Curvature-Squared Multiplets, Evanescent Effects and the U(1) Anomaly in $\mathcal{N} = 4$ Supergravity

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Abstract

We evaluate one-loop amplitudes of $\mathcal{N} = 4$ supergravity in D dimensions using the double-copy procedure that expresses gravity integrands in terms of corresponding ones in Yang–Mills theory. We organize the calculation in terms of a set of gauge-invariant tensors, allowing us to identify evanescent contributions. Among the latter, we find the matrix elements of supersymmetric completions of curvature-squared operators. In addition, we find that such evanescent terms and the U(1)-anomalous contributions to one-loop $\mathcal{N} = 4$ amplitudes are tightly intertwined. The appearance of evanescent operators in $\mathcal{N} = 4$ supergravity and their relation to anomalies raises the question of their effect on the known four-loop divergence in this theory. We provide bases of gauge-invariant tensors and corresponding projectors useful for Yang–Mills theories as a by-product of our analysis.

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I. INTRODUCTION AND REVIEW

Recent explicit calculations have shown that gravity theories still have perturbative secrets waiting to be revealed. We have learned a number of surprising lessons from these calculations: results in gravity theories can be obtained directly from their Yang–Mills counterparts via a double-copy procedure [1–4]; of a curious disconnect between the leading two-loop divergence of graviton amplitudes [5, 6] and the corresponding renormalization-scale dependence [7, 8]; and about the surprisingly tame ultraviolet behavior of certain supergravity theories [9–12]. These lessons augur more surprises to come. In this paper we investigate the role of evanescent effects in the one-loop four-point amplitude of $\mathcal{N} = 4$ supergravity, along with its relation to the U(1) anomaly in the duality symmetry of this theory [13–15].

Evanescent effects arise from operators whose matrix elements vanish when working strictly in four dimensions, but give rise to nonvanishing contributions in dimensional regularization. Such contributions originate from the cancellation of poles against small deviations in the four-dimensional limit; that is, they are due to ϵ/ϵ effects, where $\epsilon = (4-D)/2$ is the dimensional regulator. Although such effects might at first appear to be a mere technicality, they turn out to play an important role [7] in understanding ultraviolet divergences of Einstein gravity in the context of dimensional regularization [5, 6]. In particular, the Gauss–Bonnet operator is evanescent and appears as a one-loop counterterm whose insertion at two loops contaminates the ultraviolet divergence, but results in no physical consequences in the renormalized amplitude. An important question therefore is whether a supersymmetric version of the Gauss-Bonnet operator appears in the matrix elements of $\mathcal{N} = 4$ supergravity. If such an operator exists it would be important to determine its effects on the known four-loop divergence [16] of the theory.

On the other hand, the $\mathcal{N} = 4$ supergravity theory has an anomaly in its U(1) duality symmetry [13]. The anomaly manifests itself in the failure of certain helicity amplitudes which vanish at tree level to persist in vanishing at loop level. In the context of dimensional regularization these anomalous amplitudes arise from ϵ/ϵ effects, in much the same way as the usual chiral anomaly arises in the 't Hooft–Veltman scheme [17]. Refs. [14, 16] have suggested that the U(1) duality anomaly plays a key role in the four-loop divergence of the theory [14, 16], although a detailed explanation is still lacking. In contrast to the anomaly terms, it is unlikely that evanescent effects can alter any physical quantity derived from scattering amplitudes [7, 8]. Nevertheless, one may wonder if there any connections between the two phenomena, given that both arise from ϵ/ϵ effects.

In order to investigate these questions we compute the one-loop four-point amplitude of $\mathcal{N}=4$ supergravity in arbitrary dimensions, using the double-copy procedure based on the duality between color and kinematics [1, 18]. The corresponding helicity amplitudes were previously calculated using various methods [19–21]. Here, we use formal polarizations in order to study evanescent effects, which are hidden when four-dimensional helicity states are used. The conclusion of our study is two-fold: an evanescent contribution of the Gauss–Bonnet type does appear in the pure-graviton amplitude of $\mathcal{N}=4$ supergravity; and its effects are indeed intertwined with the U(1) duality anomaly.

We argue that the main evanescent contributions to the amplitude correspond to the supersymmetric generalization of the curvature-squared terms. Off-shell forms of curvature-squared operators are known for $\mathcal{N}=1$ and $\mathcal{N}=2$ supergravity [22, 23]; but explicit forms of a supersymmetric extension of the Gauss-Bonnet curvature-squared operator are not known off shell in $\mathcal{N}=4$ supergravity.[1] Nonetheless their matrix elements can be computed directly using standard amplitude methods, even without knowing their off-shell forms. In contrast to the nonsupersymmetric case, the coefficients of these matrix elements are finite. This turns out to be a consequence of the same ϵ/ϵ cancellation that generates the anomaly. As we will see, in the context of the double-copy construction there is a single object that has matrix elements that contribute to both the anomaly and evanescent curvature-squared terms.

The double-copy structure implies that we can write the one-loop four-point amplitude of $\mathcal{N}=4$ supergravity in terms of pure-Yang–Mills theory building blocks, up to an overall factor. We can therefore employ a set of gauge-invariant tensors written in terms of formal gluon polarization vectors to carry out the calculation. We present the results in terms of linearized field strengths, which is natural for connecting to operators in a Lagrangian and making manifest on-shell gauge invariance. In order to explore the evanescent properties we also construct tensors with definite four-dimensional helicity properties. We provide the tensors in a form natural for use in color-ordered Yang–Mills theory, as well as in a fully

[1] Curvature-squared operators have been studied in the context of conformal supergravity [24].

crossing-symmetric form natural in $\mathcal{N}=4$ supergravity. Similar gauge-invariant tensors have recently been discussed by Boels and Medina [25].

In the Appendix we give details of the gauge-invariant tensors and describe the construction of projectors for determining the coefficient of the tensors in a given amplitude. These projectors and tensors are useful not only for $\mathcal{N}=4$ supergravity but can be applied to four-gluon amplitudes at any loop order in any Yang–Mills theory, including quantum chromodynamics (QCD). Because of their more general usefulness we attach a *Mathematica* file [26] that includes the two sets of tensors with different symmetry properties, alongside the corresponding projectors.

This paper is organized as follows. In Sect. II we give the construction of the four-loop four-point amplitude of $\mathcal{N}=4$ supergravity and describe the gauge-invariant tensors in terms of which the amplitudes are constructed. In Sect. III we give the results for the one-loop supergravity amplitudes. Then in Sect. IV we identify evanescent curvature-squared terms in the amplitude. We show the connection of these terms to the U(1) anomaly in Sect. V. We give our conclusions in Sect. VI. An appendix describing the gauge-invariant tensors and projectors is included.

II. CONSTRUCTION OF THE ONE-LOOP AMPLITUDE

In this section we construct the one-loop four-point amplitude of $\mathcal{N}=4$ supergravity. Details of the gauge-invariant tensors used for expressing the results are found in the appendix.

A. Color-Kinematics Duality and the Double Copy

We apply the double-copy construction of gravity amplitudes based on the duality between color and kinematics [1, 18]. This has previously been discussed in some detail in Ref. [21] for the one-loop amplitudes of $\mathcal{N}=4$ supergravity. In contrast to the earlier construction, we use D-dimensional external states instead of four-dimensional ones, in order to have access to evanescent effects.

Amplitudes of half-maximal supergravity in D dimensions can be obtained through a double copy, where one factor is derived from maximally supersymmetric Yang–Mills theory

(MSYM), and the other from pure Yang–Mills (YM) theory. In four dimensions, this gives us amplitudes in $\mathcal{N}=4$ supergravity in terms of a product of $\mathcal{N}=4$ and pure Yang–Mills theory. Alternatively, one may also construct $\mathcal{N}=4$ supergravity amplitudes using two copies of $\mathcal{N}=2$ super-Yang–Mills (SYM) theory, as shown in Ref. [27]. This latter construction is, however, more complicated, and furthermore includes unwanted matter multiplets. We use the simpler construction.

The double-copy construction starts from the integrands of two Yang–Mills gauge-theory amplitudes, written in terms of purely cubic diagrams. In a Feynman-diagram language, four-point vertices can always be "blown up" into a product of three-point vertices, possibly with the exchange of a fictitious tensor field. The representation of one-loop amplitudes is,

$$\mathcal{A}_m^{\text{1-loop}} = ig^m \int \frac{d^D p}{(2\pi)^D} \sum_{j\in \text{ICD}} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}\,, \tag{2.1}$$

where the sum runs over the independent cubic diagrams (ICD) labeled by j, while the c_j and n_j are the color factors and kinematic numerators associated with each diagram. The factor $1/S_j$ accounts for the usual diagram symmetry factors and the product over α_j runs over the Feynman propagators $1/p_{\alpha_j}^2$ for diagram j. If the kinematic numerators can be arranged to satisfy the same algebraic properties as adjoint representation color factors, that is so that Jacobi relations hold,

$$c_i + c_j + c_k = 0 \;\Rightarrow\; n_i + n_j + n_k = 0\,, \tag{2.2}$$

along with all anti-symmetry properties, then we can obtain gravity integrands and thence amplitudes by replacing the color factors c_j in Eq. (2.1) by the second Yang–Mills theory's kinematic numerators,

$$c_i \to \tilde{n}_i\,. \tag{2.3}$$

We do this while keeping the original kinematic factors n_j of the first Yang–Mills theory. A similar procedure holds for particles in the fundamental representation [28].

