

# Non-screening of the Cosmological Background in K-mouflage modified gravity

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We describe the effects of the cosmological background on the K-mouflage screening properties of an astrophysical structure. We show that the K-mouflage screening of the spatial gradients of the scalar field, i.e. the screening of the fifth force, happens inside a dynamically generated screening radius. This radius is smaller than the location where the quasistatic approximation, i.e. where the spatial gradients exceed the time derivative, holds. Even though this quasistatic radius is much smaller than the size of the matter overdensity, spatial gradients remain well described by the quasistatic approximation up to the horizon. Cosmologically we find that the time derivatives can remain dominant at redshifts  $z \gtrsim 2$ , when the cosmic web shows a faster growth. Despite the existence of K-mouflage screening, we confirm that the values of the scalar field itself are still dominated by the cosmological background, down to the center of the matter overdensity, and that for instance the time drift of Newton's constant due to the large-scale cosmological evolution highly constrains K-mouflage models.

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## I. INTRODUCTION

Scalar models with derivative actions and a coupling to matter such as K-mouflage [1–4] and Galileon-like theories [5, 6] screen fifth force effects in the presence of matter. This is due to the non-linearities in the kinetic terms of the scalar field. This is sufficient to guarantee that most Solar System tests of gravity are fulfilled by these models. Now that the observation of the equality, up to a very high accuracy, between the speeds of gravity and light has ruled out most Horndeski models with self-tuning properties [7], K-mouflage remains a serious alternative to the  $\Lambda$ -CDM paradigm. Of course, K-mouflage models do not propose a solution to the “old” cosmological constant problem [8], but their peculiar features on the growth of structures are sufficiently compelling to motivate further studies, in particular on the influence on the large-scale cosmological evolution and its backreaction on small-scale properties [9]. This is the case of the time drift of Newton's constant, due to the absence of screening by the K-mouflage mechanism of the time dependence of the scalar field. In this paper, we characterise this property by going beyond the usual quasistatic approximation, which assumes that any slow dependence on time of the background scalar field can be added to the static profile associated with dense objects. We analyse the nonlinear regime with a fully time-dependent cosmological solution describing the matter era. We show how, when screening of the spatial gradients occurs inside an overdensity, the time drift itself is not affected.

In section II, we define the K-mouflage models by the nonlinear Klein-Gordon equation that governs the evolution of the scalar field. In section III, we consider the

situation with nonscreening, which corresponds to a standard kinetic term, and we study how the cosmological background propagates down to the center of the overdensity while spatial gradients converge to the quasistatic limit on subhorizon scales. In section IV, we investigate how the situation is modified by the screening effects due to the nonlinearities associated with large field gradients. In section V we conclude.

## II. K-MOUFFLAGE MODELS

### A. The dynamics

The scalar field  $\phi$  in K-mouflage models obeys the nonlinear Klein-Gordon equation [10]

$$\nabla_\mu [K' \nabla^\mu \phi] = \frac{\beta \rho}{M_{\text{Pl}}}, \quad (1)$$

where  $\nabla_\mu$  is the covariant derivative with respect to the Einstein-frame metric  $g_{\mu\nu}$ ,  $\rho$  the matter density and  $\beta$  the coupling constant. The function  $K$  is a function of the kinetic term  $\chi = -\frac{(\partial\phi)^2}{2\mathcal{M}^4}$ , where  $\mathcal{M}^4$  is of the order of the dark-energy scale.

For the cosmological background, or on large cosmological scales, matter density fields and the scalar field are exactly or almost homogeneous, so that  $\chi$  is dominated by the time derivative and  $\chi > 0$ . In the vicinity of static compact objects, such as stars, or in high-density regions such as the cores of galaxies, spatial gradients dominate over time derivatives and  $\chi < 0$ . Thus, the high-density cosmological background associated with the early Universe corresponds to  $\chi \rightarrow +\infty$ , whereas the high-density

regions associated with quasistatic astrophysical objects correspond to  $\chi \rightarrow -\infty$ . This corresponds to two unrelated screening regimes, if the function  $K$  is nonlinear for both large positive and negative argument.

The faster-than-linear growth of  $K$  for  $\chi \rightarrow +\infty$ , i.e.  $K' \rightarrow +\infty$ , ensures that the scalar field energy density is negligible at high redshift as compared with the matter density, so that one recovers the standard cosmology [10]. For small values of  $\chi$ , associated with low redshifts, we expand  $K = -1 + \chi + \dots$  (the unit factors define the normalization of  $\mathcal{M}^4$  and  $\phi$ ) and we recover a canonically normalized scalar field (the linear term) with a cosmological constant (the constant term  $-1$ ).

In a quasistatic high-density region, or close to a compact astrophysical object, spatial gradients become large and a screening mechanism also comes into play if  $K'$  becomes large for large negative  $\chi$  [1, 3]. This slows down the growth of the scalar field gradients with the rise of the matter density. For instance, in a static spherically symmetric overdensity, Eq.(1) gives after one integration an equation of the form  $K'd\phi/dx \propto M(<x)/x^2$ , where  $M(<x)$  is the mass inside the radius  $x$ , so that the scalar field gradient is suppressed by a factor  $1/K'$ . This gives rise to the K-mouflage screening mechanism and allows the fifth force to become negligible as compared with the Newtonian gravity in small and high-density regions.

If we assume that such a local picture fully describes the behavior of the scalar field in small-scale high-density regions, we could expect that in a similar fashion the large value of  $K'$  should suppress all derivatives of  $\phi$ , the time derivative as well as spatial derivatives. This is for instance the behavior that is obtained by multiplying  $K'$  in Eq.(1) by a large constant factor. Then, the scalar field at the center of a high matter overdensity should decouple from the cosmological background and no longer evolve inside a static matter halo. It turns out that this picture is not correct.

In this paper, we investigate in more details this issue, using simple power-law density profiles, for which we can derive explicit analytical results. We find that although spatial gradients are well predicted by the quasistatic approximation on subhorizon scales, the scalar field itself does not truly decouple from the cosmological background. Its time derivative remains greater than the spatial gradients down to scales much below the size of the matter overdensity, and its value at the center closely follows the drift of the cosmological background.

## B. Rescaled variables

Neglecting the metric fluctuations from the Friedmann-Lemaître-Robertson-Walker (FLRW) background, with scale factor  $a$ , the nonlinear Klein-Gordon equation (1) reads

$$-a^{-4}\partial_\tau(a^2K'\partial_\tau\phi) + a^{-2}\nabla(K'\nabla\phi) = \frac{\beta\rho}{M_{\text{Pl}}}, \quad (2)$$

where  $\tau$  is the conformal time and  $\nabla = \partial_{\mathbf{x}}$  the gradient with respect to the comoving coordinate  $\mathbf{x}$ . For simplicity, we consider an Einstein-de Sitter universe, i.e. matter dominated, with

$$a = \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{\tau}{\tau_0}\right)^2, \quad t_0 = \frac{2}{3H_0}, \quad \tau_0 = \frac{2}{H_0}, \quad \bar{\rho} = \frac{\bar{\rho}_0}{a^3}, \quad (3)$$

where  $t_0$  is the age of the universe at redshift  $z = 0$  and  $\tau_0$  the conformal time today. It is convenient to introduce the dimensionless coordinates

$$\tilde{\tau} = \frac{\tau}{\tau_0}, \quad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{\tau_0}, \quad \tilde{\phi} = \frac{\phi}{M_{\text{Pl}}}. \quad (4)$$

