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DRAFT Lecture Notes on
"Gravitational Waves"

Lecture #1

Note: page numbers are NOT
coherent

GRAVITATIONAL WAVES

* PLAN

1) Reminding of GR concepts that will be useful

2) Fundamentals of GWs : propagation equation, TT gauge, interaction with test masses, definition of the energy momentum tensor...

3) Emission of GWs : solution in linearised theory, quadruple formula , ~~radiated energy and angular momentum, examples of sources~~ ...

4) ~~GW detection~~ Inspiral of compact binaries

5) GWs from the early universe Applications to cosmology : definition of stochastic background , examples of sources (inflation...)

* REFERENCES :

- "GRAVITATIONAL WAVES" by Michele Maggiore
  Oxford University Press 2008

- Published papers : references given during the lectures, in particular for the last part
- Flanagan & Hughes, gr-qc/0501041

For general relativity :

- "GENERAL RELATIVITY WITH APPLICATIONS TO ASTROPHYSICS" by Norbert Straumann
  Springer 2004

- "GRAVITATION" by Misner, Thorne, Wheeler
  Freeman 1997
- "SPACETIME AND GEOMETRY" an introduction to GR - by Sean Carroll 2014 Pearson Education Limited

For cosmology :

- "THE COSMIC MICROWAVE BACKGROUND" by Ruth Durrer
  Cambridge University Press 2008

GRAVITATIONAL WAVES : SOME FACTS

* GWs emerge naturally in General Relativity, because it extends Newtonian theory and makes it compatible with Special Relativity $\Rightarrow$ it renders the theory CAUSAL. From the requirement of CAUSALITY one has that a change in the gravitational potential at the source position must be communicated to a distant observer no faster than the speed of light. Therefore, there must be some form of radiation that carry this information at the speed of light: these are GWs.

* The source of GWs are accelerated masses analogously to electromagnetic waves that are produced from accelerated charges: the differences will be explained in the course.

* The gravitational interaction is weak: it wins over all other interactions in the universe at large distances because it is always attractive. ~~The proper place where to test the gravitational interaction is space~~.

* For the same reason, it is difficult to detect GWs: they interact very feebly with a detector. In order to enhance the GW signal, one needs very large masses moving at high velocity: we are not able to produce GW sources on Earth, but we hope to detect GWs that are produced from astrophysical processes or from the very early universe.

* MOVIE

- GWs are produced by astrophysical objects, because one needs large masses moving at high velocities in order to get a sufficiently large amplitude since the gravitational interaction is weak (with respect to the electromagnetic one for example). Therefore, it is also difficult to detect GWs.

- Their wavelength is typically comparable to the size of the object which is emitting them: one cannot do an image (RESOLVE) the object using GWs, as it is possible to do with EM radiation, but they are more analogous to SOUND waves.

For example, object of mass M and radius R

natural frequency: $f_{GW} \sim \frac{\sqrt{G\bar{\rho}}}{2\pi}$ $\qquad \bar{\rho} = \frac{M}{R^3}$ mass density

$$\sim \frac{1}{2\pi}\sqrt{\frac{GM}{R^3}} \qquad \left(\sqrt{\frac{GM}{R^3}} = \underset{\text{orbital frequency}}{\omega} = \frac{v}{R} \quad \frac{v^2}{R} = \frac{GM}{R^2} \right)$$

For each astrophysical object, we have $R \geq R_s = \frac{2GM}{c^2}$ (the Schwarzschild radius)

$\downarrow$ = for BH

$$f_{GW} \lesssim \frac{1}{2\pi}\sqrt{\frac{GM}{R_s^3}} \overset{(\simeq 1/R_s)}{=} \frac{1}{4\sqrt{2}\pi}\frac{c^3}{GM} \simeq 10^4\ \text{Hz}\left(\frac{M_\odot}{M}\right)$$

for 30 $M_\odot$ BH one gets $f_{ISCO} \simeq 300$ Hz $\Rightarrow$ LIGO band

for 10^7 $M_\odot$ BH one gets $f_{ISCO} \simeq 10^{-3}$ Hz $\Rightarrow$ ~~eLi~~ LISA band

* Experimental evidence of the existance of GWs arises from indirect measurements : the observation of binaries of neutron stars. As the two stars inspiral towards each other they emit GWs ; this GW emission is strong enough that it back-reacts on the dynamics of the binary system on a time-scale short enough to be observable. The emission of GWs carries away energy and angular momentum from the system, reducing the size of the orbit

* This effect has been measured in the Hulse - Taylor binary pulsar (Nobel prize in 1993) composed of a pulsar orbiting a NS of which the pulses have not been measured. The change of the duration of the pulses is on a day basis, so by monitoring the system for a long period it was possible to measure the shrinking of the radius due to the emission of GWs, which matches very well the General Relativity prediction. (show figure)

* Direct detection by aLIGO 14/9/2015

$M_1 = 36 \pm 5\ M_\odot$

$M_2 = 29 \pm 4\ M_\odot$

$R_s = 210$ km

$z = 0.09 \pm 0.04$

* MOVIE : NECESSITY OF GR !

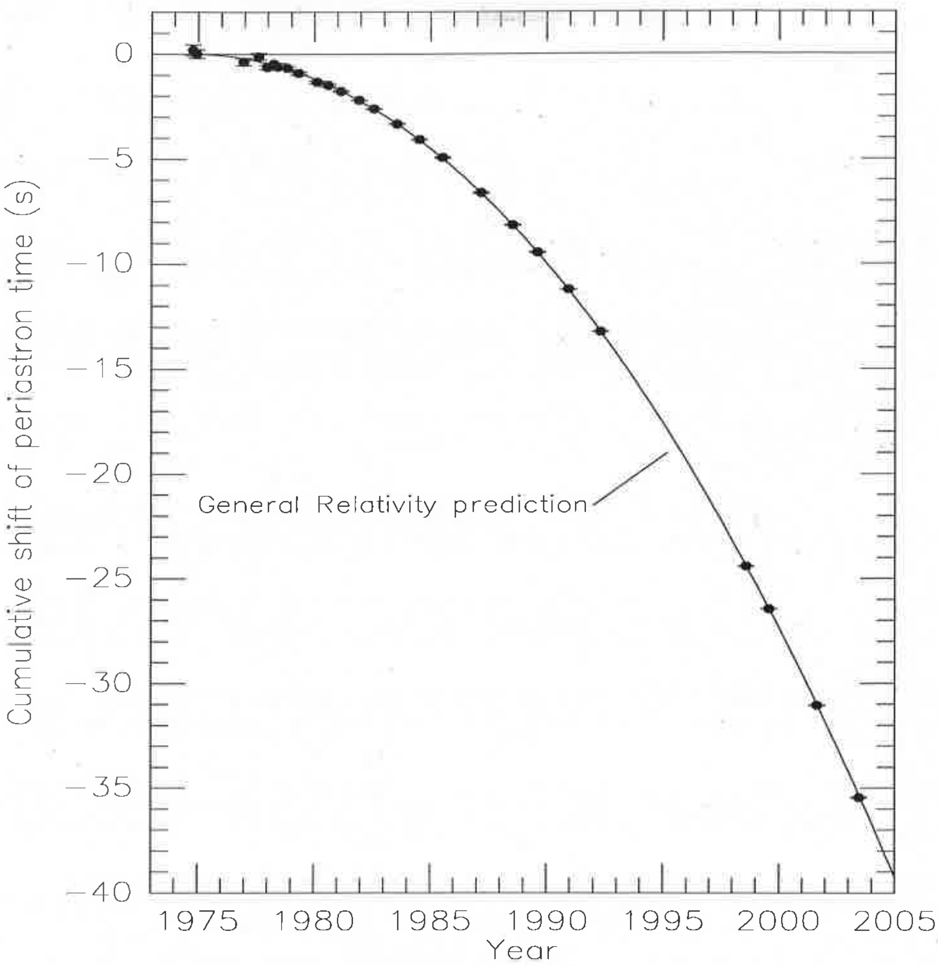

Figure 1. Orbital decay of PSR B1913+16. The data points indicate the observed change in the epoch of periastron with date while the parabola illustrates the theoretically expected change in epoch for a system emitting gravitational radiation, according to general relativity.

ELEMENTS OF GENERAL RELATIVITY

* Necessary because it is the context in which GWs have been predicted : Newtonian theory has no GWs

* Not fully rigourous, just a reminding of the basic concepts often simplified in order to be more intuitively understandable

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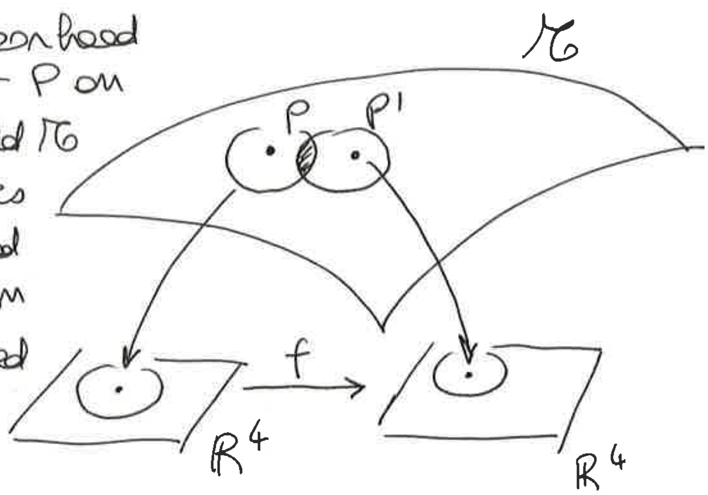
In SR : Minkowsky spacetime $\quad \eta_{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 & \\ & & & 1 \end{pmatrix}$

In GR : spacetime becomes "dynamical" in the sense that it reacts to the presence of matter $\Rightarrow$ it acquires CURVATURE.

