The holographic interpretation of $J\bar{T}$-deformed CFTs

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Abstract

Recently, a non-local yet possibly UV-complete quantum field theory has been constructed by deforming a two-dimensional CFT by the composite operator $J\bar{T}$, where J is a chiral $U(1)$ current and $\bar{T}$ is a component of the stress tensor. Assuming the original CFT was a holographic CFT, we work out the holographic dual of its $J\bar{T}$ deformation. We find that the dual spacetime is still AdS$_3$, but with modified boundary conditions that mix the metric and the Chern-Simons gauge field dual to the $U(1)$ current. We show that the energy and thermodynamics of black holes obeying these modified boundary conditions precisely reproduce the previously derived field theory spectrum and thermodynamics, provided the contribution of the current takes a particular form we motivate. The associated asymptotic symmetry group consists of two copies of the Virasoro and one copy of the $U(1)$ Kač-Moody algebra, just as before the deformation; the only effect of the latter is to modify the spacetime dependence of the right-moving Virasoro generators, whose action becomes state-dependent and effectively non-local.

Contents

1. Introduction

Recently, an interesting class of irrelevant deformations of two-dimensional quantum field theories has been uncovered [1], where the deforming operator is a certain bilinear combination of conserved currents. Thanks to this special form, the resulting deformed QFT is, in a sense, solvable - for example, it is possible to determine the finite-size spectrum and thermodynamics at finite deformation parameter in terms of the original QFT data.

The best studied of these deformations is the so-called $T\bar{T}$ deformation of two-dimensional CFTs, for which the spectrum takes a universal form [1, 2]. The $T\bar{T}$ deformation can be recast in S-matrix language [3], and the form of the resulting S-matrix suggests that the deformed theory is UV complete, even though it does not possess a usual, local UV fixed point. Theories with such high-energy behaviour have been termed "asymptotically fragile", and they open up a whole new set of possible ultraviolet behaviours of QFTs [4]. One example is provided by the worldsheet theory of the bosonic string, which has been studied from this perspective in [5–7]. The $T\bar{T}$ deformation has also found an interesting application in holography, as the field theory dual to a finite bulk cutoff for the metric fluctuations [8] (see also [9–11]) and, in a somewhat modified form, as holographic dual to a linear dilaton background [12–15]. The partition function of $T\bar{T}$ deformed QFTs on various domains has been recently studied in [16], and correlation functions in a certain large central charge limit have been computed in [17].

Another potentially interesting deformation in the Smirnov-Zamolodchikov class corresponds to a two-dimensional CFT deformed by an irrelevant double-trace operator of the schematic form [18]

$$S = S_{CFT} + \mu \int d^2 z \, J\bar{T} \tag{1.1}$$

where $\bar{T}$ represents the stress tensor and J is a $U(1)$ current. This deformation preserves an $SL(2,\mathbb{R}) \times U(1)$ subgroup of the orginal conformal group, and it may be relevant for the holographic understanding of extremal black holes [19, 20]. For J purely chiral or antichiral, the deformed spectrum again takes a universal form; the deformation is non-trivial for chiral J, while for J antichiral no modification away from the CFT spectrum was observed.

In this article, we model the $J\bar{T}$ deformation of two-dimensional CFTs in holography. The minimal ingredients in the bulk are three-dimensional Einstein gravity, which provides a holographic dual for the stress tensor, coupled to a Chern-Simons gauge-field, dual to the $U(1)$ current. By choosing the sign of the Chern-Simons coupling, the current can be made chiral or anti-chiral. We will exclusively concentrate on the non-trivial chiral J case.

Since there are no dynamical degrees of freedom in this system, all the bulk solutions are locally AdS$_3$. The only effect of the double-trace deformation is to change the asymptotic boundary conditions imposed on the bulk fields, from Dirichlet to mixed ones. The simplest way to derive the new holographic sources and expectation values is by analysing the variational principle in presence of the deformation. We find that the asymptotic boundary conditions on the metric are very similar to the "new boundary conditions for AdS$_3$" proposed in [21]; however, since our setup contains an additional gauge field, the allowed excitations are no longer restricted to be chiral. We also find that the expectation values in the deformed theory are related in a simple way to the expectation values in the original CFT in presence of non-trivial sources, which can be computed using the "usual" AdS$_3$/CFT$_2$ holographic dictionary (i.e., with Dirichlet boundary conditions on the metric). We encounter certain subtleties in evaluating the contribution of the Chern-Simons sector to the stress tensor, which we circumvent by modeling it by that of a pair of chiral fermions, as suggested by the equivalence of both sytems to a chiral boson. We check this dictionary by showing that the energy and thermodynamic properties of black holes with these boundary conditions reproduce the spectrum and thermodynamics previously derived from purely field theoretical considerations.

Note that even though the above discussion takes place in the context of holography, our calculations can also be viewed from a purely field-theoretical point of view, as a way to compute expectation values in the deformed theory at large N in terms of expectation values in the original CFT. While here we concentrate on the one-point functions of the stress tensor and the current, in principle our method can be used to compute arbitrary correlators in the deformed theory in terms of CFT ones.

As mentioned above, the $J\bar{T}$ deformation breaks the two-dimensional conformal group $SL(2,\mathbb{R})_L \times$

$SL(2,\mathbb{R})_R$ down to an $SL(2,\mathbb{R})_L \times U(1)_R$ subgroup; additionally, there is the chiral $U(1)_J$ symmetry generated by the current. An interesting question is whether these global symmetries are enhanced to infinite-dimensional ones, as is common in two dimensions. There are two ways to address this question: either by constructing the infinite set of conserved charges explicitly using the special properties of the stress tensor and the current, as in [22], or by studying the asymptotic symmetries of the dual spacetime, as in [23]. In the original CFT, either method can be used to show that the $SL(2,\mathbb{R})_L$ and $U(1)_J$ symmetries are enhanced to a left-moving Virasoro - Kač-Moody algebra, while the $SL(2,\mathbb{R})_R$ is enhanced to a right-moving Virasoro algebra. In the deformed CFT, we use both methods to show that there is a similar infinite-dimensional enhancement of the global symmetries to a Virasoro $\times$ Virasoro $\times$ $U(1)$ Kač-Moody algebra; the only change is that the argument of the right-moving Virasoro generators is shifted by a state-dependent function of the left-moving boundary coordinate.

The plan of the paper is as follows. We start section 2. by reviewing the effect of double-trace deformations in holography from a path integral approach, which is equivalent to the variational approach. We then apply the variational approach to the specific case of the $J\bar{T}$ deformation and obtain the deformed sources and expectation values in terms of the original ones. In section 3., after a quick review of the usual AdS$_3$/CFT$_2$ dictionary, we find the asymptotic expansion of the bulk fields that corresponds to the new boundary conditions and propose an expression for the holographic expectation values. In section 4., we check this holographic dictionary by showing that the thermodynamics and conserved charges of black holes obeying the new asymptotics agree with the field theoretical results of [18]. Next, we use holography to construct an infinite set of conserved charges and compute their associated asymptotic symmetry algebra. We end with a discussion and future directions.

2.   Effect of the double-trace deformation

The effect of multitrace deformations in the context of the AdS/CFT correspondence [24] has been extensively studied. At large N, as far as the low-lying single-trace operators[1] are concerned, such deformations simply correspond to changing the asymptotic boundary conditions on their dual supergravity field. At the level of the classical supergravity action, the new boundary conditions can be easily read off by studying the variational principle in presence of the deformation.

Most of the literature on the subject is concerned with deformations constructed from scalar operators of dimension smaller than $d/2$, such that the resulting multitrace operator is relevant or marginal. This ensures that the ultraviolet regime of the theory is under control; in the dual picture, the deformation of AdS is normalizable (though only visible at $1/N$ order), and so also under control. Note that the bulk field dual to such an operator in the original CFT will be quantized with Neumann, also known as alternate, boundary conditions.

The $J\bar{T}$ deformation differs in several respects from the usual case. First, the deformation is irrelevant; we will nevertheless consider it, because the resulting theory is expected to be UV complete. On the bulk side, we deform the gravitational theory in the usual quantization (i.e., with Dirichlet boundary conditions for the metric). The resulting mixed boundary conditions involve fluctuations of the non-normalizable mode of the metric and of the Chern-Simons gauge field; however, since none of these modes is dynamical, the asymptotic geometry is still locally AdS$_3$. This lack of backreaction of the deformation on the local geometry is likely related to the UV-completeness of the dual theory.

In this section, we start by reviewing the path integral derivation of the change in boundary conditions induced by the double-trace deformation, following [26], and how the same result is recovered in the variational approach. We then apply the variational approach to the $J\bar{T}$ deformed-CFT and read off the new sources and expectation values in terms of the old ones, which in principle gives us the full large N holographic dictionary for this theory.

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[1]The effect of the deformation beyond the leading order in $1/N$ and for operators other than single-trace ones has been recently studied in [25].

2.1 Review of double-trace deformations in holography

Let us review how the holographic data change under general double-trace deformations of a holographic (large N) CFT, following [26]. In the case usually considered, a term of the form

$$S_{d.tr} = \int d^d x \, \mathcal{L}_{d.tr}(\mathcal{O}_A) \tag{2.1}$$

is added to the CFT action, where the $\mathcal{O}_A$ are scalar operators dual to supergravity fields, whose correlation functions factorize at large N. We will specifically be interested in the case in which $\mathcal{L}_{d.tr}$ is a bilinear in the operators of interest

$$\mathcal{L}_{d.tr.} = \frac{1}{2}\mu^{AB}\mathcal{O}_A\mathcal{O}_B \tag{2.2}$$

where μ^{AB} is a constant matrix. The generating functional in the deformed theory is

$$e^{-W_\mu[\tilde{\jmath}^A]} = \int \mathcal{D}\varphi \, e^{-S[\varphi] + \int \tilde{\jmath}^A \mathcal{O}_A - \frac{1}{2}\int \mu^{AB}\mathcal{O}_A\mathcal{O}_B} \tag{2.3}$$

where $\tilde{\jmath}^A$ denote the sources in the deformed theory that couple to $\mathcal{O}_A$ and φ denotes the fundamental degrees of freedom in the CFT, over which the path integral is performed, weighted by the action $S[\varphi]$. Using the identity

$$1 = \sqrt{\det \mu^{-1}} \int \mathcal{D}\tilde{\sigma}^A e^{\frac{1}{2}\int \tilde{\sigma}^A (\mu^{-1})_{AB}\tilde{\sigma}^B} \tag{2.4}$$

shifting the integration variable as $\tilde{\sigma}^A = \sigma^A - \tilde{\jmath}^A + \mu^{AB}\mathcal{O}_B$ and using large N factorization, one finds

$$e^{-W_\mu[\tilde{\jmath}^A]} = \int \mathcal{D}\sigma^A \, e^{-W[\sigma^A] + \frac{1}{2}\int (\sigma^A - \tilde{\jmath}^A)(\mu^{-1})_{AB}(\sigma^B - \tilde{\jmath}^B)} \tag{2.5}$$