The one-loop four-point amplitude of $\mathcal{N}=4$ supergravity is easy to construct via the double-copy construction, because the $\mathcal{N}=4$ MSYM numerators are especially simple [29]. The numerators of triangle and bubble diagrams vanish, and the box integrals illustrated in Fig. 1 have kinematic numerators proportional to the tree amplitude,

$$n_{1234} = n_{1342} = n_{1423} = s\,t A^{\text{tree}}_{\mathcal{N}=4}(1,2,3,4)\,, \tag{2.4}$$

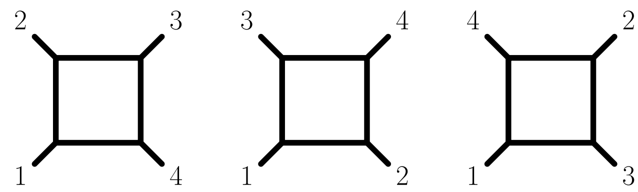

FIG. 1: Box diagrams of the one-loop four-point amplitude of $\mathcal{N} = 4$ supergravity.

where we define the usual Mandelstam invariants,

$$s = (k_1 + k_2)^2\,, \qquad t = (k_2 + k_3)^2\,, \qquad u = (k_1 + k_3)^2\,. \tag{2.5}$$

These numerators trivially satisfy the dual Jacobi identities in Eq. (2.2). Thus, the $\mathcal{N} = 4$ supergravity one-loop amplitude is

$$M^{\text{1-loop}}_{\mathcal{N}=4,\text{SG}}(1,2,3,4) = istA^{\text{tree}}_{\mathcal{N}=4}(1,2,3,4)\Big(I_{1234}[n_{1234,p}] + I_{1342}[n_{1342,p}] + I_{1423}[n_{1423,p}]\Big)\,, \tag{2.6}$$

where we have stripped the gravitational coupling, and where

$$I_{1234}[n_{1234,p}] \equiv \int \frac{d^D p}{(2\pi)^D} \frac{n_{1234,p}}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}\,, \tag{2.7}$$

is the first box integral in Fig. 1 and $n_{1234,p}$ is the pure Yang–Mills kinematic numerator given in Eq. (3.5) of Ref. [30]. We can restore the coupling to the supergravity amplitude via,

$$\mathcal{M}^{\text{tree}}_{\mathcal{N}=4,\text{SG}}(1,2,3,4) = \left(\frac{\kappa}{2}\right)^2 M^{\text{tree}}_{\mathcal{N}=4,\text{SG}}(1,2,3,4)\,, \tag{2.8}$$

at tree level, and

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{SG}}(1,2,3,4) = \left(\frac{\kappa}{2}\right)^4 M^{\text{1-loop}}_{\mathcal{N}=4,\text{SG}}(1,2,3,4)\,, \tag{2.9}$$

at one loop. The coupling is related to Newton's constant via $\kappa^2 = 32\pi G_N$. An alternate form of Eq. (2.6) is,

$$\begin{aligned} M^{\text{1-loop}}_{\mathcal{N}=4,\text{SG}}(1,2,3,4) =& istA^{\text{tree}}_{\mathcal{N}=4}(1,2,3,4) \\ & \times \Big(A^{\text{1-loop}}(1,2,3,4) + A^{\text{1-loop}}(1,3,4,2) + A^{\text{1-loop}}(1,4,2,3)\Big)\,, \end{aligned} \tag{2.10}$$

where $A^{\text{1-loop}}(1,2,3,4)$ is the color-ordered one-loop amplitude of pure Yang–Mills theory. The difference between Eqs. (2.6) and (2.10) cancels in the permutation sum. The second form makes gauge invariance manifest, as the building blocks are gauge-invariant color-ordered amplitudes. We use the form in Eq. (2.6) to evaluate the amplitude explicitly.

B. Gauge-Invariant Building Blocks

The relatively simple double-copy structure of the one-loop four-point $\mathcal{N} = 4$ supergravity amplitude displayed in Eq. (2.10) makes manifest a factorization into the product of an MSYM tree amplitude and a sum over the three distinct permutations of the one-loop color-ordered amplitude of pure Yang–Mills theory. This suggests that we can obtain a convenient organization of the supergravity amplitude by first decomposing the Yang–Mills amplitudes into gauge-invariant contributions. We do so using bases of local on-shell 'gauge-invariant tensors'. By gauge-invariant tensors here we mean polynomials in $(\varepsilon_i \cdot \varepsilon_j), (k_i \cdot \varepsilon_j)$ and $(k_i \cdot k_j)$ that vanish upon replacing ε_i by k_i. These tensors are distinct only if they differ after imposing on-shell conditions. We can build such tensors by starting with tree-level four-point scattering amplitudes for external gluons, for example, or with four-point matrix elements of local gluonic operators, and then multiplying by appropriate factors of s, t, or u to make the quantities local. Boels and Medina [25] have also recently constructed such tensors.

In the Appendix we present two different bases. In the first, we impose definite cyclic symmetry; this yields a basis natural for color-ordered Yang–Mills amplitudes. In the second, we impose definite symmetry under crossing, making them natural for supergravity. Associated with each gauge-invariant tensor is a projector built out of momenta and conjugate polarization vectors. When applied to an integrand, it yields the coefficient of the given tensor. Integrating the coefficient then yields the coefficient of the tensor in the amplitude. This type of projection to a basis of gauge-invariant tensors has been used in Ref. [31]. We stress that the first of these bases is directly useful in gauge-theory calculations. We refer the reader to the Appendix for more details about the bases, their properties, their construction and the projection techniques. We also make these tensors and projectors available in a ancillary *Mathematica* file [26].

We apply this projection technique to the integrand in Eq. (2.6). This reduces the numerators to sums of products of inverse propagators and external kinematics. The integrand is then expressed as a sum over tensors, with each coefficient expressed in terms of the scalar box and simpler triangle and bubble integrals that are easy to evaluate (via Feynman parameterization, for example). The scalar box integral is taken from Ref. [32]. As a cross-check we also evaluated the tensor integrals prior to applying the projectors, following the

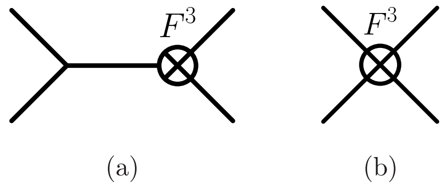

FIG. 2: Representative diagrams for (a) three- and (b) four-point F^3 insertions.

methods of Refs. [33] that express every tensor integral in terms of Schwinger parameters. These integrals are in turn expressed in terms of scalar integrals with shifted dimensions and higher powers of propagators. We use `FIRE5` [34] to reduce these integrals to elements of the standard basis of scalar integrals. The integrals are then shifted back to four dimensions using dimension-shifting formulas [33, 35]. Both methods yield identical results.

We introduce linearized field strengths corresponding to each external particle,

$$F_{i\,\mu\nu} \equiv k_{i\,\mu}\varepsilon_{i\,\nu} - k_{i\,\nu}\varepsilon_{i\,\mu}\,, \tag{2.11}$$

in order to organize the results obtained from the projection technique. We express our results using Lorentz-invariant combinations of these linearized field strengths. For four-point scattering in a parity-even theory, the only combinations at the lowest mass dimension are [36],

$$(F_iF_jF_kF_l) \equiv F_i^{\mu\nu}F_{j\,\nu\rho}F_k^{\rho\sigma}F_{l\,\sigma\mu}\,, \tag{2.12}$$

$$(F_iF_j)(F_kF_l) \equiv F_i^{\mu\nu}F_{j\,\mu\nu}F_k^{\rho\sigma}F_{l\,\rho\sigma}\,. \tag{2.13}$$

These quantities are not symmetrized over the indices i, j, k, and l.

We need only one additional tensor for four-point scattering. This tensor can be expressed as a linear combination of terms of the form D^2F^4. It is, however, more convenient to express this tensor as a matrix element with an insertion of an F^3 operator,

$$F^3 \equiv \frac{1}{3}\,\mathrm{Tr}\,F^{\mu}{}_{\nu}F^{\nu}{}_{\rho}F^{\rho}{}_{\mu}\,, \tag{2.14}$$

where the trace is over color. The gauge-invariant tensor is given by

$$T_{F^3} \equiv -istA^{\rm tree}_{F^3}(1,2,3,4)\,, \tag{2.15}$$

using the four-point tree-level color-ordered amplitude with a single insertion of the operator (2.14), as depicted in Fig. 2. As we see below, after applying the double-copy procedure, this element of our basis is the one giving rise to the curvature-squared matrix elements, as well as some of the anomalous ones.

