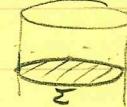


- What are asymptotic symmetries?

- global symm \rightarrow Noether current ($\partial_\mu J^\mu = 0$ on-shell)

- conserved charge $Q = \int_{\Sigma} d^{d-1}x J^0$ 

- Noether current not unique $\tilde{J}^\mu = J^\mu + \partial_\nu K^{\mu\nu}$ ^{antisym}
 $\tilde{J} = J + dK$ ^{contributes to Q on $\partial\Sigma$}

- gauge "symmetry" \rightarrow ^{bulk}Noether current $J^\mu \approx 0$ on-shell $\Rightarrow J = dK$

- conserved charge $Q = \int_{\partial\Sigma} K$ w/ $dK \approx 0$
^{bnd. of spacetime}

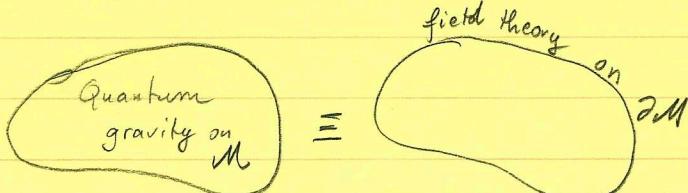
- $Q \neq 0$ if gauge parameter falls off slowly enough
 \Rightarrow the bnd. of spacetime \Rightarrow asymptotic symmetry

Asympt. symm.: gauge symm. whose parameter falls off slowly enough to lead to $Q \neq 0$.

- asympt symm are true symm: act nontrivially on phase space.

- Holography

statement:



• non-triv
 • important (non-pert. diff.)

asympt. symm. in gravity \leftrightarrow global symm of field th

(natural setup for studying asym. symm)

- universality \Rightarrow type of field th. depends only on asymt. structure of M , not on details of the gravitational theory

- e.g. AdS_{d+1}/CFT_d (very well tested)

- however, for most spacetimes other than AdS \rightarrow holography very hard!
 (e.g. Minkowski) $\text{(field theory not known)}$

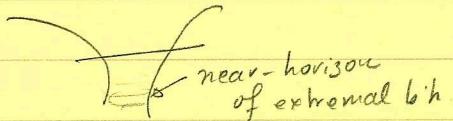
hope: if set of asympt. symm. is ∞ -dimensional (and they satisfy an interesting algebra), it may be possible to guess, or at least highly constrain, the dual field theory

Examples: AdS_{d+1}/CFT_d

$d > 2$ ASG = conformal group in d dims.

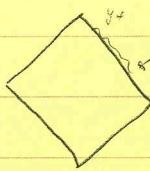
$d = 2$ (AdS_3/CFT_2) $\rightarrow \infty$ -dim-ext. (Virasoro)²
Brown-Henneaux 1986

• Kerr/CFT



ASG = Virasoro $\Rightarrow CFT_2?$

• flat space



n -dim'l ASG = BMS ('60)

• interesting connection to soft theorems
(relate amplitudes w/ & w/o an additional soft graviton) Weinberg '60

• vacuum of GR is only degenerate
(BMS commutes w/ \mathcal{H}).
 \rightarrow b.h. info paradox

Black holes

- no specific discussion, but are everywhere!

- reason for holography \rightarrow laws of black hole mechanics

$$k\delta A_n = \delta M + S \sim A_n$$

$$\delta A_n \geq 0$$

- near-horizon decoupling limits

- check holography $S = \frac{A}{4}$

Plan

- formalism for deriving asympt symm & conserved charges
- application to derivation of 1st law of b.h. mechanics (& converse)
- brief introduction to AdS/CFT
- asympt symm. of AdS_3 & simple-minded holography
- asympt. symm. of the extreme Kerr sol.

- asympt symm. of flat space
- Penrose diagram
- Bondi gauge, BMS supertransl. & superrot
- gravitational memory effect
- BMS as symmetries of the flat space S-matrix
- relation to soft theorems
 - (soft hairs on the black hole horizon)

Reading:

→ G. Compere's lecture notes 1801.07064
 + examples, derivations + refs to original literature

Conserved charges in gauge theories

- reminder: global symm. $\varepsilon = \text{const}$

Noether current $\varepsilon \rightarrow \varepsilon(x)$ $\delta_\varepsilon S = - \int J^\mu \partial_\mu \varepsilon(x)$

on-shell $\delta_\varepsilon S = 0$, $\forall \varepsilon(x) \Rightarrow \partial_\mu J^\mu = 0$

two derivations $\Rightarrow \dagger$

w/ compact support

$$\dagger) \quad \delta_\varepsilon S = \int +E \delta_\varepsilon \phi + \underbrace{\partial_\mu \Theta^\mu(\delta_\varepsilon \phi)}_{\text{o.o.m.}} , \text{ arbitrary } \delta_\varepsilon \phi \text{ (example)}$$

$$\delta_\varepsilon S = \int \partial_\mu M^\mu(\varepsilon)$$

$\varepsilon \rightarrow \text{symm of action, but not of } L$
(e.g. $x^\mu \rightarrow x^\mu + a^\mu$, $\delta L = a^\mu \partial_\mu L$)

$$J^\mu = \Theta^\mu(\delta_\varepsilon \phi) - M^\mu(\varepsilon) \text{ is conserved on-shell}$$

↓

$$\partial_\mu J^\mu = -E \delta_\varepsilon \phi$$

$$\frac{\partial L}{\partial (\partial_\mu \phi)} \delta_\varepsilon \phi$$

Exercise: Take $L = \partial_\mu \phi^* \partial^\mu \phi$ & calculate Noether current for the symmetries : i) $\phi \rightarrow e^{i\varepsilon} \phi$; ii) $x^\mu \rightarrow x^\mu + a^\mu$

- local symmetries $\varepsilon(x)$

assume for simplicity that $\delta_\varepsilon \phi = f(\phi) \varepsilon(x) + f'(\phi) \partial_\mu \varepsilon$

(so, no more than linear in derivatives of ε . E.g. $\delta \phi = i\varepsilon \phi$)

$$\delta_\varepsilon S = \int +E \delta_\varepsilon \phi + \underbrace{\partial_\mu \Theta^\mu(\delta_\varepsilon \phi, \phi)}_{\text{total driv.}} = 0 , \forall \varepsilon(x) , \text{ off-shell}$$

vanishes if
 $\varepsilon(x)$ has compact support

$$= + \int E \left(f(\phi) \varepsilon(x) + f' \partial_\mu \varepsilon \right) = + \int \underbrace{(fE - f' \partial_\mu E)}_{\substack{\text{comp. supp.} \\ 0}} \varepsilon(x)$$

e.o.m. not indep.
(constraints)

$$E \delta_\varepsilon \phi = \varepsilon(x) \left(\underbrace{f(\phi) E - f' \partial_\mu E}_{\text{Noether identities}} \right) + \partial_\mu S^\mu(E, \lambda) = \partial_\mu S^\mu(E, \lambda)$$

+ vanishes on-shell

② at the same time, the usual Noether current

$$E \delta_\varepsilon \phi = -\partial_\mu J^\mu = \partial_\mu S^\mu(E) \Rightarrow \underbrace{\Theta^\mu - M^\mu}_{\text{O.o.m.}}$$

$$S^\mu(E(\phi), \phi) = -J^\mu(\phi) - \nabla_\nu \Theta^{\mu\nu}$$

on-shell, the Noether current J^μ is a pure bnd. term.

Example: $S = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} d^d x$ $A_\mu \rightarrow A_\mu + \partial_\mu \varepsilon$

$$\delta S = - \int \partial_\mu \delta A_\nu F^{\mu\nu} = \int \underbrace{\partial_\mu F^{\mu\nu}}_{E^\nu} \delta A_\nu - \partial_\mu (F^{\mu\nu} \delta A_\nu)$$

$$E^\nu \delta_\varepsilon A_\nu = E^\nu \partial_\nu \varepsilon = \underbrace{\partial_\nu (\varepsilon E^\nu)}_{S^\nu} - \underbrace{\partial_\nu E^\nu}_{\text{Noether identity}} \varepsilon$$

$$\underbrace{\varepsilon(x) \partial_\mu F^{\mu\nu}}_{\text{on-shell}} \quad \underbrace{\partial_\nu E^\nu}_{\partial_\nu [\partial_\mu F^{\mu\nu}]} = \partial_\nu [\partial_\mu F^{\mu\nu}] = 0$$

$$J^\mu = \partial^\mu (\delta_\varepsilon \phi) - M^\mu = -F^{\mu\nu} \partial_\nu \varepsilon = -\partial_\nu (\varepsilon(x) F^{\mu\nu}) + \varepsilon \underbrace{\partial_\nu F^{\mu\nu}}_{-S^\nu}$$

$$\text{conserved charge } Q = \int \varepsilon_{\mu\nu...} g_{\sigma} Q^{S^0} = \int *F \underbrace{\varepsilon(x)}_{\text{const.}}$$

Exercise: Derive the on-shell vanishing Noether current S^k & J^μ for a complex scalar coupled to a gauge field $S = \int d\mu \phi d^\mu \phi^*$; $\partial_\mu = \partial_\mu - i A_\mu$

- Back to the Noether current, useful to introduce form notation $J^\mu \rightarrow$

$$\left\{ \begin{array}{l} J^\mu = \frac{1}{(n-1)!} \varepsilon_{\mu_1 \dots \mu_{n-1}} g_{\sigma} J^S dx^{\mu_1} \dots dx^{\mu_{n-1}} \\ Q = \frac{1}{2(n-2)!} \varepsilon_{\mu_1 \dots \mu_{n-2}} g_{\sigma} Q^{S^0} dx^{\mu_1} \dots dx^{\mu_{n-2}} \\ \mathcal{L} = \frac{1}{n!} \varepsilon_{\mu_1 \dots \mu_n} \mathcal{L} dx^{\mu_1} \dots dx^{\mu_n} \end{array} \right.$$

$$\delta \mathcal{L}(\phi) = E(\phi) \delta \phi + d \mathcal{Q}(\delta \phi, \phi)$$

\star presymplectic potential (form) only when Chern-Simons terms are present

$$\delta_\varepsilon \mathcal{L} = d M_\varepsilon(\phi) \quad M_{(\varepsilon, \Lambda)} = \varepsilon \cdot \mathcal{L}(\phi) + \Lambda \underbrace{d C_{d-2}}_{A \wedge F \wedge \dots \wedge F}$$

$$\text{so } J_\varepsilon(\phi) = \mathcal{Q}(\delta_\varepsilon \phi) - \varepsilon \cdot \mathcal{L}$$

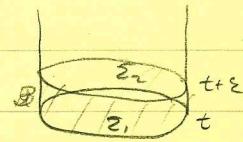
$$E(\phi) \delta_\varepsilon \phi = d S_\varepsilon(\phi) = -d J_\varepsilon(\phi)$$

$$\Rightarrow S_\varepsilon(E(\phi), \phi) = -J_\varepsilon(\phi) - d Q_\varepsilon(\phi) \quad \text{off-shell}$$

↑
on-shell vanishing Noether current

We would now like to show that dQ_ε (or some appropriate modification of it) vanishes on-shell.

$$Q(t+\varepsilon) - Q(t) = \int_{\partial\Sigma} Q_\varepsilon - \int_{\Sigma} Q_\varepsilon = \int_B dQ_\varepsilon = 0$$



- in order for the charge to be non-zero, we need $\frac{\partial Q_\varepsilon}{\partial \varepsilon} \neq 0$
- in order for the charge to be conserved, we need $dQ_\varepsilon|_B = 0$

Consider the charge difference between two field configurations $\phi, \xi \phi + \delta\phi$

$$\delta S_\varepsilon = -\delta J_\varepsilon(\phi) - \delta dQ_\varepsilon(\phi) = -\delta[\mathcal{O}(\delta_\varepsilon\phi) - \xi \cdot \mathcal{L}(\phi)] - d\delta Q_\varepsilon(\phi)$$

$$\text{on-shell } 0 = -\delta \mathcal{O}(\delta_\varepsilon\phi) + \xi \cdot d\mathcal{O}(\delta\phi) - d\delta Q_\varepsilon(\phi)$$

identity $\underset{\delta\xi}{d}\mathcal{F} = d(\xi \cdot \mathcal{F}) + \xi \cdot d\mathcal{F}$.

$$0 = \underbrace{\delta_\varepsilon \mathcal{O}(\delta\phi)}_{\omega(\delta_\varepsilon\phi, \delta\phi)} - \underbrace{\delta \mathcal{O}(\delta_\varepsilon\phi)}_{-\delta_\varepsilon \mathcal{O}(\delta\phi) + \xi \cdot d\mathcal{O}(\delta\phi)} - \underbrace{d(\xi \cdot \mathcal{O}(\delta\phi))}_{-d\delta Q_\varepsilon}$$

presymplectic form (to integrate over Cauchy surface) $\begin{cases} \mathcal{O} \propto \rho \delta\phi \\ \omega \propto \delta\rho \wedge \delta\phi \end{cases}$

$$\omega(\delta_1\phi, \delta_2\phi) = \delta_1 \mathcal{O}(\delta_2\phi) - \delta_2 \mathcal{O}(\delta_1\phi)$$

on-shell :

$$\boxed{\omega(\delta_\varepsilon\phi, \delta\phi) + dK_\varepsilon(\phi, \delta\phi) = 0}$$

where

$$\boxed{K_\varepsilon(\phi, \delta\phi) = -\delta Q_\varepsilon(\phi) - \xi \cdot \mathcal{O}(\delta\phi)}$$

conserved surface charge

$$\boxed{Q(\delta\phi) = \int_B K_\varepsilon(\phi, \delta\phi)}$$

charge $\neq 0$ between the two

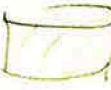
backgrounds

• Conservation

- as discussed, $Q(\delta\phi)$ will be conserved if $dK_E = 0$ on $\partial\Sigma$.