Then, the Klein-Gordon equation (2) reads

$$-\partial_{\tilde{\tau}}(K'\partial_{\tilde{\tau}}\tilde{\phi}) - \frac{4}{\tilde{\tau}}K'\partial_{\tilde{\tau}}\tilde{\phi} + \tilde{\nabla}(K'\tilde{\nabla}\tilde{\phi}) = 12\beta\frac{\rho}{\bar{\rho}\tilde{\tau}^2} \quad (5)$$

and the argument of the kinetic function  $K$

$$\chi = \frac{1}{2\tilde{\tau}^4} \left[ (\partial_{\tilde{\tau}}\tilde{\phi})^2 - (\tilde{\nabla}\tilde{\phi})^2 \right], \quad (6)$$

with the choice of normalization

$$\mathcal{M}^4 = \frac{M_{\text{Pl}}^2 H_0^2}{4}. \quad (7)$$

In the following we will omit the tildes and only work with these rescaled quantities. In this paper we focus on the response of the scalar field to the cosmological background and matter overdensities. Therefore, we discard the backreaction of the scalar field onto the cosmological expansion history and the formation of matter overdensities. This also corresponds to a small coupling constant  $\beta \ll 1$ . This is actually the case of interest as observations show that the fifth force must remain subdominant as compared with Newtonian gravity and we must recover the standard cosmological expansion up to an accuracy of a few percents at low redshifts. We will study the evolution of the scalar field for a given cosmological background, defined by the Einstein-de Sitter solution (3), and for given matter overdensities.

## III. STANDARD KINETIC TERM

### A. Cosmological background

For the homogeneous cosmological background where  $\rho = \bar{\rho}$  we have for  $\bar{\phi}(\tau)$  that Eq.(5) can be integrated once to give

$$\bar{K}'\frac{d\bar{\phi}}{d\tau} = -\frac{4\beta}{\tau}. \quad (8)$$

In this paper, we are not interested in the screening of the cosmological background at high redshifts. Therefore, we can take  $\bar{K}'$  to be constant for the cosmological

background and choose the normalization  $\bar{K}' = 1$ . This corresponds to kinetic functions with  $K' = 1$  for  $\chi \geq 0$ , or to the standard kinetic term  $K(\chi) = \chi$ . This gives the cosmological background solution

$$\bar{K}' = 1 : \quad \frac{d\bar{\phi}}{d\tau} = -\frac{4\beta}{\tau}, \quad \bar{\phi} = -4\beta \ln \tau. \quad (9)$$

## B. General linear solution

In this section, we consider the case of the standard kinetic function, where  $K' = 1$  for all positive and negative  $\chi$ . Then, the Klein-Gordon equation (5) is linear and reads as

$$-\partial_\tau^2 \phi - \frac{4}{\tau} \partial_\tau \phi + \nabla^2 \phi = 12\beta \frac{\rho}{\bar{\rho}\tau^2}. \quad (10)$$

To work with functions that vanish at infinity, we subtract the cosmological background by defining

$$\phi = \bar{\phi} + \varphi, \quad \rho = \bar{\rho}(1 + \delta), \quad (11)$$

where  $\varphi$  and  $\delta$  are not necessarily small but vanish at large distances. Indeed, in this paper we are interested in the formation of nonlinear structures, with a finite size, amidst the cosmological background. Then, the deviation  $\varphi$  obeys the linear equation

$$\mathcal{O} \cdot \varphi = 12\beta \frac{\delta}{\tau^2}, \quad (12)$$

where we have introduced the linear operator  $\mathcal{O}$  defined by

$$\mathcal{O} = -\partial_\tau^2 - \frac{4}{\tau} \partial_\tau + \nabla^2. \quad (13)$$

Using the associated retarded Green function

$$\mathcal{O} \cdot \mathcal{G}(\mathbf{x}, \tau; \mathbf{x}', \tau') = \delta_D(\mathbf{x} - \mathbf{x}') \delta_D(\tau - \tau'), \quad (14)$$

we can solve the linear equation (12) as

$$\varphi(\mathbf{x}, \tau) = 12\beta \int d\mathbf{x}' d\tau' \mathcal{G}(\mathbf{x}, \tau; \mathbf{x}', \tau') \frac{\delta(\mathbf{x}', \tau')}{\tau'^2}. \quad (15)$$

Solving Eq.(14) by using its Fourier transform, we obtain

$$\begin{aligned} \mathcal{G}(\mathbf{x}, \tau; \mathbf{x}', \tau') &= \theta(\tau - \tau') \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \frac{k\tau'^3}{\tau} \\ &\times [n_1(k\tau') j_1(k\tau) - j_1(k\tau') n_1(k\tau)], \end{aligned} \quad (16)$$

where  $\theta$  is the Heaviside function,  $j_1$  and  $n_1$  are the spherical Bessel functions of the first and second kind. Substituting the explicit expressions of  $j_1$  and  $n_1$  in terms of cosines and sines, we can easily check that in the limit of small lengths and timescales,  $|\mathbf{x} - \mathbf{x}'| \rightarrow 0$ ,  $\tau - \tau' \rightarrow 0$ ,  $k \rightarrow \infty$ , we recover the usual Green function of the 3D wave equation [11],

$$\mathcal{G} \rightarrow -\frac{\theta(\tau - \tau') \delta_D(|\mathbf{x} - \mathbf{x}'| - (\tau - \tau'))}{4\pi |\mathbf{x} - \mathbf{x}'|}. \quad (17)$$

This corresponds to the limit where the Hubble friction term in Eq.(14) is negligible.

## C. Self-similar matter density profiles

We now investigate how the scalar field reacts to the formation of an overdense region. We consider a class of simple cases where we can obtain explicit expressions, the self-similar spherical power-law density profiles

$$\delta(\mathbf{x}, \tau) = \left( \frac{x}{x_s(\tau)} \right)^{-\gamma}, \quad x_s(\tau) = x_* \tau^\alpha, \quad x_* \ll 1. \quad (18)$$

In the rescaled coordinates (4), the time  $\tau$  runs over  $0 \leq \tau \leq 1$ , and the condition  $x_* \ll 1$  ensures that the overdensity always remains far inside the Hubble radius.

The profile (18) corresponds to a halo of inner density slope  $\gamma$  and size  $x_s(\tau)$ , which grows with time in a self-similar fashion. Such a solution can be achieved for instance by the collapse of a polytropic gas with a power-law initial linear density contrast profile [12]. Then, the pressure built in the high-density core of the halo balances the gravitational pull and one obtains a static profile in physical coordinates  $\mathbf{r} = a\mathbf{x} \propto \tau^2 \mathbf{x}$ . This implies the following relation between the exponents  $\alpha$  and  $\gamma$

$$\alpha = \frac{6}{\gamma} - 2, \quad 1 < \gamma < 3, \quad \text{hence } 0 < \alpha < 4. \quad (19)$$

The lower bound  $\gamma > 1$  corresponds to the fact that for shallower slopes the core does not converge to a static profile. The mass that keeps collapsing at large radii at later times is too large and cannot be stabilized, so it continuously redistributes matter down to the center and the density at a given physical radius keeps growing with time. The upper bound  $\gamma < 3$  corresponds to the limit of a finite collapsed mass with negligible or no matter at outer radii; then,  $\alpha = 0$  and no more comoving shells turn around, i.e. decouple from the background cosmological flow and start collapsing, falling towards the central overdensity.