III. RESULT AND MAPPING TO SUPERGRAVITY

Using the tensors in Eqs. (2.12), (2.13) and (2.15), we can write the supergravity amplitude as follows[2],

$$\begin{aligned} M^{\text{1-loop}}_{\mathcal{N}=4,\text{SG}}(1,2,3,4) &= c_\Gamma st A^{\text{tree}}_{\mathcal{N}=4}(1,2,3,4) \\ &\times \Bigg[\frac{t_8 F^4}{stu}\Bigg(-\frac{2}{\epsilon^2}\sum_{i<j}^{3} s_{ij}\Big(\frac{-s_{ij}}{\mu^2}\Big)^{-\epsilon} + L_1(s,t,u)\Bigg) \\ &\quad + \frac{T_{F^3}}{stu} + \Bigg(\frac{4}{3}(F_1F_2F_3F_4)\Big(\frac{1}{st} + L_2(s,t,u)\Big) \\ &\quad + (F_1F_2)(F_3F_4)\Big(\frac{1}{s^2} + L_3(s,t,u)\Big) + \text{cyclic(2,3,4)}\Bigg)\Bigg], \end{aligned} \tag{3.1}$$

where μ is the usual scale parameter, $s_{12}=s, s_{23}=t, s_{13}=u$; where

$$c_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)}\,, \tag{3.2}$$

is the usual one-loop prefactor,

$$L_1(s,t,u) = -s\ln\Big(\frac{-s}{\mu^2}\Big) - \frac{(2s^2+st+2t^2)}{2u}\Big(\ln^2\Big(\frac{-s}{-t}\Big)+\pi^2\Big) + \text{cyclic}(s,t,u)\,, \tag{3.3}$$

$$\begin{aligned} L_2(s,t,u) &= \Bigg[-\frac{2s}{t^2u}\ln\Big(\frac{-s}{-u}\Big) + \frac{1}{4u^2}\Big(\ln^2\Big(\frac{-s}{-t}\Big)+\pi^2\Big) \\ &\quad + \frac{(s-2t)}{t^3}\Big(\ln^2\Big(\frac{-s}{-u}\Big)+\pi^2\Big)\Bigg] + (s\leftrightarrow t)\,, \end{aligned} \tag{3.4}$$

$$\begin{aligned} L_3(s,t,u) &= \frac{1}{stu}\Big(-s\ln\Big(\frac{-s}{\mu^2}\Big) - t\ln\Big(\frac{-t}{\mu^2}\Big) - u\ln\Big(\frac{-u}{\mu^2}\Big)\Big) \\ &\quad + \frac{(t-u)}{s^3}\ln\Big(\frac{-t}{-u}\Big) + \frac{(2s^2-tu)}{s^4}\Big(\ln^2\Big(\frac{-t}{-u}\Big)+\pi^2\Big)\,, \end{aligned} \tag{3.5}$$

and where we have used the combination

$$t_8F^4 = 2(F_1F_2F_3F_4) - \frac{1}{2}(F_1F_2)(F_3F_4) + \text{cyclic}(2,3,4)\,, \tag{3.6}$$

familiar from the four-point one-loop type-I superstring amplitude. The rank-8 tensor t_8 arises from the trace over the fermionic zero-modes (see for instance[3] Ref. [37]). The combination in Eq. (3.6) is crossing symmetric and is related to the Yang–Mills tree amplitude

[2] We write our results in the unphysical region where $s,t,u<0$; one can analytically continue to the physical region where $s>0$ and $t,u<0$ using $\ln(-s)\to\ln(s)-i\pi$.

[3] The t_8 tensor used here differs from the one in Ref. [37] by an overall factor of 4.

via

$$t_8F^4 = -istA^{\text{tree}}(1,2,3,4) = -isuA^{\text{tree}}(1,2,4,3) = -ituA^{\text{tree}}(1,3,2,4)\,. \tag{3.7}$$

The amplitude in Eq. (3.1) is ultraviolet-finite; the poles in ϵ in Eq. (3.1) are infrared ones.

We have carried out a number of checks of the amplitude. A simple check is that the infrared singularity in Eq. (3.1) matches the known form [38],

$$M^{\text{1-loop}}_{\mathcal{N}=4,\text{SG}}\Big|_{\text{IR}} = -M^{\text{tree}}_{\mathcal{N}=4,\text{SG}}\frac{2c_\Gamma}{\epsilon^2}\sum_{i<j}^{3} s_{ij}\left(\frac{-s_{ij}}{\mu^2}\right)^{-\epsilon}. \tag{3.8}$$

To see this we express the factors in front of the $1/\epsilon^2$ in Eq. (3.1) in terms of the supergravity tree amplitude,

$$stA^{\text{tree}}_{\mathcal{N}=4}(1,2,3,4)\frac{t_8F^4}{stu} = -isA^{\text{tree}}_{\mathcal{N}=4}(1,2,3,4)A^{\text{tree}}(1,2,4,3) = M^{\text{tree}}_{\mathcal{N}=4,\text{SG}}(1,2,3,4)\,, \tag{3.9}$$

where the last step uses the Kawai–Lewellen–Tye (KLT) relation [39] between tree-level gravity and Yang–Mills amplitudes. We have also compared the finite parts of all the amplitudes with external scalars and gravitons to the results in Ref. [14, 19, 21] and found agreement. The remaining fermionic amplitudes are related by supersymmetry Ward identities. We have checked that, prior to specializing to $D = 4$, the ultraviolet divergence cancels for $D < 8$, as expected [11]. In $D = 8$, we match the prediction from the heterotic string (see section 3.A.1 of Ref. [40]) as well as the calculation in Ref. [11]. It may also be possible to compare our D-dimensional expression to the recent $D = 10$ prediction in Ref. [41] obtained from M-theory. However, performing this comparison would be nontrivial as the divergences are quadratic in this dimension and hence depend on the regulator. It would be interesting to study this connection further.

The form in which we presented the amplitude in Eq. (3.1) makes the supersymmetry completely manifest, because it acts only on the MSYM side of the double copy. In addition, this form makes the translation to gravity transparent.

We now show in some detail how this works for the case of external gravitons. In the double-copy construction, amplitudes with four external gravitons can be built from integrands with purely gluonic external states on both sides of the double copy. As discussed in the previous section, it is convenient to use linearized field strengths in Eqs. (2.12) and (2.13) to write the answer. In order to translate to gravity we do this on both sides of the double copy. From this form, we can easily convert the linearized field strengths F in our formulas to a linearized Riemann tensor R using the relation,

$$\frac{2}{\kappa}R_{i\,\mu\nu\rho\sigma} = F_{i\,\mu\nu}F_{i\,\rho\sigma} = (k_{i\,\mu}\varepsilon_{i\,\nu} - k_{i\,\nu}\varepsilon_{i\,\mu})\,(k_{i\,\rho}\varepsilon_{i\,\sigma} - k_{i\,\sigma}\varepsilon_{i\,\rho})\,, \tag{3.10}$$

where the index i refers to the particle label, just as in Eq. (2.11). In this equation the product of Yang–Mills polarization vectors is identified as a graviton polarization tensor via the replacement $\varepsilon_{i\,\mu}\varepsilon_{i\,\nu} \to \varepsilon_{i\,\mu\nu}$. The graviton is related to the metric via $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, as in Ref. [36]. The factor of $2/\kappa$ is included in Eq. (3.10) so that $R_{i\,\mu\nu\rho\sigma}$ is given by the linearized Riemann tensor with the field $h_{\mu\nu}$ replaced by a polarization tensor $\varepsilon_{i\,\mu\nu}$.

The contribution from the pure-gluon factor from MSYM is always a factor of $stA^{\text{tree}} = it_8F^4$. Once we multiply the tensors from both sides of the double-copy we then obtain the following combinations,

$$t_8F^4t_8F^4 \to t_8t_8R^4\,, \tag{3.11}$$

$$t_8F^4(F_iF_jF_kF_l) \to t_8(R_iR_jR_kR_l)\,, \tag{3.12}$$

$$t_8F^4(F_iF_j)(F_kF_l) \to t_8(R_iR_j)(R_kR_l)\,, \tag{3.13}$$

where

$$(R_iR_j)^{\mu_1\mu_2\mu_3\mu_4}(R_kR_l)^{\ \mu_5\mu_6\mu_7\mu_8} \equiv R_i^{\ \mu_1\mu_2\nu\lambda}R_j^{\ \mu_3\mu_4}{}_{\nu\lambda}R_k^{\ \mu_5\mu_6\rho\sigma}R_l^{\ \mu_7\mu_8}{}_{\rho\sigma}\,, \tag{3.14}$$

$$(R_iR_jR_kR_l)^{\mu_1\mu_2\mu_3\mu_4\mu_4\mu_5\mu_6\mu_7\mu_8} \equiv R_i^{\ \mu_1\mu_2\nu\lambda}R_j^{\ \mu_3\mu_4}{}_{\lambda\rho}R_k^{\ \mu_5\mu_6\rho\sigma}R_l^{\ \mu_7\mu_8}{}_{\sigma\nu}\,. \tag{3.15}$$

In ten dimensions Eq. (3.11) is a component of the only $\mathcal{N} = 2$ superinvariant, whereas Eqs. (3.12) and (3.13) are components of the two $\mathcal{N} = 1$ superinvariants [41, 42].