Note that $\sigma^A \equiv \jmath^A$ plays the role of source in the undeformed theory. Evaluating the above integral via saddle point, one finds the latter occurs at

$$-\frac{\delta W[\sigma^A]}{\delta \sigma^A}\bigg|_{\sigma^A_*} = \langle \mathcal{O}_A \rangle = -(\mu^{-1})_{AB}(\sigma^B_* - \tilde{\jmath}^B) \tag{2.6}$$

and

$$\sigma^A_* = \jmath^A = \tilde{\jmath}^A + \mu^{AB}\langle \mathcal{O}_B \rangle \quad \text{or} \quad \tilde{\jmath}^A = \jmath^A - \mu^{AB}\langle \mathcal{O}_B \rangle \tag{2.7}$$

The relation between the generating functionals is then

$$W_\mu[\tilde{\jmath}^A] = W[\jmath^A] - \frac{1}{2}\int \mu^{AB}\langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle \tag{2.8}$$

In the above derivation, it was assumed that the operator $\mathcal{O}_A$ is the same in the original and the deformed theory, and we used large N factorization to effectively replace the $\mathcal{O}_A$ by their expectation values at the various steps. Note that while one adds $S_{d.tr.}$ to the CFT action, one effectively subtracts it from the generating functional, at least for the special case of a double-trace deformation.

Passing to holography, the generating functional $W[\jmath^A]$ is mapped to the renormalised on-shell action $S[\phi_A]$, which depends on the boundary values of the fields $\phi^A \leftrightarrow \jmath^A$, viewed as generalized coordinates. The operators $\mathcal{O}_A$ are to be identified with the generalized conjugate momenta, $\langle \mathcal{O}_A \rangle \leftrightarrow \pi_A$. The variation of the on-shell action as the sources are varied is

$$\delta S[\phi^A] = \int d^d x \, \pi_A \delta \phi^A \tag{2.9}$$

The relation (2.7) between the sources in the deformed and undeformed theory can be simply reproduced by considering the variational principle for the total on-shell action in presence of the deformation. Following the above discussion, we are instructed to subtract the functional of the expectation values $\mathcal{L}_{d.tr}(\mathcal{O}_A)$ from the generating functional, which amounts to performing a canonical transformation on the system. The variation of the total on-shell action is

$$\delta S_{tot} = \int d^d x \left(\pi_A \delta \phi^A - \delta \mathcal{L}_{d.tr.}(\pi^A) \right) = \int d^d x \, \tilde{\pi}_A \delta \tilde{\phi}^A \qquad (2.10)$$

For $\mathcal{L}_{d.tr.}$ given by (2.2), which only depends on the $\mathcal{O}_A$ but not the sources, the effect of this canonical transformation is to shift the sources by an amount proportional to the expectation values

$$\tilde{\pi}_A = \pi_A \,, \qquad \tilde{\phi}^A = \phi^A - \mu^{AB} \pi_B \qquad (2.11)$$

while leaving the expectation values unchanged. This is of course equivalent, almost by definition, to the previous manipulations at the level of the generating functional.

The variational approach is useful for obtaining the deformed holographic data in more complicated situations, e.g. when the deformation depends on both the operators and the sources for them. A common situation occurs when one is interested in the expectation value of the stress tensor in the deformed theory. In this case, one should include the coupling of the deforming operator to a general background metric, *i.e.*, a source for $T_{\alpha\beta}$. As shown in [27], the variational approach yields the correct shift in the expectation value of the stress tensor due to the deformation. This is a much easier computation than the direct manipulation of the generating functional.

The situation we have at hand is even more complicated, because both the stress tensor and its source (*i.e.*, the boundary metric or vielbein needed to covariantize the deformation) appear simultaneously in the deforming operator. Thus, we expect a change both in the source and in the expectation value of the stress tensor as we perform the deformation. Having convinced ourselves that the variational approach should give equivalent results to the generating functional, we use it to greatly simplify the computation.

2.2 Sources and expectation values in the $J\bar{T}$-deformed theory

The minimal set of phase space variables that we need to consider are the stress tensor $T^a{}_\alpha$ and current J^α, which are canonically conjugate to the boundary vielbein $e^\alpha{}_a$ and gauge field, a_α. Here, latin indices denote the tangent space and greek ones are spacetime indices. The reason we prefer the vielbein formulation is that the deformed theory is not Lorentz invariant; consequently, the conserved stress tensor is not symmetric and it naturally couples to the vielbein, and not the metric. The variation of the original CFT action reads

$$\delta S_{CFT} = \int d^2 x \, e \, (T^a{}_\alpha \, \delta e^\alpha_a + J^\alpha \delta a_\alpha) \qquad (2.12)$$

where $e = \det e^a_\alpha$ and from now on we will omit the brackets from the expectation values of the various operators.

Next, we add the double-trace $J\bar{T}$ deformation, appropriately covariantized. Since the coupling parameter is a dimensionful null vector, in an arbitrary background it makes the most sense to keep the coupling with tangent space indices fixed, so $\mu_a = \mu \, \delta^+_a$, where μ is a constant with dimensions of length and x^+ is a null direction along the boundary. The covariantized multitrace operator is then

$$S_{J\bar{T}} = \int d^2 x \, e \, \mu_a T^a_\alpha J^\alpha \qquad (2.13)$$

The problem that we would like to solve is to find new canonical variables such that the variation of the total action, including the multitrace contribution, can be written in the form $\tilde{\pi}_A \delta \tilde{\phi}^A$ for some new canonical variables $\tilde{\pi}_A$, $\tilde{\phi}^A$. That this should be possible is guaranteed by the fact that we are performing a canonical transformation. The variation of the action including the multitrace is

$$\begin{aligned}
\delta S - \delta S_{J\bar{T}} &= \int d^2 x \left[e \, T^a_\alpha \delta e^\alpha_a + e \, J^\alpha \delta a_\alpha - \delta(e \, \mu_a T^a_\alpha J^\alpha) \right] \qquad (2.14) \\
&= \int d^2 x \, e \left[T^a_\alpha (\delta e^\alpha_a - \mu_a \delta J^\alpha) + J^\alpha (\delta a_\alpha - \mu_a \delta T^a_\alpha) + e^a_\alpha \delta e^\alpha_a \, \mu_b T^b_\beta J^\beta \right] \\
&= \int d^2 x \, e \left[(T^a_\alpha + e^a_\alpha \, \mu_b T^b_\beta J^\beta)(\delta e^\alpha_a - \mu_a \delta J^\alpha) + J^\alpha (\delta a_\alpha - \mu_a \delta T^a_\alpha) + \mu_a e^a_\alpha \delta J^\alpha \, \mu_b T^b_\beta J^\beta \right]
\end{aligned}$$

To proceed, we use the fact that J is purely chiral $J = J_+(x^+)$, and so $\mu_a J^a = 0$. The above expression can then be manipulated into

$$\delta S - \delta S_{J\bar{T}} = \int d^2x\, e\left[(T^a_\alpha + (e^a_\alpha + \mu_\alpha J^a)\, \mu_b T^b_\beta J^\beta)(\delta e^\alpha_a - \mu_a \delta J^\alpha) + J^\alpha(\delta \mathrm{a}_\alpha - \mu_a \delta T^a_\alpha)\right] \tag{2.15}$$

We can easily read off the modified sources and expectation values from the above expression[2]

$$\boxed{\tilde{e}^\alpha_a = e^\alpha_a - \mu_a J^\alpha\,, \qquad \tilde{\mathrm{a}}_\alpha = \mathrm{a}_\alpha - \mu_a T^a_\alpha} \tag{2.16}$$

$$\boxed{\tilde{T}^a_\alpha = T^a_\alpha + (\mu_b T^b_\beta J^\beta)\,(e^a_\alpha + \mu_\alpha J^a)\,, \qquad \tilde{J}^\alpha = J^\alpha} \tag{2.17}$$

This is one of our main results. The tilded quantities are evaluated in the deformed theory with parameter μ_a, whereas the untilded quantities belong to the original CFT at $\mu = 0$. We have defined $\mu_\alpha = \mu_a e^a_\alpha$ and $J^a = e^a_\alpha J^\alpha$. Note that the expression for the sources coincides with the naive expression (2.7), for μ_{AB} off-diagonal.

One possible objection to our manipulations above is that, according to its definition, the $J\bar{T}$ deformation is supposed to involve the instantaneous stress tensor in the theory deformed by μ, and not the stress tensor of the original CFT, as written in (2.14). However, it is easy to check that $\tilde{e}\,\mu_a \tilde{T}^a_\alpha \tilde{J}^\alpha = e\,\mu_a T^a_\alpha J^\alpha$, so this distinction does not matter for our derivation. Alternatively, one could implement the instantaneous deformation by considering the infinitesimal version of the equations (2.16) - (2.17), obtained from the last line of (2.14) by discarding the $\mathcal{O}(\mu^2)$ term, and then integrating with respect to μ. While one naively obtains different expressions[3] for $\tilde{\mathrm{a}}_\alpha$ and $\tilde{T}^a_\alpha$, it is possible to show that the extra term in $\tilde{\mathrm{a}}_\alpha$ can be moved to $\tilde{T}^a_\alpha$ without affecting the variational principle. This freedom is due to the fact that (2.14) does not uniquely determine the flow equations for the field theory data, but some additional criteria are needed, such as requiring that the deformed expectation values satisfy the appropriate Ward identities.