One can derive exact self-similar solutions of the Newtonian gravitational collapse, for both collisional and collisionless matter [12–14]. The self-similarity means that the nonlinear density, velocity, and pressure profiles at different times are identical up to a rescaling of the radial coordinates and of the characteristic density, velocity and pressure. This symmetry allows one to transform the 2D problem, which involves partial differential equations over time and radius, into a 1D problem, which involves ordinary differential equations over a radial coordinate. This enables detailed analytical studies. The profile (18) is a simple approximation to such solutions, where we extend to all radii the power-law behavior of the density contrast that is only reached in the nonlinear core of the exact solutions. In all cases for  $x \gg x_s$  we simply recover the background density  $\bar{\rho}$  for the density  $\rho$ . Finally as we are not interested in the formation of the matter overdensity itself, we could extend the range of  $\gamma$  to  $0 < \gamma < 3$ .

The profile (18) is sufficient for our purposes, since we are not interested in building an exact solution to the

gravitational collapse of matter overdensities. Instead, we only wish to study how the scalar field reacts to the formation of matter overdensities. The power-law form (18) allows us to derive explicit analytical results for a realistic range of density profiles, parametrized by the

exponent  $\gamma$ .

Thanks to the simple form of the profile (18), we can perform the integrations in Eq.(15) and we obtain for  $0 < x < \tau$

$$\begin{aligned} \varphi(x, \tau) = & \frac{12\beta x_*^\gamma}{(\gamma-2)(\gamma-3)} \tau^{\alpha\gamma-2} x^{2-\gamma} + \frac{6\beta x_*^\gamma}{\Gamma(\gamma-1)} \left\{ -(\alpha\gamma+1)\Gamma(\gamma-5)\tau^{\alpha\gamma-4} x^{4-\gamma} \left[ (\gamma-5)\frac{\tau}{x} \left( {}_2F_1\left(1, 1-\alpha\gamma; 5-\gamma; -\frac{x}{\tau}\right) \right. \right. \right. \\ & \left. \left. \left. - {}_2F_1\left(1, 1-\alpha\gamma; 5-\gamma; \frac{x}{\tau}\right)\right) + {}_2F_1\left(1, 1-\alpha\gamma; 6-\gamma; -\frac{x}{\tau}\right) + {}_2F_1\left(1, 1-\alpha\gamma; 6-\gamma; \frac{x}{\tau}\right) \right] + \frac{\pi\Gamma(\alpha\gamma)(\alpha\gamma+1)}{\Gamma(\alpha\gamma+4-\gamma)\sin(\gamma\pi)} \tau^{\alpha\gamma-\gamma+1} \right. \\ & \left. \times x^{-1} \left[ \left(1 + \frac{x}{\tau}\right)^{\alpha\gamma+3-\gamma} - \left(1 - \frac{x}{\tau}\right)^{\alpha\gamma+3-\gamma} - (\alpha\gamma+5-\gamma) \left( \left(1 + \frac{x}{\tau}\right)^{\alpha\gamma+4-\gamma} - \left(1 - \frac{x}{\tau}\right)^{\alpha\gamma+4-\gamma} \right) \right] \right\}. \end{aligned} \quad (20)$$

The solution is not analytic at  $x = \tau$ . The solution for  $x > \tau$  can also be expressed in terms of hypergeometric functions but we do not give its expression here as we focus on subhorizon scales. This explicitly shows the critical role played by the horizon,  $x = \tau$ , which is expected on general grounds. Indeed, we typically expect the scalar field to relax inside the horizon, where information has time to propagate, but not beyond the horizon. Expanding Eq.(20) in  $x/\tau$ , we obtain the solution

$$\begin{aligned} \varphi(x, \tau) = & \varphi_s(x, \tau) \left[ 1 + \left(\frac{x}{\tau}\right)^2 + \dots \right] \\ & + \varphi_s(x = \tau, \tau) \left[ 1 + \left(\frac{x}{\tau}\right)^2 + \dots \right], \end{aligned} \quad (21)$$

where the dots stand for higher orders in  $(x/\tau)^2$ , and we omitted numerical factors except for the first term. We introduced the leading term  $\varphi_s$ , given by the first term in Eq.(20),

$$\begin{aligned} \varphi_s(x, \tau) = & \frac{12\beta x_*^\gamma}{(\gamma-2)(\gamma-3)} \tau^{\alpha\gamma-2} x^{2-\gamma} \\ = & \frac{12\beta}{(\gamma-2)(\gamma-3)} \frac{x^2}{\tau^2} \delta(x, \tau), \end{aligned} \quad (22)$$

and

$$\varphi_s(x = \tau, \tau) \sim \beta \delta(\tau, \tau) = \beta \left( \frac{x_s(\tau)}{\tau} \right)^\gamma \ll 1. \quad (23)$$

The term  $\varphi_s(x, \tau)$  actually corresponds to the quasistatic approximation, where we only keep the spatial derivatives in the Klein-Gordon equation (12). Indeed, we can check that it obeys

$$\nabla^2 \varphi_s = 12\beta \frac{\delta}{\tau^2}. \quad (24)$$

We can check that the spatial gradients of the exact solution (21) are governed by the quasistatic solution at small radii because  $\gamma > 0$ ,

$$x \ll \tau: \quad \nabla \varphi \simeq \nabla \varphi_s + \varphi_s(\tau, \tau) \frac{x}{\tau^2} \simeq \nabla \varphi_s \propto x^{1-\gamma}. \quad (25)$$

However, the Poisson equation (24) only defines  $\varphi_s$  up to a constant, if we do not add boundary conditions at large radii. The explicit solution (20) shows that such a term is indeed generated and can be explicitly calculated. It becomes time dependent, following the slowly evolving matter overdensity, and its magnitude is of order  $\varphi_s(x = \tau, \tau)$ , as may be expected (since there are no other scales in the problem).

Indeed, as we noticed above, the scalar field cannot have the time to relax on scales greater than the horizon,  $x > \tau$ . Therefore, the quasistatic solution (22), which corresponds to a fully relaxed solution (only exactly reached if we freeze the Hubble expansion and the growth of the matter overdensity, while waiting an infinite time for the scalar field to settle down), can only apply up to the horizon,  $x \lesssim \tau$ . Then, we can expect it to break down at  $x \simeq \tau$ . This indeed happens, through both the higher-order corrections in the first bracket in (21) and the additional term associated with the second bracket. The magnitude of these terms is set by the condition that they become of the same order as the relaxed quasistatic approximation at  $x \simeq \tau$ .

The time derivative is

$$\begin{aligned} x \ll \tau: \quad \partial_\tau \varphi \simeq & \partial_\tau \varphi_s + \frac{d}{d\tau} \varphi_s(x = \tau, \tau) \\ \sim & \tau^{\alpha\gamma-3} x^{2-\gamma} + \tau^{\alpha\gamma-1-\gamma}. \end{aligned} \quad (26)$$

For small radii,  $x \ll \tau$ , for  $\gamma < 2$  it is dominated by the second term and converges to a nonzero value, whereas for  $\gamma > 2$  it is governed by the first term and goes to infinity. Comparing with Eq.(25), we can see that spatial gradients dominate over time derivatives at small radii if  $\gamma > 1$ ,

$$\gamma > 1: \quad |\nabla \varphi| \gg |\partial_\tau \varphi| \quad \text{for } x \ll \tau. \quad (27)$$

For shallower density profiles,  $\gamma < 1$ , the time derivative associated with the second term in Eq.(21) dominates. This means that there is no true quasistatic regime in this case, in the sense that the kinetic term  $\chi$  is always dominated by time derivatives.

