arXiv:1802.09164v1 [math-ph] 26 Feb 2018

MATRIX PRODUCT SOLUTIONS TO THE REFLECTION EQUATION FROM THREE DIMENSIONAL INTEGRABILITY

ATSUO KUNIBA AND VINCENT PASQUIER

Abstract

We formulate a quantized reflection equation in which q-boson valued L and K matrices satisfy the reflection equation up to conjugation by a solution to the Isaev-Kulish 3D reflection equation. By forming its n-concatenation along the q-boson Fock space followed by suitable reductions, we construct families of solutions to the reflection equation in a matrix product form connected to the 3D integrability. They involve the quantum R matrices of the antisymmetric tensor representations of $U_p(A^{(1)}_{n-1})$ and the spin representations of $U_p(B^{(1)}_n)$, $U_p(D^{(1)}_n)$ and $U_p(D^{(2)}_{n+1})$.

1. Introduction

The tetrahedron equation [22] is a three dimensional (3D) analogue of the Yang-Baxter equation [2]. Consider the following version of the tetrahedron equation often referred to as the $RLLL = LLLR$ relation:

$$L_{124}L_{135}L_{236}\mathcal{R}_{456} = \mathcal{R}_{456}L_{236}L_{135}L_{124}.$$

Here L and $\mathcal{R}$ are linear operators on $V \otimes V \otimes F$ and $F \otimes F \otimes F$ respectively for some vector spaces V and F. The above equality is to hold between the operators on $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{4}{F} \otimes \overset{5}{F} \otimes \overset{6}{F}$, where the superscripts specify the components on which L and $\mathcal{R}$ act nontrivially. We call the latter as 3D $\mathcal{R}$.

To grasp the structure let us suppress the indices for the space F regarding it as *auxiliary* and write the above equation as

$$(L_{12}L_{13}L_{23})\mathcal{R} = \mathcal{R}(L_{23}L_{13}L_{12}).$$

It manifests that the tetrahedron equation is a Yang-Baxter equation up to conjugation by the 3D $\mathcal{R}$. One may also view it as a *quantized* Yang-Baxter equation in the sense that the Boltzmann weights for a 2D vertex model encoded in L become $\mathrm{End}(F)$ valued. This observation, though tautological, is known to lead to infinite families of R matrices, i.e. solutions to the Yang-Baxter equation, in the form of *matrix product* [19, 4, 16]. The method is first to form the *n-concatenation* of the original equation by replacing the space 1 with the copies $1_1, \ldots, 1_n$ and similarly for 2 and 3 as

$$(L_{1_12_1}L_{1_13_1}L_{2_13_1})\cdots(L_{1_n2_n}L_{1_n3_n}L_{2_n3_n})\mathcal{R} = \mathcal{R}(L_{2_13_1}L_{1_13_1}L_{1_12_1})\cdots(L_{2_n3_n}L_{1_n3_n}L_{1_n2_n}).$$

One can reduce this, after inserting spectral parameters, to the Yang-Baxter equation by evaluating $\mathcal{R}$ out suitably, for example by taking the trace over the auxiliary space $\overset{4}{F} \otimes \overset{5}{F} \otimes \overset{6}{F}$. The objects that remain after the reduction necessarily possess the structure of n-matrix product over the auxiliary space.

This construction based on the 3D integrability is known to work efficiently for the local L matrix in (5). It corresponds to the choice $V = \mathbb{C}^2$ and F= q-boson Fock space F_{q^2}, which may be viewed as a q-boson valued six vertex model. The resulting solutions to the Yang-Baxter equation are expressed in the matrix product forms (49) and (55). They live in $\mathrm{End}(\mathbf{V} \otimes \mathbf{V})$ with $\mathbf{V} = (\mathbb{C}^2)^{\otimes n}$, and cover the quantum R matrices for the antisymmetric tensor representations of $U_p(A^{(1)}_{n-1})$ [4] and the spin representations of $U_p(B^{(1)}_n)$, $U_p(D^{(1)}_n)$, $U_p(D^{(2)}_{n+1})$ for some p [16]. See Appendix B for the precise identification.

The purpose of this paper is to launch a similar 3D approach to the reflection equation [20, 13]. We propose the *quantized* reflection equation

$$(L_{12}K_2L_{21}K_1)\mathcal{K} = \mathcal{K}(K_1L_{12}K_2L_{21}),$$

which is the traditional (or 2D) reflection equation up to conjugation by $\mathcal{K}$. As with the preceding illustration on the tetrahedron equation, it actually means $L_{123}K_{24}L_{215}K_{16}\mathcal{K}_{3456} = \mathcal{K}_{3456}K_{16}L_{125}K_{24}L_{213}$, where $3,4,5,6$ are labels of the auxiliary spaces suppressed in the notation. We employ the same L as before and take the local K to be the q-boson valued 2×2 matrix as in (6). With these choices, the quantized reflection equation may be viewed as specifying the *auxiliary linear problem* for the conjugation matrix $\mathcal{K}$. In the present setting it should be a linear operator on the Fock space $F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q$.

Our first finding is that such $\mathcal{K}$ is provided exactly by the first nontrivial solution [15] to the Isaev-Kulish 3D reflection equation [7]. We call it 3D $\mathcal{K}$. Originally the 3D $\mathcal{K}$ was characterized as the intertwiner of the representations of the Hopf algebra known as the *quantized coordinate ring* $A_q(sp_4)$. See Section 3 for a quick exposition of the background [6, 18, 21] including the application to the tetrahedron equation [11]. Our result here shows that the quantized reflection equation and the intertwining relation for the $A_q(sp_4)$ modules in [15] are equivalent as auxiliary linear problems.

The reformulation of the 3D $\mathcal{K}$ is quite beneficial since the quantized reflection equation admits, like the $RLLL = LLLR$ relation, the n-concatenation with respect to the auxiliary space:

$$(L_{1_1 2_1} K_{2_1} L_{2_1 1_1} K_{1_1}) \cdots (L_{1_n 2_n} K_{2_n} L_{2_n 1_n} K_{1_n}) \mathcal{K} = \mathcal{K} (K_{1_1} L_{1_1 2_1} K_{2_1} L_{2_1 1_1}) \cdots (K_{1_n} L_{1_n 2_n} K_{2_n} L_{2_n 1_n}).$$

It is again possible to reduce this to the usual reflection equation by evaluating $\mathcal{K}$ away by taking the trace or matrix elements with respect to certain eigenvectors. These procedures are called *trace reduction* and *boundary vector reduction*, respectively[1]. The resulting solutions to the reflection equation involve the previously mentioned R matrices for $U_p(A^{(1)}_{n-1})$, $U_p(B^{(1)}_n)$, $U_p(D^{(1)}_n)$ and $U_p(D^{(2)}_{n+1})$. The companion K matrices are trigonometric[2] and are expressed in the matrix product form as in (72) and (80). They are 2^n by 2^n matrices on $\mathbf{V}$ which are neither diagonal in general nor associated with the well-studied R matrices for the vector representation [3, 9]. Therefore they are distinct from those obtained in [1, 17] for generic n and constitute new systematic solutions to the reflection equation.

The idea and maneuver in this paper demonstrate a new approach to 2D integrable systems with boundaries. We hope that it fuels further applications to related subjects, e.g. special functions, quantum many body systems, stochastic processes (cf. [5, 14]) and so forth in the presence of boundaries.

The layout of the paper is as follows. In Section 2 q-boson valued local L and K matrices are introduced and the quantized reflection equation is formulated. The conjugation matrix contained therein is identified with the 3D $\mathcal{K}$ [15] by writing out the auxiliary linear problem explicitly.

In Section 3 we briefly review the 3D $\mathcal{R}$ and 3D $\mathcal{K}$ based on [11, 4, 15]. These objects are destined to be eliminated in the reduction procedures in later sections. However they essentially control the construction behind the scene in that they guide precisely how the local L and K should be combined, how the spectral parameters are to be arranged and what kind of boundary vectors are acceptable.

In Section 4 we recall the reduction procedures to get solutions to the Yang-Baxter equation from the n-concatenation of the $RLLL = LLLR$ relation (34). This idea has a long history, see for example [19, 12, 4, 16] and references therein. The prescription is to eliminate the 3D $\mathcal{R}$ either by taking trace (trace reduction) or evaluating matrix elements between certain eigenvectors of $\mathcal{R}$ (boundary vector reduction) as already mentioned. In our setting they reproduce the solutions $S^{\mathrm{tr}}(z)$ [4] and $S^{s,s'}(z)\,(s,s'=1,2)$ [16]. They both act on $\mathbf{V} \otimes \mathbf{V}$ whose representation theoretical origin is explained in Appendix B.

In Section 5 we demonstrate that the same machinery works perfectly also for the quantized reflection equation. It produces the K matrices $K^{\mathrm{tr}}(z)$ and $K^{k,k'}(z)\,(k,k'=1,2)$ that act on $\mathbf{V}$. Together with the $S^{\mathrm{tr}}(z)$ and $S^{s,s'}(z)$ derived in Section 4, they constitute solutions to the usual (2D) reflection equation. This is the main result of the paper.

In Section 6 we give a short summary and mention some future problems.

Appendix A contains explicit formulas of 3D $\mathcal{R}$ and 3D $\mathcal{K}$. Appendix B recalls the precise identification of the $S^{\mathrm{tr}}(z)$ and $S^{s,s'}(z)$ with the quantum R matrices for the antisymmetric tensor representations of $U_p(A^{(1)}_{n-1})$ [4] and the spin representations of $U_p(D^{(2)}_{n+1})$,$U_p(B^{(1)}_n)$, $U_p(D^{(1)}_n)$ [16], respectively. Appendix C presents a couple of examples of $S^{\mathrm{tr}}(z), S^{s,s'}(z)$ and the K matrices $K^{\mathrm{tr}}(z)$ and $K^{k,k'}(z)$.

Throughout the paper we assume that q is generic and use the following notations:

$$(z;q)_m = \prod_{k=1}^{m}(1-zq^{k-1}), \quad (q)_m = (q;q)_m, \quad \binom{m}{k}_q = \frac{(q)_m}{(q)_k (q)_{m-k}},$$

$$\theta(\mathrm{true}) = 1, \quad \theta(\mathrm{false}) = 0, \quad \mathbf{e}_j = (0,\ldots,0,\overset{j}{1},0,\ldots,0) \in \mathbb{Z}^n \ (1 \le j \le n).$$

[1] Our boundary vector reduction is based on the conjectural property (78).

[2] Up to overall normalization, matrix elements are rational in q and the multiplicative spectral parameter z.

2. Quantized reflection equation

2.1. ***q*-boson valued *L* and *K* matrices.** Let $F_q = \bigoplus_{m\geq 0}\mathbb{C}|m\rangle$ and $F_q = \bigoplus_{m\geq 0}\mathbb{C}\langle m|$ be the Fock space and its dual equipped with the inner product $\langle m|m'\rangle = (q^2)_m\delta_{m,m'}$. We define the q-boson operators $\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$ on them by

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle, \quad \mathbf{k}|m\rangle = q^{m+\frac{1}{2}}|m\rangle,$$
$$\langle m|\mathbf{a}^- = \langle m+1|, \quad \langle m|\mathbf{a}^+ = \langle m-1|(1-q^{2m}), \quad \langle m|\mathbf{k} = \langle m|q^{m+\frac{1}{2}}.$$

They satisfy $(\langle m|X)|m'\rangle = \langle m|(X|m'\rangle)$ and

$$\langle m|X_1\cdots X_j|m'\rangle = \langle m'|\overline{X_j}\cdots\overline{X_1}|m\rangle, \tag{1}$$

where $\overline{(\cdots)}$ is defined by $\overline{\mathbf{a}^{\pm}} = \mathbf{a}^{\mp}, \quad \overline{\mathbf{k}} = \mathbf{k}$.

Let $F_{q^2}, F^*_{q^2}$ and $\mathbf{A}^+, \mathbf{A}^-, \mathbf{K}$ denote the same objects with q replaced by q^2, i.e.,

$$\mathbf{A}^+|m\rangle = |m+1\rangle, \quad \mathbf{A}^-|m\rangle = (1-q^{4m})|m-1\rangle, \quad \mathbf{K}|m\rangle = q^{2m+1}|m\rangle,$$
$$\langle m|\mathbf{A}^- = \langle m+1|, \quad \langle m|\mathbf{A}^+ = \langle m-1|(1-q^{4m}), \quad \langle m|\mathbf{K} = \langle m|q^{2m+1}.$$

The inner product in F_{q^2} is given by $\langle m|m'\rangle = (q^4)_m\delta_{m,m'}$ differing from the F_q case. However we write the base vectors as $\langle m|, |m\rangle$ either for $F^*_{q^2}, F_{q^2}$ or F^*_q, F_q since their distinction will always be evident from the context. Note the q-boson commutation relations

$$\mathbf{k}\,\mathbf{a}^{\pm} = q^{\pm 1}\mathbf{a}^{\pm}\mathbf{k}, \quad \mathbf{a}^{\pm}\mathbf{a}^{\mp} = 1 - q^{\mp 1}\mathbf{k}^2, \tag{2}$$
$$\mathbf{K}\,\mathbf{A}^{\pm} = q^{\pm 2}\mathbf{A}^{\pm}\mathbf{K}, \quad \mathbf{A}^{\pm}\mathbf{A}^{\mp} = 1 - q^{\mp 2}\mathbf{K}^2. \tag{3}$$

We will also use the number operator $\mathbf{h}$ defined by

$$\mathbf{h}|m\rangle = m|m\rangle, \qquad \langle m|\mathbf{h} = \langle m|m \tag{4}$$

either for F_q or F_{q^2}. One may regard $\mathbf{k} = q^{\mathbf{h}+\frac{1}{2}}$ and $\mathbf{K} = q^{2\mathbf{h}+1}$. The extra 1/2 in the spectrum of $\log_q \mathbf{k}$ is the *zero point energy*. It makes the forthcoming equations (13)–(28) and (87) totally free from the apparent q[3].

