

Some approaches to three-point correlators in $\mathcal{N} = 4$ super Yang-Mills

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Résumé

$\mathcal{N} = 4$ SYM a attiré beaucoup d'attention au cours des deux dernières décennies en raison des deux aspects : la correspondance AdS/CFT et intégrabilité. La correspondance AdS/CFT est la première réalisation précise de la dualité jauge/gravité dont l'histoire commence dans les années 60, quand la théorie des cordes a été considérée comme un candidat pour décrire les interactions fortes.

En 1997 Maldacena a fait une proposition sur la dualité entre certaines théories conformes des champs (CFT) et théories des cordes défini sur le produit de AdS espace et une certaine variété compacte, qui implique une correspondance entre les observables de théorie des cordes et théorie de jauge. Jusqu'à maintenant la correspondance AdS/CFT reste une conjecture. La dualité de $\mathcal{N} = 4$ SYM et la théorie des cordes est un exemple le plus notable de correspondance AdS/CFT. Un des obstacles principaux à l'explorer est le fait que le régime de couplage faible pour la théorie de jauge est le régime de couplage fort pour la théories des cordes et vice versa. Par conséquent, aussi longtemps que les méthodes perturbatives sont appliquées, on ne peut pas comparer les observables de deux cotés de la correspondance directement en dehors de quelques cas particuliers. A ce stade, l'énorme symétrie de $\mathcal{N} = 4$ SYM joue un rôle important en permettant le calcul exact des observables de la théorie au moins dans la limite planaire. Cette propriété de la théorie est appelée intégrabilité.

Les observables de $\mathcal{N} = 4$ SYM sont les boucles de Wilson et des fonctions de corrélation construite des opérateurs invariants de jauge. La dépendance de l'espace-temps des corrélateurs à deux et trois points est déterminée par la symétrie conforme jusqu'à certains paramètres : les dimensions des opérateurs dans le cas de fonctions à deux points et les dimensions des opérateurs et de constantes de structure dans le cas des fonctions à trois points. Il est communément accepté de se référer à le problème de trouver les dimensions des opérateurs comme le problème spectral. Au niveau classique, la dimension de l'opérateur est égale à la somme des dimensions des champs fondamentaux des quels l'opérateur est composé. Lorsque l'interaction est activée, la dimension conforme obtient la correction quantique. Afin de calculer les fonctions à trois points on a besoin calculer les constantes de structure. En CFT le calcul des corrélateurs supérieurs peuvent éventuellement être réduit au calcul de fonctions à deux et trois points au moyen de l'expansion du produit de l'opérateur. Par conséquent, la connaissance des corrélateurs à deux et à trois points permettent de construire tous les autres corrélateurs dans la théorie.

Cette thèse est consacrée au calcul des fonctions à trois points et est composée de deux parties. Dans la première partie nous considérons l'approche générale pour le calcul des fonctions à trois points sur la base de soi-disant vertex de spin, qui est inspiré de la théorie de champs des cordes. Dans la deuxième partie, nous considérons un type spécifique de fonctions à trois points appelés lourd-lourd-léger, qui sont caractérisés par la propriété que la longueur de l'un des opérateurs est beaucoup plus petite des longueurs de deux autres. Il s'avère que ces fonctions de corrélations peuvent être identifiées à des facteurs de forme diagonaux et ainsi on peut appliquer les résultats concernant les facteurs de forme.

Abstract

$\mathcal{N} = 4$ SYM theory has been drawing the attention of a lot of physicists during two last decades mainly due to the two aspects: AdS/CFT correspondence and integrability. AdS/CFT correspondence is the first precise realization of the gauge/string duality whose history starts in the 60's, when a string theory was considered as a candidate for describing the strong interactions. In 1997 Maldacena made a proposal [1] about the duality between certain conformal field theories (CFT) and string theories defined on the product of AdS space and some compact manifold, which implies a one to one map between the observables of the gauge and string counterparts. Up to now AdS/CFT correspondence still remains a conjecture. The duality of $\mathcal{N} = 4$ SYM and the appropriate string counterpart, type IIB string theory on the $AdS_5 \times S^5$ manifold, is the most notable example of the AdS/CFT correspondence. One of the main obstructions to exploring it is the fact that weak coupling regime for the gauge theory is the strong coupling regime for the string theory and vice versa. Therefore as long as perturbative methods are applied, one can not compare the observables of dual counterparts directly apart from some specific cases. At this point the huge symmetry of $\mathcal{N} = 4$ SYM plays an important role allowing exact computation of the theory observables at least in the planar limit. This property of the theory is called integrability.

The observables of the $\mathcal{N} = 4$ SYM are Wilson loops and correlation functions built out of gauge invariant operators. The space-time dependence of the two- and three-point correlators is fixed by the conformal symmetry up to some parameters: dimensions of the operators in the case of two-point functions and dimensions of the operators and structure constants in the case of three-point functions. It's commonly accepted to refer to the problem of finding the dimensions of the operators as the spectral problem. On the classical level the operator dimension is equal to the sum of the dimensions of the fundamental fields out of which the operator is composed. When the interaction is turned on, the conformal dimension gets quantum correction. In order to compute three-point functions, apart from the conformal dimensions of corresponding operators one needs to compute the structure constants. In CFT computation of the higher-point correlators eventually can be reduced to computation of two- and three-point functions by means of the operator product expansion. Therefore two- and three-point functions appear to be building blocks of any correlator of the theory.

This thesis is devoted to computation of three-point functions and consists of two parts. In the first part we consider the general approach for computing three-point functions based on the so-called spin vertex, which is inspired from the string field theory. In the second part we consider a specific kind of three-point functions called heavy-heavy-light, which are characterized by the property that the length of one of the operators is much smaller the lengths of other two. It happens that this kind of correlators can be considered as diagonal form factors which supposes that in this case one can apply the results obtained in the form factor theory.

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Chapter 1

Introduction

1.1 Gauge/string duality and AdS/CFT correspondence

According to the concept of the modern physics there are four interactions in the nature: *electromagnetic*, *weak*, *strong* and *gravitational*. On the quantum level first three of these interactions are successfully described by quantum gauge field theories. On the other hand, in spite of the numerous attempts of describing the gravitational force at the quantum level, up to now none of them resulted in the theory of quantum gravity verified by the experiments.

Another important problem of the modern physics is related to our understanding of the physics of the strongly coupled fields. When a coupling constant becomes large, and accordingly the corresponding interaction becomes strong in literal sense, the perturbation theory, our main technique of computing the observables based on the expansion in a small parameter, becomes useless. Due to this issue we are still not able to describe observables of the quantum chromodynamics (QCD), a theory of the strong interactions, satisfactory at low energies.

A phenomenon of the duality between gauge and string theories allows to shed some light on both of these issues. The relation between strings and gauge theories originates in the old idea (1960's) of describing the strong interactions in terms of strings, when e.g. quark and anti-quark are connected to each other by means of the gluon string and form a meson. This model was even able to explain certain phenomena inherent to hadrons such as linear dependence between the spin and the mass squared (Regge trajectory). However the presence of tachyon in the spectrum, some other inconsistencies and then discovery of QCD led to abandon this idea.

The next important evidence of the relation between gauge and string theories is due to Gerard 't Hooft. In [2] he considered the generalization of QCD with arbitrary number of colors N (for QCD $N = 3$). By analyzing Feynman diagrams in this case he noticed that their expressions can be organized into a double series with respect to the two parameters: 't Hooft constant $\lambda = g_{YM}^2 N$ and $1/N$. Moreover he realized that each diagram can be identified with an appropriate surface and that the power of the color number N with which it appears in the expression for a given diagram, is defined by the corresponding to the surface Euler characteristics $\chi = 2 - 2g$, where g is the genus of the surface. It means that when taking the limit $N \rightarrow \infty$ (planar limit) only

those diagrams survive which correspond to a sphere (planar diagrams). Discovering of this double expansion brought new interest to the idea of the duality between gauge and string theories, since the expansion with respect to $1/N$ clearly reminds the genus expansion of the string theory.

However the first precise realization of the gauge/string duality was discovered much later. This realization appeared in the framework of the AdS/CFT correspondence proposed by Maldacena. In [1] he conjectured that there should exist a one-to-one correspondence between observables of the string theory defined on the product of AdS space with some compact manifold and the conformal field theory (CFT) defined on the boundary of the AdS space. At that, the following identification between the theory parameters is implied

$$\lambda = 4\pi^2 T^2, \quad \frac{1}{N} = \frac{g_s}{4\pi^2 T^2}, \quad (1.1)$$

where T and g_s are the string tension and string coupling constant correspondingly. Existence of such a correspondence is very interesting since, it relate two completely different theories defined on a spaces with different dimensions.

At this moment AdS/CFT correspondence still remains a conjecture. One of the main difficulties to exploring this phenomenon is that weak coupling regime of a string theory corresponds to a strong coupling regime of a CFT and vice versa. Therefore one can not directly compare the observables from both sides of the duality as long as perturbative methods are applied. Nevertheless, due to the numerous evidences, some of which will be mentioned below, nowadays the AdS/CFT correspondence is strongly believed to be true. In this sense the difficulty mentioned above can be turned into an advantage. Because accepting Maldacena's conjecture to be valid allows us to study a string coupling regime of the gauge theory by considering weak coupling regime of the corresponding string theory, and also in an opposite way. This concept looks promising for understanding a way to describe QCD at low energy level. On the other hand, a string theory allows to unify all the interactions under the single concept including a graviton, two-spin particle describing the excitations of the gravity field, and in this sense appears to be one of the main candidate on the role of the theory describing quantum gravity. From this point of view deep connection between gauge and string theories is very intriguing and deserves to be thoroughly studied.

The most notable example of the AdS/CFT correspondence is the duality between $\mathcal{N} = 4$ SYM and type II B string theory defined on $AdS_5 \times S^5$ space. $\mathcal{N} = 4$ SYM is a superconformal theory obtained by dimensional reduction of the $\mathcal{N} = 1$ SYM in ten dimensions to four dimensions. Due to conformal symmetry all the excitations are massless and the observables of the theory are presented by correlation functions of the gauge invariant operators and Wilson loops. All the gauge invariant operators can be diagonalized with respect to the action of the dilatation operator. Each operator is characterized by its eigenvalue, conformal dimension of the operator, which gets the corrections in terms of the anomalous dimension, when the interaction is taken into account. The conformal symmetry allows to fix the expression of the two-point correlation functions up to the value of the conformal dimensions of the operators involved. On the other hand, and again due to the conformal symmetry, the *operator product expansion* (OPE) becomes a very powerful tool for computing the correlation functions. By subsequently applying it to an n -point correlator, one can reduce its computation to the computation of the two-point functions, under assumption that the structure constants are known. In its turn,

the structure constants are encoded in the expressions of the three-point functions, which should be computed separately. Therefore, all the informations about the observables of $\mathcal{N} = 4$ SYM is encoded in two- and three-point correlation functions.

As a consequence of the duality there is one-to-one correspondence between the operators of the theory and the states of the string theory. The first identification between string theory and gauge theory observables was performed for the so-called *half-BPS* operators of $N = 4$ SYM. This terminology originates from the fact that these operators are annihilated by the half of the supercharges of the theory. The half-BPS operators are identified with the supergravity modes of the string theory. Then the energies of the supergravity states are identified with conformal dimensions of appropriate half-BPS operators. Due to the symmetry, the conformal dimensions of the half-BPS operators do not get corrections when the interaction is turned on, and therefore coincide with their classical values.

Another, less trivial case, is the BMN limit [3], which is characterized by the states with large conformal dimensions carrying few low-lying excitations. From the point of view of the string theory, the BMN limit corresponds to the point-like string moving with a speed close to the speed of light along the equator of the S^5 and sitting on the center of the AdS_5 space. Such string states can be described by the so-called pp-wave background, which can be obtained from the original string theory by taking a special kind of the Penrose limit [4, 5]. The theory obtained has a great advantage compared to the string theory on the $AdS_5 \times S^5$ space, namely it can be quantized. Another useful property of the BMN limit is that one can introduce there an effective coupling constant $\lambda' = \lambda/J^2$ (where J scales as the length of the operator), which can be used as an expansion parameter at weak coupling as well as at strong coupling. This allows to compare the data obtained on both sides of the duality directly on the basis of perturbative methods! Finally, non-planar corrections survive the BMN limit [6, 7] and thus can be identified with the genus expansion of the string theory on pp-wave.

However the half-BPS sector and the BMN limit are restricted to very specific class of operators and thus can not give us the whole picture illustrating the duality between $\mathcal{N} = 4$ SYM and its string counterpart. The main feature drawing attention to these theories and giving us a hope to understand the phenomenon of the AdS/CFT correspondence better is their *integrability*, which simply means that it is possible to compute all the observables of the theory exactly. Integrability is a quite a common phenomenon for the two dimensional sigma-models. In this sense the integrability of the string counterpart of the duality is not a big surprise. On the other hand, the reasoning for the fact that $\mathcal{N} = 4$ SYM should be integrable was not obvious at all until some explicit computations was done. In this context the integrability of $\mathcal{N} = 4$ SYM, which at the moment is well-established only in the planar limit ($N \rightarrow \infty$), can be considered as one of the evidences confirming that the conjecture about AdS/CFT correspondence is a true statement.

The first explicit computation indicating the integrability of the planar $\mathcal{N} = 4$ SYM was performed in [8], where the authors considered the anomalous dimension (the part of the conformal dimension appearing when the interaction is turned on) of the operators from the so-called scalar sector, which consist of the operators made just of the scalar fields, at one loop. The action of the one loop dilatation operator is closed on the set of such operators, therefore its action can diagonalized on these operators independently

on the others.¹ By performing the explicit computation of the appropriate Feynman diagrams the authors established that the action of the one-loop dilatation operator inside the scalar sector exactly coincides with the action of the $so(6)$ spin chain Hamiltonian in periodic boundary condition.

The spin chains are the discrete models defined on the space consisting normally of the finite number of sites and broadly used in statistical physics. The fields living on each site of a spin chain form representation of the corresponding symmetry group. The most known example of the spin chain system is $XXX_{1/2}$ spin chain also referred to as Heisenberg spin chain. $XXX_{1/2}$ spin chain enjoys integrability and the solution of its spectral problem is given by the famous technique of the Bethe ansatz, proposed by Hans Bethe in [9]. Bethe-ansatz technique is broadly used now from condensed matter and statistical physics to high energy theory and since then was significantly developed and generalized. A great contribution at this point belongs to the Leningrad group in mathematical physics (see e.g. [10]).

The $so(6)$ spin chain enjoys integrability as well and the eigenstates of its Hamiltonian can be constructed by the nested algebraic Bethe ansatz (NABA). It means that the one loop spectral problem in the scalar sector of $\mathcal{N} = 4$ SYM is indeed integrable! Discovery of this fact underlies the long, but exciting way of understanding the integrable structure of $\mathcal{N} = 4$ SYM.

A great simplification in analyzing the spectral problem and in understanding the integrability of $\mathcal{N} = 4$ SYM was achieved by analyzing the dilatation operator of the theory. The systematic study of the dilatation operator was initiated in [11], where the authors considered the action of the dilatation operator on the operators consisting of the scalar fields up to two loops. Then in further works results were generalized to other type of operators and for some specific sectors to higher loops. In [12] the author established the complete one loop dilatation operator acting in the representation of the theory symmetry group $psu(2, 2|4)$. At the same time it was shown that the complete one loop dilatation operator coincide with the Hamiltonian of the $su(2, 2|4)$ spin chain [13], by that proving the one loop integrability of the theory. Further on the basis of the symmetry analysis the dilatation operator and its integrable structure were obtained for some particular sectors at higher loops [14, 15]. Finally the results on the study of the dilatation operator were systematized and summarized in [16].

An important progress was made by elaborating on the idea of the BMN limit. In [17], the agreement between the string energy and gauge theory anomalous dimension was obtained at two loop order. However, the prediction of [11] made from the gauge theory side for the three loop anomalous dimension didn't match with the corresponding quantity obtained from the string theory computation [18]². On the other hand, in [19] the authors proposed a way to identify some semi-classical highly excited strings states with the corresponding gauge theory operators. This idea, further elaborated in [20, 21, 22, 23], inspired the authors of [24], where they argued that the BMN scaling with the expansion parameter λ' should hold for the energies of the certain semi-classical states at weak as well as at strong coupling, despite violation of the dilute gas approximation. The

1. We will discuss later that there is not just the scalar sector, but plenty of sectors, which can be considered separately at one loop or even at all loop

2. It is worth mentioning here that the discrepancy between anomalous dimension and string energy appears not in the strict BMN limit, but only when the finite size corrections in powers of $1/J$ are taken into account

corresponding semi-classical limit is normally referred to as Frolov-Tseytlin limit. Similar to the BMN proposal, this statement got solid confirmation at one [25, 26] and two [27] loops, but failed at three loop order [27, 28]. The possible resolution to this paradox was proposed in [29], where the authors assumed that the reason of the three loop mismatch can be caused by the non-commutativity of the limits, which one has to take according to the BMN and Frolov-Tseytlin prescriptions. Indeed, when taking the BMN limit in gauge theory, one has to first expand an observable in the 't Hooft constant λ and then put the length of the operator to be large. At strong coupling the procedure is opposite: we first take the semi-classical limit, which implies the large angular momentum of the string (equivalently large length of the corresponding operator), and only after we can do expansion in λ' .

In the same paper [29] another very important proposal was made. Namely, the authors could generalize the one loop Bethe ansatz, diagonalizing the dilatation operator of the $su(2)$ sector³ to all loops in the *asymptotic* approximation by generalizing the $XXX_{1/2}$ spin chain to a special inhomogeneous long-range interacting spin chain. In this context, the asymptotic approximation means that the result is supposed to be obtained up to the order of $O(\lambda^L)$, where L is the length of the gauge theory operator. As it is mentioned in [29] (see also [30]), in order to obtain the exact result, one has to take into account the so-called *wrapping* corrections. The point is that when going higher order by order in perturbation theory the sites of the spin chain identified with the dilatation operator become to interact on larger distances. Finally the range of the interaction length reaches the length of the operator L , and at this point wrapping effects start to contribute.

In the meantime another significant result concerning the identification of the gauge theory operators and string theory states was obtained. As it has been mentioned above a lot of semi-classical string solutions were constructed to be identified with the corresponding gauge theory operators. In addition to already mentioned papers one can append [31, 32, 33, 34]. However the systematic approach was missing. In order to fill this gap the authors of [35] proposed a way to relate the integrability of both sides of the duality. They considered the reduction of the full world-sheet sigma model to the string living on $S^3 \times R$, that corresponds to the $su(2)$ sector of $\mathcal{N} = 4$ SYM. By exploiting the finite gap method [36] they showed that the general semi-classical solution to the string sigma-model in this case can be reduced to solving a certain Riemann-Hilbert problem of finding a function defined on some algebraic curve with known discontinuities. On the other hand, they took the semi-classical limit of the two-loop Bethe equations, arising at weak coupling. They showed that the problem of finding the solutions to these equations can be reduced to exactly the same Riemann-Hilbert problem as the one obtained in string theory and expanded up to two loops with a certain redefinitions of some parameters. This achievement allowed to identify the gauge theory operators of $su(2)$ sector with the corresponding semi-classical string solution systematically and in a straightforward way giving a solid justification of the AdS/CFT correspondence. Later it was developed further in [37], where other kind of operator were considered belonging to so-called $sl(2)$ sector of $\mathcal{N} = 4$ SYM.

The next important achievement to be mentioned is related to the extension of the asymptotic Bethe ansatz proposed in [29]. In [38] the authors generalized the result of

3. The $su(2)$ sector is formed by the operators consisting of the two scalar fields related by the $su(2)$ transformation

[29] to rank-1 sectors of $\mathcal{N} = 4$ SYM and then conjectured it for the whole theory. The main component of their proposal was the expression for the all loop S -matrix which they conjectured for the scattering of all the elementary fields. The confirmation of their ansatz was obtained in [39], where by using the symmetry constrains the expression for the all loop S -matrix was fixed up to a scalar factor (*dressing phase*). The study of the dressing phase was initiated in [40]. Further from the assumption that the crossing relation should exist for the string theory on $AdS_5 \times S^5$ the functional equations for the dressing phase were derived in [41]. The solution to this equation was found in [42]. Finally the expression for the dressing phase at weak coupling was conjectured in [43] resulting in corrected asymptotic Bethe ansatz.

The asymptotic Bethe ansatz provided us with ability to compute the spectrum of $\mathcal{N} = 4$ SYM in the limit of large length of the operator, however the implementation of the wrapping corrections still remained an outstanding problem. One of the steps in this direction was undertaken in [44], where, by following the ideas of [45], the authors derived the leading finite-size corrections (Luscher corrections) for the case of the *giant magnon* at strong coupling. However a more efficient way to reach the wrapping corrections turned out to be due to *thermodynamic Bethe ansatz* (TBA) [46]. By exploiting the TBA approach the so-called Y -system, containing the information about exact anomalous dimension of any local operator of $\mathcal{N} = 4$ SYM, was first conjectured [47] and then derived in [48, 49, 50]. The obtained system got the confirmation from the explicit computations [51, 52], which gave the same result as the one predicted by [53], where the Luscher corrections for Konishi operator were computed up to five loops. Nevertheless due to the complexity of the Y -system its applications were significantly restricted. Fortunately it appeared to be possible to turn the infinite set of the integral non-linear equations of Y -system into a finite system [54]. The final form of the equations allowing computation of the exact anomalous dimension were presented in [55], in the form of non-linear system of Riemann-Hilbert equations called the $P\mu$ -system.

Apart from the great achievements in solving the spectral problem, the impressive results were obtained in computing the planar scattering amplitudes, which in $\mathcal{N} = 4$ SYM coincide with the null polygon Wilson loops. In [56] the authors proposed to express the planar scattering amplitudes in terms of some building blocks which they call the *pentagon transition* and which can be computed exactly by means of the bootstrap procedure. The similar approach was recently applied for the computation of the three-point functions. In [57] the authors managed to express the three-point functions in terms of the objects which they called *hexagon*. As well as pentagon transitions, the hexagons happened to be computable exactly by means of integrability. In the section 1.4 of the chapter we will give a historical overview on the achievements concerning the computation of the three-point functions in more details.

The thesis is based on the following works.

1. Y. Jiang, I. Kostov, A. Petrovskii, and D. Serban, String Bits and the Spin Vertex, Nucl. Phys. B897 (2015) 374404, [arXiv:1410.8860].
2. Y. Jiang and A. Petrovskii, From Spin Vertex to String Vertex, JHEP 06 (2015) 172, [arXiv:1412.2256].
3. L. Hollo, Y. Jiang, and A. Petrovskii, Diagonal Form Factors and Heavy-Heavy-

Light Three-Point Functions at Weak Coupling, JHEP 09 (2015) 125, [arXiv:1504.0713]

4. Y. Jiang and A. Petrovskii, Diagonal form factors and hexagon form factors, [arXiv:1511.0619].

1.2 $\mathcal{N} = 4$ SYM

In this section we will review some basic information about the $\mathcal{N} = 4$ SYM. The $\mathcal{N} = 4$ SYM is a maximally supersymmetric gauge theory defined on the four-dimensional Minkowski space. The fields are presented by six scalars, four fermions and the gauge field:

$$\phi_m, \quad \psi_{ia}, \quad \bar{\psi}_{ia}, \quad \mathcal{A}_\mu, \quad (1.2)$$

and interacts through the lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{1}{4} \text{Tr} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} + \frac{1}{2} \text{Tr} \mathcal{D}^\mu \phi_m \mathcal{D}_\mu \phi_m - \frac{g^2}{4} \text{Tr} [\phi_m, \phi_n] [\phi_m, \phi_n] + \text{Tr} \bar{\psi}_{ia} \sigma_\mu^{ab} \mathcal{D}^\mu \psi_{ib} - \\ & - \frac{ig}{2} \text{Tr} \psi_{ia} \sigma_m^{ab} \epsilon^{ij} [\phi_m, \psi_{jb}] - \frac{ig}{2} \text{Tr} \bar{\psi}_{ia} \sigma_m^{ab} \epsilon^{ij} [\phi_m, \bar{\psi}_{bj}]. \end{aligned} \quad (1.3)$$

All the fields live in the adjoint representation of the gauge group $SU(N)$. The scalars form six dimensional representation of the R-symmetry $SU(4)$. The fermions form with respect to one of the indexes the fundamental representations of the R-symmetry and with respect to another the representation of the $su(2)_L$ (or $su(2)_R$) copy of the Lorentz group.

The bosonic symmetry of the theory is presented by the product of the conformal group in four dimensions $SO(4, 2)$ and the R-symmetry $SU(4)$, which are formed by the following generators: Lorentz generators L_{ij} , \bar{L}_{ij} , shifts P_μ , special conformal transformations K_μ , the dilatation operator D and fifteen R-symmetry generators R_{ij} . The supersymmetric transformations are given by 16 supercharges Q_{ia} , \bar{Q}_{ia} , S_{ia} , \bar{S}_{ia} . Together all the generators form $PSU(2, 2|4)$ group, the complete symmetry of the theory. All the symmetries still hold at the quantum level including the conformal symmetry, due to the fact that β -function is identically equal to zero.

Since the symmetry group is non-compact, the unitary representations, which a physicist is interested in, are infinite-dimensional. The physical representations are defined on the space spanned by the local gauge invariant operators (in other words, states) in general presented by a product of the *single-trace* operators

$$\mathcal{O}(x) = \text{Tr} W_1(x) \dots W_L(x), \quad (1.4)$$

where by $W_i(x)$ we understand the fields transforming homogeneously under the gauge group, such as $\mathcal{D}^k \phi$, $\mathcal{D}^k \psi$, $\mathcal{D}^k \bar{\psi}$, $\mathcal{D}^k \mathcal{F}_{\mu\nu}$, $k=0,1,2,\dots$. In what follows we will refer to the fields W_i as *elementary* fields.

In order to form a highest weight representation we will define the *primary* states $\mathcal{O}(x)$, as those which are annihilated at the point $x = 0$ by the special conformal generators:

$$K_\mu \mathcal{O}(0) = 0. \quad (1.5)$$

Then the full representation module can be formed by successive action of the lowering operators on the highest weight state $\mathcal{O}(x)$. The operators which are not primaries are called *descendants* with respect to the primary operator they are obtained from.

Since the theory is conformal, the particles are massless and, in the conventional sense, there is no spectral problem. However there is an analog of it, namely the dilatation eigenvalue Δ of a local operator $\mathcal{O}(x)$, or, in other words, its *conformal* dimension. On the classical level it is given just by the sum of dimensions of all the fields the operator is made of. When the interaction is switched on the conformal dimension gets the correction γ which is called an *anomalous* dimension. We will refer to the problem of computing the anomalous dimension of an operators as a spectral problem. As it has been already mentioned we are going to work in the planar limit of $\mathcal{N} = 4$ SYM, when the number of color N is taken to be large, so we are interested only in the leading order in the expansion of the observables in $1/N$. In the planar limit the conformal dimension of the multi-trace operator is given by the sum of its single-trace operators. Therefore one can consider just single-trace operators.

Due to the conformal symmetry of the theory, the expression of the properly normalized two-point correlator is completely defined by its conformal dimension. E.g. in the case of the primary scalar operators the two-point correlation functions is given as follows

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{\delta_{ij}}{(x-y)^{2\Delta_i}}. \quad (1.6)$$

Thus the spectral problem is equivalent to the problem of computing the two-point correlation function of the corresponding operator.

A significant simplification in the spectral analysis can be achieved by noticing that the action of the dilatation operator is closed on certain subsets of the states up to some loop level or sometimes even exactly. These subsets are referred to as *sectors* (or *subsectors*) of the theory. The simplest example is the $su(2)$ sector consisting of the two elementary fields related to each other by one of the $su(2)$ transformations of the symmetry, e.g. $Z = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)$ and $X = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. The $su(2)$ sector is closed in all orders in perturbation theory. At tree level the action of the dilatation operator on it coincides with the Hamiltonian of the Heisenberg $XX_{1/2}$ spin chain.

Another example we will be considering in this thesis is $so(6)$ sector, or scalar sector, presented by all the scalar fields $Z, \bar{Z}, X, \bar{X}, Y, \bar{Y}$ forming the fundamental representation of $so(6)$ group and given as follows

$$\begin{aligned} Z &= \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6), & \bar{Z} &= \frac{1}{\sqrt{2}}(\phi_5 - i\phi_6), \\ X &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), & \bar{X} &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2), \\ Y &= \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4), & \bar{Y} &= \frac{1}{\sqrt{2}}(\phi_3 - i\phi_4). \end{aligned} \quad (1.7)$$

The scalar sector is closed with respect to the action of one loop dilatation operator.

Other observables of the theory appear to be the rest of n -point correlation functions. Among them the three-point functions draws particular attention. The reason is that the *operator product expansion* (OPE) appears to be an exact statement in conformal field theory. The OPE states that when two operators taken within a correlation function are defined on the close enough space-time points, their product can be replaced by a sum of the local operators with the coefficients called the structure constants

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(x + \delta x) \dots \rangle = \langle \sum_k C_{12k} e_{12k}(\delta x) \mathcal{O}_k(x) \dots \rangle, \quad (1.8)$$

where e_{12k} are the conformal partial waves defined by the operators involved. Therefore by successively applying the OPE to an arbitrary n -point correlator, one can reduce it to a sum of two-point functions. At the same time it is not hard to see that the structure constants are given by the three-point functions. This means that all the information about the observables of the theory is encoded in two- and three-point correlators.

The form of the three-point correlators is fixed by the conformal symmetry as well, for scalar primary operators it is given as follows

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z) \rangle = \frac{C_{123}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|x-z|^{\Delta_1+\Delta_3-\Delta_2}|y-z|^{\Delta_2+\Delta_3-\Delta_1}}, \quad (1.9)$$

where C_{123} is a structure constant.

1.3 Basic tools of integrability

In this section we will review basic tools of integrability which we will be using in the following.

1.3.1 $XXX_{1/2}$ spin chain. Coordinate Bethe Ansatz

In this section we will describe the method of coordinate Bethe-ansatz allowing the diagonalization of the $XXX_{1/2}$ spin chain corresponding to the $su(2)$ sector of $\mathcal{N} = 4$ SYM. For a more detailed review we recommend to see [58].

The $XXX_{1/2}$ spin chain model is defined on the Hilbert space \mathcal{H} :

$$\mathcal{H} = h_1 \otimes \dots \otimes h_L \quad (1.10)$$

where $h_k = \mathbb{C}^2$, and has the following Hamiltonian:

$$H = \sum_{\alpha,k} (1 - \sigma_k^\alpha \sigma_{k+1}^\alpha). \quad (1.11)$$

Here σ_k^α are Pauli matrices and the periodicity condition is implied: $\sigma_{L+1}^\alpha = \sigma_1^\alpha$.

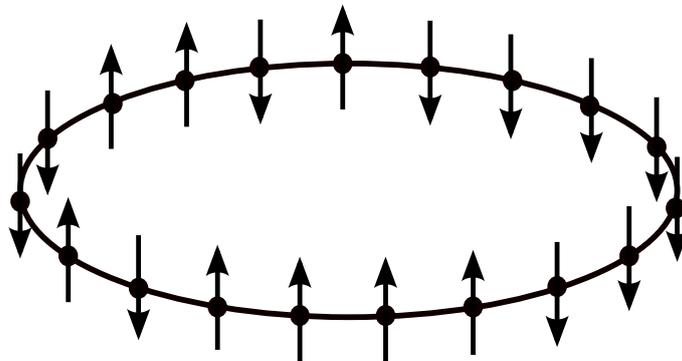


Figure 1.1 – The spin chain.

We will refer to a subspace h_k as the k -th site of the spin chain. For the basis of a one-site space h_k we choose

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1.12)$$

The full space is spanned by the states of the form

$$|\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow \dots \downarrow\uparrow\uparrow\rangle, \quad (1.13)$$

with the possibility for each site to have either spin *up* or *down*. We define the vacuum to be

$$|\Omega\rangle = |\uparrow \dots \uparrow\rangle. \quad (1.14)$$

It is not hard to see that it is an eigenstate of the Hamiltonian (1.11). We will denote the excited states with N spins *down* at positions n_1, \dots, n_N as

$$|n_1, \dots, n_N\rangle. \quad (1.15)$$

The spin operators $S^\alpha = \frac{1}{2} \sum_{k=1}^L \sigma_k^\alpha$ commutes with the Hamiltonian, thus these operators can be diagonalized simultaneously. Accordingly the ansatz for the Hamiltonian eigenstates with the total spin $S = \frac{L-2N}{2}$ should have the form

$$|\psi_N\rangle = \sum_{1 \leq n_1 \dots \leq n_N \leq L} f(n_1, \dots, n_N) |n_1, \dots, n_N\rangle, \quad (1.16)$$

where $f(n_1, \dots, n_N)$ is a factor to be defined. The solution to this problem is given by the coordinate Bethe ansatz [9] given as follows

$$\begin{aligned} |\psi_N\rangle = & \sum_{1 \leq n_1 \dots \leq n_N \leq L} \left(e^{ip_1 n_1 + ip_2 n_2 + ip_3 n_3 + \dots + ip_N n_N} + S(p_2, p_1) e^{ip_2 n_1 + ip_1 n_2 + ip_3 n_3 + \dots + ip_N n_N} + \right. \\ & \left. S(p_2, p_1) S(p_3, p_1) e^{ip_2 n_1 + ip_3 n_2 + ip_1 n_3 + \dots + ip_N n_N} + \dots \right) |n_1, \dots, n_N\rangle, \end{aligned} \quad (1.17)$$

with p_1, \dots, p_N satisfying the condition (Bethe equations):

$$e^{ip_k L} = \prod_{j \neq k} S(p_k, p_j), \quad (1.18)$$

where

$$S(p_k, p_j) = -\frac{e^{i(p_k + p_j)} - 2e^{ip_k} + 1}{e^{i(p_k + p_j)} - 2e^{ip_j} + 1}. \quad (1.19)$$

The eigenstates (1.17) are conventionally called Bethe-states. The ansatz (1.17) allows the following simple physical interpretation. In analogy with a plane wave we associate the state $e^{ipn} |n\rangle$, to which we shall refer as to *one magnon* state, with a free particle moving around the circle with a momentum p . Further the scattering is diagonal and absolutely elastic which gives that the wave function should be given by a superposition of the states of all possible permutations of the particles times a phase factor composed of the product of the corresponding set of the S -matrices $S(p, q)$. Since the system is defined in finite space, the momenta should be quantized, which is expressed by the Bethe equations (1.18).

1.3.2 $XXX_{1/2}$ spin chain. Algebraic Bethe Ansatz

In this section we will describe the algebraic Bethe-ansatz [10] serving as an alternative way for diagonalizing spin chain Hamiltonians. The idea of the algebraic Bethe-ansatz is based on introducing a connection along the spin chain, which we shall conventionally call Lax operator. The Lax operator is defined on the product of one-site quantum space h and an auxiliary space A isomorphic to a quantum space ($A = \mathbb{C}^2$) and has the form

$$L_{k,a}(u) = (u - \theta_k)\mathbb{I}_k \otimes \mathbb{I}_a + i \sum_{\alpha} S_k^{\alpha} \otimes \sigma_a^{\alpha}, \quad (1.20)$$

or

$$L_{k,a}(u) = \begin{pmatrix} u - \theta_k + iS_k^3 & iS_k^- \\ iS_k^+ & u - \theta_k - iS_k^3 \end{pmatrix}, \quad (1.21)$$

where $S^i = \frac{\sigma^i}{2}$, u is a complex variable and θ 's are the so-called inhomogeneities. In order to get the homogeneous $XXX_{1/2}$ spin chain corresponding to the one loop dilatation operator of the $su(2)$ sector we need to put the inhomogeneities to zero. The Lax-operator also can be rewritten in terms of the permutation operator $P_{a,b} \phi_a^i \otimes \phi_b^j = \phi_a^j \otimes \phi_b^i$ as

$$L_{k,a}(u) = (u - \theta_k - \frac{i}{2})I_{k,a} + iP_{k,a}, \quad (1.22)$$

$$P_{a,b} = \frac{1}{2} \left(\mathbb{I}_a \otimes \mathbb{I}_b + \sum_{\alpha} \sigma_a^{\alpha} \otimes \sigma_b^{\alpha} \right). \quad (1.23)$$

It is not hard to see that the Lax operator satisfies Yang-Baxter equation:

$$R_{a,b}(u-v)L_{k,a}(u)L_{k,b}(v) = L_{k,b}(v)L_{k,a}(u)R_{a,b}(u-v), \quad (1.24)$$

where $R_{a,b}$ is called R -matrix and given as follows

$$R_{a,b}(u) = (u - \theta_k)I_{a,b} + iP_{a,b}. \quad (1.25)$$

We then construct the monodromy matrix $T(u)$ as follows:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = L_{1,a}(u)L_{2,a}(u)\dots L_{L,a}(u). \quad (1.26)$$

From the Yang-Baxter equation (1.24) and from the property $[L_{k_1,a}(u), L_{k_2,b}] = 0$ ($k_1 \neq k_2$, $a \neq b$) one can derive the so-called RTT relation

$$R_{a,b}(u-v)T_{k,a}(u)T_{k,b}(v) = T_{k,b}(v)T_{k,a}(u)R_{a,b}(u-v). \quad (1.27)$$

It can be written elementwise as follows

$$\begin{aligned}
A(v)B(u) &= f(u-v)B(u)A(v) + g(v-u)B(v)A(u), \\
B(v)A(u) &= f(u-v)A(u)B(v) + g(v-u)A(v)B(u), \\
D(v)B(u) &= f(v-u)B(u)D(v) + g(u-v)B(v)D(u), \\
B(v)D(u) &= f(v-u)D(u)B(v) + g(u-v)D(v)B(u), \\
C(v)A(u) &= f(v-u)A(u)C(v) + g(u-v)A(v)C(u), \\
A(v)C(u) &= f(v-u)C(u)A(v) + g(u-v)C(v)A(u), \\
C(v)D(u) &= f(u-v)D(u)C(v) + g(v-u)D(v)C(u), \\
D(v)C(u) &= f(u-v)C(u)D(v) + g(v-u)C(v)D(u), \\
[C(v), B(u)] &= g(u-v) [A(v)D(u) - A(u)D(v)] = g(u-v) [D(u)A(v) - D(v)A(u)], \\
[D(v), A(u)] &= g(u-v) [B(v)C(u) - B(u)C(v)] = g(u-v) [C(u)B(v) - C(v)B(u)], \\
[A(u), A(v)] &= [B(u), B(v)] = [C(u), C(v)] = [D(u), D(v)] = 0,
\end{aligned} \tag{1.28}$$

where $f(u) = 1 + \frac{i}{u}$, $g(u) = \frac{i}{u}$. From the RRT relation we immediately get that

$$[\mathcal{T}(u), \mathcal{T}(v)] = 0, \tag{1.29}$$

where $\mathcal{T}(u)$ is the transfer matrix defined as the trace of the monodromy matrix over the auxiliary space $\mathcal{T}(u) = \text{Tr}_a T(u)$. By performing some manipulations with the permutation operator $P_{a,b}$, it is not hard to express the Hamiltonian as follows

$$H = L + i \frac{d\mathcal{T}(u)}{du} \mathcal{T}(u)^{-1} \Big|_{u=i/2}, \tag{1.30}$$

Hence

$$[H, \mathcal{T}(u)] = 0. \tag{1.31}$$

Thus we can see that by expanding the transfer matrix as a polynomial of u , we get L conserved charges, which coincide with number of degrees of freedom, giving rise to the integrable structure of the $XXX_{1/2}$ spin chain.