2.3   Ward identities

We end with a note on the Ward identities satisfied by the deformed stress tensor and the current, which will be useful in the later sections. The Ward identities in the original CFT can be obtained from the variation of the CFT action (2.12), specialized to gauge transformations and diffeomorphisms, for which $\delta S_{CFT} = 0$. Invariance under gauge transformations $\mathrm{a}_\alpha \to \mathrm{a}_\alpha + \partial_\alpha \lambda$ imposes that $\nabla_\alpha J^\alpha = 0$, though for J chiral there will generally be an anomaly, $\nabla_\alpha J^\alpha = \frac{k}{4\pi} \mathrm{f}_{\alpha\beta} \epsilon^{\alpha\beta}$. Invariance under diffeomorphisms, under which

$$\delta_\xi e^\alpha_a = -\xi^\lambda \partial_\lambda e^\alpha_a + \partial_\lambda \xi^\alpha e^\lambda_a\,, \qquad \delta_\xi \mathrm{a}_\alpha = -\xi^\lambda \mathrm{f}_{\lambda\alpha} - \partial_\alpha(\xi^\lambda \mathrm{a}_\lambda) \tag{2.18}$$

implies the conservation equation

$$\nabla_\beta(T^a_\alpha e^\beta_a) + T^a_\beta \nabla_\alpha e^\beta_a - T_{ab}\,\omega_\alpha{}^{ab} + \mathrm{f}_{\alpha\beta} J^\beta - \mathrm{a}_\alpha \nabla_\beta J^\beta = 0 \tag{2.19}$$

The second term vanishes by the tetrad postulate, the third when T_{ab} is symmetric, the fourth if the gauge connection is flat and so does the last one, using current conservation. Thus, we find that the stress tensor is conserved for any boundary metric, as long as it is symmetric and $\mathrm{f}_{\alpha\beta} = 0$.

After adding the deformation, the variation of the action is given by (2.15). In terms of the new "tilded" variables defined in (2.16) - (2.17), the Ward identity is identical with the one above

$$\tilde{\nabla}_\lambda(\tilde{T}^a_\alpha \tilde{e}^\lambda_a) + \tilde{T}^a_\lambda \tilde{\nabla}_\alpha \tilde{e}^\lambda_a - \tilde{T}_{ab}\,\tilde{\omega}_\alpha{}^{ab} + \tilde{\mathrm{f}}_{\alpha\beta} J^\beta - \tilde{\mathrm{a}}_\alpha \tilde{\nabla}_\lambda J^\lambda = 0 \tag{2.20}$$

If we assume that the original $\mathrm{f}_{\alpha\beta} = 0$, then $\tilde{e} = e$ implies that $\tilde{\nabla}_\alpha J^\alpha = \nabla_\alpha J^\alpha = 0$. The second term can be dropped as before, while the fourth term $\tilde{\mathrm{f}}_{\alpha\beta} = \mathrm{f}_{\alpha\beta} - \mu_a(\partial_\alpha T^a_\beta - \partial_\beta T^a_\alpha)$ will vanish provided that $\mu_a(\partial_\alpha T^a_\beta - \partial_\beta T^a_\alpha) = 0$. This will indeed be the case for the backgrounds we will consider. Thus, the new stress tensor (which in general will not be symmetric) will be conserved with respect to the new background $\tilde{e}^a_\alpha$, provided the spin connection vanishes.

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[2]Note that the first equation and the condition $\mu_a J^a = 0$ imply that $\tilde{e} = e$ and $\tilde{e}^a_\alpha = e^a_\alpha + \mu_\alpha J^a$.

[3]Naively integrating one obtains $\tilde{T}^b_\beta = T^b_\beta + (\mu_a T^a_\alpha J^\alpha)\left(e^b_\beta + \frac{1}{2}\mu_\beta J^b\right)$ and $\tilde{\mathrm{a}}_\beta = \mathrm{a}_\beta - \mu_b T^b_\beta - \frac{1}{2}(\mu_a T^a_\alpha J^\alpha)\,\mu_b e^b_\beta$.

3. The holographic dictionary

The analysis of the previous section reveals a very simple way to construct the holographic dictionary for the $J\bar{T}$-deformed CFT, starting from the usual AdS$_3$/CFT$_2$ holographic dictionary. Namely, the holographic dictionary in presence of sources $\tilde{e}^a_\alpha$, $\tilde{a}_\alpha$ for the stress tensor and current in the deformed theory can be constructed in two steps:

i) First, one works out the usual AdS/CFT dictionary in presence of the boundary sources

$$e^\alpha_a = \tilde{e}^\alpha_a + \mu_a J^\alpha \ , \qquad a_\alpha = \tilde{a}_\alpha + \mu_a T^a_\alpha \tag{3.1}$$

where $\tilde{e}^a_\alpha$, $\tilde{a}_\alpha$ are held fixed and T^a_α and J^α are determined by holographically computing the expectation values of the CFT stress tensor and current in the above background and feeding them back into the sources

ii) To find the expectation values $\tilde{T}^a_\alpha$, $\tilde{J}^\alpha$ in the deformed theory, one simply plugs in the values of T^a_α, J^α found at step i) into (2.17).

The dual geometry will simply be given by the asymptotically locally AdS$_3$ solution that obeys the boundary conditions (3.1). Note that the only role of holography in this procedure is to provide a simple means to compute the holographic expectation values in the undeformed CFT with non-trivial boundary sources; in particular, step ii) of the procedure is purely field-theoretical.

The above procedure is perfectly well-defined, provided we have full control over the usual AdS/CFT dictionary with arbitrary boundary sources. For Einstein gravity in AdS$_3$, the dictionary is extremely well understood [28, 29]. On the other hand, the holographic dictionary for $U(1)$ Chern-Simons turns out to be quite subtle in presence of non-zero boundary sources, especially in what concerns their contribution to the boundary stress tensor [30–32]. In this article, we circumvent these subtleties by invoking the expected equivalence between $U(1)$ Chern-Simons theory and a pair of chiral fermions. This will allow us to compute the contribution of the current sector to the stress tensor, with results that pass various consistency checks.

We start this section by reviewing the holographic dictionary for AdS$_3$ with Dirichlet boundary conditions, in presence of arbitrary boundary sources. In 3.2, we write down the most general asymptotic solution that corresponds to the deformed theory with all sources set to zero. In 3.3, we make a proposal for the current sector contribution to the stress tensor. In 3.4 we present the final answer we obtain for the holographic one-point functions of the stress tensor and the current.

3.1 Review of the AdS$_3$/CFT$_2$ holographic dictionary

We review herein the standard AdS$_3$/CFT$_2$ dictionary, namely with Dirichlet boundary conditions on the fields. In the bulk, our system consists of Einstein gravity with a negative cosmological constant coupled to a $U(1)$ Chern-Simons gauge field,

$$S_{bulk} = \int d^3x \sqrt{g} \left[\frac{1}{16\pi G} \left(R + \frac{2}{\ell^2} \right) + \frac{k}{8\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right] \tag{3.2}$$

The most general solution for the metric in radial gauge is given by the usual Fefferman-Graham expansion

$$ds^2 = \ell^2 \frac{dz^2}{z^2} + \left(\frac{g^{(0)}_{\alpha\beta}}{z^2} + g^{(2)}_{\alpha\beta} + z^2 g^{(4)}_{\alpha\beta} \right) dx^\alpha dx^\beta \tag{3.3}$$

which terminates in three dimensions [33]. The boundary metric, $g^{(0)}$, is arbitrary; the asymptotic equations of motion fix the trace and divergence of $g^{(2)}$ in terms of $g^{(0)}$, which will yield the holographic Ward identities; the component $g^{(4)}$ is entirely determined by the previous two. As for the gauge field, the Chern-Simons equations of motion require that the connection be flat. In radial gauge ($A_z = 0$), the most general solution for A_μ is then

$$A = A_\alpha(x^\alpha)dx^\alpha \ , \qquad dA = 0 \tag{3.4}$$

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where the A_α, with $\alpha = \pm$ (using null coordinates on the boundary), are z-independent.

The first step in finding the holographic dictionary is to ensure that the variational principle - in this case, with Dirichlet boundary conditions at the AdS$_3$ boundary - is well defined. For this, the gravitational bulk action (3.2) needs to be supplemented by a Gibbons-Hawking boundary term

$$S_{bnd,D}^{grav} = \frac{1}{8\pi G} \int d^2x \sqrt{\gamma}\, K \tag{3.5}$$

An additional boundary counterterm $S_{ct} = -\frac{1}{8\pi G\ell} \int d^2x \sqrt{\gamma}$ is needed to render the expectation value of the holographic stress tensor finite. The variation of the total gravitational on-shell action is then[4]

$$\delta S_{tot}^{grav} = \frac{1}{16\pi G} \int_{z=\epsilon} d^2x \sqrt{\gamma} \left(K_{\alpha\beta} - \gamma_{\alpha\beta}\, K + \frac{1}{\ell}\, \gamma_{\alpha\beta} \right) \delta\gamma^{\alpha\beta} = -\int d^2x \sqrt{g_{(0)}} \frac{1}{2} T_{\alpha\beta}^{grav} \delta g_{(0)}^{\alpha\beta} \tag{3.6}$$

Plugging in the explicit expression for the extrinsic curvature in terms of the asymptotic expansion for the metric, we find

$$T_{\alpha\beta}^{grav} = \frac{1}{8\pi G\ell} \left(g_{\alpha\beta}^{(2)} - g_{\alpha\beta}^{(0)} \frac{\ell^2}{2} R^{(0)} \right) \tag{3.7}$$

The equations obeyed by $g^{(2)}$ ensure that $T^{(grav)}$ obeys the holographic Ward identities, *i.e.*, it is conserved with respect to the boundary metric for any $g^{(0)}$, and its trace is in agreement with the holographic conformal anomaly.

The usual treatment of the Chern-Simons term is as follows (see e.g. [34]). Plugging in the asymptotic expansion of the gauge field into the on-shell variation of the Chern-Simons action, one notes that the two components $A_\pm$ of the gauge field on the boundary are canonically conjugate to each other. Therefore, in order for the variational principle to be well-defined, only one of them can be fixed. By adding the counterterm

$$S_{bnd}^{\text{CS}} = \mp \frac{k}{16\pi} \int d^2x \sqrt{\gamma}\, \gamma^{\mu\nu} A_\mu A_\nu \tag{3.8}$$

one has $\delta S_{CS} \propto \delta A_\mp$; the upper sign corresponds to fixing A_- on the boundary, and the lower one to fixing A_+. From now on, we will choose the upper sign, which yields to a chiral (as opposed to an antichiral) boundary current. The variation of the total on-shell Chern-Simons action takes the form[5]

$$\delta S_{tot}^{\text{CS}} = -\int d^2x \sqrt{\gamma} \left(\frac{1}{2} T_{\alpha\beta}^{\text{CS}} \delta\gamma^{\alpha\beta} + J^\alpha \delta A_\alpha \right) \tag{3.9}$$

where the contribution to the stress tensor is due to explicit dependence of the counterterm (3.8) on the boundary metric. The expectation value of the current is given by

$$J^\alpha = \frac{k}{8\pi}(A^\alpha + \epsilon^{\alpha\beta} A_\beta) \tag{3.10}$$

while

$$T_{\alpha\beta}^{\text{cs}} = \frac{k}{8\pi}(A_\alpha A_\beta - \frac{1}{2}\gamma_{\alpha\beta} A^2) \tag{3.11}$$

In the absence of sources ($A_- = 0$), the Chern-Simons contribution to the holographic stress tensor reads

$$T_{++}^{\text{CS}} = \frac{2\pi}{k} J^2 , \qquad T_{+-}^{\text{CS}} = T_{-+}^{\text{CS}} = T_{--}^{\text{CS}} = 0 \tag{3.12}$$

which yields the correct shift consistent with spectral flow in the dual CFT [30]. However, note that if the boundary source A_- is nonzero, then there is a nontrivial, quadratic contribution of the source to the stress tensor. We will come back to discuss this point in section 3.3.