The mapping of the final T_{F^3} tensor to gravity may appear more complicated than for the F^4-class tensors, because the former is generated from a scattering amplitude with an F^3 insertion, as previously illustrated in Fig. 2. A relatively simple way to obtain this tensor is to use KLT relations for amplitudes extended to include insertions of this higher-dimensional operator [43, 44]. This extension is in line with expectations from string-theory KLT relations [45, 46], where the operator appears in the low-energy effective action. In Refs. [43, 44] it was established that the KLT relations apply to F^3 operators as,

$$sA^{\text{tree}}(1,2,3,4) \times A^{\text{tree}}_{F^3}(1,2,4,3) = iM^{\text{tree}}_{R^2}(1,2,3,4)\,, \tag{3.16}$$

where all particles are gluons on left-hand side of the equation, and all are gravitons on the right-hand side when the helicities of each pair of gluons align. Direct checks using Feynman diagrams, starting from the Einstein action, confirm that the Gauss–Bonnet insertion into a four-point gravity tree amplitude indeed satisfies Eq. (3.16) [47]. Hence we see that the tensor T_{F^3} maps into the curvature-squared matrix elements in gravity as follows,

$$st A^{\rm tree}(1,2,3,4) T_{F^3} = -isu A^{\rm tree}(1,2,4,3) st A^{\rm tree}_{F^3}(1,2,3,4) = stu M^{\rm tree}_{R^2}(1,2,3,4)\,, \qquad (3.17)$$

where we used the crossing symmetry of $stA^{\rm tree}(1,2,3,4)$ and the KLT relation in Eq. (3.16).

After the complete map to linearized Riemann tensors, the graviton amplitude takes the form,

$$\begin{aligned} M^{\text{1-loop}}_{\mathcal{N}=4,\rm SG} = c_\Gamma \Bigg[& M^{\rm tree}_{\mathcal{N}=4,\rm SG} \bigg(-\frac{2}{\epsilon^2} \sum_{i<j}^{3} s_{ij} \Big(\frac{-s_{ij}}{\mu^2}\Big)^{-\epsilon} + L_1(s,t,u) \bigg) \\ & + M^{\rm tree}_{R^2} + \bigg(\frac{4}{3} t_8(R_1R_2R_3R_4) \bigg(\frac{1}{st} + L_2(s,t,u) \bigg) \\ & + t_8(R_1R_2)(R_3R_4) \bigg(\frac{1}{s^2} + L_3(s,t,u) \bigg) + \text{cyclic}(2,3,4) \bigg) \Bigg]\,. \end{aligned} \qquad (3.18)$$

The same construction works for any supergravity state. For all states in the supergravity multiplet, the same pure Yang–Mills tensors feed into the corresponding supergravity expressions; the differences are solely on the MSYM side of the double copy.

It is remarkable that the coefficient of the curvature-squared matrix element $M^{\rm tree}_{R^2}$ appearing in Eq. (3.18) is just a simple number. If the theory had a nonvanishing trace anomaly [48], the coefficient of $M^{\rm tree}_{R^2}$ would have contained a $1/\epsilon$ divergence [5, 7, 49]. In our calculation the divergences are suppressed by an explicit factor of $D-4=2\epsilon$, (see, for example, Eq. (2.11) of Ref. [50]) leaving a finite rational contribution. From the perspective of the double copy, this ϵ/ϵ effect also generates the nonvanishing all-plus and single-minus one-loop amplitudes associated with the U(1) duality anomaly [14]. We comment on this below.

IV. CURVATURE-SQUARED MULTIPLETS AND DIVERGENCES IN SUPERGRAVITY

In the previous section we found curvature-squared contributions to the effective action. In this section we describe these contribution in more detail.

A. Curvature-Squared Multiplets with Half-Maximal Supersymmetry

In the full superamplitude, we find a term proportional to,

$$sA^{\text{tree}}_{\mathcal{N}=4}(1,2,4,3)A^{\text{tree}}_{F^3}(1,2,3,4)\,, \tag{4.1}$$

which, as described in the previous section, contains the evanescent matrix element of curvature operators. In general dimensions there exist several off-shell curvature-squared operators in gravity theories. The two most important ones are the Gauss–Bonnet density and the square of the Weyl tensor[4] which respectively are given by,

$$E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\,, \tag{4.2}$$

$$W^2 = W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\,. \tag{4.3}$$

The difference between the two is,

$$W^2 - E_4 = 2(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2)\,, \tag{4.4}$$

which vanishes on shell. The single on-shell independent operator is usually chosen to be the Gauss–Bonnet combination (4.2). It is however a total derivative in four dimensions, which implies that all curvature-squared matrix elements are evanescent in this dimension [51]. A consequence of this is the finiteness of pure-graviton amplitudes at one loop [51] in Einstein gravity, as these operators are the only available counterterms. (When matter is added to the theory—even supersymmetric matter multiplets—generic divergences do appear at one loop starting with amplitudes for four matter particles [52].)

Off-shell R^2 supermultiplets were constructed long ago for $\mathcal{N}=1$ supergravity in four dimensions [22], and more recently for $\mathcal{N}=2$ supergravity [23] using a version of $\mathcal{N}=2$ superspace. Very recently an $\mathcal{N}=4$ supersymmetric completion of the Weyl-squared operator has been discussed in Ref. [15] in terms of linearized superfields in four dimensions. However, at the nonlinear level no fully off-shell versions have been constructed to date for any of the curvature-squared multiplets. This is unsurprising in light of the more general unsolved problem of constructing an off-shell $\mathcal{N}=4$ superspace.

[4] There is another interesting curvature-squared operator, the Pontryagin density ${}^*R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$; but it is parity odd and hence it cannot appear in the amplitudes of parity-conserving theories.

Eq. (4.1) also contains matrix elements related by supersymmetry to the one corresponding to curvature-squared operators. These must arise from the $\mathcal{N} = 4$ supersymmetric completion of the curvature operators in Eqs. (4.2) and (4.3). Therefore the existence of such matrix element implies the existence of the corresponding $\mathcal{N} = 4$ curvature-squared multiplets. In particular, these matrix elements should correspond to the single insertion of the operator discussed in Ref. [15] in four dimensions as all curvature-squared operators are equivalent on shell. However, we cannot analyze such matrix elements strictly in four dimensions, because they will vanish identically.

The double-copy construction provides additional information, because it implies that completions of curvature-squared operators with half-maximal supersymmetry should exist in any integer dimension $D \leq 10$ and that their on-shell matrix elements are given by the KLT product of the F^3 operator insertion and ordinary MSYM amplitudes. The restriction to $D \leq 10$ arises because that is the maximum dimension for a super-Yang–Mills theory.

The double-copy perspective also shows that an $\mathcal{N} \geq 5$ supersymmetric completion of curvature-squared operators [53] cannot exist. We have an overall factor of $stA^{\text{tree}}_{\mathcal{N}=4}$ from the MSYM amplitude on the one side of the double copy. On the other side we would have an $\mathcal{N} \geq 1$ super-Yang–Mills amplitude. From the double-copy perspective, in any dimension the R^2 terms correspond to an F^3 operator on this latter side. We would then need a supersymmetric completion of the F^3 operator, to make it compatible with $\mathcal{N} = 1$ supersymmetry. We know, however, that no such completion exists in four dimensions because F^3 matrix element contributes only to all-plus and single-minus helicity configurations; and these are forbidden by a supersymmetric Ward identity [54]. This also rules out supersymmetric completions for these theories in any dimension $D > 4$ because on shell there is only a single independent curvature-squared invariant and one can choose the external momenta and states to live in a four-dimensional subspace, and hence the same argument applies.

B. Possible Effects at Higher Loops

In the context of dimensional regularization, evanescent R^2 contributions such as the ones described here play a crucial role in the two-loop divergences of pure gravity [5, 6]. This happens because the evanescent R^2 terms appear at one loop with a divergent coefficient proportional to the trace anomaly. While such terms do not contribute in four dimensions,

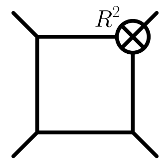

FIG. 3: Representative diagram for the insertion of the evanescent R^2 counterterm, affecting the two-loop divergence in pure-graviton amplitudes [7].

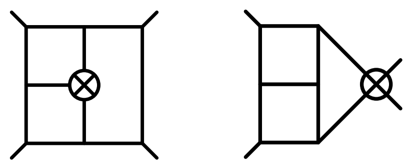

FIG. 4: Representative diagrams for insertions of the supersymmetric R^2 operator at three loops that could affect the four-loop divergence.

they do appear at two loops as subdivergences in the dimensionally regulated amplitude, directly affecting the value of the two-loop divergence [7]. One must then subtract a one-loop R^2 counterterm insertion, as illustrated in Fig. 3. This evanescent contribution becomes nonvanishing in dimensional regularization where it modifies the two-loop divergence. The net result is a curious disconnect between the coefficient of the dimensionally-regulated two-loop R^3 ultraviolet divergence of these theories and the corresponding renormalization-scale dependence. The coefficient of the divergence depends on details of the regularization, while the renormalization scale dependence is simple and robust [7, 8].