This is true if $w(\delta\phi, \delta\phi)_{\partial\Sigma} = 0$

• if ξ is a Killing vector, then $\delta_\xi\phi = 0 \Rightarrow w(\delta_\xi\phi, \delta\phi) = 0$

$Q(\delta\phi)$ is not only conserved  but also

can be integrated over any radial surface

$$\text{since } \int_{\Sigma_1} K_E = \int_{\Sigma_2} K_E = \int_{\text{m}} dK = 0$$


• if ξ is such that $\delta_\xi\phi = 0$ asymptotically, then need $w(\delta_\xi\phi, \delta\phi)_{\partial\Sigma} = 0$. Then, Q is conserved, but must

be evaluated at the spatial boundary (r-dependent)

Asymptotic symmetries: $\delta_\xi\phi \rightarrow 0$ asympt., but $\xi \neq 0$.

• Definition

- so far, we have only defined the charge difference between two infinitesimally close backgrounds $\phi \approx \bar{\phi} + \delta\phi$. The charge of a finitely-deformed background ϕ w.r.t a reference background $\bar{\phi}$ is defined as

$$Q_\xi(\phi, \bar{\phi}) = \int \int_{\partial\Sigma} K_\xi(\delta\phi, \phi) + N_\xi(\bar{\phi}) \quad \begin{matrix} \leftarrow \text{unfixed by this formula} \\ \phi \end{matrix}$$

where γ is a path in field space from $\bar{\phi}$ to ϕ

• Integrability

- Q_ξ only makes sense if indep. of γ , so κ needs to satisfy integrability condition

$$\delta_1 Q(\delta_2 \phi, \phi) - \delta_2 Q(\delta_1 \phi, \phi) = 0, \quad \forall \delta_1 \phi, \delta_2 \phi$$



Asymptotic symmetry group

- consider a spacetime w/ a set of asymptotic bnd. cond. \mathcal{B} . on the metric
 - a vector ξ^μ is an allowed diffeomorphism if $\delta_\xi \phi \in \mathcal{B}$.
 - algebra \rightarrow Lie bracket $[\xi_a, \xi_b] = C_{ab}^c \xi_c$.
 - assume bnd. cond have been chosen such that Q_ξ are finite, integrable & conserved. Then the asympt. symmetry group is
- $$ASG = \frac{\text{Allowed diffeos}}{\text{Trivial diffeos}}$$
- diffeos for which $Q_\xi = 0$
(fall off too fast).
- if not satisfied, then need more stringent bnd. cond.
 - ASG \rightarrow those diffeos that act non-trivially on states

Representation thm

charge algebra

$$\{Q_X, Q_Y\} \equiv \delta_X Q_Y = \int \mathcal{K}_X (\delta_Y \phi, \phi)$$

proof uses $[\delta_{\xi_1}, \delta_{\xi_2}] \phi = \delta_{[\xi_1, \xi_2]} \phi$

& integrability

$$= Q_{[X, Y]} + \mathcal{K}_{X, Y} [\bar{\phi}]$$

linear: $\int \mathcal{K}_X [\delta_Y \bar{\phi}, \bar{\phi}]$

central charge

cannot be abs into def^h of $Q_{[X, Y]}$

Residual ambiguities

- boundary term $\mathcal{L} \rightarrow \mathcal{L} + d\mu \Rightarrow \mathcal{Q} \rightarrow \mathcal{Q} + \delta\mu$ b.d. w invar
 - \mathcal{Q} defined up to $\mathcal{Q} \rightarrow \mathcal{Q} + dY$,
- $$J = \mathcal{Q}(\delta_\varepsilon \phi) - \xi \cdot \mathcal{L} \rightarrow J + \underbrace{\delta_\varepsilon \mu}_{\delta \cdot d\mu = d(\delta \cdot \mu)} + dY(\delta_\varepsilon \phi) - \xi \cdot d\mu = J + dY + d\xi \cdot \mu$$

$$Q \rightarrow Q + \xi \cdot \mu + Y + dZ ; \quad k = \delta(Q + \xi \cdot \mu + \delta Y / \delta \phi) + d\delta Z - \xi \cdot \delta \mu$$

new ambiguity, (corner terms). $- \xi \cdot dY / \delta \phi$

Examples ERM $\Rightarrow Q = \int *F$

$$GR: Q^{\mu} = \frac{\sqrt{g}}{16\pi G} (\nabla_{\nu} h^{\mu\nu} - \nabla^{\mu} h)$$

$$L = \frac{1}{16\pi G} \int \sqrt{g} R$$

$$\begin{aligned} J^{\mu} &= Q^{\mu} (\delta_{\mu} \phi) - \xi \cdot L \\ &\stackrel{\text{"on shell"}}{=} \frac{\sqrt{g}}{16\pi G} \left[\nabla_{\nu} (\nabla^{\nu} \xi^{\mu} + \nabla^{\mu} \xi^{\nu}) - 2J^{\mu} \nabla_{\nu} \xi^{\nu} \right] \\ &= \frac{\sqrt{g}}{16\pi G} \nabla_{\nu} \underbrace{(\nabla^{\nu} \xi^{\mu} - \nabla^{\mu} \xi^{\nu})}_{Q^{\mu\nu}} \end{aligned}$$

$$Q = \frac{1}{2} \int_{\text{long}}^{\infty} (\nabla^{\mu} \xi^{\nu} - \nabla^{\nu} \xi^{\mu}) \epsilon_{\mu\nu\rho\sigma} \text{ Komar integral for conserved charge in spt. w/ a K.V. (Exercise 1)}$$

$$k_{\xi} = \delta Q - \xi \cdot \delta \Theta =$$

$$= \frac{\sqrt{g}}{8\pi G} \epsilon_{\mu_1 \dots \mu_{d-2} \mu \nu} (\xi^{\mu} \nabla_{\nu} h^{\mu \nu} - \xi^{\mu} \nabla^{\mu} h + \xi_{\mu} \nabla^{\mu} h^{\mu \nu} + \frac{1}{2} h \nabla^{\mu} \xi^{\nu} - 2(d-2)!$$

useful review + formulae 0902.1001

$$- h^{\mu \nu} \nabla_{\mu} \xi^{\nu}) dx^{\mu} \dots dx^{d-1}$$

(Exercise 2)

Conclusions

- full-fledged formalism for computing conserved charges
- very powerful, works in many spacetimes w/ asymmetries
- applications: slick derivation of the 1st law of black hole mechanics \rightarrow generalized gravitational entropy
- slick derivation of the linearized Einstein eqns from entanglement

Exercise 1: Show that the Komar integral yields the correct M, J of the Kerr black hole for $\xi = \partial_t, \partial_\phi$

Exercise 2: Compute the mass (Q_M) of a Schwarzschild black hole by integrating the infinitesimal $k_{\xi=\partial_t}$

Derivation of the 1st law of black hole mechanics

refs: gr-qc/9307038, gr-qc/9403028

Setup

- we consider a **stationary** black hole spacetime. For virtually all such black holes, the event horizon is a **Killing horizon**, i.e. \exists a Killing vector ξ^μ of the spacetime $\ni \xi^\mu \alpha \ell^\mu =$ the normal to the horizon. Note that $\xi^2 = 0$ on the horizon.

- the only properties of Killing horizons we neede are the fact that

$$\cdot \xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu$$

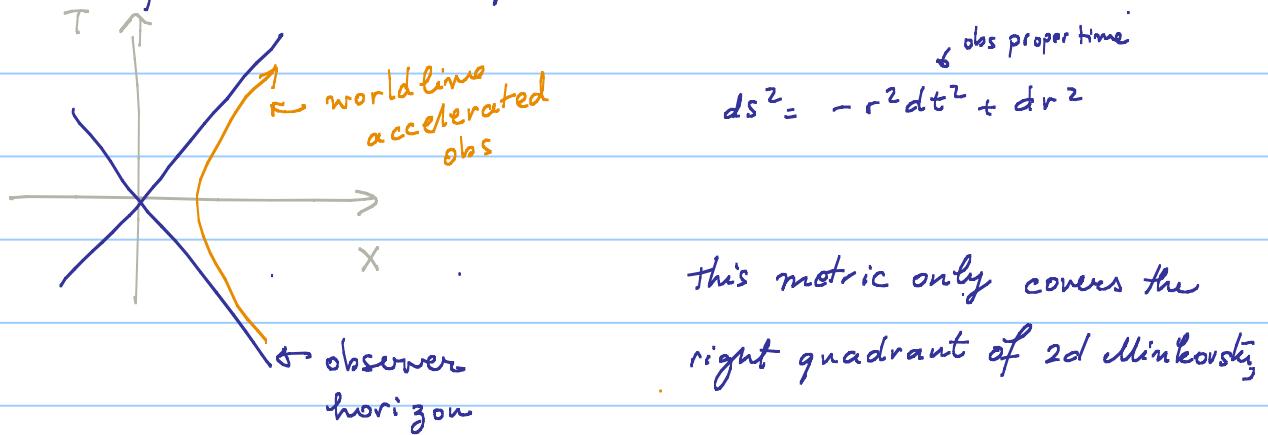
κ surface gravity & is constant

- for $\kappa \neq 0$, the horizon is **bifurcate**, i.e. ξ^μ vanishes on some codimension two surface \mathcal{B}

- more details about Killing horizons can be found in Townsend's notes gr-qc/9707012

- e.g. for a rotating black hole $\xi = \partial_t + \Omega_{\text{re}} \partial_\phi$ is the Killing vect \perp horizon.

a simple example of a bifurcate Killing horizon is the Rindler horizon (though this is not an event horizon). An observer moving w/ constant acceleration on 2d flat space will only have access to part of Minkowski space-time, and sees a metric



this metric only covers the right quadrant of 2d Minkowski

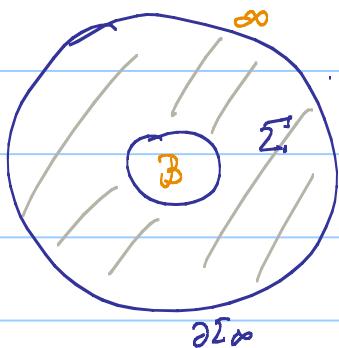
the Rindler metric has Killing vector $\partial_t = X\partial_T - T\partial_X$ when expressed in usual Minkowski coord ($ds^2 = -dT^2 + dX^2$)
The Rindler horizon is @ $X = \pm T$, so notice
 $g^{tt} = -X^2 + T^2 = 0$ on \mathcal{H} , and $\xi^\mu = 0$ @ the bifurcation point $X = T = 0$

Proof of the 1st law:

consider a stationary black hole backgd Φ w/ K.V ξ
($\delta_\xi \phi = 0$) + a perturbation $\delta\phi$

- start from the relation $\underbrace{\omega(\delta\phi, \delta\phi)}_0 + dK_f(\delta\phi) = 0$

b/c f is a f.v. of the backgd.