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[4]Note the sign difference with respect to (2.12) in the definition of the stress tensor. This is due to the differece of definitions of the stress tensor in Euclidean versus Lorentzian signature, which follows from $-S_E = iS_L$ and $\tau_E = it_L$. The expectation values of the two stress tensors are nevertheless the same, after analytic continuation of the time coordinate. The same comments apply to the variation of the Chern-Simons action below.

[5]Note this current differs by a factor of 2π from the usual definition.

3.2 Asymptotic expansion dual to the $J\bar{T}$-deformed theory

Starting from this section, we will fix zero boundary sources in the deformed theory, namely we take $\tilde{e}_b^\beta = \delta_b^\beta$ and $\tilde{a}_\beta = 0$. This will allow us to compute general one-point functions in the deformed theory, but no higher-point correlators. Using the dictionary (3.1), the asymptotic expansion of the dual bulk fields in AdS$_3$ will be given by the usual Fefferman-Graham expansion for the particular case when the boundary sources take the form

$$e_a^\alpha = \delta_a^\alpha + \mu_a J^\alpha , \qquad a_\alpha = \mu_a T^a{}_\alpha \tag{3.13}$$

At the level of the metric, the new boundary condition corresponds to

$$g_{\alpha\beta}^{(0)} = \eta_{\alpha\beta} - \mu_\alpha J_\beta - \mu_\beta J_\alpha \tag{3.14}$$

or, in components[6]

$$g_{++}^{(0)} = -2\mu J(x^+) \equiv P'(x^+) , \qquad g_{+-}^{(0)} = \frac{1}{2} \tag{3.15}$$

Above, we have made use of the assumption that the current J^α is purely chiral, so $J^+ = 0$ and $J^- = 2J_+ = 2J(x^+)$. The quantity $P(x^+)$ has been introduced in order to make contact with the notation of [21], who considered the same boundary metric. We will oftentimes use P' instead of $-2\mu J$ throughout the text. Note that for general $J(x^+)$, this metric is induced by the coordinate transformation[7]

$$x^+ \to x^+ , \qquad x^- \to x^- - 2\mu \int J(x^+)\,dx^+ = x^- + P(x^+) \tag{3.16}$$

The most general solution to Einstein's equations satisfying these boundary conditions has

$$g_{++}^{(2)} = \mathcal{L}(x^+) + \bar{\mathcal{L}}(x^- + P(x^+))P'(x^+)^2 , \qquad g_{+-}^{(2)} = \bar{\mathcal{L}}(x^- + P(x^+))P'(x^+) \tag{3.17}$$

$$g_{--}^{(2)} = \bar{\mathcal{L}}(x^- + P(x^+)) \tag{3.18}$$

where $\mathcal{L}, \bar{\mathcal{L}}$ are two arbitrary functions of their respective arguments. Again, this can be simply obtained by applying the above coordinate transformation to the most general asymptotically AdS$_3$ solution with Dirichlet boundary conditions.

The functions $\mathcal{L}$, $\bar{\mathcal{L}}$ and P entirely determine the gravitational solution. They also entirely determine the solution for the gauge field, if we assume the latter can be decomposed into a part a_α that represents the boundary source and a part A_α^{vev} that is proportional to expectation value of the CFT current

$$A_\alpha = a_\alpha + A_\alpha^{vev} \tag{3.19}$$

The source a_α is determined by the boundary condition (3.13), while A_α^{vev} is determined through the holographic relation (3.10), which in this background reads

$$J^- = 2J(x^+) = \frac{k}{2\pi}(A_+ - P'(x^+)A_-) \tag{3.20}$$

Current conservation and the Chern-Simons equation of motion, $F_{\mu\nu} = 0$, require that

$$\partial_- A_+ = P'(x^+)\partial_- A_- = \partial_+ A_- \tag{3.21}$$

---

[6]There is a single non-vanishing Christoffel symbol, $\Gamma_{++}^- = P''(x^+)$, the boundary Ricci scalar is zero and the boundary vielbeine read

$$e^a{}_\alpha = \begin{pmatrix} 1 & 0 \\ P' & 1 \end{pmatrix} , \qquad e_{a\alpha} = \frac{1}{2}\begin{pmatrix} P' & 1 \\ 1 & 0 \end{pmatrix}$$

The associated spin connection is $\omega_\alpha{}^a{}_b = -e_b^\lambda \nabla_\alpha e_\lambda^a = 0$.

[7]This agrees with the coordinate transformation in [18], since $\mu_{there} = 4\mu_{here}$ upon considering the different definition of the stress tensor and of the deforming factor.

which implies that

$$A_- = \mathcal{A}(x^- + P(x^+)) , \qquad A_+ = \frac{4\pi}{k}J(x^+) + P'(x^+)\mathcal{A}(x^- + P(x^+)) \tag{3.22}$$

for some function $\mathcal{A}$. Using the decomposition (3.19), the terms proportional to $\mathcal{A}$ must correspond to the gauge field source, whose components read

$$\mathrm{a}_+ = P'\mathcal{A} , \qquad \mathrm{a}_- = \mathcal{A} \tag{3.23}$$

These will be identified with the stress tensor via (3.13) or, more precisely,

$$\mathrm{a}_\pm = 2\mu\, T_{-\pm} \tag{3.24}$$

which will in turn determine $\mathcal{A}$ in terms of $\bar{\mathcal{L}}$. Note that by itself, the boundary source satisfies $\mathrm{f}_{\alpha\beta} = 0$. This is consistent with (3.13) provided that $\mu_a\partial_{[\alpha}T^a_{\beta]} = 0$, which we checked.

To find the solution for $\mathcal{A}$, we need to compute $T_{a\alpha} = e^\beta_a T_{\alpha\beta}$. The stress tensor is the sum of a gravitational and a Chern-Simons contribution

$$T_{\alpha\beta} = T^{grav}_{\alpha\beta} + T^{\mathrm{CS}}_{\alpha\beta} \tag{3.25}$$

The gravitational contribution is simple to compute

$$T^{grav}_{a\alpha} = e^\beta_a T^{grav}_{\beta\alpha} = \frac{1}{8\pi G\ell}\left(g^{(2)}_{a\alpha} + \mu_a g^{(2)}_{\alpha\beta}J^\beta\right) \tag{3.26}$$

or, in components

$$T^{grav}_{++} = \frac{\mathcal{L}(x^+)}{8\pi G\ell} , \qquad T^{grav}_{+-} = 0$$

$$T^{grav}_{-+} = \frac{\bar{\mathcal{L}}(x^- + P(x^+))P'(x^+)}{8\pi G\ell} , \qquad T^{grav}_{--} = \frac{\bar{\mathcal{L}}(x^- + P(x^+))}{8\pi G\ell} \tag{3.27}$$

According to the holographic dictionary (3.11), adding the Chern-Simons contribution to the stress tensor corresponds to replacing

$$\mathcal{L} \to \mathcal{L} + \frac{16\pi^2 G\ell}{k}J^2 , \qquad \bar{\mathcal{L}} \to \bar{\mathcal{L}} + kG\ell\mathcal{A}^2 \tag{3.28}$$

in the holographic expectation values (3.27). Note that the coordinate dependence of the stress tensor components and their ratio is consistent with the coordinate dependence and ratio of $\mathrm{a}_\pm$.

If we include the Chern-Simons contribution as above, we find a quadratic equation determining $\mathcal{A}$ in terms of $\bar{\mathcal{L}}$, whose solution only makes sense if $\bar{\mathcal{L}}$ is small enough[8]. Since there was no indication from the field theory analysis of [18] that there should exist such a maximum value for the right-moving energy, we take this as a sign that the contribution of the current sector of the CFT to the energy differs from (3.11). In the following subsection, we will make the case for a different contribution, for which $T^{\mathrm{CS}}_{-\alpha} = 0$. Consequently, the relation between the function $\mathcal{A}$ determining the gauge field source and the expectation value of the stress tensor becomes

$$\mathcal{A}(x^- + P(x^+)) = \frac{\mu\bar{\mathcal{L}}(x^- + P(x^+))}{4\pi G\ell} \tag{3.29}$$

With this, the bulk solution is completely specified[9]. It is parametrized by as many free functions as in the case of Dirichlet boundary conditions. In fact, the full bulk solution can be obtained by applying the field-dependent coordinate transformation (3.16), together with a field-dependent gauge transformation

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[8]More precisely, the solution that is smooth as $\mu \to 0$ is $\mathcal{A} = \frac{2\pi}{k\mu}\left(1 - \sqrt{1 - \frac{\mu^2 k\bar{\mathcal{L}}}{4\pi^2 G\ell}}\right)$.

[9] With the caveat that, since we do not currently have a holographic derivation of the Chern-Simons stress tensor (3.39), we cannot be entirely sure whether the holographic formulae (3.10) and (3.19) can be used. However, the nice fit between the structure of the bulk gauge field and the asymptotic boundary conditions (3.13) indicates that they still hold.

$$\lambda = 2\mu \int dx^- T_{--}(x^- + P(x^+)) \tag{3.30}$$

to the most general AdS$_3$ solution with Dirichlet boundary conditions and zero gauge field source. Note that both transformations break the fields' periodicities.

3.3 Current sector contribution to the stress tensor: a proposal

In the previous section, we have used the standard holographic dictionary to model the $U(1)$ current present in the boundary CFT by a Chern-Simons term in the bulk. The Chern-Simons description naturally captures the chirality of the current, its conservation (including its breaking by anomalies, which we do not study in this paper) and the shift of the stress tensor under spectral flow. However, as pointed out in [35], certain of the Chern-Simons predictions, such as its contribution to the partition function in presence of non-trivial sources, must be handled with care when comparing to the boundary CFT result.