The main insight of the algebraic Bethe ansatz is that the construction of the Hamiltonian eigenstates (Bethe-states) is performed by successive action of the element $B(u)$ of the monodromy matrix on the ferromagnetic vacuum (1.14) :

$$|u_1, \dots, u_N\rangle = B(u_1) \dots B(u_N) |\Omega\rangle. \tag{1.32}$$

The values u_1, \dots, u_N are called rapidities. In case of the $XXX_{1/2}$ spin chain they are connected to the particles momenta by the expression

$$e^{ip} = \frac{u + i/2}{u - i/2}. \tag{1.33}$$

The rapidities satisfy the Bethe equations:

$$\frac{a(u_k)}{d(u_k)} = \prod_{j \neq k}^N \frac{u_k - u_j + i}{u_k - u_j - i}, \tag{1.34}$$

In the following we will be using the expressions of how the monodromy matrix elements act on the eigenstates of the Hamiltonian. They can be derived from the algebra (1.28):

$$\begin{aligned}
A(v)|\mathbf{u}\rangle &= a(v)\frac{Q_{\mathbf{u}}(v-i)}{Q_{\mathbf{u}}(v)}|\mathbf{u}\rangle + \sum_{n=1}^N M_n(v)|\{\mathbf{u}, v\} \setminus \{u_n\}\rangle, \\
D(v)|\mathbf{u}\rangle &= d(v)\frac{Q_{\mathbf{u}}(v+i)}{Q_{\mathbf{u}}(v)}|\mathbf{u}\rangle + \sum_{n=1}^N N_n(v)|\{\mathbf{u}, v\} \setminus \{u_n\}\rangle, \\
C(v)|\mathbf{u}\rangle &= \sum_{n=1}^N K_n|\{\mathbf{u}\} \setminus \{u_n\}\rangle + \sum_{k>n} K_{kn}|\{\mathbf{u}, v\} \setminus \{u_k, u_n\}\rangle,
\end{aligned} \tag{1.35}$$

where $Q_{\mathbf{u}}(u) = \prod_{k=1}^N (u - u_k)$, $a(u) = \prod_{k=1}^L (u - \theta_k + i/2)$, $d(u) = \prod_{k=1}^L (u - \theta_k - i/2)$ and

$$\begin{aligned}
M_n(v) &= \frac{ia(u_n)}{v - u_n} \prod_{j \neq n}^N \frac{u_n - u_j - i}{u_n - u_j}, \\
N_n(v) &= \frac{id(u_n)}{u_n - v} \prod_{j \neq n}^N \frac{u_n - u_j + i}{u_n - u_j}. \\
K_n &= \frac{ia(v)d(u_n)}{u_n - v} \prod_{j \neq n}^N \frac{u_j - u_n - i}{u_j - u_n} \cdot \frac{u_j - v + i}{u_j - v} + \\
&\quad \frac{ia(u_n)d(v)}{v - u_n} \prod_{j \neq n}^N \frac{u_j - u_n + i}{u_j - u_n} \cdot \frac{u_j - v - i}{u_j - v} \\
K_{kn} &= \frac{d(u_k)a(u_n)}{(u_k - v)(u_n - v)} \frac{u_k - u_n + i}{u_k - u_n} \prod_{j \neq k, n} \frac{u_j - u_k - i}{u_j - u_k} \cdot \frac{u_j - u_n + i}{u_j - u_n} + \\
&\quad \frac{d(u_n)a(u_k)}{(u_n - v)(u_k - v)} \frac{u_k - u_n - i}{u_k - u_n} \prod_{j \neq k, n} \frac{u_j - u_k + i}{u_j - u_k} \cdot \frac{u_j - u_n - i}{u_j - u_n}.
\end{aligned} \tag{1.36}$$

By writing the state $|\mathbf{u} \setminus \{u_{i_1}, \dots, u_{i_k}\}\rangle$ we mean that the rapidities u_{i_1}, \dots, u_{i_k} should be removed from the list \mathbf{u} .

Finally from (1.35) it is not hard to derive the expression for the transfer matrix eigenvalue:

$$\mathcal{T}(u)|u_1, \dots, u_N\rangle = (A(u) + D(u))|u_1, \dots, u_N\rangle = t_{\mathbf{u}}(u)|u_1, \dots, u_N\rangle, \tag{1.37}$$

$$t_{\mathbf{u}}(u) = a(u)\frac{Q_{\mathbf{u}}(u-i)}{Q_{\mathbf{u}}(u)} + d(u)\frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u)}. \tag{1.38}$$

1.3.3 The gauge theory operators and the spin chains

In order to apply the integrability technique of the spin chains to $\mathcal{N} = 4$ SYM one makes identification between the gauge invariant operators of the field theory with the

spin chains. For example in the case of $su(2)$ sector the operators consist of the two fields connected by the $su(2)$ transformation, let's say Z and X . Then we identify the spin up site of the spin chain with the field Z , and spin down site with the field X :

$$\uparrow \rightarrow Z, \quad \downarrow \rightarrow X. \quad (1.39)$$

The state $\text{Tr}(Z^L)$ then is considered as a vacuum state, and the impurities X as excitations put on top of it. Under the following identification the action of the one loop dilatation operator on the operators from $su(2)$ sector coincide with the Hamiltonian of the $XX_{1/2}$ spin chain (1.11). So we can construct an eigenstate of the dilatation operator by use of the Bethe ansatz discussed above. It is worth mentioning that besides the Bethe equations (1.18), the momenta p_i parameterizing the Bethe states should satisfy the so-called *zero momentum condition* coming from the cyclicity of the trace:

$$e^{i(p_1+\dots+p_N)} = 1. \quad (1.40)$$

1.3.4 Scalar product of the Bethe-states

In this section we will briefly review the issue of a scalar product of two Bethe-states, which is extensively used in the computation of the three-point functions. In what follows we will refer to the states of the form (1.32) as to the Bethe-states, among which we will distinguish *on-shell* Bethe-states (with rapidities u_1, \dots, u_N satisfying the Bethe equations (1.34)) and *off-shell* Bethe-states (with rapidities being an arbitrary set of the complex numbers).

Due to the property

$$B(u)^\dagger = -C(u^*), \quad (1.41)$$

for every Bethe-state $B(u_1)\dots B(u_N)|\Omega\rangle$ one can define a *dual* Bethe-state by

$$\langle u_1, \dots, u_N | = (-1)^N \langle \Omega | C(u_1^*) \dots C(u_N^*), \quad (1.42)$$

where we have used the fact that $C(u)$'s commute with each other. The set of the Bethe roots \mathbf{u} possess the property to turn into itself under the action of the complex conjugation: $\mathbf{u}^* = \mathbf{u}$. It means that the definition of the dual states the signs of the complex conjugation can be omitted:

$$\langle u_1, \dots, u_N | = (-1)^N \langle \Omega | C(u_1) \dots C(u_N). \quad (1.43)$$

Now we can define a scalar product between two Bethe-states as $\langle \mathbf{u} | \mathbf{v} \rangle$. In this thesis we will be interested in the absolute value of the scalar product meaning that the factor of $(-1)^N$ can be dropped out:

$$\langle \mathbf{u} | \mathbf{v} \rangle = \langle \Omega | \prod_{j=1}^N C(u_j) \prod_{k=1}^N B(v_k) | \Omega \rangle. \quad (1.44)$$

The simplest case of the scalar product (1.44) correspond to the option when the sets \mathbf{u} and \mathbf{v} coincide. Then the scalar product reduces to the computation of the Bethe-state norm:

$$\langle \mathbf{u} | \mathbf{u} \rangle = \langle \Omega | \prod_{j=1}^N C(u_j) \prod_{k=1}^N B(u_k) | \Omega \rangle. \quad (1.45)$$

The compact expression of the norm for the on-shell Bethe-states was obtained by Gaudin in [59]:

$$\langle \mathbf{u} | \mathbf{u} \rangle = \left(\prod_{j=1}^N a(u_j) d(u_j) \prod_{j < k}^N \frac{1 + (u_j - u_k)^2}{(u_j - u_k)^2} \right) \rho_N(\{1, \dots, N\}), \quad (1.46)$$

with

$$\rho_N(\{1, \dots, N\}) = \left| \frac{\partial \Phi_k(\mathbf{u})}{\partial u_j} \right|, \quad (1.47)$$

where

$$\Phi_k(\mathbf{u}) = p(u_k)L - i \sum_{j \neq k}^N \log S(u_j, u_k). \quad (1.48)$$

Here $p(u)$ and $S(u, v)$ are the particle momentum and S -matrix correspondingly.

Another case we are interested in is when one of the states is an on-shell and another is off-shell Bethe-states. Then the corresponding expression is given by Slavnov determinant [60]:

$$\langle \mathbf{u} | \mathbf{v} \rangle = \prod_{j=1}^N a(v_j) d(u_j) \mathcal{S}_{\mathbf{u}, \mathbf{v}}, \quad (1.49)$$

where

$$\mathcal{S}_{\mathbf{u}, \mathbf{v}} = \frac{\det_{jk} \Omega(u_j, v_k)}{\det_{jk} \frac{1}{u_j - v_k + i}} \quad (1.50)$$

The matrix element $\Omega(u_j, v_k)$ is given by

$$\begin{aligned} \Omega(u_j, v_k) &= t(u_j - v_k) - e^{2ip_{\mathbf{u}}(v_k)} t(v_k - u_j) \\ &= i \frac{(u_j - v_k - i) - (u_j - v_k + i) e^{2ip_{\mathbf{u}}(v_k)}}{(u_j - v_k)[(u_j - v_k)^2 + 1]} \end{aligned} \quad (1.51)$$

where $p_{\mathbf{u}}(v)$ is the so-called pseudomomentum

$$e^{2ip_{\mathbf{u}}(u)} \equiv \frac{d(u) Q_{\mathbf{u}}(u + i)}{a(u) Q_{\mathbf{u}}(u - i)}. \quad (1.52)$$

and $t(u)$ is

$$t(u) = \frac{1}{u} - \frac{1}{u + i}. \quad (1.53)$$

1.3.5 Quantum inverse scattering problem

An important issue for computing the form factors of local spin operators, which is relevant to our thesis and will be considered in the chapter 3 is to establish the relation between the local spin operators σ^i and the elements of the monodromy matrix A, B, C, D , which action on the Bethe-states are given by the known expressions listed in the previous

section. This problem is referred to as Quantum Inverse Scattering Problem (QISP) which solution we will review in this section.

For the convenience we will define the operators in the quantum space $h = \mathbb{C}^2$

$$E_{ij}^{ab} = \delta_i^a \delta_j^b. \quad (1.54)$$

Then the solution to QISP can be written as follows

$$E_n^{ab} = \left\{ \prod_{k=1}^{n-1} \mathcal{T}(\theta_k + i/2) \right\} T^{ab}(\theta_n + i/2) \left\{ \prod_{k=1}^n \mathcal{T}(\theta_k + i/2) \right\}^{-1} \quad (1.55)$$

where

$$T^{11}(u) = A(u), \quad T^{12}(u) = B(u), \quad T^{21}(u) = C(u), \quad T^{22}(u) = D(u), \quad (1.56)$$

and the indexes k, n stand for the site numbers.

1.4 Three-point functions and integrability

In this section we will briefly review some results on the three-point functions in $\mathcal{N} = 4$ SYM.

The first computations of the three-point functions was restricted by the simplest configurations involving just half-BPS operators, which don't receive any quantum corrections [61, 62, 63, 64]. As it was expected the precise matching between string and gauge theory sides was obtained.

As in the case with the spectral problem, a lot of attention was drawn by the BMN limit. The comparison between gauge and string sides of the duality became possible after starting from [65] the string vertex describing the interactions between the strings in the pp-wave background was built. The comparison with weak coupling gave precise match at the leading order in the BMN limit (see e.g. [66]). We will talk in more details about the string vertex in pp-wave string theory in the chapter 2.

A systematic approach to the computation of the three-point functions from integrability at weak coupling was initiated in [67], where the authors considered the non-extremal⁴ three-point functions with the operators each of which belongs to one of the $su(2)$ sectors of $\mathcal{N} = 4$ SYM at tree level. Here it is worth mentioning that due to the trivial action of the tree level dilatation operator leading to high degeneracy of the spectrum, in order to compute a structure constant at tree level, one has to consider the eigenstates of the one loop dilatation operator. The main result which the authors of [67] obtained was to reduce the computation of the structure constant to the computation of the scalar products between off-shell Bethe states. It was achieved in two steps. The first, *cutting*, is the representation of the on-shell Bethe states, corresponding to the gauge theory operators, as a sum of products of two off-shell Bethe states. The second, *sewing*, performing the contractions between the operators by means of the scalar product between the corresponding off-shell Bethe states. Unlike to the case of the on-shell/off-shell scalar products, discussed in the section (1.3.4), off-shell/off-shell scalar products can not be

4. *Non-extremal* three-point functions correspond to the case when each operator is contracted with other two. When two of the operators are not contracted the three-point correlator is called *extremal*

presented in the compact determinant-like form. Nevertheless the authors proposed some recursive way of computing the off-shell/off-shell scalar products quite efficiently.

The result of [67] was improved in [68], where the author managed to obtain a compact determinant-like form for the structure constant by exploiting some tricks based on the analogy of the problem to the computation of the partition function of the six vertex model.

Further these results were extended to one loop in [69], where the authors used the results of the explicit computations of the Feynman diagrams for the case of the operators from the scalar sector $so(6)$ performed in [70, 71]. The final expression for the one loop structure constant again was presented in the determinant-like form, however much more complicated than in tree level. Later an alternative way of computing one loop structure constant allowing to get the semi-classical limit in a straightforward fashion was proposed in [72].

Another way to extend the method of [67] was considered in [73], where the tree level structure constant of the operators belonging to the $su(3)$ sector was examined. In some cases the authors was able to obtain the result in the determinant form. Further the method was also generalized at one loop to the $sl(2)$ sector⁵ in [74] and $su(1|1)$ sector in [75].

A different approach to the three-point correlators was proposed in [76, 77]. In these works the authors considered the so-called *heavy-heavy-light* (HHL) correlators. This configuration assumes that the two of the operators (the heavy operators) are taken to be very large compared to the third one (the light operator). In this sense HHL correlator is close to the two-point function of the heavy operators, since the influence of the third operator is small. By considering the third operator as supergravity modes emitted by the classical string corresponding to the heavy operators the authors of [76, 77] computed some HHL correlators at string theory side. The precise match of the string theory computation for HHL configuration with the weak coupling computation in $\mathcal{N} = 4$ SYM was obtained in [78], where the authors approximated the heavy states in the classical limit by the coherent states of the Landau-Lifshitz model (see [79, 80] for more details). HHL correlators is one of the main interest of this thesis and will be discussed further in the chapter 3.

A lot of works are devoted to the three-point functions in the semi-classical limit, when all three of the operators are identified with some classical string solutions. At weak coupling the three-point correlator in the classical limit was first considered in [81], where two of the operators were taken to be half-BPS and the third to be non-BPS, all from the $su(2)$ sector. Further this result was generalized to the case with three non-BPS operators in [82, 83]. In [72] this result was extended to one loop. The semi-classical limit of the three-point correlators with operators belonging to the $su(3)$ sector was considered in [73]. The computation in the semi-classical limit is interesting in the first place due to the possibility of comparison with the string theory computations, initiated in [84] and then further developed in [85, 86, 87, 88], giving the desired match between both sides of the duality [89, 90].

An alternative approach to the three-point functions in $\mathcal{N} = 4$ SYM is due to the *spin vertex*, which was inspired by the string field theory. The idea is to build an object for computing the three-point correlators which would play the same role as the string vertex

5. The configuration considered in [74] was restricted to the case with only one operator being from the $sl(2)$ sector. Other two were taken to be half-BPS states from the $su(2)$ sector

in the string field theory approach. This kind of object was first constructed in [91] and then developed in [92], where the Yangian invariance of the spin vertex was realized. The second chapter of this thesis is devoted to the spin vertex approach.

Quite recently in [57] the authors proposed an all loop based on integrability approach called *the hexagon program* for computing the non-extremal three-point functions suitable in principle for any three-point correlator. We will describe this method in more details in the chapter 3, where we apply it for the computation of the HHL correlators.

Finally what we would like to mention here is *the separation of variables* (SoV) approach due to Sklyanin [93, 94]. An interesting feature of this method is that in the basis of the separated variables the wave functions of the dilatation operator eigenstates are defined by the Q -functions, which appears to be the basic object in the quantum spectral curve approach. In this sense the SoV can provide us with a link between quantum spectral curve technique and three-point functions. Some progress in accommodating the SoV technique for computing the three-point correlators was made in [95, 96, 97], however the results are at the moment restricted to the tree level approximation.

Chapter 2

The spin vertex

In this chapter we will consider an approach for computing the three-point functions in $\mathcal{N} = 4$ SYM inspired from the string field theory (SFT). We will review the string field theory and application of these approach to the plane wave limit of the type IIB string theory defined on the $AdS_5 \times S^5$ background. The object describing the interaction between strings is called string vertex. We will construct its analog, which we call spin vertex, in the BMN limit of the $\mathcal{N} = 4$ SYM theory. Then we will show that the computation of the tree level three-point functions in the BMN limit based on the spin vertex gives precisely the same result as the computation on the string theory side. Then we will construct the spin vertex at tree level of $\mathcal{N} = 4$ SYM for any kind of the operators and show that it is invariant with respect to an infinite number of the conserved charges.

2.1 Light-cone string field theory on pp-wave background

The SFT realizes the idea of describing the string interactions in the same way as quantum field theory does for the particles. The light-cone string field theory is the first SFT which appeared. It was introduced and developed in [98, 99, 100, 101, 102, 103] and then generalized to superstrings in [104, 105, 106]. Due to an obvious drawback - breakdown of the Lorentz invariance - the light-cone SFT was abandoned for some time in favor of the covariant approach (see e.g. [107]). However the AdS/CFT correspondence drew new attention to the subject after the discovery of the pp-wave background as a special limit of $AdS_5 \times S^5$ space [4]. The reason is the particular simplicity of the string theory on the pp-wave background in the light-cone gauge. In this section we will first briefly describe how to get the pp-wave background by taking the Penrose limit of $AdS_5 \times S^5$ space and also the gauge theory operators in the BMN limit dual to the string states defined on the pp-wave space. Further we will discuss the main concepts of the light-cone SFT and in particular the light-cone SFT on the pp-wave space.

2.1.1 PP-wave background and the BMN limit of the $\mathcal{N} = 4$ SYM

The metric of $AdS_5 \times S^5$ is given as follows

$$\frac{ds^2}{R^2} = -dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + d\psi^2 \cos^2 \theta + d\theta^2 + \sin^2 \theta d\Omega_3^2, \quad (2.1)$$

where $\rho = 0$ is the center of the AdS_5 and $\rho = \infty$ is the boundary, and ψ is the coordinate along the equator of the S^5 . The pp-wave background can be obtained by taking a special kind of Penrose limit of the $AdS_5 \times S^5$ space [4, 5]. One can imagine this limit by considering a particle in the vicinity of the center of AdS moving with a speed close to the speed of light along the equator of the S^5 . In order to perform the limit it's convenient to rescale the coordinates as follows

$$x^+ = \tilde{x}^+, \quad x^- = R^2 \tilde{x}^-, \quad \rho = \frac{r}{R}, \quad \theta = \frac{y}{R}, \quad (2.2)$$

where $\tilde{x}^\pm = \frac{1}{2}(t \pm \psi)$, and after to take the limit $R \rightarrow \infty$. Then the metric will take the form

$$ds^2 = -4dx^+ dx^- - \mu^2 x^2 (dx^+)^2 + dx_I dx^I, \quad (2.3)$$

where $I = 1, \dots, 8$. The parameter μ can be eliminated by rescaling $x^\pm \rightarrow x^\pm \mu^{\pm 1}$.

The energy and the angular momentum in global coordinates of the string theory background are given as $E = i\partial_t$, $J = -i\partial_\psi$ respectively. In terms of CFT the energy corresponds to the conformal dimension of the operator. The angular momentum will correspond to one of the components of the R -charge. By choosing one of the components we break the R -symmetry from $SO(6)$ to $SO(2) \times SO(4)$, which on the string theory side correspond to choosing the plane in which the particle is moving. The relation between the string and gauge magnitudes is given as follows

$$\begin{aligned} 2p^- &= -p_+ = i\partial_{x^+} = i\partial_{\tilde{x}^+} = i(\partial_t + \partial_\psi) = \Delta - J \\ 2p^+ &\equiv -p_- = -\frac{\tilde{p}_-}{R^2} = \frac{1}{R^2} i\partial_{\tilde{x}^-} = \frac{1}{R^2} i(\partial_t - \partial_\psi) = \frac{\Delta + J}{R^2}. \end{aligned} \quad (2.4)$$

The states surviving in the Penrose limit should have the finite light-cone energy and the light-cone momentum. The relation (2.4) means that the corresponding operators on the gauge theory side should have finite value of $\Delta - J$, and $\Delta + J$ should be of order of R^2 , where R is considered to be large. These conditions define the BMN operators [3].

In order to write the BMN operators explicitly, we have to first choose the vacuum state which fixes the R -charge component of the BMN operator. Conventionally we choose the vacuum state to be $\text{Tr} Z^J$, where $Z = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)$. This is a half-BPS state, which has conformal dimension equal to its classical value $\Delta_0 = J$. In order to build the excited states we insert some other fields between the vacuum fields Z , such as four scalars ϕ_i , $i = 1, \dots, 4$, fermions $\psi_{a\alpha}$, $\bar{\psi}_{a\alpha}$, derivatives \mathcal{D}_μ , $\mu = 1, \dots, 4$ of the scalars and fermions, or by considering the derivatives of the vacuum fields Z itself. So we get the operators of the following form

$$\text{Tr}(ZZ\mathcal{D}_\mu^k \phi_i ZZ \dots Z\mathcal{D}_\nu^l \psi_{a\alpha} ZZ \dots \mathcal{D}_\rho^m ZZZ). \quad (2.5)$$

In order to keep the operator in the BMN regime we have to take the number of the excitations much smaller than the number of the vacuum fields $J \sim \sqrt{N}$. This condition is referred to as *dilute gas* approximation.

Finally, in the BMN limit other expansion parameters appear

$$\lambda' = \frac{g_{YM}^2 N}{J^2} = \frac{1}{(\mu p^+ \alpha')^2}, \quad \frac{J^2}{N} = 4\pi g_s (\mu p^+ \alpha')^2, \quad (2.6)$$

with $\alpha' = \frac{1}{2\pi T}$. From this we can see that in the BMN limit we can have 't Hooft constant λ to be in principle of any value covering the range from weak to strong coupling keeping at the same time λ' to be small. This allows to compare the observables from the both sides of the duality explicitly.

2.1.2 The concept of the SFT

The string field theory is the formulation of string theory where the main object is the field operator Φ which creates and destroys strings. It has been formulated in light-cone gauge for flat [104, 105, 106] and pp-wave [65, 108, 109, 110] backgrounds.

The field operator Φ acts in the Hilbert space \mathcal{H} :

$$\mathcal{H} = |vac\rangle \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots, \quad \mathcal{H}_n = \underbrace{\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_1}_n \quad (2.7)$$

$$\Phi : \mathcal{H}_n \rightarrow \mathcal{H}_{n\pm 1},$$

(\mathcal{H}_1 is a Hilbert space of one string) and can be expanded in the sum of creation/annihilation operators as follows (the fermionic part is suppressed for the sake of simplicity):

$$\Phi(p(\sigma)) \sim \sum_{\vec{N}} A_{\vec{N}}(p^+) \prod_{n=-\infty}^{\infty} \psi_{N_n}(p_n). \quad (2.8)$$

Here $A_{\vec{N}}(p^+)$ is an operator which creates ($p^+ < 0$) or annihilates ($p^+ > 0$) string with the set of excitations given by the occupation vector \vec{N} (each component of \vec{N} indicates the excitation level of corresponding string mode) and $\psi_{N_n}(p_n)$ is just a harmonic oscillator wave-function in momentum representation with the excitation level N_n .

The Hamiltonian of SFT can be expanded in coupling constant g_s :

$$H = H_2 + g_s H_3 + g_s^2 H_4 + \dots, \quad (2.9)$$

where H_2 is a free Hamiltonian, which in terms of Φ , has the form:

$$H_2 \sim \int p^+ dp^+ D^8(p(\sigma)) D^8(\lambda(\sigma)) \Phi^\dagger h \Phi, \quad (2.10)$$

where $p(\sigma)$ and $\lambda(\sigma)$ are momentum densities for bosonic and fermionic sectors respectively and h is the free string theory hamiltonian. The first correction H_3 to the hamiltonian is the three-point string vertex (see the Fig. 2.1) describing the interactions between the strings by means of the matrix element $\langle 3|H_3|1\rangle|2\rangle$. The main principle of getting the expression for H_3 is to require that the corrected generators should satisfy the supersymmetry algebra of the theory. After turning on the interaction, the set of generators can be divided in two groups: kinematical (those which symmetry are not affected by adding to the action terms of higher orders in g_s) and dynamical (those which has to be corrected after taking into account the interaction). In this sense all the constraints appearing due to the symmetry algebra can be divided in two groups: kinematical (commutation relations of dynamical generators with kinematical) and dynamical (commutation relations of dynamical generators between each other).

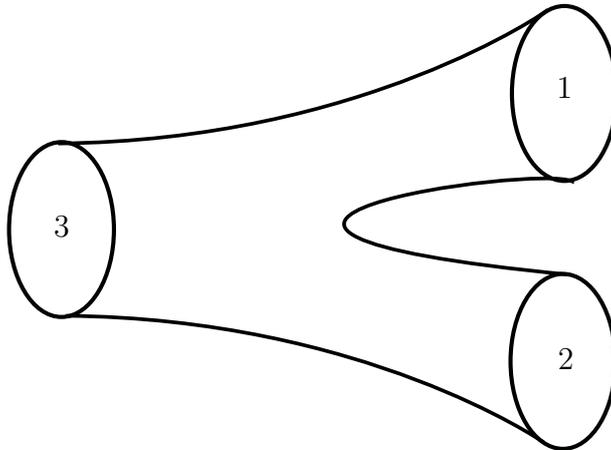


Figure 2.1

2.1.3 SFT in the pp-wave background

In the case of bosonic string the dynamical constraints are absent, since the only generator which gets corrected is the Hamiltonian. The kinematical constraints then reduce to requiring worldsheet continuity at interacting point, which can be realized by imposing a delta functional $\Delta(X_1(\sigma) - X_2(\sigma) - X_3(\sigma))$ in the functional integral over all possible configurations of three strings. The integral can be computed straightforwardly, leading to the following form of the bosonic string vertex¹

$$|V\rangle = \exp\left(-\frac{1}{2} \sum_{m,n=-\infty}^{\infty} \sum_{r,s=1}^3 \sum_{i=1}^8 a_m^{(r)i\dagger} N_{mn}^{rs} a_n^{(s)i\dagger}\right) |0\rangle\rangle. \quad (2.11)$$

Here $|0\rangle\rangle$ denotes the vacuum of three-string Hilbert space $|0\rangle\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$. The indices r and s denote the label of the strings, i denotes the polarization of the excitation and m, n are the mode numbers of the excitations. The quantities N_{mn}^{rs} are called Neumann coefficients and characterize the interactions between excitations of different strings.

For superstrings, in addition to worldsheet continuity, one also needs to require that supersymmetry is respected by the string vertex. This can be achieved by acting a new operator \mathcal{P} on the exponential part (2.11). This operator can be written as a quadratic polynomial of creation operators $a_m^{(r)i}$ (bosonic as well as fermionic, but since we consider the bosonic sector, we will drop the fermionic part) and is called the *prefactor*. The string vertex for superstring thus takes the following form

$$|H_3\rangle = \mathcal{P}|V\rangle. \quad (2.12)$$

However, it turns out that supersymmetry is not restrictive enough to fix the prefactor uniquely and there have been several proposals in the literature originating from different motivations. We will use the prefactor proposal by Dobashi and Yoneya [111]. The

1. Note that we use different notations from the ones in [66]. Our creation operator $a_m^{(r)i\dagger}$ is denoted by $\alpha_m^{(r)i\dagger}$ in [66] and our Neumann coefficient N_{mn}^{rs} is denoted by \tilde{N}_{mn}^{rs} in [66].

reason is that their proposal has the virtue to work for both extremal and non-extremal² correlation functions [66]. Interestingly, the prefactor of Dobashi and Yoneya is a half sum of the two prefactors \mathcal{P}_1 and \mathcal{P}_2 proposed in [65, 108, 109, 110] and [112] respectively

$$\mathcal{P}_h = \frac{1}{2}\mathcal{P}_1 + \frac{1}{2}\mathcal{P}_2, \quad (2.13)$$

more explicitly

$$\mathcal{P}_h = \sum_{r=1}^3 \left(\sum_{i=5}^8 \sum_{m=0}^{\infty} \frac{\omega_m^{(r)}}{\alpha^{(r)}} a_m^{(r)i\dagger} a_m^{(r)i} + \sum_{i=1}^4 \sum_{m=0}^{\infty} \frac{\omega_m^{(r)}}{\alpha^{(r)}} a_{-m}^{i(r)\dagger} a_{-m}^{(r)i} \right), \quad (2.14)$$

where $\omega_n^{(r)} = \sqrt{n^2 + \mu^2 \alpha_{(r)}^2}$ and $\alpha_{(r)} = \alpha' p_{(r)}^+$. After one fixes the string vertex, the matrix elements of H_3 can be computed straightforwardly. According to [111], the holographic relation between matrix element of the H_3 and OPE coefficients in BMN limit is given by

$$C_{123} = \frac{\sqrt{J_1 J_2 J_3}}{N} \frac{G(\Delta_1, \Delta_2, \Delta_3)}{\mu(\Delta_2 + \Delta_3 - \Delta_1)} \langle 1 | \langle 2 | \langle 3 | H_3 \rangle. \quad (2.15)$$

Here C_{123} is the structure constant of the three-point correlation function

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{13}|^{\Delta_1 + \Delta_3 - \Delta_2} |x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}} \quad (2.16)$$

where $x_{ij}^\mu = x_i^\mu - x_j^\mu$, Δ_i is the conformal dimension of the operator \mathcal{O}_i and the function $G(\Delta_1, \Delta_2, \Delta_3)$ reads

$$G(\Delta_1, \Delta_2, \Delta_3) = \left(f \frac{J_2 J_3}{J_1} \right)^{-(\Delta_2 + \Delta_3 - \Delta_1)/2} \Gamma \left(\frac{\Delta_2 + \Delta_3 - \Delta_1}{2} + 1 \right), \quad (2.17)$$

where the function f is defined in [66]. Finally, we want to emphasize that the holographic relation between the matrix elements of H_3 and the OPE coefficient in $\mathcal{N} = 4$ is not completely understood. The holographic relation (2.15) works well at the leading order [66, 113]. However, at higher loop order, the large μ expansion of the function $G(\Delta_1, \Delta_2, \Delta_3)$ give rises to non-perturbative terms such as $\log \mu$, the interpretation of which is still unclear.

2.2 The spin vertex in the BMN limit

In this section we will consider some aspects of the non-extremal three-point functions in the BMN limit of the scalar sector of $\mathcal{N} = 4$ SYM theory. The scalar sector means that we will consider the gauge invariant operators constructed as a product of only the scalar fields ϕ_i , $i = 1, \dots, 6$, forming the fundamental representation of the $so(6)$ group. Namely we will construct the object, which maps any three operators of the considered sector into a corresponding three-point correlator, in a way similar to the string vertex. In this sense we will call this object spin vertex. We examine the BMN limit of the spin vertex and show that it reproduces the string vertex of SFT (constructed in [111, 66]) at the leading order of λ' expansion. The presented material is based on [114].

2. According to the terminology of [66] the extremal and non-extremal correlation functions correspond to the impurity preserving and impurity non-preserving processes respectively.

2.2.1 The spin vertex

We will consider the following configuration:

$$\mathcal{O}_1 : \{Z, \phi^i\}, \quad \mathcal{O}_2 : \{\bar{Z}, \phi^i\}, \quad \mathcal{O}_3 : \{\bar{Z}, \phi^i\}, \quad i = 1, 2, 3, 4, \quad (2.18)$$

where

$$Z = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6), \quad \bar{Z} = \frac{1}{\sqrt{2}}(\phi^5 - i\phi^6). \quad (2.19)$$

The corresponding diagram is represented on the Fig. (2.2). The field Z is a vacuum for the first operator, and \bar{Z} for the second and the third operators. The excitations are presented by four transverse fields $\phi_i, i = 1, \dots, 4$.

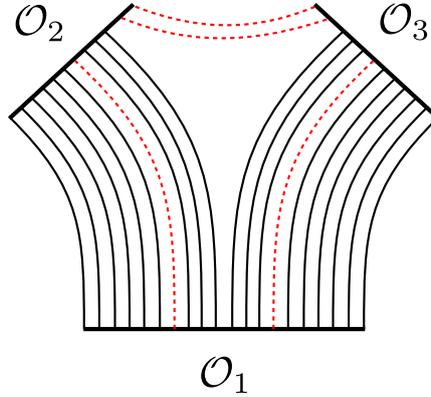


Figure 2.2 – The configuration for three-point functions. The black lines correspond to the contractions of Z and \bar{Z} and the red dashed lines correspond to the contractions of excitations $\phi^i, i = 1, 2, 3, 4$.

In order to compute the three-point functions³

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{\sqrt{L_1 L_2 L_3}}{N} \frac{c_{123}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{13}|^{\Delta_1 + \Delta_3 - \Delta_2} |x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}}. \quad (2.20)$$

at tree level we introduce the following generating fields:

$$\begin{aligned} F_1 &= \text{Tr} \prod_{k=1}^{L_1} \left(Z_k + \sum_{i=1}^4 x_k^i \phi_k^i \right) \\ F_2 &= \text{Tr} \prod_{k=1}^{L_2} \left(\bar{Z}_k + \sum_{i=1}^4 y_k^i \phi_k^i \right) \\ F_3 &= \text{Tr} \prod_{k=1}^{L_3} \left(\bar{Z}_k + \sum_{i=1}^4 z_k^i \phi_k^i \right). \end{aligned} \quad (2.21)$$

3. Here we separated the symmetry factor $\sqrt{L_1 L_2 L_3}$ (see e.g. [67]) from the structure constant in order to accommodate our notations with notations of [115]

Then any three operators can be obtained in the following way

$$\mathcal{O}_1 = \Psi_1(\partial_x)F_1|_{x^i=0}, \quad \mathcal{O}_2 = \Psi_2(\partial_y)F_2|_{y^i=0}, \quad \mathcal{O}_3 = \Psi_3(\partial_z)F_3|_{z^i=0}, \quad (2.22)$$

where $\Psi_1(\partial_x)$, $\Psi_2(\partial_y)$ and $\Psi_3(\partial_z)$ are three differential operators of the following form

$$\Psi(\partial_x) = \sum_{\mathbf{n}, I} c_{\mathbf{n}}^I \prod_{k=1}^L (\partial_k^{i_k})^{n_k}, \quad \partial_k^{i_k} \equiv \frac{\partial}{\partial x_k^{i_k}}, \quad n_k = 0, 1 \quad (2.23)$$

Here $\mathbf{n} = \{n_1, \dots, n_L\}$ and I is a collective index indicating the polarizations of the excitations, and the set of the coefficients $c_{\mathbf{n}}^I$ define the corresponding eigenstate of the dilatation operator. In order to compute the correlator between given operators \mathcal{O}_1 , \mathcal{O}_2 , \mathcal{O}_3 , we should compute the correlator of the generating functions F_1 , F_2 , F_3 , act on the result with the differential operators Ψ_1 , Ψ_2 , Ψ_3 , and then put all the variables x_k^i , y_k^i , z_k^i to zero:

$$\begin{aligned} c_{123} &= \Psi_1(\partial_x)\Psi_2(\partial_y)\Psi_3(\partial_z)\langle F_1(\mathbf{x})F_2(\mathbf{y})F_3(\mathbf{z}) \rangle|_{x^i, y^i, z^i=0} \\ &= \Psi_1(\partial_x)\Psi_2(\partial_y)\Psi_3(\partial_z)V_3(\mathbf{x}, \mathbf{y}, \mathbf{z})|_{x^i, y^i, z^i=0}, \end{aligned} \quad (2.24)$$

where

$$V_3(\mathbf{x}^i, \mathbf{y}^i, \mathbf{z}^i) = \prod_{k=1}^{L_{12}} (1 + y_k^i x_{L_1-k+1}^i) \prod_{k=1}^{L_{13}} (1 + x_k^i z_{L_3-k+1}^i) \prod_{k=1}^{L_{23}} z_k^i y_{L_2-k+1}^i, \quad (2.25)$$

which we will call the spin vertex. Note that the commutation relations of $\partial/\partial x$ and x are the same as commutation relations of bosonic creation and annihilation operators. Therefore, we can map the auxiliary variables and the corresponding derivatives into creation and annihilation operators

$$\begin{aligned} x_k^i &\rightarrow \alpha_k^{(1)i\dagger}, & y_k^i &\rightarrow \alpha_k^{(2)i\dagger}, & z_k^i &\rightarrow \alpha_k^{(3)i\dagger} \\ \frac{\partial}{\partial x_k^i} &\rightarrow \alpha_k^{(1)i}, & \frac{\partial}{\partial y_k^i} &\rightarrow \alpha_k^{(2)i}, & \frac{\partial}{\partial z_k^i} &\rightarrow \alpha_k^{(3)i}, \end{aligned} \quad (2.26)$$

and the spin vertex can be written as

$$|V_3\rangle_B \equiv V_3\left(\alpha_k^{(1)i\dagger}, \alpha_k^{(2)i\dagger}, \alpha_k^{(3)i\dagger}\right)|0\rangle_B. \quad (2.27)$$

The new Fock vacuum is defined to be the state that is annihilated by all the bosonic annihilation operators

$$\alpha_k^{(r)i}|0\rangle_B = 0. \quad (2.28)$$

The corresponding states can be written as

$$\langle 1| \equiv \langle 0|\Psi_1\left(\alpha_k^{(1)i}\right), \quad \langle 2| \equiv \langle 0|\Psi_2\left(\alpha_k^{(2)i}\right), \quad \langle 3| \equiv \langle 0|\Psi_3\left(\alpha_k^{(3)i}\right), \quad (2.29)$$

and the structure constant is given by

$$c_{123} = \langle 1|\langle 2|\langle 3|V_3\rangle_B, \quad (2.30)$$

where both the spin vertex and the states are now constructed by bosonic oscillators. The formulation we described up to now is applicable to any three states in our set-up. In the next section, we will take the BMN limit of (2.27) and show that it reproduces the string vertex of SFT constructed in [111, 66].

2.2.2 The BMN limit of the spin vertex

In this section we will take the BMN limit of the spin vertex constructed in the previous section. As it has been already discussed before the BMN limit of the $\mathcal{N} = 4$ SYM reduces to two approximations. The first says that the number of excitations should be much smaller than the length of the operator. The second - that the momenta of the excitations are small and scales like $\sim 1/L$, where L is the length of the spin chain, which is taken to be very large. The length of each spin chain is equal to the sum of the number of the vacuum fields Z or \bar{Z} and the number of excitations. We denote the number of the vacuum fields by J_i and the number of excitations by N_i so that the length of the i -th spin chain L_i is given by $L_i = J_i + N_i$. In the BMN limit we have $J_i \gg N_i$, and hence $L_i \simeq J_i$. Due to the charge conservation, we have $J_1 = J_2 + J_3$, thus we introduce the following notations: $J_1 = J$, $J_2 = rJ$, $J_3 = (1-r)J$, where $0 < r < 1$. The number of contractions between different operators are approximately given by

$$L_{12} \simeq J_2 = rJ, \quad L_{13} \simeq J_3 = (1-r)J, \quad L_{23} = M = \frac{1}{2}(N_2 + N_3 - N_1). \quad (2.31)$$

In what follows, we use M to denote the number of contractions between the two ‘outgoing’ operators. By BMN assumption, $M \ll J$. The three-point functions for $M = 0$ are called *impurity preserving*, or *extremal*, while for $M \neq 0$ they are called *impurity non-preserving* or *non-extremal*. For the extremal correlator, when diagonalizing anomalous dimension matrix one have to deal with the mixing between single trace and double trace operators [116]. On the contrary, for the non-extremal cases, the contribution from double trace operators are $1/N$ -suppressed and can be neglected in the planar limit, which makes the computation much simpler from gauge theory aspect. On the other hand, the earlier proposals for string field theory and duality relations work only for the extremal cases. A string field theory applicable to both extremal and non-extremal cases is the holographic string field theory proposed by Dobashi and Yoneya [111, 66]. In this thesis we consider only the non-extremal cases, so we always assume $M \neq 0$.