In GR spacetime is described by a pseudo-Riemannian manifold

DIFFERENTIABLE MANIFOLD : topological space such that one can define coordinates

to a neighborhood of a point P on the manifold $\mathcal{M}$ one associates a neighborhood of a point in $\mathbb{R}^4$ described by coordinates



One does the same for another point P' : in the region which is the intersection of the two neighborhoods one can connect the two coordinate choices through a function f

$$P : (x^0, x^1, x^2, x^3) = x^\mu$$

$$P' = (y^0, y^1, y^2, y^3)$$

the two coordinate choices are related by a funcion

$$y^\mu = f^\mu(x^\mu)$$

which must be a DIFFEOMORFISM $\left\{ \begin{array}{l} \text{differentiable} \\ \text{invertible} \\ \text{with inverse diff.} \end{array} \right.$

The principle of GR is that the laws of the theory (the eqions ...) cannot depend on the choice of coordinate: the physics is invariant under coordinate transformation

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On the manifold one defines:

SCALAR FIELD: function that to P associates a number

$$F(P): \quad \mathcal{M} \to \mathbb{R}$$

$$P \to F(P) = f(x^\mu) \equiv f'(y^\mu)$$

it does not change under a coordinate transformation.

VECTOR FIELD: first, vector: intuitively: something that connects two points close by

$$\vec{V} = \frac{\vec{PP'}}{\varepsilon}$$

It leaves on the tangent space to the manifold at point P.

$$P \xrightarrow{\text{associate coord.}} x^\mu$$

$$P' \longrightarrow x^\mu + dx^\mu$$

$$\vec{V} \to \frac{dx^\mu}{\varepsilon}$$

now, under a change of coordinates $y^\mu = f^\mu(x^\nu)$

$$dy^\mu = \left(\frac{\partial y^\mu}{\partial x^\nu}\right)_P dx^\nu$$

ν indeces are summed because repeated

so one infers the transformation law for a vector:

$\vec{V} \to \frac{dy^\mu}{\varepsilon}$ the components of the vector $\vec{V}$ change under coordinate transformation as

$$\boxed{\bar{V}^\mu = \left(\frac{\partial y^\mu}{\partial x^\nu}\right)_P V^\nu}$$

(Jacobian matrix)

what are the vector components?

A vector can be seen as an operation : the derivative of a scalar function F in the direction of the vector $\vec{V}$

$$\vec{V}(F) = \frac{F(P')-F(P)}{\varepsilon} = \underbrace{V^\mu}_{\text{vector components}} \underbrace{\frac{\partial}{\partial x^\mu}}_{\text{BASIS of the tangent space.}} F$$

$$\vec{e}_\mu = \frac{\partial}{\partial x^\mu} = \partial_\mu$$

basis of the tangent space associated with the coordinate choice x^μ:

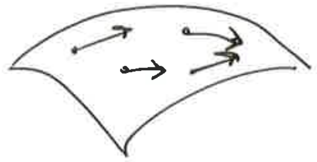
$P : (x^0, x^1, x^2, x^3)$ $P' : (x^0, x^1, x^2+\varepsilon, x^3)$

$$\vec{e}_\mu = \frac{\vec{PP'}}{\varepsilon}$$

with components $(\vec{e}_\mu)^\nu = (0,0,1,0) = \delta_\mu^\nu$

$$= \frac{\partial}{\partial x^\mu} x^\nu$$

VECTOR FIELD: operation that assigns to each point of the manifold a vector



$$\vec{V} = V^{\mu}(x)\frac{\partial}{\partial x^{\mu}} = V^{\mu}\partial_{\mu}$$

$$V^{\mu}(y) = \frac{\partial y^{\mu}}{\partial x^{\nu}} V^{\nu}(x)$$

Now, before introducing the tensors, it is useful to define the cotangent space:

the cotangent space is the dual of the tangent space, i.e. the space of all mappings which are linear and go from the tangent space to $\mathbb{R}$.

One example: $(dF)_P(\vec{V}) := \vec{V}(F)$

the differential of F in the direction of $\vec{V}$

$$(dF)_P(\vec{e}_{\mu}) := \frac{\partial F}{\partial x^{\mu}}$$

this mapping is useful to define a basis in the cotangent space, the dual basis:

F assigns to a point P one of its coordinates x^{ν}:

$$F : P \longrightarrow x^{\nu}(P)$$

$$(dx^{\nu})_P(\vec{e}_{\mu}) = \delta^{\nu}{}_{\mu}$$

we have found the dual basis

element of the cotangent space:

$$\omega = \omega_\nu \, dx^\nu \qquad \left(V = V^\mu \frac{\partial}{\partial x^\mu} \right)$$

and transforms:
$$\boxed{\bar{\omega}_\mu (y) = \left(\frac{\partial x^\nu}{\partial y^\mu} \right)_p \omega_\nu (x)}$$

$\omega(\partial_\nu) = \omega_\mu \, dx^\mu (\partial_\nu)$
$= \omega_\nu$

$$\left(\bar{V}^\mu = \left(\frac{\partial y^\mu}{\partial x^\nu} \right)_p V^\nu \right)$$

the transformation of a vector is called CONTRAVARIANT, the one of the co-vector is called COVARIANT.

As we will see, the presence of a METRIC on the manifold IDENTIFIES the two spaces.

TENSOR FIELD: linear map that to each point P of the manifold assigns a set of numbers: $T^{\mu\nu\dots m}_{\rho\sigma\dots n}$

(map operating n times on the tangent space and m times on the cotangent space)

that transforms under coordinate transformations as

$$\bar{T}^{\mu\nu\dots m}_{\rho\sigma\dots n}(y) = \underbrace{\frac{\partial y^\mu}{\partial x^{\mu'}} \frac{\partial y^\nu}{\partial x^{\nu'}}}_{\substack{m \text{ times} \\ \text{contravariant} \\ \text{(transforms as} \\ \text{a vector)}}} \cdots \underbrace{\frac{\partial x^{\rho'}}{\partial y^\rho} \frac{\partial x^{\sigma'}}{\partial y^\sigma}}_{\substack{n \text{ times} \\ \text{covariant} \\ \text{(transforms as} \\ \text{the dual vector)}}} \cdots T^{\mu'\nu'\dots m}_{\rho'\sigma'\dots n}(x)$$

we see an example of tensor: the metric

Spacetime of GENERAL RELATIVITY is defined as a pseudo-Riemannian Lorentzian manifold: manifold with a $\boxed{\text{metric}}$

$g_{\mu\nu}(x)$ twice covariant tensor

which satisfies

- $g_{\mu\nu}(x) = g_{\nu\mu}(x)$
- $\det g_{\mu\nu}(x) \neq 0$
- signature $(-,+,+,+)$

} generalisation of $\eta_{\mu\nu}$

and expresses infinitesimal distances:

$$ds^2 = \underbrace{g_{\mu\nu}(x)}_{g(\partial_\mu, \partial_\nu)} \underbrace{dx^\mu dx^\nu}_{\text{basis vector of the cotangent space}}$$

The contravariant metric tensor is such that $g_{\mu\nu} g^{\nu\delta} = \delta^\delta_\mu$ (the inverse)

and the metric connects covariant and contravariant vectors and tensors:

$$g_{\mu\nu} V^\nu = V_\mu \qquad g_{\mu\nu} T^{\nu\sigma\ldots m}_{\alpha\beta\ldots n} = T^{\sigma\ldots m}_{\mu\alpha\beta\ldots n}$$

[Sketched demonstration: the metric can be seen as a linear map $g_p(X,Y)$ from which one derives one element of the cotangent space $g_p(X,\cdot) = X_\mu dx^\mu$

$$g_p(X,Y) = g_p(X^\mu \partial_\mu, Y^\nu \partial_\nu) = X^\mu Y^\nu g_{\mu\nu}$$

$$= X_\mu dx^\mu (Y^\nu \partial_\nu) = X_\mu Y^\nu dx^\mu(\partial_\nu)$$]

$$\boxed{\begin{aligned} X^\mu Y^\nu g_{\mu\nu} &= X_\mu Y^\nu dx^\mu(\partial_\nu) = X_\mu Y^\nu \delta^\mu_\nu = X_\nu Y^\nu \\ \Rightarrow \quad X_\nu &= X^\mu g_{\mu\nu} \end{aligned}}$$

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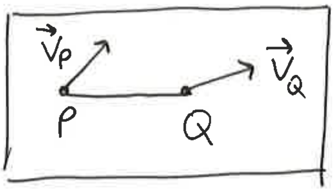
On a space time given as a Lorentzian manifold one needs to GENERALISE THE CONCEPT OF DERIVATIVE for structures as vectors and tensors

In flat space-time we know how to compare directions for a vector field



connect P and Q with a straight line and parallel transport $\vec{V}_Q$ in P

To do this on a manifold, we need the concept of

CONNECTION : operation defined locally, that to two vector fields associates another vector field:

so called because it connects vectors: transport them from one tg space to another

$$X, Y \longrightarrow \nabla_X Y$$

The components of this new vector field are called the CHRISTOFFEL SYMBOLS:

$$\nabla_{\partial_\mu}(\partial_\nu) = \Gamma^{\sigma}_{\mu\nu}\, \partial_\sigma$$

(they are not tensors, for their transformation rules see one of the books on GR) from this, one defines the

COVARIANT DERIVATIVE : a once covariant, once contravariant tensor

$$X \longrightarrow \nabla X$$

such that $\nabla X(Y, \omega) := \omega(\nabla_Y X)$

element of the cotangent space

(element of the cotangent space acts on a vector)

the components of this tensor are

$$\nabla_\nu X^\mu = X^\mu{}_{;\nu} = \partial_\nu X^\mu + \Gamma^\mu_{\nu\alpha} X^\alpha \Longrightarrow$$

the combination of the two terms transforms as a tensor

on a curved manifold, the normal derivative is not a tensor

contribution to the fact that spacetime is not flat.

