trapping potential can be scrutinized. Amongst many possible observables, the return probability of a quantum system to its initial state (also referred to as fidelity or quantum Loschmidt echo), originally proposed by Peres as an indicator of chaos in quantum systems [9], is fully relevant to monitor the dynamics after a quench. The decay of the quantum return probability with time – that can intuitively be related to decoherence – is found to be of various types: generically exponential in chaotic systems [10], it can be shown by semi-classical arguments to be slower in regular systems [11]. The situation is more subtle, though, as various time scales are involved [12], and a crossover occurs around the Ehrenfest time (see [13] for a review). For many-particle systems, decay laws ranging from power-law to super-exponential have been predicted [14, 15, 16, 17, 18, 19, 20].

In a recent work [21] we studied the quantum return probability for a system of N free fermions hopping on a discrete infinite or semi-infinite lattice, launched from a compact configuration where the fermions occupy neighboring sites. We showed that this probability decays algebraically with time, with an exponent that exhibits an intriguing dependence on the parity of N, due to the combined effects of quantum interferences and discreteness. We also determined the exact decay amplitudes, thanks to a mapping to the Selberg-Mehta integrals of random matrix theory [22].

The aim of present work is to investigate the return probability of N fermions after free expansion in a different setting, namely when the system is prepared in the lowest energy state in a 1D trapping potential. We first consider the case where the fermions expand over the full line (section 2). At time $t = 0$ the confining potential is instantaneously removed. We show that the return probability falls off as a negative power of time, with a universal exponent N^2. The associated amplitude is expressed in terms of a finite determinant of moments of the one-body bound-state wavefunctions in the potential (sections 2.1 and 2.2). It is evaluated exactly in the cases of harmonic (section 2.3) and square-well (section 2.4) potentials. When the fermion number N is large, the return probability is shown to assume a scaling form involving the ratio $N/(E_\mathrm{F}t)$, where E_F is the Fermi energy of the initial state (section 2.5). All these results are then extended in section 3, with the same setup, to the situation where the fermions expand only over a semi-infinite line. The universal decay exponent of the return probability is now $N(2N + 1)$. Section 4 contains a brief discussion of our findings. In Appendix A we give a self-consistent treatment of the problem of non-colliding classical random walkers, using an approach that parallels the quantum calculations. Appendix B and Appendix C are respectively devoted to Mehta integrals and to the Barnes G-function.

2. Non-interacting fermions released on the infinite line

In this section we consider a system of N non-interacting spinless fermions on the infinite continuous line. The system is prepared in the lowest energy state in an arbitrary confining potential. At time $t = 0$, a quantum quench is performed by releasing the confining potential. We are interested in the return probability $R_N(t)$

that the particles are back to their initial state at a later time t, and especially in the asymptotic decay of this probability in the long-time regime.

2.1. Generalities

The one-body Hamiltonian reads

$$\mathcal{H} = \frac{p^2}{2} + V(x), \tag{2.1}$$

with $p = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$. The Planck constant and the fermion mass have been set to unity. The potential $V(x)$ is confining, i.e., $V(x) \to +\infty$ as $x \to \pm\infty$. Let E_n $(n = 0, 1, \ldots)$ be the ordered eigenvalues of $\mathcal{H}$ and $\psi_n(x)$ the associated normalized wavefunctions obeying

$$-\frac{1}{2}\psi_n''(x) + V(x)\psi_n(x) = E_n\psi_n(x). \tag{2.2}$$

We consider the situation where the system is prepared in its lowest energy state. The fermions therefore occupy the N lowest bound states $(n = 0, \ldots, N-1)$. The corresponding many-body wavefunction is the Slater determinant‡

$$\langle\Psi(0)|\mathbf{x}\rangle = \frac{1}{\sqrt{N!}}\det\left(\psi_{m-1}(x_n)\right), \tag{2.3}$$

with $\mathbf{x} = (x_1, \ldots, x_N)$. At time $t = 0$ the confining potential is released, and so the fermions undergo free expansion over the infinite line. The return probability reads

$$R_N(t) = |A_N(t)|^2, \tag{2.4}$$

with

$$A_N(t) = \langle\Psi(0)|\Psi(t)\rangle. \tag{2.5}$$

The free expansion dynamics is conveniently described in momentum space. In analogy with the tight-binding case studied in [21], we have

$$A_N(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{n=1}^{N}\left(\frac{\mathrm{d}q_n}{2\pi}\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}tq_n^2}\right)|\langle\Psi(0)|\mathbf{q}\rangle|^2, \tag{2.6}$$

$$\langle\Psi(0)|\mathbf{q}\rangle = \frac{1}{\sqrt{N!}}\det\left(\widehat{\psi}_{m-1}(q_n)\right), \tag{2.7}$$

with $\mathbf{q} = (q_1, \ldots, q_N)$ and

$$\widehat{\psi}_m(q) = \int_{-\infty}^{\infty}\psi_m(x)\,\mathrm{e}^{-\mathrm{i}qx}\,\mathrm{d}x. \tag{2.8}$$

We are mainly interested in the asymptotic decay of $R_N(t)$. In the long-time regime, the integral entering (2.6) is dominated by the region where all the momenta q_n are small. In the latter region, the determinant in (2.7) can be estimated by expanding each wavefunction in momentum space as

$$\widehat{\psi}_m(q) = \sum_{k\ge 0} M_{m,k}q^k, \tag{2.9}$$

‡ Throughout this work, unless specified explicitly, determinants are of size $N \times N$, with indices in the range $1 \le m, n \le N$.

with

$$M_{m,k} = \frac{(-\mathrm{i})^k}{k!} \int_{-\infty}^{\infty} \psi_m(x)\, x^k\, \mathrm{d}x. \tag{2.10}$$

The leading contribution is obtained by truncating the expansion (2.9) at order $k = N-1$. The array M thus becomes a square matrix of size $N \times N$, and the sum over its row index can be read as a matrix product. We thus obtain

$$\det\left(\widehat{\psi}_{m-1}(q_n)\right) \approx \det\left(\sum_{k=1}^{N} M_{m-1,k-1} q_n^{k-1}\right) = C_N \Delta_N(\mathbf{q}), \tag{2.11}$$

where the amplitude

$$C_N = \det\left(M_{m-1,k-1}\right) \tag{2.12}$$

depends on the confining potential, whereas the universal second factor

$$\Delta_N(\mathbf{q}) = \det\left(q_n^{m-1}\right) = \prod_{1\le m<n\le N} (q_n - q_m) \tag{2.13}$$

is the Vandermonde determinant of the momenta.

When time is large, the expression (2.6) for the amplitude $A_N(t)$ thus simplifies to

$$A_N(t) \approx \frac{|C_N|^2}{N!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{n=1}^{N} \left(\frac{\mathrm{d}q_n}{2\pi}\, \mathrm{e}^{-\frac{1}{2}\mathrm{i}tq_n^2}\right) \Delta_N^2(\mathbf{q}). \tag{2.14}$$

The key observation is that the above integral is the analytical continuation to $a = \frac{1}{2}\mathrm{i}t$ of the Mehta integral (B.2). We thus obtain the following prediction for the asymptotic decay of the quantum return probability in the long-time regime:

$$R_N(t) \approx \frac{|C_N|^4 G(N+1)^2}{(2\pi)^N t^{N^2}}, \tag{2.15}$$

where G denotes the Barnes G-function (see Appendix C).

The universal decay exponent N^2 can be recovered by the following heuristic argument [21]. The wavefunction of N non-interacting fermions can be estimated by expressing that the particles spread on a ballistic scale $L(t) \sim t$ and that the wavefunction vanishes when two particles occupy the same position. Thus one can write

$$|\Psi(\mathbf{x},t)| \sim K_N(t) \prod_{1\le m<n\le N} |x_m - x_n| \tag{2.16}$$

if $|x_i| \le L(t)$, whereas Ψ essentially vanishes otherwise. Dimensional analysis implies that the normalization scales as $K_N(t) \sim t^{-N^2/2}$, whence the decay exponent N^2 derived above by more quantitative means for the return probability.

The expression (2.15) only depends on the confining potential $V(x)$ through the prefactor C_N, given by (2.12). Hereafter we shall calculate C_N exactly for harmonic and square-well potentials in sections 2.3 and 2.4. We shall also estimate its large-N asymptotics for a general confining potential $V(x)$ in section 2.5, using a heuristic semi-classical analysis.