Set $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \simeq \mathbb{C}^2$ and introduce the q-boson valued L matrices and K matrices by

$$L = \begin{pmatrix} L^{0,0}_{0,0} & L^{0,0}_{0,1} & L^{0,0}_{1,0} & L^{0,0}_{1,1} \\ L^{0,1}_{0,0} & L^{0,1}_{0,1} & L^{0,1}_{1,0} & L^{0,1}_{1,1} \\ L^{1,0}_{0,0} & L^{1,0}_{0,1} & L^{1,0}_{1,0} & L^{1,0}_{1,1} \\ L^{1,1}_{0,0} & L^{1,1}_{0,1} & L^{1,1}_{1,0} & L^{1,1}_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{K} & \mathbf{A}^- & 0 \\ 0 & \mathbf{A}^+ & -\mathbf{K} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{End}(V\otimes V\otimes F_{q^2}), \tag{5}$$

$$K = \begin{pmatrix} K^0_0 & K^0_1 \\ K^1_0 & K^1_1 \end{pmatrix} = \begin{pmatrix} \mathbf{a}^+ & -\mathbf{k} \\ \mathbf{k} & \mathbf{a}^- \end{pmatrix} \in \mathrm{End}(V\otimes F_q). \tag{6}$$

We let them act on the base vectors by the following rule:

$$L(v_\alpha\otimes v_\beta\otimes|m\rangle) = \sum_{0\leq\gamma,\delta\leq 1} v_\gamma\otimes v_\delta\otimes L^{\gamma,\delta}_{\alpha,\beta}|m\rangle, \quad K(v_\alpha\otimes|m\rangle) = \sum_{0\leq\beta\leq 1} v_\beta\otimes K^\beta_\alpha|m\rangle.$$

Explicitly they read

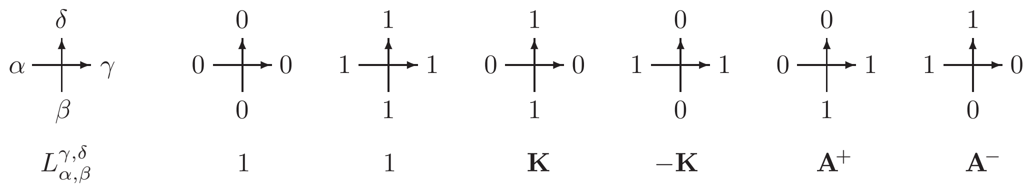
$$L(v_0\otimes v_0\otimes|m\rangle) = v_0\otimes v_0\otimes|m\rangle, \quad L(v_1\otimes v_1\otimes|m\rangle) = v_1\otimes v_1\otimes|m\rangle,$$
$$L(v_0\otimes v_1\otimes|m\rangle) = v_0\otimes v_1\otimes\mathbf{K}|m\rangle + v_1\otimes v_0\otimes\mathbf{A}^+|m\rangle = q^{2m+1}v_0\otimes v_1\otimes|m\rangle + v_1\otimes v_0\otimes|m+1\rangle,$$
$$L(v_1\otimes v_0\otimes|m\rangle) = v_0\otimes v_1\otimes\mathbf{A}^-|m\rangle - v_1\otimes v_0\otimes\mathbf{K}|m\rangle = (1-q^{4m})v_0\otimes v_1\otimes|m-1\rangle - q^{2m+1}v_1\otimes v_0\otimes|m\rangle,$$
$$K(v_0\otimes|m\rangle) = v_0\otimes\mathbf{a}^+|m\rangle + v_1\otimes\mathbf{k}|m\rangle = v_0\otimes|m+1\rangle + q^{m+\frac{1}{2}}v_1\otimes|m\rangle,$$
$$K(v_1\otimes|m\rangle) = -v_0\otimes\mathbf{k}|m\rangle + v_1\otimes\mathbf{a}^-|m\rangle = -q^{m+\frac{1}{2}}v_0\otimes|m\rangle + (1-q^{2m})v_1\otimes|m-1\rangle.$$

Note the obvious properties

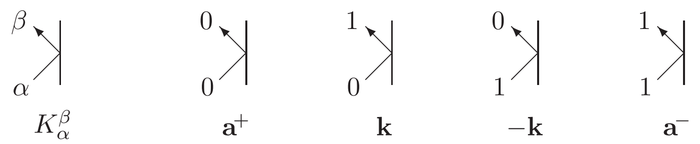
$$L^{\gamma,\delta}_{\alpha,\beta} = 0 \quad \text{unless } \alpha+\beta = \gamma+\delta, \tag{7}$$
$$\mathbf{h}L^{\gamma,\delta}_{\alpha,\beta} = L^{\gamma,\delta}_{\alpha,\beta}(\mathbf{h}+\beta-\delta), \qquad \mathbf{h}K^\beta_\alpha = K^\beta_\alpha(\mathbf{h}+1-\alpha-\beta), \tag{8}$$

[3]This is an indication of a parallel story in the modular double setting.

which will be referred to as *weight conservation*. We depict L as



So L may be regarded as defining a q-boson valued six vertex model in which the latter relation of (3) plays the role of "free-fermion" condition. See eq. $(10.16.5)|_{d=0}$ in [2]. Similarly K is pictured as



(9)

Here the lines without an arrow signifies a reflecting boundary to which no physical degree of freedom is assigned. One may imagine that each vertex as a q-boson operator acting in the direction perpendicular to these planar diagrams from the back to the front. Up to conventional difference, the q-boson valued L matrix (5) appeared in [4].

2.2. **Quantized reflection equation.** Let $\mathcal{K} \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)$ be a linear operator so normalized as

$$\mathcal{K}(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle. \tag{10}$$

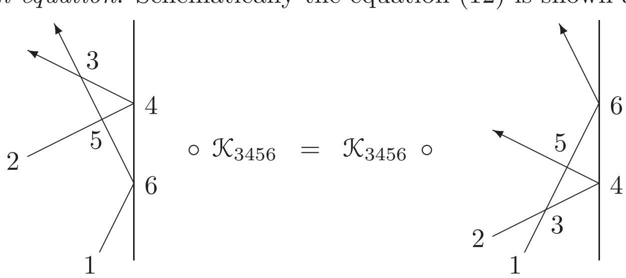
For L and K defined in (5) and (6), we propose the q-boson valued reflection equation that holds up to $\mathcal{K}$-conjugation as follows:

$$(L_{12}K_2L_{21}K_1)\mathcal{K} = \mathcal{K}(K_1L_{12}K_2L_{21}). \tag{11}$$

This is an equality of linear operators on $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{F}_{q^2} \otimes \overset{4}{F}_q \otimes \overset{5}{F}_{q^2} \otimes \overset{6}{F}_q$, where the superscripts are just temporal labels for explanation. If they are all exhibited (11) reads as

$$L_{123}K_{24}L_{215}K_{16}\mathcal{K}_{3456} = \mathcal{K}_{3456}K_{16}L_{125}K_{24}L_{213}. \tag{12}$$

Here L_{125} for example stands for the operator that acts as L on $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{5}{F}_{q^2}$ and as the identity elsewhere. We have also set $L_{215} = P_{12}L_{125}P_{12}$ and $L_{213} = P_{12}L_{123}P_{12}$, where $P_{12} : u \otimes v \mapsto v \otimes u$ is the interchange of the components of $\overset{1}{V} \otimes \overset{2}{V}$. Put in words, the factors $L_{12}K_2L_{21}K_1$ and $K_1L_{12}K_2L_{21}$ in (11) mean the compositions of operators on $\overset{1}{V} \otimes \overset{2}{V}$ and simultaneously represent the tensor product corresponding to the Fock part $\overset{3}{F}_{q^2} \otimes \overset{4}{F}_q \otimes \overset{5}{F}_{q^2} \otimes \overset{6}{F}_q$. The multiplication of $\mathcal{K}$ is with respect to the Fock part. When $\mathcal{K}$ is trivial and L and K become scalars on $\overset{3}{F}_{q^2} \otimes \overset{4}{F}_q \otimes \overset{5}{F}_{q^2} \otimes \overset{6}{F}_q$, the equation (11) reduces to the usual reflection equation without a spectral parameter $L_{12}K_2L_{21}K_1 = K_1L_{12}K_2L_{21}$. In this sense we call (11) the *quantized reflection equation*. Schematically the equation (12) is shown as follows:



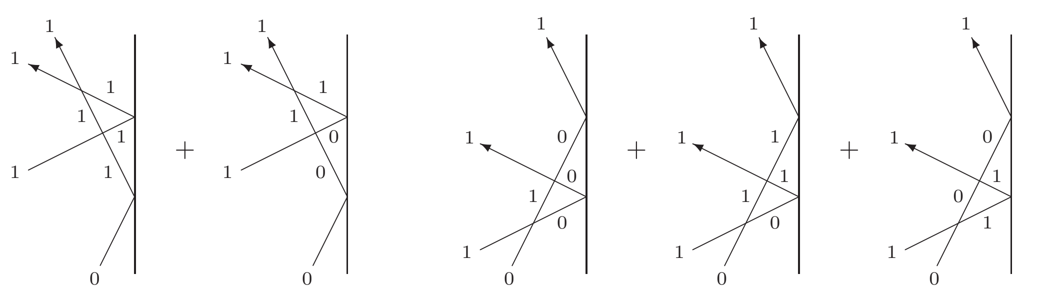
Here the indices $1, 2$ are assigned to the lines, whereas $3, 4, 5, 6$ are attached to the vertices. One may rather regard $\mathcal{K}_{3456}$ in the left (right) hand side as a point in the back (front) of the diagram where the four arrows going toward (coming from) the vertices $3, 4, 5, 6$ intersect[4].

[4] We did not have the graphical skill to draw such a nice figure.

Let us write down (11) explicitly. Let $(abij)$ denote its matrix element corresponding to the transition $v_j \otimes v_i \mapsto v_b \otimes v_a$ in $\overset{1}{V} \otimes \overset{2}{V}$. They read as

$$
\begin{aligned}
&(1111): && [1\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{a}^- - 1\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{k},\, \mathcal{K}] = 0, && (13)\\
&(1110): && (1\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+)\mathcal{K} && \\
& && = \mathcal{K}(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{A}^-\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^- - \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k}), && (14)\\
&(1101): && (1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-)\mathcal{K} = \mathcal{K}(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^-), && (15)\\
&(1100): && [1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{k}, \mathcal{K}] = 0, && (16)\\
&(1011): && (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{A}^-\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^- - \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k})\mathcal{K} && \\
& && = \mathcal{K}(1\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+), && (17)\\
&(1010): && [\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - \mathbf{A}^-\otimes\mathbf{k}\otimes 1\otimes\mathbf{k} - \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+,\, \mathcal{K}] = 0, && (18)\\
&(1001): && (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^- + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k})\mathcal{K} && \\
& && = \mathcal{K}(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+ + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k}), && (19)\\
&(1000): && (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+)\mathcal{K} = \mathcal{K}(1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^+), && (20)\\
&(0111): && (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^-)\mathcal{K} = \mathcal{K}(1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-), && (21)\\
&(0110): && (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+ + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k})\mathcal{K} && \\
& && = \mathcal{K}(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^- + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k}), && (22)\\
&(0101): && [\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - \mathbf{A}^+\otimes\mathbf{k}\otimes 1\otimes\mathbf{k} - \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^-,\, \mathcal{K}] = 0, && (23)\\
&(0100): && (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{A}^+\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+ - \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k})\mathcal{K} && \\
& && = \mathcal{K}(1\otimes\mathbf{a}^+\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{a}^-), && (24)\\
&(0011): && [1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{k},\, \mathcal{K}] = 0 \quad \text{(same as (1100))}, && (25)\\
&(0010): && (1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^+)\mathcal{K} = \mathcal{K}(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+), && (26)\\
&(0001): && (1\otimes\mathbf{a}^+\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{a}^-)\mathcal{K} && \\
& && = \mathcal{K}(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{A}^+\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+ - \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k}), && (27)\\
&(0000): && [1\otimes\mathbf{a}^+\otimes 1\otimes\mathbf{a}^+ - 1\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{k},\, \mathcal{K}] = 0. && (28)
\end{aligned}
$$

As an illustration the second equation (1110) originates in the matrix element for the transition $v_0 \otimes v_1 \mapsto v_1 \otimes v_1$. The corresponding factors $L_{123}K_{24}L_{215}K_{16}$ and $K_{16}L_{125}K_{24}L_{213}$ in (12) are calculated as



$$1\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{k} \quad + \quad 1\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+ \qquad \mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} \quad + \quad \mathbf{A}^-\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^- \quad + \quad -\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k}$$

One of our main findings in this paper is that the quantized reflection equation (11) listed in (13)–(28) exactly reproduces the characterization condition (42) of the 3D reflection matrix as the intertwiner of the quantized coordinate ring $A_q(sp_4)$ in [15, App.A][5]. Therefore $\mathcal{K}$ *is* the 3D reflection matrix. We will make use of this connection and consequent properties efficiently in Section 4 and Section 5. The next section is meant to be a preparation for it recalling basic facts from [11, 4, 15]. In short we have found a solution to the quantized reflection equation.

Remark. The set of equations (13)–(28) is invariant under the exchange $\mathbf{a}^+ \leftrightarrow \mathbf{a}^-, \mathbf{A}^+ \leftrightarrow \mathbf{A}^-$. Therefore $(L', K') = (L, K)|_{\mathbf{a}^+\leftrightarrow\mathbf{a}^-, \mathbf{A}^+\leftrightarrow\mathbf{A}^-}$ can also be employed to formulate the quantized reflection equation to

[5]The q-boson operators $\mathbf{K}$ and $\mathbf{k}$ in [15] do not contain the zero point energy, hence there are powers of q around.

characterize the same $\mathcal{K}$. The reduction procedure in Section 4 and 5 works even for the *mixture* of L, L', K, K'. However the resulting degree of freedom which apparently generalizes (49), (55), (72) and (80) can be absorbed into a suitable redefinition of the bases and gauges.

3. Brief summary on 3D R and 3D K

3.1. $A_q(sl_3)$, 3D $\mathcal{R}$ and tetrahedron equation. The quantized coordinate ring $A_q(sl_3)$ is a Hopf algebra realized by 9 generators $(t_{ij})_{1\le i,j\le 3}$ with relations [18]. Their explicit form is available in [15, Sec.2]. The maps

$$\pi_1 : (t_{ij})_{1\le i,j\le 3} \mapsto \begin{pmatrix} \mathbf{a}^- & c_1\mathbf{k} & 0 \\ d_1\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pi_2 : (t_{ij})_{1\le i,j\le 3} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{a}^- & c_2\mathbf{k} \\ 0 & d_2\mathbf{k} & \mathbf{a}^+ \end{pmatrix} \tag{29}$$

with $c_1d_1 = c_2d_2 = -1$ give irreducible representations $\pi_i : A_q(sl_3) \to \mathrm{End}(F_q)$. Let us denote $\pi_i\otimes\pi_j\otimes\pi_j$ by π_{iji} for short. According to the general theory [21] (see [15, Th.2.2]), the tensor product representations $\pi_{121}\circ\Delta$ and $\pi_{212}\circ\Delta : A_q(sl_3) \to \mathrm{End}((F_q)^{\otimes 3})$ are both irreducible and equivalent. Here $\Delta : A_q(sl_3) \to A_q(sl_3)^{\otimes 3}$ is the coproduct specified for the generators as $\Delta(t_{ij}) = \sum_{l_1 l_2} t_{il_1}\otimes t_{l_1l_2}\otimes t_{l_2j}$. Therefore there is a unique map $\Phi : (F_q)^{\otimes 3} \to (F_q)^{\otimes 3}$ characterized by the intertwining relation and the normalization as follows [11]:

$$\pi_{212}(\Delta(g))\circ\Phi = \Phi\circ\pi_{121}(\Delta(g)) \quad (\forall g \in A_q(sl_3)), \tag{30}$$

$$\Phi(|0\rangle\otimes|0\rangle\otimes|0\rangle) = |0\rangle\otimes|0\rangle\otimes|0\rangle. \tag{31}$$

We define $\hat{\mathcal{R}}$ and $\mathcal{R}$ as

$$\hat{\mathcal{R}} = \Phi P_{13} : (F_q)^{\otimes 3} \to (F_q)^{\otimes 3}, \qquad \mathcal{R} = \hat{\mathcal{R}}|_{q\to q^2} : (F_{q^2})^{\otimes 3} \to (F_{q^2})^{\otimes 3},$$

where $P_{13} : x_1\otimes x_2\otimes x_3 \mapsto x_3\otimes x_2\otimes x_1$ is the linear operator reversing the order of the tensor product. The conditions (30) and (31) are cast into

$$\pi_{212}(\Delta(g))\circ\hat{\mathcal{R}} = \hat{\mathcal{R}}\circ\pi_{121}(\Delta'(g)) \quad (\forall g \in A_q(sl_3)), \tag{32}$$

$$\hat{\mathcal{R}}(|0\rangle\otimes|0\rangle\otimes|0\rangle) = |0\rangle\otimes|0\rangle\otimes|0\rangle, \tag{33}$$

where $\Delta' = P_{13}\Delta P_{13}$ is the opposite coproduct. The equations (32) are actually independent of the parameters c_i, d_i in (29) as long as $c_id_i = -1$. We present them and an explicit formula of $\hat{\mathcal{R}}$ in Appendix A.