- integrate the remaining dK_f over a spacelike surface Σ , which stretches between the bifurcation surface B & "infinity" (not specified)

$$0 = \int_{\Sigma} dK_f(\delta\phi) = \int_{\partial\Sigma_\infty} k_f(\delta\phi) - \int_B k_f(\delta\phi)$$

- $\xi = \partial_t + \Omega_K \partial_\varphi$. By definition of the surface charge,

$$\int_{\partial\Sigma_\infty} k_f(\delta\phi) = \delta Q_{\partial_t} + \Omega_K \delta Q_{\partial_\varphi} = \delta M + \Omega_K \delta J$$

in any theory we may be working with, and for any asymptotics, since mass = charge assoc. w/ ∂_t and J \rightarrow charge associated w/ ∂_φ . Depending on the theory @ hand & the asymptotics, the expressions we obtain for $\delta M, \delta J$ may be very complicated.

$$dQ = -J$$

remember $k_\xi(\delta\phi) = \delta Q_\xi(\phi) - \xi \cdot \partial(\delta\phi)$. Since $\xi^u=0$

on \mathcal{B} =>

$$\int_{\mathcal{B}} k_\xi(\delta\phi) = \int_{\mathcal{B}} \delta Q_\xi(\phi) = \delta \int_{\mathcal{B}} Q_\xi(\phi)$$

this identifies a quantity $\int_{\mathcal{B}} Q_\xi(\phi)$, intrinsic to the horizon, whose variation enters the 1st law

$$\delta \int_{\mathcal{B}} Q_\xi(\phi) = \delta M + \Omega_K \delta J$$

we want to identify w/ $\frac{k}{2\pi} \frac{\delta A_{\mathcal{B}}}{4G}$.

in Einstein gravity, we have (see e.g. eqn (4.11) in 1801.07064)

$$Q_\xi = \frac{1}{16\pi G} (\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu) \epsilon_{\mu\nu\alpha_1\dots\alpha_{d-2}} dx^{\alpha_1} \dots dx^{\alpha_{d-2}}$$

$\times \frac{1}{2(d-2)!}$

@ the bifurcation surface $\xi^u=0$, but $\nabla_\mu \xi^\nu = K \epsilon_{\mu\nu}$
 $(\forall t^\mu \text{ tg to } \mathcal{B}, \underbrace{t^\mu \nabla_\mu \xi_\nu}_{b/c \xi=0 \text{ on } \mathcal{B}} = t^\mu g_{\mu\nu} = 0)$ binormal to \mathcal{B}

therefore

$$Q_\xi = \frac{1}{16\pi G} 2K \underbrace{\epsilon^{\mu\nu}}_{\perp \mathcal{B} \parallel \mathcal{B}} \underbrace{\epsilon_{\mu\nu\alpha_1\dots\alpha_{d-2}}}_{\text{volume form}} dx^{\alpha_1} \dots$$

is proportional to the volume form on \mathcal{B} , so $Q_\xi = \frac{k}{8\pi G} d\Omega_K$

- if κ is the same before & after the perturbation, then

$$\boxed{\frac{K}{2\pi} \delta \left(\frac{A\kappa}{4G} \right) = \delta M + \Omega_R \delta J}$$

1st law of black hole mechanics for Einstein gravity

- more generally, consider a theory of gravity that may contain arbitrary higher derivative terms

$$L(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \phi, T_{\mu\nu}, \nabla\phi)^*$$

* to also include $\nabla_{\mu} \dots \nabla_{\nu} R_{\rho\sigma\tau\omega}$, see Wald & Iyer

- then it can be shown that the Noether charge Q must take the form

$$Q = W_\mu(\phi) \xi^\mu + X^{\mu\nu} \nabla_\mu \xi^\nu + Y(\phi, \xi^\mu \partial_\mu \phi) + dz$$

this can be shown by using $\nabla_\mu \nabla_\nu \xi^\rho = R^\rho{}_{\sigma\mu\nu} \xi^\sigma$, true for Killing vectors.

- on \mathcal{B} , $\xi^\mu = 0 \Rightarrow 1^{st}$ term = 0; $\int_B dz = 0$, also

$$\delta Y = Y(\phi, \xi^\mu \delta \phi) = \xi^\mu Y(\phi, \delta \phi) = \xi^\mu d\phi + d(\xi^\mu Y), \text{ does not contribute}$$

- Wald - Iyer show that the d-2 form $X^{\mu\nu}$

$$(X^{\mu\nu})_{\mu_1 \dots \mu_{d-2}} = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu_1 \dots \mu_{d-2}}$$

for an arbitrary th. of gravity (w/ no $\nabla R_{\mu\nu}$)

- this yields the following proposal for the black hole

"entropy"

$$S_W = 2\pi \int \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma}$$

Wald entropy formula

which, by construction, satisfies the 1st law

$$\frac{K}{2\pi} \delta S_W = \delta M + \Omega_R \delta J$$

in any theory of (higher-derivative) gravity.

- e.g. use for Einstein gravity $\mathcal{L} = \frac{1}{16\pi G} R$

$$\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} = \frac{1}{16\pi G} \underbrace{g^{\mu\rho} g^{\nu\sigma}}_{-2} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} = \frac{1}{8\pi G}$$

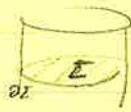
ok, up to signs

so. $S_W = \frac{1}{4G} dt_B \quad \checkmark$

Comments:

- w/ this definition of S_w , the 1st law holds in arbitrary th. of gravity & various asymptotics
- expression for generalized b.h. entropy \mapsto checked in very non-trivial examples in string theory
(where a microscopic description of the b.h. is available)
 \rightarrow perfect match! Highly non-trivial precision test for string theory (see e.g. lecture notes by A. Dabholkar "Quantum black holes")
- entropy is a Noether charge

Last topic : construction of conserved charges in the Lagrangian formalism $\delta Q_\xi = \int k_\xi(\delta\phi) \frac{\partial L}{\partial \dot{\phi}}$



$$\text{conservation : } \omega(\delta_\xi \phi, \delta\phi) + dK_\xi(\delta\phi) = 0$$

$\therefore \omega \rightarrow 0 \text{ asympt}$

This lecture : application of the formalism to obtain a simple derivation of the 1st law of b.h. mechanics

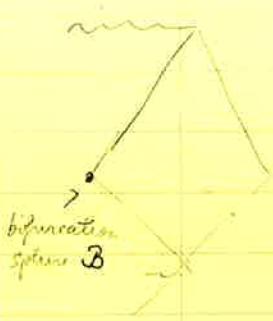
i) generalization of b.h. entropy in higher derivative theories of gravity

(Wald gr-qc/9307038; Wald & Iyer gr-qc/9403028)

ii) converse of the 1st law (Einstein eqns. from 1st law)

- asymptotic symmetries of AdS₃ (simplest non-trivial exemplification)
(intro prop CFT₂ : symmetries, Cardy)
- basic but universal check of holography ($\frac{\partial A}{\partial G} = S_{\text{micro}}$)
 - most black holes whose entropy is understood microscopically fall into this class

Derivation of the 1st law of black hole mechanics

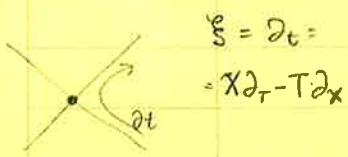


- stationary b.h. spt : horizon = Killing horizon for ξ^μ

ξ^μ : Killing vect. \perp horizon $\xi^\mu = f \ell^\mu$ ($\xi^\mu = 0$)
 ↑
 normal to Σ
 (geodetic)

e.g. Kerr $\xi = \partial_t + \Omega \sin \varphi \partial_\varphi$

e.g. Rindler



the Killing vect. satisfies $\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu$
 ↑
 surface gravity
 (const.)

- bifurcate Killing horizon ($\xi^\mu = 0$)

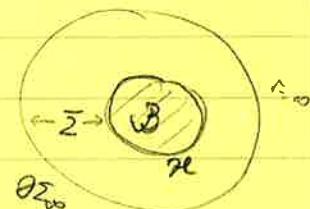
(see lectures by Townsend gr.qc/9707012)

1st law : $\frac{\kappa}{2\pi} \delta A_{\text{ext}} = \delta M + \Omega_h \delta J$ where $\delta\phi$ on-shell

Proof :

$$\omega(\delta_\xi f, \delta\phi) + d k_\xi(\delta\phi) = 0$$

b/c ξ^μ is a Killing vect. \Rightarrow



$$\therefore 0 = \int \limits_{\Sigma} d k_\xi = \int \limits_{\partial\Sigma_{\text{ext}}} k_\xi(\delta\phi) - \int \limits_{\Sigma} k_\xi(\delta\phi)$$

$$\underbrace{\delta M}_{\Sigma} + \underbrace{\Omega_h \delta J}_{\int \limits_{\Sigma} \delta Q(\phi)}$$

(remember
 $k_\xi = \delta Q_\xi - \frac{f}{2} \cdot \delta \phi$
 on B)

↑ is this $k \delta A_{\text{ext}}$?

for Einstein gravity $Q_{\mu\nu} = \frac{1}{16\pi G} \underbrace{e_{\mu\nu} \delta^\sigma}_{K E_{\mu\nu}} \underbrace{U_\sigma \xi_\sigma}_{\text{bimormal}} = \gamma_{\text{rel}}(B) \cdot k$

\Rightarrow area law

for an arbitrary, higher-deriv theory of gravity

$$L(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \phi, \partial_\mu \phi, \dots)$$

$$\nabla_\mu \nabla_\nu \xi^\delta = R^\delta_{\mu\nu\rho\sigma} \xi^\rho \xi^\sigma$$

$$Q = W_\mu(\phi) \xi^\mu + X^{\mu\nu} \nabla_\mu \xi_\nu + Y(\phi, \partial_\mu \phi) + dZ$$

$$\underbrace{\text{on } B}_{\frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} \text{ does not}} \quad \underbrace{\text{but } S=0}_{\text{does not}} \quad \text{contribute}$$

$$(\delta Y = Y(\phi, \partial_\mu \delta \phi))$$

generalized
B.H. entropy
(Wald formula)

$$S = 2\pi \int \limits_{\Sigma} \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} E_{\mu\nu} E_{\rho\sigma}$$

matches

string.th. (entropy thermal
check)

$$\left. \begin{aligned} & \text{using } \delta \phi = 0 \\ & = \int \xi \cdot Y(\phi, \delta \phi) \end{aligned} \right\} = \int \xi \cdot dY + d(S, Z)$$

$$S_{\text{ext}} = 2\pi \int \frac{\partial S}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma}$$

Bonus: 1st law holds in arbitrary theory of gravity

- provides expression for the generalized black hole entropy
- entropy is a Noether charge

Why black hole entropy?

- b.h. solution smooth \rightarrow naively no entropy
- thermodynamics $\rightarrow S \propto A_E$, but d.o.f. invisible in spacetime description

- in theories of g. gravity  \rightarrow excitations of membranes

$$\frac{A_E}{4G} = S = k_B S_L$$

- corrections also match on both sides

(but will provide universal counting later)

- check of Wald's formula

- very non-trivial precision test for string theory

Note

$$\omega(\delta\phi, \delta\phi) + dk_{\xi}(\delta\phi) = 0$$

$\stackrel{0}{\sim}$
on-shell
K.V.

$\delta\phi$ on-shell $\delta(s = -J - dQ)$

$= \xi^{\mu} E_{\mu\nu}(\delta\phi) \cdot \epsilon^{\nu}$ (off-shell)
 $(\phi$ on-shell)



$$\sum_S \delta \cdot \delta E \cdot \epsilon = \int_S dk_{\xi}(\delta\phi) = \int_{\text{in}} k_{\xi}(\delta\phi) - \int_{\text{out}} k_{\xi}(\delta\phi)$$

\sim \sim \sim
 $\delta M + \sigma \delta J$ $K \delta A_n$

Q: suppose \exists a microscopic interpretation of gravity in which $T\delta S = \delta E$, as expected, & $S \propto A_n$. Then, can one derive the Einstein eqn from the 1st law \rightarrow egn. of state?

Jacobson '95

Yes \Rightarrow gravity not a fundamental

(gr-qc/9504004)

force, should probably not be quantized.

Jacobson : local Rindler horizons \rightarrow accelerated obs.

$T \propto a$, $S \propto A_n$

- microscopic meaning of the entropy of a local Rindler horizon ??

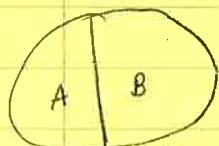
- higher-derivative corrections ?

- possible to make precise statements in the AdS/CFT context.