For a single particle, (2.15) reads

$$R_1(t) \approx \frac{|M_{00}|^4}{2\pi t}, \tag{2.17}$$

where

$$M_{00} = \int_{-\infty}^{\infty} \psi_0(x)\,\mathrm{d}x \tag{2.18}$$

is the integral of the ground-state wavefunction. This quantity can be used to define the spatial extent ℓ of the ground state by setting $|M_{00}| = \sqrt{\ell}$. The resulting expression,

$$R_1(t) \approx \frac{\ell^2}{2\pi t}, \tag{2.19}$$

can be read as $R_1(t) \sim \ell/L(t)$, where the dynamical length $L(t) \sim t/\ell$ represents the ballistic spreading of the quantum particle, whose momentum scale $p \sim 1/\ell$ is dictated by the uncertainty principle. An alternative interpretation of (2.19) is that it exhibits the formal diffusive scaling of the Schrödinger equation.

2.2. Considerations about symmetry

In the situation where the confining potential is symmetric, i.e., $V(-x) = V(x)$, the eigenstates have a definite parity, i.e.,

$$\psi_n(-x) = (-1)^n \psi_n(x), \tag{2.20}$$

and so the matrix element $M_{m,k}$ vanishes if $m+k$ is odd. The expression (2.12) of C_N therefore splits into the product of two determinants corresponding to each parity sector, namely

$$C_{2p} = c_p^{(\mathrm{even})}\, c_p^{(\mathrm{odd})}, \tag{2.21}$$

$$C_{2p+1} = c_{p+1}^{(\mathrm{even})}\, c_p^{(\mathrm{odd})}, \tag{2.22}$$

with

$$c_p^{(\mathrm{even})} = \det\left(M_{2k,2l}\right)_{0\le k,l\le p-1}, \tag{2.23}$$

$$c_p^{(\mathrm{odd})} = \det\left(M_{2k-1,2l-1}\right)_{1\le k,l\le p}. \tag{2.24}$$

Let us illustrate this in the case of two fermions. For an arbitrary potential, we have $C_2 = M_{00}M_{11} - M_{10}M_{01}$, and so

$$R_2(t) \approx \frac{|M_{00}M_{11} - M_{10}M_{01}|^4}{4\pi^2 t^4}. \tag{2.25}$$

For a symmetric potential, we have $c_1^{(\mathrm{even})} = M_{00}$, $c_1^{(\mathrm{odd})} = M_{11}$, hence $C_2 = M_{00}M_{11}$, and so

$$R_2(t) \approx \frac{|M_{00}M_{11}|^4}{4\pi^2 t^4}. \tag{2.26}$$

2.3. Harmonic potential

In this section we consider the case of a harmonic potential well:

$$V(x) = \frac{\omega^2 x^2}{2}. \tag{2.27}$$

The quantum harmonic oscillator is a textbook example of an exactly solvable system. This is actually the only example where the full time dependence of the return probability can be worked out explicitly.

The energy levels read

$$E_n = \left(n + \frac{1}{2}\right)\omega \qquad (n = 0, 1, \ldots). \tag{2.28}$$

The corresponding wavefunctions have essentially the same form in position space and in momentum space, namely

$$\psi_n(x) = \frac{(\omega/\pi)^{1/4}}{\sqrt{2^n n!}}\, H_n(x\sqrt{\omega})\, \mathrm{e}^{-\omega x^2/2}, \tag{2.29}$$

$$\widehat{\psi}_n(q) = (-\mathrm{i})^n \frac{(4\pi/\omega)^{1/4}}{\sqrt{2^n n!}}\, H_n(q/\sqrt{\omega})\, \mathrm{e}^{-q^2/(2\omega)}, \tag{2.30}$$

where

$$H_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{k!(n-2k)!} (2z)^{n-2k} \tag{2.31}$$

are the Hermite polynomials [23]. Using (2.30), and introducing the notation $z_n = q_n/\sqrt{\omega}$, the expression (2.7) can be recast as

$$\langle \Psi(0)|\mathbf{q}\rangle = \left(-\frac{\mathrm{i}}{\sqrt{2}}\right)^{N(N-1)/2} \frac{(4\pi/\omega)^{N/4}}{\sqrt{G(N+2)}} \times \exp\left(-\frac{1}{2}\sum_{n=1}^{N} z_n^2\right) \det\left(H_{m-1}(z_n)\right). \tag{2.32}$$

The latter determinant can be evaluated as follows. By subtracting from the mth line a suitably chosen linear combination of the previous ones, the Hermite polynomial $H_{m-1}(z_n)$ can be replaced by its leading term $(2z_n)^{m-1}$. The determinant is therefore proportional to a Vandermonde determinant:

$$\det\left(H_{m-1}(z_n)\right) = 2^{N(N-1)/2} \Delta_N(\mathbf{z}), \tag{2.33}$$

with $\mathbf{z} = (z_1, \ldots, z_N)$.

The expression (2.6) of the amplitude $A_N(t)$ therefore reads

$$A_N(t) = \frac{2^{N(N-1)/2}}{\pi^{N/2} G(N+2)} \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{n=1}^{N} \left(\mathrm{d}z_n\, \mathrm{e}^{-(1+\frac{1}{2}\mathrm{i}\omega t)z_n^2}\right) \Delta_N^2(\mathbf{z}). \tag{2.34}$$

The above integral is a Mehta integral of the form (B.2) at any time. This is a peculiar feature of the harmonic oscillator. We thus obtain the remarkably simple exact expressions§

$$A_N(t) = \left(1 + \frac{\mathrm{i}\omega t}{2}\right)^{-N^2/2}, \tag{2.35}$$

$$R_N(t) = \left(1 + \frac{\omega^2 t^2}{4}\right)^{-N^2/2}, \tag{2.36}$$

§ The normalization $R_N(0) = 1$ for all N provides a useful check of the formalism.

which hold for all fermion numbers N and all times t. The return probability therefore exhibits the power-law decay

$$R_N(t) \approx \left(\frac{2}{\omega t}\right)^{N^2}, \tag{2.37}$$

with exponent N^2, in agreement with (2.15), and a simple prefactor.

Anticipating the analysis of the large-N regime presented in section 2.5, we recast the above formula in terms of the Fermi energy E_F, defined as the energy of the last occupied one-particle state. In the case of the harmonic oscillator, we have $E_\mathrm{F} \approx N\omega$, hence

$$R_N(t) \sim \left(\frac{2N}{E_\mathrm{F} t}\right)^{N^2}. \tag{2.38}$$

2.4. Square-well potential

In this section we consider the case of a square-well potential:

$$V(x) = \begin{cases} 0 & (|x| < L/2), \\ +\infty & (|x| > L/2). \end{cases} \tag{2.39}$$

The particles are thus confined between two impenetrable walls at $x = \pm L/2$. This is another textbook example of an exactly solvable system. The energy levels read

$$E_n = (n+1)^2 \frac{\pi^2}{2L^2} \qquad (n = 0, 1, \ldots). \tag{2.40}$$

The corresponding wavefunctions are as follows.

- Even sector ($n = 2p$, $p = 0, 1, \ldots$):

$$\psi_{2p}(x) = \sqrt{\frac{2}{L}} \cos \frac{(2p+1)\pi x}{L}, \tag{2.41}$$

$$\widehat{\psi}_{2p}(q) = (-1)^p \, \pi\sqrt{2L} \, \frac{2(2p+1)}{(2p+1)^2\pi^2 - q^2L^2} \cos \frac{qL}{2}. \tag{2.42}$$

- Odd sector ($n = 2p-1$, $p = 1, 2, \ldots$):

$$\psi_{2p-1}(x) = \sqrt{\frac{2}{L}} \sin \frac{2p\pi x}{L}, \tag{2.43}$$

$$\widehat{\psi}_{2p-1}(q) = (-1)^p \, \mathrm{i}\pi\sqrt{2L} \, \frac{4p}{4p^2\pi^2 - q^2L^2} \sin \frac{qL}{2}. \tag{2.44}$$

At variance with the case of the harmonic oscillator, the return probability cannot be evaluated exactly at finite time.