By (2.26), we map x_k^i, y_k^i and z_k^i to creation operators. In order to obtain creation operators in the momentum space, we perform the mode expansion of the bosonic oscillator

$$\alpha_k^{(r)i\dagger} = \frac{1}{\sqrt{J_r}} \sum_{n=-L_r/2}^{L_r/2} e^{\frac{2\pi i n k}{J_r}} a_n^{(r)i\dagger}, \quad r = 1, 2, 3 \quad (2.32)$$

Let us investigate the part of spin vertex corresponding to the contractions between operators 1 and 2.

$$\begin{aligned} V_{12} &= \prod_{k=1}^{L_{12}} (1 + \alpha_{L_1-k+1}^{(1)i} \alpha_k^{(2)i}) \approx \exp \left(\frac{1}{J\sqrt{r}} \sum_{n_i^{(1)}, n_i^{(2)}} \sum_{k=0}^{J_2} e^{\frac{2\pi i n_i^{(2)} k}{J_2} - \frac{2\pi i n_i^{(1)} k}{J_1}} a_{n_i^{(1)}}^{(1)i\dagger} a_{n_i^{(2)}}^{(2)i\dagger} \right) \\ &= \exp \left(- \sum_{n_i^{(1)}, n_i^{(2)}} \mathcal{N}_{n_i^{(1)} n_i^{(2)}}^{12} a_{n_i^{(1)}}^{(1)i\dagger} a_{n_i^{(2)}}^{(2)i\dagger} \right). \end{aligned} \quad (2.33)$$

In the first line, the summation over k gives

$$\sum_{k=0}^{J_2} e^{\frac{2\pi i n_i^{(2)} k}{J_2} - \frac{2\pi i n_i^{(1)} k}{J_1}} \simeq J e^{-\pi i r n_i^{(1)}} \frac{\sin \pi r n_i^{(1)}}{\pi(n_i^{(1)} - n_i^{(2)}/r)} = J \sqrt{r} (-1)^{n_i^{(1)} + n_i^{(2)}} e^{-\pi i r n_i^{(1)}} N_{n_i^{(1)} n_i^{(2)}}^{12} \quad (2.34)$$

Therefore our Neumann coefficient from spin vertex is related to the Neumann coefficient in SFT [117, 118] by a simple phase factor ⁴

$$\mathcal{N}_{n_i^{(1)} n_i^{(2)}}^{12} = (-1)^{n_i^{(1)} + n_i^{(2)}} e^{-\pi i r n_i^{(1)}} N_{n_i^{(1)} n_i^{(2)}}^{12}. \quad (2.35)$$

Similarly, for the contractions between operators 1 and 3, we have

$$\begin{aligned} V_{13} &= \prod_{k=1}^{L_{13}} (1 + \alpha_k^{(1)i\dagger} \alpha_{L_3 - k + 1}^{(3)i\dagger}) \simeq \exp \left(\frac{\sqrt{1-r}}{J} \sum_{n_i^{(1)}, n_i^{(3)}} \sum_{k=0}^{J_3} e^{\frac{2\pi i n_i^{(1)} k}{J_1} - \frac{2\pi i n_i^{(3)} k}{J_3}} a_{n_i^{(1)}}^{(1)i\dagger} a_{n_i^{(3)}}^{(3)i\dagger} \right) \\ &= \exp \left(- \sum_{n_i^{(1)}, n_i^{(3)}} \mathcal{N}_{n_i^{(1)} n_i^{(3)}}^{13} a_{n_i^{(1)}}^{(1)i\dagger} a_{n_i^{(3)}}^{(3)i\dagger} \right), \end{aligned} \quad (2.36)$$

where our Neumann coefficient is related to the SFT Neumann coefficient by

$$\mathcal{N}_{n_i^{(1)} n_i^{(3)}}^{13} = (-1)^{n_i^{(1)}} e^{-i\pi r n_i^{(1)}} N_{n_i^{(1)} n_i^{(3)}}^{13}. \quad (2.37)$$

For the contractions between operators 2 and 3, we have

$$V_{23} = \prod_{k=1}^M \alpha_k^{(3)i\dagger} \alpha_{L_2 - k + 1}^{(2)i\dagger} = \prod_{k=1}^M \frac{1}{J \sqrt{r(1-r)}} \sum_{n_i^{(2)}, n_i^{(3)}} e^{\frac{2\pi i k n_i^{(3)}}{J_3} - \frac{2\pi i k n_i^{(2)}}{J_2}} a_{n_i^{(2)}}^{(2)i\dagger} a_{n_i^{(3)}}^{(3)i\dagger}. \quad (2.38)$$

Since $M \ll J_i$ and $n_i^{(2)} \ll J_2$, $n_i^{(3)} \ll J_3$, the phase factor in (2.38) becomes trivial

$$e^{\frac{2\pi i k n_i^{(3)}}{J_3} - \frac{2\pi i k n_i^{(2)}}{J_2}} \approx 1 \quad (2.39)$$

and V_{23} simplifies to

$$V_{23} = \left(\frac{1}{J \sqrt{r(1-r)}} \sum_{n_i^{(2)}, n_i^{(3)}} a_{n_i^{(2)}}^{\dagger(2)i} a_{n_i^{(3)}}^{\dagger(3)i} \right)^M = \left(\frac{J}{4\pi\mu|\alpha_{(1)}|} \right)^{-M} \left(- \sum_{n_i^{(2)}, n_i^{(3)}} \mathcal{N}_{n_i^{(2)} n_i^{(3)}}^{23} a_{n_i^{(2)}}^{\dagger(2)i} a_{n_i^{(3)}}^{\dagger(3)i} \right)^M, \quad (2.40)$$

where our Neumann coefficient is given by

$$\mathcal{N}_{n_i^{(2)} n_i^{(3)}}^{23} = (-1)^{n_i^{(2)}} N_{n_i^{(2)} n_i^{(3)}}^{23}. \quad (2.41)$$

4. The explicit form of Neumann coefficient at the leading order of large μ expansion can be found in appendix A

Let us consider the following vertex

$$\tilde{V}_{23} = \exp \left(- \sum_{n_i^{(2)}, n_i^{(3)}} \mathcal{N}_{n_i^{(2)} n_i^{(3)}}^{23} a_{n_i^{(2)}}^{\dagger(2)i} a_{n_i^{(3)}}^{\dagger(3)i} \right), \quad (2.42)$$

which can be expanded as

$$\tilde{V}_{23} = \sum_{M=0}^{\infty} \frac{1}{M!} \left(- \sum_{n_i^{(2)}, n_i^{(3)}} \mathcal{N}_{n_i^{(2)} n_i^{(3)}}^{23} a_{n_i^{(2)}}^{\dagger(2)i} a_{n_i^{(3)}}^{\dagger(3)i} \right)^M. \quad (2.43)$$

For a given configuration, the number M is fixed and the action of states will automatically pick out the term in \tilde{V}_{23} with the corresponding M . Hence we can replace V_{23} by

$$V_{23} \longrightarrow \left(\frac{J}{4\pi\mu|\alpha_{(1)}|} \right)^{-M} M! \cdot \tilde{V}_{23}. \quad (2.44)$$

Recalling that at the leading order

$$\frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1) = \frac{1}{2}(L_2 + L_3 - L_1) = M, \quad (2.45)$$

and

$$f \frac{J_2 J_3}{J_1} = \frac{J}{4\pi\mu|\alpha_{(1)}|} + \mathcal{O}\left(\frac{1}{\mu^2}\right), \quad (2.46)$$

we find that the factors in front of \tilde{V}_{23} at the leading order coincides with the expression (2.17) obtained in [66]

$$\left(\frac{J}{4\pi\mu|\alpha_{(1)}|} \right)^{-M} M! \simeq \left(f \frac{J_2 J_3}{J_1} \right)^{-(\Delta_2 + \Delta_3 - \Delta_1)/2} \Gamma \left(\frac{\Delta_2 + \Delta_3 - \Delta_1}{2} + 1 \right) \quad (2.47)$$

From our derivation, it is clear that the function $G(\Delta_1, \Delta_2, \Delta_3)$ is intimately related to the interaction between the two outgoing states. Therefore it is crucial for the matching between SFT calculation and the non-extremal three-point functions.

We can define our spin vertex operator as

$$V_{\text{BMN}} = G(\Delta_1, \Delta_2, \Delta_3) V_{12} V_{13} \tilde{V}_{23}. \quad (2.48)$$

This is very close to the 3-point vertex from string theory including the correct G -factor, except that our Neumann coefficients seem to be different from those of SFT by some phase factors. We will show that these phase factors are trivial provided the physical states satisfy the level matching conditions. When acting physical states on the spin vertex, we obtain the product of Neumann coefficients of the following type

$$\begin{aligned} & (-1)^{L_{12}+L_{23}+L_{13}} \prod_{12} \mathcal{N}_{p_i^{(1)} p_i^{(2)}}^{12} \prod_{13} \mathcal{N}_{q_i^{(1)} q_i^{(3)}}^{13} \prod_{23} \mathcal{N}_{r_i^{(2)} r_i^{(3)}}^{23} \\ & = \text{phase} \cdot (-1)^{L_{12}+L_{23}+L_{13}} \prod_{12} N_{p_i^{(1)} p_i^{(2)}}^{12} \prod_{13} N_{q_i^{(1)} q_i^{(3)}}^{13} \prod_{23} N_{r_i^{(2)} r_i^{(3)}}^{23}. \end{aligned} \quad (2.49)$$

From (2.35),(2.37) and (2.41), the phase factor is

$$\begin{aligned} \text{phase} &= \prod_{12} (-1)^{p_i^{(1)}+p_i^{(2)}} e^{-\pi i r p_i^{(1)}} \prod_{13} (-1)^{q_i^{(1)}} e^{-\pi i r q_i^{(1)}} \prod_{23} (-1)^{r_i^{(2)}} \\ &= (-1)^{\sum_i p_i^{(1)}+q_i^{(1)}} (-1)^{\sum_i p_i^{(2)}+r_i^{(2)}} e^{-\sum_i \pi i (p_i^{(1)}+q_i^{(1)})} = 1, \end{aligned} \quad (2.50)$$

where we have used the level matching conditions

$$\sum_i p_i^{(1)} + q_i^{(1)} = \sum_i p_i^{(2)} + r_i^{(2)} = 0. \quad (2.51)$$

This means that if we consider the physical states, we can replace our Neumann coefficients $\mathcal{N}_{m,n}^{rs}$ by the Neumann coefficients of SFT $\tilde{N}_{m,n}^{rs}$, since the phase factor just cancels. To sum up, from the spin vertex in the BMN limit, we obtain at the leading order the following cubic vertex

$$|V_{\text{BMN}}\rangle = G(\Delta_1, \Delta_2, \Delta_3) \exp \left[-\frac{1}{2} \sum_{\substack{r,s=1 \\ r \neq s}}^3 a_m^{(r)i\dagger} N_{mn}^{rs} a_n^{(s)i\dagger} \right] |0\rangle. \quad (2.52)$$

Notice that in the exponent we impose the condition $r \neq s$, while in SFT the Neumann coefficients \tilde{N}_{mn}^{rr} , which corresponds to interactions between the excitations of the same string, are non-zero. However these Neumann coefficients will appear only at higher orders in the large μ expansion. It is an interesting question whether we can obtain this kind of Neumann coefficients from weak coupling at higher loops, which we leave for future investigation. Therefore at the leading order, we have

$$|V_{\text{BMN}}\rangle = G(\Delta_1, \Delta_2, \Delta_3) |V\rangle. \quad (2.53)$$

To complete our derivation, we also need to show that the eigenstates at weak coupling in the BMN limit can be presented in the same form as the ones in SFT. In the BMN limit, the scattering phases are zero which means there is no interaction between excitations. Therefore, the wave functions of the spin chain states are simply given by plane waves. In terms of bosonic oscillators, a BMN state at the leading order can be represented by

$$\langle 0 | \Psi(\alpha_k^i) = \frac{1}{\sqrt{L^N}} \sum_{k_1 \neq \dots \neq k_N = 1}^L \langle 0 | \alpha_{k_1}^{i_1} \dots \alpha_{k_N}^{i_N} e^{-\frac{2\pi i}{L} (n_1 k_1 + \dots + n_N k_N)} \quad (2.54)$$

where L is the length of the spin chain, α_k^i is the bosonic oscillator introduced in (2.26) and creates an excitation at position k with polarization i . Here n_1, \dots, n_N are the mode numbers of the excitations. By performing the same Fourier transform as in (2.32), we have simply

$$\langle 0 | \Psi(\alpha_k^i) = \langle 0 | a_{n_1}^{i_1} \dots a_{n_N}^{i_N} = \langle n_1, \dots, n_N |, \quad (2.55)$$

which takes the form of a standard string theory state.

There is another difference between the spin vertex and string vertex. In SFT, one has to take into account the non-trivial prefactor while we do not have similar prefactor in the spin vertex formalism. However, as argued by Dobashi and Yoneya [66], at the

leading order, the action of prefactor on the states gives rise to a simple factor, which cancels neatly the holographic factor $1/\mu(\Delta_2 + \Delta_3 - \Delta_1)$. Let us briefly review how this works. The prefactor for scalar excitations at the leading order reads

$$\begin{aligned} \mathcal{P}_h = & \mu(a_0^{(2)\dagger} a_0^{(2)i} + a_0^{(3)\dagger} a_0^{(3)i} - a_0^{(1)\dagger} a_0^{(1)i}) \\ & + \frac{\mu}{2} \sum_{m=1}^{\infty} (a_m^{(2)\dagger} a_m^{(2)i} + a_m^{(3)\dagger} a_m^{(3)i} - a_m^{(1)\dagger} a_m^{(1)i} + [m \rightarrow -m]) \\ & + \frac{\mu}{2} \sum_{m=1}^{\infty} (a_m^{(2)\dagger} a_{-m}^{(2)i} + a_m^{(3)\dagger} a_{-m}^{(3)i} - a_m^{(1)\dagger} a_{-m}^{(1)i} + [m \rightarrow -m]) \end{aligned} \quad (2.56)$$

The operators in the first two lines take the form of counting operators $a_m^\dagger a_m$. On the other hand, the operators in the third line $a_m^\dagger a_{-m}$ changes the sign of mode numbers. But due to the symmetry of the Neumann coefficients with respect to exchange of the lower indexes: $N_{-n,m}^{rs} = N_{n,-m}^{rs}$, its action reduces to the counting operator as well. Hence once acted on the physical states, the prefactor (2.57) just counts twice the difference between the excitation numbers of the incoming state 1 and out-going states 2 and 3, and its action reduces just to multiplication of the factor $\mu(N_2 + N_3 - N_1) = 2\mu M$. At the leading order it can be written as

$$2\mu M = \mu(\Delta_2 + \Delta_3 - \Delta_1). \quad (2.57)$$

Therefore, we have

$$\frac{1}{\mu(\Delta_2 + \Delta_3 - \Delta_1)} \langle 1 | \langle 2 | \langle 3 | H_3 \rangle = \frac{1}{\mu(\Delta_2 + \Delta_3 - \Delta_1)} \langle 1 | \langle 2 | \langle 3 | \mathcal{P}_h | V \rangle = \langle 1 | \langle 2 | \langle 3 | V \rangle \quad (2.58)$$

Finally, gathering all the pieces, we get

$$\begin{aligned} C_{123} = & \frac{\sqrt{J_1 J_2 J_3}}{N} \langle 1 | \langle 2 | \langle 3 | V_{\text{BMN}} \rangle = \frac{\sqrt{J_1 J_2 J_3}}{N} G(\Delta_1, \Delta_2, \Delta_3) \langle 1 | \langle 2 | \langle 3 | V \rangle = \\ & \frac{\sqrt{J_1 J_2 J_3}}{N} \frac{G(\Delta_1, \Delta_2, \Delta_3)}{\mu(\Delta_2 + \Delta_3 - \Delta_1)} \langle 1 | \langle 2 | \langle 3 | \mathcal{P}_h | V \rangle = \frac{G(\Delta_1, \Delta_2, \Delta_3)}{\mu(\Delta_2 + \Delta_3 - \Delta_1)} \langle 1 | \langle 2 | \langle 3 | \frac{\sqrt{J_1 J_2 J_3}}{N} | H_3 \rangle \end{aligned} \quad (2.59)$$

where the first line is from the *spin vertex* while the last line is exactly the holographic relation (2.15) proposal by Dobashi and Yoneya. Therefore we have shown that gauge theory computation exactly reproduces SFT result at tree level.

2.3 The spin vertex for the full theory

In this section we will generalize the notion of the spin vertex to the full theory of $\mathcal{N} = 4$ SYM. For this purpose we will be extensively using the so-called oscillator representation of the symmetry algebra $psu(2, 2|4)$. The procedure to build such a representation for non-compact supergroups is well explained in [119]. In the oscillator representation the generators of the algebra are formed as bilinears of bosonic and fermionic oscillators. We will construct the full spin vertex at tree level in oscillator representation and show that it satisfies a monodromy relation with appropriately constructed monodromy matrix. A very similar to the full vertex object was constructed in [91], but the authors didn't realize its Yangian invariance. Finally we will show the relation between the spin vertex (2.25) introduced in the previous chapter and the full spin vertex built in the oscillator representation. The content of this section is based on [92].

2.3.1 Oscillator representation and the free $\mathcal{N} = 4$ SYM

In this section we will introduce the oscillator representation of the free $\mathcal{N} = 4$ SYM [120, 16] by and large following [91]. Let us first discuss the oscillator representation of the compact version of $psu(2, 2|4)$, $psu(4|4)$. It uses four copies of bosonic oscillators, a_i , $i = 1, \dots, 4$ and four copies of fermionic oscillators, c_k , $k = 1, \dots, 4$,

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad \{c_k, c_l^\dagger\} = \delta_{kl}, \quad i, j, k, l = 1, \dots, 4. \quad (2.60)$$

We organize the oscillators in a eight-dimensional vector

$$\phi = (a_i \quad c_k) \quad (2.61)$$

such that the generators of $u(4|4)$ can be written as

$$E_{\text{compact}}^{AB} = \phi^{A\dagger} \phi^B \quad \text{with} \quad E_{\text{compact}}^{AB\dagger} = E_{\text{compact}}^{BA}. \quad (2.62)$$

It is straightforward to check that they satisfy the commutation relations of the $u(4|4)$ algebra,

$$[E^{AB}, E^{CD}] = \delta^{BC} E^{AD} - (-1)^{(|A|+|B|)(|C|+|D|)} \delta^{AD} E^{CB}, \quad (2.63)$$

with $[\cdot, \cdot]$ meaning commutator or anti-commutator, depending on the grading of the generators ($|A| = 0, 1$ for bosonic and fermionic indices respectively). The non-compact form $u(2, 2|4)$ can be obtained after a particle-hole transformation for two of the bosonic oscillators:

$$b_i = a_{i+2}^\dagger, \quad b_i^\dagger = -a_{i+2} \quad i = 1, 2. \quad (2.64)$$

The commutation relations (2.63) are preserved by the particle-hole transformation, but the Hermitian conjugate of the generators are now

$$E^{AB\dagger} = \gamma E^{BA} \gamma, \quad \gamma = \text{diag}(1_2, -1_2, 1_4). \quad (2.65)$$

Sometimes, for the sake of symmetry, it is convenient to perform also a particle-hole transformation of the fermionic oscillators

$$d_i = c_{i+2}^\dagger, \quad d_i^\dagger = c_{i+2} \quad i = 1, 2. \quad (2.66)$$

Unlike the bosonic particle-hole transformation, the fermionic one is unitary and therefore it does not change the real form of the algebra. We will use alternatively the two notations. The Lie-algebra generators are expressed in terms of these oscillators as

$$E^{AB} = \bar{\psi}^A \psi^B, \quad (2.67)$$

with

$$\psi = (a_i \quad -b_i^\dagger \quad c_i \quad d_i^\dagger), \quad \bar{\psi} = \psi^\dagger \gamma = (a_i^\dagger \quad b_i \quad c_i^\dagger \quad d_i). \quad (2.68)$$

The projective condition of $psu(2, 2|4)$ is obtained by imposing the central charge of the algebra, $\sum_A E^{AA} = \bar{\psi} \psi$, to be zero,

$$\sum_A E^{AA} = \sum_{i=1,2} (N_{a_i} - N_{b_i} + N_{c_i} + N_{c_{i+2}} - 1) = \sum_{i=1,2} (N_{a_i} - N_{b_i} + N_{c_i} - N_{d_i}) = 0, \quad (2.69)$$

where N_a, N_b, N_c, N_d are the number of the respective types of bosons and fermions in the two types of representations.

Let us now concentrate on the conformal subalgebra in four dimensions $so(2, 4) \simeq su(2, 2)$. It consists of the Lorentz transformation $M_{\mu\nu}$ ($\mu, \nu = 0, \dots, 3$), translations P_μ , special conformal transformations K_μ and dilatation D . Once the following identification is implied

$$M_{\mu 5} = \frac{1}{2}(P_\mu - K_\mu), \quad M_{\mu 6} = \frac{1}{2}(P_\mu + K_\mu), \quad M_{56} = -D, \quad (2.70)$$

the commutation relations can be written in the form

$$[M_{PQ}, M_{RS}] = i(\eta_{QR}M_{PS} - \eta_{PR}M_{QS} - \eta_{QS}M_{PR} + \eta_{PS}M_{QR}), \quad (2.71)$$

where, according to the notations from [91], $P, Q, R, S = 0, 1, 2, 3, 5, 6$ with time-like directions 0, 6 and space-like directions 1, 2, 3, 5.

The conformal subgroup has a natural grading with respect to the maximal compact subalgebra $u(1) \otimes su(2) \otimes su(2)$. The grading is given by the value of the $u(1)$ generator E

$$\begin{aligned} [E, L^\pm] &= \pm L^\pm, & [E, L^0] &= 0, \\ [L^0, L^\pm] &= \pm L^\pm, & [L^+, L^-] &\in L^0, \end{aligned}$$

where

$$E = M_{06} = \frac{1}{2}(P_0 + K_0). \quad (2.72)$$

Thus, in order to build a unitary representation of the conformal subgroup we need to find the state annihilated by all the generators L^- and then to act on it with the generators L^+ .

We can express the generators M_{AB} in terms of the oscillators in the following way

$$\hat{M}_{AB} = \bar{\phi}_b M_{AB} \phi_b, \quad (2.73)$$

$$\phi_b = (a_1 \quad a_2 \quad -b_1^\dagger \quad -b_2^\dagger), \quad \bar{\phi}_b^\dagger = \phi_b \gamma_0 = (a_1^\dagger \quad a_2^\dagger \quad b_1 \quad b_2), \quad (2.74)$$

with

$$\gamma_0 = \text{diag}(1, 1, -1, -1). \quad (2.75)$$

The generators M_{AB} form the four dimensional representation of $so(2, 4)$ algebra and can be expressed in terms of the Dirac γ matrices:

$$\begin{aligned} M^{\mu\nu} &= -\frac{i}{4}[\gamma^\mu, \gamma^\nu], & M^{\mu 5} &= -\frac{i}{2}\gamma^\mu \gamma^5, \\ M^{\mu 6} &= \frac{1}{2}\gamma^\mu, & M^{56} &= \frac{1}{2}\gamma^5. \end{aligned} \quad (2.76)$$

Due to the properties

$$\gamma^0 M_{AB} = M_{AB}^\dagger \gamma^0, \quad [\hat{M}_{AB}, \hat{M}_{CD}] = \bar{\psi} [M_{AB}, M_{CD}] \psi, \quad (2.77)$$

the generators \hat{M}_{AB} are Hermitian and satisfy the commutation relations (2.71), and thus form the unitary representation of the conformal subgroup $so(2, 4)$. From the identification (2.73) we can get (we will be using these extensively in the following)

$$E = 1 + \frac{1}{2}(N_a + N_b) = \frac{1}{2}(a^\dagger a + b b^\dagger), \quad (2.78)$$

$$L_\mu^+ = -a^\dagger \bar{\sigma}_\mu b^\dagger, \quad L_\mu^- = b \sigma_\mu a, \quad (2.79)$$

where $\sigma_\mu = (-1, \vec{\sigma})$, and $\bar{\sigma}_\mu = (-1, -\vec{\sigma})$ and summation over indices of the bosonic operators is understood.

We will denote the unitary representation which is built by acting with L^+ 's on the vacuum $|0\rangle$ such that $L^-|0\rangle = 0$ as \mathcal{H}_1 .

On the other hand, when computing the correlation functions, we are interested in the primary operators, those, which are annihilated by the special conformal transformations K_μ . Then all the descendants are generated by acting on the primaries with momentum generators P_μ . We will denote the representation obtained after such manipulations by \mathcal{H}_2 . In [91] the authors noticed that in order to go from the representation \mathcal{H}_1 to the representation \mathcal{H}_2 one needs to perform the following transformation

$$U = \exp -\frac{\pi}{2} M_{05} = \exp -\frac{\pi}{4} (P_0 - K_0), \quad (2.80)$$

such that

$$|state\rangle_2 = U |state\rangle_1, \quad J_2 = U J_1 U^{-1}, \quad (2.81)$$

for states $|state\rangle$ and generators J . In particular

$$U^{-1} K_\mu U = L_\mu^-, \quad U^{-1} P_\mu U = L_\mu^+, \quad U^{-1} D U = iE. \quad (2.82)$$

At tree level we can get the expression for the U operator in terms of the oscillators. From (2.82) we have that

$$P_0 - K_0 = U(P_0 - K_0)U^{-1} = (L_0^+ - L_0^-), \quad (2.83)$$

so

$$U = \exp -\frac{\pi}{4} (a^\dagger b^\dagger + ab). \quad (2.84)$$

For the later convenience we introduce the fermionic copy of U , which we denote as U_F :

$$U_F = \exp -\frac{\pi}{4} (c^\dagger d^\dagger + cd), \quad U_F^\dagger = U_F^{-1}. \quad (2.85)$$

2.3.2 Correlation functions and the two-vertex

In this section we will show how to compute the correlation functions of gauge invariant operators at tree level in terms of oscillator representation introduced above. For an operator of length L we will need the Fock vacuum $|0\rangle$:

$$|0\rangle = |0\rangle_1 \otimes \dots \otimes |0\rangle_L. \quad (2.86)$$

Accordingly the action of the generators of the superconformal algebra on the state in the Fock space will be defined as

$$E^{AB} = \sum_{s=1}^L E_s^{AB}, \quad (2.87)$$

where for each site s of the state we introduced a copy of the algebra generators E_s^{AB} .

We will be interested in the states satisfying the condition:

$$L_\mu^- |\mathcal{O}\rangle = 0, \quad (2.88)$$

then the state $U|\mathcal{O}\rangle$ will correspond to the primary operator $\mathcal{O}(0)$, since

$$K_\mu U|\mathcal{O}\rangle = 0. \quad (2.89)$$

Translating the operators to a different space-time point can be done with the help of the momentum operator,

$$\mathcal{O}(x) = e^{iPx} \mathcal{O}(0) e^{-iPx}, \quad (2.90)$$

with corresponding Fock space representative

$$e^{iPx} U|\mathcal{O}\rangle. \quad (2.91)$$

This mapping was used by the authors of [91] to write the two point function in terms of the Fock space representation

$$\langle \bar{\mathcal{O}}_2(y) \mathcal{O}_1(x) \rangle = \langle \mathcal{O}_2 | U^\dagger e^{iP(x-y)} U | \mathcal{O}_1 \rangle = \langle \mathcal{O}_2 | U^2 e^{iL^+(x-y)} | \mathcal{O}_1 \rangle, \quad (2.92)$$

where we used (2.82) and that $U = U^\dagger$. The authors of [91] also verified that if \mathcal{O} is any elementary field, for example Z , the tree-level representation of the operators in the Fock space gives the desired result of the Wick contraction

$$\langle \bar{Z}(x) Z(y) \rangle = \langle Z | U^2 e^{iL^+(y-x)} | Z \rangle = \frac{\langle Z | Z \rangle}{(x-y)^2} = \frac{1}{(x-y)^2}. \quad (2.93)$$

In order to get the second equality one has to use the following identity

$$U_t = \exp t(a^\dagger b^\dagger + ba) = \exp(a^\dagger b^\dagger \tan t) \exp(-(a^\dagger a + bb^\dagger) \ln \cos t) \exp(ab \tan t), \quad (2.94)$$

which can be proved by taking derivative of both sides of the equation and taking into account that $U_0 = 1$. Then after substituting U_t instead of U^2 one can get that

$$\langle \bar{Z}(x) Z(y) \rangle = \lim_{t \rightarrow -\pi/2} \frac{1}{\cos^2 t} \frac{1}{1 - 2(x^0 - y^0) \tan t + (x-y)^2 \tan^2 t} = \frac{1}{(x-y)^2}. \quad (2.95)$$

(See [91] for more details). A similar representation can be used for the special case of the extremal three point function⁵, when the length of the first chain is equal to the sum of the lengths of the second and the third, $L_1 = L_2 + L_3$,

$$\langle \bar{\mathcal{O}}_2(y) \bar{\mathcal{O}}_3(z) \mathcal{O}_1(x) \rangle_{\text{ext}} = \langle \bar{\mathcal{O}}_2 | \otimes \langle \bar{\mathcal{O}}_3 | U_2 U_3 e^{iP_1 x} e^{-iP_3 z} e^{-iP_2 y} U_1 | \mathcal{O}_1 \rangle_{\text{ext}}, \quad (2.96)$$

where the index on the operators shows now the space on which they act. At tree level for the extremal correlator $U_1 = U_2 U_3$ and $P_1 = P_2 + P_3$.

5. This example is only illustrative since we are not computing an extremal correlation function even at tree level, because of the mixing of single-trace and double-trace states [116].

Here we would like to draw attention of the reader to the fact that the recipe for computing the correlators presented above has some drawback. Namely, the recipe assumes the following identification

$$\bar{\mathcal{O}}(x) \longleftrightarrow \langle \mathcal{O} | U^\dagger e^{-iPx} . \quad (2.97)$$

However, if we write the expression for $\bar{\mathcal{O}}(x)$ explicitly in terms of the oscillators we will see that this is not the case, since the conjugated elementary fields are described with different sets of oscillators (the expressions of the elementary fields in terms of the oscillators is presented in the App. B) and thus the scalar product (2.92) doesn't really correspond to Wick contraction. In order to avoid this problem, we would like to introduce the object which would mimic the operation of Wick contraction. This object will be the spin vertex. First we will introduce elementary two-vertex $|V_{12}\rangle$ which we will define as an object translating each elementary field $|\mathcal{E}\rangle$ into its what we call "flipped" version $|\bar{\mathcal{E}}\rangle$. That is, it will translate scalars to its conjugated, like $Z \rightarrow \bar{Z}$, fermions to antifermions $\psi_{i\alpha} \rightarrow \bar{\psi}_{i\alpha}$, and $F_{\alpha\beta} \rightarrow \bar{F}_{\alpha\beta}$. Thus the elementary two-vertex can be defined as a state living in the product of two Fock spaces (1) and (2) and having the following form:

$$U_F^2 |V_{12}\rangle = |\bar{S}_n^{(2)}\rangle |S_n^{(1)}\rangle + |S_n^{(2)}\rangle |\bar{S}_n^{(1)}\rangle + |\bar{\Psi}_{ia}^{(2)}\rangle |\Psi_{ia}^{(1)}\rangle + |\Psi_{ia}^{(2)}\rangle |\bar{\Psi}_{ia}^{(1)}\rangle + |\bar{F}_{ij}^{(2)}\rangle |F_{ij}^{(1)}\rangle + |F_{ij}^{(2)}\rangle |\bar{F}_{ij}^{(1)}\rangle + \dots, \quad (2.98)$$

where $|S\rangle$ stands for the scalar and (1), (2) denotes the corresponding Fock space. So we have

$$\langle V_{12} | \mathcal{E} \rangle^{(2)} = {}^{(1)}\langle \bar{\mathcal{E}} |. \quad (2.99)$$

In case of the state $|\mathcal{O}\rangle$ with the length L , we would like that the vertex also inverse the order of the elementary fields:

$$\langle V_{12} | U_F^{-2} \mathcal{E}_L \mathcal{E}_{L-1} \dots \mathcal{E}_1 \rangle^{(2)} = {}^{(1)}\langle \bar{\mathcal{E}}_L \bar{\mathcal{E}}_{L-1} \dots \bar{\mathcal{E}}_1 |, \quad (2.100)$$

or

$${}^{(1)}\langle \mathcal{O} | = \langle V_{12} | \sigma_{(1)} | \bar{\mathcal{O}} \rangle^{(2)}, \quad \sigma \equiv U_F^{-2}. \quad (2.101)$$

Now using vertex we can rewrite the expression (2.92) for the two-point function as the following

$$\begin{aligned} \langle \bar{\mathcal{O}}_2(y) \mathcal{O}_1(x) \rangle &= \langle V_{12} | U_{F(1)}^{-2} | \bar{\mathcal{O}}_2 \rangle^{(2)} | U_{(1)}^2 e^{iL_{(1)}^+(x-y)} | \mathcal{O}_1 \rangle^{(1)} \\ &= \langle V_{12} | U_{F(1)}^{-2} U_{(1)}^2 e^{iL_{(1)}^+(x-y)} | \bar{\mathcal{O}}_2 \rangle^{(2)} \otimes | \mathcal{O}_1 \rangle^{(1)} \\ &= \langle V_{12} | U_{(1)}^{\dagger 2} e^{iL_{(1)}^+(x-y)} | \bar{\mathcal{O}}_2 \rangle^{(2)} \otimes | \mathcal{O}_1 \rangle^{(1)} \\ &= \langle \mathcal{V}_{12} | e^{iL_{(1)}^+(x-y)} | \bar{\mathcal{O}}_2 \rangle^{(2)} \otimes | \mathcal{O}_1 \rangle^{(1)} \\ &= \langle \mathcal{V}_{12} | e^{i[L_{(1)}^+ x + L_{(2)}^+ y]} | \bar{\mathcal{O}}_2 \rangle^{(2)} \otimes | \mathcal{O}_1 \rangle^{(1)}, \end{aligned} \quad (2.102)$$

where in order to simplify the notations we introduced the operator $U = U U_F$ and another vertex $|\mathcal{V}_{12}\rangle = U_{(1)}^2 |V_{12}\rangle$. In the following we will be using $|\mathcal{V}_{12}\rangle$ as well as $|V_{12}\rangle$. In order to get the last equality of (2.102) we used the property

$$\langle V_{12} | U_{(1)}^{\dagger 2} (L_{(1)}^+ + L_{(2)}^+) \equiv \langle \mathcal{V}_{12} | (L_{(1)}^+ + L_{(2)}^+) = 0, \quad (2.103)$$

which will be proved later in the section 2.3.3.

In App. C we show that the two-vertex given by the expression (2.104) satisfies the expansion (2.98) and gives right propagators for all the elementary fields.

$$\begin{aligned}
|\mathcal{V}_{12}\rangle &\equiv U_{(1)}^2 |V_{12}\rangle \\
&= U_{(1)}^2 \exp \left(\sum_{s=1}^L \sum_{i=1,2} \left(b_{i,s}^{(1)\dagger} a_{i,s}^{(2)\dagger} - a_{i,s}^{(1)\dagger} b_{i,s}^{(2)\dagger} - d_{i,s}^{(1)\dagger} c_{i,s}^{(2)\dagger} - c_{i,s}^{(1)\dagger} d_{i,s}^{(2)\dagger} \right) \right) |0\rangle^{(2)} \otimes |0\rangle^{(1)} \\
&= \exp \left(- \sum_{s=1}^L \sum_{i=1,2} \left(a_{i,s}^{(1)} a_{i,s}^{(2)\dagger} - b_{i,s}^{(1)} b_{i,s}^{(2)\dagger} + d_{i,s}^{(1)} d_{i,s}^{(2)\dagger} - c_{i,s}^{(1)} c_{i,s}^{(2)\dagger} \right) \right) |0\rangle^{(2)} \otimes |\bar{0}\rangle^{(1)},
\end{aligned} \tag{2.104}$$

where the index s is related to the number of the corresponding Fock space (site number), $|\bar{0}\rangle = U^2|0\rangle$ and

$$|0\rangle^{(2)} \otimes |0\rangle^{(1)} = \left(|0\rangle_L^{(2)} \otimes \dots \otimes |0\rangle_1^{(2)} \right) \otimes \left(|0\rangle_1^{(1)} \otimes \dots \otimes |0\rangle_L^{(1)} \right). \tag{2.105}$$

Here the Fock vacua $|0\rangle_i^{(2)}$ are written in the reversed order intentionally in order to show our convention of numbering the sites of the state (2), which is $|\mathcal{O}_2\rangle = |\mathcal{E}_L \dots \mathcal{E}_1\rangle$. This convention ensures that the property (2.100) holds. Finally, it is worth mentioning

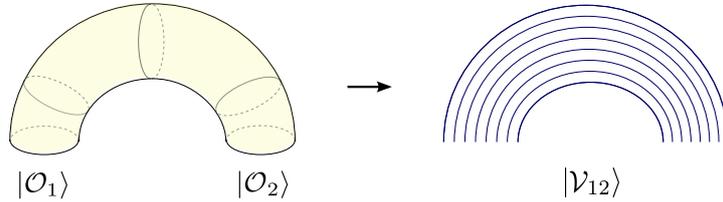


Figure 2.3 – The two point correlation function and $|\mathcal{V}_{12}\rangle$

that the expansion of the vertex (2.104) apart from the "right" terms, corresponding to the expansion (2.98), gives rise to some other terms. But these terms do not satisfy the condition of the zero central charge and as a consequence cancel when the vertex is applied to the states.

2.3.3 The properties of the vertex

Here we will consider some properties of the vertex, which significantly simplify the manipulations with it and which we use in the following. First we notice that the two-vertex translates the oscillators from the one space to another in the following way:

$$\begin{aligned}
(a_{i,s}^{(1)\dagger} + b_{i,s}^{(2)})|V_{12}\rangle &= (b_{i,s}^{(1)\dagger} - a_{i,s}^{(2)})|V_{12}\rangle = (a_{i,s}^{(1)} + b_{i,s}^{(2)\dagger})|V_{12}\rangle = (b_{i,s}^{(1)} - a_{i,s}^{(2)\dagger})|V_{12}\rangle = 0, \\
(c_{i,s}^{(1)} + d_{i,s}^{(2)\dagger})|V_{12}\rangle &= (d_{i,s}^{(1)} + c_{i,s}^{(2)\dagger})|V_{12}\rangle = (d_{i,s}^{(1)\dagger} - c_{i,s}^{(2)})|V_{12}\rangle = (c_{i,s}^{(1)\dagger} - d_{i,s}^{(2)})|V_{12}\rangle = 0.
\end{aligned} \tag{2.106}$$

It is interesting that the operator U operator has the similar behavior (see the App. D). Now let us look at the effect of the vertex on the generators of the $psu(2,2|4)$ algebra.