reduces to ∂_ν for flat ST

this can be proven:

$$\nabla X \equiv X^\mu{}_{;\nu}\, dx^\nu \partial_\mu$$

$$X^\mu{}_{;\nu} = \nabla X(\partial_\nu, dx^\mu) := dx^\mu(\nabla_{\partial_\nu}(X^\alpha \partial_\alpha)) =$$

$$= dx^\mu(X^\alpha{}_{,\nu}\partial_\alpha + X^\alpha \nabla_{\partial_\nu}(\partial_\alpha)) =$$

property of the connection analogous to Leibniz rule

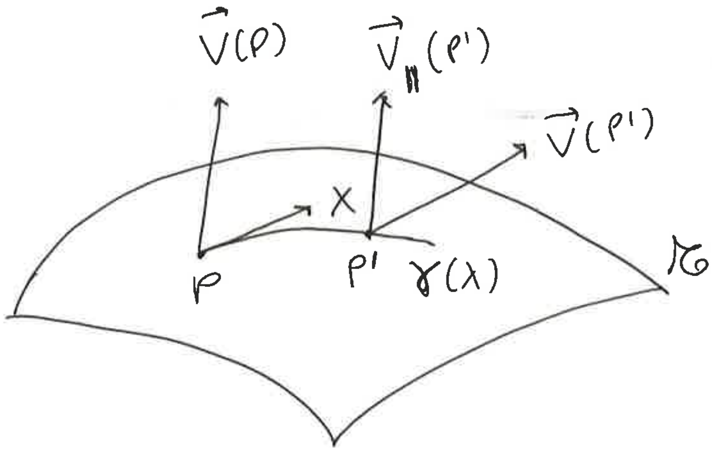
$$= dx^\mu(X^\alpha{}_{,\nu}\partial_\alpha + X^\alpha \Gamma^\sigma_{\nu\alpha}\partial_\sigma) =$$

$(dx^\mu \partial_\sigma) = \delta^\mu_\sigma$

$$= X^\mu{}_{,\nu} + \Gamma^\mu_{\nu\alpha} X^\alpha$$

what is the meaning of this? To understand it better, one needs to introduce the concept of

PARALLEL TRANSPORT ALONG A CURVE $\gamma(\lambda)$



$$\gamma(\lambda): \quad (x^0(\lambda), x^1(\lambda), x^2(\lambda), x^3(\lambda))$$

$$\frac{d\gamma}{d\lambda} = \dot{\gamma}(\lambda): \quad (\dot{x}^0(\lambda), \dot{x}^1(\lambda), \dot{x}^2(\lambda), \dot{x}^3(\lambda))$$

the tangent vector to the curve in P is:

$$X = \dot{\gamma} = \dot{x}^\mu \partial_\mu$$

there is a vector field $\vec{V}$, in P and P' close to P:

the vector $\vec{V}$ is PARALLEL TRANSPORTED along $\gamma(\lambda)$

if ITS COVARIANT DERIVATIVE ALONG $\gamma(\lambda)$ VANISHES:

$$\boxed{\nabla_{\dot{\gamma}} V = 0} \Rightarrow X^\mu \nabla_\mu V^\nu = 0$$

which means that the covariant derivative of $\vec{V}$ is the difference between $\vec{V}$ at a point close-by to P and the "parallel" vector $\vec{V}_{\parallel}$, which is parallel transported along $\gamma(\lambda)$.

The equation which is satisfied by the components of a parallel transported vector is: (14)

$$\boxed{\frac{dV^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\sigma} V^\sigma \frac{dx^\mu}{d\lambda} = 0}$$

Demonstration:

the connection is linear

"Leibniz rule"

$$\nabla_{\dot\gamma} V = \nabla_{\dot x^\mu \partial_\mu}(V^\alpha \partial_\alpha) = \dot x^\mu \nabla_{\partial_\mu}(V^\alpha \partial_\alpha) =$$

$$= \dot x^\mu \left[V^\alpha \nabla_{\partial_\mu}(\partial_\alpha) + \partial_\mu V^\alpha \partial_\alpha\right] =$$ exercise

$$= \dot x^\mu \left[V^\alpha \Gamma^\sigma_{\mu\alpha} \partial_\sigma + \partial_\mu V^\alpha \partial_\alpha\right] =$$

rename mute indices

$$= \dot x^\mu V^\sigma \Gamma^\alpha_{\mu\sigma} \partial_\alpha + \frac{dx^\mu}{d\lambda}\frac{\partial V^\alpha}{\partial x^\mu} \partial_\alpha =$$

$$= \left[\frac{dV^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\sigma} V^\sigma \frac{dx^\mu}{d\lambda}\right] \partial_\alpha = 0$$

We can now introduce the concept of

GEODESIC

a curve $\gamma(\lambda)$ is a geodesic if the tangent vector is parallel transported along $\gamma(\lambda)$:

$$\boxed{\nabla_{\dot\gamma} \dot\gamma = 0} \Longrightarrow X^\mu \nabla_\mu X^\nu = 0$$

which in coordinates, using $V^\alpha = \frac{dx^\alpha}{d\lambda}$ becomes:

$$\boxed{\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0}$$

This is the geodesic equation. We will see other ways to define it, and attach a more physical meaning to it in the next lectures. (eg. followed by a body in free-fall, on which no external forces are acting)

The COVARIANT DERIVATIVE can be generalised to TENSORS

$$\nabla_\lambda T^{\mu,\nu\ldots m}_{\rho,\sigma\ldots n} = \partial_\lambda T^{\mu,\nu\ldots m}_{\rho,\sigma\ldots n} + \Gamma^\mu_{\lambda\alpha} T^{\alpha,\nu\ldots m}_{\rho,\sigma\ldots n} + \text{every upper index}$$

$$- \Gamma^\alpha_{\lambda\rho} T^{\mu,\nu\ldots m}_{\alpha,\sigma\ldots n} - \text{every lower index}$$

In geometrical writing:

$$\underset{\begin{pmatrix} \overset{0}{n}\text{ - covariant} \\ \underset{1}{m}\text{ - contravariant}\end{pmatrix}}{T} \longrightarrow \underset{\begin{pmatrix} \overset{1}{n+1}\text{ - covariant} \\ \underset{1}{m}\text{ - contravariant}\end{pmatrix}}{\nabla T}.$$

$$\nabla T(X_1 \ldots X_{n+1}, W_1 \ldots W_m) \equiv \nabla_{X_{n+1}} T(X_1 \ldots X_n, W_1 \ldots W_m)$$

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Now equipped with the concept of CONNEXION and COVARIANT DERIVATIVE we go back to our

PHYSICAL SPACETIME : pseudo-Riemannian manifold with metric

$$g_{\mu\nu}(x^\lambda) \; : \; \begin{array}{l} - \text{symmetric} \\ - \text{non-degenerate} \\ - \text{lorentzian signature} \end{array}$$

it expresses distances: $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$

and we impose one extra condition:

connexion metric compatible

$$\boxed{\nabla_\lambda g_{\mu\nu} \equiv 0}$$

(and torsion free)

this is equivalent to say that under parallel transport angles and lengths are INVARIANT

there exists only one connexion for which this is satisfied, and it is the Levi-Civita connexion: we choose this particular connexion on the physical space-time.

With this condition, it is possible to demonstrate that

$$\boxed{C^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)}$$

expression of the Christoffel symbols in terms of the metric $g_{\mu\nu}$.

[The Christoffel symbols represent the "strength" of the gravitational field]

For a scalar field, the covariant derivatives commute:

$$\nabla_\mu \varphi = \partial_\mu \varphi \qquad \nabla_\mu \nabla_\nu \varphi = \nabla_\nu \nabla_\mu \varphi$$

But this is not true in general. This leads to the definition of the

RIEMANN (CURVATURE) TENSOR

true for any connection, also with torsion or non-metric-compatible

$$\boxed{R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\sigma\alpha} \Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta} \Gamma^{\sigma}_{\nu\alpha}}$$

vanishes if $g_{\mu\nu}$ = constant

This tensor describes all there is to know of ~~a metric~~ the curvature, all that is independent on the choice of coordinates.

It represents the fact that COVARIANT DERIVATIVES DO NOT COMMUTE:

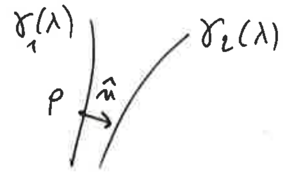
$$\boxed{(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu = R^{\mu}{}_{\nu\alpha\beta} V^\nu}$$

parallel transport of a vector depends on the path taken

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} \qquad R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$$
$$R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} \qquad \text{(properties)}$$

We will see also that it enters the equation of geodesic deviation: if we have two nearby geodesics connected by a vector n^μ, their relative acceleration is given by

$$\frac{d^2 n^i}{d\lambda^2} = n^j R^i{}_{00j}$$



so it expresses the effect of the curvature, i.e. the VARIATIONS of the gravitational field strenghts

Other very important quantities defined from the Riemann tensor are:

RICCI TENSOR: $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha}$

RICCI SCALAR: $R = g^{\mu\nu} R_{\mu\nu}$

EINSTEIN TENSOR: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$

(trace free parts are Weyl tensors)

One important property is the **BIANCHI IDENTITY**

$$\boxed{\nabla_\mu G_{\mu\nu} = 0}$$

~~which is connected with energy-momentum conservation as we will see~~.