As shown in Eq. (3.18), in $\mathcal{N} = 4$ supergravity the R^2 contribution appears with a finite coefficient, so it cannot contribute to possible two-loop divergences. One may nonetheless expect it to modify divergences at yet-higher loops. Explicit calculations reveal no divergences in $\mathcal{N} = 4$ supergravity through three loops [9], but unveil them at four loops [16]. The addition of supersymmetrization of a curvature-squared operator as a local counterterm to the action is not expected to have any physical consequences in the scattering amplitudes, because it is evanescent. The analysis in Ref. [7] shows that it can however affect divergences. It would be interesting to study the effect of such local counterterms on the known four-loop divergence calculated in Ref. [16]. One may wonder whether such a finite counterterm can be used to modify or even remove the four-loop divergence. The answer to this question would require a three-loop computation with insertions of this operator, as illustrated in

Pure YM		$\mathcal{N}=4$ MSYM	$\mathcal{N}=4$ Supergravity
$\langle g^- g^- g^+ g^+\rangle$	$\otimes$	$\langle g^- g^- g^+ g^+\rangle$	$\langle h^{--} h^{--} h^{++} h^{++}\rangle$
$\langle g^- g^+ g^+ g^+\rangle$	$\otimes$	$\langle g^- g^- g^+ g^+\rangle$	$\langle h^{--} \phi^{-+} h^{++} h^{++}\rangle$
$\langle g^+ g^+ g^+ g^+\rangle$	$\otimes$	$\langle g^- g^- g^+ g^+\rangle$	$\langle \phi^{-+} \phi^{-+} h^{++} h^{++}\rangle$

TABLE I: Top components of three of the five independent superamplitudes. The other two are obtained from CPT conjugation.

Fig. 4.

V. EVANESCENT EFFECTS AND THE U(1) ANOMALY

We now show that from the vantage point of the double copy that the U(1) anomalous contributions cannot be separated from the evanescent R^2 matrix elements, described in the previous section. We first review the anomaly and its manifestation in one-loop matrix elements [14], before explaining how these effects are intertwined.

In order to describe the anomaly we recall some basic facts about the spectrum of four-dimensional $\mathcal{N}=4$ supergravity and the associated superamplitudes. We focus here on pure $\mathcal{N}=4$ supergravity with no matter multiplets. The states of pure $\mathcal{N}=4$ supergravity fall into two supermultiplets. One contains the positive-helicity graviton and its superpartners [55]:

$$(h^{++}, \psi_a^+, A_{ab}^+, \chi_{abc}^+, \phi^{-+})\,, \tag{5.1}$$

where h^{++} is the positive-helicity graviton, ψ_a^+ are the four positive-helicity gravitinos, and so forth until the complex scalar ϕ^{-+}. The indices a, b, c are $SU(4)$ R symmetry indices. The other supermultiplet is the CPT conjugate to the one above, containing the negative-helicity graviton h^{--} and the conjugate scalar ϕ^{+-}. Seen through the lens of the double-copy, each multiplet corresponds to the supermultiplet of MSYM multiplied by either a positive- or negative-helicity gluon on the pure Yang–Mills side. For instance the positive-helicity graviton arises from a positive-helicity gluon on both sides of the double copy, and the complex scalars come from negative-helicity gluons on one side and positive-helicity gluons on the other side.

Because not all the states of this theory are in a single supermultiplet, the amplitudes are organized into different sectors not directly related by supersymmetry. For each one of these sectors there is an associated superamplitude. A simple way to understand this organization is via the double-copy construction. The supersymmetry Ward identities imply that the only nonvanishing helicity amplitudes in MSYM are those in the maximally-helicity-violating (MHV) sector corresponding to amplitudes with two negative-helicity and two positive-helicity gluons ($g^-g^-g^+g^+$) and their superpartners, which all sit in a single superamplitude. On the pure Yang–Mills side of the double copy, however, there are three distinct types of amplitudes: all-plus ($g^+g^+g^+g^+$), single-minus ($g^-g^+g^+g^+$), and two-minus or MHV ($g^-g^-g^+g^+$), together with their parity conjugates. Hence there are three distinct sectors of supergravity super-amplitudes, inherited from each of the pure-Yang–Mills helicity configurations. In the all-plus and single-minus pure Yang–Mills sectors the gluons do not have the same number of negative or positive helicities as the gluons in the MSYM amplitude. Because of this the corresponding $\mathcal{N}=4$ supergravity superamplitudes do not contain four-graviton amplitudes, but have mixed graviton–scalar amplitudes as their top components, as illustrated in Table I.

Ref. [13] showed that there exists an anomaly in an abelian U(1) subgroup of the SU(1, 1) duality group of $\mathcal{N}=4$ supergravity. This anomaly is manifested in the nonvanishing of the amplitudes,

$$
\begin{aligned}
M_{\mathcal{N}=4,\mathrm{SG}}(1_{h^{--}},2_{\phi^{-+}},3_{h^{++}},4_{h^{++}}) &= \frac{i}{(4\pi)^2}\frac{\langle 1\,2\rangle^2\,\langle 1\,3\rangle^2\,[2\,3]^2\,[3\,4]^4}{stu}\,,\\
M_{\mathcal{N}=4,\mathrm{SG}}(1_{\phi^{-+}},2_{\phi^{-+}},3_{h^{++}},4_{h^{++}}) &= \frac{i}{(4\pi)^2}\,[3\,4]^4\,,
\end{aligned}
\tag{5.2}
$$

as well as those related by supersymmetry [14]. The spinor inner products $\langle a\,b\rangle$ and $[a\,b]$ follow the standard conventions in Ref. [56]. The scalars carry a charge under the U(1) subgroup whereas the gravitons are uncharged and hence these amplitudes violate conservation of this charge. At tree level the charges are conserved because the amplitudes all vanish, but at loop level they do not. This anomaly can be traced back to $\mathcal{O}(\epsilon)$ terms which interfere with a would-be $1/\epsilon$ divergence, leaving behind a rational term. This is similar to the way the chiral anomaly arises in dimensional regularization [17].

As explained above, our calculation reveals evanescent contributions in Eq. (4.1), which are related to the supersymmetric completion of the R^2 operator. Mixed graviton–scalar amplitudes also receive non-evanescent contributions from the same terms. A simple way to see this is by expressing the F^3 matrix element in a basis of gauge-invariant tensors that

$\mathcal{N}=4$ Supergravity	$-isA^{\rm tree}_{\mathcal{N}=4}(1,2,4,3)A^{\rm tree}_{F^3}(1,2,3,4)$
$\langle h^{--}h^{--}h^{++}h^{++}\rangle$	0
$\langle h^{--}\phi^{-+}h^{++}h^{++}\rangle$	$-i\frac{\langle 1\,2\rangle^2\langle 1\,3\rangle^2[2\,3]^2}{stu}\delta^{(8)}(\overline{Q})$
$\langle \phi^{-+}\phi^{-+}h^{++}h^{++}\rangle$	$2i\,\delta^{(8)}(\overline{Q})$

TABLE II: Top components of the three independent sectors in four dimensions and corresponding superamplitudes.

has definite four-dimensional helicity properties. We give two such bases in the Appendix. In the basis with tensors that have definite crossing-symmetry properties, we find that the F^3 matrix element is given by,

$$T_{F^3} = \frac{2stu}{(s^2+t^2+u^2)}\,H^{(++++)} - H^{(-+++)} + \frac{2(s-t)(s-u)(t-u)}{3(s^2+t^2+u^2)^2}\,H^{\rm ev1} - \frac{6stu}{(s^2+t^2+u^2)^2}\,H^{\rm ev2}\,, \tag{5.3}$$

where the nonlocal denominators all cancel to give a local expression for T_{F^3}. This decomposition explicitly shows that T_{F^3} has nonvanishing contributions to the all-plus and single-minus helicity configurations, with the rest of the tensor being evanescent in four dimensions. This gives some additional insight into the evanescent nature of the R^2 matrix element in gravity. The only nonvanishing amplitudes on the MSYM side of the double copy have an MHV helicity configuration $(--++)$, whereas Eq. (5.3) shows that the F^3 matrix element does not contribute to MHV amplitudes on the pure Yang–Mills side. This implies that the pure-graviton matrix elements vanish in four dimensions. More importantly, we see that this matrix element contributes to the all-plus and single-minus helicities, thus generating anomalous mixed graviton-scalar matrix elements after applying the double-copy construction.

An alternative way to understand the different contributions of this matrix element is to recall that in general dimension, a pair of gluons is mapped via the double copy to a graviton, a dilaton and an antisymmetric tensor. In four dimensions the antisymmetric tensor is dual to a pseudoscalar that together with the dilaton combines into the complex scalar discussed above. The intertwining of the anomalous and evanescent contributions in Eq. (3.16) therefore follows from the entanglement of the graviton, dilaton and an antisymmetric tensor in the double-copy construction.