The intertwining relation (32) admits an alternative formulation in a spirit closer to "$RLL = LLR$" [4]. In fact in terms of L in (5) the set of equations (32) turn out to be equivalent with the tetrahedron equation of $RLLL = LLLR$ type mentioned in the introduction:

$$L_{124}L_{135}L_{236}\,\mathcal{R}_{456} = \mathcal{R}_{456}L_{236}L_{135}L_{124} \in \mathrm{End}(\overset{1}{V}\otimes\overset{2}{V}\otimes\overset{3}{V}\otimes\overset{4}{F}_{q^2}\otimes\overset{5}{F}_{q^2}\otimes\overset{6}{F}_{q^2}), \tag{34}$$

where the notation is similar to (12).

One has the symmetry $\mathcal{R}_{123} = \mathcal{R}_{321} (:= P_{13}R_{123}P_{31})$. Some other notable properties of $\mathcal{R}$ are

$$\text{tetrahedron eq. } : \mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}, \tag{35}$$

$$\text{inversion relation } : \mathcal{R} = \mathcal{R}^{-1}, \tag{36}$$

$$\text{weight conservation } : [x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{R}] = 0, \tag{37}$$

where x, y are generic parameters and $\mathbf{h}_1 = \mathbf{h}\otimes 1\otimes 1$, $\mathbf{h}_2 = 1\otimes\mathbf{h}\otimes 1$, $\mathbf{h}_3 = 1\otimes 1\otimes\mathbf{h}$ in terms of $\mathbf{h}$ defined in (4). The weight conservation (37) will be the source of introducing *spectral parameters* in the reduction procedure. It originates in the factor $\delta^{a+b}_{i+j}\delta^{b+c}_{j+k}$ in (89).

The solution $\hat{\mathcal{R}}$ to the tetrahedron equation was obtained in [11][6] based on the representation theory of the quantized coordinate ring [21]. Later it was also found from (34) via a quantum geometry consideration [4]. The two $\mathcal{R}$'s were identified in [15, eq.(2.27)]. Here we simply call it the 3D $\mathcal{R}$.

[6] The formula for it on p194 in [11] contains a misprint unfortunately. Eq. (89) here is a correction of it.

3.2. **$A_q(sp_4)$, 3D $\mathcal{K}$ and 3D reflection equation.** The quantized coordinate ring $A_q(sp_4)$ is a Hopf algebra realized by 16 generators $(t_{ij})_{1\le i,j\le 4}$ with relations [18]. Their explicit form is available in [15, Sec.3]. The maps

$$\pi_1 : (t_{ij})_{1\le i,j\le 4} \mapsto \begin{pmatrix} \mathbf{a}^- & c_1\mathbf{k} & 0 & 0 \\ d_1\mathbf{k} & \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & \mathbf{a}^- & d_1^{-1}\mathbf{k} \\ 0 & 0 & c_1^{-1}\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \quad \pi_2 : (t_{ij})_{1\le i,j\le 4} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & c_2\mathbf{K} & 0 \\ 0 & d_2\mathbf{K} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{38}$$

with $c_1d_1 = c_2d_2 = -1$ give irreducible representations $\pi_i : A_q(sp_4) \to \mathrm{End}(F_{q^i})$ [15]. Coexistence of the q-boson and the q^2-boson originates in the two distinct length of the simple roots of sp_4. Let us denote $\pi_i\otimes\pi_j\otimes\pi_i\otimes\pi_j$ by π_{ijij} for short. According to the general theory [21], the tensor product representations $\pi_{1212}\circ\Delta : A_q(sp_4) \to \mathrm{End}(F_q\otimes F_{q^2}\otimes F_q\otimes F_{q^2})$ and $\pi_{2121}\circ\Delta : A_q(sp_4) \to \mathrm{End}(F_{q^2}\otimes F_q\otimes F_{q^2}\otimes F_q)$ are both irreducible and equivalent. Here $\Delta : A_q(sp_4) \to A_q(sp_4)^{\otimes 4}$ is the coproduct specified for generators as $\Delta(t_{ij}) = \sum_{l_1,l_2,l_3} t_{il_1}\otimes t_{l_1l_2}\otimes t_{l_2l_3}\otimes t_{l_3j}$. Therefore there is a unique map $\Psi : F_q\otimes F_{q^2}\otimes F_q\otimes F_{q^2} \longrightarrow F_{q^2}\otimes F_q\otimes F_{q^2}\otimes F_q$ characterized by the intertwining relation and the normalization:

$$\pi_{2121}(\Delta(g))\circ\Psi = \Psi\circ\pi_{1212}(\Delta(g)) \quad (\forall g\in A_q(sp_4)), \tag{39}$$

$$\Psi(|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle) = |0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle. \tag{40}$$

We define 3D $\mathcal{K}$ by

$$\mathcal{K} = \Psi P_{1234} : \ F_{q^2}\otimes F_q\otimes F_{q^2}\otimes F_q \longrightarrow F_{q^2}\otimes F_q\otimes F_{q^2}\otimes F_q, \tag{41}$$

where $P_{1234} : x_1\otimes x_2\otimes x_3\otimes x_4 \mapsto x_4\otimes x_3\otimes x_2\otimes x_1$ is the linear operator reversing the order of the tensor product. The condition (39) is cast into

$$\pi_{2121}(\Delta(g))\circ\mathcal{K} = \mathcal{K}\circ\pi_{2121}(\Delta'(g)) \quad (\forall g\in A_q(sp_4)), \tag{42}$$

where $\Delta' = P_{1234}\Delta P_{1234}$ is the opposite coproduct. With the choice $g = t_{ij}$ $(1\le i,j\le 4)$, (42) produces 16 equations. They are independent of the parameters c_i, d_i in (38) as long as $c_id_i = -1$. As stated in the end of Section 2, they coincide exactly with (13)–(28). Moreover the normalization conditions (40) becomes (10) under the correspondence (41) of $\mathcal{K}$ and Ψ. Therefore we conclude that the conjugation operator $\mathcal{K}$ in our quantized reflection equation (11) is nothing but the 3D $\mathcal{K}$, justifying the usage of the same symbol for it. See Appendix A for an explicit formula of $\mathcal{K}$. Here we pick some notable properties.

$$\text{3D reflection eqs.}\ \ \hat{\mathcal{R}}_{456}\hat{\mathcal{R}}_{489}\mathcal{K}_{3579}\hat{\mathcal{R}}_{269}\hat{\mathcal{R}}_{258}\mathcal{K}_{1678}\mathcal{K}_{1234} = \mathcal{K}_{1234}\mathcal{K}_{1678}\hat{\mathcal{R}}_{258}\hat{\mathcal{R}}_{269}\mathcal{K}_{3579}\hat{\mathcal{R}}_{489}\hat{\mathcal{R}}_{456},) \tag{43}$$

$$\text{inversion relation}\ :\ \mathcal{K} = \mathcal{K}^{-1}, \tag{44}$$

$$\text{weight conservation}\ :\ [(xy^{-1})^{\mathbf{h}_1}x^{\mathbf{h}_2}(xy)^{\mathbf{h}_3}y^{\mathbf{h}_4}, \mathcal{K}] = 0, \tag{45}$$

where x, y are generic parameters and $\mathbf{h}_i$ is defined similarly to those in (37). The weight conservation (45) originates in the factor $\delta^{a+b+c}_{i+j+k}\,\delta^{b+2c+d}_{j+2k+l}$ in (91). The $\hat{\mathcal{R}}$ in (43) is the 3D $\mathcal{R}$ described in the previous subsection.

The 3D reflection equation was proposed by Isaev and Kulish [7]. The above solution $(\hat{\mathcal{R}}, \mathcal{K})$ is due to [15]. In this paper we will not directly concern the 3D equations (35) and (43) but rather utilize their auxiliary linear problems (34) and (12).

4. Reduction to Yang-Baxter equation

4.1. **Concatenation of the tetrahedron equation.** Consider n copies of (34) in which the spaces labeled with $1, 2, 3$ are replaced by $1_i, 2_i, 3_i$ with $i = 1, 2, \ldots, n$:

$$(L_{1_i2_i4}L_{1_i3_i5}L_{2_i3_i6})\,\mathcal{R}_{456} = \mathcal{R}_{456}\,(L_{2_i3_i6}L_{1_i3_i5}L_{1_i2_i4}).$$

Sending $\mathcal{R}_{456}$ to the left by repeatedly applying this relation, we get

$$\begin{aligned}&(L_{1_12_14}L_{1_13_15}L_{2_13_16})\cdots(L_{1_n2_n4}L_{1_n3_n5}L_{2_n3_n6})\,\mathcal{R}_{456}\\ &\quad = \mathcal{R}_{456}\,(L_{2_13_16}L_{1_13_15}L_{1_12_14})\cdots(L_{2_n3_n6}L_{1_n3_n5}L_{1_n2_n4}).\end{aligned} \tag{46}$$

Set $\mathbf{V} = V^{\otimes n} \simeq (\mathbb{C}^2)^{\otimes n}$ in general and $\overset{\mathbf{1}}{\mathbf{V}} = \overset{1_1}{V}\otimes\cdots\otimes\overset{1_n}{V}$ with labels. The notations $\overset{\mathbf{2}}{\mathbf{V}}, \overset{\mathbf{3}}{\mathbf{V}}$ are to be understood similarly. The equality (46) holds in $\mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}\otimes\overset{\mathbf{2}}{\mathbf{V}}\otimes\overset{\mathbf{3}}{\mathbf{V}}\otimes\overset{4}{F}_{q^2}\otimes\overset{5}{F}_{q^2}\otimes\overset{6}{F}_{q^2})$.

The above maneuver is just a 3D analogue of deriving commutation relations of monodromy matrices for length n chain from concatenation of the local $RLL = LLR$ relation. It is possible to rearrange (46) without changing the order of any two operators sharing common labels as

$$\begin{aligned}&(L_{1_12_14}\cdots L_{1_n2_n4})(L_{1_13_15}\cdots L_{1_n3_n5})(L_{2_13_16}\cdots L_{2_n3_n6})\,\mathcal{R}_{456}\\ &\quad=\mathcal{R}_{456}\,(L_{2_13_16}\cdots L_{2_n3_n6})(L_{1_13_15}\cdots L_{1_n3_n5})(L_{1_12_14}\cdots L_{1_n2_n4}).\end{aligned}\tag{47}$$

4.2. **Trace reduction.** Write (37) in the form $\mathcal{R}_{456}^{-1}x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6} = x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}\mathcal{R}_{456}^{-1}$. Multiplying this to (47) from the left and taking the trace over $\overset{4}{F}_{q^2}\otimes\overset{5}{F}_{q^2}\otimes\overset{6}{F}_{q^2}$ we get

$$\begin{aligned}&\mathrm{Tr}_4(x^{\mathbf{h}_4}L_{1_12_14}\cdots L_{1_n2_n4})\mathrm{Tr}_5((xy)^{\mathbf{h}_5}L_{1_13_15}\cdots L_{1_n3_n5})\mathrm{Tr}_6(y^{\mathbf{h}_6}L_{2_13_16}\cdots L_{2_n3_n6})\\ &=\mathrm{Tr}_6(y^{\mathbf{h}_6}L_{2_13_16}\cdots L_{2_n3_n6})\mathrm{Tr}_5((xy)^{\mathbf{h}_5}L_{1_13_15}\cdots L_{1_n3_n5})\mathrm{Tr}_4(x^{\mathbf{h}_4}L_{1_12_14}\cdots L_{1_n2_n4}).\end{aligned}\tag{48}$$

All the factors appearing here possess the same structure as

$$S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(z)=\varrho^{\mathrm{tr}}(z)\mathrm{Tr}_a(z^{\mathbf{h}_a}L_{1_12_1a}\cdots L_{1_n2_na})\in\mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}\otimes\overset{\mathbf{2}}{\mathbf{V}}),\tag{49}$$

where a is a dummy label for the auxiliary Fock space $\overset{a}{F}_{q^2}$. We have inserted a scalar $\varrho^{\mathrm{tr}}(z)$ which will be specified in (67). Now the relation (48) is stated as the Yang-Baxter equation:

$$S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(x)S^{\mathrm{tr}}_{\mathbf{1},\mathbf{3}}(xy)S^{\mathrm{tr}}_{\mathbf{2},\mathbf{3}}(y)=S^{\mathrm{tr}}_{\mathbf{2},\mathbf{3}}(y)S^{\mathrm{tr}}_{\mathbf{1},\mathbf{3}}(xy)S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(x).\tag{50}$$

In the sequel, we will often suppress the labels of the spaces like $\mathbf{1},\mathbf{2}$ etc without notice if they are unnecessary. The above construction of $S^{\mathrm{tr}}(z)$ is due to [4], where it was claimed to yield the quantum R matrix for the antisymmetric tensor representations of $U_p(A^{(1)}_{n-1})$ with some p. A precise description adapted to the present convention is available in Appendix B.