- global Rindler horizons \leftrightarrow entanglement entropy

Entanglement entropy

(see e.g. review by van Raamsdonk 1609.00026)



total state
pure

$|1\rangle$

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$S_A = -\text{Tr}_{\mathcal{H}_B} |1\rangle \langle 1|$$

reduced density matrix

for subsystem A

$$S_A = -\text{Tr}_{\mathcal{H}_B} \rho_A \ln \rho_A$$

entanglement entropy
(e.g. for $|1\rangle = A|1\rangle + B|1\rangle$)

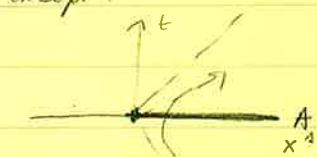
1st law

$$\delta S_A = -\text{Tr}(\delta \rho_A \ln \rho_A) = \delta \langle H_A \rangle$$

$$\rho_A = e^{-H_A} \approx \text{modular Hamiltonian}$$

$\mathcal{H}_A \rightarrow$ usually non-local & not known, except:

- QFT $|A\rangle = |0\rangle$ $A = \{x^{\mu} \in \mathbb{R}^d, x^d > 0\}$



then $\mathcal{H}_A = \int d^{d-1}x x^d T_{00}$ Rindler Hamilton.

- CFT $|A\rangle = |0\rangle$ $A =$ Ball radius R

$$\mathcal{H}_A = \int_B d^{d-1}x \frac{R^2 - |x|^2}{2R} T_{00} = \int \epsilon_{\mu\nu\rho\sigma} \delta^\nu_R$$



- the modular Hamiltonian generates a flow $S \rightarrow e^{iS} S$ in the causal development of the region $\mathcal{H} = Q_S$

Holographic entanglement entropy

AdS/CFT

$$ds^2 = \frac{g_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2} \quad z > 0$$

$$S_A = \frac{\text{area bulk minimal surface } \tilde{A}}{4G} + \text{higher order terms}$$

where $\partial \tilde{A} = \partial A$

Note S_A is divergent \rightarrow UV/IR

The 4th law. Take $A =$ Ball radius R

\tilde{A} = Bifurcation surface



$$ds^2 = -\left(\frac{z^2}{r^2} - 1\right)dx^2 + \frac{dz^2}{\frac{z^2}{r^2} - 1} + r^2 (\underbrace{du^2 + \sinh^2 u d\Omega_{d-2}^2}_{K d-1})$$

(can find coord \Rightarrow metric takes the form)

- now, consider vacuum AdS + metric perturbation $\epsilon_{\mu\nu}$, not necessarily on-shell. If $S \propto A_R$

$$\int_{\partial A_R} \delta Q_S - \int_{\partial A_R} \delta Q_S = \int \epsilon^{\mu\nu} \delta \epsilon_{\mu\nu} \epsilon_{d-2,d-1} dx^d / A_R dx^{d-1} + B(R, x^0)$$

$\int \epsilon^{\mu\nu} \delta \epsilon_{\mu\nu} \epsilon_{d-2,d-1}$ \Rightarrow the linearized Einstein eqns must hold.

\downarrow states
 \Leftrightarrow geometry

Asymptotic symmetries of AdS_3 and match to CFT_2

Plan: start by defining $AdS_{d+1} \cong CFT_d$ (see chap 2. of 9905111)

- properties of CFT_2 (di Francesco)
- asymptotically AdS_3 solutions
- parenthesis: the Brown-York stress tensor
- BTZ black holes
- the ASG of AdS_3
- universal match to CFT_2 (Strominger 9712.251)

AdS_{d+1} : maximally symmetric spacetime of constant negative curvature

$$R_{\mu\nu\rho\sigma} = -\frac{1}{c^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

- the number of Killing vectors of an n -dim'l maximally symmetric spt. is $\frac{n(n+1)}{2}$
- for AdS_{d+1} , the isometry group is $SO(d, 2)$

Exercise : check $SO(d, 2)$ has the correct number of generators to correspond to the isometries of AdS_{d+1}

- simplest construction : embed into $\mathbb{R}^{d, 2}$, w/ metric

$$ds^2 = \tilde{\eta}_{MN} dx^M dx^N \quad \tilde{\eta}_{MN} = \begin{pmatrix} - & - & + \\ - & + & \\ + & & + \end{pmatrix}$$

manifest $SO(d, 2)$ invariance

and restrict to the hyperboloid $\tilde{\eta}_{MN} X^M X^N = -l^2$

- writing explicit embedding coord. that solve, can obtain metric on AdS_{d+1}

$$\cdot \text{e.g. } X^0 = l \cosh g \cos r \quad X^i = l \sinh g S^i$$

$$X^1 = l \cosh g \sin r \quad \sum_{i=2}^{d+1} S_i^2 = 1$$

global coord

$$\Rightarrow ds^2 = l^2 (-\cosh^2 g dr^2 + dg^2 + \sinh^2 g dS_{d-1}^2)$$

(after decompactifying the r coord)

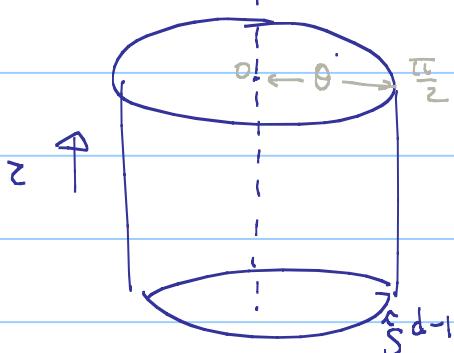
Penrose diagram : note that multiplying the metric by a conformal factor, $g_{\mu\nu}(x) \rightarrow S^2(x) g_{\mu\nu}(x)$, does not affect the causal structure of the spacetime (determined by the trajectories of null rays). The idea is then to choose $S^2(x)$ so that "infinity" ($g \rightarrow \infty$ in the example above) is brought to finite distance.

- to achieve this, intro new radial coord. θ via

$$\cosh g = \frac{1}{\cos \theta} \quad g \in [0, \infty) \Rightarrow \theta \in (0, \frac{\pi}{2})$$

$$ds^2 = \frac{l^2}{\cos^2 \theta} \left(-dr^2 + d\theta^2 + \sin^2 \theta dS_{d-1}^2 \right)$$

metric on $1/2 S^d \approx$ disk

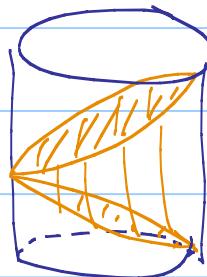


Penrose diagram AdS_{d+1} = infinite solid cylinder ($S^{d-1} \times \mathbb{R}_z$ bnd)
 $(S^2 = \cos \theta)$

- other coord system are also v. convenient, e.g.

Poincaré coordinates

$$ds^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2}$$



bad at $z \rightarrow 0$

(only cover a patch of
global AdS)

CFTd:

- conformally invariant QFTd
- conformal transf: coord. transf. that leave the Minkowski metric invar. up to an overall factor
- infinitesimally, $x^\mu \rightarrow x^\mu + \xi^\mu$. $\delta \eta_{\mu\nu} \propto \eta_{\mu\nu}$, or

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \delta \Omega \eta_{\mu\nu} \quad w/ soln$$

$$\cdot \xi^\mu = a^\mu = \text{const} \quad \text{translations } P^\mu \quad (\delta \Omega = 0)$$

$$\cdot \xi^\mu = \omega^{\mu\nu} x_\nu, \quad \omega^{\mu\nu} = -\omega^{\nu\mu} \quad \text{Lorentz transf } M^{\mu\nu} \quad (\delta \Omega = 0)$$

- $\xi^\mu = \lambda x^\mu$ dilatations D
- $\xi^\mu = b^\mu x^2 - 2x^\mu b_\nu x^\nu$ special conformal, K^μ
- these symmetries match the isometries of vacuum AdS_{d+1}
 (e.g. $P^\mu, M^{\mu\nu}, D$ are very simply implemented
 @ the level of the Poincaré AdS metric)

Obs! Note that the symmetry argument for AdS/CFT is not that the symm. of the **vacuum** AdS_{d+1} solution should match the symm. of the CFT_d, but rather that the **asymptotic symmetries** of AdS_{d+1} match those of the CFT_d. However, for $d > 2$ the ASG of AdS_{d+1} is the same as the isometries of the vacuum solution (i.e. SO(d, 2)), so this distinction does not matter. For AdS₃, they are different, and it is the ASG that matches the CFT₂ symmetries

- these symmetries impose strong constraints on the form of correlation functions of the CFT_d

Two-dimensional CFTs

(chaps 5,6 of di Francesco)

- conformal transformations

$$\text{let } x^\pm = x \pm t \Rightarrow ds^2 = dx^+ dx^- \Rightarrow \eta_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

the conformal Killing vector eqn $\partial_\mu \xi^\nu + \partial_\nu \xi^\mu \propto \eta_{\mu\nu}$
becomes

$$\partial_+ \xi^+ = \frac{1}{2} \partial_+ \xi^- = 0 \Rightarrow \xi^- = \xi^-(x^-) \quad \text{arbitrary}$$

$$\partial_- \xi^- = \frac{1}{2} \partial_- \xi^+ = 0 \Rightarrow \xi^+ = \xi^+(x^+) \quad \text{functions}$$

\Rightarrow 2d conformal gp. is ∞ -dim'l, generated by

$$\xi_L^m = -(x^+)^{m+1} \partial_+ \quad \xi_R^m = -(x^-)^{m+1} \partial_-$$

$\overset{\circ}{\ell}_m \qquad \overset{\circ}{\ell}_m$

- for $m = \pm 1, 0 \rightarrow$ global conformal generators

$$SO(2,2) \simeq SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$$

$$\ell_{-1} = -\partial_+ \rightarrow \text{transl. in } x^+ = q + t$$

$$\ell_{-1} = -\partial_- \quad -/- \quad x^- = q - t$$

$\ell_0, \bar{\ell}_0$ correspond to combinations of a

dilatation ($x^\mu \rightarrow \lambda x^\mu$) and a boost ($x^\pm \rightarrow e^{\pm \sigma} x^\pm$)

- the algebra of these diffeos (Lie bracket) is

$$[l_m, l_n] = (m-n) l_{m+n} \quad [\bar{l}_m, \bar{l}_n] = (m-n) \bar{l}_{m+n}$$

Witt alg.

but quantum-mechanically, there is an *anomaly*

\Rightarrow central extension (only affects $m \neq 0, \pm 1$)

$$[L_m, L_n] = (m-n) \delta_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}$$

Virasoro algebra

c = central charge characterizes the CFT

Casimir energy of CFT on a cylinder of rad. R

$$\langle T_{++} \rangle_{\text{cyl}} = - \frac{c}{24R^2}$$

(change in vacuum energy density due to bnd. cond.)

of degrees of freedom in the CFT, since @ high. temperatures $S \propto c T$

Cardy's formula. In a 2d CFT, the entropy @ high temperature is given by a universal formula, i.e. which holds in any CFT_2 , no matter how strongly coupled.