From the discussion above, we conclude that the F^3 KLT product in Eq. (3.16) not only gives the evanescent curvature-squared matrix elements, but it necessarily results in an anomalous contribution to the amplitude. It is striking that contributions to both can be traced back to precisely the same term in the double copy. The anomalous contributions arising from T_{F^3} are summarized in Table II. In this table the supermomentum delta function can be expanded as [57]

$$\delta^{(8)}(\overline{Q}) = \delta^{(8)}\Big(\sum_{j=1}^{4} \tilde{\lambda}_j^{\dot{\alpha}} \tilde{\eta}_{ja}\Big) = \prod_{a=1}^{4} \sum_{i<j}^{4} [i\, j]\, \tilde{\eta}_{ia} \tilde{\eta}_{ja}, \tag{5.4}$$

where we take the top component to be the one containing the factor $[3\,4]^4$. Comparing these to the anomalous amplitudes in Eq. (5.2) we see that, while the amplitudes in the single-scalar sector are fully contained in this term, those in the two-scalar sector are off by an overall factor and receive additional contributions that change the overall coefficient.

Finally, it is interesting to note that such anomalous and evanescent effects will not appear in the one-loop amplitudes of $\mathcal{N} \geq 5$ supergravity. The lack of anomalous one-loop amplitudes in $N \geq 5$ supergravity has been recently explained from the vantage point of super-invariants [15]. This, together with the absence of evanescent effects, is understood in the double-copy procedure as a consequence of the vanishing of the one-loop all-plus and single-minus amplitudes in super-Yang–Mills theories.

VI. CONCLUSION

In this paper we identified terms in the dimensionally regulated one-loop four-point amplitude of pure $\mathcal{N} = 4$ supergravity that can be written as insertions of curvature-squared operators into matrix elements. Such terms are evanescent and vanish for four-dimensional external states. We also showed that these evanescent terms are intertwined with contributions generated by the U(1) duality anomaly [13, 14]. These two effects both arise from rational pieces that result from an ϵ/ϵ cancellation, where $\epsilon = (4 - D)/2$ is the dimensional regularization parameter.

Both the anomaly and the evanescent curvature-squared terms may play a central role in the ultraviolet properties of gravity theories. As explained in Ref. [14] the anomaly in $\mathcal{N} = 4$ supergravity gives contributions with a poor ultraviolet behavior. We also know that beyond one loop, evanescent effects contribute to dimensionally regulated ultraviolet divergences in gravity theories [7].

We carried out our analysis using the double-copy construction [1, 18] of $\mathcal{N} = 4$ supergravity [21] in terms of the corresponding pure Yang–Mills and $\mathcal{N} = 4$ MSYM amplitudes. The double-copy construction makes the on-shell supersymmetry manifest, because $\mathcal{N} = 4$ supergravity inherits the well-understood on-shell superspace of MSYM theory. By using formal polarization vectors on the pure-Yang–Mills side of the double copy, we were able to evaluate all one-loop four-point amplitudes of $\mathcal{N} = 4$ supergravity simultaneously. In the graviton sector we gave explicit conversion formulas from gauge theory to gravity, using relations between linearized Riemann tensors and Yang–Mills field strengths. The double-copy construction implies that completions of curvature-squared operators with half-maximal supersymmetry should exist in any dimension with $D \leq 10$ and that their on-shell matrix elements are given by the KLT product of the F^3 operator insertion and ordinary MSYM amplitudes.

There are a number of interesting avenues for future research. Although it is is not known how to write the super-Gauss–Bonnet in an off-shell superspace, our paper provides all components of four-point matrix elements of single insertions of these operators. For the pure-graviton amplitude the Gauss–Bonnet operator is the correct one for generating these matrix elements. For amplitudes with other external states, one would first need to systematically write down a set of evanescent operators of the same dimension, feed them through a tree-level matrix-element computation and then match them to our evanescent matrix elements. Once the combination of operators leading to our evanescent matrix elements are found, one can try to appropriately package the components into superfields.

We organized the one-loop amplitude in terms of gauge-invariant tensors. These and their associated projectors are described in the appendix and given in the *Mathematica* attachement [26]. They are useful, not only for $\mathcal{N} = 4$ supergravity, but for any gauge-theory four-gluon amplitude at any loop order.

In pure gravity the evanescent one-loop curvature-squared terms enter with a coefficient proportional to $1/\epsilon$. Because of this, when inserted as counterterms in a two-loop calculation they affect the leading ultraviolet divergence [7]. In $\mathcal{N} = 4$ supergravity these evanescent terms appear with a finite coefficient. This means that they cannot affect divergences until three loops or higher. Direct calculations show that the three-loop divergences cancel [9] and the first divergence occurs at four loops [16]. It is important to understand the effect of evanescent and anomalous contribution on higher-loop amplitudes, especially to see whether their contributions can account for the four-loop divergence of $\mathcal{N} = 4$ supergravity. A direct study requires a three-loop computation. An important step in this direction would be to analyze the anomalous sector at two loops in $\mathcal{N} = 4$ supergravity and its relation to evanescent effects. In the longer term, understanding the role of anomalies and evanescent effects more generally at higher loops appears to be crucial in order to unravel the ultraviolet properties of supergravity theories.

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Appendix A: Gauge-Invariant Tensors for Yang–Mills Four-Point Amplitudes

In this appendix, we describe two independent sets of Yang–Mills kinematic tensors built out of physical polarization vectors ε_i and on-shell momenta k_i. In both sets, the tensors are constrained to be on-shell gauge invariant, that is vanishing under the substitution $\varepsilon_i \to k_i$ for each external leg independently. The tensors are polynomials in the dot products $k_i \cdot \varepsilon_j$, $\varepsilon_i \cdot \varepsilon_j$, and the Mandelstam invariants s and t. They are thus free of poles by construction. We also organize the tensors to have definite symmetry properties under a relevant symmetry, and to be diagonal in a four-dimensional helicity basis. The tensors are dimension-agnostic, and so the sets are not in general diagonal in a basis of external states outside of four dimensions. Both sets have seven tensors.

In the first set, each tensor represents kinematic parts of a color-ordered amplitude, up to a function of s and t. Such amplitudes are invariant under a cyclic permutation of the external indices, $i \to (i+1) \mod 4$, so we choose the tensors to have definite symmetry properties under the cyclic shift. An arbitrary function can be split up into symmetric and antisymmetric combinations, $f_{\pm}(s,t) = \frac{1}{2}[f(s,t) \pm f(t,s)]$, so we choose the tensors to be symmetric or antisymmetric. It might seem simpler to choose them to be symmetric; but for some of them, an antisymmetric form is simpler. In an amplitude, such antisymmetric tensors would then appear multiplied by an antisymmetric function of s and t. We present this set in the first subsection.

For the second set, each tensor represents one Yang–Mills copy in a double-copy construction of an $\mathcal{N} = 4$ supergravity amplitude, where the other copy is given by the tree-level tensor. These tensors then suffice to construct the $\mathcal{N} = 4$ supergravity four-point amplitude at one and two loops. These tensors are required to have definite symmetry properties under the full permutation group acting on the external indices. We are interested only in the one-dimensional representations of this group, so again each tensor will either be completely invariant, or will change sign according to the signature of a permutation. We present this set in the second subsection.

In the third subsection, we describe set of projection operators that can be applied to an expression given in terms of polarization vectors and momenta to obtain the (scalar) coefficients of the different basis tensors.

1. Tensors with Definite Cyclic Symmetry

We take the first element of the set of tensors with definite cyclic properties to be the tensor of engineering dimension 4 that appears in the tree amplitude,

$$\begin{aligned}
T^{\rm tree} = t_8 F^4 = {} & s\,(s+t)\,\epsilon_1\cdot\epsilon_4\,\epsilon_2\cdot\epsilon_3 - s\,t\,\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4 + t\,(s+t)\,\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4 \\
& - 2\,(s+t)\,\epsilon_1\cdot\epsilon_4\,k_1\cdot\epsilon_2\,k_1\cdot\epsilon_3 - 2\,(s+t)\,\epsilon_1\cdot\epsilon_4\,k_1\cdot\epsilon_2\,k_2\cdot\epsilon_3 \\
& - 2\,s\,\epsilon_1\cdot\epsilon_3\,k_1\cdot\epsilon_2\,k_2\cdot\epsilon_4 - 2\,t\,\epsilon_1\cdot\epsilon_2\,k_1\cdot\epsilon_3\,k_2\cdot\epsilon_4 - 2\,(s+t)\,\epsilon_1\cdot\epsilon_2\,k_2\cdot\epsilon_3\,k_2\cdot\epsilon_4 \\
& - 2\,t\,\epsilon_2\cdot\epsilon_4\,k_1\cdot\epsilon_3\,k_3\cdot\epsilon_1 - 2\,t\,\epsilon_2\cdot\epsilon_4\,k_2\cdot\epsilon_3\,k_3\cdot\epsilon_1 - 2\,s\,\epsilon_2\cdot\epsilon_3\,k_2\cdot\epsilon_4\,k_3\cdot\epsilon_1 \\
& - 2\,(s+t)\,\epsilon_1\cdot\epsilon_3\,k_1\cdot\epsilon_2\,k_3\cdot\epsilon_4 - 2\,(s+t)\,\epsilon_1\cdot\epsilon_2\,k_2\cdot\epsilon_3\,k_3\cdot\epsilon_4 \\
& - 2\,(s+t)\,\epsilon_2\cdot\epsilon_3\,k_3\cdot\epsilon_1\,k_3\cdot\epsilon_4 - 2\,(s+t)\,\epsilon_3\cdot\epsilon_4\,k_1\cdot\epsilon_2\,k_4\cdot\epsilon_1 \\
& - 2\,t\,\epsilon_2\cdot\epsilon_4\,k_1\cdot\epsilon_3\,k_4\cdot\epsilon_1 - 2\,(s+t)\,\epsilon_2\cdot\epsilon_4\,k_2\cdot\epsilon_3\,k_4\cdot\epsilon_1 \\
& - 2\,(s+t)\,\epsilon_2\cdot\epsilon_3\,k_3\cdot\epsilon_4\,k_4\cdot\epsilon_1 - 2\,s\,\epsilon_1\cdot\epsilon_4\,k_1\cdot\epsilon_3\,k_4\cdot\epsilon_2 - 2\,s\,\epsilon_1\cdot\epsilon_3\,k_2\cdot\epsilon_4\,k_4\cdot\epsilon_2 \\
& - 2\,t\,\epsilon_3\cdot\epsilon_4\,k_3\cdot\epsilon_1\,k_4\cdot\epsilon_2 - 2\,s\,\epsilon_1\cdot\epsilon_3\,k_3\cdot\epsilon_4\,k_4\cdot\epsilon_2 - 2\,(s+t)\,\epsilon_3\cdot\epsilon_4\,k_4\cdot\epsilon_1\,k_4\cdot\epsilon_2\,.
\end{aligned} \tag{A1}$$