4.3. **Boundary vector reduction.** For $s = 1, 2$, introduce the vectors

$$\langle\chi_s| = \sum_{m\geq 0}\frac{\langle sm|}{(q^{2s^2})_m}\in F^*_{q^2},\qquad |\chi_s\rangle=\sum_{m\geq 0}\frac{|sm\rangle}{(q^{2s^2})_m}\in F_{q^2},\tag{51}$$

$$\langle\eta_s| = \sum_{m\geq 0}\frac{\langle sm|}{(q^{s^2})_m}\in F^*_{q},\qquad |\eta_s\rangle=\sum_{m\geq 0}\frac{|sm\rangle}{(q^{s^2})_m}\in F_{q}.\tag{52}$$

Obviously $|\chi_s\rangle = |\eta_s\rangle|_{q\to q^2}$ and $\langle\chi_s| = \langle\eta_s||_{q\to q^2}$ hold. The following relations are proved in [16, Prop.4.1]:

$$(\langle\chi_s|\otimes\langle\chi_s|\otimes\langle\chi_s|)\mathcal{R}=\langle\chi_s|\otimes\langle\chi_s|\otimes\langle\chi_s|,\quad \mathcal{R}(|\chi_s\rangle\otimes|\chi_s\rangle\otimes|\chi_s\rangle)=|\chi_s\rangle\otimes|\chi_s\rangle\otimes|\chi_s\rangle.\tag{53}$$

Sandwich the relation (47) between the bra vector $(\langle\overset{4}{\chi_s}|\otimes\langle\overset{5}{\chi_s}|\otimes\langle\overset{6}{\chi_s}|)x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}$ and the ket vector $|\overset{4}{\chi_{s'}}\rangle\otimes|\overset{5}{\chi_{s'}}\rangle\otimes|\overset{6}{\chi_{s'}}\rangle$. Using (53) and (37) we find

$$\begin{aligned}&\langle\overset{4}{\chi_s}|x^{\mathbf{h}_4}L_{1_12_14}\cdots L_{1_n2_n4}|\overset{4}{\chi_{s'}}\rangle\langle\overset{5}{\chi_s}|(xy)^{\mathbf{h}_5}L_{1_13_15}\cdots L_{1_n3_n5}|\overset{5}{\chi_{s'}}\rangle\langle\overset{6}{\chi_s}|y^{\mathbf{h}_6}L_{2_13_16}\cdots L_{2_n3_n6}|\overset{6}{\chi_{s'}}\rangle\\ &=\langle\overset{6}{\chi_s}|y^{\mathbf{h}_6}L_{2_13_16}\cdots L_{2_n3_n6}|\overset{6}{\chi_{s'}}\rangle\langle\overset{5}{\chi_s}|(xy)^{\mathbf{h}_5}L_{1_13_15}\cdots L_{1_n3_n5}|\overset{5}{\chi_{s'}}\rangle\langle\overset{4}{\chi_s}|x^{\mathbf{h}_4}L_{1_12_14}\cdots L_{1_n2_n4}|\overset{4}{\chi_{s'}}\rangle.\end{aligned}\tag{54}$$

All the factors appearing here have the form

$$S^{s,s'}_{\mathbf{1},\mathbf{2}}(z)=\varrho^{s,s'}(z)\langle\overset{a}{\chi_s}|z^{\mathbf{h}_a}L_{1_12_1a}\cdots L_{1_n2_na}|\overset{a}{\chi_{s'}}\rangle\in\mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}\otimes\overset{\mathbf{2}}{\mathbf{V}})\qquad(s,s'=1,2),\tag{55}$$

where the notation is similar to (49). The scalar $\varrho^{s,s'}(z)$ will be specified in (67). Now (54) is stated as the Yang-Baxter equation:

$$S^{s,s'}_{\mathbf{1},\mathbf{2}}(x)S^{s,s'}_{\mathbf{1},\mathbf{3}}(xy)S^{s,s'}_{\mathbf{2},\mathbf{3}}(y)=S^{s,s'}_{\mathbf{2},\mathbf{3}}(y)S^{s,s'}_{\mathbf{1},\mathbf{3}}(xy)S^{s,s'}_{\mathbf{1},\mathbf{2}}(x).\tag{56}$$

This construction of the four solutions corresponding to the choice $1\leq s,s'\leq 2$ is due to [16], where the cases $(s,s')=(1,1),(2,1)$ and $(2,2)$ were identified with the quantum R matrices for the spin representations of $U_p(D^{(2)}_{n+1}), U_p(B^{(1)}_n)$ and $U_p(D^{(1)}_n)$ with some p. See Appendix B for a precise description including the case $(s,s')=(1,2)$.

4.4. **Basic properties of $S^{\mathrm{tr}}(z)$ and $S^{s,s'}(z)$.** We write the base vectors of $\mathbf{V} = V^{\otimes n}$ as $|\boldsymbol{\alpha}\rangle = v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n}$ in terms of an array $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$[7]. Set

$$S(z)(|\boldsymbol{\alpha}\rangle \otimes |\boldsymbol{\beta}\rangle) = \sum_{\boldsymbol{\gamma},\boldsymbol{\delta}\in\{0,1\}^n} S(z)^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} |\boldsymbol{\gamma}\rangle \otimes |\boldsymbol{\delta}\rangle \qquad (S = S^{\mathrm{tr}}, S^{s,s'}).$$

Then the formulas (49) and (55) imply the matrix product structure as

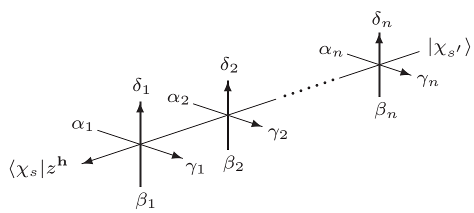
$$S^{\mathrm{tr}}(z)^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \varrho^{\mathrm{tr}}(z)\mathrm{Tr}\big(z^{\mathbf{h}} L^{\gamma_1,\delta_1}_{\alpha_1,\beta_1} \cdots L^{\gamma_n,\delta_n}_{\alpha_n,\beta_n}\big), \tag{57}$$

$$S^{s,s'}(z)^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \varrho^{s,s'}(z)\langle\chi_s| z^{\mathbf{h}} L^{\gamma_1,\delta_1}_{\alpha_1,\beta_1} \cdots L^{\gamma_n,\delta_n}_{\alpha_n,\beta_n} |\chi_{s'}\rangle, \tag{58}$$

where $L^{\gamma,\delta}_{\alpha,\beta}$ is given by (5) and $\mathrm{Tr}(\cdots)$ and $\langle\chi_s|(\cdots)|\chi_{s'}\rangle$ are taken over F_{q^2}. They are evaluated by using the commutation relations (3), the formulas in (81) with $q \to q^2$ and

$$\mathrm{Tr}(z^{\mathbf{h}}\mathbf{K}^r(\mathbf{A}^+)^s(\mathbf{A}^-)^{s'}) = \delta_{s,s'}\frac{q^r(q^4;q^4)_s}{(zq^{2r};q^4)_{s+1}}. \tag{59}$$

The expression (58) is shown diagrammatically as the barbeque stick with n X-shape sausage[8]



In view of this, we call (51) and (52) the *boundary vectors*, and the procedure in Section 4.3 the *boundary vector reduction*. Intriguingly the boundary vectors are known to reflect the end shape of the Dynkin diagrams of the relevant quantum affine algebras. See Appendix B and [16, Rem.7.2]. The vectors $\langle\eta_s|$ and $|\eta_s\rangle$ will play a similar role for the quantized reflection equation in Section 5.

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$ set

$$|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n, \qquad \mathbf{V}_k = \bigoplus_{\boldsymbol{\alpha}\in\{0,1\}^n,\, |\boldsymbol{\alpha}|=k} \mathbb{C}|\boldsymbol{\alpha}\rangle, \qquad \mathbf{V}^{\pm} = \bigoplus_{\boldsymbol{\alpha}\in\{0,1\}^n,\, (-1)^{|\boldsymbol{\alpha}|}=\pm 1} \mathbb{C}|\boldsymbol{\alpha}\rangle. \tag{60}$$

By the definition the following direct sum decomposition holds:

$$\mathbf{V} = V^{\otimes n} = \mathbf{V}_0 \oplus \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_n, \qquad \mathbf{V} = \mathbf{V}^+ \oplus \mathbf{V}^-.$$

From (1), (7), (8) and (51) one can show

$$S^{\mathrm{tr}}(z)^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = z^{|\boldsymbol{\beta}|-|\boldsymbol{\delta}|} S^{\mathrm{tr}}(z)^{\boldsymbol{\alpha}^\vee,\boldsymbol{\beta}^\vee}_{\boldsymbol{\gamma}^\vee,\boldsymbol{\delta}^\vee}, \quad S^{s,s'}(z)^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = z^{|\boldsymbol{\beta}|-|\boldsymbol{\delta}|} S^{s',s}(z)^{\boldsymbol{\alpha}^\vee,\boldsymbol{\beta}^\vee}_{\boldsymbol{\gamma}^\vee,\boldsymbol{\delta}^\vee}, \tag{61}$$

$$S(z)^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0 \ \text{ unless } \ \boldsymbol{\alpha}+\boldsymbol{\beta} = \boldsymbol{\gamma}+\boldsymbol{\delta} \in \mathbb{Z}^n \qquad (S = S^{\mathrm{tr}}, S^{s,s'}), \tag{62}$$

$$S^{\mathrm{tr}}(z)^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0 \ \text{ unless } \ |\boldsymbol{\alpha}| = |\boldsymbol{\gamma}| \ \text{ and } \ |\boldsymbol{\beta}| = |\boldsymbol{\delta}|, \tag{63}$$

$$S^{2,2}(z)^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0 \ \text{ unless } \ |\boldsymbol{\alpha}| \equiv |\boldsymbol{\gamma}| \ \text{ and } \ |\boldsymbol{\beta}| \equiv |\boldsymbol{\delta}| \quad \mod 2, \tag{64}$$

where $\boldsymbol{\alpha}^\vee = (\alpha_n, \ldots, \alpha_1)$ signifies the reversal of the array $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$. The properties (63) and (64) imply the decomposition

$$S^{\mathrm{tr}}(z) = \bigoplus_{0\le l,m\le n} S^{\mathrm{tr}}_{l,m}(z), \qquad S^{\mathrm{tr}}_{l,m}(z) \in \mathrm{End}(\mathbf{V}_l \otimes \mathbf{V}_m), \tag{65}$$

$$S^{2,2}(z) = \bigoplus_{\sigma,\sigma'=+,-} S^{2,2}_{\sigma,\sigma'}(z), \qquad S^{2,2}_{\sigma,\sigma'}(z) \in \mathrm{End}(\mathbf{V}^\sigma \otimes \mathbf{V}^{\sigma'}). \tag{66}$$

The Yang-Baxter equations (50) and $(56)|_{s=s'=2}$ are valid within the subspaces $\mathbf{V}_k \otimes \mathbf{V}_l \otimes \mathbf{V}_m$ and $\mathbf{V}^\sigma \otimes \mathbf{V}^{\sigma'} \otimes \mathbf{V}^{\sigma''}$ of $\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{2}}{\mathbf{V}} \otimes \overset{\mathbf{3}}{\mathbf{V}}$, respectively. The scalar prefactors in (57) and (58) may be specified

[7] $|\boldsymbol{\alpha}\rangle \in \mathbf{V}$ should not be confused with the base $|m\rangle$ of the Fock space containing a single integer.

[8] Each sausage carries $V \otimes V$ whereas the stick does F_{q^2}. The trace (57) corresponds to a ring shape stick. Description due to Sergey Sergeev.

depending on the summands in (65) and (66). We take them as

$$\begin{aligned}&\varrho^{\mathrm{tr}}_{l,m}(z)=q^{-|l-m|}(1-zq^{2|l-m|}),\qquad \varrho^{s,s'}(z)=\frac{(z^u;q^{2ss'})_\infty}{(-z^uq^2;q^{2ss'})_\infty}\quad ((s,s')\neq(2,2)),\\ &\varrho^{2,2}_{\pm,\pm}(z)=\frac{(z^2;q^8)_\infty}{(z^2q^4;q^8)_\infty},\qquad \varrho^{2,2}_{\pm,\mp}(z)=\frac{(z^2q^4;q^8)_\infty}{q(z^2q^8;q^8)_\infty},\end{aligned}\tag{67}$$

where $u=\max(s,s')$. This choice makes all the matrix elements of $S^{\mathrm{tr}}_{l,m}(z)$ and $S^{s,s'}(z)$ rational functions of z and q. It also simplifies some "typical" elements so that

$$\begin{aligned}&S^{\mathrm{tr}}_{l,m}(z)(|\mathbf{e}_1+\cdots+\mathbf{e}_l\rangle\otimes|\mathbf{e}_1+\cdots+\mathbf{e}_m\rangle)=(-1)^{\max(l-m,0)}|\mathbf{e}_1+\cdots+\mathbf{e}_l\rangle\otimes|\mathbf{e}_1+\cdots+\mathbf{e}_m\rangle,\\ &S(z)(|\mathbf{0}\rangle\otimes|\mathbf{0}\rangle)=|\mathbf{0}\rangle\otimes|\mathbf{0}\rangle\quad (S=S^{1,1},S^{1,2},S^{2,1},S^{2,2}_{+,+}),\qquad S^{2,2}_{-,-}(z)(|\mathbf{e}_1\rangle\otimes|\mathbf{e}_1\rangle)=|\mathbf{e}_1\rangle\otimes|\mathbf{e}_1\rangle,\\ &S^{2,2}_{+,-}(z)(|\mathbf{0}\rangle\otimes|\mathbf{e}_1\rangle)=|\mathbf{0}\rangle\otimes|\mathbf{e}_1\rangle,\qquad S^{2,2}_{-,+}(z)(|\mathbf{e}_1\rangle\otimes|\mathbf{0}\rangle)=-|\mathbf{e}_1\rangle\otimes|\mathbf{0}\rangle.\end{aligned}\tag{68}$$

5. Reduction to reflection equation

Starting from the quantized reflection equation (11), one can perform two kinds of reductions similar to Section 4 to construct the 2D K matrices systematically in the matrix product form. This is the main result of the paper which we are going to present in this section.