For general modified-gravity scenarios involving an additional scalar field, the quasistatic approximation is usually understood as  $|\nabla\varphi| \gg |\partial_\tau\varphi|$ , that is, the spatial gradient of the scalar field perturbation is greater than its time derivative. Assuming  $\nabla\varphi \sim \varphi/x$  and  $\partial_\tau\varphi \sim \mathcal{H}\varphi \sim \varphi/\tau$ , one naturally expects this quasistatic regime to hold on subhorizon scales,  $x \ll \tau$ . Of course, this also requires that the sound horizon of the scalar field is of the order of the Hubble radius [15], i.e. its propagation speed is of order unity, so that the scalar field has the time to relax on scales  $x \ll \tau$ . The validity of this quasistatic regime on subhorizon scales has been checked for various modified-gravity scenarios, from analytical studies and numerical simulations, at both the linear [16] and nonlinear [17–19] levels.

The condition  $\gamma > 1$  in (27) shows that for cosmological structures that grow too fast this quasistatic regime may not be reached, even though the gradients of the scalar field are already well described by the quasistatic approximation. In practice, such regimes of fast growth may only occur in transient events, such as mergings of collapsed halos. On the other hand, in Cold Dark Matter cosmologies, the variance of the linear matter density fluctuations behaves as  $\sigma^2(x, z) \propto a^2 x^{-(n+3)}$ , where  $n$  runs from 1 to  $-3$  from large to small scales, and  $n \simeq -2$  on galaxy scales. This gives for the scale  $x_{\text{NL}}(\tau)$  that enters the nonlinear regime

$$x_{\text{NL}}(\tau) \propto \tau^{4/(n+3)}, \quad \text{hence } \alpha = \frac{4}{n+3}. \quad (28)$$

In the stable-clustering ansatz [20], this gives a slope in the nonlinear regime for the two-point correlation function

$$x \ll x_{\text{NL}} : \quad \xi(x) \propto x^{-3(n+3)/(n+5)}, \quad \text{hence } \gamma = \frac{3(n+3)}{n+5}. \quad (29)$$

These exponents  $\alpha$  and  $\gamma$  satisfy the relationship (19). The stable-clustering ansatz (29) is not very accurate [21], and in practice it has been replaced by halo models [22], or numerical simulations. However, it suggests that for  $n \leq -2$  and for redshifts  $z \gtrsim 2$  the quasistatic condition  $|\nabla\varphi| \gg |\partial_\tau\varphi|$  may not be always fulfilled as the cosmic web shows a fast build-up. On the other hand, as the fifth force (i.e. the scalar field gradients) remains well predicted by the quasistatic approximation and the impact of dark energy typically becomes negligible at high redshifts, these deviations from the usual quasistatic condition are unlikely to have important effects.

The full solution to the Klein-Gordon equation (10) is  $\phi = \bar{\phi} + \varphi$ . The background term  $\bar{\phi}$  does not contribute to the spatial gradients but it contributes to the time derivative. In particular, for  $\gamma < 2$  and  $\tau \sim 1$  it dominates over the time derivative  $\partial_\tau\varphi$  at small radii, and for all  $\gamma$  it dominates for  $x \sim \tau$ . This means that the spatial gradients  $\nabla\phi$  dominate over the time derivative  $\partial_\tau\phi$  over a smaller range than in (27). Comparing Eqs.(9) and (25) we obtain

$$\gamma > 1 : \quad |\nabla\phi| \gg |\partial_\tau\phi| \quad \text{for } x \ll x_{\text{qs}}(\tau), \quad (30)$$

with

$$x_{\text{qs}}(\tau) = x_s(\tau) \left( \frac{x_s(\tau)}{\tau} \right)^{1/(\gamma-1)} \ll x_s(\tau), \quad (31)$$

where  $x_s(\tau)$  is the size of the overdensity, defined in Eq.(18). As the overdense region must always remain far inside the Hubble radius,  $x_s(\tau) \ll \tau$ , we find that  $x_{\text{qs}} \ll x_s$ . Thus, the fully quasistatic regime, defined as  $|\nabla\phi| \gg |\partial_\tau\phi|$ , only applies far inside the overdense region. This is consistent with the fact that clusters of galaxies are not screened, as found in [23].

In the outer parts,  $x_{\text{qs}} \ll x \ll x_s$ , where the density contrast is already much greater than unity and the density profile has converged to its static limit in physical coordinates, the scalar field  $\phi$  has not yet converged to a full quasistatic regime in the sense that  $|\nabla\phi| \ll |\partial_\tau\phi|$ . However, its spatial gradients have already converged to the quasistatic prediction, in fact as soon as  $x \ll \tau$ , that is, far beyond the size of the overdensity. For  $\gamma < 2$ , the value at the center of the scalar field is dominated by the background,

$$\gamma < 2 : \quad \phi(0) = \varphi(0) + \bar{\phi} \simeq \bar{\phi} = -4\beta \ln \tau, \quad (32)$$

whereas for  $\gamma > 2$  it is dominated by the quasistatic solution  $\varphi_s$ , which goes to infinity. In realistic cases, the matter density and the scalar field remain finite inside collapsed structures and the central value of the scalar field follows the cosmological drift.

The two conditions  $|\nabla\varphi| \gg |\partial_\tau\varphi|$  and  $|\nabla\phi| \gg |\partial_\tau\phi|$  define two different quasistatic regimes. The first condition, which has a greater range of validity, is often used to study linear perturbations. However, once we take into account nonlinearities and screening mechanisms, the second condition is more adequate, as it is a necessary condition for local screening of the fifth force and for a local analysis, where the computation of the fifth force does not depend on the cosmological background and the history of the scalar field evolution.

Thus, even in the simple case where the kinetic term  $K'$  is constant and the Klein-Gordon equation is linear, the quasistatic limit is not so simple. As expected, spatial gradients converge to the quasistatic prediction as soon as  $x \ll \tau$ , i.e. inside the horizon. Indeed, as the scalar field propagation speed is unity, it has time to relax and follow the slow cosmological evolution of the density field on subhorizon scales. The same convergence to the quasistatic limit was found in studies of modified-gravity models that display the Vainshtein mechanism, which also involves a wave equation and a similar derivative screening [19].

However, time derivatives remain dominant down to the much smaller radius  $x_{\text{qs}}$ , far inside the nonlinear overdense region, where they are dominated by the cosmological background. If  $\gamma < 2$  and the quasistatic solution is finite at the center, which is the case in realistic matter overdensities, the value of the scalar field at the center remains governed by the cosmological background.

This shows that the quasistatic approximation predicts the spatial gradients, hence the fifth force, with great accuracy on all subhorizon scales. However, the scalar field does not decouple from the cosmological background, except at the very center for steep density profiles in the particular case where it becomes infinite. This also shows that both the nonlinear transition and the quasistatic regime of the scalar field differ from their counterparts for the matter density field.