It vanishes, of course, for the $(+\,+\,+\,+)$ and $(-\,+\,+\,+)$ classes of helicities, and is nonvanishing for MHV helicities $(-\,-\,+\,+)$. It is invariant under cyclic shifts of the external legs. We choose the remaining tensors to have definite helicity properties as well. We can give compact expressions for the tensors in terms of the following combinations of the linearized field-strength tensors defined in Eq. (2.11),

$$\begin{aligned}
F^4_{st} &\equiv (F_1F_2F_3F_4)\,, & F^4_{tu} &\equiv (F_1F_4F_2F_3)\,, & F^4_{us} &\equiv (F_1F_3F_4F_2)\,, \\
(F^2_s)^2 &\equiv (F_1F_2)(F_3F_4)\,, & (F^2_t)^2 &\equiv (F_1F_4)(F_2F_3)\,, & (F^2_u)^2 &\equiv (F_1F_3)(F_4F_2)\,,
\end{aligned} \tag{A2}$$

Tensor	Dimension	Symmetry	Nonvanishing $D=4$ Helicity	$D=4$ Value
T^{tree}	4	$+$	$(--++)$	$\langle 1\,2\rangle^2\ [3\,4]^2$
			$(-+-+)$	$\langle 1\,3\rangle^2\ [2\,4]^2$
$T^{(++++)}$	4	$+$	$(++++)$	$[1\,3]^2\ [2\,4]^2$
$T^{(-+++)}$	6	$+$	$(-+++)$	$\langle 1\,2\rangle^2\ [2\,3]^2\ [2\,4]^2$
$T^{(--++)}$	4	$-$	$(--++)$	$\langle 1\,2\rangle^2\ [3\,4]^2$
$T^{(-+-+)}$	4	$+$	$(-+-+)$	$\langle 1\,3\rangle^2\ [2\,4]^2$
T^{ev1}	6	$+$	—	0
T^{ev2}	6	$-$	—	0

TABLE III: Nonvanishing helicities and values for the color-ordered tensor basis. Each tensor is also nonvanishing on the cyclic permutations and parity conjugates of the indicated helicity states. The evanescent tensors vanish for all four-dimensional helicities but are included in the table.

along with the T_{F^3} tensor defined in Eq. (2.15). In terms of these quantities, the basis tensors have the following expressions,

$$\begin{aligned}
T^{\text{tree}} &= -\frac{1}{2}((F_s^2)^2+(F_t^2)^2+(F_u^2)^2)+2\,(F_{st}^4+F_{tu}^4+F_{us}^4)=t_8F^4\,,\\
T^{(++++)} &= -2\,F_{st}^4+\frac{1}{2}\,((F_s^2)^2+(F_t^2)^2+(F_u^2)^2)\,,\\
T^{(-+++)} &= -T_{F^3}-(F_{tu}^4-F_{us}^4)\,(s-t)+(F_{st}^4-\frac{1}{4}((F_s^2)^2+(F_t^2)^2+(F_u^2)^2))\,(s+t)\,,\\
T^{(--++)} &= (F_s^2)^2-(F_t^2)^2+2\,(F_{tu}^4-F_{us}^4)\,,\\
T^{(-+-+)} &= 2\,F_{st}^4-\frac{1}{2}((F_s^2)^2+(F_t^2)^2-(F_u^2)^2)\,,\\
T^{\text{ev1}} &= -(2\,F_{st}^4+\frac{3}{2}\,((F_s^2)^2+(F_t^2)^2+(F_u^2)^2))\,(s+t)+2\,(F_{us}^4\,(3\,s+t)+F_{tu}^4\,(s+3\,t))\\
&= -4\,(F_{tu}^4\,s+F_{us}^4\,t)-(s+t)\,(8\,F_{st}^4-3\,T^{\text{tree}})\,,\\
T^{\text{ev2}} &= -(2\,F_{st}^4-\frac{1}{2}((F_s^2)^2+(F_t^2)^2+(F_u^2)^2))\,(s-t)+2\,(F_{tu}^4-F_{us}^4)\,(s+t)\\
&= 4\,(F_{tu}^4\,s-F_{us}^4\,t)-(s-t)\,T^{\text{tree}}\,.
\end{aligned}\tag{A3}$$

The first tensor is the tree-level tensor given above in Eq. (A1). The subsequent four tensors each are labeled by the class of four-dimensional helicity configuration on which they are nonvanishing. The final two tensors are nontrivial formal objects, but vanish for all four-dimensional helicities. Outside of four dimensions, they do not vanish, however, as demonstrated, for example, by the nonvanishing value of the sum over states of each tensor multiplied by its conjugate. They represent the kinematic part of evanescent operators in Yang–Mills theory. In a slight abuse of language, we will therefore call them evanescent tensors. Three other gauge-invariant tensors can be constructed, but these do not have the correct symmetry properties to appear in color-ordered physical amplitudes. The properties of all the tensors, as well as their values in four-dimensional helicity are summarized in Table III. The expressions for the tensors are also given in a companion *Mathematica* file, tensors-ym.m. The notation there is,

$$\mathsf{ee[i,j]} = \varepsilon_i \cdot \varepsilon_j \,, \qquad \mathsf{ke[i,j]} \ = k_i \cdot \varepsilon_j \,, \qquad \mathsf{dot[i,j]} = k_i \cdot k_j \,. \tag{A4}$$

The seven tensors in Eq. (A3) sequentially correspond to T[[i]] in the file for $i = 1, \ldots, 7$. The spinor-valued expressions for the tensors in four dimensions are also given that file, with the seven values for each four-dimensional helicity configuration recorded in value[*helicity-string*], for example value["++++"]. These expressions employ the notation,

$$\mathsf{spa[i,j]} = \langle i\, j \rangle \;, \qquad \mathsf{spb[i,j]} = [i\, j] \;. \tag{A5}$$

Conversely, we can express the linearized combinations (A2) in terms of the color-ordered tensors,

$$\begin{aligned}
F^4_{st} &= -\frac{T^{\rm ev1}}{8\,(s+t)} - \frac{(s-t)\,T^{\rm ev2}}{8\,(s+t)^2} + \frac{1}{4}T^{\rm tree} - \frac{s\,t\,T^{(++++)}}{2\,(s+t)^2}\,, \\
F^4_{tu} &= \frac{T^{\rm ev2}}{4\,(s+t)} + \frac{1}{4}T^{\rm tree} + \frac{t\,T^{(++++)}}{2\,(s+t)}\,, \\
F^4_{us} &= -\frac{T^{\rm ev2}}{4\,(s+t)} + \frac{1}{4}T^{\rm tree} + \frac{s\,T^{(++++)}}{2\,(s+t)}\,, \\
(F^2_s)^2 &= -\frac{T^{\rm ev1}}{4\,(s+t)} - \frac{(3\,s+t)\,T^{\rm ev2}}{4\,(s+t)^2} + \frac{1}{2}T^{\rm tree} + \frac{1}{2}T^{(--++)} - \frac{1}{2}T^{(-+-+)} + \frac{s^2\,T^{(++++)}}{(s+t)^2}\,, \\
(F^2_t)^2 &= -\frac{T^{\rm ev1}}{4\,(s+t)} + \frac{(s+3\,t)\,T^{\rm ev2}}{4\,(s+t)^2} + \frac{1}{2}T^{\rm tree} - \frac{1}{2}T^{(--++)} - \frac{1}{2}T^{(-+-+)} + \frac{t^2\,T^{(++++)}}{(s+t)^2}\,, \\
(F^2_u)^2 &= T^{(-+-+)} + T^{(++++)}\,, \\
T_{F^3} &= -\frac{(s-t)\,T^{\rm ev2}}{2\,(s+t)} - T^{(-+++)} - \frac{2\,s\,t\,T^{(++++)}}{s+t}\,.
\end{aligned} \tag{A6}$$

2. Tensors with Definite Permutation Symmetry

In this subsection, we present four-gluon kinematic tensors with definite properties under the full permutation group. These are ultimately useful for decomposing $\mathcal{N}=4$ supergravity amplitudes at one and two loops in a double-copy approach. The tree tensor (A1) is already fully crossing invariant, so we take it to be the first tensor in this set as well, here calling it H^{tree}. The remaining tensors are either invariant under all permutations of external labels, or are multiplied by the signature of the permutation (± 1). We will call the latter signature-odd.