In general, the vertex transforms generators acting in one of the Fock spaces, $G_{(1)}$, into operators acting in the other space, $\tilde{G}_{(2)}$, by

$$\begin{aligned} G_{(1)}|V_{12}\rangle &\equiv -\tilde{G}_{(2)}|V_{12}\rangle, \\ G_{(1)}H_{(1)}|V_{12}\rangle &= (-1)^{|G||H|}\tilde{H}_{(2)}\tilde{G}_{(2)}|V_{12}\rangle, \end{aligned} \quad (2.107)$$

with $|G|$ denoting the grading of the operator G , *i.e.* the number of fermions it contains modulo 2. The transformation above is an anti-morphism, because it changes the order of the operators. Let us consider the generators of the $psu(2, 2|4)$ algebra (or rather $u(2, 2|4)$, since we prefer not to factor out the central element and the super identity) $E_{(1)}^{AB}$ which obey the commutation relations (2.63). According to (2.107), they are transformed by the vertex into another set of generators, $\tilde{E}_{(2)}^{AB}$, also obeying the commutation relations⁶ of $psu(2, 2|4)$, and a priori different from $E_{(2)}^{AB}$. We deduce that the vertex obeys the local symmetry condition

$$\left(E_{s,(1)}^{AB} + \tilde{E}_{s,(2)}^{AB}\right)|V_{12}\rangle = 0, \quad s = 1, \dots, L. \quad (2.108)$$

The explicit form of \tilde{E}^{AB} can be determined using (2.106) and (2.107). On the other hand, by inspection one can show that

$$\tilde{E}^{AB} = U^2(E^{AB} + (-1)^{|B|}\delta^{AB})U^{-2} \quad (2.109)$$

for all the generators, even and odd, with $|B| = 0, 1$ for bosonic and fermionic indices respectively. We therefore conclude that the symmetry of the vertex $|\mathcal{V}_{12}\rangle$, at tree level, can be expressed as

$$\left(E_{s,(1)}^{AB} + E_{s,(2)}^{AB} + (-1)^{|B|}\delta^{AB}\right)|\mathcal{V}_{12}\rangle = 0, \quad s = 1, \dots, L. \quad (2.110)$$

The term $(-1)^{|B|}\delta^{AB}$ is proportional to the identity in the oscillator space and it can be incorporated into a shift of the Cartan generators, $E^{AA} \rightarrow E^{AA} + \frac{(-1)^{|A|}}{2}$, which does not affect the $u(2, 2|4)$ commutation relations. We therefore conclude that the vertex possess local $u(2, 2|4)$ symmetry.

2.3.4 Three-point functions and the spin vertex

In this section we will generalize the notion of the vertex to the three-point functions. The generalization to the case of extremal correlator straightforwardly follows from the expression (2.96) in the same way as in the case with the two-point functions:

$$\begin{aligned} \langle \bar{\mathcal{O}}_2(y)\bar{\mathcal{O}}_3(z)\mathcal{O}_1(x) \rangle_{\text{ext}} &= \text{ext} \langle V_{123} | U_{(1)}^{+2} e^{i[L_{(1)}^+ x + L_{(2)}^+ y + L_{(3)}^+ z]} | \bar{\mathcal{O}}_2 \rangle \otimes | \bar{\mathcal{O}}_3 \rangle \otimes | \mathcal{O}_1 \rangle \\ &= \text{ext} \langle \mathcal{V}_{123} | e^{i[L_{(1)}^+ x + L_{(2)}^+ y + L_{(3)}^+ z]} | \bar{\mathcal{O}}_2 \rangle \otimes | \bar{\mathcal{O}}_3 \rangle \otimes | \mathcal{O}_1 \rangle, \end{aligned} \quad (2.111)$$

where the extremal vertex $|\mathcal{V}_{123}\rangle_{\text{ext}}$ is built from two pieces connecting each the operators \mathcal{O}_2 and \mathcal{O}_3 with \mathcal{O}_1 ,

$$|\mathcal{V}_{123}\rangle_{\text{ext}} = |\mathcal{V}_{12}\rangle \otimes |\mathcal{V}_{13}\rangle. \quad (2.112)$$

6. We have introduced the minus sign in the first line of (2.107) to get the right commutation relations for $\tilde{E}_{(2)}^{AB}$.

In this case, at tree level there are Wick contractions only between the operators 1 and 2 and 1 and 3 and there are no contractions between the operators 2 and 3. At this point we are starting to see that in the vertex formulation the operators can be treated more democratically,

$$\langle \mathcal{O}_2(y)\mathcal{O}_3(z)\mathcal{O}_1(x) \rangle_{\text{ext}} = \text{ext} \langle \mathcal{V}_{123} | e^{i[L_{(1)}^+ x + L_{(2)}^+ y + L_{(3)}^+ z]} | \mathcal{O}_2 \rangle \otimes | \mathcal{O}_3 \rangle \otimes | \mathcal{O}_1 \rangle . \quad (2.113)$$

This helps to define the slightly more complicated case of a non-extremal three-point function, where the operators \mathcal{O}_2 and \mathcal{O}_3 are also connected by Wick contractions. At tree level, we can split any of the operators \mathcal{O}_i into pieces \mathcal{O}_{ij} which are contracted to pieces \mathcal{O}_{ji} of operator \mathcal{O}_j . At the level of the states we have ⁷

$$\begin{aligned} |\mathcal{O}_1\rangle &= |\mathcal{O}_{13}\rangle \otimes |\mathcal{O}_{12}\rangle , \\ |\mathcal{O}_2\rangle &= |\mathcal{O}_{21}\rangle \otimes |\mathcal{O}_{23}\rangle , \\ |\mathcal{O}_3\rangle &= |\mathcal{O}_{32}\rangle \otimes |\mathcal{O}_{31}\rangle . \end{aligned} \quad (2.114)$$

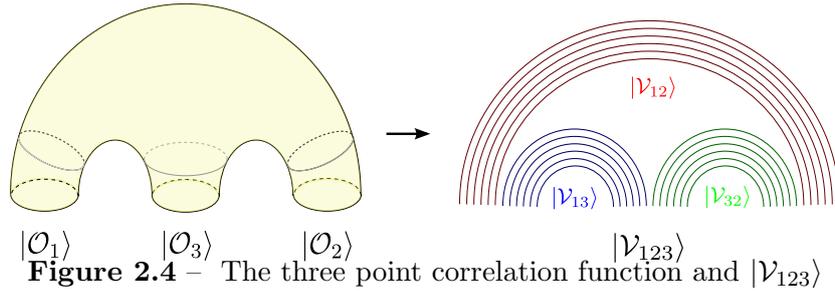
The non-extremal three point function, at tree level, can be then written in the same way as non-extremal

$$\langle \mathcal{O}_2(y)\mathcal{O}_3(z)\mathcal{O}_1(x) \rangle = \langle \mathcal{V}_{123} | e^{i[L_{(1)}^+ x + L_{(2)}^+ y + L_{(3)}^+ z]} | \mathcal{O}_2 \rangle \otimes | \mathcal{O}_3 \rangle \otimes | \mathcal{O}_1 \rangle , \quad (2.115)$$

but with the different vertex $|\mathcal{V}_{123}\rangle$ built out as

$$|\mathcal{V}_{123}\rangle = |\mathcal{V}_{12}\rangle \otimes |\mathcal{V}_{13}\rangle \otimes |\mathcal{V}_{32}\rangle = U_{(12)}^2 |V_{12}\rangle \otimes U_{(13)}^2 |V_{13}\rangle \otimes U_{(32)}^2 |V_{32}\rangle . \quad (2.116)$$

Numbering of the sites for the states in (2.114) subjects to the same convention as for



the case of the two-point correlators. Which means that if one uses the vertex $|\mathcal{V}_{ij}\rangle$, then the state belonging to the i -th operator is numbered in the natural order while the state belonging to the j -th state is numbered oppositely. Thus according to (2.116)

$$\begin{aligned} |\mathcal{O}_{12}\rangle &= |\mathcal{E}_1 \dots \mathcal{E}_{L_{12}}\rangle, & |\mathcal{O}_{13}\rangle &= |\mathcal{E}_1 \dots \mathcal{E}_{L_{13}}\rangle, \\ |\mathcal{O}_{21}\rangle &= |\mathcal{E}_{L_{12}} \dots \mathcal{E}_1\rangle, & |\mathcal{O}_{23}\rangle &= |\mathcal{E}_{L_{23}} \dots \mathcal{E}_1\rangle, \\ |\mathcal{O}_{31}\rangle &= |\mathcal{E}_{L_{13}} \dots \mathcal{E}_1\rangle, & |\mathcal{O}_{32}\rangle &= |\mathcal{E}_1 \dots \mathcal{E}_{L_{23}}\rangle. \end{aligned} \quad (2.117)$$

7. The writing below does not imply that the state associated to the operator i is a product, just that it belongs to the tensor product of the Fock spaces denoted by ij and ik .

2.3.5 Tree level correlation function in the $so(6)$ sector

In the scalar sector $so(6)$ the operators are built only from the fermionic oscillators and one can divide the vertex in bosonic and fermionic parts. The bosonic part would give the right space-time dependence and the fermionic part, which we will denote by $\langle v_{12}|$ will play the role of the vertex reduced to the scalar sector:

$$\langle \mathcal{O}_2(x)\mathcal{O}_1(y) \rangle = \langle \mathcal{V}_{12}| e^{i[L_{(1)}^+ x + L_{(2)}^+ y]} |\mathcal{O}_2\rangle^{(2)} \otimes |\mathcal{O}_1\rangle^{(1)} = \frac{\langle v_{12}|\mathcal{O}_2\rangle^{(2)} \otimes |\mathcal{O}_1\rangle^{(1)}}{(x-y)^{2\Delta_1}}. \quad (2.118)$$

The same is valid for the three point function at tree level,

$$\begin{aligned} \langle \mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_1(x_1) \rangle &= \\ &= \langle V_{123}| U_{(13)}^2 U_{(12)}^2 U_{(32)}^2 e^{i[L_{(1)}^+ x_1 + L_{(2)}^+ x_2 + L_{(3)}^+ x_3]} |\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle \otimes |\mathcal{O}_1\rangle \\ &= \frac{\langle v_{123}|\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle \otimes |\mathcal{O}_1\rangle}{|x_{12}|^{\Delta_{12}} |x_{13}|^{\Delta_{13}} |x_{23}|^{\Delta_{23}}}, \end{aligned} \quad (2.119)$$

where $\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k$ with $\{i, j, k\} = \{1, 2, 3\}$. So, the structure constant would be given as

$$C_{123} = \langle v_{123}|\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle \otimes |\mathcal{O}_1\rangle, \quad (2.120)$$

where we suppose that the states $|\mathcal{O}_i\rangle$ are normalized, $\mathcal{N}_i = \langle \mathcal{O}_i|\mathcal{O}_i\rangle = 1$. If this is not the case, one has to divide out $\sqrt{\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3}$.

2.3.6 Tree level correlation function in the $su(2)$ sector

In this section we will concentrate on the $su(2)$ sector of $\mathcal{N} = 4$ SYM. This sector is investigated in the literature quite extensively and will show how to get the results obtained previously in [67] using spin vertex approach and consider some other cases. For this purpose we will have to translate the spin vertex formalism to the language of the spin chains and algebraic Bethe-ansatz.

As it has been mentioned before, due to the large degeneracy of trace states at tree level, one prefers to use a pre-diagonalization and use as basis of states the eigenstates of the one-loop dilatation operator, which is conveniently given by (nested) algebraic Bethe ansatz. Suppose that we have built the one-loop Lax matrix

$$L_s(u) = u - i/2 - i(-1)^{|A|} E_0^{AB} E_s^{BA}. \quad (2.121)$$

where the generators in the auxiliary space $E_{0,d}^{AB}$ belong to the defining (4|4 dimensional) representation of $\mathfrak{psu}(2, 2|4)$ and E_s^{AB} are the generators in the actual physical representation, *e.g.* the oscillators representation. Using the property (2.110) of the vertex it is straightforward to show that

$$L^{(1)}(u)|\mathcal{V}_{12}\rangle = -L^{(2)}(-u)|\mathcal{V}_{12}\rangle. \quad (2.122)$$

The change of sign in the Lax matrix can be absorbed in the normalization, and we will tacitly assume in the following that we have done so. Let us now consider the monodromy matrices of the two chains

$$T^{(1)}(u) = L_1^{(1)}(u) \dots L_L^{(1)}(u), \quad T^{(2)}(u) = L_L^{(2)}(u) \dots L_1^{(2)}(u). \quad (2.123)$$

Now we act with $T^{(1)}$ on the spin vertex and then repeatedly apply the relation (2.122). The result is

$$L_1^{(1)}(u) \dots L_L^{(1)}(u) |\mathcal{V}_{12}\rangle = L_1^{(2)}(-u) \dots L_L^{(2)}(-u) |\mathcal{V}_{12}\rangle . \quad (2.124)$$

The right hand side is not exactly the monodromy matrix for the second chain $T^{(2)}(u)$, because the Lax matrices are in reverse order. This mismatch can be cured by taking an operation which reverses the order of the operators, like the (super) transposition t in the auxiliary space. In $so(4) \simeq su(2)_L \otimes su(2)_R$ ⁸ sector one can correlate the change of the signs of the supertraceless generators E^{ab} with the transposition

$$E^{ab} = -\sigma E^{ab,t} \sigma^{-1} . \quad (2.125)$$

where t denotes the (super) transposition in the quantum space and $\sigma = \sigma^{-1} = -\sigma_{2,L} \sigma_{2,R} = U_F^2$. As one can check on (2.121), in any of the $su(2)$ sectors we have

$$L(u) = L^{t_0,t}(u) = -\sigma L^{t_0}(-u) \sigma^{-1} = -\sigma_0 L^{t_0}(-u) \sigma_0^{-1} , \quad (2.126)$$

where $\sigma_0 = i\sigma_{2,0}$. The last equality sign comes from the invariance of the Lax matrix $[L_s(u), E_0^{ab} + E_s^{ab}] = 0$. Substituting one of the last two equalities above into the r.h.s. of in (2.124) we obtain⁹

$$T^{(1)}(u) |\mathcal{V}_{12}\rangle = \sigma T^{(2),t_0}(u) \sigma^{-1} |\mathcal{V}_{12}\rangle = \sigma_0 T^{(2),t_0}(u) \sigma_0^{-1} |\mathcal{V}_{12}\rangle . \quad (2.127)$$

or in matrix form

$$\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}^{(1)} |\mathcal{V}_{12}\rangle = \begin{pmatrix} \sigma A(u) \sigma^{-1} & \sigma C(u) \sigma^{-1} \\ \sigma B(u) \sigma^{-1} & \sigma D(u) \sigma^{-1} \end{pmatrix}^{(2)} |\mathcal{V}_{12}\rangle = \begin{pmatrix} D(u) & -B(u) \\ -C(u) & A(u) \end{pmatrix}^{(2)} |\mathcal{V}_{12}\rangle . \quad (2.128)$$

We will exemplify now the consequence of these relation in a given $su(2)$ sub-sector. The eigenvectors of the dilatation operator can be constructed by the action of the B operators on the vacuum state $|Z^L\rangle$ followed by an arbitrary $so(6)$ rotation \mathcal{R} in the quantum space,

$$|\mathcal{O}\rangle = \mathcal{R} B(u_1) \dots B(u_M) |Z^L\rangle . \quad (2.129)$$

Since we are working with operators which do not have components outside the $so(6)$ sector, we are going to use a version of the vertex $\langle v_{12}|$ truncated to $so(6)$. By equation (2.128) we obtain the rule which transfers the Bethe operators from one space to the other through the vertex,

$$\begin{aligned} \langle v_{12}| [\mathcal{R} B(u_1) \dots B(u_M)]^{(1)} &= \langle v_{12}| [B(u_M) \dots B(u_1) \mathcal{R}^{-1}]^{(2)} \\ &= \langle v_{12}| [\sigma C(u_M) \dots C(u_1) \sigma \mathcal{R}^{-1}]^{(2)} . \end{aligned} \quad (2.130)$$

8. In conventional sense the $so(4)$ sector doesn't exist, because the action of the one-loop dilatation operator is not closed on it. What we mean by $so(4)$ sector is the case when any elementary fields of the operators involved can be presented in the form $a_1 Z + a_2 \bar{Z} + a_3 X + a_4 \bar{X}$, but at the same time each of the operators corresponds to the $su(2)$ spin chain, $su(2)_L$ or $su(2)_R$. As an example one can consider the configuration described in [67]

9. We neglect again an overall normalization.

This relation is fundamental in exploiting the vertex, and it prescribes in particular how to characterize the flipped states

$$\begin{aligned}
{}^{(2)}\langle \bar{\mathcal{O}} | &= \langle v_{12} | \mathcal{O} \rangle^{(1)} = & (2.131) \\
&= \langle v_{12} | [B(u_M) \dots B(u_1) \mathcal{R}^{-1}]^{(2)} | Z^L \rangle^{(1)} = {}^{(2)}\langle \bar{Z}^L | [B(u_M) \dots B(u_1) \mathcal{R}^{-1}]^{(2)} \\
&= \langle v_{12} | [\sigma C(u_M) \dots C(u_1) \sigma \mathcal{R}^{-1}]^{(2)} | Z^L \rangle^{(1)} = {}^{(2)}\langle Z^L | [C(u_M) \dots C(u_1) \sigma \mathcal{R}^{-1}]^{(2)}
\end{aligned}$$

Using $B(u)^\dagger = -C(u^*)$ and considering distributions of rapidities which are self-conjugate, $\{u\} = \{u^*\}$ we conclude that, up to an overall sign,

$$|\bar{\mathcal{O}}\rangle = \mathcal{R} C(u_1) \dots C(u_M) |\bar{Z}^L\rangle = \mathcal{R} \sigma B(u_1) \dots B(u_M) |Z^L\rangle. \quad (2.132)$$

Now we would like to consider some concrete configurations of the three-point correlators from what we call $so(4)$ sector with two $su(2)$ orbits corresponding to (Z, X) ($su(2)_R$) and (Z, \bar{X}) ($su(2)_L$) rotations. As it has been mentioned above, although each of the operators we are going to consider will belong to the $su(2)$ sector (either right or left), the spin vertex needed for the computation of the correlators should be composed of all the elementary fields appearing in the operators. Thus from the spin vertex point of view the sector of interest is the $so(4)$ sector. According to our conventions

$$|Z\rangle = |0\rangle, \quad |\bar{Z}\rangle = c_1^\dagger d_1^\dagger c_2^\dagger d_2^\dagger |0\rangle, \quad |X\rangle = c_1^\dagger d_1^\dagger |0\rangle, \quad |\bar{X}\rangle = -c_2^\dagger d_2^\dagger |0\rangle. \quad (2.133)$$

Thus the L sector is generated by c_1, d_1 and the R sector by c_2, d_2 . Obviously, the generators in the two sectors commute, and the operators X, \bar{X}, Z, \bar{Z} can be seen as basis vectors in the bi-fundamental representation of $su(2)_R \otimes su(2)_L$,

$$\begin{aligned}
|Z\rangle &= |\uparrow\rangle_L \otimes |\uparrow\rangle_R \equiv |\uparrow\uparrow\rangle, & |\bar{Z}\rangle &= |\downarrow\rangle_L \otimes |\downarrow\rangle_R \equiv |\downarrow\downarrow\rangle, & (2.134) \\
|X\rangle &= |\uparrow\rangle_L \otimes |\downarrow\rangle_R \equiv |\uparrow\downarrow\rangle, & |\bar{X}\rangle &= -|\downarrow\rangle_L \otimes |\uparrow\rangle_R \equiv -|\downarrow\uparrow\rangle.
\end{aligned}$$

Then the full vertex will take the form

$$\begin{aligned}
|v_{12}\rangle^{so(4)} &= |Z\rangle^{(2)} \otimes |\bar{Z}\rangle^{(1)} + |X\rangle^{(2)} \otimes |\bar{X}\rangle^{(1)} + |\bar{Z}\rangle^{(2)} \otimes |Z\rangle^{(1)} + |\bar{X}\rangle^{(2)} \otimes |X\rangle^{(1)} \\
&= |v_{12}\rangle^{su(2)_L} \otimes |v_{12}\rangle^{su(2)_R}, & (2.135)
\end{aligned}$$

where

$$|v_{12}\rangle^{su(2)_{L,R}} = |\uparrow\rangle_{L,R}^{(2)} \otimes |\downarrow\rangle_{L,R}^{(1)} - |\downarrow\rangle_{L,R}^{(2)} \otimes |\uparrow\rangle_{L,R}^{(1)}. \quad (2.136)$$

We can have two different cases (see [121] for more details):

i) The $I - I - I$ case, when all the three operators are in the same sector, say R . In this case, the three operators can be chosen as

$$\begin{aligned}
|\mathcal{O}_1\rangle &= \mathcal{R}_1 B_R(u_1) \dots B_R(u_{M_1}) |Z^{L_1}\rangle, & (2.137) \\
|\mathcal{O}_2\rangle &= \mathcal{R}_2 \sigma B_R(v_1) \dots B_R(v_{M_2}) |Z^{L_2}\rangle, \\
|\mathcal{O}_3\rangle &= \mathcal{R}_3 \sigma B_R(w_1) \dots B_R(w_{M_3}) |Z^{L_3}\rangle.
\end{aligned}$$

The convention is such that $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = 1$ reduces to the extremal case. Exploring this case goes beyond the scope of this thesis. An interested reader is encouraged to look

at [97], where the authors applied the separation of variables technique for computing the $I - I - I$ correlators.

ii) The $I - I - II$ case, when two operators, say \mathcal{O}_1 and \mathcal{O}_2 , are in the sector $R(L)$ and \mathcal{O}_3 is in another sector $L(R)$. In this case we choose

$$\begin{aligned} |\mathcal{O}_1\rangle &= \mathcal{R}_1 B_R(u_1) \dots B_R(u_{M_1}) |Z^{L_1}\rangle, \\ |\mathcal{O}_2\rangle &= \mathcal{R}_2 \sigma B_R(v_1) \dots B_R(v_{M_2}) |Z^{L_2}\rangle, \\ |\mathcal{O}_3\rangle &= \mathcal{R}_3 B_L(w_1) \dots B_L(w_{M_3}) |Z^{L_3}\rangle. \end{aligned} \quad (2.138)$$

Again, our choice is such that $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = 1$ is the case originally considered in [67]. In this case the left and right sector decouple

$$\begin{aligned} C_{123}^{EGSV} &= {}^{so(4)}\langle v_{123} | B_L(w) |Z^{L_3}\rangle \otimes \sigma_{(2)} B_R(v) |Z^{L_2}\rangle \otimes B_R(u) |Z^{L_1}\rangle \\ &= {}^{so(4)}\langle v_{123} | \sigma_{(32)} B_L(w) |Z^{L_3}\rangle \otimes \sigma_{(21)} B_R(v) |Z^{L_2}\rangle \otimes B_R(u) |Z^{L_1}\rangle. \\ &= \text{SIMPLE} \times \text{INVOLVED} \end{aligned} \quad (2.139)$$

The SIMPLE part is given by the contribution of the L sector (as before we don't pay attention to the signs):

$$\begin{aligned} \text{SIMPLE} &= {}^{su(2)_L}\langle v_{123} | \sigma_{(32)L} B_L(w) | \uparrow^{L_3}\rangle \otimes \sigma_{(21)L} | \uparrow^{L_2}\rangle \otimes | \uparrow^{L_1}\rangle \\ &= \langle \downarrow^{L_3} | \sigma_{(32)L} B_L(w) | \uparrow^{L_3}\rangle = \langle \downarrow^{L_{31}} | \langle \uparrow^{L_{32}} | B_L(w) | \uparrow^{L_3}\rangle, \end{aligned} \quad (2.141)$$

while INVOLVED is given by the contribution of the R sector

$$\begin{aligned} \text{INVOLVED} &= {}^{su(2)_R}\langle v_{123} | \sigma_{(32)R} | \uparrow^{L_3}\rangle \otimes \sigma_{(21)R} B_R(v) | \uparrow^{L_2}\rangle \otimes B_R(u) | \uparrow^{L_1}\rangle \\ &= {}^{su(2)_R}\langle v_{12} | \sigma_{(21)R} \langle \uparrow^{L_{23}} | B_R(v) | \uparrow^{L_2}\rangle \otimes \langle \downarrow^{L_{13}} | B_R(u) | \uparrow^{L_1}\rangle = \\ &= {}^{su(2)_R}\langle \tilde{v}_{12} | \langle \uparrow^{L_{23}} | B_R(v) | \uparrow^{L_2}\rangle \otimes \langle \downarrow^{L_{13}} | B_R(u) | \uparrow^{L_1}\rangle, \end{aligned} \quad (2.142)$$

where $\langle \tilde{v}_{12} | = {}^{(2)}\langle \uparrow | \otimes {}^{(1)}\langle \uparrow | + {}^{(2)}\langle \downarrow | \otimes {}^{(1)}\langle \downarrow |$. This gives the known result of [67] combined with the freezing trick from [68] (see [69] for comparison). The case when the global rotations $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are arbitrary is considered in [121].

2.3.7 Monodromy condition on the spin vertex

In this section we are going to show that the local symmetry condition (2.110) of the spin vertex can be reformulated as an extended symmetry. This is the same Yangian symmetry, satisfied by the tree-level amplitudes in $\mathcal{N} = 4$ SYM [122].

The spin vertex is an invariant of the Yangian. We are going first to show this on the two-vertex, and then extend it to the three-vertex which is needed for the computation of the three point function. There are two types of monodromy matrices which are interesting for us. The first is the monodromy matrix where the auxiliary space is in the defining, $4|4$ dimensional, representation. This monodromy matrix is useful to build the Yangian generators and the for the nested Bethe ansatz procedure. The second type of monodromy matrix, useful for getting the local conserved quantities, contains the same physical representation in the auxiliary and quantum spaces. Here we construct the monodromy matrix with the auxiliary space in the defining representation. For the monodromy matrix with the auxiliary space in the physical representation, the construction of the $so(6)$ sector is

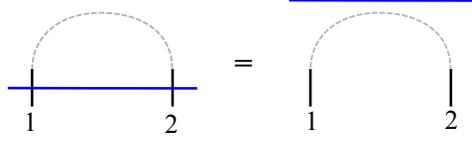


Figure 2.5 – The basic monodromy relation of the spin vertex.

relatively straightforward, however the construction in the $sl(2)$ sector is more subtle and we are not doing it here.

Let us take the $psu(2, 2|4)$ R -matrix living in the tensor product of the defining and the physical representations

$$R_{01}(u) = u - i\Pi_{01} , \quad \Pi_{01} = (-1)^{|A|} E_0^{AB} E_1^{BA} , \quad (2.143)$$

where E_0^{AB} are $4|4 \times 4|4$ super matrices, the generators in the quantum space are in the oscillator representation $E_1^{BA} = \bar{\psi}^A \psi^B$ and the summation over the repeated indexes is implied. When E_1^{BA} are also in the defining representation, Π_{01} is a super-permutation. In the representation we are considering

$$\begin{aligned} \Pi_{01}^2 &= (-1)^{|A|+|C|} E_0^{AB} E_1^{BA} E_0^{CD} E_1^{DC} = (-1)^{|A|+|B|+(|A|+|B|)(|B|+|D|)} E_0^{AD} E_1^{BA} E_1^{DB} \\ &= (-1)^{|A|} E_0^{AD} E_1^{DA} (E_1^{BB} - 1) + E_1^{BB} = \Pi_{01} (E_1^{BB} - 1) + E_1^{BB} = -\Pi_{01} . \end{aligned} \quad (2.144)$$

Here we have used the (anti)commutation relations $[\psi^A, \bar{\psi}^B]_{\pm} = \delta^{AB}$ and that in the physical representation the central charge should be zero ($c = E_1^{BB} = \bar{\psi}^B \psi^B = 0$)¹⁰ and in the auxiliary representation $E_0^{BB} = 1$. The R -matrix above satisfies the unitarity condition

$$R_{01}(u)R_{01}(-i - u) = -u(i + u) . \quad (2.145)$$

For a representation with arbitrary central charge c , the unitarity condition would be

$$R_{01}(u)R_{01}(i(c - 1) - u) = -u(i(1 - c) + u) - c . \quad (2.146)$$

We are now going to build the monodromy condition for the two-site vertex $|\mathcal{V}_{12}\rangle$,

$$R_{01}(u)R_{02}(u)|\mathcal{V}_{12}\rangle = -R_{01}(u)R_{01}(-i - u)|\mathcal{V}_{12}\rangle = u(u + i)|\mathcal{V}_{12}\rangle . \quad (2.147)$$

Here we have used that the R matrix is related to the Lax matrix defined in (2.121) by $R_{01}^{(2)}(u) = L^{(2)}(u + i/2)$, and then used the property (2.122) of the vertex

$$R_{02}(u)|\mathcal{V}_{12}\rangle = -R_{01}(-i - u)|\mathcal{V}_{12}\rangle . \quad (2.148)$$

The property (2.147) can be nicely presented by drawing some kind of diagrams with the R -matrix corresponding to the crossing of the solid lines, and the vertex presented by the dashed line (see Fig. 2.5). By means of this diagrams it is easy to realize that (2.147) can be lifted to the two-vertex with an arbitrary number of sites:

$$t^{(12)}(u)|\mathcal{V}_{12}\rangle = R_{01}^{(1)}(u) \dots R_{0L}^{(1)}(u) R_{0L}^{(2)}(u) \dots R_{01}^{(2)}(u) |\mathcal{V}_{12}\rangle = (u(u + i))^L |\mathcal{V}_{12}\rangle , \quad (2.149)$$

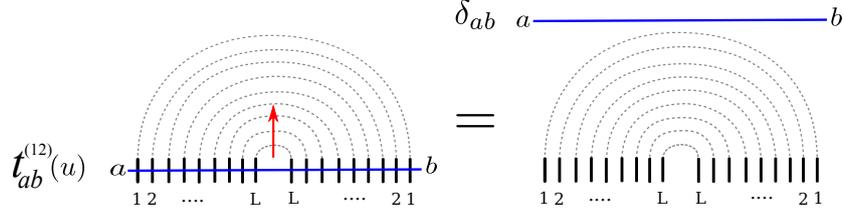


Figure 2.6 – The two chain monodromy matrix $t_{ab}^{(12)}(u)$ and its action on the vertex $|\mathcal{V}_{12}\rangle$

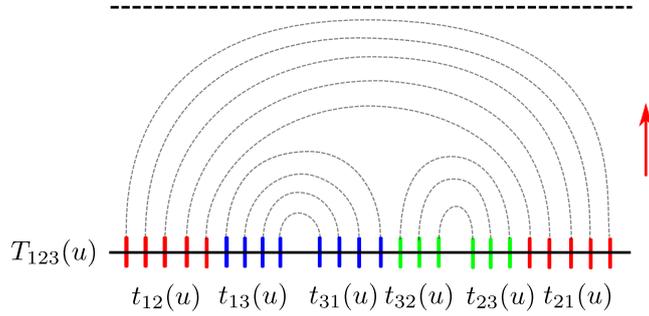


Figure 2.7 – The three-point spin vertex and monodromy condition.

as depicted in Fig. 2.6. Finally the relation (2.147) can be generalized for the three-vertex (Fig. 2.7) with the monodromy matrix T_{123} .

$$T_{123}(u) = t^{(12)}(u)t^{(13)}(u)t^{(31)}(u)t^{(32)}(u)t^{(23)}(u)t^{(21)}(u) , \quad (2.150)$$

$$T_{123}(u)|\mathcal{V}_{123}\rangle = (u(u+i))^{\frac{L_1+L_2+L_3}{2}}|\mathcal{V}_{123}\rangle. \quad (2.151)$$

The subsectors:

The $psu(2, 2|4)$ R matrix can be readily reduced to different subsectors, just by restricting the sum in the definition of the central charge (2.69) to the corresponding subsector. As a result, the central charge can take non-zero value $c = E_1^{BB}$.

- In the $su(1|1)$, $su(2|3)$ and $su(2)$ sector, where the fields belong to the fundamental representation, $c = 1$, so that the unitarity condition is slightly modified,

$$\Pi_{01}^2 = 1 , \quad R_{01}(u)R_{01}(-u) = -(u^2 + 1) . \quad (2.152)$$

The monodromy condition will be

$$R_{01}(u)R_{02}(u-i)|\mathcal{V}_{12}\rangle = -R_{01}(u)R_{01}(-u)|\mathcal{V}_{12}\rangle = (u^2 + 1)|\mathcal{V}_{12}\rangle . \quad (2.153)$$

- In the $sl(2)$ sector, $c = 0$, so the unitarity and monodromy conditions are the same as for $psu(2, 2|4)$.

10. The condition $c = 0$ should be understood as a constraint imposed on the states, which projects on the irreducible representation we are interested in. This constraint can be implemented in the definition of the spin vertex, but then the vertex will lose its nice exponential form.

– In the $so(6)$ sector we have $c = 2$, so that

$$\Pi_{01}^2 = \Pi_{01} + 2, \quad R_{01}(u)R_{01}(i-u) = u(i-u) - 2. \quad (2.154)$$

The monodromy condition is then

$$R_{01}(u)R_{02}(u-2i)|\mathcal{V}_{12}\rangle = -R_{01}(u)R_{01}(i-u)|\mathcal{V}_{12}\rangle = (u(u-i)+2)|\mathcal{V}_{12}\rangle. \quad (2.155)$$

2.3.8 The relation between the full vertex $|\mathcal{V}_{123}\rangle$ and the polynomial vertex (2.25)

In this section we will relate the full vertex $|\mathcal{V}_{123}\rangle$ with the polynomial vertex (2.25). First of all we have to reduce the full vertex $|\mathcal{V}_{123}\rangle$ to the scalar sector, which means that we will consider its shorter form $|v_{123}\rangle$ with elementary one-site two-vertex given by¹¹

$$\begin{aligned} |v_{ij}\rangle = & |X\rangle^{(i)} \otimes |\bar{X}\rangle^{(j)} + |\bar{X}\rangle^{(i)} \otimes |X\rangle^{(j)} + |Y\rangle^{(i)} \otimes |\bar{Y}\rangle^{(j)} + |\bar{Y}\rangle^{(i)} \otimes |Y\rangle^{(j)} + \\ & + |Z\rangle^{(i)} \otimes |\bar{Z}\rangle^{(j)} + |\bar{Z}\rangle^{(i)} \otimes |Z\rangle^{(j)} + \dots, \end{aligned} \quad (2.156)$$

where the three dots as usual stand for the terms which don't satisfy the condition that the central charge should be zero. These terms cancel when computing the correlators and we drop it in the following. Recalling that $X = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $Y = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)$, $Z = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_5)$, we rewrite the vertex as follows

$$|v_{ij}\rangle = |Z\rangle^{(i)} \otimes |\bar{Z}\rangle^{(j)} + |\bar{Z}\rangle^{(i)} \otimes |Z\rangle^{(j)} + \sum_{i=1}^4 |\phi_i\rangle^{(i)} \otimes |\phi_i\rangle^{(j)}. \quad (2.157)$$

Then we the full three-vertex will take the form

$$\begin{aligned} |\tilde{v}_{123}\rangle = & \prod_{k=1}^{L_{12}} \left(|Z\rangle_{L_1-k+1}^{(1)} \otimes |\bar{Z}\rangle_k^{(2)} + |\bar{Z}\rangle_{L_1-k+1}^{(1)} \otimes |Z\rangle_k^{(2)} + \sum_{i=1}^4 |\phi_i\rangle_{L_1-k+1}^{(1)} \otimes |\phi_i\rangle_k^{(2)} \right) \otimes \\ & (2.158) \\ & \prod_{k=1}^{L_{13}} \left(|Z\rangle_k^{(1)} \otimes |\bar{Z}\rangle_{L_3-k+1}^{(3)} + |\bar{Z}\rangle_k^{(1)} \otimes |Z\rangle_{L_3-k+1}^{(3)} + \sum_{i=1}^4 |\phi_i\rangle_k^{(1)} \otimes |\phi_i\rangle_{L_3-k+1}^{(3)} \right) \otimes \\ & \prod_{k=1}^{L_{23}} \left(|Z\rangle_{L_2-k+1}^{(2)} \otimes |\bar{Z}\rangle_k^{(3)} + |\bar{Z}\rangle_{L_2-k+1}^{(2)} \otimes |Z\rangle_k^{(3)} + \sum_{i=1}^4 |\phi_i\rangle_{L_2-k+1}^{(2)} \otimes |\phi_i\rangle_k^{(3)} \right), \end{aligned}$$

where we changed a bit the convention of numbering the sites in the vertex (that's why the vertex is marked with tilde) in order to adjust the convention of the section 2.2.1.¹²

11. According to our convention the indexes (i, j) , related to the states connected by the vertex, should be ordered oppositely, e.g. $|Z\rangle^{(j)} \otimes |\bar{Z}\rangle^{(i)}$, however in this case we deal just with the scalar fields, which can be permuted without picking any additional signs, which we cared about when described the most general case.

12. Here the vertex contracts for example the site number i of one operator with the site number $L-i+1$ of another unlike the original formulation (2.104), where the vertex contracts those sites of the operators which has the same numbers. This convention has been used in the previous sections in order to ensure the writing of some of the vertex properties in the nice way. The price which we paid is the weird way of numbering the sites in the operators like in (2.117).

Further we will rewrite the generating fields (2.22) in the oscillator representation, we will get

$$\begin{aligned}
|F_1\rangle &= \left(|Z_1\rangle + \sum_{i=1}^4 x_1^i |\phi_1^i\rangle \right) \otimes \dots \otimes \left(|Z_{L_1}\rangle + \sum_{i=1}^4 x_{L_1}^i |\phi_{L_1}^i\rangle \right), \\
|F_2\rangle &= \left(|\bar{Z}_1\rangle + \sum_{i=1}^4 y_1^i |\phi_1^i\rangle \right) \otimes \dots \otimes \left(|\bar{Z}_{L_2}\rangle + \sum_{i=1}^4 y_{L_2}^i |\phi_{L_2}^i\rangle \right), \\
|F_3\rangle &= \left(|\bar{Z}_1\rangle + \sum_{i=1}^4 z_1^i |\phi_1^i\rangle \right) \otimes \dots \otimes \left(|\bar{Z}_{L_3}\rangle + \sum_{i=1}^4 z_{L_3}^i |\phi_{L_3}^i\rangle \right).
\end{aligned} \tag{2.159}$$

Then the polynomial vertex $V_{123}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ can be obtained by the following expression

$$V_{123}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \langle \tilde{v}_{123} | F_1 \rangle | F_2 \rangle | F_3 \rangle. \tag{2.160}$$

Chapter 3

HHL correlators and diagonal form factors

This section is devoted to considering some specific types of the three-point functions in $\mathcal{N} = 4$ SYM, namely, heavy-heavy-light (HHL) correlators (the case when the length of one of the operators is significantly smaller than the length of the two others). It turns out that the computation of such correlators can be related to the computation of the form factors in the 2D quantum integrable field theory in the finite volume. At strong coupling this idea gets the confirmation from the observation that the corresponding HHL correlator at semi-classical approximation can be computed as an integral of the vertex operator (defined by the light operator) over the classical states corresponding to the heavy operators. From the weak coupling point of view this relation originates from the connection between the gauge invariant operators and the integrable spin chains. Due to this relation it appears that the computation of the correlator can be formulated as the computation of the spin chain form factor between the two spin chain states corresponding to the two of the operators with the spin chain operator defined by the third operator [123]. The most convenient probably is the case when the two operators are conjugated to each other. In [124] the authors considered such a correlator at strong coupling and noticed on the example of some concrete computations that the dependence of the correlator on the length of the heavy operator L is defined by the volume dependence of the diagonal form factor in 2D integrable models formulated in [125] at any coupling. From this observation the authors conjectured that the L -dependence of the HHL correlator should be defined by the volume dependence of the diagonal form factor established in [125] at any coupling. In the first part of this chapter we will prove the conjecture of [124] at weak coupling on the example of the HHL correlators with heavy states being from the $su(2)$ sector. In the second part by exploiting the recently proposed hexagon approach we will extend our prove to finite coupling in the asymptotic regime. The material is based on the results of the papers [126, 127]

3.1 HHL correlator at weak coupling

3.1.1 Form factors in integrable models

Form factors in infinite volume

Here we will review some background related to the form factors in integrable models necessary for the following. We consider a 1 + 1 dimensional integrable quantum field theory defined by its S -matrix. For simplicity we restrict ourselves to diagonally scattering theories with a self-conjugated particle, the generalization to any diagonally scattering theory is straightforward.