In this paper, rather than working out the subtleties of the Chern-Simons holographic dictionary in presence of sources and finding a way to recover the boundary CFT prediction, we use a shortcut that brings us immediately to the expected answer. Namely, it is well known that the dynamics of *pure* $U(1)$ Chern-Simons theory, in AdS or on any manifold with a boundary, reduces to that of a chiral boson on the boundary [36, 37]. The latter is equivalent to a pair of chiral fermions via bosonisation. When coupling each of these systems to an external gauge field, even a pure gauge one of the form (3.23), the naive shift of the stress tensor away from its free value is different among the three descriptions. The question then becomes which of the three predictions should best model the boundary CFT result. This turns out to be the description in terms of chiral fermions, which is the closest in spirit to the actual CFT realisation of the chiral current for known examples such as the D1-D5 CFT. Of course, in principle there should be a way to relate the contribution obtained in the fermionic picture to the one in terms of the chiral boson[10] and in turn to the Chern-Simons one, but we leave the detailed analysis to future work.

The action for fermions coupled to an external gauge field a_α is

$$S = \int d^2x \, e \, i\bar{\Psi}\gamma^a e_a^\alpha (\mathcal{D}_\alpha - i a_\alpha)\Psi \tag{3.31}$$

where $\mathcal{D}_\alpha$ includes the spin connection and the fermions are chiral, $\gamma^3\Psi = \Psi$. We are interested in computing the stress tensor defined in (2.12), which couples to the vielbein e_a^α. The result should then be evaluated on the background (3.13), in presence of the pure gauge source (3.23), and should yield the "Chern-Simons" contribution to the stress-tensor one-point function in (3.25).

This stress tensor is obtained by varying the action (3.31) with respect to the vielbein; however, in order to better understand how the presence of the gauge field affects the physical left-moving solution for the fermions, we will plug in the background (3.13) directly into the action and compute the stress tensor via the canonical method. Due to the explicit time dependence of the gauge field, the latter is not conserved; however, it is related to the gauge-invariant stress tensor via

$$T_{a\alpha}^{gauge\,inv.} = T_{a\alpha}^{can} + J_a a_\alpha \tag{3.32}$$

which is conserved because $f_{\alpha\beta} = 0$. This gauge-invariant stress tensor coincides with the one obtained by varying with respect to the vielbein. Plugging in the explicit two-dimensional gamma matrices and $\Psi = (\psi\,0)$, the action simplifies to

$$S = -i \int d^2x \, \psi^\star (\partial_- - i a_-)\psi \tag{3.33}$$

The equation of motion is $\partial_-\psi = i a_- \psi$, with solution

$$\psi(x^+, x^-) = e^{i\lambda(x^+, x^-)}\psi^{(0)}(x^+)\,, \qquad \partial_-\lambda = a_- \tag{3.34}$$

---

[10]For example, treating the chiral boson in the Hamiltonian formalism, with $H = \pi(\partial_x\varphi - 2a_-)$, yields a shift in the canonical energy which is identical to the one we obtain in the fermionic picture.

where $\psi^{(0)}$ is the solution in absence of the gauge field, which is purely left-moving. The only non-zero component of the gauge-invariant stress tensor $T_{a\alpha}$, evaluated on the solution, is

$$T_{++} = -\frac{i}{2}\psi^\star(\partial_+ - ia_+)\psi = -\frac{i}{2}\psi^{(0)\star}\partial_+\psi^{(0)} - \frac{1}{2}\psi^\star\psi\,(a_+ - \partial_+\lambda) \tag{3.35}$$

Note that even though the external source is pure gauge, a_+ need not exactly equal $\partial_+\lambda$, since the gauge parameter is constrained to respect the fermion's periodicity condition, i.e. Neveu-Schwarz or Ramond[11]. Rewriting the above equation in terms of the undeformed stress tensor $T_{++}^{(0)}$ and the current, the change in the stress tensor due to adding the external source is

$$T_{++} = T_{++}^{(0)} + J(x^+)(a_+ - \partial_+\lambda) \tag{3.36}$$

where the normalizations have been fixed by requiring that they map to the standard normalization for a chiral boson. More generally, one would also expect contributions to the stress tensor that are quadratic in the "transverse" gauge field (see e.g. [38] for a recent review and a list of references); however, these terms vanish in our case because the field strength of the external gauge source vanishes.

We would now like to evaluate the expectation value of the above stress tensor on a cylinder of radius R, when the gauge field we are coupling to has components

$$a_- = \frac{\mu\bar{\mathcal{L}}(x^- + P(x^+))}{4\pi G\ell} \;,\quad a_+ = \frac{\mu P'\bar{\mathcal{L}}(x^- + P(x^+))}{4\pi G\ell} \tag{3.37}$$

This form of the gauge field follows from (3.24), after using the above-derived fact that the current contribution to $T_{-\pm}$ is zero. Expanding the function $\bar{\mathcal{L}}(x^- + P(x^+))$ in Fourier modes $\bar{\mathcal{L}}_n$, the solution for λ defined in (3.34) is

$$\lambda(x^+, x^-) = \frac{\mu}{4\pi G\ell}\left[\bar{\mathcal{L}}_0(x^- - x^+) - iR'\sum_{n\neq 0}\frac{\bar{\mathcal{L}}_n}{n}e^{\frac{in}{R'}(x^- + P(x^+))}\right] \tag{3.38}$$

where, importantly, the first term multiplying the $\bar{\mathcal{L}}$ zero mode is fixed by requiring that λ have no winding mode, which is the same as requiring the correct periodicity of the deformed fermion, and $2\pi R'$ is the periodicity of the coordinate $x^- + P(x^+)$. It is then easy to check that $\partial_+\lambda = a_+$ for all the non-zero modes, but the zero mode contribution is non-vanishing, yielding

$$T_{++}^{\text{cs}} = T_{++}^{(0)\,\text{cs}} + J(x^+)\frac{\mu\bar{\mathcal{L}}_0}{4\pi G\ell}(1 + P'(x^+)) \tag{3.39}$$

$$T_{+-}^{\text{cs}} = T_{-+}^{\text{cs}} = T_{--}^{\text{cs}} = 0$$

Thus, we find that the contribution of the current sector to the stress tensor in presence of the external gauge field is almost identical to its contribution in absence of the gauge field, except for a term involving the zero mode. This is not surprising, given that the field strength of the external source is zero. Note that our argument is almost identical to the one of [18] for the case of $J\bar{T}$ - deformed free fermions, since performing a gauge transformation of the form (3.30) on the free fermion action is the same as deforming by $J\bar{T}$. The zeroth order stress tensor is given by (3.12) and follows from the usual Chern-Simons holographic dictionary. As we will see in section 4.2, the above contribution of the current to the stress tensor exactly reproduces the field theory result.

3.4 The holographic expectation values

We now have all the necessary ingredients to write down the holographic one-point functions in the deformed CFT. As already explained, the expectation values of the stress tensor $\tilde{T}$ and current $\tilde{J}$ are given in terms of the expectation values in the original CFT in presence of sources, using (2.17). The expectation value of the current is trivially the same $\tilde{J}_+ = J(x^+)$. The new stress tensor is given by

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[11]Note that the parameter λ in (3.34) is *not* the same as the one in (3.30), but they differ at the level of winding modes.

$$\tilde{T}_\alpha^a = T_\alpha^a + (e_\alpha^a + \mu_\alpha J^a)\, \mu_b T_\beta^b J^\beta \tag{3.40}$$

where T_α^a includes both the gravitational and the Chern-Simons contribution. Plugging in (3.27) and (3.39) into the above, we find that the components of $\tilde{T}_{a\alpha}$ are

$$\tilde{T}_{++} = \frac{\mathcal{L}(x^+)}{8\pi G\ell} + \frac{2\pi J^2(x^+)}{k} + J(x^+)\frac{\mu\bar{\mathcal{L}}_0}{4\pi G\ell}(1 - 2\mu J(x^+))\,, \qquad \tilde{T}_{-+} = 0$$

$$\tilde{T}_{+-} = -\frac{\bar{\mathcal{L}}(x^- + P(x^+))P'(x^+)}{8\pi G\ell}\,, \qquad \tilde{T}_{--} = \frac{\bar{\mathcal{L}}(x^- + P(x^+))}{8\pi G\ell} \tag{3.41}$$

This is our proposed holographic dictionary for the stress tensor. It is easy to check that it satisfies the Ward identity (2.20) for $\tilde{e}_\alpha^a = \delta_\alpha^a$ and $\tilde{\mathrm{a}}_\alpha = 0$, namely

$$\partial_\lambda(\tilde{T}_\alpha^a \delta_a^\lambda) = 0 \tag{3.42}$$

Note it is crucial that the "Chern-Simons" contribution only contains the zero mode $\bar{\mathcal{L}}_0$, as otherwise the conservation law would be violated. The vanishing of the component $\tilde{T}_{-+} = 0$ is precisely what we expect from the $SL(2,\mathbb{R})$ symmetry of the deformed theory. We will now show that the above values are also in perfect agreement with the one-point functions one obtains from the field theory.

4. Checks and predictions

In [18], the Smirnov-Zamolodchikov method was used to compute the spectrum of energy-momentum-charge eigenstates in the $J\bar{T}$-deformed theory on a cylinder of radius R in terms of the original CFT spectrum[12]

$$E_R = \frac{h_R - \frac{c}{24}}{R - \mu Q/\pi}\,, \qquad E_L = \frac{h_L - \frac{c}{24}}{R} + \frac{\mu Q\left(h_R - \frac{c}{24}\right)}{\pi R(R - \mu Q/\pi)} \tag{4.1}$$

where $E_{L,R} = \frac{1}{2}(E - \mathcal{J})$, $h_{L,R}$ are the CFT left/right conformal dimensions and Q is the $U(1)$ charge. Since the spectrum is continuously deformed, the degeneracy of states, and thus the entropy, should be unchanged when written in terms of $h_{L,R}$. For $h_{L,R} \gg c$, the entropy is given by Cardy's formula

$$S = 2\pi\sqrt{\frac{c}{6}\left(h_L - \frac{c}{24} - \frac{Q^2}{k}\right)} + 2\pi\sqrt{\frac{c}{6}\left(h_R - \frac{c}{24}\right)} \tag{4.2}$$

Replacing $h_{L,R}$ by their expressions in terms of the energy, one can easily derive the thermodynamic properties of the deformed theory. In two dimensions, it is natural to introduce the left/right temperatures $T_{L,R}$, which are conjugate to $E_{L,R}$

$$\delta S = \frac{1}{T_L}\delta E_L + \frac{1}{T_R}\delta E_R \tag{4.3}$$

Using (4.1), one finds that the left/right temperatures $\tilde{T}_{L,R}$ in the $J\bar{T}$ - deformed CFT are related to their undeformed counterparts via

$$\tilde{T}_L = T_L\,, \qquad \frac{1}{\tilde{T}_R} = \frac{R - \mu Q/\pi}{R}\frac{1}{T_R} - \frac{\mu Q}{\pi R}\frac{1}{T_L} \tag{4.4}$$

at fixed $h_{L,R}, Q$. Note the above implies that the usual temperature T_H (conjugate to the total energy $E = E_L + E_R$) is given by

$$\tilde{T}_H = \frac{T_H}{1 - \mu Q/(\pi R)} \tag{4.5}$$

---

[12]We have adapted the results of [18] to our new conventions, in which $E_{L,R} \to 2\pi E_{L,R}$, $R \to 2\pi R$ and $\mu \to 4\mu$.

as also found in [18]. These computations are valid when $R > \mu Q/\pi$.