The prefactor C_N entering the asymptotic decay law (2.15) of $R_N(t)$ can however be evaluated exactly, for all values of the fermion number N. The key point of the derivation resides in the following observation. The determinants $c_p^{(\mathrm{even})}$ and $c_p^{(\mathrm{odd})}$, introduced in (2.23), (2.24), can be simplified along the lines of the derivation of (2.33). In the even sector, it is legitimate to replace $\cos(qL/2)$ by unity in the expression (2.42) of $\widehat{\psi}_{2p}(q)$, as this amounts to subtracting from the rows of the matrix $M_{2k,2l}$ a suitably chosen linear combination of the previous ones. We denote by $\widetilde{M}$ the matrix simplified in this way. Similarly, for the odd sector, $\sin(qL/2)$ can be linearized to $qL/2$ in the expression (2.44) of $\widehat{\psi}_{2p-1}(q)$.

In the even sector, we thus obtain

$$c_p^{(\mathrm{even})} = \det\,(\widetilde{M}_{2k,2l})_{0\le k,l\le p-1},$$

$$\widetilde{M}_{2k,2l} = \frac{(-1)^k\,2\sqrt{2}}{((2k+1)\pi)^{2l+1}}\,L^{2l+1/2}, \tag{2.45}$$

and so

$$c_p^{(\mathrm{even})} = \frac{2^{3p/2}}{\pi^{p^2}\,(2p-1)!!}\,d_p^{(\mathrm{even})}\,L^{p(p-1/2)}, \tag{2.46}$$

with

$$d_p^{(\mathrm{even})} = (-1)^{p(p-1)/2}\det\,((2k+1)^{-2l})_{0\le k,l\le p-1}. \tag{2.47}$$

In the odd sector, we have

$$c_p^{(\mathrm{odd})} = \det\,(\widetilde{M}_{2k-1,2l-1})_{1\le k,l\le p},$$

$$\widetilde{M}_{2k-1,2l-1} = \frac{(-1)^k\,\mathrm{i}\sqrt{2}}{(2k\pi)^{2l-1}}\,L^{2l-1/2}, \tag{2.48}$$

and so

$$c_p^{(\mathrm{odd})} = \frac{(-\mathrm{i})^p\,2^{p/2}}{(2\pi)^{p^2}\,p!}\,d_p^{(\mathrm{odd})}\,L^{p(p+1/2)}, \tag{2.49}$$

with

$$d_p^{(\mathrm{odd})} = (-1)^{p(p-1)/2}\det\,(k^{-2(l-1)})_{1\le k,l\le p}. \tag{2.50}$$

The determinants $d_p^{(\mathrm{even})}$ and $d_p^{(\mathrm{odd})}$ are evaluated in Appendix C. Their explicit expressions (C.11) and (C.12) yield

$$c_p^{(\mathrm{even})} = \frac{2^{3p^2}}{\pi^{p^2}}\left(\frac{p!}{(2p)!}\right)^{2p}\sqrt{\frac{G(2p+2)}{p!}}\,L^{p(p-1/2)}, \tag{2.51}$$

$$c_p^{(\mathrm{odd})} = \frac{(-\mathrm{i})^p}{(2\pi)^{p^2}\,p!^{2p}}\sqrt{\frac{G(2p+2)}{p!}}\,L^{p(p+1/2)}. \tag{2.52}$$

Inserting the above results into (2.21), (2.22), we get

$$|C_N| = \frac{G(N+2)}{N!^N\,\Gamma(\frac{N}{2}+1)}\left(\frac{2L}{\pi}\right)^{N^2/2}, \tag{2.53}$$

irrespective of the parity of N.

We thus obtain the following asymptotic decay law for the return probability

$$R_N(t) \approx K_N\left(\frac{L^2}{\pi^2 t}\right)^{N^2}, \tag{2.54}$$

where the exponent N^2 is in agreement with (2.15), and the prefactor reads

$$K_N = \frac{2^{2N^2}\,G(N+1)^6}{(2\pi)^N\,N!^{4(N-1)}\,\Gamma(\frac{N}{2}+1)^4}. \tag{2.55}$$

In particular

$$K_1 = \frac{32}{\pi^3},\quad K_2 = \frac{4}{\pi^2},\quad K_3 = \frac{2^{21}}{3^{12}\pi^5}. \tag{2.56}$$

When the fermion number N becomes large, the leading behavior of K_N can be derived by using the asymptotic expansions (C.4) and (C.5). Keeping only terms in N^2, we obtain

$$\ln K_N \approx -N^2 \left(\ln N - 2\ln 2 + \frac{1}{2} \right). \tag{2.57}$$

The expression (2.54) therefore simplifies to

$$R_N(t) \sim \left(\frac{4L^2}{\mathrm{e}^{1/2}\pi^2 N t} \right)^{N^2}. \tag{2.58}$$

Anticipating again the analysis of section 2.5, we recast the above formula in terms of the Fermi energy E_{F}. For a square-well potential, the Fermi energy grows as $E_{\mathrm{F}} \approx N^2\pi^2/(2L^2)$ and the above formula can be rewritten as

$$R_N(t) \sim \left(\frac{2\mathrm{e}^{-1/2}\, N}{E_{\mathrm{F}} t} \right)^{N^2}. \tag{2.59}$$

2.5. Scaling at large N

We now focus our attention onto the regime where the fermion number N is large. By observing the formulas (2.38) and (2.59) found for the harmonic and the square-well potential, it is tempting to propose the following scaling Ansatz for the return probability for an arbitrary confining potential $V(x)$:

$$R_N(t) \sim \left(\frac{BN}{E_{\mathrm{F}} t} \right)^{N^2}. \tag{2.60}$$

This scaling law is meant to hold in the regime where the long-time limit is taken before the limit of a large fermion number. In the denominator, E_{F} is the Fermi energy, i.e., the energy of the last occupied one-particle state. The occurrence of the dimensionless combination $E_{\mathrm{F}} t$ is quite natural. In the numerator, B appears as a numerical constant of order unity, which depends on the confining potential $V(x)$. The above results give $B = 2$ for the harmonic oscillator and $B = 2\mathrm{e}^{-1/2} = 1.213061$ for the square-well potential.

The Ansatz (2.60) is corroborated by the following heuristic semi-classical analysis for a general symmetric power-law confining potential,

$$V(x) = g|x|^a, \tag{2.61}$$

with arbitrary growth exponent $a > 0$. Consider a highly excited bound state in this potential, with energy E_n ($n \gg 1$). The wavefunction $\psi_n(x)$ exhibits two turning points at $x = \pm x_n$, such that $E_n = g x_n^a$. The energy E_n is given by the semi-classical quantization formula [24]

$$2\int_0^{x_n} \sqrt{2(E_n - g x^a)}\, \mathrm{d}x \approx \left(n + \frac{1}{2} \right) \pi. \tag{2.62}$$

Some algebra leads to

$$x_n \approx \left(\frac{\lambda n}{\sqrt{g}} \right)^{\beta}, \qquad E_n \approx g^{\beta} (\lambda n)^{2(1-\beta)}, \tag{2.63}$$

with

$$\beta = \frac{2}{a+2}, \qquad \lambda = \sqrt{\frac{\pi}{2}}\, \frac{\Gamma\big(\frac{3}{2} + \frac{1}{a}\big)}{\Gamma\big(1 + \frac{1}{a}\big)}. \tag{2.64}$$

We have therefore $E_{\rm F} \approx g^\beta(\lambda N)^{2(1-\beta)}$. The wavefunction $\psi_n(x)$ is rapidly oscillating in the allowed region ($|x| < x_n$), and exponentially decaying in the forbidden regions ($|x| > x_n$). Its amplitude is maximal in the transition regions near the turning points. As a consequence, for large values of the integers m and n, the integral entering the expression (2.10) of $M_{m,k}$ can be expected to be dominated by the vicinity of the turning points. This heuristic argument yields the rough estimate

$$M_{m,k} \sim \frac{x_m^k}{k!}; \tag{2.65}$$

and hence

$$\ln|M_{m,k}| \approx k\left(\beta\ln\frac{\lambda m}{\sqrt{g}} - \ln k + 1\right). \tag{2.66}$$

The leading scaling behavior of C_N can be read off from the above estimate by replacing m by N in the argument of the first logarithm and summing the resulting expression over k. This yields

$$\ln|C_N| \approx \frac{N^2}{2}\left((\beta-1)\ln N + \beta\ln\frac{\lambda}{\sqrt{g}} + \cdots\right). \tag{2.67}$$

Inserting this estimate into (2.15), we find that the return probability indeed obeys the scaling law (2.60). The above line of reasoning is however too crude to predict the numerical constant B.