We are ready now to state the <u>principles of GR</u>:

* Space-time is given by a pseudo-Riemannian manifold with metric $g_{\mu\nu}$ which is symmetric, non degenerate, and with Lorentzian signature

* The metric describes <u>both</u>
  - the metric and causal structure of space-time
  - the gravitational field (which is therefore not an additional field propagating through space-time)

* $g_{\mu\nu}$ is therefore a <u>dynamical element</u>, determined by the energy-momentum content of the space-time (think in analogy with the Newtonian gravitation theory)

* <u>GR unifies geometry and gravitation</u>

* THE EQUIVALENCE PRINCIPLE:

for a point x of a Lorentzian manifold it is always possible to find a coordinate system such that

1) $g_{\mu\nu}(x) = \eta_{\mu\nu}$ $\Rightarrow$ locally, spacetime is described by Minkowsky metric

2) $g_{\mu\nu,\lambda}(x) = 0$ $\Rightarrow$ combined with the condition $\nabla_\lambda g_{\mu\nu} = 0$, this means that $\Gamma^{\mu}_{\alpha\beta}(x) = 0$

this coordinate system is called a **LOCAL INERTIAL SYSTEM AT POINT x**. CONSEQUENCES:

1) by this coordinate system, it is always possible to eliminate LOCALLY the effect of the gravitational field.

→ this suggests that the action of gravity should be attributed to curvature

2) No local experiment can distinguish a non-rotating, freely falling system from a non-accelerated system in the absence of a gravitational field.

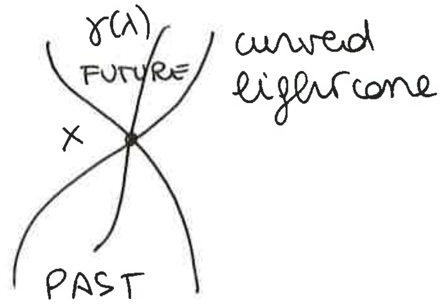
3) In this reference system SPECIAL RELATIVITY is valid. using the equivalence principle one can formulate and generalize (to the context of GR) all the equations describing physical systems in the presence of a gravitational field: start from eqs of special relativity, find a COVARIANT VERSION of them - which means write them in a form that does not depend on the choice of coordinate basis: using the appropriate geometrical language, equations can be formulated WITHOUT A CHOICE OF COORDINATES

4) We will see that the equivalence principle extends all along a world line: it is possible to choose coordinates for which the metric is flat and the christoffel vanish all along a world-line: **FREELY FALLING SYSTEM** (SR is valid all along the worldline)

5) Therefore, the equivalence principle implies the equality among inertial (m^i) and gravitational mass (m^g): in a freely falling frame, two test bodies with different mass must have the same trajectory, i.e. the same velocity, i.e. the same acceleration, i.e. $\underline{a} = \frac{m^g_1}{m^i_1} F_g = \frac{m^g_2}{m^i_2} F_g$

<u>NOTE</u>: $C^{\mu}_{\alpha\beta}(x) = 0$ the christ. symbols can be put to zero, transformed away - they represent the field strength and <u>are not tensors</u>. This cannot be done with the curvature, the Riemann <u>tensor</u>

* The metric describes also the <u>causal structure</u> of space-time:



$\gamma(\lambda)$ is a worldline:

- timelike: $g_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} < 0$
- lightlike: $g_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} = 0$
- space-like: $g_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} > 0$

Now the question arises obviously:

<u>What determines the metric</u>?

The energy-momentum content of the space-time: the metric reacts CAUSALLY to the presence of matter.

The metric $g_{\mu\nu}$ couples in a universal way to all standard model fields. One can then generalize the standard model action by:

1) define a new measure which is independent from the choice of coordinates:

$$\int \sqrt{-g}\, d^4x \qquad \text{with} \qquad g = \det g_{\mu\nu} < 0$$

$$\sqrt{-\bar{g}(\bar{x})} = \sqrt{-g(x)}\, \left| \det\left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\right)\right|$$

going from coordinates x^α to $\bar{x}^\alpha$, inverse of the Jacobian

2) replace everywhere $\eta_{\mu\nu} \to g_{\mu\nu}$, $\partial_\nu \to \nabla_\nu$

Example: $S^{SR}_{EM} = \int d^4x \left(-\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}\right)$ becomes

$$S^{GR}_{EM} = \int \sqrt{-g}\, d^4x \left(-\frac{1}{16\pi} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}\right)$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

GENERAL RELATIVITY is then described by the total action

$$S_{GR} + S_{MAT}$$

$$\boxed{S_{GR} = \frac{c^3}{16\pi G}\int \sqrt{-g}\, d^4x\, (R - 2\Lambda)}$$

Ricci scalar and a constant Λ

S_{MAT} can be defined also through the energy momentum tensor:

$$\boxed{T^{\mu\nu}_{MAT} := \frac{2}{\sqrt{-g}} \frac{\delta S_{MAT}}{\delta g_{\mu\nu}}}$$

This expression of the en. mom. tensor is obtained by a variational principle: one considers variations in the action integral which are induced by variations in the metric. This definition leads to the usual expressions for the en. mom. tensor, for example if applied to S^{GR}_{EM}

SKETCHED DERIVATION OF EINSTEIN EQS FROM A VARIATIONAL PRINCIPLE:

to find Einstein eqs, one has then to vary the total action with respect to the gravitational field

$$\delta\left(S_{GR}+S_{MAT}\right) = \frac{c^3}{16\pi G}\,\delta\int\sqrt{-g}\,d^4x\,(R-2\Lambda) + \frac{1}{2c}\int\sqrt{-g}\,d^4x\,T^{\mu\nu}_{MAT}\,\delta g_{\mu\nu} = 0$$

(Reference: Straumann section 3.3)
section 3.2: heuristic derivation

1) $\delta\int R\sqrt{-g}\,d^4x = \int\delta\left(R_{\mu\nu}\,g^{\mu\nu}\sqrt{-g}\right)d^4x$ (explicit all the fields)

$$= \underbrace{\int\delta R_{\mu\nu}\,g^{\mu\nu}\sqrt{-g}\,d^4x}_{} + \int R_{\mu\nu}\,\delta\left(g^{\mu\nu}\sqrt{-g}\right)d^4x$$

choose a local inertial system, so one can neglect the terms which are quadratic in the Christoffel (because they are zero – but the derivatives are not)

$$\delta R_{\mu\nu} = \partial_\alpha\left(\delta\Gamma^\alpha_{\mu\nu}\right) - \partial_\nu\left(\delta\Gamma^\alpha_{\mu\alpha}\right)$$

From the rules of transformation of $\Gamma^\alpha_{\mu\nu}$ under a coordinate transformation, it turns out that these $\delta\Gamma^\alpha_{\mu\nu}$ are tensors. So we rewrite them with covariant derivatives, and we can insert the metric

$$g^{\mu\nu}\,\delta R_{\mu\nu} = \nabla_\alpha\left(g^{\mu\nu}\,\delta\Gamma^\alpha_{\mu\nu}\right) - \nabla_\alpha\left(g^{\alpha\mu}\,\delta\Gamma^\beta_{\mu\beta}\right) = \nabla_\alpha V^\alpha$$

↓ tensor equation, valid in every coordinate basis

↓ rename mute indices

$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\alpha V^\alpha$ is a surphace term, so it does not contribute (Stoke's theorem)

Second term in 1):

$$R_{\mu\nu}\,\delta\left(g^{\mu\nu}\sqrt{-g}\right) = R_{\mu\nu}\,g^{\mu\nu}\,\delta\left(\sqrt{-g}\right) + R_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}$$

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\frac{1}{\sqrt{-g}}\,\delta g \overset{*}{=} -\frac{1}{2}\frac{1}{\sqrt{-g}}\,g\,g^{\alpha\beta}\,\delta g_{\alpha\beta}$$

$$= +\frac{1}{2}\sqrt{-g}\left(-g_{\alpha\beta}\,\delta g^{\alpha\beta}\right)$$

(this * needs some definitions from linear algebra)

$$R_{\mu\nu}\,\delta\left(g^{\mu\nu}\sqrt{-g}\right) = R_{\mu\nu}\,g^{\mu\nu}\left(-\frac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\,\delta g^{\alpha\beta}\right) + R_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}$$

$$= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu}\right)\delta g^{\mu\nu}$$

the Einstein tensor appears

Therefore we have that 1) is equal to:

$$\delta\int R\sqrt{-g}\,d^4x = \int G_{\mu\nu}\,\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

2) $$\delta\int(-2\Lambda)\sqrt{-g}\,d^4x = -2\Lambda\int\delta\left(\sqrt{-g}\right)d^4x =$$

$$= \Lambda\int g_{\mu\nu}\,\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

If we now put 1) and 2) together, and suppose for a second that there is no matter, we obtain:

$$\delta\int(R-2\Lambda)\sqrt{-g}\,d^4x = \int\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{-g}\,d^4x = 0$$

Therefore, varying the gravitational action S_{GR} with respect to the field $g^{\mu\nu}$ gives the Einstein equations in vacuum:

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = 0}$$

If one now includes also the matter action S_{MAT}, one gets from the total variation principle:

$$\delta (S_{GR} + S_{MAT}) = 0$$

$$= \delta \int \left[\frac{c^3}{16\pi G} (G_{\mu\nu} + \Lambda g_{\mu\nu}) \delta g^{\mu\nu} - \frac{1}{2c} T^{MAT}_{\mu\nu} \delta g^{\mu\nu} \right] \sqrt{-g}\, d^4x$$

the complete Einstein equations:

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

This set of equations are partial differential equations at second order, that represent how $g_{\mu\nu}$ reacts to the presence of matter.

•) second order: $G_{\mu\nu} \sim R_{\mu\nu} \sim \partial\Gamma + \Gamma^2 \sim \partial^2 g + (\partial g)^2$

•) ~~the conservation of energy and momentum of matter follows~~ from the Bianchi identities,

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu T_{\mu\nu} = 0$$

•) (the contribution from the cosmological constant can be put on the other side by identifying:

$$T^{\Lambda}_{\mu\nu} \equiv - \frac{c^4}{8\pi G} \Lambda g_{\mu\nu}$$

(see later))

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We have used $\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}\,(-g_{\alpha\beta}\,\delta g^{\alpha\beta})$

where does this come from?