The infinite volume states can be characterized by the set of particles momenta. The particles are labeled by their rapidities u with the corresponding values of the momenta $p(u)$ and the energy $\varepsilon(u)$.

In the remote past we have an *in* state which is defined by the corresponding order of the rapidities: the fastest one is the leftmost while the slowest is the rightmost. The particles in an *out* state are ordered oppositely:

$$|u_1, \dots, u_N\rangle = \begin{cases} |u_1, \dots, u_N\rangle^{\text{in}} & u_1 > \dots > u_N \\ |u_1, \dots, u_N\rangle^{\text{out}} & u_1 < \dots < u_N \end{cases} \quad (3.1)$$

The infinite volume states that differ only in the order of rapidities are related by the two particle S -matrix:

$$|u_1, \dots, u_i, u_{i+1}, \dots, u_N\rangle = S(u_i, u_{i+1}) |u_1, \dots, u_{i+1}, u_i, \dots, u_N\rangle \quad (3.2)$$

The energy of a multiparticle state is the sum of the one particle energies

$$E(u_1, \dots, u_N) = \sum_{i=1}^N \varepsilon(u_i). \quad (3.3)$$

In infinite volume we normalize the *in* states as

$${}^{\text{in}}\langle u'_1, \dots, u'_M | u_1, \dots, u_N \rangle^{\text{in}} = (2\pi)^N \delta_{NM} \delta(u_1 - u'_1) \dots \delta(u_N - u'_N), \quad (3.4)$$

and the norm of a general state can be determined from (3.4) by (3.2).

Let us consider the matrix elements of a *local* operator $\mathcal{O}(t, x)$ between asymptotic states. The space-time dependence can be factored out

$${}^{\text{out}}\langle u'_1, \dots, u'_M | \mathcal{O}(x, t) | u_1, \dots, u_N \rangle^{\text{in}} = e^{it\Delta E - ix\Delta P} {}^{\text{out}}\langle u'_1, \dots, u'_M | \mathcal{O}(0, 0) | u_1, \dots, u_N \rangle^{\text{in}},$$

where

$$\begin{aligned} \Delta E &= \sum_{j=1}^M \varepsilon(u'_j) - \sum_{k=1}^N \varepsilon(u_k), \\ \Delta P &= \sum_{j=1}^M p(u'_j) - \sum_{k=1}^N p(u_k), \end{aligned} \quad (3.5)$$

and we define the *form factor* of operator \mathcal{O} as

$$F_{M,N}^{\mathcal{O}}(u'_1, \dots, u'_M | u_1, \dots, u_N) = {}^{\text{out}} \langle u'_1, \dots, u'_M | \mathcal{O}(0, 0) | u_1, \dots, u_N \rangle^{\text{in}} \quad (3.6)$$

The form factors are *a priori* defined for ordered set of incoming and outgoing rapidities but can be analytically continued by (3.2). A form factor is a meromorphic function in all variables and each pole has a physical origin [128].

Suppose that the theory possesses crossing symmetry, i.e. a transformation which maps an outgoing particle with rapidity u to an incoming anti-particle with rapidity \bar{u} . The crossing symmetry implies the crossing equation for the form factors which, in case of a single self-conjugated particle, reads

$$F_{M,N}^{\mathcal{O}}(u'_1, \dots, u'_M | u_1, \dots, u_N) = F_{M-1, N+1}^{\mathcal{O}}(u'_1, \dots, u'_{M-1} | \bar{u}_M, u_1, \dots, u_N) \quad (3.7)$$

$$+ \sum_{k=1}^N \langle u'_M | u_k \rangle \prod_{l=1}^{k-1} S(u_l, u_k) F_{M-1, N-1}^{\mathcal{O}}(u'_1, \dots, u'_{M-1} | u_1, \dots, \hat{u}_k, \dots, u_N)$$

where the terms on the second line of (3.7) describe disconnected processes that occur if one of the incoming and one of the outgoing particles have the same rapidity. The hat \hat{u}_k denotes that u_k is missing from the list of rapidities. By using the crossing relation all form factors can be expressed in terms of *elementary* form factors

$$F_N^{\mathcal{O}}(u_1, \dots, u_N) = \langle 0 | \mathcal{O}(0, 0) | u_1, \dots, u_N \rangle. \quad (3.8)$$

These elementary form factors satisfy several functional relations, called the form factor axioms, which form the basis of the form factor bootstrap program [128]. Here they are.

1. Permutation

$$F_N^{\mathcal{O}}(u_1, \dots, u_i, u_{i+1}, \dots, u_N) = S(u_i, u_{i+1}) F_N^{\mathcal{O}}(u_1, \dots, u_i, u_{i+1}, \dots, u_N) \quad (3.9)$$

2. Periodicity

$$F_N^{\mathcal{O}}(u_1, \dots, u_i, u_{i+1}, \dots, u_N + 2\pi i) = F_N^{\mathcal{O}}(u_N, u_1, \dots, u_i, u_{i+1}, \dots, u_{N-1}) \quad (3.10)$$

3. Kinematical singularities

$$-i \text{Res}_{u=u'} F_{N+2}^{\mathcal{O}}(u + i\pi, u', u_1, \dots, u_N) = \left(1 - \prod_{i=1}^N S(u, u_i)\right) F_N^{\mathcal{O}}(u_1, \dots, u_N) \quad (3.11)$$

3. Dynamical singularities

$$-i \text{Res}_{u=u'} F_{N+2}^{\mathcal{O}}(u + i\bar{\theta}/2, u' - i\bar{\theta}/2, u_1, \dots, u_N) = \Gamma F_{N+1}^{\mathcal{O}}(u, u_1, \dots, u_N), \quad (3.12)$$

where u is so-called fusion angle, $\bar{\theta} = \pi - \theta$ and Γ defined from

$$S(u \sim i\theta) \sim \frac{i\Gamma^2}{u - i\theta}. \quad (3.13)$$

Diagonal form factors

The diagonal form factor of the local operator \mathcal{O} , defined as

$$\text{in}\langle u_1, \dots, u_N | \mathcal{O}(0, 0) | u_1, \dots, u_N \rangle^{\text{in}}, \quad (3.14)$$

is singular due to the disconnected terms in the crossing relation (3.7). To avoid the singularities we regularize it by slightly shifting the outgoing rapidities. After crossing we get

$$F_{2N}^{\mathcal{O}}(\bar{u}_1 + \epsilon_1, \dots, \bar{u}_N + \epsilon_N, u_N, \dots, u_1) = \langle 0 | \mathcal{O} | \bar{u}_1 + \epsilon_1, \dots, \bar{u}_N + \epsilon_N, u_1, \dots, u_N \rangle^{\text{in}} \quad (3.15)$$

The diagonal limit, $\epsilon_i \rightarrow 0$, of (3.15) is not well-defined. It was first noticed in [129] that the singular parts vanish in the limit when all $\epsilon_i \rightarrow 0$, but the result depends on the direction of the limit. Its general structure can be written as

$$\begin{aligned} & F_{2N}^{\mathcal{O}}(\bar{u}_1 + \epsilon_1, \dots, \bar{u}_N + \epsilon_N, u_N, \dots, u_1) \\ &= \prod_{i=1}^N \frac{1}{\epsilon_i} \cdot \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_N=1}^N a_{i_1 i_2 \dots i_N}(u_1, \dots, u_N) \epsilon_{i_1} \epsilon_{i_2} \cdots \epsilon_{i_N} + \cdots \end{aligned} \quad (3.16)$$

where $a_{i_1 i_2 \dots i_N}$ is a completely symmetric tensor of rank N . The three dots denote the terms which vanish in the $\epsilon_i \rightarrow 0$ limit.

There are two generally used regularization scheme in the literature. The first is the so-called *symmetric* evaluation when we set all ϵ_i to be the same

$$F_N^{\mathcal{O},s}(u_1, \dots, u_N) = \lim_{\epsilon \rightarrow 0} F_{2N}^{\mathcal{O}}(\bar{u}_1 + \epsilon, \dots, \bar{u}_N + \epsilon, u_N, \dots, u_1). \quad (3.17)$$

The second scheme is called *connected*, in which the diagonal form factors are defined as the ϵ -independent term of (3.16):

$$F_N^{\mathcal{O},c}(u_1, \dots, u_N) = N! a_{12\dots N}. \quad (3.18)$$

Both the symmetric and the connected diagonal form factors are symmetric in the rapidity variables u_1, \dots, u_N and not independent: each can be expressed with use of the other [125].

Diagonal form factors in finite volume

Here we will summarize the results about the volume dependence of the diagonal form factors following [125, 130].

In finite volume L , the rapidities are quantized and a generic multiparticle state can be labeled by the Bethe quantum numbers $\{|I_1, \dots, I_N\}_L$. Apart from the infinite volume case the states in finite volume should be symmetric in all the particles. We adapt our notation to the conventions used in [125, 130] and order the quantum numbers in a monotonly decreasing sequence, $I_1 \geq \dots \geq I_N$ ¹. The quantized rapidities with the quantum numbers $\{I_1, \dots, I_N\}$ are solutions of the corresponding Bethe Ansatz equations:

$$\Phi_j(\{u_1, \dots, u_N\}) = 2\pi I_j, \quad j = 1, \dots, N, \quad (3.19)$$

1. Apart from the free boson case all known S -matrix obey the property $S(u, u) = -1$ which is an effective Pauli exclusion. In this cases we have $I_1 > \dots > I_N$.

where we defined

$$\Phi_j(\{u_1, \dots, u_N\}) = p(u_j)L - i \sum_{\substack{k=1 \\ k \neq j}} \log S(u_k, u_j). \quad (3.20)$$

The finite volume states are orthogonal to each other

$${}_L \langle \{J_1, \dots, J_M\} | \{I_1, \dots, I_N\} \rangle_L \propto \delta_{NM} \delta_{I_1, J_1} \cdots \delta_{I_N, J_N} \quad (3.21)$$

and their normalization is a question of the convention.

One can change from the quantum number representation of states to the rapidity representation. This change of variables involves the Jacobian, which is the density of N -particle states, defined as

$$\begin{aligned} \varrho_N(u_1, \dots, u_N)_L &= \det \mathcal{J}^{(N)}(u_1, \dots, u_N)_L \\ \mathcal{J}_{k,l}^{(N)}(u_1, \dots, u_N)_L &= \frac{\partial \Phi_k(u_1, \dots, u_N)}{\partial u_l}, \quad k, l = 1, \dots, N. \end{aligned} \quad (3.22)$$

where we explicitly indicated the volume dependence of these quantities. Then the relation between the states in quantum number and rapidity representation reads as

$$|\{I_1, \dots, I_N\}\rangle_L = \frac{1}{\sqrt{\varrho_N(u_1, \dots, u_N)_L \prod_{i < j} S(u_i, u_j)}} |u_1, \dots, u_N\rangle \quad (3.23)$$

where the rapidities $\{u_i\}$ are the solutions of the Bethe Ansatz equations (3.20) corresponding to the quantum numbers $\{I_1, \dots, I_N\}$. This identification holds up to exponential corrections. The product of S -matrices in the denominator ensures that the finite volume state is indeed symmetric under the exchange of particles.

Defining the system in finite volume regularizes all the divergences appearing in the diagonal limit of form factors (3.16), thus the normalized finite volume diagonal matrix element

$$\frac{{}_L \langle \{I_1, \dots, I_N\} | \mathcal{O}(0, 0) | \{I_1, \dots, I_N\} \rangle_L}{{}_L \langle \{I_1, \dots, I_N\} | \{I_1, \dots, I_N\} \rangle_L} \quad (3.24)$$

is finite, well defined and does not depend on the normalization of the states. Further we represent the connection between finite and infinite volume form factors. This problem, challenging in the general case [131, 132], becomes considerably simpler if we neglect the exponentially small wrapping corrections.

Up to wrapping, the finite volume N -particle diagonal form factor (3.24) can be expressed as a sum over the bipartite partitions of the full set $\{1, 2, \dots, N\}$, involving the infinite volume form factors and some kind of densities of states. As the diagonal form factors in infinite volume depend on the regularization scheme, this series is also scheme dependent. In case of the connected evaluation the relation reads [133, 125]

$$\frac{{}_L \langle \{I_1, \dots, I_N\} | \mathcal{O}(0, 0) | \{I_1, \dots, I_N\} \rangle_L}{{}_L \langle \{I_1, \dots, I_N\} | \{I_1, \dots, I_N\} \rangle_L} = \frac{1}{\rho_N(\{1, \dots, N\})} \sum_{\alpha \subseteq \{1, \dots, N\}} f^{\mathcal{O}}(\{u_k\}_{k \in \bar{\alpha}}) \rho_N(\alpha) \quad (3.25)$$

where $\bar{\alpha}$ denotes the complement of α in the full set. The functions appearing on the right hand side are the connected diagonal form factors

$$f^{\mathcal{O}}(u_1, \dots, u_l) = F_l^{\mathcal{O},c}(u_1, \dots, u_l) \quad (3.26)$$

The functions ρ_N are defined as the diagonal minor determinants of the N -particle Jacobian (3.22),

$$\rho_N(\alpha) = \det_{k,l \in \alpha} \mathcal{J}_{k,l}^{(N)}(u_1, \dots, u_N)_L \quad , \quad \alpha \subseteq \{1, \dots, N\}. \quad (3.27)$$

They can also be referred to as partial Gaudin norms. As special cases we have

$$\rho_N(\{1, \dots, N\}) = \varrho_N(u_1, \dots, u_N)_L \quad ; \quad \rho_N(\emptyset) = 1. \quad (3.28)$$

We want to emphasize that the function $\rho_N(\alpha)$ depend on *all* the N rapidities. The set of rapidities $\{u_i\}$ in the right hand side of (3.25) is the solution of the Bethe Ansatz equations (3.20) corresponding to the quantum numbers $\{I_1, \dots, I_N\}$. Thus, the *explicit* volume dependence is encoded *only* into the factors ρ_N , the connected form factors $f^{\mathcal{O}}$ depend on the volume only *implicitly* via the Bethe Ansatz equations.

As the connected and symmetric diagonal form factors are not independent, we can express the finite volume matrix element in the symmetric regularization scheme. In this case the series take the form [130]

$$\frac{{}_L\langle\{I_1, \dots, I_N\}|\mathcal{O}(0,0)|\{I_1, \dots, I_N\}\rangle_L}{{}_L\langle\{I_1, \dots, I_N\}|\{I_1, \dots, I_N\}\rangle_L} = \frac{1}{\rho_N(\{1, \dots, N\})} \sum_{\alpha \subseteq \{1, \dots, n\}} F_{|\alpha|}^s(\{u_k\}_{k \in \bar{\alpha}}) \rho_{|\alpha|}(\alpha). \quad (3.29)$$

Here again, the rapidities $\{u_i\}$ are the solutions of the Bethe Ansatz equations (3.20) with the quantum numbers $\{I_1, \dots, I_N\}$. The $\rho_{|\alpha|}$ functions appearing in the sum are the $|\alpha|$ -particle densities of state (3.28,3.22) evaluated at the rapidities $\{u_i\}_{i \in \alpha}$. Note that, contrary to the connected expansion, they depend *only* on the rapidities labeled by the set α . The explicit volume dependence is still carried only by the ρ functions.

Form factor of densities of conserved charges

An important special case of local operators is the density of a conserved quantity,

$$Q = \int_0^L J(x, t) dx \quad (3.30)$$

where Q acts diagonally and additively on the multiparticle states. Its density therefore satisfies

$$\frac{{}_L\langle\{I_1, \dots, I_N\}|J(0,0)|\{I_1, \dots, I_N\}\rangle_L}{{}_L\langle\{I_1, \dots, I_N\}|\{I_1, \dots, I_N\}\rangle_L} = \frac{1}{L} \sum_{j=1}^N q(u_j), \quad (3.31)$$

where $\{u_i\}$ are the solutions of the Bethe Ansatz equations (3.20) corresponding to the quantum numbers $\{I_1, \dots, I_N\}$, and $q(u)$ is the one-particle eigenvalue of the operator Q .

A compact expression (presented in [131, 133]) for the connected diagonal form factors of these densities can be cast into the form

$$F_{2N}^{J,c}(u_1 \dots, u_N) = \sum_{\sigma \in S_N} p'(u_{\sigma(1)}) \varphi(u_{\sigma(1)}, u_{\sigma(2)}) \varphi(u_{\sigma(2)}, u_{\sigma(3)}) \dots \varphi(u_{\sigma(N-1)}, u_{\sigma(N)}) q(u_{\sigma(N)}) \quad (3.32)$$

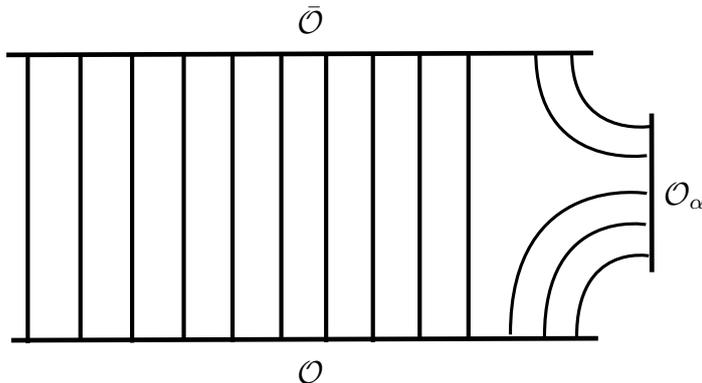


Figure 3.1 – HHL correlator at gauge theory side

where the summation runs over all the permutation of the set $\{1, \dots, N\}$. Here, p' denotes the derivative of the momentum w.r.t the rapidity,

$$p'(u) = \frac{\partial}{\partial u} p(u). \quad (3.33)$$

For massive relativistic models we have $p(u) = m \sinh u$ and $\varepsilon(u) = m \cosh u$, so that $p'(u) = \varepsilon(u)$ and (3.32) reduces to the expression presented in [131, 133]. However, in the case of the Heisenberg $XXX_{1/2}$ spin chain an extra sign appears: $p'(u) = -\varepsilon(u)$.

3.1.2 Heavy-Heavy-Light correlator at weak and strong coupling

In this section we will introduce the notion of the heavy-heavy-light operator (HHL) and review the main results relevant to the subject. As it has been mentioned above the HHL correlator is the correlator which has the length of one of the operators (light) significantly smaller then the other two. Let's consider the correlator $\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \mathcal{O}_\alpha(z) \rangle$ with the lengths L_α, L_1, L_2 such that $L_\alpha \ll L_1, L_2$. From the Fig. 3.1 it is clear that the heavy operators ($\mathcal{O}_1, \mathcal{O}_2$) in this case should be approximately the same. Further we will take $\mathcal{O}_1 = \bar{\mathcal{O}}_2 = \mathcal{O}$ (symmetric case), where by *bar* we denoted complex conjugation.

The attention to the HHL correlators was drawn by the papers [76, 77], where the author considered it at the strong coupling limit. From string theory side the heavy operators correspond to the same propagating string, whereas the third operator can be presented as a particle-like disturbance of this string corresponding to some supergravity mode (Fig. 3.2). In this case the HHL correlator in semi-classical approximation can be presented as the integral of the vertex operator corresponding to the light operator over the classical string solution corresponding to the heavy operator:

$$C_{HHL} = \int d^2\sigma V_{\mathcal{O}_\alpha}(X_H(\sigma, \tau)), \quad (3.34)$$

where by X_H we denoted the sets of the target space coordinates corresponding to the classical string solution.

After this prescription was proposed the computation of the HHL correlators at weak coupling appeared [78], where the exact match with the strong coupling computation was

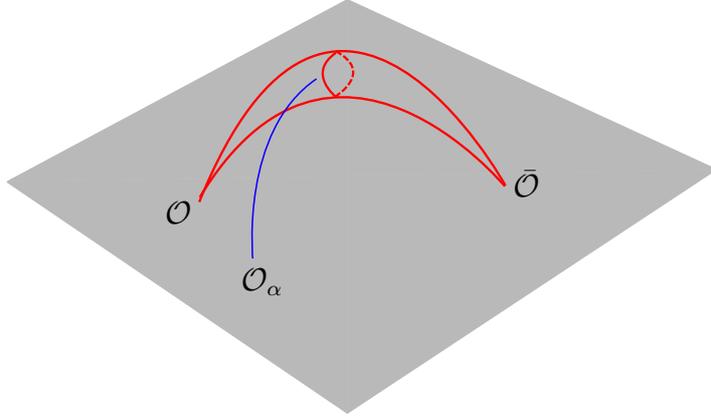


Figure 3.2 – HHL correlator at string theory side

established. However several years later some new insights related to HHL correlators were published. In [124] the authors showed that when considering more involved cases of the correlators the prescription given in [76] is not exact. The source of the problem is that the prescription assumes that the string solution over which one has to perform the integration should correspond to the two-point correlator of the heavy operators $\langle \mathcal{O} \bar{\mathcal{O}} \rangle$, but there is always at least one-parameter family of solutions for the given two-point functions. The authors of [124] claimed that the right prescription implies the integration over the whole family of solutions corresponding to the two-point function:

$$C_{HHL} = \frac{1}{V_{mod}} \int dp_1 \dots dp_{N_p} \int d^2\sigma V_{\mathcal{O}_\alpha}(X_H(\sigma, \tau, p_1, \dots, p_{N_p})), \quad (3.35)$$

where N_p stands for the number of parameters and V_{mod} for the volume of the moduli space. They considered several examples and showed that, although for some cases the integration over the whole family of solutions is trivial, in general case it changes the result and should be taken into account. On the other hand, the results obtained on the basis of the new prescription allowed the authors to conjecture that the L -dependence (dependence on the length of the heavy operator) of the HHL correlator coincides with the volume dependence of the diagonal form factor and as a consequence is given by the expression (3.25), that is:

$$C_{HHL} = \frac{1}{\rho_N(\mathbf{u})} \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} f^\mathcal{O}(\bar{\alpha}) \rho_N(\alpha) \quad (3.36)$$

This conjecture was claimed to be valid at any order in the coupling constant and the first goal of this chapter is to show that it works at weak coupling.

Apart from the two-dimensional integrable field theories in $\mathcal{N} = 4$ SYM it is not very clear what physical meaning we can give to the coefficients f . Therefore what we are going to prove is that the expression of the HHL correlator is given by (3.36) with some coefficients \mathcal{F} , which don't depend on the length of the heavy operator L .

As a set up we consider the scalar sector of $\mathcal{N} = 4$ SYM with the heavy operators belonging to the $su(2)$ sector. Thus we have

$$\mathcal{O}_1 = \mathcal{O} = \text{Tr}\{Z, X\}, \quad \mathcal{O}_2 = \bar{\mathcal{O}} = \text{Tr}\{\bar{Z}, \bar{X}\}, \quad (3.37)$$

where by $\{\dots\}$ we mean the set of the elementary fields an operator is made of. Since we concentrate on the tree level approximation the light operator should be composed as follows:

$$\mathcal{O}_\alpha = \text{Tr}\{Z, X, \bar{Z}, \bar{X}\}. \quad (3.38)$$

As it has been already mentioned in the introduction, when computing the three-point correlator at tree level one takes the operators being the eigenstates of one-loop dilatation operator. Thus the heavy operators will be taken as eigenstates of the $XX_{1/2}$ spin chain and will be constructed by the algebraic Bethe ansatz technique. On the other hand we will avoid taking the light operator to be an eigenstate, since in this case we will need more involved technique of the nested algebraic Bethe ansatz. Instead of this we will prove the statement (3.36) for any operator of the form $\text{Tr}\{Z, X, \bar{Z}, \bar{X}\}$. Since any eigenstate is given by a linear combination of these operators², the statement will obviously hold for the eigenstates as well, but just with accordingly changed coefficients f given by the appropriate linear combinations.

We will imply the following normalization convention:

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \mathcal{O}_\alpha(z) \rangle = \frac{\mathcal{N}_H C_{HHL}}{|x-y|^{2\Delta_{12}} |y-z|^{2\Delta_{23}} |x-z|^{2\Delta_{13}}}, \quad (3.39)$$

where \mathcal{N}_H is the norm of the heavy operator squared.

3.1.3 From field theory correlation functions to spin chain form factors

In order to efficiently use the technique of the algebraic Bethe-ansatz, we will translate the computation of the correlation function $\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \mathcal{O}_\alpha(z) \rangle$ to the computation of the spin chain form factors.

Let's imagine that we have the Bethe-state of the XXX spin chain and defined by the set of the rapidities $\mathbf{u} = u_1, \dots, u_N$:

$$|u_1, \dots, u_N\rangle. \quad (3.40)$$

Then the diagonal spin chain form factor of some local spin operator o_l is given by the expression

$$\langle u_1, \dots, u_N | o_l | u_1, \dots, u_N \rangle, \quad (3.41)$$

where the local spin chain operator is given by the tensor product of the algebra generators:

$$o_l = E_1^{a_1 b_1} \dots E_l^{a_l b_l}, \quad (3.42)$$

2. Generally speaking, since the sector composed of the fields Z, \bar{Z}, X, \bar{X} is not closed even at one loop, we need to consider the whole scalar sector $so(6)$. This means that the fields Y, \bar{Y} will also appear in the eigenstates. However the heavy states belong to the $su(2)$ sectors Z, X and \bar{Z}, \bar{X} . It means that at tree level the terms comprising the fields Y, \bar{Y} will not contribute in the structure constant and can be neglected

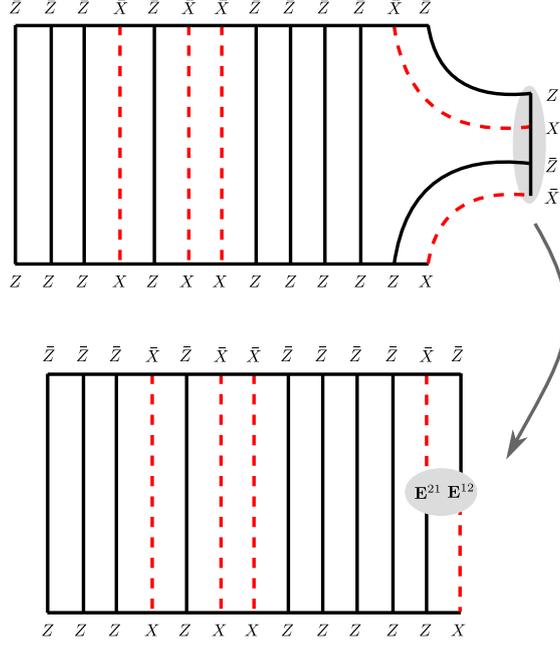


Figure 3.4 – An example of the mapping between field operators to spin chain operators. The operator in the field theory $\mathcal{O} = \text{tr}ZX\bar{Z}\bar{X}$ is mapped to the spin chain operator $\hat{\mathcal{O}} = E_{L-1}^{21}E_L^{12}$.

In the case of $L_\alpha = 2l_s = 4$ we have more possibilities (length-2 operators). In general we have sixteen operator $E_n^{ij} \otimes E_{n+1}^{kl}$, however just six correspond to the operators with zero R charge:

$$E_n^{11} \otimes E_{n+1}^{11}, \quad E_n^{11} \otimes E_{n+1}^{22}, \quad E_n^{22} \otimes E_{n+1}^{11}, \quad E_n^{22} \otimes E_{n+1}^{22}, \quad E_n^{12} \otimes E_{n+1}^{21}, \quad E_n^{21} \otimes E_{n+1}^{12}. \quad (3.47)$$

Due to the same relation $E_n^{22} = \mathbb{I}_n - E_n^{11}$ the correlators corresponding to the second, third and fourth operators can be expressed through the matrix elements $\langle \mathbf{u} | E_n^{11} | \mathbf{u} \rangle$ and $\langle \mathbf{u} | E_n^{11} \otimes E_{n+1}^{11} | \mathbf{u} \rangle$. Finally the fifth and sixth operators are related to each other by transposition, which means that their correlators are related by complex conjugation. Thus there are just three independent quantities among the diagonal form factors of length-1 and length-2 operators:

$$\mathcal{F}^{o_1} = \langle \mathbf{u} | o_1(n) | \mathbf{u} \rangle, \quad \mathcal{F}^{o_2^1} = \langle \mathbf{u} | o_2^1(n) | \mathbf{u} \rangle, \quad \mathcal{F}^{o_2^2} = \langle \mathbf{u} | o_2^2(n) | \mathbf{u} \rangle, \quad (3.48)$$

where $o_1(n) = E_n^{11}$, $o_2^1(n) = E_n^{11} \otimes E_{n+1}^{11}$, $o_2^2(n) = E_n^{12} \otimes E_{n+1}^{21}$.

The computation of the quantity (3.41) can be performed by means of the solution to QISP reviewed in the section 1.3.5. From (1.55) we get for length-1 operator⁴

$$\langle \mathbf{u} | E_n^{ab} | \mathbf{u} \rangle = \frac{\langle \mathbf{u} | T^{ab}(\theta_n + i/2) | \mathbf{u} \rangle}{t_{\mathbf{u}}(\theta_n + i/2)} \quad (3.49)$$

where $t_{\mathbf{u}}(u)$ is given by (1.38). At $u = \theta_n + i/2$, we have

$$t_{\mathbf{u}}(\theta_n + i/2) = Q_{\theta}(\theta_n + i) \frac{Q_{\mathbf{u}}^-(\theta_n)}{Q_{\mathbf{u}}^+(\theta_n)}, \quad (3.50)$$

4. We will keep the inhomogeneities θ being arbitrary values through the whole derivation, and put them to zero only when computing concrete examples. The reason is that we will need them to have non-zero values when considering HHL correlator at one loop in the section 3.1.7

where we used the generally accepted convention

$$Q_{\mathbf{u}}^+(\theta) = Q_{\mathbf{u}}(\theta + i/2), \quad Q_{\mathbf{u}}^-(\theta) = Q_{\mathbf{u}}(\theta - i/2). \quad (3.51)$$

The generalization of (3.49) to a string of $l_s + 1$ operators is straightforward

$$\langle \mathbf{u} | E_n^{a_0 b_0} \dots E_{n+l}^{a_{l_s} b_{l_s}} | \mathbf{u} \rangle = \left(\prod_{k=0}^{l_s} \frac{Q_{\mathbf{u}}^+(\theta_{n+k})}{Q_{\theta}(\theta_{n+k} + i) Q_{\mathbf{u}}^-(\theta_{n+k})} \right) \langle \mathbf{u} | \prod_{k=0}^{l_s} T_{n+k}^{a_k b_k}(\theta_{n+k} + i/2) | \mathbf{u} \rangle \quad (3.52)$$

3.1.4 Computation of a spin chain form factor. Simple example

Before proving the statement (3.36) for the most general case we will consider one simple example, namely the computation of the matrix element $\mathcal{F}^{o_2^1} = \langle \mathbf{u} | o_2^1(n) | \mathbf{u} \rangle$. From the expression (3.52) it is clear that

$$\mathcal{F}^{o_2^1} \propto F^{AA} = \langle \mathbf{u} | A(\theta_n + i/2) A(\theta_{n+1} + i/2) | \mathbf{u} \rangle. \quad (3.53)$$

In order to compute F^{AA} we first act with $A(\theta_{n+1} + i/2)$ on the state $|\mathbf{u}\rangle$, (1.35) gives

$$A(\theta_{n+1} + i/2) | \mathbf{u} \rangle = a(\theta_{n+1} + i/2) \frac{Q_{\mathbf{u}}(\theta_{n+1} - i/2)}{Q_{\mathbf{u}}(\theta_{n+1} + i/2)} | \mathbf{u} \rangle + \sum_{k=1}^N M_k(\theta_{n+1} + i/2) | \{\mathbf{u}, \theta_{n+1} + i/2\} \setminus \{u_k\} \rangle. \quad (3.54)$$

We got two terms, one of them proportional to $|\mathbf{u}\rangle$ and another proportional to $|\{\mathbf{u}, \theta_{n+1} + i/2\} \setminus \{u_k\}\rangle$. Then we act with $A(\theta_n + i/2)$ on these two terms by means of the same property (1.35). It is not hard to see that we get four terms of the form

$$\begin{aligned} \propto |\mathbf{u}\rangle, \quad & \propto |\{\mathbf{u}, \theta_n + i/2\} \setminus \{u_k\}\rangle, \quad \propto |\{\mathbf{u}, \theta_{n+1} + i/2\} \setminus \{u_k\}\rangle, \\ & \propto |\{\mathbf{u}, \theta_n + i/2, \theta_{n+1} + i/2\} \setminus \{u_k, u_l\}\rangle, \end{aligned} \quad (3.55)$$

meaning that we need to compute four following scalar products

$$\begin{aligned} \langle \mathbf{u} | \mathbf{u} \rangle, \quad \langle \mathbf{u} | \{\mathbf{u}, \theta_n + i/2\} \setminus \{u_k\} \rangle, \quad \langle \mathbf{u} | \{\mathbf{u}, \theta_{n+1} + i/2\} \setminus \{u_k\} \rangle, \\ \langle \mathbf{u} | \{\mathbf{u}, \theta_n + i/2, \theta_{n+1} + i/2\} \setminus \{u_k, u_l\} \rangle. \end{aligned} \quad (3.56)$$

3.1.5 Computation of a spin chain form factor. General case

The logic of the computation of spin chain form factors in the general case remains the same as in the example considered in the previous section. The computation of the diagonal form factor of the local operator

$$E_n^{a_1 b_1} \otimes \dots \otimes E_{n+l_s-1}^{a_{l_s} b_{l_s}} \quad (3.57)$$

between the Bethe-states $|\mathbf{u}\rangle$ reduces to the computation of the quantity

$$\langle \mathbf{u} | T^{a_1 b_1}(\theta_n + i/2) \dots T^{a_{l_s} b_{l_s}}(\theta_{n+l_s-1} + i/2) | \mathbf{u} \rangle. \quad (3.58)$$

As it should be clear from the discussion of the previous section the expression (3.58) will generate the terms of the form:

$$\begin{aligned} \langle \mathbf{u} | \mathbf{u} \rangle, \quad \langle \mathbf{u} | \{\mathbf{u}, \theta_{n_1} + i/2\} \setminus \{u_{k_1}\} \rangle, \\ \langle \mathbf{u} | \{\mathbf{u}, \theta_{n_1} + i/2, \theta_{n_2} + i/2\} \setminus \{u_{k_1}, u_{k_2}\} \rangle, \dots, \\ \langle \mathbf{u} | \{\mathbf{u}, \theta_{n_1} + i/2, \theta_{n_2} + i/2, \dots, \theta_{n_{l_s}} + i/2\} \setminus \{u_{k_1}, \dots, u_{k_{l_s}}\} \rangle. \end{aligned} \quad (3.59)$$

The scalar products listed above have the same common property that the bra state is on-shell Bethe-state. It means that they can be computed by means of the Slavnov formula (1.49). Another important thing which should be noticed that for any of these scalar products the ket states have very specific form. Their rapidities are not just some random numbers, but part of the rapidities belong to the same set as the rapidities of the bra state. Let us consider an example of such a scalar product in more details. In the Slavnov determinant (1.49), the denominator is a simple Cauchy determinant and can be readily computed. We therefore focus on the non-trivial numerator $\det_{jk} \Omega(u_j, v_k)$. To start with let us consider the scalar product $\langle \mathbf{u} | \{ \mathbf{u}, \theta_n, \theta_{n+1} \} \setminus \{ u_j, u_k \} \rangle$. The determinant takes the following form

$$\det \Omega = \begin{vmatrix} i \phi_{11} & \cdots & \Omega_{1j} & \cdots & \Omega_{1k} & \cdots & i \phi_{1N} \\ i \phi_{21} & \cdots & \Omega_{2j} & \cdots & \Omega_{2k} & \cdots & i \phi_{2N} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ i \phi_{N1} & \cdots & \Omega_{Nj} & \cdots & \Omega_{Nk} & \cdots & i \phi_{NN} \end{vmatrix}. \quad (3.60)$$

where we have defined $i \phi_{jk} = \Omega(u_j, u_k)$ and $\Omega_{ik} = \Omega_{ik}(u_i, \theta_n + i/2)$. First we perform the minor expansion of the determinant with respect to all the columns that does not have any element of the form ϕ_{nn} , that is with respect to the columns j and k in our case. After performing this expansion we will get the sum of some minors with the corresponding coefficients. Due to the structure of the determinant (3.60) it is not hard to see that one can always expand the obtained minors to the diagonal minors of the Guadin norm (3.22):

$$\rho_N(u_{i_1}, \dots, u_{i_m}) = (-1)^m \begin{vmatrix} \phi_{i_1 i_1} & \cdots & \cdots & \cdots \\ \cdots & \phi_{i_2 i_2} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \phi_{i_m i_m} \end{vmatrix} \quad (3.61)$$

apart from those minors which contains the elements ϕ_{jj} and ϕ_{kk} . Therefore, the following expansion holds

$$\langle \mathbf{u} | \{ \mathbf{u}, \theta_n, \theta_{n+1} \} \setminus \{ u_j, u_k \} \rangle = \sum_{\alpha \cup \bar{\alpha} = A} f^{\mathcal{O}}(\bar{\alpha}) \rho_N(\alpha), \quad (3.62)$$

where $A = \{ u_1, \dots, \hat{u}_j, \dots, \hat{u}_k, \dots, u_N \}$ ⁵. Here $\bar{\alpha}$ is the complement of α in A . For an explicit and simple example, see Appendix E.

From the discussion above it is clear that any scalar product appearing when computing the spin chain form factors can be presented in the form of the expansion (3.62), just with different set A and coefficients $f^{\mathcal{O}}(\bar{\alpha})$. Then when we sum over all scalar products (3.59) appearing in the spin chain form factor after applying the solution to the QISP, we will formally get

$$\sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} f^{\mathcal{O}}(\bar{\alpha}) \rho_N(\alpha). \quad (3.63)$$

Now in order to prove the statement (3.36) at tree level for the considered types of the HHL correlators we need to show that the coefficients $f^{\mathcal{O}}(\bar{\alpha})$ do not depend on the heavy operator length L .