3. Non-interacting fermions released on the semi-infinite line

We now consider the same problem on the semi-infinite line ($x > 0$). We assume that there is an impenetrable wall at the origin. The system is prepared in the lowest energy state in the presence of an arbitrary confining potential $V^{\rm w}(x)$ acting for $x > 0$.‖ At time $t = 0$ the confining potential is released but the wall at the origin is kept, so that the particles expand over the semi-infinite line. We are again interested in the return probability $R_N^{\rm w}(t)$, and especially in its asymptotic decay.

3.1. Generalities

The one-body Hamiltonian reads

$$\mathcal{H}^{\rm w} = \frac{p^2}{2} + V^{\rm w}(x) \qquad (x > 0), \tag{3.1}$$

with Dirichlet boundary condition at the origin. Let $E_n^{\rm w}$ ($n = 0, 1, \ldots$) be the ordered eigenvalues of $\mathcal{H}^{\rm w}$ and $\psi_n^{\rm w}(x)$ the associated normalized wavefunctions.

At time $t = 0$ the confining potential is released, and so the fermions undergo free expansion over the semi-infinite line. The return probability reads

$$R_N^{\rm w}(t) = |A_N^{\rm w}(t)|^2, \tag{3.2}$$

with

$$A_N^{\rm w}(t) = \langle\Psi^{\rm w}(0)|\Psi^{\rm w}(t)\rangle. \tag{3.3}$$

The free expansion dynamics is again best described in momentum space. We have

$$A_N^{\rm w}(t) = \int_0^\infty \cdots \int_0^\infty \prod_{n=1}^N \left(\frac{{\rm d}q_n}{\pi}\,{\rm e}^{-\frac{1}{2}{\rm i}tq_n^2}\right)|\langle\Psi^{\rm w}(0)|\mathbf{q}\rangle|^2, \tag{3.4}$$

‖ The superscript 'w' reminds of the permanent presence of an impenetrable wall at the origin.

with

$$\langle\Psi^{\rm w}(0)|\mathbf{q}\rangle=\frac{1}{\sqrt{N!}}\det\big(\widehat{\psi}^{\rm w}_{m-1}(q_n)\big) \tag{3.5}$$

and

$$\widehat{\psi}^{\rm w}_m(q)=\sqrt{2}\int_0^\infty\psi^{\rm w}_m(x)\,\sin qx\,{\rm d}x. \tag{3.6}$$

In the long-time regime, the integral entering (3.4) is again dominated by the region where all the momenta q_n are small. In the latter region, the determinant in (3.5) can be estimated by expanding each wavefunction in momentum space as

$$\widehat{\psi}^{\rm w}_m(q)=\sqrt{2}\sum_{k\ge0}M^{\rm w}_{m,k}q^{2k+1}, \tag{3.7}$$

with

$$M^{\rm w}_{m,k}=\frac{(-1)^k}{(2k+1)!}\int_0^\infty\psi^{\rm w}_m(x)\,x^{2k+1}\,{\rm d}x. \tag{3.8}$$

The leading contribution is again obtained by truncating the expansion (3.7) at order $k=N-1$. We thus obtain

$$\det\big(\widehat{\psi}^{\rm w}_{m-1}(q_n)\big)\approx 2^{N/2}\,C^{\rm w}_N\prod_{n=1}^N q_n\,\Delta_N(\mathbf{q}^2), \tag{3.9}$$

where

$$C^{\rm w}_N=\det\big(M^{\rm w}_{m-1,k-1}\big), \tag{3.10}$$

and $\mathbf{q}^2=(q_1^2,\dots,q_N^2)$. When time is large, the expression (3.3) for the amplitude $A^{\rm w}_N(t)$ thus simplifies to

$$A^{\rm w}_N(t)\approx\frac{2^N|C^{\rm w}_N|^2}{N!}\int_0^\infty\!\!\cdots\int_0^\infty\prod_{n=1}^N\left(\frac{{\rm d}q_n}{\pi}\,q_n^2\,{\rm e}^{-\frac12{\rm i}tq_n^2}\right)\Delta_N^2(\mathbf{q}^2). \tag{3.11}$$

The above integral is proportional to the analytical continuation to $a=\frac12{\rm i}t$ of the Mehta integral (B.6). We thus obtain the following asymptotic prediction for the return probability in the long-time regime:

$$R^{\rm w}_N(t)\approx\frac{|C^{\rm w}_N|^4G(2N+2)}{\pi^N N!\,t^{N(2N+1)}}. \tag{3.12}$$

The return probability again exhibits a universal power-law decay, with exponent $N(2N+1)$. This exponent can also be predicted by means of a heuristic argument (see [21]). The dependence of the above prediction on the confining potential $V^{\rm w}(x)$ is entirely contained in the prefactor $C^{\rm w}_N$, given by (3.10).

For one single particle, (3.12) reads

$$R^{\rm w}_1(t)\approx\frac{2|M^{\rm w}_{00}|^4}{\pi t^3}, \tag{3.13}$$

with

$$M^{\rm w}_{00}=\int_0^\infty x\,\psi^{\rm w}_0(x)\,{\rm d}x. \tag{3.14}$$

This integral can be used to define the spatial extent $\ell^{\rm w}$ of the ground state by setting $|M^{\rm w}_{00}|=(\ell^{\rm w})^{3/2}$. The resulting expression,

$$R_1(t)\approx\frac{2\,(\ell^{\rm w})^6}{\pi t^3}, \tag{3.15}$$

again reflects the formal diffusive scaling of the Schrödinger equation.

3.2. Considerations about symmetry

It is worth comparing the one-sided situation, i.e., free expansion on the semi-infinite line after preparation in the confining potential $V^{\rm w}(x)$ for $x > 0$, with the two-sided situation (section 2) with potential

$$V(x) = V^{\rm w}(|x|), \tag{3.16}$$

obtained by symmetrizing $V^{\rm w}(x)$.

The correspondence between both situations goes as follows. The Dirichlet boundary condition at the origin implies that the wavefunctions $\psi_n^{\rm w}(x)$ essentially coincide with the odd wavefunctions of the two-sided problem, namely

$$\psi_n^{\rm w}(x) = \sqrt{2}\,\psi_{2n+1}(x) \tag{3.17}$$

for $x > 0$ and $n = 0, 1, \ldots$, where the factor $\sqrt{2}$ ensures the correct normalizations. In momentum space, with the definitions (2.8) and (3.6), this reads

$$\widehat{\psi}_n^{\rm w}(q) = {\rm i}\,\widehat{\psi}_{2n+1}(q). \tag{3.18}$$

We have therefore

$$M_{m,k}^{\rm w} = \frac{\rm i}{\sqrt{2}}\,M_{2m+1,2k+1}. \tag{3.19}$$

In particular

$$C_N^{\rm w} = \left(\frac{\rm i}{\sqrt{2}}\right)^N c_N^{\rm (odd)}. \tag{3.20}$$

3.3. Harmonic potential

In this section we consider the case of a harmonic half-well:

$$V^{\rm w}(x) = \frac{\omega^2 x^2}{2} \qquad (x > 0). \tag{3.21}$$

This will again be the only example where the full time dependence of the return probability $R_N^{\rm w}(t)$ can be worked out explicitly.

We shall exploit the correspondence underlined in section 3.2. The two-sided situation is that of a harmonic well, studied in section 2.3. Using (3.18), as well as (2.30) and (C.10), and introducing the notation $z_n = q_n/\sqrt{\omega}$, the expression (3.5) can be recast as

$$\begin{aligned}\langle\Psi^{\rm w}(0)|\mathbf{q}\rangle = &\left(-\frac{1}{2}\right)^{N(N-1)/2}\left(\frac{(2\pi/\omega)^N}{N!G(2N+2)}\right)^{1/4} \\ &\times\ \exp\left(-\frac{1}{2}\sum_{n=1}^{N} z_n^2\right)\det\left(H_{2m-1}(z_n)\right).\end{aligned} \tag{3.22}$$

The latter determinant can be simplified along the lines of the derivation of (2.33). We thus obtain

$$\det\left(H_{2m-1}(z_n)\right) = 2^{N^2}\prod_{i=1}^{N} z_n\,\Delta_N(\mathbf{z}^2), \tag{3.23}$$

with $\mathbf{z}^2 = (z_1^2, \ldots, z_N^2)$.