Laplace expansion for the determinant of a matrix:

$$g = \Sigma_\alpha\, g_{\beta\alpha}\, \tilde{g}_{\beta\alpha} \qquad \underset{\text{cofactor}}{\tilde{g}_{\beta\alpha}} = (-1)^{\beta+\alpha}\, \underset{\text{minor matrix}}{M_{\beta\alpha}}$$

now the inverse is given by (also a formula in linear algebra)

$$g^{\beta\alpha} = \frac{\tilde{g}_{\beta\alpha}}{g}$$

and therefore:

$$\frac{\delta g}{\delta g_{\beta\alpha}} = \frac{\delta\left(\Sigma_\alpha\, g_{\beta\alpha}\,\tilde{g}_{\beta\alpha}\right)}{\delta g_{\beta\alpha}} = \tilde{g}_{\beta\alpha} = g\, g^{\beta\alpha}$$

$$\delta g = g\, g^{\beta\alpha}\, \delta g_{\beta\alpha} = -g\, g_{\beta\alpha}\, \delta g^{\beta\alpha}$$

[because $g_{\alpha\beta}\, g^{\beta\sigma} = \delta_\alpha{}^\sigma$

$\delta g_{\alpha\beta}\, g^{\beta\sigma} + g_{\alpha\beta}\, \delta g^{\beta\sigma} = 0$]

$$\delta(\sqrt{-g}) = -\frac{1}{2}\frac{1}{\sqrt{-g}}\,\delta g = -\frac{1}{2}\sqrt{-g}\; g_{\beta\alpha}\,\delta g^{\beta\alpha}$$

$\nabla_\mu T^{\mu\nu} = 0$ is a consequence of diffeomorphism invariance (it gives four identities)
(for demonstration, see CARROLL appendix B)

Straumann, par. 3.3.6 and 3.4

AND it is the covariant generalization of energy-momentum conservation in flat space-time $\partial_\mu T^{\mu\nu} = 0$

Noether's theorem (consequence of ~~space-time~~ invariance with respect to space and time translations)

BUT it is not a conservation law in GR in general

(example: Friedmann Universe)
because "total energy and momentum" are not defined in general

1) Equivalence principle: there is no way to localise in general the gravitational field, it can always be transformed away locally

2) translations do not act as isometries on a Lorentzian manifold, except on special cases

Example:
total energy and total momentum of an isolated system can be defined for an asymptotically flat geometry.

Intuitively: $T_{\mu\nu}$ only describes matter and not the gravitational field: the system still exchange energy and mom.

to see whether a conservation law follows from $\nabla_\mu T^{\mu\nu} = 0$ one has to look for isometries of the space-time, i.e. the existence of Killing vectors.

A Killing vector satisfies $\boxed{\nabla_{(\mu} K_{\nu)} = 0}$ (Carroll Appendix B)

and therefore

$$\mathcal{L}_K g_{\mu\nu} = \nabla_{(\mu} K_{\nu)} = 0$$

Lie derivative of the metric: represents how the metric changes on the flow of K. Any vector generates a diffeomorphism that represents the flow along the integral curves of the vector

one has therefore that the metric "does not change in the direction of K": the diffeomorphism generated by K is an isometry.

In coordinates such that $K = \partial_A$ ($K^\mu = \delta^\mu_A$) one has then $\boxed{\dfrac{\partial g_{\mu\nu}}{\partial x^A} = 0}$

[EXAMPLE: stationary spacetime is a spacetime with $K = \partial_t$ and $g_{\mu\nu}(x_i)$]

If the space-time has a Killing vector field, for any geodesic with tangent vector P^μ one has

$$P^\mu \nabla_\mu (K_\alpha P^\alpha) = 0$$

so that $K_\alpha P^\alpha$ is conserved along the geodesic

$$P^\mu (\nabla_\mu P^\alpha) K_\alpha + P^\mu P^\alpha (\nabla_\mu K_\alpha) = P^\mu P^\alpha \nabla_{(\mu} K_{\alpha)} = 0$$

A Killing vector implies the existence of a conserved quantity $\to K_\mu P^\mu$ associated with the geodesic motion (intuitively: the metric does not change in that direction so a particle does not feel a force and the component of its momentum in that direction will be conserved)

If a Killing vector exists, then a conservation law follows from $\nabla_\mu T^{\mu\nu} = 0$ for the vector
$(\nabla_\mu P^\mu = 0)$

$$P^\mu = T^{\mu\nu} K_\nu \implies$$ one can construct a constant of motion from $\nabla_\mu T^{\mu\nu} = 0$

$$\nabla_\mu P^\mu = (\nabla_\mu T^{\mu\nu}) K_\nu + T^{\mu\nu} \nabla_\mu K_\nu = T^{\mu\nu} \nabla_{(\mu} K_{\nu)} = 0$$

example: Schwarzschild

In space-times with a time-like Killing vector (stationary) one can construct a motion of conserved energy. But in general not.

This is due to the profound difference between flat and curved space-times.

~~we will see how~~

flat ST: $\partial_\mu T^{\mu\nu} = 0 \to$ invariance under translations $\to$ Noether theorem, exists a conserved current $\frac{dP^\mu}{dt} = 0$ which is

$$P^\nu = \int_{t=\text{const}} d^3x \; T^{0\nu}(\underline{x}, t)$$

Three examples of solutions of Einstein eqs which will be useful for the following:

- Newtonian limit
- Friedmann Robertson Walker spacetime
- Schwarzschild spacetime

NEWTONIAN LIMIT

(weak gravitational field, almost stationary, slow motion of the source)

First rewrite Einstein eqs as:

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right]$$

Straumann, sec. 2.5

$$\left(R - \frac{1}{2} R\, g^{\mu}_{\mu} = \frac{8\pi G}{c^4} T \Rightarrow R = -\frac{8\pi G}{c^4} T \right)$$

In the Newtonian limit, the gravitational field must be weak, so we can rewrite the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1 \quad \left(\begin{array}{c}\text{in the}\\ \text{chosen}\\ \text{coord.}\end{array}\right)$$

at first order in h one has $\Gamma \sim \partial h$ (we will see this limit precisely in the context of GWs) and therefore we neglect the quadratic terms in the Ricci tensor:

$$R_{\mu\nu} \sim \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} \quad \text{with}$$

$$\Gamma^\alpha_{\mu\nu} \sim \frac{1}{2}\left(\partial_\mu h^\alpha_{\ \nu} + \partial_\nu h^\alpha_{\ \mu} - \partial^\alpha h_{\mu\nu}\right)$$

Furthermore, the gravitational field in this limit is also almost stationary (independent on time) so that

$$\partial_0 h = \frac{1}{c}\partial_t h \ll \partial_i h$$

We therefore write the Ricci tensor as:

$$R_{00} \sim \partial_\alpha \Gamma^\alpha_{\ 00} - \partial_0 \Gamma^\alpha_{\ 0\alpha} = \partial_i \Gamma^i_{\ 00} + \cancel{\partial_0 \Gamma^0_{\ 00}} - \cancel{\partial_0 \Gamma^\alpha_{\ 0\alpha}} \sim$$ (exchange derivatives)

$$\sim \frac{1}{2}\partial_i \left[\cancel{2\partial_0 h^i_{\ 0}} - \partial^i h_{00}\right] \simeq -\frac{1}{2}\Delta h_{00} \qquad (\Delta = \delta^{ij}\partial_i\partial_j)$$

Inserting into Einstein eqs:

$$R_{00} \sim \frac{8\pi G}{c^4}\left[T_{00} + \frac{1}{2}\left(-T_{00} + T_{ii}\right)\right] \sim \frac{4\pi G}{c^4} T_{00} \sim -\frac{1}{2}\Delta h_{00}$$

here we use only the Minkowsky metric

slow motion of the source

$v \ll c$

and therefore

$T_{00} \simeq \rho c^2 \gg T_{ii} \simeq \rho v^2$

$$\boxed{\Delta\left(\frac{c^2}{2} h_{00}\right) \sim -4\pi G \rho}$$

We therefore obtain Poisson equation, by identifying the Newtonian gravitational potential

$$\begin{cases} U = -\dfrac{c^2}{2} h_{00} \\ g_{00} \simeq -1 - \dfrac{2}{c^2} U \end{cases}$$

Is the Newtonian limit commonly valid? YES!

$$-\frac{U}{c^2} = \frac{GM}{Rc^2} = \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right) 2 \cdot 10^{-6}$$

units $c=1$

$$\begin{cases} G = 6.7\cdot 10^{-39}\ \text{GeV}^{-2} \\ R_\odot = 3.527\cdot 10^{24}\ \text{GeV}^{-1} = 6.96\cdot 10^{5}\ \text{km} \\ M_\odot = 1.116\cdot 10^{57}\ \text{GeV} = 2\cdot 10^{33}\ \text{gr} \end{cases}$$

So one gets

$$-\frac{U}{c^2} \simeq \begin{cases} 10^{-9} & \text{Earth} \\ 2\cdot 10^{-6} & \text{Sun} \\ 10^{-1} & \text{Neutron star} \\ \frac{GM}{R_s} \simeq 1 & \text{BH} \end{cases}$$

$$R_s = \frac{2GM}{c^2}$$

typically the potential is weak except for extreme conditions

The geodesic equation in the Newtonian limit becomes:

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

choose proper time as the affine parameter
$c\,d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}$

$$\frac{d^2x^i}{dt^2} \simeq -\Gamma^i_{00}c^2 = +\frac{1}{2}\partial^i h_{00}c^2 \Rightarrow$$

to get this, use Minkowsky metric (weak field limit)

$$(c\,d\tau)^2 = dx^{0\,2} + dx^{i\,2}$$

$$\left(\frac{d\tau}{dt}\right)^2 = \left(\frac{dx^0}{c\,dt}\right)^2 + \left(\frac{dx^i}{c\,dt}\right)^2$$

$$= 1 + \frac{v^2}{c^2} \simeq 1$$

limit of slow motion in the Newtonian approximation

$$\boxed{\frac{d^2x^i}{dt^2} \simeq -\partial_i U}$$

this is Newton's second law of motion for the conservative gravitational force.