5. Here \hat{j} and \hat{k} mean these two indices are absent.

In the approach of algebraic Bethe-ansatz, the diagonal matrix elements are given in terms of the following functions: the eigenvalue of the diagonal elements of the transfer matrix, $a(u)$ and $d(u)$, the products of functions $f(u-v)$ and $g(u-v)$ (see the section 1.3.2), and the matrix elements in the Slavnov formula Ω_{jk} and ϕ_{jk} . Under the normalization we described in the section 3.1.2, the functions $a(u)$ and $d(u)$ either cancel or appear in the expression as the ratio $a(u)/d(u) = e^{ipL}$, which can be replaced by products of scattering matrices using the Bethe Ansatz equations. The products of $f(u-v)$ and $g(u-v)$ functions do not depend on L . The matrix element Ω_{jk} defined in (1.51) also has no dependence on L . Finally, ϕ_{jk} with $j \neq k$ reads

$$\phi_{jk} = \phi(u_j, u_k) = \frac{2}{(u_j - u_k)^2 + 1}, \quad j \neq k, \quad (3.64)$$

again, do not depend on L . The only dependence on L is hidden in the diagonal element ϕ_{nn} . Recall that we have

$$\phi_{nn} = \sum_{m=1}^L \frac{1}{(u_n - \theta_m)^2 + 1/4} - \sum_{\substack{l=1 \\ l \neq n}}^N \phi_{nl} \quad (3.65)$$

In the homogeneous limit, where $\theta_m = 0$ ($m = 1, \dots, L$), the first term of (3.65) becomes $L/(u_n^2 + 1/4)$ which depends linearly on L . When we perform the minor expansion, we carefully avoid expansion with this kind of terms and they only appear in the diagonal minors $\rho_N(\alpha)$. Therefore, the dependence on the length L is completely contained in $\rho_N(\alpha)$. This finishes the proof of the conjecture (3.36), which we will call here *finite volume expansion* due to the fact that this conjecture originates from the analogy with the finite volume expansion of the diagonal form factors in the 2D quantum field theories described in the section 3.1.1. However, we want to emphasize that in our proof we didn't get any restriction to the coefficients $f^{\mathcal{O}}(\bar{\alpha})$, whereas in the original paper [134], where the conjecture was formulated, the authors claimed that these coefficients should be given by the form factors in the infinite volume. It is not clear to us what physical meaning we could assign to these coefficient at weak coupling, but maybe it could be done from the string theory side. In [135] the authors formulated the set of axioms, analogous to those listed in the section 3.1.1, for the world-sheet form factors of the light-cone gauge fixed string theory on $AdS_5 \times S^5$. Solving these axioms at finite coupling in case of the diagonal form factors could provide us with data, which we would be able to compare with the coefficient of finite volume expansion obtained at weak coupling.

Despite the fact that we wasn't able to come up with the clear physical interpretation of the coefficients $f^{\mathcal{O}}(\bar{\alpha})$, we noticed that they have some interesting structure analogously to the infinite volume form factors in the 2D QFT's. In the following few sections we will compute some examples of the diagonal spin chain matrix elements and describe the structure of the corresponding coefficients $f^{\mathcal{O}}(\bar{\alpha})$ explicitly.

3.1.6 The coefficients $f^{\mathcal{O}}(\bar{\alpha})$

In this section we will perform the finite volume expansion for the diagonal matrix elements of length-1 and length-2 operators. We will give a simple example of this procedure for the case of length-1 operator with 2 magnons in detail and then present the results for more complicated correlators.

An example: length-1 operator with -2 magnons

We consider the finite volume diagonal matrix element for the operator $o_1(n) = E_n^{11}$ with two magnons

$$\mathcal{F}_L^{o_1}(u_1, u_2) = \frac{\langle u_1, u_2 | o_1(n) | u_1, u_2 \rangle}{\langle u_1, u_2 | u_1, u_2 \rangle}. \quad (3.66)$$

It has the following structure in finite volume

$$\mathcal{F}_L^{o_1}(u_1, u_2) = \frac{1}{\rho_2(u_1, u_2)} (\rho_2(u_1, u_2) + f^{o_1}(u_2) \rho_2(u_1) + f^{o_1}(u_1) \rho_2(u_2) + f^{o_1}(u_1, u_2)). \quad (3.67)$$

We proceed as described in the previous sections. Using the solution to QISP, we have

$$\mathcal{F}_L^{o_1}(u_1, u_2) = \frac{1}{t_{\mathbf{u}}(\theta_n^+)} \frac{\langle u_1, u_2 | A(\theta_n^+) | u_1, u_2 \rangle}{\langle u_1, u_2 | u_1, u_2 \rangle}, \quad (3.68)$$

where the denominator is the Gaudin norm (1.46),

$$\langle u_1, u_2 | u_1, u_2 \rangle = \left(\prod_{j=1}^2 a(u_j) d(u_j) \right) \frac{1 + (u_1 - u_2)^2}{(u_1 - u_2)^2} \rho_2(u_1, u_2), \quad (3.69)$$

and $\theta_n^+ = \theta_n + i/2$. From (1.35) we obtain

$$\begin{aligned} \langle u_1, u_2 | A(\theta_n^+) | u_1, u_2 \rangle &= M_0(\theta_n^+) \langle u_1, u_2 | u_1, u_2 \rangle \\ &+ M_1(\theta_n^+) \langle u_1, u_2 | u_2, \theta_n^+ \rangle + M_2(\theta_n^+) \langle u_1, u_2 | u_1, \theta_n^+ \rangle. \end{aligned} \quad (3.70)$$

We introduce some notations in order to simplify the expressions. Let us define

$$\mathcal{C}_{\mathbf{u}, \mathbf{v}} = \frac{\prod_{j=1}^N a(v_j) d(u_j)}{\det_{jk} \frac{1}{u_j - v_k + i}}, \quad (3.71)$$

thus

$$\langle \mathbf{v} | \mathbf{u} \rangle = \mathcal{C}_{\mathbf{u}, \mathbf{v}} \det_{jk} \Omega(u_j, v_k). \quad (3.72)$$

By performing the finite volume expansion for the three scalar products, we obtain

$$\begin{aligned} \langle u_1, u_2 | A(\theta_n^+) | u_1, u_2 \rangle &= -\mathcal{C}_{\{u_1, u_2\}, \{u_1, u_2\}} M_0(\theta_n^+) \rho_2(u_1, u_2) \\ &- i \mathcal{C}_{\{u_1, u_2\}, \{u_1, \theta_n^+\}} M_2(\theta_n^+) \Omega(u_2, \theta_n^+) \rho_2(u_1) \\ &- i \mathcal{C}_{\{u_1, u_2\}, \{u_2, \theta_n^+\}} M_1(\theta_n^+) \Omega(u_1, \theta_n^+) \rho_2(u_2) \\ &- \phi_{12} \left(\mathcal{C}_{\{u_1, u_2\}, \{u_2, \theta_n^+\}} \Omega(u_2, \theta_n^+) M_1(\theta_n^+) + \mathcal{C}_{\{u_1, u_2\}, \{u_1, \theta_n^+\}} \Omega(u_1, \theta_n^+) M_2(\theta_n^+) \right) \end{aligned} \quad (3.73)$$

Plugging (3.73) into (3.68) and comparing to the expansion (3.67), we obtain the expression for the coefficients

$$\begin{aligned}
f^{o_1}(u_1) &= i \frac{\mathcal{C}_{\{u_1, u_2\}, \{u_2, \theta_n^+\}}}{\mathcal{C}_{\{u_1, u_2\}, \{u_1, u_2\}}} \frac{M_1(\theta_n^+)}{M_0(\theta_n^+)} \Omega(u_1, \theta_n^+) \\
f^{o_1}(u_2) &= i \frac{\mathcal{C}_{\{u_1, u_2\}, \{u_1, \theta_n^+\}}}{\mathcal{C}_{\{u_1, u_2\}, \{u_1, u_2\}}} \frac{M_2(\theta_n^+)}{M_0(\theta_n^+)} \Omega(u_2, \theta_n^+) \\
f^{o_1}(u_1, u_2) &= \frac{\phi_{12}}{\mathcal{C}_{\{u_1, u_2\}, \{u_1, u_2\}} M_0(\theta_n^+)} \left(\mathcal{C}_{\{u_1, u_2\}, \{u_2, \theta_n^+\}} \Omega(u_2, \theta_n^+) M_1(\theta_n^+) \right. \\
&\quad \left. + \mathcal{C}_{\{u_1, u_2\}, \{u_1, \theta_n^+\}} \Omega(u_1, \theta_n^+) M_2(\theta_n^+) \right)
\end{aligned} \tag{3.74}$$

Substituting the explicit expressions in (3.74) and, at the end, taking the homogeneous limit $\theta_n \rightarrow 0$, we obtain very compact results

$$\begin{aligned}
f^{o_1}(u_k) &= \frac{1}{u_k^2 + 1/4}, \quad k = 1, 2 \\
f^{o_1}(u_1, u_2) &= \left(\frac{1}{u_1^2 + 1/4} + \frac{1}{u_2^2 + 1/4} \right) \frac{2}{1 + (u_1 - u_2)^2}.
\end{aligned} \tag{3.75}$$

Length-1 operator with N magnons

We can perform the same calculation as in the previous subsection and extract the form factors with more magnons. The process becomes cumbersome for higher number of particles. However, from the first few magnon cases, we are able to observe a nice pattern for the coefficients $f^{\mathcal{O}}$. The N -magnon connected diagonal form factor for $o_1(n)$ is given as

$$f^{o_1}(u_1, \dots, u_N) = \varepsilon_1 \phi_{12} \phi_{23} \dots \phi_{N-1, N} + \text{permutations} \tag{3.76}$$

where ε_k is the energy of the magnon with rapidity u_k and ϕ_{jk} can be seen as some ‘‘propagator’’ defined as

$$\varepsilon_k = \varepsilon(u_k) = \frac{1}{u_k^2 + 1/4}, \quad \phi_{jk} = \frac{2}{1 + (u_j - u_k)^2}, \quad j \neq k. \tag{3.77}$$

The structure of f^{o_1} is exactly the structure of the connected form factors of conserved charge densities (3.32). This is not surprising, since $o_1(n) = \frac{1}{2}(\mathbb{I} + \sigma_n^z)$ is indeed a length-1 conserved charge density of the Heisenberg spin chain. The nice feature is that once we know the one particle eigenvalue $q(u)$ of the charge, we can immediately write down the expression for the corresponding infinite volume form factors. We remark here that our result (3.76) is consistent with the determinant formula of [136].

Length-2 form factors of N magnons

The calculation of the diagonal matrix elements for the case of length-2 operators can be performed by following the same logic, but the process is more involved. Nevertheless, we again found some patterns for the various matrix elements. In summarizing, the infinite

volume diagonal matrix element of a length-2 operator, $f^O(u_1, \dots, u_N)$, is characterized by two functions $f^O(u)$ and $\psi^O(u, v)$. The result for N -magnon is given by

$$f^O(u_1, \dots, u_N) = \left(\varepsilon_1 \phi_{12} \dots \phi_{N-1, N} f_N^O + \text{permutations} \right) + \left(\sum_{i=1}^{N-1} \varepsilon_1 \phi_{12} \dots \psi_{i, i+1}^O \dots \phi_{N-1, N} \varepsilon'_N + \text{permutations} \right) \quad (3.78)$$

We list the coefficients for o_2^i , since any diagonal form factor of a length-2 operator can be expressed through the corresponding form factors of o_2^1 , o_2^2 and o_1 , as it was discussed in the section 3.1.3:

$$\begin{aligned} f^{o_2^1}(u) &= 2 & \psi^{o_2^1}(u, v) &= -(u-v)(uv-1/4)\phi(u, v) \\ f^{o_2^2}(u) &= -\frac{u-i/2}{u+i/2} & \psi^{o_2^2}(u, v) &= (u-v)(u-i/2)(v-i/2)\phi(u, v) \end{aligned} \quad (3.79)$$

These data for the operators can be read off simply from the computation of 2 magnon case. Once the functions in (3.79) are known, we can write down the diagonal form factor of length-2 operators for any magnon number of the Bethe-state.

Examples of length-2 operators

Here we compute the matrix elements of the length-2 conserve charge density, which is the permutation operator $P_{k, k+1}$, or equivalently the Hamiltonian density $H_{k, k+1} = L_{k, k+1} - P_{k, k+1}$. We will see that the data for the permutation operator simplifies and the final result takes exactly the form predicted in (3.32).

The permutation operator $P_{k, k+1}$ is a length-2 operator of the Heisenberg spin chain. It has the following form

$$P_{k, k+1} = \sum_{i, j=1}^2 E_k^{ij} E_{k+1}^{ji}. \quad (3.80)$$

After expressing the $P_{k, k+1}$ in terms of the operators o_2^1 , o_2^2 , it is not hard to get $o_1 \otimes \mathbb{I}$, $\mathbb{I} \otimes o_1$

$$f^P(u) = \varepsilon(u) = \frac{1}{u^2 + 1/4}, \quad \psi^P(u, v) = 0, \quad f^P(\emptyset) = 1. \quad (3.81)$$

The coefficients f^P (3.78) with the entires (3.81) has the structure as a conserved charge should have (3.32), with the one particle eigenvalue of the corresponding charge being $-\varepsilon(u)$.

3.1.7 Matrix elements at one loop

In this section we generalize the above considerations from tree level to one loop. We will not give here a rigorous proof of the statement (3.36) at one loop. Our argumentation will have a minor gap which we will discuss as well. However in the end of this section

we will give a simple example on which we will demonstrate that the conjecture (3.36) works at one loop. The examination of (3.36) at any loop order will be given in the next sections, where we exploit the technique of the hexagon form factor [57].

We are still considering the set up from the section 3.1.2. It means that that the heavy operators belong to the $su(2)$ sectors Z, X and \bar{Z}, \bar{X} correspondingly and the third belongs to the scalar sector $so(6)$. However there are several new features appearing when computing the three-point correlator at one loop.

First of all at one loop we need to consider the operators being the eigenstates of two-loop dilatation operator. Therefore the dilatation operator in the $su(2)$ sector is no longer the Hamiltonian of the $XX_{1/2}$ spin chain, but becomes long-range interacting spin chain (the BDS spin chain [29]). Therefore the two heavy operators correspond to the eigenvectors of the BDS spin chain. The eigenstates of the BDS spin chain can be obtained from the eigenstates of a special inhomogeneous $XX_{1/2}$ spin chain by performing the unitary transformation, namely $|\mathbf{u}\rangle_{\text{BDS}} = \text{S}|\mathbf{u}; \boldsymbol{\theta}^{\text{BDS}}\rangle$. Where the inhomogeneities θ was put to some special value $\boldsymbol{\theta}^{\text{BDS}}$ given by

$$\theta_k^{\text{BDS}} = 2g \sin \frac{2\pi k}{L}, \quad k = 1, \dots, L \quad (3.82)$$

where g is the coupling constant. The unitary operator S has been worked out in [72] up to g^2 order

$$\text{S} = \exp i \sum_{k=1}^L \left[\nu_k \text{H}_k + \frac{i}{2} \rho_k [\text{H}]_{k-1} \right], \quad (3.83)$$

where $\text{H}_k \equiv \text{I}_{k,k+1} - \text{P}_{k,k+1}$ and $[\text{H}]_k \equiv [\text{H}_k, \text{H}_{k+1}]$. The parameters ν_k and ρ_k are related to the inhomogeneities by

$$\nu_k = - \sum_{j=1}^k \theta_j, \quad \rho_k = 2g^2 k - \theta_k \nu_k - \sum_{j=1}^k \theta_j^2, \quad k = 1, \dots, L. \quad (3.84)$$

Concerning the light operator, we still will not be considering it as an eigenstate of the two-loop dilatation operator, but rather as an arbitrary term of an eigenstate as we did for the tree level case. At two loops the scalar sector is not closed anymore. Apart from the scalars another kind of fields, such as fermions and covariant derivatives of scalars, will appear in the eigenstates (see e.g. [137]).

Finally the contributions of the one-loop Feynman diagrams should be taken into account. In what follows for the sake of simplicity we will put some constrains on the light operator. Namely, we will consider the terms which consist only of the scalar fields: $Z, \bar{Z}, X, \bar{X}, Y, \bar{Y}$. However our argumentation can be easily generalized to the light operators containing the fermionic fields and derivatives.

The effects of one-loop Feynman diagrams

In the case when all the fields of which the operators are made appear to be scalars the contribution from the one-loop Feynman diagrams can be computed by means of the *operator insertions* at the splitting points [70, 71], as is shown in Fig. (3.5). The insertions

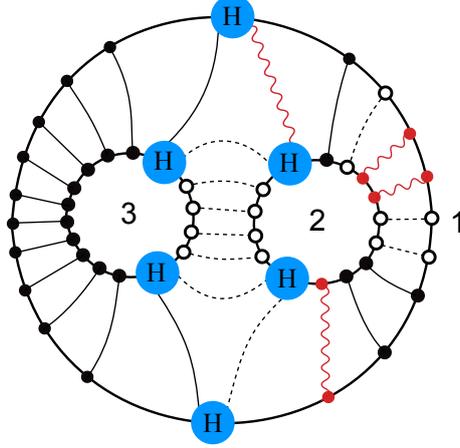


Figure 3.5 – The quantum corrections are taken into account by operator insertions at the splitting points. At one-loop in the $\mathfrak{so}(6)$ sector, the insertion takes the form of one-loop Hamiltonian density.

take the form of the $so(6)$ spin chain Hamiltonian density coming with the factor of $-\frac{g^2}{2}$:

$$\mathcal{H}_{l,l+1} = -\frac{g^2}{2} \mathbb{H}_l^{\mathfrak{so}(6)} = -\frac{g^2}{2} (\mathbb{K}_{l,l+1} + 2\mathbb{I}_{l,l+1} - 2\mathbb{P}_{l,l+1}) \quad (3.85)$$

where $\mathbb{I}_{l,l+1}$, $\mathbb{P}_{l,l+1}$ and $\mathbb{K}_{l,l+1}$ are the identity, permutation and trace operators, and $g^2 = \frac{\lambda}{16\pi^2}$. They act on the $so(6)$ spin chain states as

$$\begin{aligned} \mathbb{I}_{l,l+1} |\dots \phi_l^i \phi_{l+1}^j \dots\rangle &= |\dots \phi_l^i \phi_{l+1}^j \dots\rangle, \\ \mathbb{P}_{l,l+1} |\dots \phi_l^i \phi_{l+1}^j \dots\rangle &= |\dots \phi_l^j \phi_{l+1}^i \dots\rangle, \\ \mathbb{K}_{l,l+1} |\dots \phi_l^i \phi_{l+1}^j \dots\rangle &= \delta_{ij} \sum_{k=1}^6 |\dots \phi_l^k \phi_{l+1}^k \dots\rangle. \end{aligned} \quad (3.86)$$

As it can be seen from the Fig. (3.5) there are six insertions, two for each operator. We divide them in two groups: insertions to the light operator and insertions to the heavy operators. It is not hard to see that we can present the light operator with insertions as some local spin operator acting on the heavy states. We will demonstrate it on a simple example. Let us consider the light operator containing the fields Y , and \bar{Y} , which unlike to the tree level case we have to take into account at one loop. If an operator has the terms comprising these fields the only way they can contribute is through the trace operator from the insertions, since this is the only option to transform the fields Y, \bar{Y} into Z, \bar{Z} or X, \bar{X} . It is clear that since the heavy operators still belong to $su(2)$ sector they have the same form of the $su(2)$ spin chain states. Therefore according to the same logic as described in the section 3.1.3 the light operator with insertions can be presented as an appropriate local spin chain operator $\hat{O}(\sigma^\pm, \sigma^z)$ acting on the heavy states (see the Fig. 3.6). When we compute the insertions to the heavy operators the same logic works. The only difference is that the length of the local operators increases when we take the insertions into account (see the Fig. (3.7))

To summarize, the one-loop structure constants can be recast to the calculation of matrix elements of the BDS spin chain ${}_{\text{BDS}}\langle \mathbf{u} | \hat{O}(\sigma^\pm, \sigma^z; g^2) | \mathbf{u} \rangle_{\text{BDS}}$.

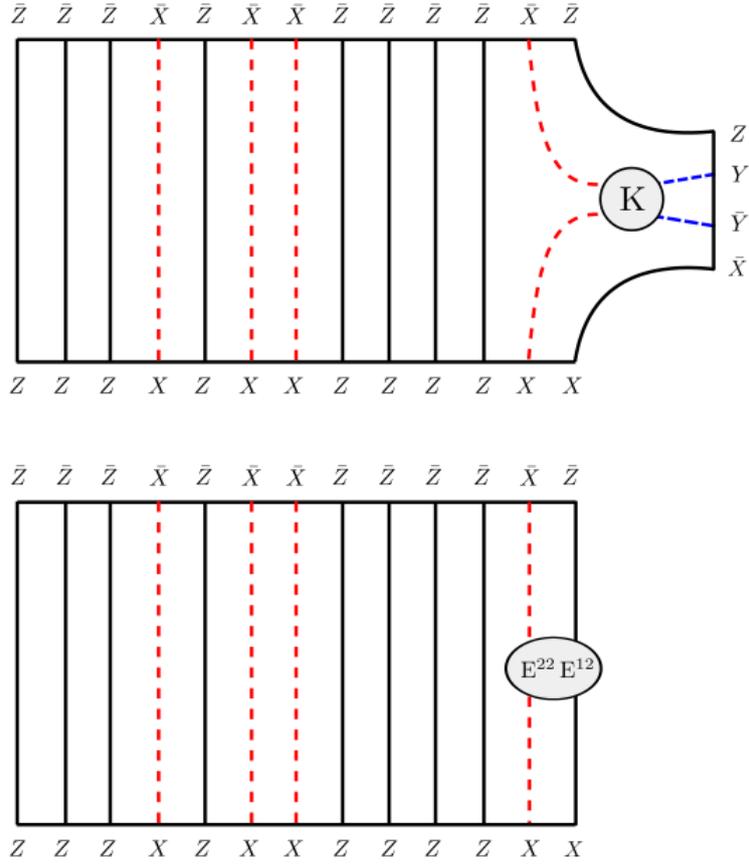


Figure 3.6 – An example of mapping the light operator to the spin chain operator in the presence of one-loop insertion.

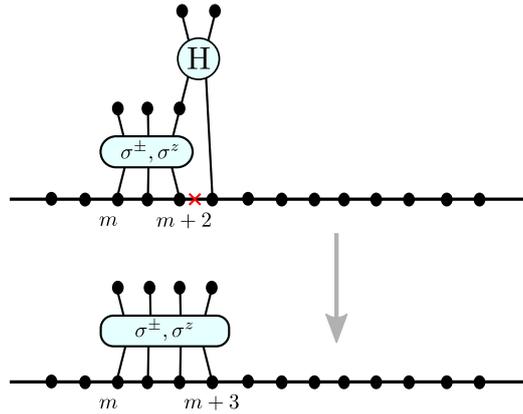


Figure 3.7 – The effect of operator insertions for the heavy operators. They increase the length of the spin operator by 1. The red cross denotes the splitting point.

The effects of the operator S

After we take into account the effect of insertions and modify the states the HHL three-point correlators can be written as follows

$$\langle \mathbf{u}; \boldsymbol{\theta}^{\text{BDS}} | S^{-1} \hat{O}_{l+1}(\sigma^\pm, \sigma^z) S | \mathbf{u}; \boldsymbol{\theta}^{\text{BDS}} \rangle, \quad (3.87)$$

where the index $l + 1$ stands for the length of the local operator after taking insertions into account. The S operator takes an exponential form $S = \exp \hat{F}$, thus we have

$$S^{-1} \hat{O}_{l+1}(\sigma^\pm, \sigma^z) S = \hat{O}_{l+1}(\sigma^\pm, \sigma^z) - [\hat{F}, \hat{O}_{l+1}(\sigma^\pm, \sigma^z)] + \frac{1}{2} [\hat{F}, [\hat{F}, \hat{O}_{l+1}(\sigma^\pm, \sigma^z)]] + \mathcal{O}(g^3). \quad (3.88)$$

The action of S operator on the spin chain operator can be divided into two types. The first type is length preserving, it originates from the operators H_k and $[H]_k$ that act within the range of the spin chain operator \hat{O}_{l+1} , which gives rise to an operator with the same length, this is depicted in Fig. (3.8).

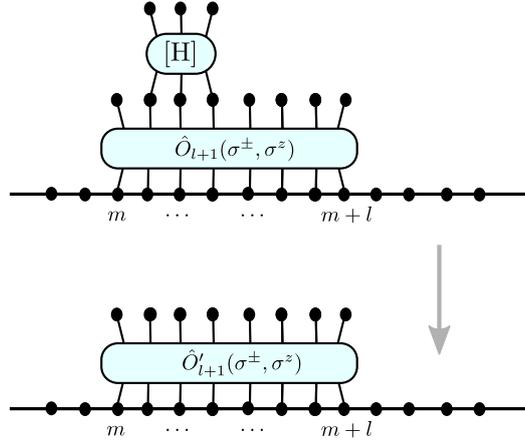


Figure 3.8 – The length preserving action generated by $[H]_k$ on the spin chain operator.

The other type of the action increases the length of the operator by 1 or 2, which are generated from the operators H_k and $[H]_k$ acting at the boundary of the spin chain operator (see Fig. (3.9),(3.10)).

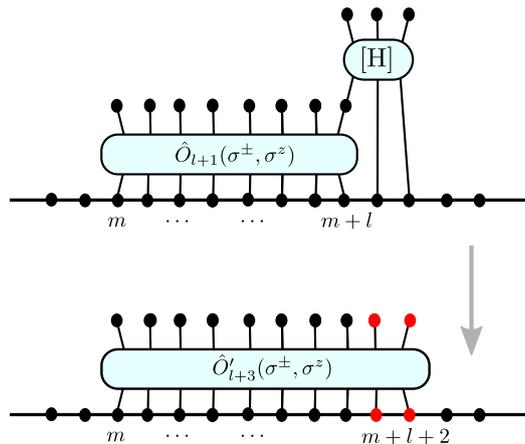


Figure 3.9 – The length changing action generated by $[H]_k$ on the spin chain operator. In this example it increases the length of the spin chain operator by 2.

From our analysis we see that the action of the S operator on the spin chain operators, in general, increases the length of the spin chain operator:

$$S^{-1} \hat{O}_l(\sigma^\pm, \sigma^z) S = \hat{O}'_l(\sigma^\pm, \sigma^z) + \hat{O}'_{l+1}(\sigma^\pm, \sigma^z) + \hat{O}'_{l+2}(\sigma^\pm, \sigma^z) + \mathcal{O}(g^3) \quad (3.89)$$

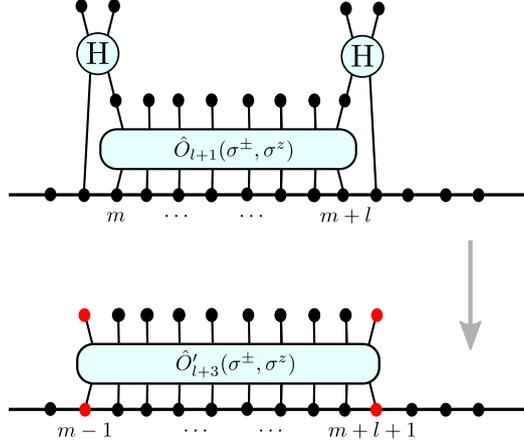


Figure 3.10 – The length changing action generated by two H_k 's on both ends of the spin chain operator. IN this example it increases the length of the spin operator by 2.

This implies that in order to compute the form factor of length l operator for BDS spin chain, we need to compute the form factors of length $l + 2$, $l + 1$ and l operators in the inhomogeneous $XXX_{1/2}$ spin chain.

Since the argumentation in the section (3.1.5) is given for the general inhomogeneous spin chain, once we write the three-point function in terms of matrix elements of the inhomogeneous $XXX_{1/2}$ spin chain, we can perform the finite volume expansion and organize the results in the form conjectured in [124]. The difference is that in the end we do not put the θ 's to zero, but put them to the BDS values. It assumes the modification of the Gaudin norm. The equations (1.48), and (1.47) are still valid, but the eigenvalues $a(u)$ and $d(u)$ are corrected

$$a(u) = \prod_{k=1}^L (u - \theta_k^{\text{BDS}} + i/2) = x(u + i/2)^L + \mathcal{O}(g^{2L}), \quad (3.90)$$

$$d(u) = \prod_{k=1}^L (u - \theta_k^{\text{BDS}} - i/2) = x(u - i/2)^L + \mathcal{O}(g^{2L}),$$

where $x(u)$ is the Zhukowsky map given by

$$x(u) + \frac{g^2}{x(u)} = u. \quad (3.91)$$

By replacing

$$p(u_k) = \frac{u_k + i/2}{u_k - i/2} \longrightarrow \frac{x(u_k + i/2)}{x(u_k - i/2)} \quad (3.92)$$

in (1.48) and expanding the result up to $\mathcal{O}(g^2)$ order, we obtain the Gaudin norm at one-loop, $\rho_n^{1\text{-loop}}$. In fact, the replacement (3.92) gives the correct Gaudin norm up to the wrapping corrections [69, 72].

From the discussion above it should be clear that the presence of the fermions and derivatives in the light operator will not change the argumentation, since the heavy operators anyway have the form of the $su(2)$ spin chain states, and therefore any kind of the

light operator can be finally presented as some local $su(2)$ spin chain operator acting on the heavy states. Moreover, since the $su(2)$ sector is closed at any loop order we can hope that the conjecture (3.36) really holds asymptotically at any loop order. We will see that this is indeed the case in the following sections where we exploit the hexagon program approach.

However, before going to the all loop consideration we will discuss a minor gap in our one-loop argumentation. Namely, the conjecture (3.36) states that the coefficients $f^\mathcal{O}$ do not depend on the length of the heavy operators L . But if we look at the expression (3.88) for the local operator after we take the action of the operator S into account, we will see that the coefficient in front of the spin chain operators will depend on L through the functions ν_k and ρ_k (3.84). This means that when we compute the matrix element (3.87) it is not guaranteed that after performing the finite volume expansion the coefficients $f^\mathcal{O}$ will not depend on L .

Here we are not going to prove rigorously that when putting everything together the explicit dependence of the coefficients $f^\mathcal{O}$ on L will cancel. But we will give a simple example of one-loop computation for which we show that this is true.

A simple example of the one-loop computation

As an example we will consider the HHL correlator with the light operator being a simple half-BPS state $\mathcal{O}_\alpha = \text{Tr}(\tilde{Z}^2)$, where $\tilde{Z} = Z + \bar{Z} + Y - \bar{Y}$. And the heavy operators will be taken to be a one magnon state. Let us first consider the contribution from the insertions.

As it has been mentioned above we have two kinds of insertions: to the light operator and to the heavy operators. In our case they are schematically presented on the Fig.(3.11),

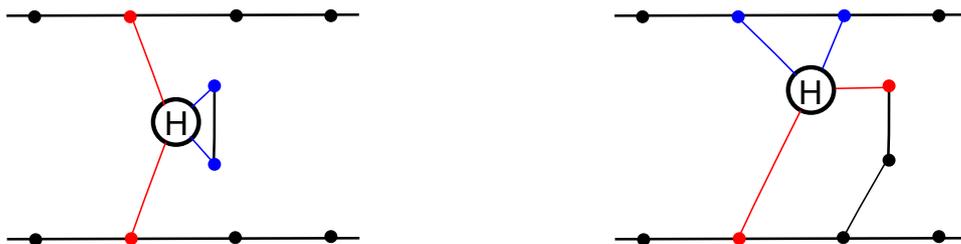


Figure 3.11 – Insertions to the light (on the left) and to the heavy (on the right) operators

Now we will translate the action of the insertions into diagonal matrix element of the lengthened light operator. Let us first consider the left picture on the Fig. (3.11). Due to the symmetry of the problem it is clear that the only terms of the light operator which will contribute are $\text{Tr}(Z\bar{Z})$ and $-\text{Tr}(Y\bar{Y})$. The action of the insertions on the first can

be divided in two parts: $\mathcal{H}_{12}(Z_1 \otimes \bar{Z}_2)$ and $\mathcal{H}_{12}(\bar{Z}_1 \otimes Z_2)$, they give

$$\begin{aligned}
1. \mathcal{H}_{12}(Z_1 \otimes \bar{Z}_2) &\rightarrow -(Z_1 \otimes \bar{Z}_2 + \bar{Z}_1 \otimes Z_2 + X_1 \otimes \bar{X}_2 + \bar{X}_1 \otimes X_2 + Y_1 \otimes \bar{Y}_2 + \bar{Y}_1 \otimes Y_2) \\
&\quad - 2(Z_1 \otimes \bar{Z}_2) + 2(\bar{Z}_1 \otimes Z_2). \\
2. \mathcal{H}_{12}(\bar{Z}_1 \otimes Z_2) &\rightarrow -(Z_1 \otimes \bar{Z}_2 + \bar{Z}_1 \otimes Z_2 + X_1 \otimes \bar{X}_2 + \bar{X}_1 \otimes X_2 + Y_1 \otimes \bar{Y}_2 + \bar{Y}_1 \otimes Y_2) \\
&\quad + 2(Z_1 \otimes \bar{Z}_2) - 2(\bar{Z}_1 \otimes Z_2).
\end{aligned} \tag{3.93}$$

The expression above means that after taking into account the action of the insertions the term $\text{Tr}(Z\bar{Z})$ of the light operator transforms into two terms on the *rhs* of the expressions (3.93). Analogous result we have for the term $-\text{Tr}(Y\bar{Y})$. The only difference is that the action of the unity and permutation operators can be neglected since they preserve the fields of the operator:

$$\begin{aligned}
1. \mathcal{H}_{12}(Y_1 \otimes \bar{Y}_2) &\rightarrow Z_1 \otimes \bar{Z}_2 + \bar{Z}_1 \otimes Z_2 + X_1 \otimes \bar{X}_2 + \bar{X}_1 \otimes X_2 + Y_1 \otimes \bar{Y}_2 + \bar{Y}_1 \otimes Y_2. \\
2. \mathcal{H}_{12}(\bar{Z}_1 \otimes Z_2) &\rightarrow Z_1 \otimes \bar{Z}_2 + \bar{Z}_1 \otimes Z_2 + X_1 \otimes \bar{X}_2 + \bar{X}_1 \otimes X_2 + Y_1 \otimes \bar{Y}_2 + \bar{Y}_1 \otimes Y_2.
\end{aligned} \tag{3.94}$$

It is not hard to see that when putting all the terms of the expression (3.93), (3.94) together we will get zero. Thus the left diagram from (3.11) does not contribute to the correlator.

Now we consider the contribution of the insertions to the heavy operators. It is slightly more complicated. The hamiltonian density acts on the two site of the heavy operator, which means that the action of the trace operator K can be neglected. In addition one site of the light operator is contracted with heavy operator. This means that the terms comprising Y or \bar{Y} will not contribute as well. Let us consider one of the configurations corresponding to the case depicted on the Fig. (3.12) So we have that the hamiltonian

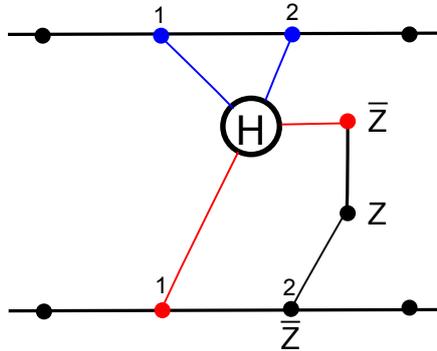


Figure 3.12

density acts on the two sites of the heavy operator $\mathcal{O}_h : (Z, X)$ and the result is contracted with the two sites of the heavy operator $\bar{\mathcal{O}}_h : (\bar{Z}, \bar{X})$ with the second site being \bar{Z} . It's not

hard to see that the computation of this diagram can be computed as the matrix element of the following spin chain operator:

$$I_1 = E_k^{11}(\mathcal{H}_{k,k+1}) = -g^2(\mathbb{I}_k \otimes E_{k+1}^{11} - E_k^{11} \otimes E_{k+1}^{11} - E_k^{21} \otimes E_{k+1}^{12}), \quad (3.95)$$

There are three more diagrams corresponding to the right picture of the Fig.(3.11), one of them, I_2 , coincide with I_1 , and two others are equal to transposed of I_1 :

$$I_3 = I_4 = -g^2(\mathbb{I}_k \otimes E_{k+1}^{11} - E_k^{11} \otimes E_{k+1}^{11} - E_k^{12} \otimes E_{k+1}^{21}). \quad (3.96)$$

Since the operator insertions have the order of $O(g^2)$ we can but the inhomogeneities θ_k to zero. Therefore the computation of the contribution from the insertions reduced to the computation of the following spin chain matrix elements:

$$\langle \mathbf{u} | E^{11} | \mathbf{u} \rangle, \quad \langle \mathbf{u} | E^{11} \otimes E^{11} | \mathbf{u} \rangle, \quad \langle \mathbf{u} | E^{12} \otimes E^{21} | \mathbf{u} \rangle, \quad \langle \mathbf{u} | E^{21} \otimes E^{12} | \mathbf{u} \rangle, \quad (3.97)$$

which we already learnt how to do. Putting all the contributions together we get that the contribution from the insertions δ_H takes the form

$$\delta_H = \frac{1}{\rho_1(u)} \frac{32g^2}{(4u^2 + 1)^2}. \quad (3.98)$$

The expression (3.98) should come with the factor of 2, since there are two terms $\text{Tr}(Z\bar{Z})$ in the light operator $\text{Tr}(\tilde{Z}^2)$, but we normalize by $2\langle \mathbf{u} | \mathbf{u} \rangle$.