The expression (3.4) of the amplitude $A_N^{\rm w}(t)$ therefore reads

$$A_N^{\rm w}(t) = \frac{2^{N(2N+1)/2}}{\sqrt{\pi^N N!\,G(2N+2)}} \times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{n=1}^{N}\left({\rm d}z_n\, z_n^2\, {\rm e}^{-(1+\frac{1}{2}{\rm i}\omega t)z_n^2}\right)\Delta_N^2(\mathbf{z}^2). \tag{3.24}$$

The above integral is a Mehta integral of the form (B.6) at any time. We again obtain remarkably simple exact expressions¶

$$A_N^{\rm w}(t) = \left(1+\frac{{\rm i}\omega t}{2}\right)^{-N(2N+1)/2}, \tag{3.25}$$

$$R_N^{\rm w}(t) = \left(1+\frac{\omega^2 t^2}{4}\right)^{-N(2N+1)/2}, \tag{3.26}$$

for all fermion numbers N and all times t. The return probability therefore exhibits the power-law decay

$$R_N^{\rm w}(t) \approx \left(\frac{2}{\omega t}\right)^{N(2N+1)}, \tag{3.27}$$

with exponent $N(2N+1)$, in agreement with (3.12), and a simple prefactor.

3.4. Square-well potential

In this section we consider the case of the square-well potential:

$$V(x) = \begin{cases} 0 & (0 < x < L/2), \\ +\infty & \text{else.} \end{cases} \tag{3.28}$$

The particles are thus confined between two impenetrable walls at $x = 0$ and $x = L/2$. The right wall is removed instantaneously at time $t = 0$, whereas the left one is maintained permanently. This setting can be viewed as a special case of the problem studied long ago by Doescher and Rice [25], namely a square well with a wall moving at constant velocity. The present problem of an instantaneous release corresponds to the infinite velocity limit.

We shall again make use of the correspondence underlined in section 3.2. The two-sided situation is that considered in section 2.4. Inserting the expression (2.52) of $c_N^{\rm (odd)}$ into (3.20), we readily obtain

$$C_N^{\rm w} = \frac{1}{(2\pi)^{N^2} N!^{2N}}\sqrt{\frac{G(2N+2)}{2^N N!}}\, L^{N(N+1/2)}, \tag{3.29}$$

and finally

$$R_N^{\rm w}(t) \approx K_N^{\rm w}\left(\frac{L^2}{4\pi^2 t}\right)^{N(2N+1)}, \tag{3.30}$$

where the exponent $N(2N+1)$ is again in agreement with (3.12), and the prefactor reads

$$K_N^{\rm w} = \frac{\pi^N G(2N+2)^3}{N!^{8N+3}}. \tag{3.31}$$

¶ The normalization $R_N^{\rm w}(0) = 1$ again provides a useful check of the formalism.

In particular

$$K_1^{\mathrm{w}} = 8\pi, \quad K_2^{\mathrm{w}} = \frac{3^6\pi^2}{2^4}, \quad K_3^{\mathrm{w}} = \frac{2^9 5^6 \pi^3}{3^{12}}. \tag{3.32}$$

When the fermion number N becomes large, the leading decay law of K_N^{w} can again be derived by using the asymptotic expansions (C.4) and (C.5). Keeping only terms in N^2, we obtain

$$\ln K_N^{\mathrm{w}} \approx -2N^2 \left(\ln N - 3\ln 2 + \frac{1}{2} \right). \tag{3.33}$$

The expression (3.30) therefore simplifies to

$$R_N^{\mathrm{w}}(t) \sim \left(\frac{2L^2}{\mathrm{e}^{1/2}\pi^2 N t} \right)^{N(2N+1)}. \tag{3.34}$$

3.5. Scaling at large N

In the regime where the fermion number N is large, along the lines of section 2.5, we propose the following scaling Ansatz for the return probability:

$$R_N^{\mathrm{w}}(t) \sim \left(\frac{B^{\mathrm{w}} N}{E_{\mathrm{F}}^{\mathrm{w}} t} \right)^{N(2N+1)}. \tag{3.35}$$

Let us again begin by revisiting the two exactly solvable examples considered above. For the harmonic oscillator (section 3.3), the decay of the return probability is given by (3.27), whereas $E_{\mathrm{F}}^{\mathrm{w}} \approx 2N\omega$. This is in agreement with the above Ansatz, with $B^{\mathrm{w}} = 4$, whereas we had $B = 2$ in the two-sided situation. For the square-well potential (section 3.4), the decay of the return probability is given by (3.34) at large N, whereas $E_{\mathrm{F}}^{\mathrm{w}} \approx 2N^2\pi^2/L^2$. This, too, is in agreement with the above Ansatz, with $B^{\mathrm{w}} = 4\mathrm{e}^{-1/2}$, whereas we had $B = 2\mathrm{e}^{-1/2}$ in the two-sided situation.

The relation

$$B^{\mathrm{w}} = 2B \tag{3.36}$$

between the constants pertaining to the one-sided and two-sided situations, in the sense of section 3.2, is in fact quite general. This can be shown as follows. Assume the Ansatz (2.60) holds in the two-sided situation, in the presence of the symmetrized potential (3.16). Using (2.15), this yields the estimate

$$\ln |C_N| \approx \frac{N^2}{4} \left(\ln B - \ln E_{\mathrm{F}}(N) + \frac{3}{2} \right), \tag{3.37}$$

where we have emphasized the dependence of the Fermi energy on the fermion number N. Consider now the one-sided situation. First, the correspondence (3.17) implies that the Fermi energy reads approximately

$$E_{\mathrm{F}}^{\mathrm{w}}(N) \approx E_{\mathrm{F}}(2N). \tag{3.38}$$

Second, (3.20) yields

$$\ln |C_N^{\mathrm{w}}| \approx \ln |c_N^{(\mathrm{odd})}| \approx \frac{1}{2} \ln |C_{2N}|. \tag{3.39}$$

The last estimate is obtained by expressing that both sectors equally contribute to the expression (2.21) of C_{2N}, to leading order for large N. Combining the two above results with (3.12) and (3.37), we readily obtain the aforementioned relation (3.36).

4. Discussion

We have investigated the quantum return probability for a system of N non-interacting fermions prepared in the ground state of a 1D confining potential and submitted to an instantaneous quench consisting in releasing the trapping potential.

Our main finding is that the quantum return probability falls off as a power law in the long-time regime, with a universal exponent which only depends on the fermion number and on the geometry, i.e., whether the fermions expand over the full line or only over a half-line. Table 1 presents a comparison of the values of this decay exponent in both geometries with the decay exponent defined similarly in two different analogous situations, namely tight-binding lattice fermions launched from a compact configuration, investigated in our recent work [21], and non-colliding classical random walkers, whose survival and return probabilities are derived in Appendix A. These exponents share many common features. Their dependence on the particle number N is a quadratic polynomial with simple coefficients. Their growth at large N is twice larger in the half-line geometry than on the full line.

Model		Exponent (full line)	Exponent (half-line)
Continuum fermions		N^2	$N(2N+1)$
Lattice fermions	N even	$\frac{1}{2}N^2$	$N(N+1)$
	N odd	$\frac{1}{2}(N^2+1)$	N^2+N+1
Classical walkers		$\frac{1}{4}N(N+1)$	$\frac{1}{2}N(N+1)$

Table 1. Decay exponent of the return probability for a system of N particles moving either on the full line or on a half-line. First row: Non-interacting fermions in the continuum, prepared in the ground state of a confining potential and instantaneously released (body of this work). Second row: Non-interacting tight-binding lattice fermions launched from a compact configuration (Reference [21]). There, the exponent depends on the parity of the fermion number N. Third row: Non-colliding classical walkers (Appendix A of this work). There, the exponent governs the power-law falloff of the return probability conditioned on survival.

The amplitudes of the power-law decay of the quantum return probability in both geometries have been shown to depend on the confining potential only through the quantities C_N and $C_N^{\rm w}$, expressed in (2.12) and (3.10) as $N\times N$ determinants of moments of the one-body bound-state wavefunctions in the potential. These amplitudes have been worked out explicitly for the harmonic and square-well potentials (see (2.37), (2.54) and (3.27), (3.30)). The return probabilities have also been demonstrated to simplify at large fermion numbers, where they obey scaling laws involving the ratio $N/(E_{\rm F}t)$, with $E_{\rm F}$ being the Fermi energy of the initial state (see (2.60), (3.35)).