For $h_{L,R} \gg c$, we expect that these energy-momentum-charge eigenstates[13] are modeled by charged black holes in the bulk. The goal of this section is to check whether the energy and thermodynamics of black holes obeying the boundary conditions presented in section 3.2 agree with the above field-theoretical expressions. In particular, we would like to check our proposed expression (3.41) for the holographic stress tensor against the field theory result (4.1).

For this, we must first construct the black hole solutions obeying the modified boundary conditions that correspond to the states labeled by $h_{L,R}$ in (4.1). To ensure this, we match their parametres to those in the field theory by requiring that their thermodynamic properties (calculated from the horizon area and the identifications of the euclidean solution) agree. Having fixed the parameters this way, we show in 4.2 that the energy and angular momentum of these black holes also match the field theory expectation, as required by the first law of black hole mechanics, thus providing a non-trivial consistency check for our method and the proposed holographic dictionary (3.41).

In section 4.3 we use the special properties of the holographic stress tensor (3.41) to construct an infinite set of conserved charges in the deformed theory and we compute on the algebra they satisfy.

4.1 Black holes and their thermodynamics

As shown in section 3.2, gravitational solutions dual to classical states in the deformed theory are obtained by simply applying the coordinate transformation (3.16) to the known AdS$_3$ solutions. To obtain a black hole, the seed AdS$_3$ solution is the BTZ black hole, with metric

$$ds^2 = -\frac{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)}{\rho^2}dt^2 + \frac{\ell^2\rho^2 d\rho^2}{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)} + \rho^2\left(d\hat{\varphi} + \frac{\rho_+\rho_-}{\rho^2}dt\right)^2 \tag{4.6}$$

where $\hat{\varphi} \sim \hat{\varphi} + 2\pi$ and $\rho_\pm$ are the horizon radii. The coordinate $\hat{\varphi}$ is related to the compact boundary coordinate φ via $\varphi = R\hat{\varphi}$. To make the connection with black hole thermodynamics, it is useful to introduce the dimensionless temperatures $\hat{T}_{L,R}$ via

$$\hat{T}_{L,R} = \frac{\rho_+ \pm \rho_-}{2\pi\ell} = R\,T_{L,R} \tag{4.7}$$

which are the conjugate potentials to the left/right energies of the black hole defined as in (4.3). These temperatures can be easily read off from the identifications of the euclidean BTZ solution [39], obtained by the analytic continuation $t \to i\tau$

$$\hat{\varphi} + i\tau \sim \hat{\varphi} + i\tau + 2\pi m + \frac{in}{\hat{T}_L}, \qquad m,n \in \mathbb{Z} \tag{4.8}$$

In euclidean BTZ, ρ_- is purely imaginary, so $\hat{T}_{L,R}$ become complex conjugate to each other. Analytically continuing back to the Lorentzian solution, $\hat{\varphi}+i\tau \to \hat{x}^+$ and $\hat{\varphi}-i\tau \to \hat{x}^-$, where $\hat{x}^\pm = \hat{\varphi} \pm t$ are the rescaled null coordinates. The above identifications of the euclidean coordinates formally act on the Lorenzian null coordinates as

$$\hat{x}^+ \sim \hat{x}^+ + 2\pi m + \frac{in}{\hat{T}_L}, \qquad \hat{x}^- \sim \hat{x}^- + 2\pi m - \frac{in}{\hat{T}_R} \tag{4.9}$$

where $\hat{T}_{L,R}$ have gone back to being real and independent.

To read off the conserved charges, it is useful to put the metric (4.6) in Fefferman-Graham form[14]

$$ds^2 = \frac{\ell^2 dz^2}{z^2} + \frac{dx^+ dx^-}{z^2 R^2} + 4G\ell\hat{h}_L\frac{(dx^+)^2}{R^2} + 4G\ell\hat{h}_R\frac{(dx^-)^2}{R^2} + \frac{z^2}{R^2}16G^2\ell^2\hat{h}_L\hat{h}_R\,dx^+ dx^- \tag{4.10}$$

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[13]Or, rather, collections of such eigenstates in a narrow energy band.

[14]The relationship between ρ and z is $\rho = \frac{1}{z}\sqrt{1 + \frac{z^2}{2}(\rho_+^2 + \rho_-^2) + \frac{z^4}{16}(\rho_+^2 - \rho_-^2)^2}$; the horizon is at $z = \frac{2}{\sqrt{\rho_+^2 - \rho_-^2}}$.

where $x^\pm = R\hat{x}^\pm = \varphi \pm t$ and t has been rescaled with respect to the one in the previous equation by a factor of R. We have defined

$$\hat{h}_{L,R} = \frac{(\rho_+ \pm \rho_-)^2}{16G\ell} = \frac{\pi^2 c}{6}\hat{T}_{L,R}^2 , \qquad c = \frac{3\ell}{2G} \tag{4.11}$$

Computing the left/right energies by integrating the holographic stress tensor (3.7) over φ, we find that $\hat{h}_{L,R} = E_{L,R}R$, so $\hat{h}_{L,R}$ are related to the CFT conformal dimensions via $\hat{h}_{L,R} = h_{L,R} - c/24$. If the black hole also carries a left-moving $U(1)$ charge Q, then the relationship between the left-moving energy and $\hat{h}_L$ is shifted by the Chern-Simons contribution (3.12) as

$$E_L R = \hat{h}_L + Q^2/k = h_L - c/24 \tag{4.12}$$

While the metric (4.10) differs from the usual Fefferman-Graham form by a rescaling of z by R, this form is extremely useful for tracking the R dependence of the various quantities we read off from the geometry, which will be important later. In particular, one can easily check, for example, that the line element is completely independent of R, as can be seen by rewriting it in terms of the rescaled coordinates $\hat{x}^\pm$, which are identified mod 2π. This implies, in particular, that the black hole entropy is R-independent, and reads

$$S = \frac{\mathcal{A}_{horizon}}{4G} = \frac{\pi\rho_+}{2G} = 2\pi\sqrt{\frac{c\,\hat{h}_L}{6}} + 2\pi\sqrt{\frac{c\,\hat{h}_R}{6}} \tag{4.13}$$

which is of course identical to (4.2).

We would now like to construct black holes in the deformed theory such that their entropy is identical to that of BTZ, when written in terms of $h_{L,R}$. Such black holes are obtained by performing the coordinate transformation (3.16) on the undeformed solution. For the energy-momentum-charge eigenstates, J is a constant, $J = Q/(2\pi R)$, so the coordinate transformation simplifies to

$$x'^+ = x^+ , \qquad x'^- = x^- - \frac{\mu Q}{\pi R}x^+ \tag{4.14}$$

These coordinates have identifications

$$x'^+ \sim x'^+ + 2\pi R , \qquad x'^- \sim x'^- + 2\pi R - 2\mu Q \tag{4.15}$$

We will also find it useful to introduce the rescaled coordinates $\hat{x}'^\pm$

$$\hat{x}'^+ = \frac{x'^+}{R} , \qquad \hat{x}'^- = \frac{x'^-}{R - \mu Q/\pi} \tag{4.16}$$

which are identified mod 2π. Since the black hole entropy only depends on $\hat{h}_{L,R}$, we should require that the deformed metric be the same as (4.10), when written in terms of the coordinates $\hat{x}'^\pm$

$$ds^2 = \frac{\ell^2 dz^2}{z^2} + \frac{d\hat{x}'^+ d\hat{x}'^-}{z^2} + 4G\ell\hat{h}_L(d\hat{x}'^+)^2 + 4G\ell\hat{h}_R(d\hat{x}'^-)^2 + 16G^2\ell^2\hat{h}_L\hat{h}_R\,z^2\,d\hat{x}'^+ d\hat{x}'^- \tag{4.17}$$

By construction, the above black hole has the same horizon area and thus the same entropy as the original BTZ solution. When rewritten in terms of the $x^\pm$ boundary coordinates (4.14) and after a simple rescaling of z, this metric obeys the modified boundary conditions of section 3.2.

We would now like to check whether the left/right temperatures that we can read off from the identification of the euclideanized solution (4.17) are in agreement with the field theory prediction. Using the fact that, in terms of the hatted coordinates, the metric is the same as (4.10), smoothness of the euclidean solution requires that

$$\hat{x}'^+ \sim \hat{x}'^+ + \frac{in}{\hat{T}_L} , \qquad \hat{x}'^- \sim \hat{x}'^- - \frac{in}{\hat{T}_R} \tag{4.18}$$

This translates into the following identifications of $x^\pm$

$$x^+ \sim x^+ + \frac{inR}{\hat{T}_L} \ , \qquad x^- \sim x^- - \frac{in(R - \mu Q/\pi)}{\hat{T}_R} + \frac{\mu Q}{\pi} \frac{in}{\hat{T}_L} \tag{4.19}$$

which imply that the left/right temperatures of these black holes are identical to the left/right temperatures (4.4) in the deformed theory. This confirms that (4.17) are the correct black hole solutions, whose energies we should compare with (4.1).

Note that the charge Q was kept fixed thoughout the above discussion; in particular, we did not consider the electric potential contribution to the first law. Including such a term is possible in principle and it would lead to an additional check on the holographic dictionary: namely, whether the gauge field (3.1), in presence of appropriate boundary sources $\tilde{a}_\alpha$ proportional to the chemical potential, has vanishing holonomy along the contractible (time) circle of the euclidean black hole geometry. We leave this more complete analysis for future work.