Proof: modular invariance of the CFT partition function

$$Z(\beta) = \text{Tr } e^{-\beta H} \quad \text{for CFT on circle of radius 1}$$

= path integral on euclidean torus w/ identification

$$t_E \sim t_E + m\beta \quad \varphi \sim \varphi + 2\pi n \quad m, n \in \mathbb{Z}$$

$$Z(\beta) = \int_{-\pi}^{\pi} \int_0^{2\pi} e^{-\beta (\varphi - \varphi')^2 / 2} d\varphi d\varphi' = \left(\frac{2\pi}{\beta} \right)^2 = \frac{4\pi^2}{\beta}$$

We can invert what we view as time & space, so

$$\varphi' = \frac{2\pi}{\beta} t_E \quad t'_E = \frac{2\pi}{\beta} \varphi$$

where the rescaling factor is chosen such that $\varphi' \sim \varphi + 2\pi$

- Since we are dealing w/ a CFT, which is invariant under rescalings, the part. f. is invariant, so

$$Z(\beta) = Z\left(\frac{4\pi^2}{\beta}\right)$$

low \leftrightarrow high temperature
duality

- this can now be used to derive a universal formula for the high-temperature ($\beta \rightarrow 0$) behaviour of Z

$$Z(\beta)_{\beta \rightarrow 0} = Z\left(\frac{4\pi^2}{\beta}\right) \underset{\beta' \rightarrow \infty}{\sim} e^{-\beta'E_0} \underset{\substack{\downarrow \\ \text{gnd-state energy}}}{\sim} e^{-\frac{4\pi^2}{\beta}E_0}$$

however, the gnd-state energy on the cylinder is given by the universal Casimir energy, so

$$E_0 = (L_0 + \bar{L}_0)_{\text{cyl}} = \text{gnd st. } 2 \times \left(-\frac{C}{24}\right) = -\frac{C}{12}$$

$$\Rightarrow Z(\beta)_{\beta \rightarrow 0} \simeq e^{\frac{\pi^2 C}{3\beta}} \Rightarrow F = -T \ln Z = -\frac{\pi^2 C}{3} T^2$$

$$\Rightarrow S = -\frac{\partial F}{\partial T} = 2\frac{\pi^2 C}{3} T \text{ as promised}$$

- in the microcanonical ensemble

$$E = F + TS = \frac{\pi^2 c}{3} T^2 = \frac{3 S^2}{4\pi^2 c}$$

$$\Rightarrow S = 2\pi \sqrt{\frac{c E}{3}}$$

- more generally, we can consider states that also carry angular momentum. Letting

$$E = L_o + \bar{L}_o - \frac{c}{12} \quad J = L_o - \bar{L}_o$$

we find

$$S = 2\pi \sqrt{\frac{c}{6} \left(L_o - \frac{c}{24} \right)} + 2\pi \sqrt{\frac{c}{6} \left(\bar{L}_o - \frac{c}{24} \right)}$$

Caroly's formula

- holds universally when $L_o, \bar{L}_o \gg c$
- regime of validity can be extended by considering CFTs w/ a sparse spectrum (see e.g. 1405.5137)
- L_o, \bar{L}_o refer to eigenvalues of these operators on the plane, which become $E \pm J$ on the cylinder, up to the Casimir shift of $-\frac{c}{24}$.

Exercise: Derive the Cardy formula for general E, J by considering instead $Z(\beta, \theta) = \text{Tr } e^{-\beta H} + i\theta J$

Summary CFT₂:

$SO(2,2)$

- the global conformal group $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$
is enhanced to an ∞ -dimil Virasoro_L \times Virasoro_R symm

- algebra of generators

central ext (quantum)
anomaly

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

- entropy @ high energy is universal & given by Cardy's formula

Next: How to see all these properties from the point of view of dual AdS_3 gravity?

$SO(2,2) \leftrightarrow$ isometries AdS_3

$(Virasoro)^2 \leftrightarrow$ asymptotic symmetries of AdS_3

thermal ensemble \rightarrow black hole, whose entropy ($\frac{A_R}{4G}$) matches perfectly the CFT Cardy entropy \Rightarrow universal check of holography

AdS₃

- Consider Einstein gravity w/ negative cosmological const.

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left(R + \frac{2}{l^2} \right)$$

How to find the ASG (asympt. symm. gp).

- want to consider spacetimes that are "asymptotically AdS₃" in an appropriate sense (i.e. AdS₃ + deformations that carry finite charges) \rightarrow specify bnd. cond. on the allowed metric perturbations
- find the diffeos ξ^μ that preserve these bnd. cond.
- compute Q_ξ & check finiteness, conservation & integrability
(if $Q_\xi \propto$, the bnd. cond are too lax; if generically zero, then they are too stringent \rightarrow find the most lax bnd. cond that still lead to finite, well-defined charges). A good guide for choosing the bnd. cond is to look @ the asympt. falloffs of the sol. to the eqn
- compute the symmetry algebra $\{Q_\xi, Q_\eta\}$ & look for potential central extensions

In the following, we will discuss

- solutions to pure Einstein gravity in AdS_3 + falloffs
- formalism for computing conserved charges in AdS_3
w/o using Mathematica (Brown-York)

Solutions to pure gravity in AdS_3

- all solutions to $G_{\mu\nu} = \frac{1}{l^2} g_{\mu\nu}$ are locally diffeomorphic to the vacuum. This is b/c in 3d the Weyl tensor vanishes, so $R_{\mu\nu\rho\sigma}$ is determined by $R_{\mu\nu}, R$; Einstein's eqns set $R_{\mu\nu} \propto g_{\mu\nu}$, so finally

$$R_{\mu\nu\rho\sigma} = -\frac{1}{l^2} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \quad \text{everywhere}$$

- however, \exists particles (conical defects) and black holes, which are obtained via non-trivial identifications of the vacuum spacetime.

$$\begin{aligned} \text{remember global } AdS_3 \quad ds^2 &= l^2 (-\cosh^2 \rho dr^2 + d\rho^2 + \sinh^2 \rho d\varphi^2) \\ (r = sh \rho) \quad &= l^2 \left(-(r^2 + 1) dr^2 + \frac{dr^2}{r^2 + 1} + r^2 d\varphi^2 \right) \end{aligned}$$

Conical defects. Consider instead the metric

$$ds^2 = l^2 \left[-(r^2 + \alpha^2) dz^2 + \frac{dr^2}{r^2 + \alpha^2} + r^2 d\varphi^2 \right]$$

letting $r = \tilde{r} \alpha$ $\varphi = \frac{\tilde{\varphi}}{\alpha}$ $z = \frac{\tilde{z}}{\alpha}$, this yields the same metric as global AdS_3 above, except that

$$\tilde{\varphi} \sim \tilde{\varphi} + 2\pi\alpha \quad \alpha \neq 1 \text{ is a conical defect}$$

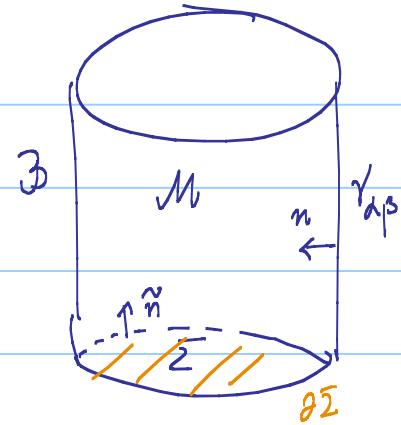
(see e.g. Deser & Jackiw)

Let us now compute the energy of these spacetimes. While the method presented in the 1st course is very powerful & general, for AdS spacetimes it is useful to introduce another method that lets us read off the energy directly from the metric. This is the construction of conserved charges using the holographic stress tensor, which is similar to an older method by Brown & York.

The main idea (very heuristically) stems from the fact that in classical mechanics $H = - \frac{\partial S_{\text{on-shell}}}{\partial T}$. The on-shell (gravitational) action is computed w/ some fixed bnd. cond. for the metric g_{ab} .

β
 $\underbrace{}$
on $\partial\Sigma \times \text{time}$. Given that

the boundary is multi-dimensional,
it is natural to introduce the
bnd. stress tensor



$$T_{\alpha\beta} = -\frac{2}{\sqrt{g}} \frac{\delta S_{\text{on-shell}}[\gamma]}{\delta g^{\alpha\beta}}$$

in order for this object to be well-defined, we
need that $\delta S_{\text{on-shell}} \propto \delta g^{\alpha\beta}$, i.e. δS does not
involve $\delta \partial_n \gamma^{\alpha\beta}$, the variation of the normal deriv.
of $\gamma^{\alpha\beta}$. In classical mechanics terms, we would
like to only fix q @ the endpoints of the interval.

As discussed e.g. in appendix E of Wald GR, $\delta S_{\text{Einstein-Hilbert}}$
has terms proportional to both $\delta g^{\alpha\beta}$ & $\delta \partial_n g^{\alpha\beta}$.
We can add a boundary term (which does not
affect the e.o.m) such that $\delta(S_{\text{E-H}} + S_{\text{bdy}}) \propto \delta g^{\alpha\beta}$

$$S_{\text{tot}} = \frac{1}{16\pi G} \int_M d^{d+1}x R \sqrt{g} + \frac{1}{8\pi G} \int_{\partial\Sigma} d^d x K \sqrt{g}$$

Gibbons - Hawking
term

where $K = \gamma^{\alpha\beta} K_{\alpha\beta}$, $K_{\alpha\beta} = \frac{1}{2} \mathcal{L}_n \gamma_{\alpha\beta}$ is the extrinsic curvature of the surface \mathcal{B} (Lie deriv. w.r.t. the outward pointing normal vector to \mathcal{B})

Taking the variation of the on-shell action, the expr. for the Brown-York stress tensor is

$$T_{\alpha\beta} = -\frac{1}{8\pi G} (K_{\alpha\beta} - K \gamma_{\alpha\beta})$$

(minus sign
comes from def'n
of $T_{\alpha\beta}$)

The conserved charges are computed as usual
induced metric on $\partial\mathcal{B}$

$$Q_\xi = \int d^{d-1}x \sqrt{\sigma} \hat{n}^\alpha T_{\alpha\beta} \xi^\beta$$

↗
(timelike) normal to $\mathcal{I} @ \mathcal{B}$ ↙ killing vect.
generating the symm.

Back to AdS_3 , we would like to compute $T_{\alpha\beta}$ for a general, asymptotically AdS_3 metric. The most general form of the metric in radial coord is given by the Fefferman-Graham expansion (see, e.g. hep-th/9902121)

$$ds^2 = \ell^2 \frac{dz^2}{z^2} + \left(\frac{g^{(0)}_{\alpha\beta}}{z^2} + g^{(2)}_{\alpha\beta} + z^2 g^{(4)}_{\alpha\beta} \right) dx^\alpha dx^\beta$$

$\underbrace{\phantom{g^{(0)}_{\alpha\beta}}}_{T_{\alpha\beta}}$

which actually terminates. $g^{(0)}_{\alpha\beta}$ is arbitrary, $g^{(2)}_{\alpha\beta}$ obey

$$\text{Tr}(g^{(0)-1} g^{(2)}) = \frac{\ell^2}{2} R[g^{(0)}] \quad \nabla_{(0)\alpha}^{\beta} g^{(2)}_{\alpha\beta} = \nabla_\beta \text{tr} g^{(2)}$$

and $g^{(4)}_{\alpha\beta} = \frac{1}{4} g^{(2)}_{\alpha\delta} g^{(0)}_{\beta\delta} \delta^\delta_\alpha g^{(2)}_{\delta\beta}$ const. from $g^{(0)}$

the outward pointing normal to the bnd @ $z=0$ is

$$n = -\frac{z}{\ell} \hat{z}, \text{ and the extrinsic curvature is}$$

$$K_{\alpha\beta} = -\frac{1}{2\ell} z \hat{z} \gamma_{\alpha\beta} = \left(\frac{g^{(0)}_{\alpha\beta}}{z^2} - z^2 g^{(4)}_{\alpha\beta} \right) \frac{1}{\ell}$$

$$\begin{aligned} K &= \gamma^{\alpha\beta} K_{\alpha\beta} = z^2 \left(g^{(0)\alpha\beta} - z^2 g^{(2)\alpha\beta} + \dots \right) \left(\frac{g^{(0)}_{\alpha\beta}}{z^2} - z^2 g^{(4)}_{\alpha\beta} \right) \\ &= \left(2 - z^2 \text{tr } g^{(2)} + \mathcal{O}(z^4) \right) \frac{1}{\ell} \end{aligned}$$

we can now compute $T_{\alpha\beta}$. Some quantities diverge as $z \rightarrow 0$, we can imagine regulating by a cutoff at $z=\epsilon$

divergent !!

$$T_{\alpha\beta} = -\frac{1}{8\pi G \ell} \left(\frac{g^{(0)}_{\alpha\beta}}{z^2} - \left(\frac{g^{(0)}_{\alpha\beta}}{z^2} + g^{(2)}_{\alpha\beta} \right) \left(2 - z^2 \text{tr } g^{(2)} \right) + \mathcal{O}(z^2) \right)$$

$$= \frac{1}{8\pi G l} \left(\frac{g_{\alpha\beta}^{(0)}}{z^2} + 2g_{\alpha\beta}^{(2)} - g_{\alpha\beta}^{(0)} \text{tr } g^{(2)} + O(z^2) \right)$$

on $z=0$

However, we can "renormalize" the stress tensor by adding a boundary "counterterm" $S_{ct}[\gamma]$ that respects the variational principle $\delta S_{tot} \propto \delta g^{\alpha\beta}$ and absorbs the divergence in $T_{\alpha\beta}$. In our case

$$S_{ct} = -\frac{1}{8\pi G l} \int d^2x \sqrt{\gamma}$$

$$T_{\alpha\beta}^{\text{ren}} = T_{\alpha\beta} - \frac{1}{8\pi G l} \gamma_{\alpha\beta}$$

and so

$$T_{\alpha\beta}^{\text{ren}} = \frac{1}{8\pi G l} \left(g_{\alpha\beta}^{(2)} - g_{\alpha\beta}^{(0)} \text{tr } g^{(2)} \right)$$