Finally, the investigations pursued in [21] and in the present work reveal similarities between the dynamics of free fermionic systems and random matrix theory. This resemblance, which has also been put forward recently in a static context [26], is essentially due to the effective repulsion felt both by the eigenvalues of a random matrix and by 1D fermions. More specifically, Selberg-Mehta integrals stemming from random matrix theory have been instrumental in deriving most key results. The outcomes often involve the Barnes G-function, which is also ubiquitous in random matrix theory, whereas the scaling in N^2 of the exponents listed in table 1 is reminiscent of the scaling of the free energy of matrix models.

Acknowledgments

We are grateful to Jean-Marie Stéphan for very interesting discussions.

Appendix A. Classical analogue: non-colliding random walkers

A classical analogue of the problem considered in the body of this work is a collection of N independent random walkers on the line, or on the half-line, conditioned to never collide. This system has been studied by various approaches [27, 28, 29, 30, 31]. The goal of this appendix is to derive in a self-consistent way many results on the survival and return probabilities, some of which are already known but scattered in the literature. The analogies and the differences between the classical and the quantum situations and between the geometries of the line and of the half-line are briefly summarized in section 4. Hereafter we follow the approach initiated by Karlin and McGregor [27], and pursued by Lindström [32] and Gessel and Viennot [33], yielding to determinantal formulas such as (A.1).

Appendix A.1. Walkers on the line

Consider N independent random walkers on the line starting at time $t = 0$ from the positions $\mathbf{x} = (x_1, \ldots, x_N)$, with $x_1 < \ldots < x_N$. At any subsequent time t, the probability (or probability density) $P(\mathbf{x}, \mathbf{y}, t)$ that the walkers are at the positions $\mathbf{y} = (y_1, \ldots, y_N)$, with $y_1 < \ldots < y_N$, and that their trajectories have not intersected, is given by the Karlin-McGregor formula [27]

$$P(\mathbf{x}, \mathbf{y}, t) = \det\, p(x_i, y_j, t), \tag{A.1}$$

where $p(x, y, t)$ is the transition kernel (probability or probability density) for one single walker. The above determinantal formula holds for several kinds of microscopic realizations of a random walk, either discrete or continuous. The only condition is that no particle can jump over another one. Here are two important examples.

• For Brownian particles, with diffusion coefficient $D = 1/2$, the transition probability density reads

$$p(x, y, t) = \frac{\mathrm{e}^{-(y-x)^2/(2t)}}{\sqrt{2\pi t}}. \tag{A.2}$$

• For particles executing continuous-time random walks on the lattice of integers, with jumps to neighboring sites at unit rate, so that again $D = 1/2$, the transition probability reads

$$p(x, y, t) = \mathrm{e}^{-t}\, I_{y-x}(t), \tag{A.3}$$

where I_{y-x} is the modified Bessel function. In the continuum limit, i.e., for t large and distances $|y - x|$ not much greater than the diffusive scale $\sqrt{t}$, the discrete expression (A.3) becomes the continuous one (A.2).

For a long time and for fixed (i.e., bounded) initial positions x_i, and arbitrary final positions y_j, using (A.2) allows us to simplify (A.1) to

$$P(\mathbf{x},\mathbf{y},t) \approx (2\pi t)^{-N/2} \det\left(\mathrm{e}^{x_i y_j/t}\right) \exp\left(-\frac{1}{2t}\sum_j y_j^2\right). \qquad \text{(A.4)}$$

Furthermore, the determinant can be evaluated along the lines of the derivation of (2.11). We have indeed, to leading order in the regime where all the variables x_i are small,

$$\det\left(F_j(x_i)\right) \approx \det\left(F_{j,k-1}\right)\Delta_N(\mathbf{x}), \qquad \text{(A.5)}$$

where

$$F_j(x) = \sum_{k\ge 0} F_{j,k} x^k, \qquad F_{j,k} = \frac{F_j^{(k)}(0)}{k!}, \qquad \text{(A.6)}$$

whereas $\Delta_N(\mathbf{x})$ is the Vandermonde determinant of the x_i (see (2.13)). In the present case, $F_j(x) = \mathrm{e}^{x y_j/t}$, and so $F_{j,k} = (y_j/t)^k$. We are thus left with

$$P(\mathbf{x},\mathbf{y},t) \approx \frac{\Delta_N(\mathbf{x})\Delta_N(\mathbf{y})}{(2\pi)^{N/2} G(N+1)\, t^{N^2/2}} \exp\left(-\frac{1}{2t}\sum_j y_j^2\right). \qquad \text{(A.7)}$$

The first quantity of interest is the survival probability $S_N(\mathbf{x},t)$, i.e., the probability that no two trajectories of the N walkers have intersected up to time t. The behavior of this quantity in the long-time regime is obtained by integrating (A.7) over the allowed range of final positions ($-\infty < y_1 < \ldots < y_N < +\infty$). The result is proportional to a Mehta integral of the form (B.3). We thus obtain

$$S_N(\mathbf{x},t) \approx \frac{\sigma_N\, \Delta_N(\mathbf{x})}{G(N+2)\, t^{N(N-1)/4}}, \qquad \text{(A.8)}$$

where σ_N is given by (B.4). The decay exponent $N(N-1)/4$ of the survival probability can be found in [28, 29, 30, 31]. Reference [30] also contains the expression of the prefactor.

The second quantity of interest is the return probability $R_N(\mathbf{x},t)$, i.e., the probability (or probability density) that the walkers return at (or close to) their initial positions and that no two trajectories of the N walkers have intersected up to time t. This quantity can be directly read off from (A.7):

$$R_N(\mathbf{x},t) \approx \frac{\Delta_N^2(\mathbf{x})}{(2\pi)^{N/2} G(N+1)\, t^{N^2/2}}. \qquad \text{(A.9)}$$

The classical analogue of the quantum return probability studied in the body of this work is the return probability conditioned on survival, i.e.,

$$\widetilde{R}_N(\mathbf{x},t) = \frac{R_N(\mathbf{x},t)}{S_N(\mathbf{x},t)}, \qquad \text{(A.10)}$$

whose decay is predicted to be

$$\widetilde{R}_N(\mathbf{x},t) \approx \frac{N!\, \Delta_N(\mathbf{x})}{(2\pi)^{N/2}\, \sigma_N\, t^{N(N+1)/4}}. \qquad \text{(A.11)}$$

The above predictions for both return probabilities seem to be novel.

In the case of continuous-time lattice walks, if the walkers are launched from any N consecutive sites, we have

$$\Delta_N(\mathbf{x}) = G(N+1), \tag{A.12}$$

and so the above expressions read

$$\begin{aligned} S_N(t) &\approx \frac{\sigma_N}{N!\, t^{N(N-1)/4}}, \\ R_N(t) &\approx \frac{G(N+1)}{(2\pi)^{N/2}\, t^{N^2/2}}, \\ \widetilde{R}_N(t) &\approx \frac{G(N+2)}{(2\pi)^{N/2}\, \sigma_N\, t^{N(N+1)/4}}, \end{aligned} \tag{A.13}$$

where σ_N is given by (B.4).

Finally, when the number of walkers becomes large, the above expressions can be further simplified by means of the expansions (C.4) and (C.5). Keeping only leading terms in N^2 in the exponentials, we obtain

$$\begin{aligned} S_N(t) &\sim \left(\frac{N}{2\mathrm{e}^{3/2}t}\right)^{N(N-1)/4}, \\ R_N(t) &\sim \left(\frac{N}{\mathrm{e}^{3/2}t}\right)^{N^2/2}, \\ \widetilde{R}_N(t) &\sim \left(\frac{2N}{\mathrm{e}^{3/2}t}\right)^{N(N+1)/4}. \end{aligned} \tag{A.14}$$

Appendix A.2. Walkers on the half-line

Consider now N independent random walkers on the half-line ($x > 0$), starting at time $t = 0$ from the positions $\mathbf{x} = (x_1, \ldots, x_N)$, with $0 < x_1 < \ldots < x_N$. The probability (or probability density) $P^{\mathrm{w}}(\mathbf{x}, \mathbf{y}, t)$ that the walkers are at the positions $\mathbf{y} = (y_1, \ldots, y_N)$ at time t, with $0 < y_1 < \ldots < y_N$, and that their trajectories have neither intersected nor gone through the origin, is again given by (A.1), albeit with

$$p^{\mathrm{w}}(x, y, t) = \frac{1}{\sqrt{2\pi t}} \left(\mathrm{e}^{-(y-x)^2/(2t)} - \mathrm{e}^{-(y+x)^2/(2t)}\right) \tag{A.15}$$

for Brownian particles, and

$$p^{\mathrm{w}}(x, y, t) = \mathrm{e}^{-t} \left(I_{y-x}(t) - I_{y+x}(t)\right) \tag{A.16}$$

for particles executing continuous-time random walks on the lattice. In the continuum limit, the discrete expression (A.16) again becomes the continuous one (A.15).