5.1. **Concatenation of quantized reflection equation.**

Consider n copies of (12) in which the spaces labeled with $1,2$ are replaced by $1_i,2_i$ with $i=1,2,\ldots,n$:

$$L_{1_i2_i3}K_{2_i4}L_{2_i1_i5}K_{1_i6}\mathcal{K}_{3456}=\mathcal{K}_{3456}K_{1_i6}L_{1_i2_i5}K_{2_i4}L_{2_i1_i3}.\tag{69}$$

As commented after (28) we know that the 3D $\mathcal{K}$ characterized by (42) and detailed in (91), (92) makes this relation hold. Using (69) successively, one can let $\mathcal{K}_{3456}$ penetrate $L_{1_i2_i3}K_{2_i4}L_{2_i1_i5}K_{1_i6}$ through to the left converting it into $K_{1_i6}L_{1_i2_i5}K_{2_i4}L_{2_i1_i3}$ $(i=1,2,\ldots,n)$ as

$$\begin{aligned}&(L_{1_12_13}K_{2_14}L_{2_11_15}K_{1_16})\cdots(L_{1_n2_n3}K_{2_n4}L_{2_n1_n5}K_{1_n6})\mathcal{K}_{3456}\\ &\quad=\mathcal{K}_{3456}(K_{1_16}L_{1_12_15}K_{2_14}L_{2_11_13})\cdots(K_{1_n6}L_{1_n2_n5}K_{2_n4}L_{2_n1_n3}).\end{aligned}$$

One can rearrange this without changing the order of operators sharing common labels as

$$\begin{aligned}&(L_{1_12_13}\cdots L_{1_n2_n3})(K_{2_14}\cdots K_{2_n4})(L_{2_11_15}\cdots L_{2_n1_n5})(K_{1_16}\cdots K_{1_n6})\mathcal{K}_{3456}\\ &\quad=\mathcal{K}_{3456}(K_{1_16}\cdots K_{1_n6})(L_{1_12_15}\cdots L_{1_n2_n5})(K_{2_14}\cdots K_{2_n4})(L_{2_11_13}\cdots L_{2_n1_n3}).\end{aligned}\tag{70}$$

5.2. **Trace reduction.**

Write (45) as $\mathcal{K}^{-1}_{3456}(xy^{-1})^{\mathbf{h}_3}x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}=(xy^{-1})^{\mathbf{h}_3}x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}\mathcal{K}^{-1}_{3456}$. Multiplying this to (70) from the left and taking the trace over $\overset{3}{F}_{q^2}\otimes\overset{4}{F}_q\otimes\overset{5}{F}_{q^2}\otimes\overset{6}{F}_q$, we obtain

$$\begin{aligned}&\mathrm{Tr}_3\big((xy^{-1})^{\mathbf{h}_3}L_{1_12_13}\cdots L_{1_n2_n3}\big)\mathrm{Tr}_4\big(x^{\mathbf{h}_4}K_{2_14}\cdots K_{2_n4}\big)\times\\ &\qquad\times\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5}L_{2_11_15}\cdots L_{2_n1_n5}\big)\mathrm{Tr}_6\big(y^{\mathbf{h}_6}K_{1_16}\cdots K_{1_n6}\big)\\ &=\mathrm{Tr}_6\big(y^{\mathbf{h}_6}K_{1_16}\cdots K_{1_n6}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5}L_{1_12_15}\cdots L_{1_n2_n5}\big)\times\\ &\qquad\times\mathrm{Tr}_4\big(x^{\mathbf{h}_4}K_{2_14}\cdots K_{2_n4}\big)\mathrm{Tr}_3\big((xy^{-1})^{\mathbf{h}_3}L_{2_11_13}\cdots L_{2_n1_n3}\big).\end{aligned}\tag{71}$$

Here $\mathrm{Tr}_3(\cdots)$ and $\mathrm{Tr}_5(\cdots)$ are identified with $S^{\mathrm{tr}}(z)$ in (49). The other factors emerging from K have the form

$$K^{\mathrm{tr}}_{\mathbf{1}}(z)=\kappa^{\mathrm{tr}}(z)\mathrm{Tr}_a\big(z^{\mathbf{h}_a}K_{1_1a}\cdots K_{1_na}\big)\in\mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}),\tag{72}$$

where $\overset{\mathbf{1}}{\mathbf{V}}=\overset{1_1}{V}\otimes\cdots\otimes\overset{1_n}{V}\simeq(\mathbb{C}^2)^{\otimes n}$ as before. The trace is taken over $\overset{a}{F}_q$ and evaluated by means of (2) and $(59)|_{q\to q^{1/2}}$. The scalar $\kappa^{\mathrm{tr}}(z)$ will be specified in (77). Now the relation (71) is the usual reflection equation in 2D:

$$S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(xy^{-1})K^{\mathrm{tr}}_{\mathbf{2}}(x)S^{\mathrm{tr}}_{\mathbf{2},\mathbf{1}}(xy)K^{\mathrm{tr}}_{\mathbf{1}}(y)=K^{\mathrm{tr}}_{\mathbf{1}}(y)S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(xy)K^{\mathrm{tr}}_{\mathbf{2}}(x)S^{\mathrm{tr}}_{\mathbf{2},\mathbf{1}}(xy^{-1}).\tag{73}$$

The construction (72) implies the matrix product formula for each element as

$$\begin{aligned}&K^{\mathrm{tr}}(z)|\boldsymbol{\alpha}\rangle=\sum_{\boldsymbol{\beta}\in\{0,1\}^n}K^{\mathrm{tr}}(z)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}|\boldsymbol{\beta}\rangle,\\ &K^{\mathrm{tr}}(z)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}=\kappa^{\mathrm{tr}}(z)\mathrm{Tr}\big(z^{\mathbf{h}}K^{\beta_1}_{\alpha_1}\cdots K^{\beta_n}_{\alpha_n}\big)\end{aligned}\tag{74}$$

in terms of K^{β}_{α} specified in (6).

To derive the selection rule of (74), suppose the number of pairs $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ in the multiset $\{(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)\}$ is r,s,t,u, respectively in (74). By the definition (60) we have $|\boldsymbol{\alpha}|=t+u$, $|\boldsymbol{\beta}|=s+u$ and $n=r+s+t+u$. Moreover in order to have a non-vanishing matrix element (74), there must be as many creation operators as annihilation operators. From (6) or (9), this imposes the constraint $r=u$. These relations force $|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|=n$. Namely we have the "dual weight" conservation:

$$K^{\mathrm{tr}}(z)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}=0 \quad \text{unless} \;\; |\boldsymbol{\alpha}|+|\boldsymbol{\beta}|=n \tag{75}$$

or equivalently, the direct sum decomposition:

$$K^{\mathrm{tr}}(z)=\bigoplus_{0\le l\le n} K^{\mathrm{tr}}_l(z), \qquad K^{\mathrm{tr}}_l(z):\mathbf{V}_l\to\mathbf{V}_{n-l}. \tag{76}$$

The space $\mathbf{V}_l$ (60) is naturally regarded as a fundamental $U_p(A^{(1)}_{n-1})$-module. See Appendix B. The above result suggests that a natural representation theoretical formulation of the boundary reflection is $\mathbf{V}_l\to\mathbf{V}_{n-l}$ rather than $\mathbf{V}_l\to\mathbf{V}_l$.

The equality (73) holds in a finer manner, i.e. as the identity of linear operators $\mathbf{V}_l\otimes\mathbf{V}_m\to\mathbf{V}_{n-l}\otimes\mathbf{V}_{n-m}$ for each pair $(l,m)\in\{0,1,\dots,n\}^2$. The scalar in (72) can be specified depending on l as $\kappa^{\mathrm{tr}}_l(z)$. We take it as

$$\kappa^{\mathrm{tr}}_l(z)=(-1)^l q^{-\frac{n}{2}}(1-zq^n). \tag{77}$$

Then applying (9) and $(59)|_{q\to q^{1/2}}$ to (74), it is easy to check

$$K^{\mathrm{tr}}_l(z)|\mathbf{e}_1+\cdots+\mathbf{e}_l\rangle=|\mathbf{e}_{l+1}+\cdots+\mathbf{e}_n\rangle+\cdots \qquad (0\le l\le n).$$

5.3. **Boundary vector reduction.** Supported by computer experiments we conjecture

$$\begin{aligned}
(\langle\chi_s|\otimes\langle\eta_k|\otimes\langle\chi_s|\otimes\langle\eta_k|)\mathcal{K}&=\langle\chi_s|\otimes\langle\eta_k|\otimes\langle\chi_s|\otimes\langle\eta_k| \qquad (1\le s\le k\le 2),\\
\mathcal{K}(|\chi_s\rangle\otimes|\eta_k\rangle\otimes|\chi_s\rangle\otimes|\eta_k\rangle)&=|\chi_s\rangle\otimes|\eta_k\rangle\otimes|\chi_s\rangle\otimes|\eta_k\rangle \qquad (1\le s\le k\le 2),
\end{aligned} \tag{78}$$

where the components are defined in (51) and (52). Sandwich the relation (70) between the bra vector $(\langle\overset{3}{\chi_s}|\otimes\langle\overset{4}{\eta_k}|\otimes\langle\overset{5}{\chi_s}|\otimes\langle\overset{6}{\eta_k}|)(xy^{-1})^{\mathbf{h}_3}x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}$ and the ket vector $|\overset{3}{\chi_{s'}}\rangle\otimes|\overset{4}{\eta_{k'}}\rangle\otimes|\overset{5}{\chi_{s'}}\rangle\otimes|\overset{6}{\eta_{k'}}\rangle$. Thanks to (78) the result reduces to

$$\begin{aligned}
&\langle\overset{3}{\chi_s}|(xy^{-1})^{\mathbf{h}_3}L_{1_12_13}\cdots L_{1_n2_n3}|\overset{3}{\chi_{s'}}\rangle\langle\overset{4}{\eta_k}|x^{\mathbf{h}_4}K_{2_14}\cdots K_{2_n4}|\overset{4}{\eta_{k'}}\rangle\times\\
&\qquad\times\langle\overset{5}{\chi_s}|(xy)^{\mathbf{h}_5}L_{2_11_15}\cdots L_{2_n1_n5}|\overset{5}{\chi_{s'}}\rangle\langle\overset{6}{\eta_k}|y^{\mathbf{h}_6}K_{1_16}\cdots K_{1_n6}|\overset{6}{\eta_{k'}}\rangle\\
&=\langle\overset{6}{\eta_k}|y^{\mathbf{h}_6}K_{1_16}\cdots K_{1_n6}|\overset{6}{\eta_{k'}}\rangle\langle\overset{5}{\chi_s}|(xy)^{\mathbf{h}_5}L_{1_12_15}\cdots L_{1_n2_n5}|\overset{5}{\chi_{s'}}\rangle\times\\
&\qquad\times\langle\overset{4}{\eta_k}|x^{\mathbf{h}_4}K_{2_14}\cdots K_{2_n4}|\overset{4}{\eta_{k'}}\rangle\langle\overset{3}{\chi_s}|(xy^{-1})^{\mathbf{h}_3}L_{2_11_13}\cdots L_{2_n1_n3}|\overset{3}{\chi_{s'}}\rangle.
\end{aligned} \tag{79}$$

The factors $\langle\chi_s|(\cdots)|\chi_{s'}\rangle$ involving L are identified with $S^{s,s'}(z)$ in (55). The other factors emerging from K have the form

$$K^{k,k'}_{\mathbf{1}}(z)=\kappa^{k,k'}(z)\langle\overset{a}{\eta_k}|z^{\mathbf{h}_a}K_{1_1a}\cdots K_{1_na}|\overset{a}{\eta_{k'}}\rangle\in\mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}) \qquad (k,k'=1,2), \tag{80}$$

where the scalar $\kappa^{k,k'}(z)$ will be specified in (86). The quantities $\langle\eta_k|(\cdots)|\eta_{k'}\rangle$ are evaluated by means of (2) and the following formulas:

$$\begin{aligned}
&\langle\eta_k|z^{\mathbf{h}}(\mathbf{a}^{\pm})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_{k'}\rangle=\langle\eta_{k'}|w^{\mathbf{h}}\mathbf{k}^m(\mathbf{a}^{\mp})^j z^{\mathbf{h}}|\eta_k\rangle \quad (k,k'=1,2),\\
&\langle\eta_1|z^{\mathbf{h}}(\mathbf{a}^{+})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_1\rangle=q^{\frac{m}{2}}z^j(-q;q)_j\frac{(-q^{j+m+1}zw;q)_\infty}{(q^mzw;q)_\infty},\\
&\langle\eta_1|z^{\mathbf{h}}(\mathbf{a}^{-})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_2\rangle=q^{\frac{m}{2}}z^{-j}\sum_{i=0}^{j}(-1)^i q^{\frac12 i(i+1-2j)}\binom{j}{i}_q\frac{(-q^{2i+2m+1}z^2w^2;q^2)_\infty}{(q^{2i+2m}z^2w^2;q^2)_\infty},\\
&\langle\eta_1|z^{\mathbf{h}}(\mathbf{a}^{+})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_2\rangle=q^{\frac{m}{2}}z^{j}\sum_{i=0}^{j}q^{\frac12 i(i+1)}\binom{j}{i}_q\frac{(-q^{2i+2m+1}z^2w^2;q^2)_\infty}{(q^{2i+2m}z^2w^2;q^2)_\infty},\\
&\langle\eta_2|z^{\mathbf{h}}(\mathbf{a}^{+})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_2\rangle=\theta(j\in 2\mathbb{Z})\,q^{\frac{m}{2}}z^j(q^2;q^4)_{j/2}\frac{(q^{2j+2m+2}z^2w^2;q^4)_\infty}{(q^{2m}z^2w^2;q^4)_\infty}.
\end{aligned} \tag{81}$$

These are easily derived by only using the elementary identity

$$\sum_{j\ge 0}\frac{(w;q)_j}{(q;q)_j}z^j=\frac{(wz;q)_\infty}{(z;q)_\infty}.$$

In terms of (80) and (58), the relation (79) is stated as the reflection equation:

$$S^{s,s'}_{\mathbf{1},\mathbf{2}}(xy^{-1})K^{k,k'}_{\mathbf{2}}(x)S^{s,s'}_{\mathbf{2},\mathbf{1}}(xy)K^{k,k'}_{\mathbf{1}}(y)=K^{k,k'}_{\mathbf{1}}(y)S^{s,s'}_{\mathbf{1},\mathbf{2}}(xy)K^{k,k'}_{\mathbf{2}}(x)S^{s,s'}_{\mathbf{2},\mathbf{1}}(xy^{-1}) \tag{82}$$

for any $1\le s\le k\le 2$ and $1\le s'\le k'\le 2$. Thus we get, assuming (78), the solutions $(S^{s,s'}(z),K^{k,k'}(z))$ to the reflection equation involving the quantum R matrices for the spin representation of $U_p(D^{(2)}_{n+1})$, $U_p(B^{(1)}_n)$, $U_p(\tilde{B}^{(1)}_n)$ and $U_p(D^{(1)}_n)$.