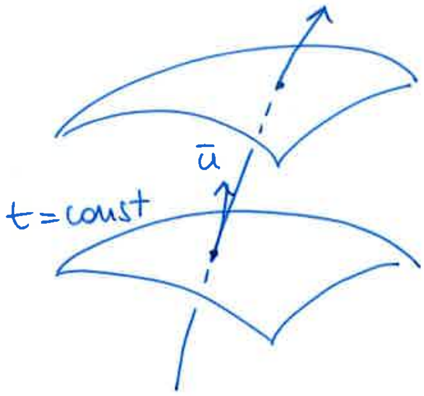
FRW METRIC and the cosmological constant

This is the cosmological solution of Einstein eqs, which describes the universe in the context of the big bang theory. It is based on the cosmological principle:

· spacetime is HOMOGENEOUS (invariance under translations) and ISOTROPIC (invariance under rotations).

One can therefore perform a slicing of space-time into maximally symmetric 3-spaces: the hypersurfaces of constant time and constant curvature, and a preferred geodesic time coordinate with tangent vector to the geodesic orthogonal to the hypersurfaces.

(Reference: Ruth Durrer book, chapter 1)



$$\boxed{ds^2 = -c^2 dt^2 + a^2(t)\,\gamma_{ij}\,dx^i dx^j}$$

preferred time coordinate: COSMIC TIME

scale factor: dynamics of the space-time, expansion of the universe

metric of the 3-space of constant curvature K (in polar coordinates)

$$\boxed{\gamma_{ij}\,dx^i dx^j = \frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}$$

[From observations we know now that $K \simeq 0$]

Following Einstein eqs, the energy momentum tensor must have the form

$$\boxed{T_{\mu\nu} = \begin{pmatrix} -\rho\, g_{00} & 0 \\ 0 & p\, g_{ij} \end{pmatrix}} \quad ⊛$$

otherwise the symmetries of the space-time cannot be fullfilled. The ~~conservation of energy~~ $\nabla_\mu T^{\mu\nu} = 0$ yields

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \quad ⊛$$

One can furthermore write Einstein eqs for the FRW metric: they express the dynamical evolution of the space-time

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho \\ 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} - \Lambda c^2 = -8\pi G\frac{p}{c^2} \end{cases}$$

FRIEDMANN EQUATIONS

[the components of the Einstein tensor for the FRW metric are given for example in chapter 1 of Ruth Durrer book]

THE COSMOLOGICAL CONSTANT:

We already saw that its contribution can be cast in the form of an energy-momentum tensor and transferred to the right of Einstein eqs:

$$T^{\Lambda}_{\mu\nu} := -\frac{c^4}{8\pi G}\Lambda\, g_{\mu\nu}$$

which has therefore the above mentioned form with: ⊛

$$\rho_\Lambda = \frac{c^2}{8\pi G}\Lambda \quad \text{and} \quad p_\Lambda = -c^2\rho_\Lambda$$

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$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \qquad (c=1)$$

$$\Longrightarrow \quad \frac{d\rho}{da} = -\frac{3}{a}(\rho + p)$$

$\left(E = \int d^3x\, a^3 \rho \text{ with } a^3 = \sqrt{-g} \right)$

$$\boxed{d(\rho a^3) + 3 p a^2 da = 0}$$

$\rightarrow$ not a conservation equation if $p \neq 0$

•) in the case of dust, $p = 0$: the energy density in a volume is conserved (it comes from $\rho = n m$)

(in fact, this can be derived also for a general space-time: $T^{\mu\nu} = \rho u^\mu u^\nu$ and $\nabla_\mu T^{\mu\nu} = 0$ provides $(\rho u^\mu)_{;\mu} = 0$ which is a cons. law)

•) in the case of radiation, $p = \frac{1}{3}\rho$: the energy density is not conserved!

$\rho \propto \frac{1}{a^4}$ $\quad$ ($\frac{1}{a^3}$ for particle conservation $\times \frac{1}{a}$ for redshift) from Friedmann equations, if we had a universe with no matter, zero curvature and only a cosmological constant we would find:

$$\frac{\ddot{a}}{a} = \frac{\Lambda c^2}{3} > 0 \qquad \left(\text{since one must have positive energy density}\right)$$

So the cosmological constant leads to ACCELERATION of the universe.

The cosmological constant was first introduced by Einstein to get a STATIC universe: it would balance the attractive force by normal matter through its repulsive effect.
It is not forbidden by the equations, but what could be its value? And its physical origin?

Today we have measured the accelerated expansion of the universe (Nobel prize in 2011 to Perlmutter, Schmidt and Riess - Supernovae)

through many cosmological measurements:

$$\Omega_\Lambda \simeq 0.7 \quad \Rightarrow \quad \rho_\Lambda^{(\text{observation})} \sim 2 \cdot 10^{-47}\ \text{GeV}^4 \qquad (c=1)$$

However, there is no widely accepted explanation for this value, or for the physical origin of the cosmol. constant.

It has been noted that the energy momentum tensor of a quantum field in the vacuum state is:

$$\langle 0 | T_{\mu\nu} | 0 \rangle = -\rho_{vac}\, g_{\mu\nu}$$

it has the same form as the one of the cosmological constant!

The "cosmological constant" could therefore be due to vacuum energy of all the fields present in the universe: assuming that for the standard model one has an upper cutoff of the theory at the Planck scale, one can derive the (very heuristic) value: ($\rho_{vac} \sim \hbar k^4_{max}$)

$$\rho_{vac} \sim (M_{pl})^4 \simeq (10^{19}\ GeV)^4$$

But this is different from the observed value by

effective energy scale associated to the observed value of ρ_Λ $\rightarrow$

$$\frac{M_\Lambda^{observed}}{M_{pl}} \simeq \frac{10^{-12}\ GeV}{10^{19}\ GeV} \simeq 10^{-31}$$

This is the so-called COSMOLOGICAL CONSTANT PROBLEM

SCHWARZSCHILD METRIC | definition and a list of facts

This is the solution of Einstein eqs in vacuum that describes for example the exterior of a static star.

It has spherical symmetry (Killing vector ∂_ϕ) and it is static (Killing vector ∂_t, hypersurface orthogonal):

↳ the time like Killing vector is orthogonal to time hypersurfaces

static metric: $ds^2 = g_{00}(\underline{x})\,dt^2 + g_{ij}(\underline{x})\,dx^i dx^j$

stationary metric: can have a mixed term $dt\,dx^i$

↳ exists a timelike killing vector

$$\boxed{ds^2 = -\left(1-\frac{2m}{r}\right)c^2dt^2 + \frac{dr^2}{\left(1-\frac{2m}{r}\right)} + r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)}$$

• The parameter m can be interpreted through an analogy with the Newtonian case:

for $r \gg m$ i.e. far away from the star where the metric is almost flat:

$$g_{00} = -1 - \frac{2}{c^2}U(\underline{x}) = -1 + \frac{2}{c^2}\frac{GM}{r} \overset{\text{(Schwarzschild)}}{=} -1 + \frac{2m}{r}$$

therefore one finds $\boxed{m = \frac{GM}{c^2}}$ with M the mass of the star.

• The quantity $\boxed{R_s = 2\frac{GM}{c^2}}$ is the Schwarzschild radius

At the coordinate value $r = R_s$ there is an apparent singularity of space-time; it is only apparent because it is possible to find coordinates that properly cover the limit $r \to R_s$.

- When using these coordinates, one can see that R_s has the meaning of an horizon : it represents the boundary of the region which is causally connected to a distant observer.

  No information can be exchanged between the regions $r < R_s$ and $r > R_s$.

- A test particle falling in this metric reaches the coordinate position $r = R_s$ at infinite value of the coordinate time t. However, this does not happen for the test particle proper time. Therefore, the exterior observer will never see the test particle crossing the horizon ; but for the particle itself nothing special happens when it crosses the horizon.
  (locally the space-time geometry is the same as everywhere else)
- If the radius of a star becomes smaller than R_s, the star collapses to a black hole. For a distant observer, the star then "freezes" at the Schwarzschild radius : it effectively becomes invisible because the redshift increases exponentially and the luminosity decreases correspondingly.
- After the star has collapsed, we are dealing with a BH : the interior is not relevant from the point of view of astrophysics, it just looks like a very massive object with radius R_s.

Example for the SUN: we are safe...

$$R_S = 2\frac{GM_\odot}{c^2} = 3\,km \ll R_\odot = 6\cdot 10^5\,km$$

- we nowadays know that at the center of galaxies there are very massive BHs of $10^6 - 10^9\,M_\odot$. In our galaxy it has a mass of $4\cdot 10^6\,M_\odot$ and its existence has been inferred from the motion of stars around it.

- Astrophysical objects "falling" in such kind of BHs can be powerful sources of GWs; it is also believed that the collapse of two galaxies can put their respective BHs in inspiral motion around each other, generating BH binaries which are also a powerful source of GWs.

GRAVITATIONAL WAVES IN LINEARIZED THEORY

EXPANSION AROUND FLAT SPACE : WAVE EQUATION

we study GWs in the context of LINEARISED THEORY meaning that we make the assumption

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \qquad (1)$$

here we have picked a reference frame, and we have therefore broken the invariance of GR under coordinate transformation.

the values of $h_{\mu\nu}$ depend on the coordinate choice: we are saying that, for the physical situation we want to describe, there exists a frame for which (1) is realised. However, even after having chosen this frame, there remains a residual gauge symmetry.