For a long time and for fixed (i.e., bounded) initial positions x_i, and arbitrary final positions y_j, the expression (A.1) simplifies to

$$P^{\mathrm{w}}(\mathbf{x}, \mathbf{y}, t) \approx \left(\frac{2}{\pi t}\right)^{N/2} \det\left(\sinh\frac{x_i y_j}{t}\right) \exp\left(-\frac{1}{2t}\sum_j y_j^2\right). \tag{A.17}$$

Furthermore, the determinant can be evaluated along the lines of the derivation of (3.9). In the case where all the functions $F_j(x)$ are odd, (A.5) becomes

$$\det\left(F_j(x_i)\right) \approx \det\left(F_{j,k-1}\right) \prod_i x_i\, \Delta_N(\mathbf{x}^2), \tag{A.18}$$

with

$$F_j(x) = \sum_{k\ge 0} F_{j,k} x^{2k+1}, \qquad F_{j,k} = \frac{F_j^{(2k+1)}(0)}{(2k+1)!}, \tag{A.19}$$

and $\mathbf{x}^2 = (x_1^2, \dots, x_N^2)$. In the present situation, we have $F_j(x) = \sinh(xy_j/t)$, and so $F_{j,k} = (y_j/t)^{2k+1}$. Using (C.10), we obtain

$$P^{\rm w}(\mathbf{x},\mathbf{y},t) \approx 2^N \sqrt{\frac{N!}{\pi^N G(2N+2)}} \frac{\prod_i (x_i y_i) \Delta_N(\mathbf{x}^2)\Delta_N(\mathbf{y}^2)}{t^{N(2N+1)/2}} \times \exp\left(-\frac{1}{2t}\sum_j y_j^2\right). \tag{A.20}$$

The survival probability $S_N^{\rm w}(\mathbf{x},t)$ is now the probability that no trajectory has either crossed the origin or intersected another one up to time t. The behavior of this quantity in the long-time regime is obtained by integrating (A.20) over the allowed range of final positions ($0 < y_1 < \ldots < y_N < +\infty$). The result is proportional to a Mehta integral of the form (B.7). We thus obtain

$$S_N^{\rm w}(\mathbf{x},t) \approx \frac{2^N G(N+2)\prod_i x_i\,\Delta_N(\mathbf{x}^2)}{\sqrt{\pi^N N! G(2N+2)}\, t^{N^2/2}}. \tag{A.21}$$

The return probability $R_N^{\rm w}(\mathbf{x},t)$ is now the probability (or probability density) that the walkers return at (or close to) their initial positions and that no trajectory has either crossed the origin or intersected another one up to time t. This quantity can be directly read off from (A.20):

$$R_N^{\rm w}(\mathbf{x},t) \approx 2^N \sqrt{\frac{N!}{\pi^N G(2N+2)}} \frac{\prod_i x_i^2\,\Delta_N^2(\mathbf{x}^2)}{t^{N(2N+1)/2}}. \tag{A.22}$$

The return probability conditioned on survival scales as

$$\widetilde{R}_N^{\rm w}(\mathbf{x},t) \approx \frac{\prod_i x_i\,\Delta_N(\mathbf{x}^2)}{G(N+1)\, t^{N(N+1)/2}}. \tag{A.23}$$

In the case of continuous-time lattice walks, if the walkers are launched from the first N sites of the half-infinite chain ($x_i = i$), we have

$$\prod_i x_i = N!, \qquad \Delta_N(\mathbf{x}^2) = \sqrt{\frac{G(2N+2)}{2^N N!^3}}. \tag{A.24}$$

The second equality is equivalent to (C.12). The above expressions become

$$\begin{aligned} S_N^{\rm w}(t) &\approx \frac{2^{N/2} G(N+1)}{\pi^{N/2}\, t^{N^2/2}}, \\ R_N^{\rm w}(t) &\approx \sqrt{\frac{G(2N+2)}{\pi^N N!}} \frac{1}{t^{N(2N+1)/2}}, \\ \widetilde{R}_N^{\rm w}(t) &\approx \sqrt{\frac{G(2N+2)}{2^N N!}} \frac{1}{G(N+1)\, t^{N(N+1)/2}}. \end{aligned} \tag{A.25}$$

Finally, when the number of walkers becomes large, the above expressions can be further simplified by means of the expansions (C.4) and (C.5). Keeping only leading terms in N^2 in the exponentials, we obtain

$$\begin{aligned} S_N^{\mathrm{w}}(t) &\sim \left(\frac{N}{\mathrm{e}^{3/2}t}\right)^{N^2/2}, \\ R_N^{\mathrm{w}}(t) &\sim \left(\frac{2N}{\mathrm{e}^{3/2}t}\right)^{N(2N+1)/2}, \\ \widetilde{R}_N^{\mathrm{w}}(t) &\sim \left(\frac{4N}{\mathrm{e}^{3/2}t}\right)^{N(N+1)/2}. \end{aligned} \tag{A.26}$$

Appendix B. Mehta integrals

In this appendix we give the expressions of two of the well-known Mehta integrals [22]. These multiple integrals, which can be derived as limiting values of the Selberg integral and play a central part in random matrix theory, are used in the present work at several places. Reference [34] provides a comprehensive historical overview of the Selberg and related integrals.

• First Mehta integral [22, Eq. (17.6.7)]:

$$\begin{aligned} I_1(N,a,\gamma) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{n=1}^{N}\left(\mathrm{d}x_n\,\mathrm{e}^{-ax_n^2}\right)|\Delta_N(\mathbf{x})|^{2\gamma} \\ &= \frac{(2\pi)^{N/2}}{(2a)^{N(1+(N-1)\gamma)/2}}\prod_{j=1}^{N}\frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)}. \end{aligned} \tag{B.1}$$

We have in particular

$$I_1(N,a,1) = \frac{(2\pi)^{N/2}G(N+2)}{(2a)^{N^2/2}} \tag{B.2}$$

for $\gamma = 1$, where G denotes the Barnes G-function (see Appendix C), and

$$I_1(N,a,1/2) = \frac{(2\pi)^{N/2}\,\sigma_N}{(2a)^{N(N+1)/4}}, \tag{B.3}$$

with

$$\sigma_N = \frac{\sqrt{\Gamma(\frac{N}{2}+1)G(N+2)}}{2^{N(N-3)/4}\pi^{N/4}}, \tag{B.4}$$

for $\gamma = 1/2$.

• Second Mehta integral [22, Eq. (17.6.6)]:

$$\begin{aligned} I_2(N,a,\beta,\gamma) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{n=1}^{N}\left(\mathrm{d}x_n\,|x_n|^{2\beta-1}\mathrm{e}^{-ax_n^2}\right)|\Delta_N(\mathbf{x}^2)|^{2\gamma} \\ &= \frac{1}{a^{N(\beta+(N-1)\gamma)}}\prod_{j=1}^{N}\frac{\Gamma(1+j\gamma)\Gamma(\beta+(j-1)\gamma)}{\Gamma(1+\gamma)}. \end{aligned} \tag{B.5}$$

We have in particular

$$I_2(N,a,3/2,1) = \frac{\sqrt{\pi^N}N!G(2N+2)}{(2a)^{N(2N+1)/2}} \tag{B.6}$$

for $\beta = 3/2$ and $\gamma = 1$, and

$$I_2(N, a, 1, 1/2) = \frac{2^N G(N+2)}{(2a)^{N(N+1)/2}} \tag{B.7}$$

for $\beta = 1$ and $\gamma = 1/2$.