The construction (80) implies the matrix product formula for each element as

$$\begin{aligned}K^{k,k'}(z)|\boldsymbol{\alpha}\rangle&=\sum_{\boldsymbol{\beta}\in\{0,1\}^n}K^{k,k'}(z)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}|\boldsymbol{\beta}\rangle,\\ K^{k,k'}(z)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}&=\kappa^{k,k'}(z)\langle\eta_k|z^{\mathbf{h}}K^{\beta_1}_{\alpha_1}\cdots K^{\beta_n}_{\alpha_n}|\eta_{k'}\rangle\end{aligned} \tag{83}$$

in terms of K^β_α in (6). From (1) and the fact $\kappa^{k,k'}(z)=\kappa^{k',k}(z)$ in (86), it can be shown that

$$K^{k,k'}(z)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}=z^{n-|\boldsymbol{\alpha}|-|\boldsymbol{\beta}|}K^{k',k}(z)^{\mathbf{e}_1+\cdots+\mathbf{e}_n-\boldsymbol{\alpha}^\vee}_{\mathbf{e}_1+\cdots+\mathbf{e}_n-\boldsymbol{\beta}^\vee}, \tag{84}$$

where $\vee$ is the same as in (61). Noting the factor $\theta(j\in 2\mathbb{Z})$ in the last formula in (81), one can show

$$K^{2,2}(z)^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}=0\quad\text{unless}\ \ |\boldsymbol{\alpha}|+|\boldsymbol{\beta}|\equiv n\mod 2 \tag{85}$$

by an argument similar to that given after (74). Consequently the direct sum decomposition

$$K^{2,2}(z)=K^{2,2}_+(z)\oplus K^{2,2}_-(z),\qquad K^{2,2}_\sigma(z):\mathbf{V}^\sigma\to\mathbf{V}^{\sigma(-1)^n}$$

holds, where $\mathbf{V}^\pm$ was defined in (60). As for $K^{k,k'}(z)$ with $(k,k')\ne(2,2)$, there is no selection rule like (75) nor (85). We choose the scalar $\kappa^{k,k'}(z)$ as

$$\kappa^{k,k'}(z)=q^{-\frac{n}{2}}\frac{((zq^n)^u;q^{kk'})_\infty}{((-q)^r(zq^n)^u;q^{kk'})_\infty},\qquad r=\min(k,k'),\ \ u=\max(k,k'), \tag{86}$$

which is the inverse of $\langle\eta_k|z^{\mathbf{h}}\mathbf{k}^n|\eta_{k'}\rangle$ calculated from (81). It leads to the normalization

$$K^{k,k'}(z)|\mathbf{e}_1+\cdots+\mathbf{e}_l\rangle=(-1)^l|\mathbf{e}_{l+1}+\cdots+\mathbf{e}_n\rangle+\cdots\qquad(0\le l\le n,\ 1\le k,k'\le 2).$$

6. Concluding remarks

In this paper we have proposed the quantized reflection equation (11) and presented a solution in terms of the q-boson values L and K matrices in (2) and (3) and most notably the intertwiner of $A_q(sp_4)$ module known as the 3D $\mathcal{K}$ [15] in (90)–(92). From its n-concatenation the pair $(S^{\rm tr}(z),K^{\rm tr}(z))$ is constructed by the trace reduction in (49), (74) and $(S^{s,s'}(z),K^{k,k'}(z))$ by the boundary vector reduction in (55), (80). They are all expressed in the matrix product form and yield new solutions to the reflection equation as in (73) and (82). Our boundary vector reduction is based on the yet conjectural property (78)[9].

The matrices $S(z)=S^{\rm tr}(z)$ and $S^{s,s'}(z)$ satisfy the Yang-Baxter equation by themselves. In fact, as detailed in Appendix B, they are quantum R matrices for finite dimensional representations of quantum affine algebras $U_p(\mathfrak{g})$ with $\mathfrak{g}=A^{(1)}_{n-1},B^{(1)}_n,\tilde{B}^{(1)}_n,D^{(1)}_n$ and $D^{(2)}_{n+1}$. To summarize our solutions, we list $\mathfrak{g}$, the associated $S(z)$ in (99)–(103) and those $K(z)$'s that can be paired with the $S(z)$ to jointly constitute a solution $(S(z),K(z))$ to the reflection equation.

$\mathfrak{g}$	R matrix	K matrix
$A^{(1)}_{n-1}$	$S^{\rm tr}(z)$	$K^{\rm tr}(z)$,
$D^{(2)}_{n+1}$	$S^{1,1}(z)$	$K^{1,1}(z)$, $K^{1,2}(z)$, $K^{2,1}(z)$, $K^{2,2}(z)$
$B^{(1)}_n$	$S^{2,1}(z)$	$K^{2,1}(z)$, $K^{2,2}(z)$
$\tilde{B}^{(1)}_n$	$S^{1,2}(z)$	$K^{1,2}(z)$, $K^{2,2}(z)$
$D^{(1)}_n$	$S^{2,2}(z)$	$K^{2,2}(z)$

[9] Besides (78), the relevant reflection equation (82) have been verified for $n=2$ and for many examples from $n=3$.

This paper achieves the first systematic solutions to the reflection equation by a method of matrix product connected to the 3D integrability. It suggests a number of future problems.

(i) Prove (78) and more generally classify the eigenvectors of $\mathcal{R}$ and $\mathcal{K}$ which are factorized as in (53) and (78). Such vectors will serve as boundary vectors to produce further solutions to the Yang-Baxter and the reflection equations.

(ii) Study the commuting transfer matrices with boundary associated with the solutions in this paper. The routine construction of the double row transfer matrices acquires 3D interpretation. They are actually double *layer* transfer matrices with boundary where the rank n specifies a length of the layer in one direction.

(iii) Explore further solutions or versions of the quantized reflection equation. For instance it is natural to consider a counterpart of (12) in which $V = \mathbb{C}^2$ is replaced by F_{q^2}. It will generate a large family of matrix product solutions to the reflection equation. The resulting systems are expected to possess rich contents both in physics and mathematics related to special functions, combinatorics in the crystal limit $q \to 0$, stochastic processes (cf. [5, 14]) and so forth.

Appendix A. Explicit form of 3D $\mathcal{R}$ and 3D $\mathcal{K}$

It suffices to impose (32) for the generators $g = t_{ij}$ with $1 \leq i, j \leq 3$. The resulting nine equations read as

$$\begin{aligned}
&\hat{\mathcal{R}}(\mathbf{a}^{\pm} \otimes \mathbf{k} \otimes 1) = (\mathbf{a}^{\pm} \otimes 1 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{a}^{\pm} \otimes \mathbf{a}^{\mp})\hat{\mathcal{R}},\\
&\hat{\mathcal{R}}(1 \otimes \mathbf{k} \otimes \mathbf{a}^{\pm}) = (\mathbf{k} \otimes 1 \otimes \mathbf{a}^{\pm} + \mathbf{a}^{\mp} \otimes \mathbf{a}^{\pm} \otimes \mathbf{k})\hat{\mathcal{R}},\\
&\hat{\mathcal{R}}(1 \otimes \mathbf{a}^{\pm} \otimes 1) = (\mathbf{a}^{\pm} \otimes 1 \otimes \mathbf{a}^{\pm} - \mathbf{k} \otimes \mathbf{a}^{\pm} \otimes \mathbf{k})\hat{\mathcal{R}},\\
&\hat{\mathcal{R}}(\mathbf{a}^{+} \otimes \mathbf{a}^{-} \otimes \mathbf{a}^{+} - \mathbf{k} \otimes 1 \otimes \mathbf{k}) = (\mathbf{a}^{-} \otimes \mathbf{a}^{+} \otimes \mathbf{a}^{-} - \mathbf{k} \otimes 1 \otimes \mathbf{k})\hat{\mathcal{R}},\\
&[\hat{\mathcal{R}}, \mathbf{k} \otimes \mathbf{k} \otimes 1] = [\hat{\mathcal{R}}, 1 \otimes \mathbf{k} \otimes \mathbf{k}] = 0.
\end{aligned} \tag{87}$$

These are analogue of (13)–(28) for $\mathcal{K}$ and yield recursion relations on the matrix elements of $\mathcal{R}$. With the normalization (33) the solution is unique and given by

$$\hat{\mathcal{R}}(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c \geq 0} \hat{\mathcal{R}}^{a,b,c}_{i,j,k} |a\rangle \otimes |b\rangle \otimes |c\rangle, \tag{88}$$

$$\hat{\mathcal{R}}^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k} \sum_{\lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}, \tag{89}$$

where $\delta^j_k = \theta(j = k)$ just to save the space. The sum (89) is over $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ satisfying $\lambda + \mu = b$, which is also bounded by the condition $\mu \leq i$ and $\lambda \leq j$. For instance, the following is the list of all the nonzero $\hat{\mathcal{R}}^{a,b,c}_{3,1,2}$:

$$\begin{aligned}
&\mathcal{R}^{1,3,0}_{3,1,2} = -q^2(1-q^4)(1-q^6), \quad &&\mathcal{R}^{2,2,1}_{3,1,2} = (1+q^2)(1-q^6)(1-q^2-q^6),\\
&\mathcal{R}^{1,3,0}_{3,1,2} = q^6, &&\mathcal{R}^{3,1,2}_{3,1,2} = -q^2(-1-q^2+q^6+q^8+q^{10}).
\end{aligned}$$

From (89) we see $\hat{\mathcal{R}}^{a,b,c}_{i,j,k} \in q^{\xi}\mathbb{Z}[q^2]$, where $\xi = 0, 1$ is specified by $\xi \equiv (a-j)(c-j) \bmod 2$. See [15, Sec.2] for further properties.

Let us turn to an explicit formula for the 3D $\mathcal{K}$ which belongs to $\mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)$. See (41). We set

$$\mathcal{K}(|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle) = \sum_{a,b,c,d \geq 0} \mathcal{K}^{a,b,c,d}_{i,j,k,l} |a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle. \tag{90}$$

From (13)–(28) and the normalization (10) the matrix element is uniquely determined [15, Th.2.4] as

$$\begin{aligned}
\mathcal{K}^{a,b,c,d}_{i,j,k,l} &= \delta^{a+b+c}_{i+j+k}\,\delta^{b+2c+d}_{j+2k+l} \frac{(q^4)_i}{(q^4)_a} \sum_{\alpha,\beta,\gamma} \frac{(-1)^{\alpha+\gamma}}{(q^4)_{c-\beta}} q^{\phi_1}\\
&\quad \times \mathcal{K}^{i,j+k-\alpha-\beta-\gamma,0,l+k-\alpha-\beta-\gamma}_{a,b+c-\alpha-\beta-\gamma,0,c+d-\alpha-\beta-\gamma} \begin{Bmatrix} k, c-\beta, j+k-\alpha-\beta, k+l-\alpha-\beta \\ \alpha, \beta, \gamma, b-\alpha, d-\alpha, k-\alpha-\beta, c-\beta-\gamma \end{Bmatrix},
\end{aligned} \tag{91}$$

$$\phi_1 = \alpha(\alpha + 2c - 2\beta - 1) + (2\beta - c)(b + c + d) + \gamma(\gamma - 1) - k(j + k + l),$$

where the sum is over $\alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}$. The special case $\mathcal{K}^{\bullet,\bullet,0,\bullet}_{\bullet,\bullet,0,\bullet}$ appearing in the sum is given by

$$\mathcal{K}^{a,b,0,d}_{i,j,0,l} = \delta^{a+b}_{i+j}\, \delta^{b+d}_{j+l} \sum_{\lambda} (-1)^{b+\lambda} \frac{(q^4)_{a+\lambda}}{(q^4)_a} q^{\phi_2} \begin{Bmatrix} j, l \\ \lambda, l-\lambda, b-\lambda, j-b+\lambda \end{Bmatrix}, \tag{92}$$
$$\phi_2 = (i+a+1)(b+l-2\lambda) + b - l,$$

where the sum is over $\lambda \in \mathbb{Z}_{\geq 0}$. In (91) and (92) we have used the notation

$$\begin{Bmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_s \end{Bmatrix} = \begin{cases} \frac{\prod_{k=1}^r (q^2)_{i_k}}{\prod_{k=1}^s (q^2)_{j_k}} & \forall i_k, j_k \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise} \end{cases}$$

without requiring $\sum_{k=1}^r i_k = \sum_{k=1}^r j_k$. Due to the definition of the symbol $\{\cdots\}$ the sums $\sum_{\alpha,\beta,\gamma}$ in (91) and $\sum_\lambda$ in (92) are both finite ones. It has been shown [15, Th.3.5] that $K^{a,b,c,d}_{i,j,k,l} \in q^\eta \mathbb{Z}[q^2]$ holds, where $\eta = 0, 1$ is specified by $\eta \equiv bd + jl \mod 2$. For instance the following is the list of all the nonzero $\mathcal{K}^{1,1,1,1}_{i,j,k,l}$:

$$\begin{aligned}
\mathcal{K}^{1,1,1,1}_{0,2,1,0} &= q^5(1+q^2)(1-q^2-q^6), \\
\mathcal{K}^{1,1,1,1}_{0,3,0,1} &= -q^2(1-q^6)(1-q^2-q^4-q^6-q^8), \\
\mathcal{K}^{1,1,1,1}_{1,0,2,0} &= -q(1+q^2)(1+q^4)(1-q^4+q^{10}), \\
\mathcal{K}^{1,1,1,1}_{1,1,1,1} &= (1-q^4-q^8)(1-q^2-q^4+q^8+q^{10}), \\
\mathcal{K}^{1,1,1,1}_{1,2,0,2} &= -q^5(1+q^2)(1-q^4)(2-q^2+q^4-2q^6-q^{10}), \\
\mathcal{K}^{1,1,1,1}_{2,0,1,2} &= q(1+q^2)(1-q^8)(1-q^4-q^8+q^{10}+q^{14}), \\
\mathcal{K}^{1,1,1,1}_{2,1,0,3} &= q^2(1+q^2)(1+q^4)(1-q^6)^2(1-q^2-q^8), \\
\mathcal{K}^{1,1,1,1}_{3,0,0,4} &= q^5(1+q^2)(1+q^4)(1-q^6)(1-q^8)(1-q^{12}).
\end{aligned}$$

See [15, Sec.3] for further properties.

Appendix B. $S^{\mathrm{tr}}(z)$ and $S^{s,s'}(z)$ as quantum R matrices

The solutions to the Yang-Baxter equation $S^{\mathrm{tr}}(z)$ (49), (65) and $S^{s,s'}(z)$ (55) are identified with the quantum R matrices which are characterized by the commutativity with quantum groups.

B.1. **Quantum affine algebras.** Consider the Drinfeld-Jimbo quantum affine algebra (without derivation) $U_p(A^{(1)}_n), U_p(D^{(2)}_{n+1}), U_p(B^{(1)}_n), U_p(\tilde{B}^{(1)}_n)$ and $U_p(D^{(1)}_n)$. They are Hopf algebras generated by $e_i, f_i, k_i^{\pm 1}$ $(0 \leq i \leq n)$ satisfying

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \;\; [k_i, k_j] = 0, \;\; k_i e_j k_i^{-1} = p_i^{a_{ij}} e_j, \;\; k_i f_j k_i^{-1} = p_i^{-a_{ij}} f_j, \;\; [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{p_i - p_i^{-1}} \tag{93}$$

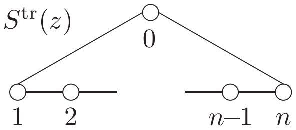
together with the p-Serre relations [6, 8]. We employ the coproduct Δ of the form

$$\Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta e_i = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i. \tag{94}$$

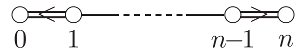
We follow the convention in [10] to determine the Cartan matrix $(a_{ij})_{0 \leq i,j \leq n}$ from the Dynkin diagrams[10]:

[10] The solutions to the Yang-Baxter equation $S^{s,s'}(z)$ which will be linked in (99)–(103) are also shown. One observes that the end shape of the Dynkin diagrams is reflected in s, s', namely, the boundary vectors $\langle \chi_s |$, $|\chi_{s'}\rangle$ in (51).