- the renormalized stress tensor is finite, and its value unambiguously matches to the value of the stress tensor in the dual CFT (up to possible finite counterterms not present here)

- it is conserved due to the e.o.m for $g^{(2)}$

- $\text{tr}(T_{\alpha\beta}^{\text{ren}}) = -\frac{1}{8\pi G l} \text{tr } g^{(2)} = -\frac{l}{16\pi G} R \{ g^{(0)} \}$

from this, we can read off the central charge, since when placing a CFT on curved sp $g_{ab}^{(0)}$

$$\text{tr } T = - \frac{c}{24\pi} R[g^{(0)}] \Rightarrow c = \frac{3l}{2G}$$

Let us now use this formula to compute the (absolute) energy of global AdS_3 & the conical deficit backgrounds (after making sure to use the correct bnd. coord t, φ)

$$ds^2 = l^2 \left(- (r^2 + \alpha^2) dt^2 + \frac{dr^2}{r^2 + \alpha^2} + r^2 d\varphi^2 \right)$$

changing coord to radial gauge $\frac{d\varphi}{\delta} = \frac{dr}{\sqrt{r^2 + \alpha^2}} \Rightarrow r = \delta - \frac{\alpha^2}{\eta p}$

$$r^2 + \alpha^2 = \delta^2 + \frac{\alpha^2}{\delta^2} + \delta \left(\frac{1}{\eta p} \right)$$

$$\Rightarrow g_{rr}^{(2)} = - \frac{l^2 \alpha^2}{2} \quad \tilde{m}^2 = \frac{1}{\delta \eta} + \dots \quad \sqrt{\sigma} = \delta \eta + \dots$$

the energy is the charge associated w/ $\xi = 2+$

$$E = \int_0^{2\pi} d\varphi \sqrt{\sigma} \tilde{m}^2 T_{rr} \xi^2 = - \frac{1}{4G\delta} \frac{l^2 \alpha^2}{2} = - \frac{l^2 \alpha^2}{8G}$$

Letting $c = \frac{3l}{2G}$, $E_\lambda = -\frac{c}{12} \alpha^2$

$\alpha = 1$, $E = -\frac{c}{12}$ precisely matches the energy of the CFT on the cylinder

Black holes (BTZ)

$$ds^2 = -N^2(r) dt^2 + \frac{dr^2}{N^2(r)} + r^2(d\varphi + N^4 dt)^2$$

$$N^2(r) = \frac{r^2}{l^2} - 8MG + \frac{16G^2J^2}{r^2} \quad N^4 = -\frac{4GJ}{r^2}$$

• 2 param. family M, J = mass & angular momentum

• we can easily check that M is the mass when $J=0$

going to radial gauge via $\frac{df}{f} = \frac{dr}{N(r)}$ as before,
we find

$$g_{rr}^{(2)} = 4MG \quad \sqrt{\sigma} \simeq f \quad \tilde{n}^r \simeq \frac{l}{f}$$

$$Q_{BT} = \int_0^{2\pi} d\varphi \sqrt{\sigma} \tilde{n}^r T_{rr} \frac{\xi^r}{\frac{g_{rr}}{8\pi G f}} = \frac{2\pi l}{8\pi G f} 4MG = M$$

as expected

Exercise: Show that the mass & angular momentum (Q_{dt} & $Q_{d\phi}$) of the general BTZ black hole are given by M, J

the horizon is @ $N_{in}^2 = 0 \Rightarrow$

$$r_{\pm} = l \sqrt{4GM} \sqrt{1 \pm \sqrt{1 - \frac{J^2}{(Ml)^2}}} = l \sqrt{2GM} \left(\sqrt{1 + \frac{|J|}{Ml}} \pm \sqrt{1 - \frac{|J|}{Ml}} \right)$$

$$\Rightarrow |J| < Ml$$

horizon is a Killing horizon for $\xi = \partial_t + \Omega \partial_\phi$

$$\text{w/ } \Omega \text{ det by } \xi^2|_{\mathcal{H}} = 0 \Rightarrow \Omega = -N^\phi = \frac{4GJ}{r_+^2} = \frac{4r_-}{2r_+^2}$$

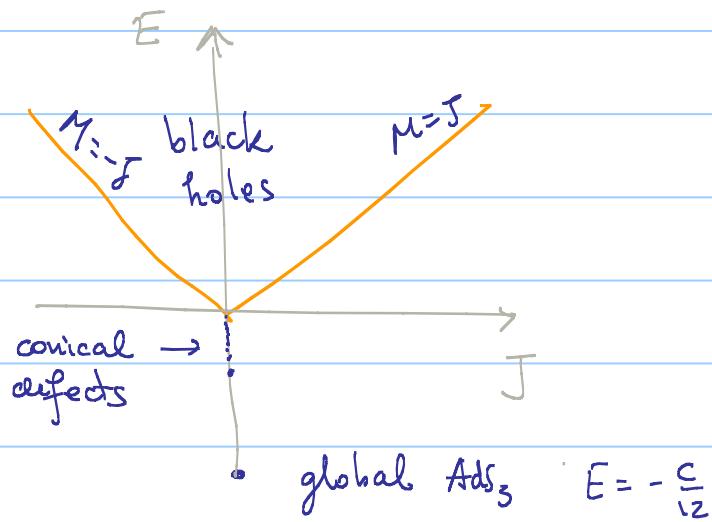
the Hawking temp is $T_H = \frac{\kappa}{2\pi} = \frac{r_+^2 - r_-^2}{2\pi l^2 r_+}$ (from $\xi^\mu \nabla_\mu \xi^\nu = \kappa g^{\mu\nu}$)

the Bekenstein-Hawking entropy reads

$$S_{BH} = \frac{A\pi}{4G} = \frac{2\pi r_+}{4G} = 2\pi \sqrt{\frac{Ml^2}{8G} \left(1 + \frac{J}{Ml} \right)} + 2\pi \sqrt{\frac{Ml^2}{8G} \left(1 - \frac{J}{Ml} \right)}$$

Exercise Show that the 1st law of black hole mechanics is satisfied

Finally, we find that the spectrum of all solutions to 3d Einstein gravity w/ a negative cosm. const is



vacuum energy $-\frac{c}{12}$ matches
Casimir energy in the
dual CFT

The ASG

- looking at the solutions above, bnd. conditions that allow them take the form (Brown - Henneaux)

$$g_{rr} = -r^2 + \mathcal{O}(1) \quad g_{r\phi} \sim \mathcal{O}(1) \quad g_{\phi\phi} \sim \frac{1}{r^3}$$

$$g_{rr} = \frac{1}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right) \quad g_{r\phi} \sim \mathcal{O}\left(\frac{1}{r^3}\right) \quad g_{\phi\phi} = r^2 + \mathcal{O}(1)$$

- the asympt. expansion of the diff. eqs that leave invariant these bnd. cond. is (w/ $x^\pm = \varphi \pm t$)

$$\xi_L = \chi_L(x^+) \partial_+ - \frac{1}{2} \chi_L'(x^+) r \partial_r - \frac{\ell^2}{2r^2} \chi_L''(x^+) \partial_- + \dots$$

$$\xi_R = \chi_R(x^-) \partial_- - \frac{1}{2} \chi_R'(x^-) r \partial_r - \frac{\ell^2}{2r^2} \chi_R''(x^-) \partial_+ + \dots$$

A basis natural for x^\pm compact is $\chi_L^m(x^+) = e^{imx^+}$
 $\chi_R^m(x^-) = e^{imx^-}$

the Lie bracket algebra of these symm is

$$[\xi_{L,R}^m, \xi_{L,R}^n] = -i(m-n) \xi_{L,R}^{m+n} \quad (\text{Witt algebra})$$

\rightarrow same as cl. 2d conf gp

as mentioned in the 1st lecture, the Lie bracket algebra of the asympt. diffeos is isomorphic to the Dirac bracket algebra of the conserved charges up to a central extension $\mathcal{K}_{\eta,\xi}$

$$\{Q_\eta, Q_\xi\} = \delta_\xi Q_\eta = Q_{[\eta, \xi]_{L-B}} + \mathcal{K}_{\eta, \xi}$$

to compute \mathcal{K} , we need to compute $\delta_\xi Q_\eta$, i.e. how the charge associated w/ an asympt. symm η changes when the diffeo ξ acts on the backgnd.

To compute this, it is useful to go to radial gauge & take the bnd. metric $g_{\alpha\beta}^{(0)}$ in the Fefferman - Graham expansion to equal $\eta_{\alpha\beta}$. Then, the asympt. eqn fix

$$\text{tr } g^{(2)} = 2g_{+-}^{(2)} = 0 \quad D^\alpha g_{\alpha\beta}^{(2)} = 0$$

$$\Rightarrow \partial_+ g_{--}^{(2)} = \partial_- g_{++}^{(2)} = 0$$

Consequently, the asympt expansion takes the form

$$ds^2 = l^2 \frac{dg^2}{g^2} + g^2 dx^+ dx^- + \mathcal{L}(x^+) (dx^+)^2 + \bar{\mathcal{L}}(x^-) (dx^-)^2 + \frac{1}{g^2} \mathcal{L}(x^+) \bar{\mathcal{L}}(x^-) dx^+ dx^-$$

The conserved charges are

$$Q_\xi = \int_0^{2\pi} d\varphi \sqrt{g} \hat{n}^\mu T_{\mu\nu} \xi^\nu = \int_0^{2\pi} d\varphi (T_{+\alpha} - T_{-\alpha}) \xi^\alpha$$

so, e.g.

$$\delta Q_{\xi_L} = \frac{1}{8\pi G l} \int_0^{2\pi} d\varphi \delta \mathcal{L}(x^+) \chi_L(x^+) \quad \text{so}$$

(for Q_{ξ_R} , careful about - signs in the def'n of x^-)

$$\{Q_\eta, Q_\xi\} = \delta_\xi Q_\eta = \frac{1}{8\pi G e} \int_0^{2\pi} d\varphi \delta_\xi \mathcal{L}(x^+) \chi_\eta(x^+)$$

the change in $\mathcal{L}(x^+)$ under the action of ξ^μ is

$$\delta_\xi \mathcal{L}(x^+) = 2 \mathcal{L} \chi'_\xi(x^+) + \chi_\xi(x^+) \mathcal{L}'(x^+) - \frac{\ell^2}{2} \chi''_\xi(x^+)$$

$\underbrace{\qquad\qquad\qquad}_{[Q(\eta, \xi)]}$ $\underbrace{\qquad\qquad\qquad}_{\text{will contr to central ext}}$

take $\chi_\eta = e^{imx^+}$, $\chi_\xi = e^{inx^+}$ then

$$Y_{\eta, \xi} = \frac{1}{8\pi G e} \int_0^{2\pi} d\varphi \left(-\frac{\ell^2}{2} \chi''_\xi \right) \chi_\eta(x^+) = + \frac{i\ell}{8G} n^3 \delta_{m+n,0}$$

so

$$\{Q_m, Q_n\}_{DB} = -i(m-n) Q_{m+n} - \frac{i\ell}{8G} m^3 \delta_{m+n,0}$$

passing from Dirac brackets to commutators $[\cdot, \cdot] = i\{\cdot, \cdot\}_{DB}$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}$$

w/ C = $\frac{3\ell}{2G}$

and we have absorbed $\frac{c}{24}$
in a shift of L_0

So, we obtain the full Virasoro algebra, including the central extension, from a purely (semi) classical calculation in G.R. ($\text{ASG} = \text{symm. of } \text{CFT}_2$)

- agrees w/ the central charge of the dual CFT_2 in known stringy examples
- can easily check that the black hole entropy takes the Cardy form for this value of c : universal check of holography!
- most black holes whose entropy we understand can be reduced to a BTZ b.h. in some duality frame

Asympt. symm. of flat sp

asympt. str. of Minkowski space

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$$

want to construct Penrose diag (can)

$$\text{let } u = t - r \quad v = t + r \quad ds^2 = -dudv + \left(\frac{u-v}{2}\right)^2 d\Omega_2^2$$

$u, v \in (-\infty, \infty)$

to make coord range compact, let $u = \tan U \quad v = \tan V$.