Next we compute the contribution coming from the correction of the state. Thus we have to calculate the expression

$$\langle u, \theta^{\text{BDS}} | S^{-1} \mathcal{O}_\alpha S | u, \theta^{\text{BDS}} \rangle. \quad (3.99)$$

It is obvious that the terms comprising the field Y will not contribute here. It means that the only term we need to consider is the term $\text{Tr}(Z\bar{Z})$, which correspond to the local operator E^{11} . As it has been just mentioned this term comes with the facto of 2, but due to the normalization this factor cancels. Thus the corrections to the states can be written as follows

$$\delta_S = \frac{\langle u, \theta^{\text{BDS}} | S^{-1} E_l^{11} S | u, \theta^{\text{BDS}} \rangle}{2\langle u, \theta^{\text{BDS}} | u, \theta^{\text{BDS}} \rangle}, \quad (3.100)$$

where the index l states for the site on with the local operator E_l^{11} acts. From (3.88) we have

$$S^{-1} E^{11} S = E^{11} + [E^{11}, \hat{F}] + \frac{1}{2}[\hat{F}, [\hat{F}, E^{11}]] + O(g^3). \quad (3.101)$$

From the (3.83) we get

$$\begin{aligned} S^{-1} E^{11} S = & E_l^{11} + i\nu_{l-1}[E_l^{11}, H_{l-1}] + i\nu_l[E_l^{11}, H_l] - \frac{1}{2}\rho_{l-1}[E_l^{11}, [H_{l-2}, H_{l-1}]] - \\ & \frac{1}{2}\rho_l[E_l^{11}, [H_{l-1}, H_l]] - \frac{1}{2}\rho_{l+1}[E_l^{11}, [H_l, H_{l+1}]] - \frac{1}{2}\left(\nu_{l-2}\nu_{l-1}[H_{l-2}, [H_{l-1}, E_l^{11}]] + \right. \\ & \nu_{l-1}^2[H_{l-1}, [H_{l-1}, E_l^{11}]] + \nu_{l-1}\nu_l[H_l, [H_{l-1}, E_l^{11}]] + \nu_{l-1}\nu_l[H_{l-1}, [H_l, E_l^{11}]] + \\ & \left. \nu_l^2[H_l, [H_l, E_l^{11}]] + \nu_l\nu_{l+1}[H_{l+1}, [H_l, E_l^{11}]]\right). \end{aligned} \quad (3.102)$$

Thus as it was discussed above we need to compute length-1,-2,-3 operators. Notably all the length-3 operators have the coefficients in front of them proportional to g^2 , which means, which means that when computing the length-3 operators, we can put all the θ to zero. Moreover, one can show that the matrix elements of only three of the length-3 operators involved are independent:

$$E^{11}E^{12}E^{21}, E^{12}E^{11}E^{21}, E^{12}E^{21}E^{11}, E^{12}E^{22}E^{21} \quad (3.103)$$

The rest can be obtained from them by conjugation or their linear combinations with already known length-1 and length-2 operators. For one magnon case we easily compute the matrix element for the length-3 operators listed above. We get

$$\begin{aligned} \frac{\langle \mathbf{u} | E^{11} E^{12} E^{21} | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle} &= -\frac{1}{\rho_1(u)} \frac{1}{(u+i/2)^2}, \\ \frac{\langle \mathbf{u} | E^{12} E^{11} E^{21} | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle} &= -\frac{1}{\rho_1(u)} \frac{u-i/2}{(u+i/2)^3}, \\ \frac{\langle \mathbf{u} | E^{12} E^{21} E^{11} | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle} &= -\frac{1}{\rho_1(u)} \frac{1}{(u+i/2)^2}, \\ \frac{\langle \mathbf{u} | E^{12} E^{22} E^{21} | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle} &= 0. \end{aligned} \quad (3.104)$$

Then by substituting the expression for all the matrix elements in (3.102) we get

$$\delta_S = \frac{1}{\rho_1^{(1)}(u)} \left(\rho_1^{(1)}(u) + \frac{1}{u^2 + 1/4} + \frac{32g^2(12u^2 - 1)}{(4u^2 + 1)^3} \right). \quad (3.105)$$

As we can see from (3.105) the dependence on L contained in the functions ν and ρ indeed cancels. Finally the HHL correlator under consideration is given by the expression

$$\delta_H + \delta_S = \frac{1}{\rho_1^{(1)}(u)} \left(\rho_1^{(1)}(u) + \frac{1}{u^2 + 1/4} + \frac{512g^2u^2}{(4u^2 + 1)^3} \right), \quad (3.106)$$

which has the form of the expansion (3.36), and the explicit dependence on the length of the heavy operator L comes only through $\rho_1^{(1)}(u)$. The expression for the coefficient $f^{\tilde{Z}^2}(u)$ at one loop takes the form

$$f^{\tilde{Z}^2}(u) = \frac{1}{u^2 + 1/4} + \frac{512g^2u^2}{(4u^2 + 1)^3}. \quad (3.107)$$

3.2 HHL correlator at all loop

In this section we will consider the HHL correlator at any loop by exploiting the approach of the hexagon form factor [57] in the asymptotic approximation. It means that we will assume that the length of the light operator although is much smaller than the length of the heavy operators, but still large enough, so we can neglect the wrapping corrections. We will consider the configuration of the correlator with the light operator being the half-BPS state $\text{Tr}(\tilde{Z}^{2l_0})$, where $\tilde{Z} = Z + \bar{Z} + Y - \bar{Y}$. The heavy states, as in the previous sections, will be taken from the $su(2)$ sector. For this configuration we

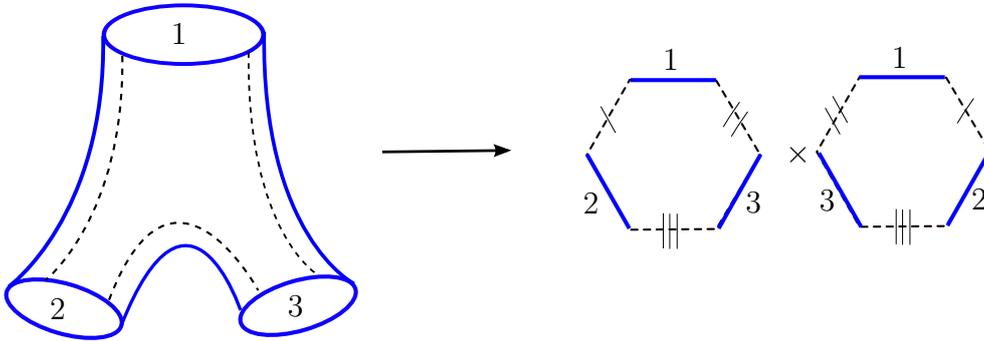


Figure 3.13 – Cutting of the three-point correlator in two hexagons. The dashed lines correspond to the cuts

will show that in the case when the heavy operators has one and two excitations the statement (3.36) works up to the leading wrapping corrections.⁶ The choice of the light operator is quite specific, but, as it has been shown in [138], the proof given in [127] can be generalized to the cases with the light operator having arbitrary excitations added upon the BMN vacuum $\text{Tr}(\tilde{Z}^{2l_0})$, and even with the heavy states taken from other rank-1 sectors, such as $sl(2)$ and $su(1|1)$.

3.2.1 The hexagon program

In this section we will review the recently proposed all-loop technique of the computation of the non-extremal three-point functions in $\mathcal{N} = 4$ SYM called *the hexagon program* [57].

This technique is very similar to the one developed for computing the scattering amplitudes [56], where the computation of the null polygon Wilson loops is reduced to the computation of some building blocks called *pentagon transition*. In its turn the computation of the pentagon transitions is possible to perform exactly by exploiting the integrability of the theory.

As it follows from the name of the method, the main building block for computing the three-point functions appears to be a *hexagon*, or hexagon form factor. Indeed the main idea of the hexagon program is to cut the three-point function pants in two hexagons (see the Fig. 3.13). We cut an each cuff of the three-point function pants in two parts. In this way we produce two hexagons. Thus each hexagon will have three *physical* edges corresponding to each of the gauge invariant operators, and three edges appeared due to the cutting. After the cutting we left some part of the excitations of the operators to be on one hexagon and the rest on another. According to the prescription of the hexagon program one has to perform a sum over all possible partitions of the excitations of each operator between the two hexagons. The summing over all the partitions is performed in the same way as it is described in [67], which implies that, as it is depicted on the Fig.

6. The proof for any number of excitations is given in [127]

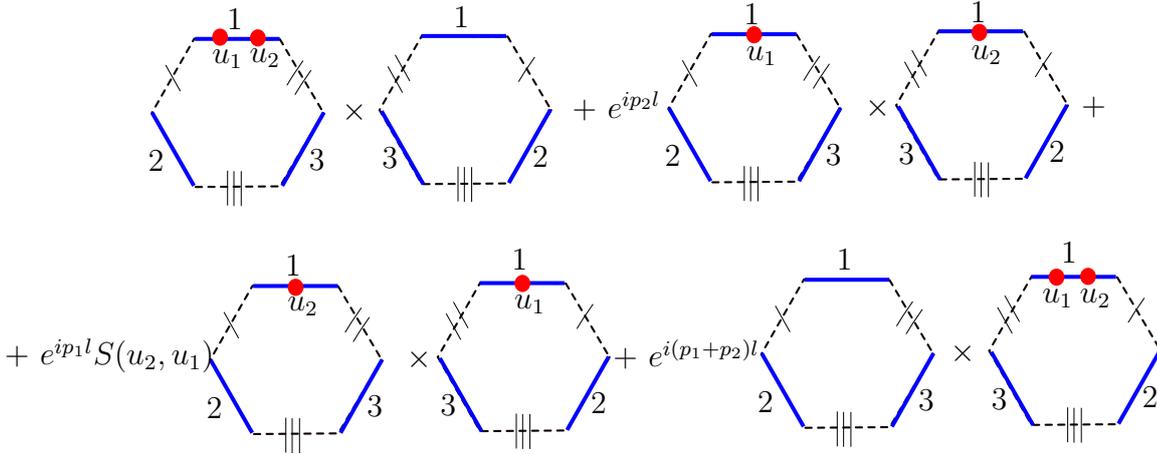


Figure 3.14 – The Sum over partitions of the excitations for the case of the two magnons

3.18, each term should come with appropriate cutting factor

$$\omega_{l_{ij}}(\alpha, \bar{\alpha}) = \prod_{u_j \in \bar{\alpha}} e^{ip(u_j)l} \prod_{u_i \in \alpha, i > j} S(u_i, u_j), \quad (3.108)$$

where $l_{ij} = \frac{L_i + L_j - L_k}{2}$ is the length of the *bridge* connecting the i -th and j -th operators. Here we would like to draw attention of the reader to the fact that here and till the end of the section 3.2, we accommodate the convention for the S -matrix notation according to the one used in [57], which means the following substitution with respect to the notation of the previous sections: $S(u, v) \rightarrow 1/S(u, v)$.

When we cut the three-point function we forbid the mirror particles to travel between the hexagons. In order to take the mirror channel into account we have to sew the pants back. It is performed by summing over all possible states living on the edges formed due to the cutting, which implies the integrating over the rapidities of the mirror excitations. The mirror excitation start contribute at the order of $O(g^{2l_{ij}})$, so in the case when all the lengths are taken to be large they can be neglected.

The main advantage of the hexagon program is that the authors of [57] proposed an all-loop expression for the hexagon. The ansatz, they gave, corresponds to the so-called *fundamental hexagon*, the configuration when all the excitations are located on one of the edges. However, as we will discuss below, it turned out that all other hexagons are related to the fundamental one by the *mirror transformation*, which translates the particle from one edge to another.

In order to fix the form of the fundamental hexagon the authors performed some symmetry analysis. When computing three-point functions in $\mathcal{N} = 4$ SYM we first choose the physical vacua of the states, and then form the operators by putting the excitations on top of them. In the quality of vacuum one can choose the generic half-BPS state:

$$\mathcal{O}_i(x_i^\mu) = \text{Tr}(y_i^k \phi_k)^{L_i}(x_i^\mu), \quad (3.109)$$

where y^k are six complex values such that $(y^k)^2 = 0$ and summation over the repeated index is implied. The symmetry of such a BMN vacuum is described by $psu(2|2)^2$ algebra. When having three different states of this kind, by use of the conformal and R -symmetry,

one can simultaneously reduce them to the following form

$$\mathcal{O}_i = \text{Tr}(Z + a_i^2 \bar{Z} + a_i(Y - \bar{Y}))^{L_i}(0, a_i, 0, 0). \quad (3.110)$$

Then the residual symmetry left at our disposal will correspond to the algebra $psu(2|2)$.⁷ Then by appealing to the idea of the vertex (see chapter 2) the authors of [57] define the fundamental hexagon through the following quantity

$$H^{A_1 \dot{A}_1 \dots A_N \dot{A}_N}(u_1, \dots, u_N) = \langle \mathfrak{h} | \chi^{A_1 \dot{A}_1}(u_1) \dots \chi^{A_N \dot{A}_N}(u_N) \rangle, \quad (3.111)$$

Here $\langle \mathfrak{h} |$ stands for the *hexagon vertex*. The excitations form the representation of $psu(2|2)^2$, so A_i, \dot{A}_i are the $su(2|2)_L \otimes su(2|2)_R$ bifundamental indexes of the i -th excitation. Finally u_i denotes as usual the rapidity of the corresponding excitation. The authors of [57] proposes that the hexagon vertex should preserve the symmetry left after reducing the states to the form (3.110). That is it should be annihilated by the generators of the residual symmetry of $psu(2|2)$ algebra. Elaborating on these ideas the authors easily fixed the expression of the fundamental hexagon for the cases of one and two magnons. Then by generalizing the obtained results they suggested the following expression for the fundamental hexagon

$$H = H^{\text{dyn}} \cdot H^{\text{mat}}, \quad (3.112)$$

where

$$H^{\text{dyn}} = \prod_{i < j} h(u_i, u_j) \quad (3.113)$$

$$H^{\text{mat}} = (-1)^f \langle \chi_N^{\dot{A}_N} \dots \chi_1^{\dot{A}_1} | \mathcal{S} | \chi_1^{A_1} \dots \chi_N^{A_N} \rangle,$$

and the factor $(-1)^f$ accommodates for the grading. The *dynamical part* H^{dyn} is simply a product of the scalar function $h(u, v)$ given by

$$h(u, v) = \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^+} \frac{1}{\sigma_{12}} \quad (3.114)$$

where the variables $x_{1,2}^\pm$ are defined as $x_1^\pm = x(u \pm i/2)$ and $x_2^\pm = x(v \pm i/2)$. Here $x(u)$ is the Zhukowsky variable satisfying $x + 1/x = u/g$ and $\sigma_{12} = \sigma(u, v)$ is the square root of BES dressing phase[43]. Finally \mathcal{S} form the matrix part H^{mat} stands for Beisert's S -matrix [39, 139] with the scalar factor S_{12}^0 set to 1.

Let us now discuss the mentioned above mirror transformation γ , which relates the fundamental hexagon to a generic one. When the excitations are distributed between more than one edges, the mirror transformation allows to move the particles from one edge to another as it is depicted on the Fig. (3.15). To be more precise, since we are not considering here the contribution from mirror excitations we will be always moving the particles with even numbers of the mirror transformations. So we will concentrate on double-mirror transformation, or in other words *crossing* transformation 2γ .

In [57] the authors introduced two frames for describing the hexagon excitations: spin chain frame and string frame. We will stick to the spin chain frame, since it is well

7. To be more precise the symmetry of the BMN vacuum includes also three central charges. Thus the full symmetry is $psu(2|2)^2 \times \mathbb{R}^3$. And when we reduce the three BMN vacua to the form (3.110), one of the central charges is preserved, so the residual symmetry is $psu(2|2) \times \mathbb{R}$

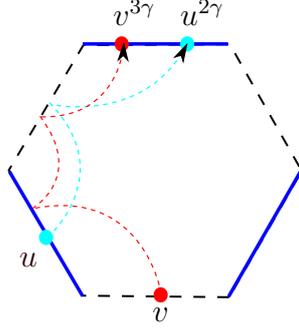


Figure 3.15 – Crossing transformation for the excitations of the two hexagons. One magnon case

suitable for comparison with result at weak coupling. However the string frame is more convenient for describing effect of the crossing transformation, so we will start with it. The rule relating the excitations connected by crossing transformation is very simple:

$$\chi^{A\dot{B}}(u) \xrightarrow{2\gamma} -\chi^{B\dot{A}}(u^{2\gamma}). \quad (3.115)$$

The dynamical part H^{dyn} also changes under the crossing. First of all the Zhukowsky variable transform as follows:

$$\begin{aligned} x^- &\xrightarrow{\gamma} x^-, & x^+ &\xrightarrow{\gamma} 1/x^+, \\ x^- &\xrightarrow{2\gamma} 1/x^-, & x^+ &\xrightarrow{2\gamma} 1/x^+, \\ x^- &\xrightarrow{3\gamma} x^-, & x^+ &\xrightarrow{3\gamma} 1/x^+, \\ x^- &\xrightarrow{4\gamma} x^-, & x^+ &\xrightarrow{4\gamma} x^+. \end{aligned} \quad (3.116)$$

In addition the BES dressing phase $\sigma(u)$ satisfies the following relation

$$\sigma(u^{2\gamma}, v)\sigma(u, v) = \frac{(1 - 1/x^+y^+)(1 - x^-/y^+)}{(1 - x^-/y^-)(1 - 1/x^+y^-)}. \quad (3.117)$$

Finally the relation connecting the hexagon form factor at string and spin chain frames read as

$$\langle \mathfrak{h}|\psi_1\rangle|\psi_2\rangle|\psi_3\rangle_{string} = e^{-\frac{i}{2}(P(l+n-m)+Q(m+l-n)+R(n+m-l))} F_p F_q F_r \langle \mathfrak{h}|\psi_1\rangle|\psi_2\rangle|\psi_3\rangle_{spin}, \quad (3.118)$$

where $|\psi_i\rangle$ stands for the spin chain state located on the i -th physical edge of the hexagon, l, m, n and P, Q, R are numbers of the excitations and total momenta of these states correspondingly, and the quantities F_p, F_q, F_r are given by the expressions

$$F_p = \prod_k \frac{e^{ip_k/2}\zeta_k}{\zeta}, \quad F_q = \prod_k \frac{e^{iq_k/2}\zeta_k}{\zeta}, \quad F_r = \prod_k \frac{e^{ir_k/2}\zeta_k}{\zeta}. \quad (3.119)$$

Here p_k, q_k, r_k are the momenta of the excitations of the corresponding spin chain state, $\zeta_{k+1} = e^{ip_k}\zeta_k, \zeta_1 = \zeta$.

3.2.2 Set-up

In this section we will discuss the correlator we are going to consider from hexagon approach point of view. As it should clear from the previous section this approach consider the correlators, which operators are built by putting the excitations on top of the vacua (3.110). By putting a_1 to 0, a_2 to ∞ and finally a_3 to 1, we get the following three vacua

$$\mathrm{Tr}(Z^{L_1}), \quad \mathrm{Tr}(\bar{Z}^{L_2}), \quad \mathrm{Tr}(\tilde{Z}^{L_3}). \quad (3.120)$$

This is the configuration was used by the authors of [57] and we will be using it in our computation. Therefore we consider two heavy operators belonging to the $su(2)$ sectors Z, X and \bar{Z}, \bar{X} correspondingly as before. The length of the heavy operators are $L_1 = L_2 = L$. In quality of the light operator we will take the generalization of the one considered in the section (3.1.7), namely a half-BPS state $\mathrm{Tr}(\tilde{Z}^{2l_0})$.

In the HHL three-point function we have $l_0 \ll L$. As it has been mentioned above we will consider the asymptotic L -dependence of the structure constant. It means that we will neglect the wrapping corrections, two kinds of which we will distinguish. The first type of the wrapping is the appearing from the corrections to the states, when we normalize the three-point function by the norm of the heavy operators. It appears at the order of $\mathcal{O}(g^{2L})$ and we can neglect it, since the conjecture (3.36) is formulated to be valid up to this kind of corrections. Another kind of wrapping, to which we will refer as to *bridge wrapping*, appears from the mirror channel of the correlator itself, and should be taken into account by correct sewing the hexagons together. As it has been already mentioned, this wrapping appears at the order of $\mathcal{O}(g^{l_{ij}})$, and in case of small length of one of the operators, which is our case, might contribute already at two loop order. We postpone the computation of this corrections to future investigation, since it is still

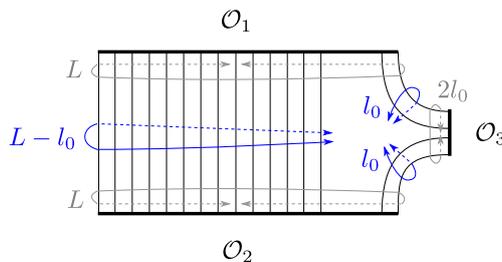


Figure 3.16 – Two kinds of wrapping corrections of the structure constant. The blue lines corresponds to the bridge wrappings and the gray lines corresponds to the state wrappings.

not very clear how to sum all the bridge wrapping corrections (see [140, 141] for more details). Thus in the following we assume $1 \ll l_0 \ll L$, so that we can trust the result up to relatively high orders without worrying about the wrapping. In this regime, we only need to consider the physical excitations and we can study the asymptotic L -dependence. We will see that the result confirms (3.36).

Let us denote the two sets of rapidities of the excitations of the heavy operators \mathcal{O}_1 and \mathcal{O}_2 to be $\{u\}_N = \{u_N, \dots, u_1\}$ and $\{v\}_N = \{v_1, \dots, v_N\}$, as is shown in Fig. 3.17. We first compute the correlator $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_\alpha \rangle$ and then take the diagonal limit $v_i \rightarrow u_i$. Then

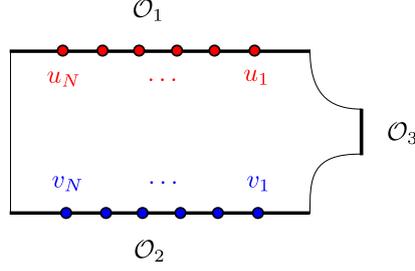


Figure 3.17 – The rapidities for the two heavy operators. Notice that the for \mathcal{O}_1 , the set rapidities is labeled by u_N, u_{N-1}, \dots, u_1 while for \mathcal{O}_2 is labeled by v_1, v_2, \dots, v_N .

the structure constant will have the following sum-over-partition expression [57]

$$\mathcal{C}_{2N} = \sum_{\substack{\alpha \cup \bar{\alpha} = \{u\}_N \\ \beta \cup \bar{\beta} = \{v\}_N}} \omega_{-l}(\alpha, \bar{\alpha}) \omega_l(\beta, \bar{\beta}) \mathbb{H}(\alpha|\beta) \mathbb{H}(\bar{\beta}|\bar{\alpha}). \quad (3.121)$$

where the two splitting factors are

$$\begin{aligned} \omega_{-l}(\alpha, \bar{\alpha}) &= \prod_{u_j \in \bar{\alpha}} (e^{-ilp(u_j)}) \prod_{\substack{u_i \in \alpha \\ i > j}} S(u_j, u_i) \\ \omega_l(\beta, \bar{\beta}) &= \prod_{v_j \in \bar{\beta}} (e^{ilp(v_j)}) \prod_{\substack{v_i \in \beta \\ i > j}} S(v_i, v_j). \end{aligned} \quad (3.122)$$

and $l = L - l_0 \sim L$. Here $\mathbb{H}(\alpha|\beta)$ and $\mathbb{H}(\bar{\beta}|\bar{\alpha})$ are the hexagon form factors which can be computed non-perturbatively by applying the prescription (3.112)-(3.114). Note that we have applied Bethe Ansatz Equations (BAE) to rewrite the splitting factor in (3.122). It turns out to be an important point which we will discuss in the next section. Then both splitting factors ω_{-l} and ω_l depend on the large size scale l . This is the origin of the explicit L -dependence of the structure constant. As we will see later, when we take the diagonal limit $\{v\}_N \rightarrow \{u\}_N$, a $\frac{0}{0}$ uncertainty appears, so we will have to take derivatives of the phase factors $e^{ilp(v)}$, which leads to the polynomial dependence of L . Another source of the L -dependence is the phase factors itself, but, after taking the limit, it can be eliminated by applying BAE.

3.2.3 One-magnon case

First let us consider the case when the heavy operators have just one excitation. The structure constant in this case is given as follows

$$\mathcal{C}_2 = \mathbb{H}(u|v) + e^{ipl_{31}} e^{iql_{12}} \mathbb{H}(v|u), \quad (3.123)$$

where $l_{12} = \frac{L_1 + L_2 - L_3}{2} = L - l_0 = l$, $l_{13} = \frac{L_1 + L_3 - L_2}{2} = l_0$ and $p = p(u)$, $q = p(v)$.

The first hexagon is given by the expression

$$\mathbb{H}(u|v) = \langle \mathfrak{h} | X(u) | \bar{X}(v) \rangle, \quad (3.124)$$

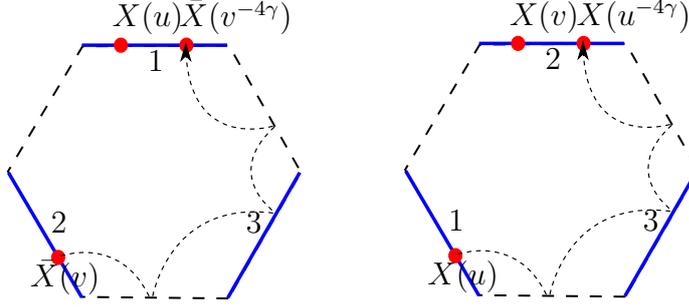


Figure 3.18 – Crossing transformation for the excitations of the two hexagons. One magnon case

with $X = \Phi^{11}$ and $\bar{X} = -\Phi^{22}$, which correspond to the notations of [57]. Now we have to perform the crossing transformation in order to put the excitations on the same edge (see the left hexagon on the Fig. 3.18). According to the prescription (3.115) in the string frame we get

$$\langle \mathfrak{h} | X(u) \rangle | \bar{X}(v) \rangle_{st} = \langle \mathfrak{h} | X_1(u) \bar{X}_2(v^{-4\gamma}) \rangle_{st}. \quad (3.125)$$

From the connection between string and spin chain description (3.118), we get

$$\begin{aligned} \langle \mathfrak{h} | X(u) \rangle | \bar{X}(v) \rangle_{st} &= e^{-iq} e^{ip/2} e^{iq/2} \langle \mathfrak{h} | X(u) \rangle | \bar{X}(v) \rangle_{sp}, \\ \langle \mathfrak{h} | X_1(u) \bar{X}_2(v^{-4\gamma}) \rangle_{st} &= e^{-i(p+q)} e^{ip/2} e^{iq/2} e^{ip} \langle \mathfrak{h} | X_1(u) \bar{X}_2(v^{-4\gamma}) \rangle_{sp}. \end{aligned} \quad (3.126)$$

which gives for the spin chain frame

$$\langle \mathfrak{h} | X(u) \rangle | \bar{X}(v) \rangle_{sp} = \langle \mathfrak{h} | X_1(u) \bar{X}_2(v^{-4\gamma}) \rangle_{sp}, \quad (3.127)$$

so in this case the factor relating two frames cancels resulting in the same expression. performing analogous computation for the second hexagon⁸ (right hexagon on the Fig. 3.18) we get

$$\langle \mathfrak{h} | \bar{X}(v) \rangle | X(u) \rangle_{sp} = \langle \mathfrak{h} | \bar{X}_1(v) X_2(u^{-4\gamma}) \rangle_{sp}. \quad (3.128)$$

Thus one can see that for both hexagons the phase factor cancel in the spin chain frame⁹.

8. We draw attention of the reader to the fact that for the first hexagon we apply -4γ transformation to the excitations of the second operator and move them to the edge corresponding to the first operator. Whereas for the second hexagon we apply the same -4γ transformation to the excitations of the first operator moving them to the second edge

9. In fact it turns out that the phase factor cancels for arbitrary number of magnons. Indeed

$$\begin{aligned} \langle \mathfrak{h} | X_1(u_1) \dots X_N(u_N) \rangle | \bar{X}_1(v_1) \dots \bar{X}_1(v_N) \rangle_{st} &= \\ = e^{-iNQ} e^{iP/2} e^{iQ/2} e^{ip_1(N-1) + ip_2(N-2) + \dots + ip_{N-1}} e^{iq_1(N-1) + iq_2(N-2) + \dots + iq_{N-1}} \times \\ \langle \mathfrak{h} | X_1(u_1) \dots X_N(u_N) \rangle | \bar{X}_1(v_1) \dots \bar{X}_1(v_N) \rangle_{sp}, \end{aligned} \quad (3.129)$$

and

$$\begin{aligned} \langle \mathfrak{h} | X_1(u_1) \dots X_N(u_N) \bar{X}_{N+1}(v_1^{-4\gamma}) \dots \bar{X}_{2N}(v_N^{-4\gamma}) \rangle_{st} &= \\ = e^{-i(P+Q)N} e^{iP/2} e^{iQ/2} e^{ip_1(2N-1) + ip_2(2N-2) + \dots + ip_N N} e^{iq_1(N-1) + iq_2(N-2) + \dots + iq_{N-1}} \times \\ \langle \mathfrak{h} | X_1(u_1) \dots X_N(u_N) \bar{X}_{N+1}(v_1^{-4\gamma}) \dots \bar{X}_{2N}(v_N^{-4\gamma}) \rangle_{sp}, \end{aligned} \quad (3.130)$$

Now in order to compute the obtained fundamental hexagons we use the prescription (3.113)

$$\langle \mathfrak{h} | \Phi_1^{a\dot{a}}(u) \Phi_2^{b\dot{b}}(v) \rangle = h(u, v) A(u, v) \epsilon^{a\dot{b}} \epsilon^{b\dot{a}} + \frac{1}{2} h(u, v) \left(A(u, v) - B(u, v) \right) \epsilon^{ab} \epsilon^{\dot{a}\dot{b}}, \quad (3.132)$$

where the coefficients $A(u, v)$, $B(u, v)$ are the elements of the Beisert's S -matrix with the scalar factor S^0 put to 1:

$$A(u, v) = \frac{x_2^+ - x_1^-}{x_2^- - x_1^+}, \quad B(u, v) = \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} \left(1 - 2 \frac{1 - \frac{1}{x_2^- x_1^+} x_2^+ - x_1^+}{1 - \frac{1}{x_2^- x_1^-} x_2^+ - x_1^-} \right). \quad (3.133)$$

Applying (3.132) to the first hexagon we get

$$\begin{aligned} \mathbb{H}(u|v) &= \langle \mathfrak{h} | \Phi_1^{1\dot{1}}(u) (-\Phi_2^{2\dot{2}}(v^{-4\gamma})) \rangle = - \left(-h(u, v^{-4\gamma}) A(u, v^{-4\gamma}) + \right. \\ &\quad \left. \frac{h(u, v^{-4\gamma})}{2} \left(A(u, v^{-4\gamma}) - B(u, v^{-4\gamma}) \right) \right), \end{aligned} \quad (3.134)$$

and analogously for the second one

$$\begin{aligned} \mathbb{H}(v|u) &= \langle \mathfrak{h} | (-\Phi_1^{2\dot{2}}(v)) \Phi_2^{1\dot{1}}(u^{-4\gamma}) \rangle = - \left(-h(v, u^{-4\gamma}) A(v, u^{-4\gamma}) + \right. \\ &\quad \left. \frac{h(v, u^{-4\gamma})}{2} \left(A(v, u^{-4\gamma}) - B(v, u^{-4\gamma}) \right) \right), \end{aligned} \quad (3.135)$$

Then by using the property (3.117) of the dressing phase $\sigma(u, v)$ to transform under the crossing transformation and noticing that

$$(x^- - y^-) \left(1 - \frac{1}{x^- y^-} \right) = (x^+ - y^+) \left(1 - \frac{1}{x^+ y^+} \right) = \frac{u - v}{g}, \quad (3.136)$$

we get

$$h(u, v^{-4\gamma}) = \frac{1}{h(v, u)}. \quad (3.137)$$

On the other hand for the coefficients $A(u, v)$ and $B(u, v)$ we have simply

$$A(u, v^{-4\gamma}) = A(u, v), \quad B(u, v^{-4\gamma}) = B(u, v). \quad (3.138)$$

By putting everything together we get the all loop result

$$\mathcal{C}_2 = \frac{1}{2h(v, u)} \left(A(u, v) + B(u, v) \right) + \frac{e^{ip(u)l_{31}} e^{ip(v)l_{12}}}{2h(u, v)} \left(A(v, u) + B(v, u) \right), \quad (3.139)$$

so

$$\begin{aligned} &\langle \mathfrak{h} | X_1(u_1) \dots X_N(u_N) | \bar{X}_1(v_1) \dots \bar{X}_1(v_N) \rangle_{sp} = \\ &\langle \mathfrak{h} | X_1(u_1) \dots X_N(u_N) \bar{X}_{N+1}(v_1^{-4\gamma}) \dots \bar{X}_{2N}(v_N^{-4\gamma}) \rangle_{sp}. \end{aligned} \quad (3.131)$$

where by $p(u)$ we as usual denotes the momenta. Now, since we are interested in the symmetric case, we need to take the diagonal limit $v \rightarrow u$. At this point some interesting issues appear. First of all we notice that the function $\frac{1}{h(u,v)}$ has a simple pole when at $u = v$:

$$\frac{1}{h(u,v)} = \frac{u-v-i}{u-v} \frac{1}{\tilde{h}(u,v)}, \quad (3.140)$$

where

$$\tilde{h}(u,v) = \frac{(1 - 1/x_1^- x_2^+)^2}{(1 - 1/x_1^- x_2^-)(1 - 1/x_1^+ x_2^+)} \frac{1}{\sigma_{12}} \quad (3.141)$$

is nonzero for coinciding rapidities ($\tilde{h}(u,u) \neq 0$). Since the structure constant in the symmetric case should be a well-defined quantity, this pole should be counterbalanced by the zero in the numerator leading to the $\frac{0}{0}$ uncertainty. Let us see how it works at tree level, we have

$$\mathcal{C}_2^{tr} = -\frac{i}{u-v} - e^{ip(u)l_0} e^{ip(v)(L-l_0)} \frac{i}{v-u}. \quad (3.142)$$

From this one can see that in order to get a zero in the numerator we have to take into account the Bethe equations (1.34). The problem is that we can do it by two different ways: applying the Bethe equations to the momentum of the first state p or to the momentum of the second state q . As it is not hard to see, the results will be different. In order to resolve this issue we will follow a simple strategy, namely by comparing with the already known computation at tree level we choose the right choice. By comparing with the tree level approximation of one-loop result (3.106) corresponding to the case $l_0 = 1$, we get the prescription for our set-up: one should apply the Bethe equations to those momenta, in front of which there is a small length l_0 in order to restore the dependence on the large length L . Thus we will be using the expression (3.123) for the structure constant, where we already took this prescription into account.

Now armed with this recipe we take the diagonal limit of the all loop result (3.139). We introduce the following quantity

$$\Omega(u,v) = \frac{u-v+i}{2\tilde{h}(v,u)} \left(A(u,v) + B(u,v) \right). \quad (3.143)$$

Then the expression (3.139) can be rewritten as follows

$$\mathcal{C}_2 = \frac{\Omega(u,v) - e^{il(p(v)-p(u))}\Omega(v,u)}{u-v}. \quad (3.144)$$

Which gives for the symmetric structure constant denoted as $\mathcal{C}_{\text{HHL}}(u)$

$$\mathcal{C}_{\text{HHL}}(u) \propto ilp'(u)\Omega(u,u) + \Omega^{(1,0)}(u,u) - \Omega^{(0,1)}(u,u), \quad (3.145)$$

where the sign \propto signals that the expression should be appropriately normalized. By recalling that the expression for $\rho_N((u))$ (we always use the connected scheme, see the section 3.1.1) is given by (1.47) up to the wrapping corrections, when assuming the substitution (3.92), we get

$$\mathcal{C}_{\text{HHL}}(u) \propto \frac{1}{\tilde{h}(u,u)} \left(\rho_1(u) - l_0 p'(u) + \tilde{h}(u,u) (\Omega^{(1,0)}(u,u) - \Omega^{(0,1)}(u,u)) \right). \quad (3.146)$$

Now we notice that $\frac{1}{\tilde{h}(u,u)} = \mu_X(u)$, where $\mu_X(u)$ is a measure introduced in [57], which is contained in the norm of the operators:

$$\mathcal{N}_H = \prod_i \mu_X(u_i) \rho_N(\mathbf{u}). \quad (3.147)$$

We stick to the convention of normalization described in the section 3.1.2, thus we normalize the structure constant dividing it by \mathcal{N}_H , and we get exactly the result (3.106):

$$\begin{aligned} \mathcal{C}_{\text{HHL}}(u) &= \frac{1}{\rho_1(u)} \left(\rho_1(u) - l_0 p'(u) + \tilde{h}(u,u) (\Omega^{(1,0)}(u,u) - \Omega^{(0,1)}(u,u)) \right) = \\ &= \frac{1}{\rho_1^{(1)}(u)} \left(\rho_1^{(1)}(u) + \frac{1}{u^2 + 1/4} + \frac{512g^2 u^2}{(4u^2 + 1)^3} \right) + O(g^4), \end{aligned} \quad (3.148)$$

where we put l_0 to 1. Thus we can see that at least for one magnon case the conjecture (3.36) holds up to the leading wrapping corrections with the all-loop coefficient $f(u)$ given as follows

$$f(u) = -l_0 p'(u) + \tilde{h}(u,u) (\Omega^{(1,0)}(u,u) - \Omega^{(0,1)}(u,u)). \quad (3.149)$$

3.2.4 Two-magnon case

In this section we will consider more complicated case with the heavy operators having two excitations. For the structure constant from (3.123) we have

$$\begin{aligned} \mathcal{C}_4 &= \text{H}(u_1, u_2 | v_1, v_2) + e^{-ilp(u_1) - ilp(u_2) + ilp(v_1) + ilp(v_2)} S(u_2, u_1) \text{H}(v_1, v_2 | u_1, u_2) \\ &+ e^{-ilp(u_2) + ilp(v_2)} \text{H}(u_1 | v_1) \text{H}(v_2 | u_2) + e^{-ip(u_2) + ilp(v_1)} S(v_2, v_1) \text{H}(u_1 | v_2) \text{H}(v_1 | u_2) \\ &+ e^{-ilp(u_1) + ilp(v_2)} S(u_1, u_2) \text{H}(u_2 | v_1) \text{H}(v_2 | u_1) \\ &+ e^{-ilp(u_1) + ilp(v_2)} S(u_1, u_2) S(v_2, v_1) \text{H}(u_2 | v_2) \text{H}(v_1 | u_1). \end{aligned} \quad (3.150)$$

Let us consider the hexagon $\text{H}(u_1, u_2 | v_1, v_2)$. Its dynamical part is given as follows

$$\text{H}^{\text{dyn}}(u_1, u_2 | v_1, v_2) = \frac{u_1 - v_1 + i}{u_1 - v_1} \frac{u_2 - v_2 + i}{u_2 - v_2} \frac{1}{\tilde{h}(v_1, u_1) \tilde{h}(v_2, u_2)} \frac{h(u_1, u_2) h(v_1, v_2)}{h(v_2, u_1) h(v_1, u_2)}. \quad (3.151)$$

As in the case with one magnon we can see that there is a pole appearing when taking diagonal limit $v_i \rightarrow u_i$. Following the logic of the previous section we separate the divergent part by introducing a function $\Omega(u_1, u_2, v_1, v_2)$ such that

$$\text{H}(u_1, u_2 | v_1, v_2) = \frac{\Omega(u_1, u_2, v_1, v_2)}{(u_1 - v_1)(u_2 - v_2)}, \quad (3.152)$$

and which therefore remains finite when taking the diagonal limit. Now we can rewrite the structure constant in terms of $\Omega(u, v)$ and $\Omega(u_1, u_2, v_1, v_2)$, we get

$$\begin{aligned} \mathcal{C}_4 &= \frac{\Omega(u_1, u_2, v_1, v_2)}{(u_1 - v_1)(u_2 - v_2)} + e^{-ilp(u_1) - ilp(u_2) + ilp(v_1) + ilp(v_2)} S(u_2, u_1) \frac{\Omega(v_1, v_2, u_1, u_2)}{(v_1 - u_1)(v_2 - u_2)} \\ &+ e^{-ilp(u_2) + ilp(v_2)} \frac{\Omega(u_1, v_1) \Omega(v_2, u_2)}{(u_1 - v_1)(v_2 - u_2)} + e^{-ip(u_2) + ilp(v_1)} S(v_2, v_1) \frac{\Omega(u_1, v_2) \Omega(v_1, u_2)}{(u_1 - v_2)(v_1 - u_2)} \\ &+ e^{-ilp(u_1) + ilp(v_2)} S(u_1, u_2) \frac{\Omega(u_2, v_1) \Omega(v_2, u_1)}{(u_2 - v_1)(v_2 - u_1)} \\ &+ e^{-ilp(u_1) + ilp(v_1)} S(u_1, u_2) S(v_2, v_1) \frac{\Omega(u_2, v_2) \Omega(v_1, u_1)}{(u_2 - v_2)(v_1 - u_1)}. \end{aligned} \quad (3.153)$$

From (3.153) we can see that there are two types of the terms appear, which we denote as T_1 and T_2 :

$$\begin{aligned}
T_1 &= e^{-ip(u_2)+ilp(v_1)} S(v_2, v_1) \frac{\Omega(u_1, v_2)\Omega(v_1, u_2)}{(u_1 - v_2)(v_1 - u_2)} \\
&+ e^{-ilp(u_1)+ilp(v_2)} S(u_1, u_2) \frac{\Omega(u_2, v_1)\Omega(v_2, u_1)}{(u_2 - v_1)(v_2 - u_1)}, \\
T_2 &= \frac{\Omega(u_1, u_2, v_1, v_2)}{(u_1 - v_1)(u_2 - v_2)} + e^{-ilp(u_1)-ilp(u_2)+ilp(v_1)+ilp(v_2)} S(u_2, u_1) \frac{\Omega(v_1, v_2, u_1, u_2)}{(v_1 - u_1)(v_2 - u_2)} \\
&+ e^{-ilp(u_2)+ilp(v_2)} \frac{\Omega(u_1, v_1)\Omega(v_2, u_2)}{(u_1 - v_1)(v_2 - u_2)} \\
&+ e^{-ilp(u_1)+ilp(v_1)} S(u_1, u_2) S(v_2, v_1) \frac{\Omega(u_2, v_2)\Omega(v_1, u_1)}{(u_2 - v_2)(v_1 - u_1)}.
\end{aligned} \tag{3.154}$$

The terms gathered in T_1 are finite when taking the diagonal limit and the terms in T_2 have a double pole and serve as a source of the $\frac{0}{0}$ uncertainty. We introduce the following notations

$$T_1^d = \lim_{v_i \rightarrow u_i} T_1, \quad T_2^d = \lim_{v_i \rightarrow u_i} T_2. \tag{3.155}$$

Then T_1^d is simply given as follows

$$T_1^d = e^{-ip(u_2)+ilp(u_1)} S(u_2, u_1) \frac{\Omega(u_1, u_2)^2}{(u_1 - u_2)^2} + e^{-ilp(u_1)+ilp(u_2)} S(u_1, u_2) \frac{\Omega(u_2, u_1)^2}{(u_2 - u_1)^2}. \tag{3.156}$$

On the other hand the expression for T_2 is more complicated and can be written in the following from

$$\begin{aligned}
T_2^d &= \frac{1}{2} \frac{\partial^2}{\partial \epsilon^2} \left(\Omega(u_1, u_2, u_1 + \epsilon, u_2 + \epsilon) \right. \\
&+ e^{-ilp(u_1)-ilp(u_2)+ilp(u_1+\epsilon)+ilp(u_2+\epsilon)} S(u_2, u_1) \Omega(u_1 + \epsilon, u_2 + \epsilon, u_1, u_2) \\
&- e^{-ilp(u_2)+ilp(u_2+\epsilon)} \Omega(u_1, u_1 + \epsilon) \Omega(u_2 + \epsilon, u_2) \\
&\left. - e^{-ilp(u_1)+ilp(u_1+\epsilon)} S(u_1, u_2) S(u_2 + \epsilon, u_1 + \epsilon) \Omega(u_2, u_2 + \epsilon) \Omega(u_1 + \epsilon, u_1) \right) \Big|_{\epsilon=0}.
\end{aligned} \tag{3.157}$$

Let us now analyze the L -dependence recalling that $l = L - 1$. We start with the term which appear with a factor of L^2 . It is the following

$$\begin{aligned}
&\frac{1}{2} \left(-L^2 p'(u_1)^2 - L^2 p'(u_2)^2 - 2L^2 p'(u_1) p'(u_2) \right) S(u_2, u_1) \Omega(u_1, u_2, u_1, u_2) \\
&+ \frac{1}{2} L^2 p'(u_2)^2 \Omega(u_1, u_1) \Omega(u_2, u_2) + \frac{1}{2} L^2 p'(u_1)^2 \Omega(u_2, u_2) \Omega(u_1, u_1).
\end{aligned} \tag{3.158}$$

Before going further we describe one of the properties of the hexagon, which we are going to use in order to prove that the form of the L -dependence indeed satisfy (3.36). Namely, it turns out that when two rapidities of the hexagon coincide the resulting hexagon form factor is proportional to the hexagon form factor with less excitations. We will refer to this relation as the factorization property:

$$\mathbb{H}^{\text{mat}}(\mathbf{u}, \{u\}_n | \{v\}_n, \mathbf{u}) = -\mathbb{H}^{\text{mat}}(\{u\}_n | \{v\}_n). \tag{3.159}$$

Note that here the relation only concern the matrix part. The factorization property is a reformulation of the decoupling condition in [57], reflecting the fact that a pair of a particle and its anti-particle with zero energy and momentum decouples from the state by forming a singlet state [39], which scatters trivially with the rest of the excitations.