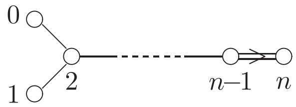
$A_n^{(1)};\ S^{\mathrm{tr}}(z)$



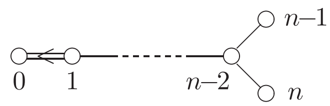
$D_{n+1}^{(2)};\ S^{1,1}(z)$



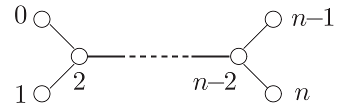
$B_n^{(1)};\ S^{2,1}(z)$



$\tilde{B}_n^{(1)};\ S^{1,2}(z)$



$D_n^{(1)};\ S^{2,2}(z)$



Here the affine Lie algebra $\tilde{B}_n^{(1)}$ is just $B_n^{(1)}$ but only with different enumeration of the nodes as shown above. We keep it for uniformity of the description although. The constants p_i $(0 \le i \le n)$ in (93) are all taken as $p_i = p^2$ except the following:

$$p_0 = p_n = p \ \text{ for } D_{n+1}^{(2)}, \qquad p_n = p \ \text{ for } B_n^{(1)}, \qquad p_0 = p \ \text{ for } \tilde{B}_n^{(1)}.$$

Thus for instance in $U_p(D_{n+1}^{(2)})$, one has $a_{01} = -2, a_{10} = -1$ and $k_0 e_1 = p^{-2} e_1 k_0$, $k_1 e_0 = p^{-2} e_0 k_1$ and $k_1 e_1 = p^4 e_1 k_1$. Forgetting the 0-th node in the Dynkin diagrams yields the classical subalgebras $U_p(A_n) \subset U_p(A_n^{(1)})$, $U_p(B_n) \subset U_p(D_{n+1}^{(2)})$, $U_p(D_n) \subset U_p(\tilde{B}_n^{(1)})$ and $U_p(D_n) \subset U_p(D_n^{(1)})$.

B.2. **Representations.** We assume that p is generic throughout. We use the notations $|\boldsymbol{\alpha}\rangle$, $|\boldsymbol{\alpha}|$, $\mathbf{V} = V^{\otimes n}$ $(V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \simeq \mathbb{C}^2)$, $\mathbf{V}_l$, $\mathbf{V}^{\pm}$ explained in the beginning of Section 4.4 and (60).

First consider $U_p(A_{n-1}^{(1)})$ (rather than $U_p(A_n^{(1)})$). For $0 \le l \le n$, the following map $\pi_{l,z} : U_p(A_{n-1}^{(1)}) \to \mathrm{End}(\mathbf{V}_l)$ defines an irreducible representation depending on the spectral parameter z[11]:

$$e_j|\boldsymbol{\alpha}\rangle = z^{\delta_{j,0}}|\boldsymbol{\alpha} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle, \quad f_j|\boldsymbol{\alpha}\rangle = z^{-\delta_{j,0}}|\boldsymbol{\alpha} + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle, \quad k_j|\boldsymbol{\alpha}\rangle = p^{2(\alpha_{j+1}-\alpha_j)}|\boldsymbol{\alpha}\rangle, \tag{95}$$

where $j \in \mathbb{Z}_n$. Any vector $|\alpha'_1, \ldots, \alpha'_n\rangle$ appearing in the right hand sides are to be understood as 0 unless $(\alpha'_1, \ldots, \alpha'_n) \in \{0,1\}^n$. (This convention should also apply to (96) below.) We call $\pi_{l,z}$ the degree-l *antisymmetric tensor* representation (or the l-th *fundamental* representation) following the terminology as the representation with respect to the classical subalgebra $U_p(A_{n-1})$.

Let us proceed to the other algebras $U_p(\mathfrak{g})$ with $\mathfrak{g} = D_{n+1}^{(2)}, B_n^{(1)}, \tilde{B}_n^{(1)}$ and $D_n^{(1)}$ under consideration. Define the map $\pi_z : U_p(\mathfrak{g}) \to \mathrm{End}(\mathbf{V})$ by (95) for $0 < j < n$ and the following formulas for $j = 0, n$ depending on $\mathfrak{g}$:

$$\begin{aligned}
&D_{n+1}^{(2)}, \tilde{B}_n^{(1)}; \quad e_0|\boldsymbol{\alpha}\rangle = z|\boldsymbol{\alpha} + \mathbf{e}_1\rangle, && f_0|\boldsymbol{\alpha}\rangle = z^{-1}|\boldsymbol{\alpha} - \mathbf{e}_1\rangle, && k_0|\boldsymbol{\alpha}\rangle = p^{2\alpha_1 - 1}|\boldsymbol{\alpha}\rangle,\\
&B_n^{(1)}, D_n^{(1)}; \quad e_0|\boldsymbol{\alpha}\rangle = z|\boldsymbol{\alpha} + \mathbf{e}_1 + \mathbf{e}_2\rangle, && f_0|\boldsymbol{\alpha}\rangle = z^{-1}|\boldsymbol{\alpha} - \mathbf{e}_1 - \mathbf{e}_2\rangle, && k_0|\boldsymbol{\alpha}\rangle = p^{2(\alpha_1+\alpha_2-1)}|\boldsymbol{\alpha}\rangle,\\
&D_{n+1}^{(2)}, B_n^{(1)}; \quad e_n|\boldsymbol{\alpha}\rangle = |\boldsymbol{\alpha} - \mathbf{e}_n\rangle, && f_n|\boldsymbol{\alpha}\rangle = |\boldsymbol{\alpha} + \mathbf{e}_n\rangle, && k_n|\boldsymbol{\alpha}\rangle = p^{1-2\alpha_n}|\boldsymbol{\alpha}\rangle,\\
&\tilde{B}_n^{(1)}, D_n^{(1)}; \quad e_n|\boldsymbol{\alpha}\rangle = |\boldsymbol{\alpha} - \mathbf{e}_1 - \mathbf{e}_2\rangle, && f_n|\boldsymbol{\alpha}\rangle = |\boldsymbol{\alpha} + \mathbf{e}_1 + \mathbf{e}_2\rangle, && k_n|\boldsymbol{\alpha}\rangle = p^{2(1-\alpha_n-\alpha_{n-1})}|\boldsymbol{\alpha}\rangle.
\end{aligned} \tag{96}$$

For $U_p(D_n^{(1)})$, one sees that the above action of the generators preserves the parity of $|\boldsymbol{\alpha}|$. Therefore π_z can be restricted to $\pi_z^{\pm} : U_p(D_n^{(1)}) \to \mathbf{V}^{\pm}$ (60). We call π_z ($\pi_z^{\pm}$ for $U_p(D_n^{(1)})$) the *spin* representation by abusing the name as a representation of the classical subalgebra $U_p(B_n)$ or $U_p(D_n)$.

B.3. **Quantum *R* matrices.** Consider $U_p = U_p(\mathfrak{g})$ with $\mathfrak{g}$ being any one of $A_{n-1}^{(1)}, D_{n+1}^{(2)}, B_n^{(1)}, \tilde{B}_n^{(1)}$ and $D_n^{(1)}$. Let $R \in \mathrm{End}(\mathbf{V}_l \otimes \mathbf{V}_m)$ for $\mathfrak{g} = A_{n-1}^{(1)}$ $(0 \le l, m \le n)$, $R \in \mathrm{End}(\mathbf{V}^{\sigma} \otimes \mathbf{V}^{\sigma'})$ for $\mathfrak{g} = D_n^{(1)}$ $(\sigma, \sigma' = +, -)$ and $R \in \mathrm{End}(\mathbf{V} \otimes \mathbf{V})$ for the other $\mathfrak{g}$. Consider the linear equation on R

$$\Delta'_{x,y}(g) R = R\, \Delta_{x,y}(g) \qquad \forall g \in U_p, \tag{97}$$

where $\Delta_{x,y}$ signifies the tensor product representation $(\pi_{l,x} \otimes \pi_{m,y}) \circ \Delta$ for $U_p(A_{n-1}^{(1)})$, $(\pi_x^{\sigma} \otimes \pi_y^{\sigma'}) \circ \Delta$ for $U_p(D_n^{(1)})$ and $(\pi_x \otimes \pi_y) \circ \Delta$ for the other algebras. Similarly $\Delta'_{x,y}$ is defined by replacing Δ with the opposite coproduct Δ' in $\Delta_{x,y}$. A little inspection tells that R actually depends on x and y only via the ratio $z = x/y$. The tensor product representation $\Delta_{x,y}$ is irreducible for generic x/y, hence

[11] In the left hand sides of (95) and (96), $\pi_{l,z}(g)$ and $\pi_z(g)$ are denoted by g for simplicity.

R is determined uniquely up to an overall scalar. Denote them by $R_{l,m}(z|A^{(1)}_{n-1})$ for $U_p(A^{(1)}_{n-1})$ and $R^{\sigma,\sigma'}(z|D^{(1)}_n)$ for $U_p(D^{(1)}_n)$ and $R(z|\mathfrak{g})$ for the other cases $\mathfrak{g} = D^{(2)}_{n+1}, B^{(1)}_n, \tilde{B}^{(1)}_n$. They all satisfy the Yang-Baxter equation.

For $\mathfrak{g} \neq A^{(1)}_{n-1}$, we introduce a slight gauge transformation retaining the Yang-Baxter equation:

$$\tilde{R}_\pm(z|\mathfrak{g}) = (\mathcal{I}_\pm^{-1} \otimes 1)R(z|\mathfrak{g})(1 \otimes \mathcal{I}_\pm). \qquad \mathcal{I}_\pm|\boldsymbol{\alpha}\rangle = (\pm \mathrm{i})^{|\boldsymbol{\alpha}|}|\boldsymbol{\alpha}\rangle. \tag{98}$$

When $\mathfrak{g} = D^{(1)}_n$ this should be applied to define $\tilde{R}^{\sigma,\sigma'}_\pm(z|D^{(1)}_n)$ for each (σ,σ').

Now the identification of $S^{\mathrm{tr}}(z)$ (49), (65) and $S^{s,s'}(z)$ (55) with the quantum R matrices is stated as follows:

$$S^{\mathrm{tr}}_{l,m}(z) = R_{l,m}(z|A^{(1)}_{n-1}) \qquad p^2 = -q^{-2}, \tag{99}$$
$$S^{1,1}(z) = \tilde{R}_\pm(z|D^{(2)}_{n+1}) \qquad p = \pm \mathrm{i}q^{-1}, \tag{100}$$
$$S^{2,1}(z) = \tilde{R}_\pm(z|B^{(1)}_n) \qquad p = \pm \mathrm{i}q^{-1}, \tag{101}$$
$$S^{1,2}(z) = \tilde{R}_\pm(z|\tilde{B}^{(1)}_n) \qquad p = \pm \mathrm{i}q^{-1}, \tag{102}$$
$$S^{2,2}_{\sigma,\sigma'}(z) = \tilde{R}^{\sigma,\sigma'}_\pm(z|D^{(1)}_n) \qquad p^2 = -q^{-2}, \tag{103}$$

where we assume that the R matrices in the right hand sides have been normalized in parallel with (68). In (99) and (102), it suffices to specify p^2 since our definition of $U_p(A^{(1)}_{n-1}), U_p(D^{(1)}_n)$[12] and their representations contain p only via p^2. In particular $\tilde{R}^{\sigma,\sigma'}_+(z|D^{(1)}_n) = \tilde{R}^{\sigma,\sigma'}_-(z|D^{(1)}_n)$ holds because the spin representation (96) always changes $|\boldsymbol{\alpha}|$ by an even number. Up to conventional difference (99) was claimed in [4]. The results (100), (101), (103) were proved in [16, Th.7.1] and (102) was suggested in [16, Rem.7.2]. The essence of the proof is to show that the matrix product forms implied by the left hand sides fulfill the characterization (97) of the R matrices.

Appendix C. Examples

Let us write down a few examples of $S^{\mathrm{tr}}(z)$, $K^{\mathrm{tr}}(z)$, $S^{s,s'}(z)$ and $K^{k,k'}(z)$ explicitly.

C.1. $S^{\mathrm{tr}}_{m,1}(z)$ and $S^{\mathrm{tr}}_{1,m}(z)$ with general m, n. The $S^{\mathrm{tr}}_{l,m}(z)$ in (65) is an elementary example of quantum R matrices associated with the antisymmetric tensor representations as noted in (99). When $\min(l,m) = 1$, its nonzero matrix elements are given as

$$S^{\mathrm{tr}}_{m,1}(z)^{\boldsymbol{\alpha},\mathbf{e}_j}_{\boldsymbol{\alpha},\mathbf{e}_j} = \begin{cases} (-1)^m \frac{q^2(1-q^{2m-2}z)}{1-q^{2m+2}z} & \alpha_j = 1, \\ (-1)^{m+1} & \alpha_j = 0, \end{cases}$$

$$S^{\mathrm{tr}}_{m,1}(z)^{\boldsymbol{\gamma},\mathbf{e}_k}_{\boldsymbol{\alpha},\mathbf{e}_j} = \begin{cases} (-1)^{m+1} \frac{z(1-q^4)}{1-q^{2m+2}z} q^{2(m-\alpha_{j+1}-\alpha_{j+2}-\cdots-\alpha_k)} & j < k, \\ (-1)^{m+1} \frac{1-q^4}{1-q^{2m+2}z} q^{2(\alpha_{k+1}+\alpha_{k+2}+\cdots+\alpha_j)} & j > k, \end{cases}$$

$$S^{\mathrm{tr}}_{1,m}(z)^{\mathbf{e}_j,\boldsymbol{\beta}}_{\mathbf{e}_j,\boldsymbol{\beta}} = \begin{cases} 1 & \beta_j = 1, \\ -\frac{q^2(1-q^{2m-2}z)}{1-q^{2m+2}z} & \beta_j = 0, \end{cases}$$

$$S^{\mathrm{tr}}_{1,m}(z)^{\mathbf{e}_k,\boldsymbol{\delta}}_{\mathbf{e}_j,\boldsymbol{\beta}} = \begin{cases} \frac{1-q^4}{1-q^{2m+2}z} q^{2(\delta_{j+1}+\delta_{j+2}+\cdots+\delta_k)} & j < k, \\ \frac{z(1-q^4)}{1-q^{2m+2}z} q^{2(m-\delta_{k+1}-\delta_{k+2}-\cdots-\delta_j)} & j > k, \end{cases}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \in \{0,1\}^n$ with $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = |\boldsymbol{\gamma}| = |\boldsymbol{\delta}| = m$. The case $m = 1$ corresponds to the well known $n(2n-1)$-vertex model associated with the vector representation. In particular the case $n = 2$ is the six-vertex model in which $S^{\mathrm{tr}}_{1,1}(z)$ acts on the base vectors as

$$|ij, ij\rangle \mapsto |ij, ij\rangle \quad (i, j \in \{0,1\}),$$
$$|01, 10\rangle \mapsto -\frac{q^2(-1+z)|01,10\rangle}{-1+q^4z} + \frac{(-1+q^4)z|10,01\rangle}{-1+q^4z},$$
$$|10, 01\rangle \mapsto \frac{(-1+q^4)|01,10\rangle}{-1+q^4z} - \frac{q^2(-1+z)|10,01\rangle}{-1+q^4z}.$$

Here we have written $(v_1 \otimes v_0) \otimes (v_1 \otimes v_1) \in V^{\otimes 2} \otimes V^{\otimes 2}$ for example as $|10, 11\rangle$ for simplicity.