U/V ranges $U, V \in (-\frac{\pi}{2}, \frac{\pi}{2})$ w/ $U \leq V$

$$dudv = \frac{dUdV}{\cos^2 U \cos^2 V}$$

$$u-v = \tan U - \tan V = \frac{\sin(U-V)}{\cos U \cos V}$$

$$ds^2 = \underbrace{\frac{1}{\cos^2 U \cos^2 V}}_{\text{drop. } 4 \frac{1}{4}} \left(-dUdV + \frac{\sin^2(U-V)}{4} d\Omega_2^2 \right)$$

let $T = U+V \in (-\pi, \pi)$ $R = V-U \in (-\pi, \pi) \ni$
 $(0, \pi)$

$$ds^2 = -dT^2 + dR^2 + \underbrace{\sin^2 R}_{S^3} d\Omega_2^2 \quad 0 < R + |T| < \pi$$

$\mathbb{R}^{1,3}$ is conformal to $\mathbb{R} \times S^3$ Einstein static universe
 a patch of

each point is an S^2 .

i^+ or future timelike

unwrapping

$T=0$

$R=0$

i^0

$R=\pi$

i^-

$-T$

i^-

$T=0$

$R=\pi$

i^0

$R=0$

i^+

$T=\pi$

$R=\pi$

i^+

$R=0$

i^0

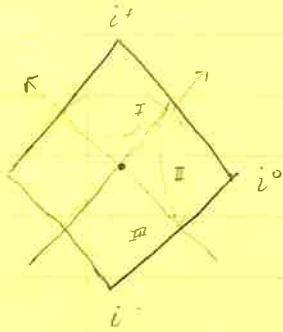
$T=\pi$

$R=\pi$

Resolving the singular points

- introduce a coordinate system that are good near i^0, i^\pm (set of)

- foliate Minkowski space w/ const. curvature slices



$$-t^2 + \sum_{i=1}^3 x_i^2 = \begin{cases} -\hat{\epsilon}^2 & (\text{regions I, III}) \\ \hat{s}^2 & (\text{region II}) \end{cases}$$

induced metric will be $\underbrace{\text{Eucl. } AdS_3}_{H_3} / ds_3$.

region I : $t = \hat{\epsilon} \cosh \hat{\phi}$ $r = \hat{\epsilon} \sinh \hat{\phi}$

$$ds^2 = -d\hat{\epsilon}^2 + \hat{\epsilon}^2 \left(d\hat{\phi}^2 + \underbrace{\sinh^2 \hat{\phi}}_{H_3} d\Omega_2^2 \right)$$

region II : $t = \hat{s} \sinh r$ $r = \hat{s} \cosh r$

$$ds^2 = d\hat{s}^2 + \hat{s}^2 \left(-d\hat{r}^2 + \underbrace{\cosh^2 r}_{ds_3} d\Omega_2^2 \right)$$

- this foliation resolves the singular str. near i^0, i^\pm

i^0 : $s \rightarrow \infty$, intersects γ^\pm as $r \rightarrow \pm\infty$

i^\pm : $\hat{\epsilon} \rightarrow \pm\infty$

departure point for flat sp holography (de Boer & Solodukhin).

Isometries of Minkowski space-time (action on γ^\pm)

retarded

• γ^+ take $t, r \rightarrow \infty$ w/ $u = t - r$ fixed ; $ds^2 = -du^2 - 2du dr + r^2 d\Omega_2^2$

• time translations $t \rightarrow t + \text{const}$ $\Rightarrow u \rightarrow u + \text{const}$

• spatial translat., e.g. $x^3 \rightarrow x^3 + c$; $x^3 = r \cos \theta$

$$\partial_{x^3} = \cos \theta \partial_r - \frac{1}{r} \sin \theta \partial_\theta \stackrel{\text{static coord}}{=} -\cos \theta \partial_u + \cos \theta \partial_r - \frac{1}{r} \sin \theta \partial_\theta$$

etc.

$\gamma_{\ell=1}^{+/-}$

• celestial sphere: sphere of all directions to which an obs @ (at ∞) $n=0$ & $t=u$ can look

• action of Lorentz gp on celestial sphere ($r \rightarrow \infty$, n)

$$x^\mu \rightarrow x'^\mu \rightarrow \Lambda^\mu_\nu x^\nu \quad \Lambda \in O(1,3)$$

• proper orthochronous $L_+^1 \cong \frac{SL(2, \mathbb{C})}{Z_2}$

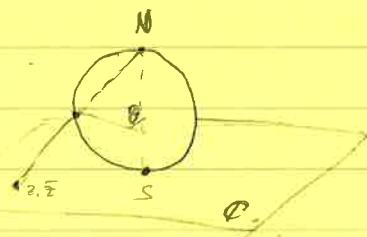
(see e.g. 1508.0092)

$$X = \begin{pmatrix} x^0 + ix^2 & -x^1 - ix^2 \\ -x^1 + ix^2 & x^0 - ix^2 \end{pmatrix} = x^0 \sigma_0 - x^i \sigma_i$$

$$\det X = x^0 x_{\infty} = \text{invar} \Rightarrow SL(2, \mathbb{C}). = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad X \rightarrow S X S^+ \quad ad - bc = 1$$

• parametrize Λ^μ_ν in terms of the $SL(2, \mathbb{C})$ matrix S (quadratic)
(e.g. $R_3 = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$, $B_3 = \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$)

• introduce stereographic coord. on the S^2



$$d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\varphi^2 = \frac{4dzd\bar{z}}{(1+z\bar{z})^2}$$

$$z = e^{i\varphi} \operatorname{ctg} \frac{\theta}{2} = \frac{x_1 + ix_2}{r - ix_3}$$

$$\cdot n^\mu = \sqrt{(x_1)^2 + x_2^2 + x_3^2} \rightarrow n' = r \underbrace{\frac{|az+b|^2 + |cz+d|^2}{1+z\bar{z}}}_{F(z, \bar{z})} + \mathcal{O}(1)$$

so Lorentz transf. preserve the limit $r \rightarrow \infty$

\rightarrow angle-dependent rescaling by $F(z, \bar{z})$

$$\cdot u' = t' - n' = \underbrace{G(z, \bar{z})}_\frac{1}{|cz+d|^2} u + \mathcal{O}\left(\frac{1}{r}\right) \quad (\mathcal{O}(r) \text{ term cancels out!})$$

$$\cdot z' = \frac{az+b}{cz+d} + \mathcal{O}\left(\frac{1}{r}\right) \quad \text{conformal transf of the celestial } S^2$$

e.g. for a boost along x^3 $z' = e^{-\chi} z$, so objects on S^2 w/ rapidity χ become closer to the direction of motion.

Asymptotically flat spt.

- easiest: fix a coord. system & specify falloff cond on the metric components (downside: int. brd. may not be visible)
- near \mathcal{I}^+ \rightarrow Bondi gauge. (avoid $\log r$ terms in the asympt. exp.)
 - def. (u, θ, φ) $\xrightarrow{x^A}$ they are const. along outgoing null geodesics $\Rightarrow g_{rr} = 0$

$$ds^2 = g_{uu} du^2 + 2g_{ur} du dr + 2g_{uA} du dx^A + g_{AB} dx^A dx^B$$

- choose r to be the luminosity dist. $\partial_r (\det \frac{g_{AB}}{r^2}) = 0$.

- so far, we just fixed a gauge. Asympt. flatness cond.

$$\begin{aligned} ds^2 = & -du^2 - 2du dr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} \quad \leftarrow \text{Minkowski} \quad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2} \\ & + \frac{2m_B}{r} du^2 + r C_{zz} dz^2 + r C_{\bar{z}\bar{z}} d\bar{z}^2 + D^2 C_{zz} du dz + D^{\bar{2}} C_{\bar{z}\bar{z}} du d\bar{z} \\ & + \frac{1}{r} \left(\frac{4}{3} (N_z + u \partial_z m_B) - \frac{1}{4} \partial_z (C_{zz} C^{zz}) \right) du dz + \text{c.c.} + \dots \end{aligned}$$

$C_{AB} \propto \partial_A \partial_B^2$

$N_A = \text{angular mom. aspect}$

or

$$g_{uu} = -1 + O(\frac{1}{r}) \quad g_{ur} = -1 + O(\frac{1}{r^2}) \quad g_{uz^*} = O(1) \quad g_{z\bar{z}} = O(r)$$

$$g_{z\bar{z}} = r^2 \gamma_{z\bar{z}} + O(1)$$

$$g_{rr} = g_{rz} = 0$$

(only in r , will also need in u later)

$$m_B(u, z, \bar{z}) = \text{Bondi mass aspect. (in Schwarzschild/Kerr)} \quad m_B = GM \quad \text{out}$$

- angular density of energy @ \mathcal{I}^+

- $\int_{S^2} m_B = M(u) = \text{Bondi mass.} \quad \partial_u M(u) \leq 0 \quad \text{for Einst. matter/NE} \Rightarrow \text{grav. rad. carries away energy}$

$\cdot N_{AB} = \partial_u C_{AB}$ Bondi news
 $N_{AB} N^{AB} \propto \text{energy flux } \mathcal{I}^+$

$\cdot C_{zz}, C_{\bar{z}\bar{z}}$ (C_{AB} w/ $\gamma^{AB} C_{AB} = 0$) \cdot grav. waves (2 polar)

- asymptotic data: $m_B(u, x^A)$, $C_{AB}(u, x^A)$, $N_A(u, x^A)$ + subleading but the o.o.m. fix some relations

$$\partial_u m_B = \frac{1}{4} D^A D^C N_{AC} - T_{uu}$$

$$T_{uu} = \frac{1}{8} N_{AB} N^{AC} + 4\pi \text{lim}_{r \rightarrow \infty} (r^2 T_{uu})$$

$$\partial_u N_A \sim f(C_{AB}, T_{uu}, T_{uA}, N_{AB})$$

$$\partial_u C_{AB} \equiv N_{AB}$$

so actual data are $N_{AB}(u, x^A)$, C_{AB} , m_B , N_A @ $u \rightarrow \infty$

Asymptotic symmetries:

- diffeos that preserve the asympt. form of the metric (lead to non-triv. cons. charges)

$$\rightarrow \text{Bondi gauge } g_{rr} = g_{rA} = 0 \quad \partial_r \text{ det} \frac{g_{AB}}{r^2} = 0$$

$g^{\mu\nu}$ can be exp in powers of $r \times f_n(u, x^A)$ (4 of them)
+ asympt grnd cond \rightarrow 3 f_n on the sp^h $T(x^A)$, $R^A(x^A)$.

$$\left. \begin{array}{l} \xi_T = T(x^c) \partial_u - \frac{1}{r} D^A T(x^c) \partial_A + \frac{1}{2} D_A D^A T(x^c) \partial_u \\ \xi_R = R^A(x^c) \partial_A + \frac{1}{2} D_A R^A(x^c) \partial_u - \frac{r+4}{2} D_A R^A(x^c) \partial_u \end{array} \right\} \begin{array}{l} \text{BMS generators} \\ (\text{RA} \rightarrow \text{globally def}) \end{array} \quad \text{(supertranslations)}$$

w/ R^A a conformal Killing vector on S^2 . $D_A R_B + D_B R_A = \gamma_{AB} D_C R^C$

- globally defined $\Rightarrow R^A \rightarrow$ Lorentz. transf. / G CKV
not $-1/-$ $R^A \rightarrow$ superrotations

Supertranslations

- angle-dependent translations $\rightarrow T(x^A) = \sum_{l,m} Y_{l,m}(x^A) e_m$

$$T(x^A) = a_0 Y_0^0(x^A) + a_m Y_1^m$$

- conserved charges not integrable \Rightarrow not conserved (the Noether $N_{AB} = 0$)

$$\delta Q_T = \delta \left[\frac{1}{4\pi G} \int d^2\Omega \sqrt{\gamma} T \cdot m \right] + \frac{i}{32\pi G} \int d^2\Omega (\overline{\delta \gamma} N^{AB} \delta C_{AB})$$

flux through \mathcal{Y} .