#### D. Static compact object

The power-law profiles (18) allowed us to study the evolution of the scalar field for a variety of matter density profile exponents and for cosmological structures that keep growing with time. It is also interesting to consider small-scale structures that no longer grow, with a constant matter density. This corresponds to compact objects such as stars, the Solar System, or an isolated galaxy. Thus, we consider the top-hat density profiles

$$\tau > \tau_* : \quad \delta(\mathbf{x}, \tau) = \theta\left(\frac{r_*}{\tau^2} - x\right) \frac{\tau^6}{\tau_*^6} \quad \text{with } r_* \ll \tau_*^3, \quad (33)$$

and  $\delta = 0$  for  $\tau < \tau_*$ . This corresponds to matter overdensities that form at time  $\tau_*$ , with a fixed physical radius  $r_*$  and density  $\rho_* \sim \bar{\rho}(\tau_*)$ , so that  $\delta$  grows as  $a^3$  at later times. The condition  $r_* \ll \tau_*^3$  means that the structure is far inside the Hubble radius at formation time. From Eq.(15) we obtain the solution as

$$\begin{aligned} \varphi(x, \tau) &= \frac{24\beta}{\pi\tau\tau_*^6} \int_{\tau_*}^{\tau} d\tau' \tau'^7 \int_0^{\infty} dk \frac{\sin(kx)}{kx} \\ &\times [\sin(k\frac{r_*}{\tau'^2}) - k\frac{r_*}{\tau'^2} \cos(k\frac{r_*}{\tau'^2})] \\ &\times [n_1(k\tau')j_1(k\tau) - j_1(k\tau')n_1(k\tau)]. \end{aligned} \quad (34)$$

We could not derive a simple explicit expression for the profile of the scalar field, but we can obtain the value at the center, which at leading order over  $r_*$  reads as

$$\varphi(0, \tau) \simeq -6\beta \frac{r_*^2}{\tau_*^6}. \quad (35)$$

Thus, as for the self-similar profiles in Eq.(32), we find that the scalar field closely follows the cosmological drift with  $\phi(0) \simeq \bar{\phi}$ .

We can now check that  $|\varphi(0)| \sim |\varphi_s(x = \tau, \tau)|$ , in agreement with the expansion (21) that we explicitly derived for the power-law profiles. For the top-hat profile (33), the quasistatic solution that corresponds to Eq.(22), normalized to zero at the center, reads as

$$\begin{aligned} 0 < x < \frac{r_*}{\tau^2} : \quad \varphi_s(x) &= \frac{2\beta\tau^4 x^2}{\tau_*^6}, \\ x > \frac{r_*}{\tau^2} : \quad \varphi_s(x) &= \frac{6\beta r_*^2}{\tau_*^6} - \frac{4\beta r_*^3}{\tau^2 \tau_*^6 x}. \end{aligned} \quad (36)$$

This gives  $\varphi_s(x = \tau, \tau) \simeq 6\beta r_*^2 / \tau_*^6$ , which is of the same order of magnitude as (35). This confirms the general behaviors found in section III C for the power-law profiles.

## IV. NONLINEAR KINETIC TERM

### A. Screening radius and quasistatic solution

We will now consider the impact of the nonlinear K-mouflage screening mechanism. As recalled in the introduction, the effects of the nonlinearity of the kinetic function  $K$  on the cosmological background and on small-scale astrophysical structures are independent as they are related to the two separate regimes  $\chi \rightarrow +\infty$  and  $\chi \rightarrow -\infty$ . The nonlinear impact on the cosmological background is simple to analyze [10, 23], and follows from the nonlinear ordinary differential equation (8). In this paper, we are interested in the nonlinearities that occur in small-scale high-density environments, associated with large negative  $\chi$ , that also screen the fifth force in the Solar System. Therefore, we keep  $K' = 1$  for positive  $\chi$  and focus on the nonlinear screening associated with large spatial gradients of the scalar field. More precisely, we consider the case where the kinetic function  $K'$  remains constant and equal to unity over all  $\chi > \chi_{\text{sc}}$ , with  $-\chi_{\text{sc}} \gg 1$ ,

$$\chi > \chi_{\text{sc}} : \quad K'(\chi) = 1, \quad \chi < \chi_{\text{sc}} : \quad K'(\chi) \gg 1. \quad (37)$$

The threshold  $\chi_{\text{sc}}$  determines the boundary  $x_{\text{sc}}(\tau)$  of the screened region, where  $K' \gg 1$  and the fifth force is damped by the K-mouflage screening mechanism,

$$\chi = \chi_{\text{sc}} \quad \text{at } x = x_{\text{sc}}(\tau). \quad (38)$$

In any case, for non-constant  $K'$  the Klein-Gordon equation (5) becomes nonlinear and we must analyse the full equation, which can be written as

$$\begin{aligned} -\partial_\tau^2 \phi - \frac{4}{\tau} \partial_\tau \phi + \nabla^2 \phi &= \frac{12\beta}{K'} \frac{1 + \delta}{\tau^2} + \partial_\tau(\ln K') \partial_\tau \phi \\ &\quad - \nabla(\ln K') \cdot \nabla \phi. \end{aligned} \quad (39)$$

For a constant  $K'$ , we recover the linear equation (10). For a nonstandard kinetic term, new source terms appear on the right-hand side, which involve derivatives of  $K'$ , while the left-hand side is identical to the linear equation (10) with the same linear operator  $\mathcal{O}$  defined in (13). We will solve this nonlinear equation of motion in a two-step procedure. We will first consider the right-hand side as an external source and obtain  $\phi$  from the same Green function (16) as in the linear case. Then, we will implement the self-consistency condition that states that the source term is given in terms of  $\phi$  by the right-hand side of Eq.(39).

However, it is useful to first consider the quasistatic solution. Indeed, as for the linear case studied in section III, we can anticipate that at small radii the radial

profile of the scalar field will be determined by the quasistatic solution. Thus, we define the quasistatic solution  $\varphi_s$  by

$$\nabla(K'\nabla\varphi_s) = 12\beta\frac{\delta}{\tau^2}. \quad (40)$$

This generalizes to the nonlinear case the previous equation (24). As in the linear case, we separate the source  $\delta$  associated with the matter overdensity from the unit factor of the term  $(1 + \delta)$ , which is related to the mean cosmological background, and we only keep the spatial derivatives in Eq.(39). For a spherically symmetric overdensity, integrating this nonlinear Poisson equation once, we obtain

$$K'\frac{d\varphi_s}{dx} = \frac{12\beta}{x^2\tau^2} \int_0^x dx x^2\delta \quad (41)$$

with

$$\chi_s = -\frac{1}{2\tau^4} \left( \frac{d\varphi_s}{dx} \right)^2. \quad (42)$$

At large radii  $x > x_{sc}$ , where  $\chi > \chi_{sc}$ , we have  $K' = 1$  and we obtain the explicit expression

$$x > x_{sc} : \frac{d\varphi_s}{dx} = \frac{12\beta}{x^2\tau^2} \int_0^x dx x^2\delta, \quad (43)$$

independently of the nonlinear behavior at smaller radii. For the self-similar density profile (18) this gives

$$\frac{d\varphi_s}{dx} = \frac{12\beta}{3-\gamma} x_*^\gamma \tau^{\alpha\gamma-2} x^{1-\gamma} \sim \beta \frac{x}{\tau^2} \delta \quad (44)$$

and

$$\chi_s = -\frac{1}{2} \left( \frac{12\beta}{3-\gamma} \right)^2 x_*^{2\gamma} \tau^{2\alpha\gamma-8} x^{2-2\gamma} \sim -\beta^2 \frac{x^2}{\tau^8} \delta^2, \quad (45)$$

which coincide with the results obtained from (22) in the case of the standard kinetic term.