4.2 Match to the field theory spectrum

We would now like to use the holographic dictionary (3.41) to read off the conserved charges associated with the metric (4.17) and check whether the energies agree with the field theory expressions (4.1). Upon a simple z rescaling, the metric (4.17) is of the form (3.17), with

$$\mathcal{L} = \frac{4\hat{h}_L G\ell}{R^2} \ , \qquad \bar{\mathcal{L}} = \bar{\mathcal{L}}_0 = \frac{4\hat{h}_R G\ell}{(R - \frac{\mu Q}{\pi})^2} \ , \qquad J = \frac{Q}{2\pi R} \tag{4.20}$$

Plugging these into the expectation value of the deformed stress tensor (3.41), we find

$$\tilde{T}_{++} = \frac{1}{2\pi R} \left(\frac{\hat{h}_L}{R} + \frac{Q^2}{kR} + \frac{\mu Q \hat{h}_R}{\pi R(R - \mu Q/\pi)} \right) \tag{4.21}$$

$$\tilde{T}_{--} = \frac{\hat{h}_R}{2\pi(R - \mu Q/\pi)^2} \ , \qquad \tilde{T}_{+-} = \frac{\mu Q}{\pi R} \cdot \frac{\hat{h}_R}{2\pi(R - \mu Q/\pi)^2} \tag{4.22}$$

and $\tilde{T}_{-+} = 0$, as before. The conserved charges associated to translations along a boundary (Killing) vector ξ^α are given by

$$Q_\xi = \int_{\mathcal{P}} d\varphi \, n^a \tilde{T}_{a\alpha} \xi^\alpha = \int_0^{2\pi R} d\varphi \, (\tilde{T}_{+\alpha} - \tilde{T}_{-\alpha}) \, \xi^\alpha \tag{4.23}$$

where $\mathcal{P}$ is a constant t slice of the cylinder on which the deformed theory is defined and $n^a = \partial_t$ is the unit vector normal to it. Note that the integral is not performed over the induced metric at the boundary of the asymptotically AdS$_3$ spacetime (3.14), but rather over an abstract boundary whose metric, previously denoted as $\tilde{\gamma}_{\alpha\beta}$, is flat.

The left-moving energy of the black hole is the conserved charge associated with $\xi = \partial_+$

$$E_L = \int_0^{2\pi R} d\varphi \, \tilde{T}_{++} = \frac{h_L - \frac{c}{24}}{R} + \frac{\mu Q(h_R - \frac{c}{24})}{\pi R(R - \mu Q/\pi)} \tag{4.24}$$

where we used (4.12). The right-moving energy is the charge associated with $\xi = -\partial_-$

$$E_R = \int_0^{2\pi R} d\varphi \, (\tilde{T}_{--} - \tilde{T}_{+-}) = \frac{h_R - \frac{c}{24}}{R - \mu Q/\pi} \tag{4.25}$$

Both expressions are in perfect agreement with the field theory result (4.1). This constitutes a rather non-trivial check of our proposed holographic dictionary (3.41).

4.3 Symmetry enhancement

An interesting question is whether the global $SL(2,\mathbb{R})_L \times U(1)_R \times U(1)_J$ symmetries of the $J\bar{T}$ - deformed CFT allow for an infinite-dimensional extension. For the case of *local* two-dimensional QFTs with an $SL(2,\mathbb{R})_L \times U(1)_R$ global symmetry, this question has been previously studied in [22], who showed that the $SL(2,\mathbb{R})_L$ symmetry is enhanced, as expected, to a left-moving Virasoro symmetry, while the $U(1)_R$ is enhanced to either a left-moving Kač-Moody symmetry, or to a right-moving Virasoro.

The case of $J\bar{T}$-deformed CFTs is somewhat different, because the theory is non-local along the $U(1)_R$, a.k.a. x^-, direction. Thus, we do not necessarily expect to find a local infinite-dimensional symmetry. To study the symmetry enhancement, there are two possible approaches: either to use the special properties of the stress tensor in the deformed theory to construct an infinite set of conserved charges, or to study the asymptotic symmetries of the dual spacetime. In the following, we will use both methods to argue that the above global symmetries are enhanced to infinite-dimensional ones, the $SL(2,\mathbb{R})_L \times U(1)_J$ to a left-moving Virasoro-Kač-Moody symmetry as before, and the $U(1)_R$ to a state-dependent, effectively non-local Virasoro symmetry.

i) Direct construction of the conserved charges

The holographic one-point functions (3.41) show that the stress tensor of the deformed CFT obeys

$$\tilde{T}_{-+} = 0 \;, \qquad \tilde{T}_{+-} = -P'\,\tilde{T}_{--} = 2\mu J \tilde{T}_{--} \tag{4.26}$$

for arbitrary CFT states with a classical bulk dual. The first equation simply follows from the $SL(2,\mathbb{R})_L$ symmetry of the theory and holds also as an operator equation; the second is derived from holography, though there may also exist a purely CFT derivation, valid for more general states. Together with the conservation equations (3.42), they imply the following spacetime dependence of the stress tensor components

$$\tilde{T}_{++} = \tilde{T}_{++}(x^+) \qquad \tilde{T}_{--} = \tilde{T}_{--}(x^- + P(x^+)) \tag{4.27}$$

Using the above, it is easy to construct an infinite family of conserved charges. These are given by

$$Q_{\chi_L} = \int_0^{2\pi R} d\varphi\, \tilde{T}_{++}\, \chi_L(x^+) \;, \qquad Q_{\chi_R} = \int_0^{2\pi R} d\varphi\, \left(\tilde{T}_{--} - \tilde{T}_{+-} \right) \chi_R(x^- + P(x^+)) \tag{4.28}$$

where $\chi_{L,R}$ are arbitrary functions of their respective arguments, and each of their Fourier modes is associated with a separate conserved charge. These charges are conserved due to conservation of the currents $\tilde{T}_{\alpha\beta}\,\xi_{L,R}^\beta$ for $\xi_L = \chi_L(x^+)\partial_+$ and $\xi_R = -\chi_R(x^- + P(x^+))\partial_-$, respectively, where the derivative is computed with respect to the flat "boundary" metric $\tilde{\gamma}_{\alpha\beta} = \eta_{\alpha\beta}$ of the deformed CFT. Note that while ξ_L is a conformal Killing vector of the "boundary" metric $\eta_{a\beta}$, as is usually the case, χ_R is not; rather, its form is dictated by charge conservation, together with the relation (4.27) between the stress tensor components. Using it, the formula for the right-moving conserved charges reduces to

$$Q_{\xi_R} = \int_0^{2\pi R} d\varphi\, (1 - 2\mu J(x^+))\, \tilde{T}_{--}(x^- + P(x^+))\, \chi_R(x^- + P(x^+)) \tag{4.29}$$

whose integrand is a total φ derivative. Note this is very similar to the formula for the right-moving Virasoro generators in a two-dimensional CFT, except that the argument has been shifted by a state-dependent function of x^+. This implies that the action of these symmetry generators on local operators is effectively non-local.

In addition to the above translational symmetries, there is an infinite enhancement of the $U(1)_J$ global symmetries, generated by

$$Q_\lambda = \int_0^{2\pi R} d\varphi\, \lambda(x^+) J(x^+) \tag{4.30}$$

for an arbitrary function $\lambda(x^+)$, as follows from the conservation of the current $\lambda(x^+)\, J^\alpha$.

Thus, we have shown that each symmetry factor acquires an infinite-dimensional extension, in a way very similar to what happens in two-dimensional CFTs. Two of these enhanced symmetries, Q_{χ_L} and Q_λ, are local, while the remaining one, Q_{χ_R} is state-dependent and appears non-local. One natural question is to find the algebra satisfied by these conserved charges, which before the deformation is Virasoro$_L \times$ Virasoro$_R \times U(1)$ Kač-Moody. In the following, we will use the complementary approach of asymptotic symmetries to argue that the algebra stays the same, even though the generators (4.29) have been deformed.

ii) Asymptotic symmetries

The enhanced symmetries of the field theory can be often obtained from an analysis of the asymptotic symmetries of the dual spacetime. These are the diffeomorphisms and gauge transformations that preserve the boundary conditions on the metric and gauge field, while leading to non-trivial conserved charges. An important result that we will be using in this section is the representation theorem [40], which states that the Dirac bracket algebra of the conserved charges is isomorphic to the (modified) Lie bracket algebra $\{\,,\,\}_*$ of the associated asymptotic symmetry generators, up to a possible central extension

$$\{Q_{g_1}, Q_{g_2}\}_{D.B.} = Q_{\{g_1,g_2\}_*} + \mathcal{K}_{g_1,g_2}[\bar{\Phi}] \tag{4.31}$$

that only depends on the reference background $\bar{\Phi}$. This means that we do not need to explicitly compute the charges to find the algebra they satisfy, but we can simply infer it from the algebra of the asymptotic symmetries, under the assumption that the charges will turn out to be integrable.

As will turn out, each asymptotic symmetry is implemented by a pair $g = (\xi, \Lambda)$ of a diffeomorphism and a gauge transformation. Additionally, both symmetry parameters can depend on the background fields $\Phi = \{\mathcal{L}, \bar{\mathcal{L}}, J\}$ that parametrize the asymptotic solution: $\xi = \xi[\chi, \Phi]$, $\Lambda = \Lambda[\chi, \Phi]$, where χ denotes the parameter of the transformation. In presence of symmetries simultaneously associated with diffeomorphisms and gauge transformations, the algebra of the asymptotic symmetries is [41]

$$\{(\xi_1, \Lambda_1), (\xi_2, \Lambda_2)\}_* = \left(\{\xi_1, \xi_2\}_*, \{\Lambda_1, \Lambda_2\}_*\right) \tag{4.32}$$

where $\{\xi_1, \xi_2\}_*$ is given by a modified Lie bracket [42] (see also [43])

$$\{\xi_1, \xi_2\}_* = [\xi_1, \xi_2]_{L.B.} - (\delta_{\chi_1}\Phi\, \partial_\Phi \xi_2 - \delta_{\chi_2}\Phi\, \partial_\Phi \xi_1) \tag{4.33}$$

which takes into account the field-dependence of the diffeomorphisms. The bracket of the gauge transformations is given by

$$\{\Lambda_1, \Lambda_2\}_* = \mathcal{L}_{\xi_1}\Lambda_2 - \mathcal{L}_{\xi_2}\Lambda_1 - (\delta_{\chi_1}\Phi\, \partial_\Phi \Lambda_2 - \delta_{\chi_2}\Phi\, \partial_\Phi \Lambda_1) \tag{4.34}$$

where we subtracted a similar correction due to field-dependence of the gauge parameter from the bracket quoted in [41].