A signature-odd tensor will be multiplied by a signature-odd polynomial in s and t in any physical amplitude. Any invariant polynomial can also appear as a tensor prefactor in an amplitude, of course. All invariant polynomials can be built out of products of two basic polynomials,

$$\begin{aligned} \sigma_2(s,t,u) &= s^2+t^2+u^2 = 2(s^2+st+t^2) = -2(st+tu+us)\,, \\ \sigma_3(s,t,u) &= s^3+t^3+u^3 = 3stu\,, \end{aligned} \tag{A7}$$

with a constant prefactor. Any signature-odd polynomial is a product of an invariant polynomial and the basic signature-odd polynomial,

$$\alpha(s,t,u) = -(s-t)(t-u)(u-s) = (s-t)(2s+t)(s+2t)\,. \tag{A8}$$

This polynomial satisfies the identity

$$2\,\alpha^2 = \sigma_2^3 - 6\,\sigma_3^2\,, \tag{A9}$$

so that we need not consider powers of α.

We can again express the tensors in terms of the linearized-field strength quantities defined in Eq. (A2),

$$\begin{aligned}
H^{\text{tree}} &= -\frac{1}{2}((F_s^2)^2 + (F_t^2)^2 + (F_u^2)^2) + 2\,(F_{st}^4 + F_{tu}^4 + F_{us}^4) = t_8 F^4\,,\\
H^{(++++)} &= \frac{3}{2}\,((F_s^2)^2 + (F_t^2)^2 + (F_u^2)^2) - 2\,(F_{st}^4 + F_{tu}^4 + F_{us}^4)\,,\\
H^{(-+++)} &= -T_{F^3} - \frac{4}{3}\,(F_{tu}^4\, s + F_{us}^4\, t - F_{st}^4\,(s+t))\,,\\
H^{\text{MHV1}} &= -((F_s^2)^2 + 2\,F_{tu}^4)\, s - ((F_t^2)^2 + 2\,F_{us}^4)\, t + (2\,F_{st}^4 + (F_u^2)^2)\,(s+t)\,,\\
H^{\text{MHV2}} &= (F_u^2)^2\,(s-t)\,(s+t) + (F_t^2)^2\, t\,(2\,s+t) - (F_s^2)^2\, s\,(s+2\,t)\,,\\
H^{\text{ev1}} &= 4\,(F_{st}^4\,(s-t)\,(s+t) + F_{us}^4\, t\,(2\,s+t) - F_{tu}^4\, s\,(s+2\,t))\,,\\
H^{\text{ev2}} &= ((F_s^2)^2 + (F_t^2)^2 + (F_u^2)^2)\,(s^2 + s\,t + t^2) - 4\,(F_{tu}^4\, t\,(s+t) - s\,(F_{st}^4\, t - F_{us}^4\,(s+t)))\,.
\end{aligned} \tag{A10}$$

The second and third tensors are again labeled by the four-dimensional helicity class for which they are nonvanishing; the fourth and fifth are both nonvanishing for all MHV helicities. The last two are again "evanescent", in the sense that they are nonvanishing outside of four dimensions but vanish for all four-dimensional helicity configurations. (As in Sect. A 1, they do not include factors of $1/\epsilon$ that would be needed to yield a nonvanishing result in four dimensions.)

Tensor	Dimension	Signature	Nonvanishing $D = 4$ Helicity	$D = 4$ Value
H^{tree}	4	even	$(-\,-\,+\,+)$	$\langle 1\,2\rangle^2\ [3\,4]^2$
$H^{(++++)}$	4	even	$(+\,+\,+\,+)$	$[1\,4]^2\ [2\,3]^2 + [1\,3]^2\ [2\,4]^2 + [1\,2]^2\ [3\,4]^2$
$H^{(-+++)}$	6	even	$(-\,+\,+\,+)$	$\langle 1\,2\rangle^2\ [2\,3]^2\ [2\,4]^2$
H^{MHV1}	6	even	$(-\,-\,+\,+)$	$\langle 1\,2\rangle^3\ [1\,2]\ [3\,4]^2$
H^{MHV2}	8	odd	$(-\,-\,+\,+)$	$(s+2\,t)\ \langle 1\,2\rangle^3\ [1\,2]\ [3\,4]^2$
H^{ev1}	8	odd	—	0
H^{ev2}	8	even	—	0

TABLE IV: Nonvanishing helicities and values for the pregravity tensor basis. Each tensor is also nonvanishing on the permutations and parity conjugates of the indicated helicity states. The evanescent tensors vanish for all four-dimensional helicities but are included in the table.

The expressions for the tensors are also given in a companion *Mathematica* file, tensors-neq4gr.m, with H[[i]], $i = 1, \ldots, 7$ corresponding in order to the tensors in Eq. (A10).

The spinor-valued expressions for the tensors in four dimensions are also given in that file; as in Sect. A 1, the seven values for each four-dimensional helicity configuration given by value[*helicity-string*]. The notation follows Eqs. (A4) and (A5). The properties of the tensors are summarized in Table IV.

Because these tensor have definite properties under permutations, we can connect them straightforwardly to matrix elements of corresponding operators after the double copy. A few examples would be,

$$
\begin{aligned}
\sigma_2 t_8 F^4 t_8 F^4 &\leftrightarrow t_8 t_8 D^4 R^4 \,, \\
t_8 F^4 (u F^4_{st} + s F^4_{tu} + t F^4_{us}) &\leftrightarrow t_8 \mathrm{tr}(D^2 R^4) \,, \\
\sigma_2 t_8 F^4 ((F^2_s)^2 + (F^2_u)^2 + (F^2_t)^2) &\leftrightarrow t_8 (\mathrm{tr}(DR)^2)^2 \,.
\end{aligned}
\tag{A11}
$$

3. Projectors for Basis Tensors

In this subsection, we present a set of projectors that can be used to obtain the scalar coefficients of the basis tensors for an expression given in terms of polarization vectors and momenta. When applied to an integrated expression for an amplitude, the resulting decomposition will reproduce the original expression; when applied to an integrand, there may be a total-derivative discrepancy that will integrate to zero.

We define an inner product $\odot$ of a polarization vector and its conjugate to be given by the sum over states,

$$
\epsilon_i^{*\mu} \odot \epsilon_i^{\nu} = \sum_{\text{states}\, h} \epsilon_i^{*(h),\mu} \epsilon_i^{(h),\nu} = -g^{\mu\nu} + \frac{k_i^\mu q^\nu + q^\mu k_i^\nu}{q \cdot k_i} \,,
\tag{A12}
$$

where q is a null reference vector not collinear to any external momentum. (It is similar to a lightcone-gauge vector.) In four dimensions, the state sum becomes,

$$
\sum_{\text{states}\, h} \epsilon_i^{*(h),\mu} \epsilon_i^{(h),\nu} = \sum_{h=\pm} \epsilon_i^{*(h),\mu} \epsilon_i^{(h),\nu} = \sum_{h=\pm} \epsilon_i^{(-h),\mu} \epsilon_i^{(h),\nu} \,,
\tag{A13}
$$

where the sum is over vector helicities.

In all dimensions, the projector onto the jth tensor is then given by,

$$
P_j = c_{ji} T_i^* \,,
\tag{A14}
$$

where the matrix c is the inverse of the (symmetric) inner product matrix m, whose elements are given by,

$$
m_{ij} = T_i^* \odot T_j \,.
\tag{A15}
$$

The coefficient of T_j in an expression X is given by $P_j \odot X$.

Each basis has a corresponding set of projectors; the projectors for the cyclicly-organized basis described in Sect. A 1 are given alongside the tensors and helicity values in tensors-ym.m, where the projector P_j onto T_j is given by P[[j]]. The expressions make use of the following notation in addition to that in Eq. (A4),

$$\mathsf{cc[i,j]} = \varepsilon_i^* \cdot \varepsilon_j^* \,, \qquad \mathsf{kc[i,j]} = k_i \cdot \varepsilon_j^* \,, \qquad \mathsf{chi} = t/s \,, \qquad \mathsf{d} = D \,. \tag{A16}$$

In four dimensions, m has rank 5, as expected from the nature of T_5 and T_6. In six dimensions, it has rank 7, showing indirectly that there are some helicities with non-vanishing values for these two tensors. The corresponding projectors for the basis of Sect. A 2 organized under the full crossing symmetry are given in tensors-neq4gr.m. The projector matrix again has rank 5 in four dimensions, and rank 7 in six dimensions.

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