In this paper, we investigate whether the nonlinearity of the kinetic function can decouple small-scale structures from the cosmological background. Therefore, we consider the case  $\gamma > 1$ , where the gradient  $d\varphi_s/dx$  and the magnitude of the argument  $\chi_s$  of the kinetic function grow at smaller radii, so that the core of the overdensity enters the nonlinear screening regime. The threshold  $\chi_{sc}$  is reached by  $\chi$  at the radius  $x_{sc}(\tau)$ , given by

$$x_{sc}(\tau) = x_{qs}(\tau) \left( \frac{12\beta}{(3-\gamma)\sqrt{-2\chi_{sc}\tau^3}} \right)^{1/(\gamma-1)}. \quad (46)$$

Since  $-\chi_{sc} \gg 1$ , at late times  $\tau \sim 1$  the screening radius  $x_{sc}$  is far inside the quasistatic region  $x_{qs}$ . However, at early times this is not the case anymore as  $x_{sc}/x_{qs}$  grows and becomes of order unity at the time  $\tau_{sc}$  given by

$$\tau_{sc} = \left( \frac{12\beta}{(3-\gamma)\sqrt{-2\chi_{sc}}} \right)^{1/3}, \quad (47)$$

which is independent of  $x_*$ . This provides a small-time cutoff, as for earlier times the quasistatic approximation no longer holds up to  $x_{sc}$  given by Eq.(46). Using the relationship (19), we can see from the expression (46) that  $x_{sc} \propto \tau^{-2}$ , that is,

$$\tau > \tau_{sc} : x_{sc}(\tau) = \frac{r_{sc}}{\tau^2} = \frac{r_{sc}}{a}, \quad (48)$$

where  $r_{sc}$  is constant. This means that in physical coordinates the screening radius  $r_{sc}$  does not depend on time. This is a direct consequence of the fact that the density profile (18) converges to a static profile in physical coordinates, in the nonlinear region  $\delta \gg 1$ . There,  $d\varphi_s/dr$  and  $\chi_s$  also converge to a static profile in physical coordinates, so that the threshold  $\chi_{sc}$  corresponds to a constant physical radius  $r_{sc}$ .

## B. Decomposition of the nonlinear solution

Because  $K' = 1$  at large radii and the universe has only a finite lifetime  $\tau \leq 1$ , beyond a finite radius we must recover the solution obtained in section III. Indeed, no information from the nonlinear region has had time to reach these large radii yet. In particular, this implies that the scalar field converges to the cosmological background  $\bar{\phi}$  at large radii. Therefore, we again split the solution as in (11),

$$\phi = \bar{\phi} + \varphi, \quad (49)$$

and the nonlinear Klein-Gordon equation (39) reads

$$\mathcal{O} \cdot \varphi = S(\mathbf{x}, \tau) \quad (50)$$

with the new source term

$$S(\mathbf{x}, \tau) = \partial_\tau(\ln K') \left[ \partial_\tau \varphi - \frac{4\beta}{\tau} \right] - \nabla(\ln K') \cdot \nabla \varphi + \frac{12\beta}{\tau^2} \left( \frac{1+\delta}{K'} - 1 \right). \quad (51)$$

For  $K' = 1$  we recover the linear equation (12) obtained for the standard kinetic term. We further split the solution of Eq.(50) as

$$\varphi = \varphi_\delta + \varphi_{K'} + \varphi_\tau, \quad (52)$$

where  $\varphi_*$  is associated with the source term  $S_*$  by  $\mathcal{O} \cdot \varphi_* = S_*$ , with

$$S_\delta = \frac{12\beta}{\tau^2} \frac{\delta}{K'} - \nabla(\ln K') \cdot \nabla \varphi, \quad (53)$$

$$S_{K'} = \frac{12\beta}{\tau^2} \left( \frac{1}{K'} - 1 \right), \quad (54)$$

and

$$S_\tau = \partial_\tau(\ln K') \left[ \partial_\tau \varphi - \frac{4\beta}{\tau} \right]. \quad (55)$$

Then, after solving each linear equation with the Green function (16), the solution  $\varphi$  of the nonlinear equation (50) is defined by the self-consistency condition that writes the source terms as functions of  $\varphi$ . This two-step procedure allows us to analyze the nonlinear solution  $\varphi$ .

### C. Quasistatic component $\varphi_\delta$

Let us first consider the part  $\varphi_\delta$ . This gives the same equation (12) as in the analysis of section III for the case of the standard kinetic term, except that the density contrast in the right-hand side is replaced by the field-dependent effective density contrast  $\delta_{\text{eff}} = \delta/K' - \nabla(\ln K') \cdot \nabla\varphi$ . Because we consider kinetic functions with  $K' = 1$  for  $\chi > \chi_{\text{sc}}$  and  $K' > 1$  for  $\chi < \chi_{\text{sc}}$ , this effective density contrast  $\delta_{\text{eff}}$  is equal to  $\delta$  at large radii and reduced at small radii in the screening region. Then, treating in a first step  $\delta_{\text{eff}}$  as an external source, we can apply to this equation the general lessons learnt in section III, which do not depend on the detailed profile of  $\delta$ . Thus, we infer that the scalar field  $\varphi_\delta$  quickly converges to the linear quasistatic solution  $\varphi_s$  associated with  $S_\delta$ , below the horizon,  $x \lesssim \tau$ , with a boundary value at  $x = \tau$  of the order of  $\varphi_s(x = \tau, \tau)$ . This offset is negligible as compared with the cosmological background  $\bar{\phi}$  and can be discarded for our purposes.

In the screening region, where  $K'$  differs from unity and  $\delta_{\text{eff}}$  deviates from  $\delta$ , we are far inside the horizon and by definition  $\chi < \chi_{\text{sc}} \ll -1$ , so that  $\chi$  is set by spatial gradients. Then, we can anticipate that the latter are dominated by the gradients of the quasistatic solution, at these small radii. This means that  $\varphi_\delta$  converges to the fully nonlinear quasistatic solution  $\varphi_s$  of Eq.(40). In other words, the argument of  $K'$  and the spatial gradient  $\nabla\varphi$  in the source  $S_\delta$  can be written in terms of  $\varphi_\delta$  itself, because the sources terms  $S_{K'}$  and  $S_\tau$  do not grow at small radii with  $\delta$  as  $S_\delta$ . Then, the quasistatic limit of  $\mathcal{O} \cdot \varphi_\delta = S_\delta$  becomes identical to the system (41)-(42), and we can write

$$x \lesssim \tau : \quad \varphi_\delta \simeq \varphi_s. \quad (56)$$

Thus, by choosing a kinetic function  $K'$  that is constant for all  $\chi > \chi_{\text{sc}}$ , with  $\chi_{\text{sc}} \ll -1$ , we ensure that both the slow time dependence that may arise from the coupling to the cosmological background and the complex behavior at large non-relaxed radii do not affect  $\varphi_\delta$ , because they only become relevant at large radii where  $\chi \geq 0$  and  $K' = 1$  is a constant, which no longer depends on its argument. Then,  $\varphi_\delta$  is simply given by the nonlinear quasistatic solution  $\varphi_s$  of the system (41)-(42), and it decouples from  $\varphi_{K'}$  and  $\varphi_\tau$ .