Appendix C. The Barnes G-function and related identities

Let us begin with a brief summary of the main properties of the Barnes G-function (see [35] and [36, Ch. 5.17]). The Euler Γ-function and the Barnes G-function are meromorphic functions of the complex variable z obeying the recursion relations

$$\Gamma(z+1) = z\Gamma(z), \qquad G(z+1) = \Gamma(z)G(z). \tag{C.1}$$

When z is a positive integer, the Γ-function becomes the usual factorial:

$$\Gamma(n+1) = n!, \tag{C.2}$$

whereas the G-function becomes the ‘superfactorial’:

$$G(n+2) = \prod_{k=1}^{n} k! = \prod_{\ell=1}^{n} \ell^{n+1-\ell} = \prod_{1\le i<j\le n+1} (j-i). \tag{C.3}$$

The Γ and G-functions have the following asymptotic expansions as $z \to +\infty$:

$$\ln\Gamma(z+1) = \left(z+\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} + \cdots, \tag{C.4}$$

$$\ln G(z+2) = \left(\frac{z^2}{2} + z + \frac{5}{12}\right)\ln z - \frac{3z^2}{4} - z + \frac{z+1}{2}\ln(2\pi) + \zeta'(-1) + \cdots, \tag{C.5}$$

where $\zeta'(-1) = -0.165421$ (ζ being the Riemann ζ-function).

The following products can be expressed in terms of values of the G-function:

$$\prod_{k=1}^{n}(n-k)! = G(n+1), \tag{C.6}$$

$$\prod_{k=1}^{n}(n+k)! = \frac{G(2n+2)}{G(n+2)}, \tag{C.7}$$

$$\prod_{k=1}^{n}(n+k-1)! = \frac{G(2n+1)}{G(n+1)}, \tag{C.8}$$

$$\prod_{k=1}^{n}(2k)! = \sqrt{2^n n! G(2n+2)}, \tag{C.9}$$

$$\prod_{k=1}^{n}(2k-1)! = \sqrt{\frac{G(2n+2)}{2^n n!}}. \tag{C.10}$$

The above identities allow us to evaluate in closed form the determinants $d_p^{(\mathrm{even})}$ and $d_p^{(\mathrm{odd})}$ introduced in (2.47) and (2.50). We have indeed

$$d_p^{(\mathrm{even})} = (-1)^{p(p-1)/2} \det\left((2k+1)^{-2l}\right)_{0\le k,l\le p-1}$$

$$
\begin{aligned}
&= \prod_{1\le k<l\le p} \left(\frac{1}{(2k-1)^2} - \frac{1}{(2l-1)^2} \right) \\
&= \prod_{1\le k<l\le p} \frac{4(l-k)(l+k-1)}{(2k-1)^2(2l-1)^2} \\
&= \frac{2^{p(p-1)}}{((2p-1)!!)^{2(p-1)}} \prod_{k=1}^{p} \frac{(p-k)!(p+k-1)!}{(2k-1)!} \\
&= 2^{p(3p-2)} \left(\frac{p!}{(2p)!} \right)^{2p-1} \sqrt{\frac{G(2p+2)}{2^p p!}}
\end{aligned}
\tag{C.11}
$$

and

$$
\begin{aligned}
d_p^{(\mathrm{odd})} &= (-1)^{p(p-1)/2} \det \left(k^{-2(l-1)} \right)_{1\le k,l\le p} \\
&= \prod_{1\le k<l\le p} \left(\frac{1}{k^2} - \frac{1}{l^2} \right) \\
&= \prod_{1\le k<l\le p} \frac{(l-k)(l+k)}{k^2 l^2} \\
&= \frac{1}{p!^{2(p-1)}} \prod_{k=1}^{p} \frac{(p-k)!(p+k)!}{(2k)!} \\
&= \frac{1}{p!^{2p-1}} \sqrt{\frac{G(2p+2)}{2^p p!}}.
\end{aligned}
\tag{C.12}
$$

References

[1] Calabrese P and Cardy J 2006 Time dependence of correlation functions following a quantum quench *Phys. Rev. Lett.* **96** 136801

[2] Caux J S and Konik R M 2012 Constructing the Generalized Gibbs ensemble after a quantum quench *Phys. Rev. Lett.* **109** 175301

[3] Fagotti M and Essler F H L 2013 Reduced density matrix after a quantum quench *Phys. Rev. B* **87** 245107

[4] Polkovnikov A, Sengupta K, Silva A and Vengalatorre M 2011 Nonequilibrium dynamics of closed interacting quantum systems *Rev. Mod. Phys.* **83** 863

[5] Eisert J, Friesdorf M and Gogolin C 2015 Quantum many-body systems out of equilibrium *Nature Phys.* **11** 124

[6] Gogolin C and Eisert J 2016 Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems *Rep. Prog. Phys.* **79** 056001

[7] D'Alessio L, Kafri Y, Polkovnikov A and Rigol M 2016 From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics *Adv. Phys.* **65** 239

[8] Borgonovi F, Izrailev F M, Santos L F and Zelevinsky V G 2016 Quantum chaos and thermalization in isolated systems of interacting particles *Phys. Rep.* **626** 1

[9] Peres A 1995 Quantum Theory: Concepts and Methods (Dordrecht: Kluwer)

[10] Peres A 1984 Stability of quantum motion in chaotic and regular systems *Phys. Rev. A* **30** 1610

[11] Jalabert R A and Pastawski H M 2001 Environment-independent decoherence rate in classically chaotic systems *Phys. Rev. Lett.* **86** 2490

[12] Manfredi G and Hervieux P A 2006 Loschmidt echo in a system of interacting electrons *Phys. Rev. Lett.* **97** 190404

[13] Gorin T, Prosen T, Seligman T H and Znidaric M 2006 Dynamics of Loschmidt echoes and fidelity decay *Phys. Rep.* **435** 33

[14] García-Calderón G and Mendoza-Luna L G 2011 Time evolution of decay of two identical quantum particles *Phys. Rev. A* **84** 032106

[15] del Campo A 2011 Long-time behavior of many-particle quantum decay *Phys. Rev. A* **84** 012113

[16] del Campo A 2016 Exact quantum decay of an interacting many-particle system: The Calogero-Sutherland model *New J. Phys.* **18** 015014
[17] Longhi S and della Valle G 2012 Many-particle quantum decay and trapping: The role of statistics and Fano resonances *Phys. Rev. A* **86** 012112
[18] Marchewka A and Granot E 2015 Role of quantum statistics in multi-particle decay dynamics *Ann. Phys.* **355** 348
[19] Stéphan J M and Dubail J 2011 Local quantum quenches in critical one-dimensional systems: Entanglement, the Loschmidt echo, and light-cone effects *J. Stat. Mech.* P08019
[20] Stéphan J M 2017 Return probability after a quench from a domain-wall initial state in the spin-1/2 XXZ chain *J. Stat. Mech.* P103108
[21] Krapivsky P L, Luck J M and Mallick K 2018 Quantum return probability of a system of N non-interacting lattice fermions *J. Stat. Mech.* 023104
[22] Mehta M 1990 *Random Matrices* 2nd ed (London: Academic)
[23] Abramowitz M and Stegun I 1965 Handbook of Mathematical Functions with Formulas (New-York: Dover)
[24] Bender C M and Orszag S A 1978 Advanced Mathematical Methods for Scientists and Engineers (New-York: McGraw-Hill)
[25] Doescher S W and Rice M H 1969 Infinite square-well potential with a moving wall *Am. J. Phys.* **37** 1246
[26] Dean D S, Le Doussal P, Majumdar S N and Schehr G 2016 Non-interacting fermions at finite temperature in a d-dimensional trap: Universal correlations *Phys. Rev. A* **94** 063622
[27] Karlin S and McGregor J 1959 Coincidence probabilities *Pacific J. Math.* **9** 1141
[28] Fisher M E 1984 Walks, walls, wetting, and melting *J. Stat. Phys.* **34** 667
[29] Huse D A and Fisher M E 1984 Commensurate melting, domain walls, and dislocations *Phys. Rev. B* **29** 239
[30] Grabiner D J 1999 Brownian motion in a Weyl chamber, non-colliding particles, and random matrices *Ann. Inst. Henri Poincaré* **35** 177
[31] Katori M and Tanemura H 2007 Noncolliding Brownian motion and determinantal processes *J. Stat. Phys.* **129** 1233
[32] Lindström B 1973 On the vector representation of induced matroids *Bull. London Math. Soc.* **5** 85
[33] Gessel I and Viennot G 1985 Binomial determinants, paths and hook length formulae *Adv. Math.* **58** 300
[34] Forrester P J and Warnaar S O 2008 The importance of the Selberg integral *Bull. Amer. Math. Soc.* **45** 489
[35] Wikipedia, The Free Encyclopedia https://en.wikipedia.org/wiki/Barnes_G-function
[36] NIST *Digital Library of Mathematical Functions* http://dlmf.nist.gov