Consider now the coordinate transformation:

$$\boxed{x^\mu \rightarrow x^{\mu'} = x^\mu + \xi^\mu(x)} \quad \boxed{\text{with } |\partial_\mu \xi^\nu| = \mathcal{O}(h)} \quad ⊛$$

applying

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

$\Rightarrow$ transformation of a twice covariant tensor under coordinate transformation

we get to LOWEST ORDER

$$h_{\mu\nu}(x) \longrightarrow h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

so that $|h'_{\mu\nu}| \ll 1$ is still valid provided that ⊛

SLOWLY VARYING coordinate transformations are a symmetry of the linearised theory, in the sense that a linearised theory remains a linearised theory (around flat space-time) ②

Now a Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is such that,

$$g'_{\mu\nu}(x') = \Lambda_\mu{}^\sigma \Lambda_\nu{}^\rho g_{\sigma\rho}(x)$$

and therefore $\Lambda_\mu{}^\sigma \Lambda_\nu{}^\rho (\eta_{\sigma\rho} + h_{\sigma\rho}) = \eta_{\mu\nu} + \underbrace{\Lambda_\mu{}^\sigma \Lambda_\nu{}^\rho h_{\sigma\rho}}$

this is invariant

this transforms as a tensor.

if we require $|\Lambda_\mu{}^\sigma \Lambda_\nu{}^\rho h_{\sigma\rho}| \ll 1$ then also Lorentz transf. are a symmetry of linearised theory: because $\eta_{\mu\nu}$ is invariant, we can still write the transformed theory as $\eta_{\mu\nu} + h'_{\mu\nu}$, with $|h'_{\mu\nu}| \ll 1$

⊛ What about constant translations? $x'^\mu \to x^\mu + a^\mu$

⊛ what about the Poincaré group, given by constant translations plus Lorentz transformations?

From the definition of the Riemann tensor, of Christoffel symbols we can find **the Riemann tensor** in the linearised theory:

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}$$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \qquad (g^{\rho\sigma} = \eta^{\rho\sigma} - h^{\rho\sigma})$$

together give: $= \frac{1}{2}(\partial_\mu h^\rho{}_\nu + \partial_\nu h^\rho{}_\mu - \partial^\rho h_{\mu\nu}) + O(h^2)$

COVARIANCE: a change of coordinate does not change the form of the equations of a theory.

EXAMPLE: Einstein equations ~~[illegible]~~ are the same in all coordinate systems, they can be written in a covariant form.

EXAMPLE: Geodesic equation, $\nabla_{\dot\gamma}\dot\gamma = 0$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

INVARIANCE: an object of the theory does not change it's form under a change of coordinates.

EXAMPLE: Riemann tensor in linearized theory under the slowly varying infinitesimal coordinate transformation seen ~~[illegible]~~ above.

(while in full GR it is COVARIANT)

linear th.

$$\boxed{R^{\mu}{}_{\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu\partial_\rho h^{\mu}{}_{\sigma} + \partial^{\mu}\partial_\sigma h_{\nu\rho} - \partial^{\mu}\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}\right)}$$

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note that here, since we are in linearised theory around a flat spacetime we raise and lower indices with $\eta_{\mu\nu}$ in quantities which are first order in h.

(INVARIANT, as the EM field under $A^{\mu} \to A^{\mu} - \partial^{\mu}\Lambda$)

Since we have the Riemann tensor, we can find the Einstein tensor, plug it in Einstein eqs and find how they look like in linearised theory around a flat background. This is better done introducing

$$h = \eta_{\mu\nu} h^{\mu\nu}$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \qquad \text{(trace-reversed perturbation)}$$

and $\bar{h} = \eta_{\mu\nu}\bar{h}^{\mu\nu} = h - 2h = -h$ since $\eta^{\mu}{}_{\mu} = 4$.

In terms of this quantity, Einstein eqs. take the form:

$$\boxed{\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^{\rho}\partial^{\sigma}\bar{h}_{\rho\sigma} - \partial^{\rho}\partial_{\nu}\bar{h}_{\mu\rho} - \partial^{\rho}\partial_{\mu}\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$

We now use the gauge freedom given by the slowly varying infinitesimal coordinate transformation, and pick the **LORENTZ GAUGE:**

$$\boxed{\partial^{\nu}\bar{h}_{\mu\nu} = 0}$$

Can we do this?

To prove that we can, we need to see how $\bar{h}_{\mu\nu}$ transforms under $x^\mu \to x^\mu + \xi^\mu(x)$: ④

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho)$$ implying

$$\partial^\nu \bar{h}_{\mu\nu} \to (\partial^\nu \bar{h}_{\mu\nu})' = \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu \qquad \left(\text{with } \Box = \partial_\mu \partial^\mu\right)$$

Therefore, if we started from a frame in which we had $\partial^\nu \bar{h}_{\mu\nu} = f_\mu(x) \neq 0$, it is enough to choose the transformation $\boxed{\Box \xi_\mu = f_\mu(x)}$ to transfer to the Lorentz gauge.

Since the above equation always admit solutions, then we are free to pick the Lorentz gauge.

$$\Box \xi_\mu = f_\mu(x) \Rightarrow \xi_\mu(x) = \int d^4y \, G(x-y) f_\mu(y)$$

($G(x)$ is the Green's function of $\Box$) with $\Box_x G(x-y) = \delta^4(x-y)$

Therefore we find the **wave equation**

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$

in linearized theory Einstein eqs take the form of a wave equation.

Before concluding that this implies the existence of GWs, we should first eliminate all the gauge freedom to reach only the physical degrees of freedom.

UNIQUELY FIX THE COORDINATES

* how many indipendent components by now?

$\bar{h}_{\mu\nu} = \bar{h}_{\nu\mu}$ (metric) $\Longrightarrow$ 10

$\partial_\mu \bar{h}^{\mu\nu} = 0 \Longrightarrow 10 - 4 = 6$

AND THEY ALL APPEAR TO BE RADIATIVE!

Are these all indipendent??

NO BACKREACTION

* Note that $\partial_\mu T^{\mu\nu} = 0$, which is equivalent to the conservation of energy and momentum in linearised theory

* $\eta_{\mu\nu}$ is the background: the dynamics of the bodies that act as sources of GWs are described in flat spacetime (i.e. Newtonian gravity)

$h_{\mu\nu}$ is the propagating wave

* the response to waves of test masses are analysed in the full metric $\boxed{\eta_{\mu\nu} + h_{\mu\nu}}$ (linearised theory cannot describe how gravity influences the sources)

THE TT GAUGE:
- outside of the source
- propagation of GWs
- removes all the gauge freedom

Outside of the source the eq. is

$$\boxed{\Box \bar{h}_{\mu\nu} = 0}$$

makes sense in the context of linearised theory

meaning that there could be gravitational waves in vacuum which move at the speed of light:

$\Box = -\frac{1}{c^2}\partial_t^2 + \partial_i \partial^i$ analogous to electrodynamics

($\Box A^\mu = j^\mu = 0$ outside the source)

However, we still did not remove all the gauge freedom.

From the transformation:

$$\partial^\nu \bar{h}_{\mu\nu} \to (\partial^\nu \bar{h}_{\mu\nu})' = \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu$$

we see that we can still re-transform with

$$x^\mu \to x^\mu + \bar{\xi}_\mu$$

provided that $\Box \bar{\xi}_\mu = 0$: we remain in fact in the Lorentz gauge, since $(\partial^\nu \bar{h}_{\mu\nu})' = 0$.

How can we use this fact? Note that also

$$\Box \left(\partial_\mu \bar{\xi}_\nu + \partial_\nu \bar{\xi}_\mu - \eta_{\mu\nu} \partial_\rho \bar{\xi}^\rho \right) = 0 \qquad \left(\begin{array}{c} \Box \text{ and } \partial_\mu \\ \text{do commute} \end{array} \right)$$

which was the transformation law of $\bar{h}_{\mu\nu}$ under the ~~[illegible]~~ coordinate transformation $x^\mu \to x^\mu + \bar{\xi}^\mu$.

We set $\bar{\xi}_{\mu\nu} = \partial_\mu \bar{\xi}_\nu + \partial_\nu \bar{\xi}_\mu - \eta_{\mu\nu} \partial_\rho \bar{\xi}^\rho$.

Since we are looking at the situation $\boxed{\text{in vacuum}}$, we have $\Box \bar{h}_{\mu\nu} = 0$. Therefore $\Box \left(\bar{h}_{\mu\nu} + \bar{\xi}_{\mu\nu} \right) = 0$

> we can therefore use $\bar{\xi}_{\mu\nu}$ to set conditions on $\bar{h}_{\mu\nu}$!

We have four independent conditions to choose

1) $\boxed{\bar{h} = 0}$ $\Rightarrow$ $\bar{h}_{\mu\nu} = h_{\mu\nu}$ $\Rightarrow$ $\partial_0 h^{00} + \partial_i h^{0i} = 0$

(1 condition) (from $\partial_\nu \bar{h}^{\mu\nu} = 0$)

(3 conditions)

2) $\boxed{h^{0i}(x) = 0}$ $\Rightarrow$ $\partial_0 h^{00} = 0$ $\Rightarrow$ $h^{00} = 0$

The second equality follows from the fact that h^{00} is the (static, in this case) Newtonian potential in the context of a linearised theory. Therefore, it is irrelevant from the point of view of GWs: it is a non-dynamical metric component that we can as well fix to zero.

To summarise, we have now fixed:

- Lorentz gauge $\partial_\mu h^{\mu\nu} = 0$
- $h^{0\mu} = 0$
- $h = 0$

This eliminates all the gauge freedom of the theory and fixes the TRANSVERSE TRACELESS GAUGE, for which

$$\boxed{h^i{}_i = 0 \qquad \partial_i h^{ij} = 0 \qquad h^{0\mu} = 0}$$

Summary of simplifications:

- $\bar{h}_{\mu\nu}$ has 16 terms ; $\bar{h}_{\mu\nu} = \bar{h}_{\nu\mu}$ eliminates 6 $\Rightarrow$ 10
- Lorentz gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$ eliminates 4 $\Rightarrow$ 6
- further coordinate transformation with $\Box \xi_\mu = 0$ eliminates 4 $\Rightarrow$ $\boxed{2 \text{ in total}}$ if we are in vacuum (outside the source)

that we can fix to be two space elements of the symmetric matrix h_{ij}.