Other useful relations we are also going to use simply express the property of the hexagon to acquire a factor of the S -matrix when permuting the particles. The relations again are written for the matrix part, which gives rise to appearing of the $h(u, v)$'s:

$$\mathbb{H}^{\text{mat}}(\star, u, v, \star|\star) = S(v, u) \frac{h(v, u)}{h(u, v)} \mathbb{H}^{\text{mat}}(\star, v, u, \star|\star), \quad (3.160)$$

$$\mathbb{H}^{\text{mat}}(\star|\star, u, v, \star) = S(v, u) \frac{h(v, u)}{h(u, v)} \mathbb{H}^{\text{mat}}(\star|\star, v, u, \star).$$

Applying the factorization property to (3.158) we get that the L^2 -term is given as follows

$$\frac{L^2 p'(u_1) p'(u_2)}{\tilde{h}(u_1, u_1) \tilde{h}(u_2, u_2)}. \quad (3.161)$$

By performing the same kind of manipulations with the L -term one can get that it reads

$$\frac{1}{\tilde{h}(u_1, u_1) \tilde{h}(u_2, u_2)} \left(- \frac{i L p'(u_1) S^{(0,1)}(u_1, u_2)}{S(u_1, u_2)} - \frac{i L p'(u_2) S^{(0,1)}(u_2, u_1)}{S(u_2, u_1)} + L p'(u_1) f(u_2) + L p'(u_2) f(u_1) \right). \quad (3.162)$$

Therefore the L -dependence of the structure constant \mathcal{C}_4 exactly reproduce the one predicted by (3.36). After putting everything together, one can get

$$\mathcal{C}_{\text{HHL}}(u_1, u_2) \propto \frac{1}{\tilde{h}(u_1, u_1) \tilde{h}(u_2, u_2)} \left(\rho_2(u_1, u_2) + f(u_1) \rho_1(u_2) + f(u_2) \rho_1(u_1) + f(u_1, u_2) \right), \quad (3.163)$$

where $f(u)$ is defined in (3.149) and $f(u_1, u_2)$ doesn't depend on L . We will not give the exact expression for $f(u_1, u_2)$ here, since it is very huge.

Finally, normalizing the structure constant by the norm of the heavy operator squared we get

$$\mathcal{C}_{\text{HHL}}(u_1, u_2) = \frac{1}{\rho_2(u_1, u_2)} \left(\rho_2(u_1, u_2) + f(u_1) \rho_1(u_2) + f(u_2) \rho_1(u_1) + f(u_1, u_2) \right). \quad (3.164)$$

When we go to the arbitrary number of magnons, the main points stay the same. The dynamical part of the hexagon, which easily can be computed at all loop as follows

$$\begin{aligned} & \mathbb{H}^{\text{dyn}}(\{u\}_n | \{v^{-4\gamma}\}_n) \quad (3.165) \\ &= \prod_{k=1}^n \frac{u_k - v_k + i}{u_k - v_k} \frac{1}{\tilde{h}(v_k, u_k)} \frac{\prod_{i<j} h(u_i, u_j) h(v_i, v_j)}{\prod_{j \neq k} h(v_i, u_j)}, \\ & \mathbb{H}^{\text{dyn}}(\{v^{4\gamma}\}_n | \{u\}_n) \\ &= \prod_{k=1}^n \frac{u_k - v_k - i}{u_k - v_k} \frac{1}{\tilde{h}(u_k, v_k)} \frac{\prod_{i<j} h(u_i, u_j) h(v_i, v_j)}{\prod_{j \neq k} h(u_i, v_j)}, \end{aligned}$$

will be always the source of the simple poles at $u_i = v_i$, generating $\frac{0}{0}$ uncertainty. Resolving this uncertainty by taking the derivatives we will be always generating the L -dependence coming from the cutting exponential factor $e^{\pm ilp(u)}$. Finally by using the factorization property (3.159) one will be able to extract all the constituents of the proposal (3.36).¹⁰

3.2.5 Coefficients f

Now let us discuss the coefficients f . We have already seen before that for one magnon case we got match with one loop calculation (3.106). For arbitrary number of excitations we can compare with the tree level result, since, if we put $l_0 = 1$, the configuration of the correlator we consider here reduces to the case, when the light operator is being $2\text{Tr}(Z\bar{Z})$, corresponding to the situation discussed in (3.1.6) up to the factor of 2, which however cancels under our normalization convention.

The comparison with the tree level gave the perfect match. We remind that in the connected scheme (see the section (3.1.1)) the expression for the coefficients f^c is given as follows

$$f^{c(0)}(\{u\}_N) = \sigma_1^{(0)} \varphi_{12}^{(0)} \varphi_{23}^{(0)} \cdots \varphi_{N-1,N}^{(0)} + \text{permutations.} \quad (3.166)$$

where

$$\sigma^{(0)}(u) = \frac{1}{u^2 + 1/4}, \quad \varphi^{(0)}(u, v) = \frac{2}{(u - v)^2 + 1}. \quad (3.167)$$

It is interesting that one loop computation indicates that the form still holds with the following corrections

$$\begin{aligned} \sigma^{(1)}(u) &= \frac{1}{u^2 + 1/4} + \frac{8g^2 u^2}{(u^2 + 1/4)^3}, \\ \varphi^{(1)}(u, v) &= \frac{2}{(u - v)^2 + 1} \\ &\quad + \frac{4g^2(u^2 - v^2)}{(u^2 + 1/4)(v^2 + 1/4)((u - v)^2 + 1)}, \end{aligned} \quad (3.168)$$

which we checked analytically up to three magnons and numerically up to four. This observation gives us a hope that the ansatz of this kind with appropriate corrections can still hold at higher loops. However, in the case when $l_0 = 1$, in order to check if it is true, even at two loop we already need to take into account the leading wrapping corrections, which goes beyond the subject of this thesis.

10. The proof of (3.36) for the arbitrary number of magnons is given in [127]

Chapter 4

Conclusion

In this thesis we considered some approaches to the three-point correlators in $\mathcal{N} = 4$ SYM theory. The body of the thesis consists of the two parts presented in the second and third chapters correspondingly. In the second chapter we considered the approach to computation of the three-point functions inspired from the string field theory (SFT). On the analogy of the string vertex which describes the interaction between the strings in SFT we constructed the object which we call spin vertex. First we constructed the spin vertex at tree level of the scalar sector of $\mathcal{N} = 4$ SYM. We examined the BMN limit of the spin vertex and showed that it reproduces the string vertex of SFT constructed in [142, 115]. Then we generalized the spin vertex to full theory at tree level by exploiting the oscillator representation of the symmetry algebra $psu(2, 2|4)$. The vertex was written in the form of exponential of bilinears of the oscillators acting on the vacuum. Then the scalar product of the spin vertex with the states corresponding to the operators gives the expression of the three-point function. By reducing the vertex to the $so(4)$ sector we reproduced the tree level result obtained in [67]. Finally we showed that the spin vertex satisfies the monodromy condition and as a consequence appears to be Yangian invariant.

In the third chapter we considered the approach to the HHL correlators based on the relation of this kind of correlators with the diagonal form factors. According to the conjecture of [124] in case when the heavy operators are conjugated to each other the dependence of the structure constant on the length of the heavy operators L should take the form proposed for the diagonal form factors in two-dimensional quantum field theories (QFT) [125]. First we examined this conjecture at tree level on the example of the three-point correlators with the heavy operators being from the $su(2)$ sector and the light operators belonging to the scalar sector $so(6)$. We showed that for the considered configuration the conjecture of [124] holds at tree level. We also presented the general argumentation in favor of the validness of the conjecture at one loop supported by the explicit computation.

According to the conjecture, the structure constant of HHL correlator takes the form of the sum over all the partitions of the heavy operator rapidities:

$$C_{HHL} = \frac{1}{\rho_N(\mathbf{u})} \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} f^{\mathcal{O}}(\bar{\alpha}) \rho_N(\alpha), \quad (4.1)$$

where the $\rho_N(\alpha)$ is a diagonal minor of a Gaudin norm (see (3.61)) and the coefficients $f^{\mathcal{O}}$ are defined by the form of the light operator \mathcal{O} and don't depend on L explicitly. In case of the diagonal form factors in two-dimensional QFT's these coefficients appear

to be the diagonal form factors of the same local operator, but in infinite volume. The infinite volume form factors are known to satisfy the property to be expressed in terms of some simple building blocks for any number of magnons. In spite of it is not clear to us what exact physical meaning is carried by the coefficients $f^{\mathcal{O}}$ appearing in the structure constant of the heavy-heavy-light correlators in $\mathcal{N} = 4$ SYM, we showed that they also can be expressed by means of some simple functions (see e.g. (3.76)), which we checked up to five magnon case.

In the second part of the third chapter we extended our consideration to all loop by exploiting the hexagon program [57], the newly proposed all loop method for computing non-extremal three-point correlators. We considered the same kind of correlator configuration as in the first part with the only difference that for the sake of simplicity we restricted ourselves by taking the light operator being a half-BPS state. On the examples with heavy operators having one and two excitations we demonstrated that the conjecture still holds at the level of any loop up to the leading wrapping corrections.

As for the tree level computation we examined the behavior of the coefficients $f^{\mathcal{O}}$. Interestingly that we found that at one loop the coefficients $f^{\mathcal{O}}$ can still be expressed by the same ansatz with the proper one loop modification of the building functions (see (3.168)), which we checked up to four magnon case. This give us a hope that the same kind of ansatz can hold at the level of any loop.

In the end we would like to say that in spite of the great progress recently made due to the hexagon program proposal, there is still a lot to be done in the field of three-point function computation. One of the unsolved questions is computation of the finite size corrections. It happened that already at four loop level it is not clear how to compute the wrapping corrections to some correlators due to the appearance of the double pole in the integrand which is not clear how to deal with [140]. Another important issue is related to efficiency of the hexagon program. Even in the asymptotic approximation the structure constant is given as a sum over partitions of the operator excitations, which makes very hard the computation of the three-point correlators of the operators with large number of excitations. Finally even after the problem with double pole is resolved, the computation of three-point correlators at finite coupling still will remain an outstanding problem even for the operators with small number of excitations. At this point further study of the HHL correlators might play an important role, since the computation of the wrapping corrections to the diagonal form factors exist in the literature [132, 143], and one can try to apply these results to $\mathcal{N} = 4$ SYM. Another idea which should be kept in mind is the application of the separation of variables (SoV) technique to three-point function computation. However, as it has been mentioned in the introduction, at the moment all the results made in this direction are restricted to the tree level approximation.

Appendix A

Large μ behavior of Neumann coefficients

In this appendix, we list the leading order of Neumann coefficients in the large μ expansion. We take the same convention as in [66]. For $(m, n) \neq (0, 0)$

$$\begin{aligned} N_{mn}^{22} &= \frac{(-1)^{m+n}}{4\pi\mu|\alpha_{(1)}|r}, & N_{mn}^{23} &= \frac{(-1)^{m+1}}{4\pi\mu|\alpha_{(1)}|\sqrt{r(1-r)}} \\ N_{mn}^{33} &= \frac{1}{4\pi\mu|\alpha_{(1)}|(1-r)}, & N_{mn}^{11} &= \frac{(-1)^{m+n+1} \sin(\pi mr) \sin(\pi nr)}{\pi\mu|\alpha_{(1)}|} \\ N_{mn}^{21} &= \frac{(-1)^{m+n+1} \sin \pi nr}{\pi\sqrt{r}(n-m/r)}, & N_{mn}^{31} &= \frac{(-1)^n \sin(\pi nr)}{\pi\sqrt{1-r}(n-m/(1-r))}. \end{aligned} \tag{A.1}$$

For $(m, n) = (0, 0)$, we have

$$\begin{aligned} N_{00}^{11} &= 0, & N_{00}^{12} &= -\sqrt{r}, & N_{00}^{13} &= -\sqrt{1-r} \\ N_{00}^{23} &= -\frac{1}{4\pi\mu|\alpha_{(1)}|\sqrt{r(1-r)}}, & N_{00}^{22} &= \frac{1}{4\pi\mu|\alpha_{(1)}|r}, & N_{00}^{33} &= \frac{1}{4\pi\mu|\alpha_{(1)}|(1-r)} \end{aligned} \tag{A.2}$$

Appendix B

The elementary fields in oscillator representation

$$\begin{aligned}
Z &= |0\rangle, & \bar{Z} &= c_1^\dagger d_1^\dagger c_2^\dagger d_2^\dagger |0\rangle, \\
Y &= c_2^\dagger d_1^\dagger |0\rangle, & \bar{Y} &= c_1^\dagger d_2^\dagger |0\rangle, \\
X &= c_1^\dagger d_1^\dagger |0\rangle, & \bar{X} &= -c_2^\dagger d_2^\dagger |0\rangle, \\
\Psi_{i1} &= b_i^\dagger c_2^\dagger |0\rangle, & \bar{\Psi}_{i1} &= -a_i^\dagger c_1^\dagger d_2^\dagger d_1^\dagger |0\rangle, \\
\Psi_{i2} &= -b_i^\dagger c_1^\dagger |0\rangle, & \bar{\Psi}_{i2} &= -a_i^\dagger c_2^\dagger d_2^\dagger d_1^\dagger |0\rangle, \\
\Psi_{i3} &= b_i^\dagger c_1^\dagger c_2^\dagger d_1^\dagger |0\rangle, & \bar{\Psi}_{i3} &= a_i^\dagger d_2^\dagger |0\rangle, \\
\Psi_{i4} &= b_i^\dagger c_1^\dagger c_2^\dagger d_2^\dagger |0\rangle, & \bar{\Psi}_{i4} &= -a_i^\dagger d_1^\dagger |0\rangle, \\
F_{ij} &= -b_i^\dagger b_j^\dagger c_1^\dagger c_2^\dagger |0\rangle, & \bar{F}_{ij} &= a_i^\dagger a_j^\dagger d_1^\dagger d_2^\dagger |0\rangle,
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
\mathcal{F}^{\mu\nu} &= (\bar{\sigma}^{\mu\nu} \epsilon)_{ij} \bar{F}_{ij} - (\epsilon \sigma^{\mu\nu})_{ij} F_{ij}, \quad i, j = 1, 2, \quad \mu, \nu = 1, \dots, 4, \\
\sigma^{\mu\nu} &= \frac{1}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu), \quad \bar{\sigma}^{\mu\nu} = \frac{1}{4} (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu), \quad \epsilon_{12} = 1.
\end{aligned} \tag{B.2}$$

Appendix C

Computing the propagators from the spin vertex

Here we will justify the expression for the spin vertex (2.104) and explain why the expressions (2.102), (2.115) give the correct expression for the two- and three-point functions.

The propagators for the elementary fields have the following form:

$$\begin{aligned}\langle \bar{S}(y)S(x) \rangle &= \frac{1}{(x-y)^2}, \quad S = X, Y, Z, \\ \langle \bar{\Psi}_{jb}(y)\Psi_{ia}(x) \rangle &= i\delta_{ab}\sigma_{ij}^\mu \partial_{x^\mu} \frac{1}{(x-y)^2}, \quad a, b = 1, \dots, 4, \quad i, j = 1, 2, \\ \langle \mathcal{F}_{\rho\sigma}(y)\mathcal{F}_{\mu\nu}(x) \rangle &= (\eta_{\nu\sigma}\partial_\mu\partial_\rho + \eta_{\mu\rho}\partial_\nu\partial_\sigma - \eta_{\mu\sigma}\partial_\nu\partial_\rho - \eta_{\nu\rho}\partial_\mu\partial_\sigma) \frac{1}{(x-y)^2}.\end{aligned}\tag{C.1}$$

We have to show that the spin vertex formalism reproduce these propagators correctly, by means of the equation

$$\langle \mathcal{O}_2(y)\mathcal{O}_1(x) \rangle = \langle \mathcal{V}_{12} | e^{i(L_1^+ x + L_2^+ y)} | \mathcal{O}_2 \rangle \otimes | \mathcal{O}_1 \rangle.\tag{C.2}$$

First we establish the rule how the vertex transform the fields from the space (2) to the space (1). Using the representation of the elementary fields from the App. B we obtain by direct computation

$$\begin{aligned}{}^{(2)}\langle S|U_F^2|V_{12} \rangle &= |\bar{S}\rangle^{(1)}, \quad {}^{(2)}\langle \bar{S}|U_F^2|V_{12} \rangle = |S\rangle^{(1)}, \quad S = X, Y, Z \\ {}^{(2)}\langle \Psi_{ia}|U_F^2|V_{12} \rangle &= |\bar{\Psi}_{ia}\rangle^{(1)}, \quad {}^{(2)}\langle \bar{\Psi}_{ia}|U_F^2|V_{12} \rangle = |\Psi_{ia}\rangle^{(1)}, \\ {}^{(2)}\langle F_{ij}|U_F^2|V_{12} \rangle &= |\bar{F}_{ij}\rangle^{(1)}, \quad {}^{(2)}\langle \bar{F}_{ij}|U_F^2|V_{12} \rangle = |F_{ij}\rangle^{(1)},\end{aligned}\tag{C.3}$$

where

$$U_F^2|V_{12} \rangle = e^{\sum_{i=1,2} (b_i^{(1)\dagger} a_i^{(2)\dagger} - a_i^{(1)\dagger} b_i^{(2)\dagger} + c_i^{(1)} c_i^{(2)\dagger} - d_i^{(1)} d_i^{(2)\dagger})} c_1^{(1)\dagger} d_1^{(1)\dagger} c_2^{(1)\dagger} d_2^{(1)\dagger} |0\rangle^{(1)} |0\rangle^{(2)}.\tag{C.4}$$

This leads to the following expansion for the vertex

$$U_F^2|V_{12} \rangle = |\bar{S}_n^{(2)}\rangle |S_n^{(1)}\rangle + |S_n^{(2)}\rangle |\bar{S}_n^{(1)}\rangle + |\bar{\Psi}_{ia}^{(2)}\rangle |\Psi_{ia}^{(1)}\rangle + |\Psi_{ia}^{(2)}\rangle |\bar{\Psi}_{ia}^{(1)}\rangle + |\bar{F}_{ij}^{(2)}\rangle |F_{ij}^{(1)}\rangle + |F_{ij}^{(2)}\rangle |\bar{F}_{ij}^{(1)}\rangle + \dots,\tag{C.5}$$

where we assume summation over repeating indexes and three dots mean other possible states appearing in the vertex expansion, including those not satisfying the zero central charge condition.

Now we are ready to compute the propagators using the (C.2). We start with the scalars.

$$\begin{aligned}
\langle \bar{S}(y)S(x) \rangle &= \langle \mathcal{V}_{12} | e^{i(L_{(1)}^+ x + L_{(2)}^+ y)} | \bar{S} \rangle_{(2)} \otimes | S \rangle_{(1)} = \langle \mathcal{V}_{12} | e^{i(L_{(1)}^+ x - L_{(1)}^+ y)} | \bar{S} \rangle_{(2)} \otimes | S \rangle_{(1)} = \\
\langle V_{12} | U_{F(1)}^2 U_{(1)}^2 e^{i(L_{(1)}^+ x - L_{(1)}^+ y)} | \bar{S} \rangle_{(2)} \otimes | S \rangle_{(1)} &= \langle S | U^2 e^{i(L^+ x - L^+ y)} | S \rangle = \\
\langle 0 | U^2 e^{i(L^+ x - L^+ y)} | 0 \rangle &= \frac{1}{(x - y)^2},
\end{aligned} \tag{C.6}$$

where in order to get the last line we used (2.93). For the fermions we'll consider one of the possible propagators, the rest can be computed absolutely analogously:

$$\begin{aligned}
\langle \bar{\Psi}_{j4}(y) \Psi_{i4}(x) \rangle &= -\langle \mathcal{V}_{12} | e^{i(L_{(1)}^+ x + L_{(2)}^+ y)} a_j^{(2)\dagger} d_1^{(2)\dagger} | 0 \rangle_{(2)} b_i^{(1)\dagger} c_1^{(1)\dagger} c_2^{(1)\dagger} d_2^{(1)\dagger} | 0 \rangle_{(1)} = \\
\langle 0 | b_i d_2 c_2 c_1 U^2 e^{i(L^+ x - L^+ y)} b_i^\dagger c_1^\dagger c_2^\dagger d_2^\dagger | 0 \rangle &= -\langle 0 | U^2 e^{-iL^+ y} a_j^\dagger b_i^\dagger e^{iL^+ x} | 0 \rangle = \frac{i}{2} \partial_\mu \sigma_{ij}^\mu \langle 0 | U^2 e^{i(L^+ x - L^+ y)} | 0 \rangle \\
&= \frac{i}{2} \partial_\mu \sigma_{ij}^\mu \frac{1}{(x - y)^2},
\end{aligned} \tag{C.7}$$

where we used the explicit expression in terms of the oscillators for the $L^{+\mu} = -a_i^\dagger \bar{\sigma}_{ij}^\mu b_j^\dagger$ and the property of the σ matrices

$$\sigma_{ij}^\mu (\bar{\sigma}_\mu)_{kl} = -2\delta_{il} \delta_{jk}. \tag{C.8}$$

Finally we compute the propagator for the strength field:

$$\begin{aligned}
\langle F^{\rho\sigma}(y) F^{\mu\nu}(x) \rangle &= \langle \mathcal{V}_{12} | e^{iL_{(1)}^+ x} e^{iL_{(2)}^+ y} \left((\bar{\sigma}^{\mu\nu} \epsilon)_{ij} a_i^{(2)\dagger} a_j^{(2)\dagger} d_1^{(2)\dagger} d_2^{(2)\dagger} + (\sigma^{\mu\nu})_{ij} b_i^{(2)\dagger} b_j^{(2)\dagger} c_1^{(2)\dagger} c_2^{(2)\dagger} \right) | 0 \rangle_{(2)} \otimes \\
\left((\bar{\sigma}^{\mu\nu} \epsilon)_{ij} a_i^{(1)\dagger} a_j^{(1)\dagger} d_1^{(1)\dagger} d_2^{(1)\dagger} + (\sigma^{\mu\nu})_{ij} b_i^{(1)\dagger} b_j^{(1)\dagger} c_1^{(1)\dagger} c_2^{(1)\dagger} \right) | 0 \rangle_{(1)} &= \\
- (\bar{\sigma}^{\mu\nu} \epsilon)_{ij} (\epsilon \sigma^{\rho\sigma})_{kl} \langle 0 | U^2 e^{-iL^+ y} a_i^\dagger a_j^\dagger b_k^\dagger b_l^\dagger e^{iL^+ x} | 0 \rangle + (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) &= \\
\frac{1}{4} (\bar{\sigma}^{\mu\nu} \epsilon)_{ij} (\epsilon \sigma^{\rho\sigma})_{kl} \sigma_{ki}^\kappa \sigma_{lj}^\omega \partial_\kappa \partial_\omega \frac{1}{(x - y)^2} + (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) &= \\
\frac{1}{8} (\bar{\sigma}^{\mu\nu} \epsilon)_{ij} (\epsilon \sigma^{\rho\sigma})_{kl} \left(\sigma_{ij}^\omega \sigma_{ki}^\kappa + \sigma_{ij}^\kappa \sigma_{ki}^\omega \right) \partial_\kappa \partial_\omega \frac{1}{(x - y)^2} + (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma). &
\end{aligned} \tag{C.9}$$

Further we use the following identity:

$$\sigma_{ij}^\mu \sigma_{kl}^\nu + (\mu \leftrightarrow \nu) = -\eta^{\mu\nu} \bar{\epsilon}_{ik} \bar{\epsilon}_{jl} + 4\eta_{\kappa\omega} (\sigma^{\kappa\mu} \bar{\epsilon})_{ik} (\bar{\epsilon}^{\omega\nu})_{jl}, \tag{C.10}$$

where $\bar{\epsilon}_{12} = -1$. It gives

$$\begin{aligned}
\frac{1}{8} (\bar{\sigma}^{\mu\nu} \epsilon)_{ij} (\epsilon \sigma^{\rho\sigma})_{kl} \left(-\eta^{\kappa\omega} \bar{\epsilon}_{ik} \bar{\epsilon}_{jl} + 4\eta_{\tau\theta} (\sigma^{\tau\kappa} \bar{\epsilon})_{lk} (\bar{\epsilon}^{\theta\omega})_{ji} \right) \partial_\kappa \partial_\omega \frac{1}{(x - y)^2} + (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) &= \\
\left(-\frac{\eta^{\kappa\omega}}{8} \text{Tr}(\sigma^{\rho\sigma}) \text{Tr}(\bar{\sigma}^{\mu\nu}) + \frac{\eta_{\tau\theta}}{2} \text{Tr}(\sigma^{\rho\sigma} \sigma^{\tau\kappa}) \text{Tr}(\bar{\sigma}^{\mu\nu} \bar{\sigma}^{\theta\omega}) \right) \partial_\kappa \partial_\omega \frac{1}{(x - y)^2} + (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma). &
\end{aligned} \tag{C.11}$$

Next, noticing that $\text{Tr}(\sigma^{\mu\nu}) = \text{Tr}(\bar{\sigma}^{\mu\nu}) = 0$ and also using the relations

$$\begin{aligned}\text{Tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}) &= -\frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} + i\epsilon^{\mu\nu\rho\sigma}\right), \\ \text{Tr}(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}) &= -\frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} - i\epsilon^{\mu\nu\rho\sigma}\right),\end{aligned}\tag{C.12}$$

we get

$$\begin{aligned}\langle F^{\rho\sigma}(y)F^{\mu\nu}(x)\rangle &= \frac{\eta_{\tau\theta}}{8}\left(\eta^{\rho\tau}\eta^{\sigma\kappa} - \eta^{\rho\kappa}\eta^{\tau\sigma} + i\epsilon^{\rho\sigma\tau\kappa}\right)\left(\eta^{\mu\theta}\eta^{\nu\omega} - \eta^{\mu\omega}\eta^{\nu\theta} - i\epsilon^{\mu\nu\theta\omega}\right)\partial_\kappa\partial_\omega\frac{1}{(x-y)^2} + \\ (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) &= \frac{1}{8}\left(\eta^{\sigma\kappa}\eta^{\mu\rho}\eta^{\nu\omega} - \eta^{\rho\nu}\eta^{\sigma\kappa}\eta^{\mu\omega} - \eta^{\rho\kappa}\eta^{\mu\sigma}\eta^{\nu\omega} + \eta^{\rho\kappa}\eta^{\mu\omega}\eta^{\nu\sigma} + i\epsilon^{\rho\sigma\mu\kappa}\eta^{\nu\omega} - i\epsilon^{\rho\sigma\nu\kappa}\eta^{\mu\omega} - \right. \\ i\epsilon^{\mu\nu\rho\omega}\eta^{\sigma\kappa} &+ \left. i\epsilon^{\mu\nu\sigma\omega}\eta^{\rho\kappa} + \eta_{\tau\theta}\epsilon^{\rho\sigma\tau\kappa}\epsilon^{\mu\nu\theta\omega}\right)\partial_\kappa\partial_\omega\frac{1}{(x-y)^2} + (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma).\end{aligned}\tag{C.13}$$

One can see that after taking into account symmetrization with respect to the permutation $(\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma)$ and also $(\kappa \leftrightarrow \omega)$, all the terms proportional to i cancel out. Decomposition of the Levi-Civita tensor contraction gives (we use convention $\epsilon^{0123} = 1$)

$$\eta_{\tau\theta}\epsilon^{\rho\sigma\tau\kappa}\epsilon^{\mu\nu\theta\omega} = \eta^{\sigma\nu}\eta^{\rho\omega}\eta^{\kappa\mu} + \eta^{\sigma\omega}\eta^{\rho\mu}\eta^{\kappa\nu} + \eta^{\rho\nu}\eta^{\sigma\mu}\eta^{\kappa\omega} - \eta^{\sigma\omega}\eta^{\rho\nu}\eta^{\kappa\mu} - \eta^{\sigma\mu}\eta^{\rho\omega}\eta^{\kappa\nu} - \eta^{\rho\mu}\eta^{\sigma\nu}\eta^{\kappa\omega}.\tag{C.14}$$

The terms proportional to $\eta^{\kappa\omega}$ cancel out due to equation of motion $\partial^2\frac{1}{(x-y)^2} = 0$. Taking all this remarks into account we get final result:

$$\langle F^{\rho\sigma}(y)F^{\mu\nu}(x)\rangle = \frac{1}{2}\left(\eta^{\sigma\kappa}\eta^{\mu\rho}\eta^{\nu\omega} - \eta^{\rho\nu}\eta^{\sigma\kappa}\eta^{\mu\omega} - \eta^{\rho\kappa}\eta^{\mu\sigma}\eta^{\nu\omega} + \eta^{\rho\kappa}\eta^{\mu\omega}\eta^{\nu\sigma}\right)\partial_\kappa\partial_\omega\frac{1}{(x-y)^2}.\tag{C.15}$$

The action of covariant derivatives in terms of oscillators is given by $\mathcal{D}_{ij} = a_i^\dagger b_j^\dagger$. Thus, in case, when an elementary field belongs to the non-compact sector, the corresponding propagator can be obtained by taking appropriate number of derivatives contracted with right component of the sigma matrices, e.g.

$$\begin{aligned}\langle \bar{Z}(y)\mathcal{D}_{ij}Z(x)\rangle &= \langle \mathcal{V}_{12}|e^{i(L_{(1)}^+x+L_{(2)}^+y)}|\bar{Z}\rangle_{(2)} \otimes |\mathcal{D}_{ij}Z\rangle_{(1)} \\ &= -\frac{i}{2}\sigma_{ji}^\mu\partial_{x^\mu}\langle \mathcal{V}_{12}|e^{i(L_{(1)}^+x+L_{(2)}^+y)}|\bar{Z}\rangle_{(2)} \otimes |Z\rangle_{(1)} = -\frac{i}{2}\sigma_{ji}^\mu\partial_{x^\mu}\frac{1}{(x-y)^2}.\end{aligned}\tag{C.16}$$

Appendix D

The operator U

In this Appendix we collect some formulas about the action of the operator $U = UU_F$ which represents a finite super-conformal transformation. The operator is a product of a $su(2, 2)$ -rotation in imaginary angle

$$U = e^{-\frac{\pi}{4}(P_0 - K_0)} = e^{-\frac{\pi}{4}(L_0^+ - L_0^-)} = e^{-\frac{\pi}{4}(a_i^\dagger b_i^\dagger + b_i a_i)} \quad (\text{D.1})$$

and a unitary $su(4)$ -rotation

$$U_F = e^{-\frac{\pi}{4}(R_{13} - R_{31} + R_{24} - R_{42})} = e^{-\frac{\pi}{4}(c_i^\dagger d_i^\dagger - d_i c_i)}. \quad (\text{D.2})$$

As it was suggested in [91], it is convenient first to compute the action of a rotation in an arbitrary angle it

$$U_t = U_t^\dagger \equiv e^{t(a_i^\dagger b_i^\dagger + b_i a_i)}. \quad (\text{D.3})$$

The action of U_t on the oscillators $a_i, a_i^\dagger, b_i, b_i^\dagger$ is

$$\begin{aligned} a_i(t) &\equiv U_t a_i U_t^{-1} = a_i \cos t - b_i^\dagger \sin t, & b_i(t) &\equiv U_t b_i U_t^{-1} = b_i \cos t - a_i^\dagger \sin t, \\ a_i^\dagger(t) &\equiv U_t a_i^\dagger U_t^{-1} = a_i^\dagger \cos t + b_i \sin t, & b_i^\dagger(t) &\equiv U_t b_i^\dagger U_t^{-1} = b_i^\dagger \cos t + a_i \sin t. \end{aligned} \quad (\text{D.4})$$

From here one easily obtains the following property

$$U^2 a U^{-2} = b^\dagger, \quad U^2 a^\dagger U^{-2} = -b, \quad U^2 b U^{-2} = a^\dagger, \quad U^2 b^\dagger U^{-2} = -a. \quad (\text{D.5})$$

Similarly one derives the analogous relations for compact piece U_F

$$U_F^2 c_i U_F^{-2} = d_i^\dagger, \quad U_F^2 c_i^\dagger U_F^{-2} = d_i, \quad U_F^2 d_i U_F^{-2} = -c_i^\dagger, \quad U_F^2 d_i^\dagger U_F^{-2} = -c_i. \quad (\text{D.6})$$

Appendix E

Finite volume expansion

In this appendix we will demonstrate how to perform the finite volume expansion of the scalar product discussed in the section 3.1.5 explicitly. Let us consider the following scalar product

$$\langle u_1, u_2, u_3, u_4 | \theta + i/2, u_2, u_3, u_4 \rangle \propto \begin{vmatrix} i\Omega_{11} & -\phi_{12} & -\phi_{13} & -\phi_{14} \\ i\Omega_{21} & -\phi_{22} & -\phi_{23} & -\phi_{24} \\ i\Omega_{31} & -\phi_{32} & -\phi_{33} & -\phi_{34} \\ i\Omega_{41} & -\phi_{42} & -\phi_{43} & -\phi_{44} \end{vmatrix}. \quad (\text{E.1})$$

By expanding the determinant with respect to the first column we get

$$\begin{aligned} & i\Omega_{11} \begin{vmatrix} -\phi_{22} & -\phi_{23} & -\phi_{24} \\ -\phi_{32} & -\phi_{33} & -\phi_{34} \\ -\phi_{42} & -\phi_{43} & -\phi_{44} \end{vmatrix} \\ & - i\Omega_{21} \begin{vmatrix} -\phi_{12} & -\phi_{13} & -\phi_{14} \\ -\phi_{32} & -\phi_{33} & -\phi_{34} \\ -\phi_{42} & -\phi_{43} & -\phi_{44} \end{vmatrix} \\ & + i\Omega_{31} \begin{vmatrix} -\phi_{12} & -\phi_{13} & -\phi_{14} \\ -\phi_{22} & -\phi_{23} & -\phi_{24} \\ -\phi_{42} & -\phi_{43} & -\phi_{44} \end{vmatrix} \\ & - i\Omega_{41} \begin{vmatrix} -\phi_{12} & -\phi_{13} & -\phi_{14} \\ -\phi_{22} & -\phi_{23} & -\phi_{24} \\ -\phi_{32} & -\phi_{33} & -\phi_{34} \end{vmatrix}. \end{aligned} \quad (\text{E.2})$$

By recalling the definition of ρ given in (3.61) it is easy to see that the first term is simply equal to

$$i\Omega_{11}\rho_3(u_2, u_3, u_4). \quad (\text{E.3})$$

Now we perform one more expansion of each determinant of the second, third and fourth terms with respect to those columns which do not contain the diagonal elements ϕ_{ii} . We get

$$- i\Omega_{21} \left(-\phi_{12}\rho_2(u_3, u_4) + \phi_{32}(-\phi_{13}\rho_1(u_4) - \phi_{14}\phi_{43}) - \phi_{42}(\phi_{13}\phi_{34} + \phi_{14}\rho_1(u_3)) \right), \quad (\text{E.4})$$

$$i\Omega_{31} \left(\phi_{13}\rho_2(u_2, u_4) + \phi_{23}(\phi_{12}\rho_1(u_4) + \phi_{14}\phi_{42}) + \phi_{43}(\phi_{12}\phi_{24} + \phi_{14}\rho_1(u_2)) \right), \quad (\text{E.5})$$

$$-i\Omega_{41} \left(-\phi_{14}\rho_2(u_2, u_3) + \phi_{24}(-\phi_{12}\rho_1(u_3) - \phi_{13}\phi_{32}) + \phi_{34}(\phi_{12}\phi_{23} + \phi_{13}\rho_1(u_2)) \right). \quad (\text{E.6})$$

By putting everything together we get the finite volume expansion for the given scalar product

$$\begin{aligned} \langle u_1, u_2, u_3, u_4 | \theta + i/2, u_2, u_3, u_4 \rangle &\propto i\Omega_{11}\rho_3(u_2, u_3, u_4) + i\Omega_{41}\phi_{14}\rho_2(u_2, u_3) \\ &+ i\Omega_{31}\phi_{13}\rho_2(u_2, u_4) + i\Omega_{21}\phi_{12}\rho_2(u_3, u_4) + (i\Omega_{31}\phi_{43}\phi_{14} - i\Omega_{41}\phi_{34}\phi_{13})\rho_1(u_2) \\ &+ (i\Omega_{21}\phi_{42}\phi_{14} + i\Omega_{41}\phi_{24}\phi_{12})\rho_1(u_3) + (i\Omega_{21}\phi_{32}\phi_{13} - i\Omega_{31}\phi_{23}\phi_{12})\rho_1(u_4) \\ &+ i\Omega_{21}\phi_{32}\phi_{14}\phi_{43} + i\Omega_{21}\phi_{42}\phi_{13}\phi_{34} + i\Omega_{31}\phi_{23}\phi_{14}\phi_{42} + i\Omega_{31}\phi_{43}\phi_{12}\phi_{24} \\ &+ i\Omega_{41}\phi_{24}\phi_{13}\phi_{32} - i\Omega_{41}\phi_{34}\phi_{12}\phi_{23}. \end{aligned} \quad (\text{E.7})$$

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