The asymptotic form of the metric is left invariant by a combination of diffeomorphisms and a gauge transformation that i) preserve the radial gauge for the metric and the gauge field and ii) leave invariant the asymptotic relations (3.13) between sources and expectation values. They are parametrized by the same three functions $\chi_L(x^+)$, $\chi_R(x^- + P(x^+))$ and $\lambda(x^+)$ that we introduced above. The diffeomorphism component of the asymptotic symmetries is

$$\xi_L = \chi_L\, \partial_+ - \frac{z^2 \ell^2}{2}\, \chi_L''\partial_- + \frac{z}{2}\, \chi_L'\, \partial_z + \dots$$

$$\xi_R = \left(\chi_R + \frac{z^2 \ell^2}{2}\, P'(x^+)\, \chi_R''\right)\partial_- - \frac{z^2 \ell^2}{2}\chi_R''\, \partial_+ + \frac{z}{2}\chi_R'\, \partial_z + \dots \tag{4.35}$$

$$\xi_\Lambda = -\frac{\mu k}{2\pi}\lambda(x^+)\, \partial_-$$

The first two diffeomorphisms are the direct analogues of Brown-Henneaux diffeomorphisms in AdS$_3$; the last one is the same as the generator of Kač-Moody symmetries of [21]. Note that ξ_R is now a conformal Killing vector of the boundary metric (3.14).

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The accompanying gauge transformations take the form

$$\Lambda_L = \frac{\mu \ell \bar{\mathcal{L}}}{8\pi G} \, \chi_L'' z^2 + \dots \tag{4.36}$$

$$\Lambda_R = \int \mathcal{A} \chi_R' - \left(\frac{\mu \ell}{8\pi G} - \frac{2\pi \ell^2}{k} z^2 J(x^+) \right) \chi_R'' + \dots \tag{4.37}$$

$$\Lambda_\Lambda = \lambda(x^+) \tag{4.38}$$

where $\mathcal{A}$ has been defined in (3.22). The $\dots$ denote terms proportional to higher powers of z, which are not expected to contribute to the conserved charges.

The combined action of the diffeomorphisms and the gauge transformation on the functions that specify solution is given by

$$\delta \mathcal{L} = 2\mathcal{L} \chi_L' + \mathcal{L}' \chi_L - \frac{\ell^2}{2} \chi_L''' \,, \qquad \delta \bar{\mathcal{L}} = 2\bar{\mathcal{L}} \chi_R' + \bar{\mathcal{L}}' \chi_R - \frac{\ell^2}{2} \chi_R''' \tag{4.39}$$

$$\delta J(x^+) = \partial_+ \left(\chi_L J(x^+) + \frac{k}{4\pi} \lambda(x^+) \right) \tag{4.40}$$

which is identical to the transformation of the analogous functions specifying a general asymptotically AdS$_3$ solution (with Dirichlet boundary conditions) under the Brown-Henneaux diffeomorphisms and a holomorphic gauge transformation. However, due to the dependence of its argument on J, $\bar{\mathcal{L}}$ additionally transforms under the left-moving symmetries parametrized by χ_L and λ as

$$\delta \bar{\mathcal{L}} = -2\mu \left(\chi_L J(x^+) + \frac{k}{4\pi} \lambda(x^+) \right) \bar{\mathcal{L}}' \tag{4.41}$$

The above equations provide all the variations $\delta_\chi \Phi$ needed to compute the algebra of the asymptotic symmetries. Denoting $(\xi_L, \Lambda_L) = \mathrm{g}_L$ etc., we find

$$\{ \mathrm{g}_L[\chi_L], \mathrm{g}_L[\eta_L] \}_* = \mathrm{g}_L[\chi_L \eta_L' - \chi_L' \eta_L] \,, \qquad \{ \mathrm{g}_L[\chi_L], \mathrm{g}_\Lambda[\lambda] \}_* = \mathrm{g}_\Lambda[\chi_L \lambda'] \tag{4.42}$$

$$\{ \mathrm{g}_R[\chi_R], \mathrm{g}_R[\eta_R] \}_* = \mathrm{g}_R[\chi_R \eta_R' - \chi_R' \eta_R] \tag{4.43}$$

$$\{ \mathrm{g}_L[\chi_L], \mathrm{g}_R[\chi_R] \}_* = \{ \mathrm{g}_\Lambda[\lambda], \mathrm{g}_R[\chi_R] \}_* = \{ \mathrm{g}_\Lambda[\lambda], \mathrm{g}_\Lambda[\lambda'] \}_* = 0 \tag{4.44}$$

The use of the modified brackets (4.33) - (4.34) is essential to show that the various cross-commutators between the symmetries vanish. To find the asymptotic symmetry algebra, we consider as usual the individual Fourier modes of the above gauge transformations, i.e. we take

$$\chi_L(x^+) = R \exp \left(\frac{imx^+}{R} \right) \equiv \chi_m \,, \qquad \lambda(x^+) = R \exp \left(\frac{imx^+}{R} \right) \equiv \lambda_m$$

$$\chi_R(x^- + P(x^+)) = (R - \mu Q/\pi) \exp \left(\frac{im(x^- + P(x^+))}{R - \mu Q/\pi} \right) \equiv \tilde{\chi}_m \tag{4.45}$$

with $m \in \mathbb{Z}$. The non-zero commutators are

$$\{ \chi_m, \chi_n \}_* = -i \, (m - n) \chi_{m+n} \,, \qquad \{ \chi_m, \lambda_n \}_* = i \, n \lambda_{m+n} \,, \qquad \{ \tilde{\chi}_m, \tilde{\chi}_n \}_* = -i \, (m - n) \tilde{\chi}_{m+n} \tag{4.46}$$

which correspond to two copies of the Witt algebra and one copy of $U(1)$ Kač-Moody. Note that in order to make sense of the $\tilde{\chi}_m$ commutators, we work in a sector of fixed total charge Q.

The modified Lie bracket of the asymptotic symmetry generators only yields the algebra of the corresponding conserved charges up to a possible central extension $\mathcal{K}_{g_1, g_2}$ - a term that only depends on the

background that cannot be absorbed into a redefinition of the conserved charges. The simplest way to compute it is from the definition of the commutator

$$\{Q_{g_1}, Q_{g_2}\} = \delta_{g_2} Q_{g_1} \tag{4.47}$$

together with the expressions (4.28), (4.30) for the charges and the variation (4.39) of the background fields. Since we are only interested in the central extensions, we only need to keep the inhomogenous term in the symmetry variations (4.39). These terms are identical to their AdS$_3$ counterparts, which implies that the central extensions are the same as before the deformation. The only potentially non-trivial additional contribution we find comes from the term linear in J in the expression for $\tilde{T}_{++}$ in (3.41), which would imply a central extension of the commutator between the left-moving Virasoro generators and the Kač-Moody ones; however, the n dependence of this term is such that it can be absorbed into a redefinition of the current. Consequently, the algebra of the symmetry generators that we find is identical, including all the central extensions, to that of a CFT with a left-moving $U(1)$ current; the only difference is that the spacetime dependence of the right-moving Virasoro generators becomes state-dependent.

5. Discussion

In this article, we have worked out the holographic interpretation of $J\bar{T}$ deformed CFTs. Even though the deforming operator is irrelevant, we have shown that the usual treatment of the double-trace deformation in terms of mixed boundary conditions for the bulk fields yields results that are in perfect agreement with the ones previously derived from field theory.

Since the variational principle we employed here should be equivalent, in principle, with the direct evaluation of the deformed generating functional at large N, we expect that our results on relating sources and expectation values before and after the deformation will be valid also at a purely field-theoretical level. This should allow us to compute arbitrary correlation functions in the deformed theory in terms of correlation functions in the original CFT. The results can then be compared with the correlators computed using conformal perturbation theory, as well as with predictions from the holographic dictionary.

Along the way, we encountered several technical points that need to be further addressed. One of them is to understand how to derive the contribution (3.39) of the current sector to the stress tensor directly from a Chern-Simons perspective, which could also confirm the holographic relations (3.10) and (3.19). In particular, it would be interesting to work out the Chern-Simons contribution to the boundary stress tensor when the gauge field source has a non-trivial field strength, which will enable us to compute higher-point functions of the current. A related technical point is to verify that the usual smoothness conditions on the euclidean continuation of the gauge field is in agreement with the holographic dictionary we proposed and with the first law of black hole mechanics.

An interesting outcome of our analysis is that the double-trace $J\bar{T}$ deformation, rather than completely breaking the right-moving Virasoro symmetries - as one would naively expect from the fact that it is irrelevant on the right - merely deforms it into a non-local version of the Virasoro algebra. In principle, it should be possible to derive its presence from a purely field-theoretical perspective, e.g. by using conformal perturbation theory. It would be interesting to better understand the significance of these "state-dependent" symmetries, their representations, and, provided they represent proper symmetries, how the states of the $J\bar{T}$-deformed theory organise themselves into these representations. A related question is whether the relation (4.27) between the stress tensor components can be understood at an operatorial level as a consequence of a similarly state-dependent *global* $SL(2,\mathbb{R})_R$ symmetry, which is then enhanced to an infinite-dimensional symmetry in the usual way[15]. Of course, the precise definition and meaning of such a global symmetry remains to be understood.

As we have already noted, the boundary conditions we have derived are very similar to the alternate boundary conditions for AdS$_3$ proposed by [21]. In that work, since there was no gauge field to compensate for the variation of the boundary metric, the second Virasoro symmetry was absent, and the asymptotic symmetry group consisted solely of the left-moving Virasoro and Kač-Moody algebras. The level of the $U(1)$ Kač-Moody algebra was found to depend on the $\bar{\mathcal{L}}_0$ eigenvalue, and was negative for black hole spacetimes;

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[15]We thank Diego Hofman for emphasizing this point.

this fact was interpreted as corresponding to ergosphere formation. It would be interesting to embed these boundary conditions in our setup and understand the interpretation of these "warped conformal" symmetries and state-dependent level from the perspective of the $J\bar{T}$-deformed CFT. In particular, it would be interesting if our setup could be used to construct an explicit holographic warped CFT.

Finally, note that our method based on the variational principle should straightforwardly apply also to more general deformations of the Smirnov-Zamolodchikov type, such as the $T\bar{T}$ deformation. Heuristically, it is easy to see that the mixed boundary conditions associated with this deformation will fix a combination of the boundary metric and the stress tensor that, for the appropriate sign of μ, corresponds to fixing the metric on a particular radial slice, in agreement with the proposal of [8]. It would be interesting to study the relation between the expression for the large N generating functional that would be obtained using the variational approach and the recent proposals of [3, 16], as well as the conformal perturbation theory results of [10].

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