### D. Coupling to the cosmological background

We have seen that  $\varphi_\delta$  and  $S_\delta$  are not coupled to the cosmological background  $\bar{\phi}$ . However, the cosmological

background, which was explicitly introduced in the splitting (49) that led to Eq.(50), also appeared in the equation of motion through the terms  $-4\beta/\tau$  and  $-1$  in the full source term  $S$  of Eq.(51). Let us now consider the term  $\varphi_{K'}$ . We will find that it indeed encapsulates a coupling to the cosmological background and that, once combined with  $\bar{\phi}$ , it determines whether the full solution  $\phi$  follows the cosmological drift or not.

To simplify the analysis, we consider the case of a two-value kinetic function  $K'$ , with

$$x < x_{\text{sc}}(\tau) : \quad K' = K'_{\text{sc}} \gg 1, \quad x > x_{\text{sc}}(\tau) : \quad K' = 1. \quad (57)$$

This is an approximation to the more realistic case where  $K'$  smoothly interpolates between two constant values for  $\chi \ll \chi_{\text{sc}}$  and  $\chi \gg \chi_{\text{sc}}$ . In particular, the nonlinear equation (41) implies that  $K'$  as a function of the radial coordinate remains smooth, with the interpolation taking place over a finite radial interval  $\Delta x$ , even if  $K'$  is a discontinuous function of  $\chi$ . However, since we are not interested in the details of the profile around the transition the simple approximation (57) is sufficient for our purposes. Then,  $\varphi_{K'}$  and  $S_{K'}$  do not depend on the detailed density profile at small and large radii, but only on the radius  $x_{\text{sc}}(\tau)$  that marks the screening region and the associated value  $K'_{\text{sc}}$  of the kinetic function. Indeed, in this approximation the source term  $S_{K'}$  takes the simple form

$$S_{K'}(x, \tau) = \frac{12\beta}{\tau^2} \left( \frac{1}{K'_{\text{sc}}} - 1 \right) \theta\left(\frac{r_{\text{sc}}}{\tau^2} - x\right). \quad (58)$$

Because the expression (58) is fully explicit (it does not involve  $\varphi$ ), the equation of motion for  $\varphi_{K'}$  becomes linear and it is solved in one step as

$$\varphi_{K'}(\mathbf{x}, \tau) = \int d\mathbf{x}' d\tau' \mathcal{G}(\mathbf{x}, \tau; \mathbf{x}', \tau') S_{K'}(\mathbf{x}', \tau'), \quad (59)$$

without the need to add a self-consistency condition in a second step. Thanks to the explicit expressions of the Green function and of the source term, we can perform most of the integrations and we can derive explicit results at the center,  $x = 0$ . Integrating over  $k$  and  $x'$ , we obtain

$$\begin{aligned} \varphi_{K'}(0, \tau) &= 4\beta \left( 1 - \frac{1}{K'_{\text{sc}}} \right) \int_{\tau_{\text{sc}}}^{\tau} d\tau' \left\{ \theta(\tau - \tau' < \frac{r_{\text{sc}}}{\tau'^2}) \right. \\ &\quad \times \tau^{-3} [\tau'^{-1}(\tau - \tau')^3 + 3\tau(\tau - \tau')] + \theta(\tau - \tau' > \frac{r_{\text{sc}}}{\tau'^2}) \\ &\quad \left. \times r_{\text{sc}}^3 \tau^{-3} \tau'^{-7} \right\}. \end{aligned} \quad (60)$$

For large values of  $r_{\text{sc}}$ , such that  $\tau - \tau' < r_{\text{sc}}/\tau'^2$  at all times (e.g., if  $r_{\text{sc}} > \tau^3$ ), only the first Heaviside factor contributes and we obtain

$$r_{\text{sc}} > \tau^3 : \quad \varphi_{K'}(0, \tau) = 4\beta \left( 1 - \frac{1}{K'_{\text{sc}}} \right) \left[ \ln \frac{\tau}{\tau_{\text{sc}}} - \frac{1}{3} + \frac{\tau_{\text{sc}}^3}{\tau^3} \right]. \quad (61)$$



Combining with  $\bar{\phi}$  this yields

$$\begin{aligned} \bar{\phi} + \varphi_{K'}(0, \tau) &= -\frac{4\beta}{K'_{\text{sc}}} \ln \tau + 4\beta \left(1 - \frac{1}{K'_{\text{sc}}}\right) \\ &\times \left[-\ln \tau_{\text{sc}} - \frac{1}{3} + \frac{\tau_{\text{sc}}^3}{\tau^3}\right]. \end{aligned} \quad (62)$$

In this case, we find that the scalar field at the center decouples from the cosmological background and its magnitude can be much smaller there. Mathematically, the artificial cosmological drift that was introduced by the splitting (49) is fully absorbed by the component  $\varphi_{K'}$  so that the full solution  $\phi$  decouples from the cosmological drift.

For small values of  $r_{\text{sc}}$ , such as  $r_{\text{sc}} \ll \tau_{\text{sc}}^3$ , the first Heaviside factor only contributes for  $\tau'$  very close to  $\tau$ . Then, we obtain at leading order over  $r_{\text{sc}}$ , with  $\tau_{\text{sc}} \ll \tau$ ,

$$r_{\text{sc}} \ll \tau^3: \varphi_{K'}(0, \tau) = \frac{2\beta}{3} \left(1 - \frac{1}{K'_{\text{sc}}}\right) \frac{r_{\text{sc}}^2}{\tau^3 \tau_{\text{sc}}^6} \left(r_{\text{sc}} + 9 \frac{\tau_{\text{sc}}^6}{\tau^3}\right). \quad (63)$$

Thus,  $\varphi_{K'}(0, \tau)$  is much smaller than  $\bar{\phi}$  and vanishes for  $r_{\text{sc}} \rightarrow 0$ . Then, the scalar field remains strongly coupled to the cosmological background down to the center of the overdensity, as the sum  $\bar{\phi} + \varphi_{K'}$  closely follows  $\bar{\phi}$ . This is the case of interest for realistic matter density structures. In particular,  $r_{\text{sc}} \ll \tau_{\text{sc}}^3$  corresponds to  $x_{\text{sc}}(\tau_{\text{sc}}) \ll \tau_{\text{sc}}$ , that is, the structure is much below the horizon at the early time  $\tau_{\text{sc}}$ .