[12] They are referred to as $U_{p^2}(A^{(1)}_{n-1})$ and $U_{p^2}(D^{(1)}_n)$ in the usual convention.

C.2. $\boldsymbol{K^{\mathrm{tr}}(z)}$ **for** $\boldsymbol{n=2,3}$. Denote $v_0\otimes v_1$ by $|01\rangle$ etc. When $n=2$, $K^{\mathrm{tr}}(z)$ acts on the base vectors as

$$|00\rangle \mapsto |11\rangle, \qquad |01\rangle \mapsto -\frac{q^{-1}(-1+q^2)z|01\rangle}{(-1+z)} + |10\rangle,$$
$$|11\rangle \mapsto |00\rangle, \qquad |10\rangle \mapsto |01\rangle - \frac{q^{-1}(-1+q^2)|10\rangle}{(-1+z)}.$$

When $n=3$, $K^{\mathrm{tr}}(z)$ acts on the base vectors as

$$|000\rangle \mapsto |111\rangle, \qquad |111\rangle \mapsto |000\rangle,$$
$$|001\rangle \mapsto -\frac{(-1+q^2)z|011\rangle}{q(-1+qz)} - \frac{(-1+q^2)z|101\rangle}{-1+qz} + |110\rangle,$$
$$|010\rangle \mapsto -\frac{(-1+q^2)z|011\rangle}{-1+qz} + |101\rangle - \frac{(-1+q^2)|110\rangle}{q(-1+qz)},$$
$$|011\rangle \mapsto -\frac{(-1+q^2)z|001\rangle}{q(-1+qz)} - \frac{(-1+q^2)z|010\rangle}{-1+qz} + |100\rangle,$$
$$|100\rangle \mapsto |011\rangle - \frac{(-1+q^2)|101\rangle}{q(-1+qz)} - \frac{(-1+q^2)|110\rangle}{-1+qz},$$
$$|101\rangle \mapsto -\frac{(-1+q^2)z|001\rangle}{-1+qz} + |010\rangle - \frac{(-1+q^2)|100\rangle}{q(-1+qz)},$$
$$|110\rangle \mapsto |001\rangle - \frac{(-1+q^2)|010\rangle}{q(-1+qz)} - \frac{(-1+q^2)|100\rangle}{-1+qz}.$$

These formulas are consistent with (76).

C.3. $\boldsymbol{S^{s,s'}(z)}$ **and** $\boldsymbol{K^{k,k'}(z)}$ **for** $\boldsymbol{n=1}$. Let us present the action of $S^{s,s'}(z)$ on $V\otimes V$. From the parity constraint (64), $S^{2,2}(z)$ becomes diagonal whose elements are already fixed by the normalization condition (68). In the remaining cases we will only cover $(s,s')=(1,1),(1,2)$ in view of (61). We write, for example, as $|0,1\rangle = v_0\otimes v_1$.

$$S^{1,1}(z):\ |0,0\rangle \mapsto |0,0\rangle, \quad |0,1\rangle \mapsto \frac{q(1-z)|0,1\rangle}{1+q^2z} + \frac{(1+q^2)z|1,0\rangle}{1+q^2z},$$
$$|1,1\rangle \mapsto |1,1\rangle, \quad |1,0\rangle \mapsto \frac{(1+q^2)|0,1\rangle}{1+q^2z} + \frac{q(-1+z)|1,0\rangle}{1+q^2z},$$
$$S^{1,2}(z):\ |0,0\rangle \mapsto |0,0\rangle, \quad |0,1\rangle \mapsto \frac{q(1-z^2)|0,1\rangle}{1+q^2z^2} + \frac{(1+q^2)z|1,0\rangle}{1+q^2z^2},$$
$$|1,1\rangle \mapsto |1,1\rangle, \quad |1,0\rangle \mapsto \frac{(1+q^2)z|0,1\rangle}{1+q^2z^2} + \frac{q(-1+z^2)|1,0\rangle}{1+q^2z^2}.$$

So $S^{1,1}(z)$ and $S^{1,2}(z)$ define just six vertex models in some gauge.

For $K^{k,k'}(z)$, we will again cover $(k,k')=(1,1),(1,2)$ and $(2,2)$ only by virtue of (84).

$$K^{1,1}(z): |0\rangle \mapsto -\frac{q^{-\frac{1}{2}}(1+q)z|0\rangle}{-1+z} + |1\rangle, \qquad |1\rangle \mapsto -|0\rangle - \frac{q^{-\frac{1}{2}}(1+q)|1\rangle}{-1+z},$$
$$K^{1,2}(z): |0\rangle \mapsto -\frac{q^{-\frac{1}{2}}(1+q)z|0\rangle}{-1+z^2} + |1\rangle, \qquad |1\rangle \mapsto -|0\rangle - \frac{q^{-\frac{1}{2}}(1+q)z|1\rangle}{-1+z^2},$$
$$K^{2,2}(z): |0\rangle \mapsto |1\rangle, \qquad |1\rangle \mapsto -|0\rangle.$$

C.4. **$S^{1,1}(z)$ and $K^{k,k'}(z)$ for $n=2$.** We set $|10,11\rangle = (v_1\otimes v_0)\otimes(v_1\otimes v_1)$ etc as before. The $S^{1,1}(z)$ acts on the base vectors of $V^{\otimes 2}\otimes V^{\otimes 2}$ as follows:

$$\begin{aligned}
&|ij,ij\rangle \mapsto |ij,ij\rangle \quad (i,j\in\{0,1\}),\\
&|00,01\rangle \mapsto -\frac{q(-1+z)|00,01\rangle}{1+q^2z}+\frac{(1+q^2)z|01,00\rangle}{1+q^2z},\\
&|00,10\rangle \mapsto -\frac{q(-1+z)|00,10\rangle}{1+q^2z}+\frac{(1+q^2)z|10,00\rangle}{1+q^2z},\\
&|00,11\rangle \mapsto \frac{q^2(-1+z)(-1+q^2z)|00,11\rangle}{(1+q^2z)(1+q^4z)}-\frac{q^3(1+q^2)(-1+z)z|01,10\rangle}{(1+q^2z)(1+q^4z)}\\
&\qquad -\frac{q(1+q^2)(-1+z)z|10,01\rangle}{(1+q^2z)(1+q^4z)}+\frac{(1+q^2)(1+q^4)z^2|11,00\rangle}{(1+q^2z)(1+q^4z)},\\
&|01,00\rangle \mapsto \frac{(1+q^2)|00,01\rangle}{1+q^2z}+\frac{q(-1+z)|01,00\rangle}{1+q^2z},\\
&|01,10\rangle \mapsto -\frac{q(1+q^2)(-1+z)|00,11\rangle}{(1+q^2z)(1+q^4z)}-\frac{q^2(-1+z)(-1+q^2z)|01,10\rangle}{(1+q^2z)(1+q^4z)}\\
&\qquad +\frac{(1+q^2)z(1+q^2-q^2z+q^4z)|10,01\rangle}{(1+q^2z)(1+q^4z)}+\frac{q(1+q^2)(-1+z)z|11,00\rangle}{(1+q^2z)(1+q^4z)},\\
&|01,11\rangle \mapsto -\frac{q(-1+z)|01,11\rangle}{1+q^2z}+\frac{(1+q^2)z|11,01\rangle}{1+q^2z},\\
&|10,00\rangle \mapsto \frac{(1+q^2)|00,10\rangle}{1+q^2z}+\frac{q(-1+z)|10,00\rangle}{1+q^2z},\\
&|10,01\rangle \mapsto -\frac{q^3(1+q^2)(-1+z)|00,11\rangle}{(1+q^2z)(1+q^4z)}+\frac{(1+q^2)(1-q^2+q^2z+q^4z)|01,10\rangle}{(1+q^2z)(1+q^4z)}\\
&\qquad -\frac{q^2(-1+z)(-1+q^2z)|10,01\rangle}{(1+q^2z)(1+q^4z)}+\frac{q^3(1+q^2)(-1+z)z|11,00\rangle}{(1+q^2z)(1+q^4z)},\\
&|10,11\rangle \mapsto -\frac{q(-1+z)|10,11\rangle}{1+q^2z}+\frac{(1+q^2)z|11,10\rangle}{1+q^2z},\\
&|11,00\rangle \mapsto \frac{(1+q^2)(1+q^4)|00,11\rangle}{(1+q^2z)(1+q^4z)}+\frac{q^3(1+q^2)(-1+z)|01,10\rangle}{(1+q^2z)(1+q^4z)}\\
&\qquad +\frac{q(1+q^2)(-1+z)|10,01\rangle}{(1+q^2z)(1+q^4z)}+\frac{q^2(-1+z)(-1+q^2z)|11,00\rangle}{(1+q^2z)(1+q^4z)},\\
&|11,01\rangle \mapsto \frac{(1+q^2)|01,11\rangle}{1+q^2z}+\frac{q(-1+z)|11,01\rangle}{1+q^2z},\\
&|11,10\rangle \mapsto \frac{(1+q^2)|10,11\rangle}{1+q^2z}+\frac{q(-1+z)|11,10\rangle}{1+q^2z}.
\end{aligned}$$

$K^{1,1}(z)$ acts on the base vectors of $V^{\otimes 2}$ as follows:

$$\begin{aligned}
&|00\rangle \mapsto \frac{q^{-1}(1+q)(1+q^2)z^2|00\rangle}{(-1+z)(-1+qz)}-\frac{q^{-\frac12}(1+q)z|01\rangle}{(-1+qz)}-\frac{q^{\frac12}(1+q)z|10\rangle}{-1+qz}+|11\rangle,\\
&|01\rangle \mapsto \frac{q^{-\frac12}(1+q)z|00\rangle}{-1+qz}+\frac{q^{-1}(1+q)z(1+q-qz+q^2z)|01\rangle}{(-1+z)(-1+qz)}-|10\rangle-\frac{q^{-\frac12}(1+q)|11\rangle}{-1+qz},\\
&|10\rangle \mapsto \frac{q^{\frac12}(1+q)z|00\rangle}{-1+qz}-|01\rangle+\frac{q^{-1}(1+q)(1-q+qz+q^2z)|10\rangle}{(-1+z)(-1+qz)}-\frac{q^{\frac12}(1+q)|11\rangle}{-1+qz},\\
&|11\rangle \mapsto |00\rangle+\frac{q^{-\frac12}(1+q)|01\rangle}{-1+qz}+\frac{q^{\frac12}(1+q)|10\rangle}{-1+qz}+\frac{q^{-1}(1+q)(1+q^2)|11\rangle}{(-1+z)(-1+qz)}.
\end{aligned}$$

$K^{1,2}(z)$ acts on the base vectors of $V^{\otimes 2}$ as follows:

$$|00\rangle \mapsto \frac{q^{-1}(1+q)z^2(1+q^2-q^2z^2+q^3z^2)|00\rangle}{(-1+z^2)(-1+q^2z^2)} - \frac{q^{-\frac{1}{2}}(1+q)z|01\rangle}{-1+q^2z^2} - \frac{q^{\frac{1}{2}}(1+q)z|10\rangle}{-1+q^2z^2} + |11\rangle,$$

$$|01\rangle \mapsto \frac{q^{-\frac{1}{2}}(1+q)z|00\rangle}{-1+q^2z^2} + \frac{q^{-1}(1+q)z^2(1+q^2-q^2z^2+q^3z^2)|01\rangle}{(-1+z^2)(-1+q^2z^2)} - |10\rangle - \frac{q^{\frac{1}{2}}(1+q)z|11\rangle}{-1+q^2z^2},$$

$$|10\rangle \mapsto \frac{q^{\frac{1}{2}}(1+q)z|00\rangle}{-1+q^2z^2} - |01\rangle + \frac{q^{-1}(1+q)(1-q+qz^2+q^3z^2)|10\rangle}{(-1+z^2)(-1+q^2z^2)} - \frac{q^{\frac{3}{2}}(1+q)z|11\rangle}{-1+q^2z^2},$$

$$|11\rangle \mapsto |00\rangle + \frac{q^{\frac{1}{2}}(1+q)z|01\rangle}{-1+q^2z^2} + \frac{q^{\frac{3}{2}}(1+q)z|10\rangle}{-1+q^2z^2} + \frac{q^{-1}(1+q)(1-q+qz^2+q^3z^2)|11\rangle}{(-1+z^2)(-1+q^2z^2)}.$$

$K^{2,2}(z)$ acts on the base vectors of $V^{\otimes 2}$ as follows:

$$|00\rangle \mapsto \frac{q^{-1}(-1+q^2)z^2|00\rangle}{-1+z^2} + |11\rangle, \qquad |01\rangle \mapsto \frac{q^{-1}(-1+q^2)z^2|01\rangle}{-1+z^2} - |10\rangle,$$

$$|10\rangle \mapsto -|01\rangle + \frac{q^{-1}(-1+q^2)|10\rangle}{-1+z^2}, \qquad |11\rangle \mapsto |00\rangle + \frac{q^{-1}(-1+q^2)|11\rangle}{-1+z^2}.$$

Acknowledgments

The authors thank the organizers of MATRIX Program *Non-Equilibrium Systems and Special Functions* at University of Melbourne (Creswick, 8 January 2018 – 2 February 2018), which triggered this work. AK is supported by Grants-in-Aid for Scientific Research No. 15K13429 from JSPS.

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E-mail address: atsuo.s.kuniba@gmail.com

Institute of Physics, University of Tokyo, Komaba, Tokyo 153-8902, Japan

E-mail address: vincent.pasquier@ipht.fr

Institut de Physique Théorique, Université Paris Saclay, CEA, CNRS, F-91191 Gif-sur-Yvette, France