- charge algebra $\{Q_{S_1}, Q_{S_2}\}_{\mathcal{Y}} = -\delta_{S_2} Q_{S_1}[\mathcal{Y}] + Q_{S_2}(-\delta_{S_1} \chi, \chi)$ trivial (ASG commutes)

- act non-trivially on the geom.

$$\delta_T C_{AB} = T \partial_u C_{AB} - 2 D_A D_B T + \gamma_{AB} D_c D^c T$$

$$\delta_T m = T \partial_u m + \frac{1}{4} (N^{AB} D_A D_B T + 2 D_A N^{AB} D_B T) \quad \star \text{ e.g. } m \text{ changes}$$

$$\delta_T N_{AB} = T \partial_u N_{AB}$$

- interesting example: flat sp ($m = c_{AB} = N_{AB} = 0$)

$$\delta_T C_{AB} = -2(D_A D_B T - \frac{1}{2} \gamma_{AB} D_c D^c T) \neq 0$$

$$R_{\mu\nu\rho\sigma} = 0 \Rightarrow C_{AB} = -2 D_A D_B C + \gamma_{AB} D^c D^c C$$

\star memory field.

so $\delta_T C = T$ (transf. similarly to a Goldstone boson - of spont. broken supertransl. invariance, except the usual transl.)

- this has zero supertranslation charge ($m=0$), but non-zero superrotation charges (if these make sense)

$$Q_{R^A} = \frac{1}{16\pi G} \int d^2\Omega R^A (2N_A + \frac{1}{16} \partial_A (C_{BC} C^{BC}))$$

- translations \subset ideal of BMS^+ (supertransl. \times Lorentz) $[P^M, T] = 0$
 $\{P^\mu, N^\nu\} = 0$

- Lorentz does not commute w/ supertransl.; but in every vac \exists a Poinc. subgr. that leaves it inv.

Lorentz algebra & superrotations

$$\mathcal{L}_R = R^A(x) \partial_A - \frac{v+u}{2} D_A R^A \partial_v + \frac{u}{2} D_A R^A \partial_u$$

$R^A \rightarrow$ conformal Killing $\Rightarrow \partial_{\bar{z}} R^{\bar{z}} = \partial_z R^{\bar{z}} = 0$. $R^z(z), R^{\bar{z}}(\bar{z})$
globally defined for $a + bz + cz^2$ (infinitesimal version
of $SU(2)$)

- in the case of CFTs, considering not globally defined CKV.
led to the Virasoro alg \rightarrow how about here?

for $g_{z\bar{z}} = \underbrace{\partial r^2}_{\text{if } Y^z \text{ only meromorphic}} \gamma_{z\bar{z}} \partial_{\bar{z}} Y^z + O(r)$.

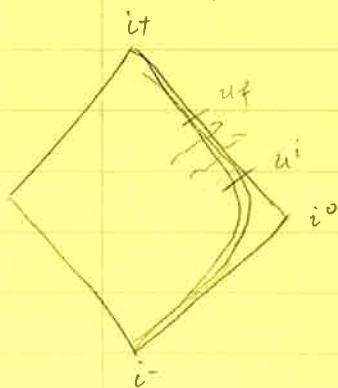
γ^z only meromorphic, then $\partial_{\bar{z}} Y^z = \delta^z(z-w)$
e.g. $\frac{1}{z-w}$.

\Rightarrow asymptotic fall-off condition violated @ $z=w$.

- ok to consider infinitesimal superrot, but finite ones \rightarrow modify
of the phase sp. to asympt. locally flat + defects

The memory effect

- \exists physical, observable effect of BMS?



- consider 2 inertial obs near i^+ , moving along $\gamma_{2,1}$
- no radiation for $u < u_i$ & $u > u_p$ w/dist \gg
- displacement memory effect: change in sup.
due to the radiation (supertranslations)

- geodetic deviation eqn

$$\frac{D^2 S^\mu}{d\tau^2} = R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

$$= T^\mu \nabla_\alpha (T^\beta \nabla_\beta S^\alpha)$$

assume $S^\nu = 0$

$$\partial_\mu S^\mu = R^A_{\mu\nu B} S^\nu$$

$$r^2 \gamma_{AB} \partial_\mu S^\mu = -R_{\mu A \nu B} S^\nu = \frac{r}{2} \partial_\mu^2 C_{AB} S^\mu$$

Integrating from z_i to z_f , find

$$\gamma_{AB} \Delta z^B = \frac{1}{2r} \Delta C_{AB} z^B + O(\frac{1}{r^2})$$

grav. memory effect. (physical manif. of S-trace)

- what causes ΔC_{AB} / how is it determined?

$$\partial_m m = \frac{1}{4} D^A D^B N_{AB} - T_{uu}$$

(grav. $\int_{u_i}^{u_f} \frac{1}{8} N_{AB} N^{AB} + 4\pi (r^2 T_{uu})_{r=\infty}$)

$$\Rightarrow \Delta m = \frac{1}{4} D^A D^B \Delta C_{AB} - \int_{u_i}^{u_f} T_{uu}.$$

if. spt. is stationary outside (u_i, u_f)

$$\Delta C_{AB} = -2 D_A D_B \Delta C + \gamma_{AB} D^2 \Delta C. ; \Delta C_{zz} = -2 D_z^2 \Delta C$$

$$\Rightarrow + \frac{1}{2} (D^2)^2 D_z^2 \Delta C = \Delta m + \int_{u_i}^{u_f} \partial_u \left\{ \frac{1}{8} N_{AB} N^{AB} + 4\pi (r^2 T_{uu}) \right\}$$

grav. waves. mat. matter

w/ soln

$$\Delta C(z, \bar{z}) = 2 \int d^2 z' G(z, \bar{z}, z', \bar{z}') \int_{u_i}^{u_f} \partial_u T_{uu}(z', \bar{z}') + \Delta m \quad \text{if } \delta z'$$

$$D_z^2 D_{\bar{z}}^2 G = - \gamma_{z\bar{z}} \delta^{(2)}(z - z')$$

$$G = -\frac{i}{\pi} \sin^2 \frac{\theta}{2} \log \sin^2 \frac{\theta}{2} \quad \text{angle betw 2 det. } A(z, \bar{z})$$

$$z^{(2)}_z = \frac{(z - z')^2}{(1+z)(1+\bar{z})}$$

\rightarrow highly non-local on S^2 . if $T_{uu} \sim$ localized near $z = \infty$

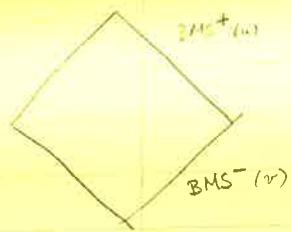
then G vanishes $\frac{1-z^2}{1+z^2} \circ \frac{z=0}{z=\infty}$

but large on equator, apparently

- spin memory effect \rightarrow (time delay between clockwise / counter-clockwise light rays)

BMS symmetries of gravitational scattering

$v = t + r$



$$\text{near } \mathcal{S}^+ \quad ds^2 = -dt^2 + r^2 d\Omega^2 \quad \text{near } \mathcal{S}^- \quad ds^2 = -dr^2 + r^2 d\Omega^2$$

advanced coord

how to match them?

- Lienard - Wiechart soln for the el field of a moving charge

$$F_{rt} = \frac{e^2}{4\pi} \frac{Q \gamma (r - t \hat{x} \cdot \vec{\beta})}{(\gamma^2(t - r \hat{x} \cdot \vec{\beta})^2 - t^2 + r^2)^{3/2}}$$

- this continuous @ $r = \infty$.

$$y^+ : u = t - r \quad F_{rt} = F_{ru} = \frac{e^2 Q \gamma (r - (u + r) \hat{x} \cdot \vec{\beta})}{4\pi \left(\gamma^2(u + r - r \hat{x} \cdot \vec{\beta})^2 - (u + r)^2 \right)^{3/2}}$$

$$\lim_{r \rightarrow \infty} F_{ru} = \frac{e^2}{4\pi r^2} \frac{Q \gamma}{\gamma^2(1 - \hat{x} \cdot \vec{\beta})^2} \quad \text{valid at } y^+$$

$$y^- : v = t + r \quad F_{rt} = F_{rv} = \frac{e^2 Q \gamma (r - (v - r) \hat{x} \cdot \vec{\beta})}{4\pi \left(\gamma^2((v - r - r \hat{x} \cdot \vec{\beta}))^2 - (v - r)^2 \right)^{3/2}}$$

$$\lim_{r \rightarrow 0} F_{rv} = \frac{e^2}{4\pi r^2} \frac{Q}{\gamma^2(1 + \hat{x} \cdot \vec{\beta})^2} \quad \text{valid at } y^-$$

- not equal, but related by an antipodal matching cond.

$$\lim_{r \rightarrow \infty} r^2 F_{ru}(x) \Big|_{y^+} = \lim_{r \rightarrow 0} r^2 F_{rv}(-x) \Big|_{y^-}$$

- only one consistent w/ Lorentz invar

Choose conventions in which the s^2 @ \mathcal{S}^- is antipodally related to the one @ \mathcal{S}^+

up

$$ds^2 = -du^2 - 2du dr + \frac{4r^2 dt^2}{(1 + z\bar{z})^2}$$

$$x^1 + i x^2 = \frac{2rz}{1 + z\bar{z}} \quad x^3 = r \frac{1 - z\bar{z}}{1 + z\bar{z}}$$

$$ds^2 = -dv^2 + 2dv dr + \frac{4r^2 dz d\bar{z}}{(1 + z\bar{z})^2}$$

$$x^1 + i x^2 = \frac{-2rz}{1 + z\bar{z}} \quad x^3 = -r \frac{1 - z\bar{z}}{1 + z\bar{z}}$$

in these conventions, the matching cond are.

$$C(\vec{r}, \vec{s})|_{\vec{r}=\vec{s}} = Q(\vec{r}, \vec{s})|_{\vec{r}=\vec{s}}$$

$$m_2 f(\vec{r}, \vec{s})|_{\vec{r}=\vec{s}} = m_2 \bar{f}(\vec{r}, \vec{s})|_{\vec{r}=\vec{s}}$$

cons. charges $Q_f^+ = \frac{1}{4\pi G} \int_{S^2} d^2\Omega f(x^*) m_B(z, \bar{z})$

\bar{f} also term equal.

$$Q_f^- = \frac{1}{4\pi G} \int_{S^2} d^2\Omega \bar{f}(x^*) m_B(z, \bar{z})$$

f was rotated T before.

then the charges will be conserved.

$$\partial_\mu m = \frac{1}{4} D^A D^B N_{AB} - T_{uu} = \frac{1}{4} (D_z^2 N^{zz} + D_{\bar{z}}^2 \bar{N}^{\bar{z}\bar{z}}) - T_{uu}$$

so the charges can also be written as.

$$Q_f^\pm = \frac{1}{4\pi G} \int_{S^2} dm d^2\Omega z \bar{z} f(z, \bar{z}) [T_{uu} - \frac{1}{4} (D_z^2 N^{zz} + D_{\bar{z}}^2 \bar{N}^{\bar{z}\bar{z}})]$$

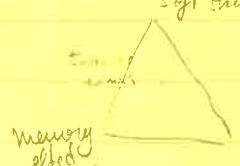
take e.g. $f(z, \bar{z}) = \delta^2(z - w)$. Then $Q_f^+ = Q_f^-$ \Rightarrow soft terms (total α' deriv.)

integrated energy over S^2 @ every angle = int eng. over S^2

quantum theory, sym. commute w/ S-matrix.

- equivalent to Weinberg's soft grav. th.

$$M_{uv}(q, p_{out}, p_{in}) = \frac{1}{4\pi G} \left(\sum_{k=1}^{N_{out}} \frac{p_k^u p_k^v}{p_k \cdot q} - \sum_{k=1}^{N_{in}} \frac{p_k u p_k v}{p_k \cdot q} \right) dU(p_k)$$



Ward. id. on S^2 , look like CFT Ward id.

Black holes info paradox

- no hair thm. \rightarrow b.h. soln is unique \rightarrow universal form of Hawking rad \rightarrow info loss
- if b.h. w/ M, J, Q, up to diff \rightarrow can carry ∞ amount of B.H. hair, not in part, sending a shockwave into fl. b.h. changes induces a supertransl. can be measured by superrot. charges

- not clear if this supertransl. laws can be viewed as intrinsic to the b.h., or rather to the esp. field
- possible new way to store info, though not clear if yet a low-ent.

To conclude

1. • AdS flat sp \rightarrow oo-dimil. supertranslations (grav. vacuum
only degenerat.)
+ ? superrotations? \rightarrow Virasoro, symm. central ext?
2. • possible to exprin scattering ampt. as CFT correlators.
using conf. basis. \rightarrow CFT₂ connection?
3. • effect of asymt. symm. measurable @ future detectors
4. • fascinating link to soft thms
5. • possibility to explain b.h. entropy in flat sp?
6. • implications for the b.h. info paradox?