We now check that the third contribution  $\varphi_{\tau}$  does not invalidate these conclusions. In the same approximation (57) of a two-value kinetic function, the source term  $S_{\tau}$  reads as

$$S_{\tau}(x, \tau) = s_{\tau}(\tau) \delta_D\left(x - \frac{r_{\text{sc}}}{\tau^2}\right), \quad (64)$$

with

$$s_{\tau}(\tau) = (\ln K'_{\text{sc}}) 2 \frac{r_{\text{sc}}}{\tau^3} \left(\frac{4\beta}{\tau} - \partial_{\tau} \varphi\right)_{x=r_{\text{sc}}/\tau^2}. \quad (65)$$

As in (59), we can again express  $\varphi_{\tau}$  in terms of  $S_{\tau}$  through the Green function. Integrating over  $k$  and  $x'$  we obtain

$$\begin{aligned} \varphi_{\tau}(0, \tau) &= -r_{\text{sc}}^2 \tau^{-3} \int_{\tau_{\text{sc}}}^{\tau} d\tau' s_{\tau}(\tau') \tau'^{-3} \theta(\tau - \tau' > \frac{r_{\text{sc}}}{\tau'^2}) \\ &- r_{\text{sc}}^{1/2} \tau^{-2} \int_{\tau_{\text{sc}}}^{\tau} d\tau' s_{\tau}(\tau') \tau' \sqrt{\tau - \tau'} \delta(\tau - \tau' - \frac{r_{\text{sc}}}{\tau'^2}). \end{aligned} \quad (66)$$

For large values of  $r_{\text{sc}}$ , such that  $\tau - \tau' < r_{\text{sc}}/\tau'^2$  at all times, we obtain at once  $\varphi_{\tau}(0, \tau) = 0$ , hence

$$r_{\text{sc}} > \tau^3: \varphi_{\tau}(0, \tau) = 0. \quad (67)$$

Then, the full solution reads as  $\phi(0, \tau) = \bar{\phi} + \varphi_{\delta} + \varphi_{K'}$ , which decouples from the cosmological background as seen above from Eq.(62).

For small values of  $r_{\text{sc}}$ , such as  $r_{\text{sc}} \ll \tau_{\text{sc}}^3$ , we obtain

$$\begin{aligned} r_{\text{sc}} \ll \tau_{\text{sc}}^3: \varphi_{\tau}(0, \tau) &= -r_{\text{sc}}^2 \tau^{-3} \int_{\tau_{\text{sc}}}^{\tau} d\tau' s_{\tau}(\tau') \tau'^{-3} \\ &- r_{\text{sc}} \tau^{-2} s_{\tau}(\tau). \end{aligned} \quad (68)$$

For such small-scale matter overdensities, we have already seen from Eqs.(56) and (63) that the components  $\varphi_{\delta}$  and  $\varphi_{K'}$  are small and give a time dependence that is negligible as compared with the cosmological drift. In particular, they vanish if  $r_{\text{sc}} \rightarrow 0$ . We can see that this will also be the case of the component  $\varphi_{\tau}$ , because of the prefactors  $r_{\text{sc}}$ . Therefore, we can neglect the factor  $\partial_{\tau} \varphi$  as compared with the background time derivative  $4\beta/\tau$  in the source (65). This gives at leading order over  $r_{\text{sc}}$  and  $\tau_{\text{sc}}$ ,

$$r_{\text{sc}} \ll \tau_{\text{sc}}^3: \varphi_{\tau}(0, \tau) = -\frac{4\beta}{3} (\ln K'_{\text{sc}}) \frac{r_{\text{sc}}^2}{\tau^3 \tau_{\text{sc}}^6} \left(r_{\text{sc}} + 6 \frac{\tau_{\text{sc}}^6}{\tau^3}\right). \quad (69)$$

The comparison with Eq.(63) shows that  $\varphi_{\tau}(0, \tau) \sim -(\ln K'_{\text{sc}}) \varphi_{K'}(0, \tau)$ . Therefore, we also find that  $\varphi_{\tau}(0, \tau)$  is much smaller than  $\bar{\phi}$  and vanishes for  $r_{\text{sc}} \rightarrow 0$ . This confirms that the scalar field remains strongly coupled to the cosmological background for such small-scale matter overdensities.

Thus, we find that the naive local analysis of the equation of motion (1), which could suggest that in screened regions where  $K'$  is very large the scalar field  $\phi$  no longer evolves and remains constant in space and time, is not correct. In fact, the only size that can be considered local is the Hubble radius, independently of the variations and nonlinearities of  $K'$ . This could be expected from the fact that the propagation speed remains of order unity, even in nonlinear domains, and that there is no damping of the amplitude of the scalar field as the equation of motion only involves its derivatives. Indeed, in small-scale nonlinear environments the propagation speed reads [3]

$$c_{\phi}^2 = \frac{K' + 2\chi K''}{K'} \sim 1, \quad (70)$$

as for power-law kinetic functions we have  $\chi K'' \sim K'$  whereas  $\chi K'' \ll K'$  in regimes where  $K'$  is almost constant.

## V. CONCLUSION

The value of the scalar field deep inside a collapsed region of the Universe is highly relevant as it determines the value of Newton's constant, which is proportional to  $A^2(\phi)$  where  $A(\phi) \sim e^{\beta\phi/M_{\text{Pl}}}$  is the coupling function to matter and  $\beta = \mathcal{O}(1)$  the coupling to matter. In screened regions where the K-mouflage mechanism is at play, the spatial gradients of the scalar field are large, much larger than the time derivatives, and the fifth force induced by the scalar is largely depleted. On the other hand, it is well known that a linear time drift  $H_0 t$  still allows for

static solutions around a time-independent astrophysical object and can provide an approximate matching with the large-scale cosmological evolution of the scalar field. This induces then a cosmological time drift of Newton's constant, jeopardizing the viability of many models of the K-mouflage type.

In this paper, we have investigated the influence of the background cosmology on the short-distance physics within a collapsed structure of the Universe. We have taken it to be described by a self-similar power-law density profile, which allows us to provide an almost exact treatment. We find that inside the structure there is a critical radius  $x_{qs}$  within which the quasistatic approximation holds, in the sense that spatial derivatives are greater than time derivatives. This radius is much smaller than the size  $x_s$  of the matter overdensity, where the matter density contrast becomes of order unity. However, spatial gradients are well described by the quasistatic approximation up to the horizon, hence up to much larger scales, as found for other modified-gravity scenarios in previous studies. We also find that for structures that grow fast with time, which could apply to transient mergings but also to the fast building of the cosmic web at redshifts  $z \gtrsim 2$ , the time derivative of the scalar field perturbations remains greater than its spatial gradient.

Screening of the fifth force takes place only well-inside

the quasistatic radius, where  $\nabla\phi \gg \partial_\tau\phi$ . However, inside the screening radius  $x_{sc}$  and down to the center of the overdensity, the values of the scalar field remain strongly dependent on the background cosmological evolution: no screening of the time drift of Newton's constant takes place. We have explicitly shown that the scalar field only decouples from the cosmological background if the matter structure extends up to the horizon, which is not the case for realistic astrophysical and cosmological structures. Of course, this result does not invalidate K-mouflage models and simply implies that the strong constraints deduced in [9] must be taken seriously. Thus, the K-mouflage screening mechanism only damps the spatial gradients of the scalar field, reducing the fifth force in small-scale high-density environments, while following the large-scale drift of the cosmological background. We can expect that this behavior extends to other derivative screening mechanisms, such as Vainshtein screening.

Thus, we have shown that the dynamics of screening in K-mouflage models are more complex than can be deduced by a fully quasistatic approximation. In particular, the appearance of two radii: the quasistatic and screening radii is a new feature. It would be extremely interesting to see if N-body simulations of K-mouflage models could reveal other new dynamical characteristics of K-mouflage, for instance around fast-growing structures. This is left for future work.

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