CHOSING THE TT GAUGE IS NOT NECESSARY, BUT STILL VERY CONVENIENT : THE METRIC PERTURBATION h_{ij}^{TT} ONLY CONTAINS THE PHYSICAL INFORMATION ABOUT THE RADIATION. It also exhibits the fact that GWs have two polarisation components.

Note that the TT gauge cannot be chosen <u>inside the source</u>, because in this case $\Box \bar{h}_{\mu\nu} \neq 0$.

Suppose we are inside the source: we choose the Lorentz gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$. We then make another coordinate transformation

$$x^\mu \to x^\mu + \bar{\xi}^\mu \qquad \text{with} \quad \Box \bar{\xi}^\mu = 0$$

we remain in the Lorentz gauge, and still

$$\Box \bar{\xi}_{\mu\nu} = \Box \left(\partial_\mu \bar{\xi}_\nu + \partial_\nu \bar{\xi}_\mu - \eta_{\mu\nu} \partial_\rho \bar{\xi}^\rho \right) = 0 .$$

However, now we cannot use this fact to impose conditions on $\bar{h}_{\mu\nu}$, because

$$\Box \bar{h}_{\mu\nu} \neq 0 .$$

Therefore, we cannot set to zero any further component of $\bar{h}_{\mu\nu}$ by subtracting from it a function $\bar{\xi}_{\mu\nu}$ which satisfies $\Box \bar{\xi}_{\mu\nu} = 0$.

[<u>ELECTRODYNAMICS</u>: $\partial_\mu F^{\mu\nu} = j^\nu$ 4-current

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) \underset{\uparrow}{=} \Box A^\mu = j^\nu$$

Lorentz gauge $\partial_\mu A^\mu = 0$

with residual gauge freedom $A_\mu \to A_\mu - \partial_\mu \theta$ with $\Box \theta = 0$

if $j^\nu = 0$ can use further condition $\Box(A^\mu - \theta) = 0$ to set $A^0 = 0$]

FORM OF A GRAV. WAVE IN THE TT GAUGE

the solution of the wave equation

$$\Box\, h_{ij}^{TT}(\underline{x},t) = 0$$

are plane waves
$$\boxed{h_{ij}^{TT}(\underline{x},t) = e_{ij}(\underline{k})\, e^{i k_\mu x^\mu}} \quad (*)$$

with wave vector $k^\mu = \left(\frac{\omega}{c}, \underline{k}\right)$ and $|\underline{k}| = \frac{\omega}{c}$

(propagate at the speed of light)

and $e_{ij}(\underline{k})$ is the polarization tensor.

For a single wave with wave-vector $\underline{k} = |\underline{k}|\,\hat{n}$

(OR different wave-numb k but same direction $\hat{n}$)

the tensor h_{ij}^{TT} has non-zero components only in the plane transverse to $\hat{n}$ because of the L.G. condition $\partial^j h_{ij} = 0 \implies n^j h_{ij} = 0$

from the plane wave solution $(*)$

For $\hat{n} \parallel z$ axis, with $h_{ij} = h_{ji}$ and $h_i{}^i = 0$

$$\boxed{h_{ij}^{TT}(z,t) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos\left[\omega\left(t - \frac{z}{c}\right)\right]}$$

therefore one has:

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu =$$
$$= -c^2 dt^2 + dz^2 + \left[1 + h_+ \cos\left(\omega\left(t - \tfrac{z}{c}\right)\right)\right] dx^2$$
$$+ \left[1 - h_+ \cos\left(\omega\left(t - \tfrac{z}{c}\right)\right)\right] dy^2 + 2 h_\times \cos\left(\omega\left(t - \tfrac{z}{c}\right)\right) dx\, dy$$

We can define a tensor that projects on the TT gauge a plane wave propagating in the direction $\hat{n}$, in vacuum, and which is already in the L.G.

First we define the projector in the plane orthogonal to $\hat{n}$:
$$P_{ij}(\hat{n}) = \delta_{ij} - \hat{n}_i \hat{n}_j$$

$$P_{ij}(\hat{n})\, n_j = 0\,, \quad \underbrace{P_{ik} P_{kj} = P_{ij}}_{\text{projector}}\,, \quad P_{ii} = 2$$

we then define:

$$\boxed{\Lambda_{ij,kl}(\hat{n}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}}$$

$$\underbrace{\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}}_{\text{projector}}\,, \qquad \underbrace{\Lambda_{ii,kl} = \Lambda_{ij,kk}}_{\text{traceless}} = 0$$

$$n^i \Lambda_{ij,kl} = \underbrace{n^j \Lambda_{ij,kl} = n^k \Lambda_{ij,kl}}_{\text{transverse}} = n^l \Lambda_{ij,kl} = 0$$

Now, if applied to a plane wave $h_{\mu\nu}$ in the Lorentz gauge, so that it satisfies $\Box h_{\mu\nu} = 0$, this Λ-tensor extracts the TT part (so it projects it on the TT gauge):

$$h_{ij}^{TT} = \Lambda_{ij,kl}\, h_{kl} \Longrightarrow \text{transverse and traceless in } ij$$

$$\text{and} \quad \Box\, h_{ij}^{TT} = 0$$

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The explicit form of the Λ tensor in terms of $\hat{n}$ is:

$$\Lambda_{ij,kl}(\hat{n}) = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik}$$
$$- n_i n_k \delta_{jl} + \frac{1}{2} n_k n_l \delta_{ij} + \frac{1}{2} n_i n_j \delta_{kl}$$
$$+ \frac{1}{2} n_i n_j n_k n_l$$

We will have sometimes to integrate this tensor over angles. Then we can use the identities:

$$\int \frac{d\Omega}{4\pi} n_i n_j = \frac{1}{3}\delta_{ij}$$

(mixed products $i \neq j$ go to zero as one can easy verify)

and

$$\underbrace{\int \frac{d\Omega}{4\pi} n_i n_j n_k n_l}_{\text{totally symmetric}} = \frac{1}{15}\underbrace{(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})}_{\text{totally symmetrized products of } \delta\text{s}}$$

So that

$$\int d\Omega \, \Lambda_{ij,kl} = \frac{2\pi}{15}\left(11\,\delta_{ik}\delta_{jl} - 4\,\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\right)$$

FOURIER EXPANSION

Any linear combination of the solutions of the wave equation is still a solution. We can therefore expand a gravitational wave in vacuum (i.e. outside the source) in terms of plane waves:

$$\boxed{h_{ij}^{TT}(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3} \left(A_{ij}(\underline{k})\, e^{i k_\mu x^\mu} + A_{ij}^*(\underline{k})\, e^{-i k_\mu x^\mu} \right)}$$

in terms of the frequency $f = \frac{\omega}{2\pi}$ we can rewrite: $|\underline{k}| = \frac{\omega}{c}$

$$d^3k = k^2 dk\, d\sin\theta\, d\varphi = \frac{(2\pi)^3}{c^3} f^2 df\, d^2\hat{n} \; : \quad (\underline{k} \parallel \hat{n})$$

$$h_{ij}^{TT}(\underline{x},t) \doteq \frac{1}{c^3} \int_0^\infty f^2 df \int d^2\hat{n} \left[A_{ij}(f,\hat{n})\, e^{-2\pi i f \left(t - \frac{\hat{n}\cdot\underline{x}}{c}\right)} + \text{c.c.} \right]$$

(1)

where the expansion coefficients are such that

$$A_{ii}(\underline{k}) = 0 \;, \quad k^i A_{ij}(\underline{k}) = 0 \,.$$

If the wave propagates in the definite direction $\hat{n}_0$:

$$A_{ij}(\underline{k}) = A_{ij}(f)\, \delta^2(\hat{n} - \hat{n}_0)$$

In this case, the expansion above can be further simplified:

$$h_{\underbrace{ab}}^{(TT)}(\underline{x},t) = \int_0^\infty df \left[\tilde{h}_{ab}(f,\underline{x})\, e^{-2\pi i f t} + \tilde{h}_{ab}^*\, e^{2\pi i f t} \right]$$

$ab = 1,2$ are the indexes spanning the plane orthogonal to $\hat{n}_0$, because of the transversality condition.

and the coefficients are:

$$\tilde{h}_{ab}(f, \underline{x}) = \frac{f^2}{c^3} \int d^2\hat{n}\; A_{ab}(f)\, \delta^2(\hat{n}-\hat{n}_0)\, e^{2\pi i f \underbrace{\frac{\hat{n}\cdot \underline{x}}{c}}}$$

$$= \frac{f^2}{c^3} A_{ab}(f)\, e^{2\pi i f \frac{\hat{n}_0 \cdot \underline{x}}{c}}$$

if the size of the detector is much smaller than the reduced wave lenght of the GW, $x/\lambda = x 2\pi f \ll 1$ this goes to 1

we can also extend to negative frequencies (unphysical) and compactify the form even more:

$$\tilde{h}_{ab}(-f, \underline{x}) := \tilde{h}^*_{ab}(f, \underline{x})$$

(note that here we can have a general $\hat{n}$)

so that

$$h^{(TT)}_{ab}(\underline{x}, t) = \int_{-\infty}^{+\infty} df\; \tilde{h}_{ab}(f, \underline{x})\, e^{-2\pi i f t}$$ with inverse

$$\tilde{h}^{(TT)}_{ab}(\underline{x}, f) = \int_{-\infty}^{+\infty} dt\; h_{ab}(t, \underline{x})\, e^{2\pi i f t}$$

We can also rewrite the POLARISATION TENSORS:

$$\boxed{e^+_{ij} = \hat{u}_i \hat{u}_j - \hat{v}_i \hat{v}_j \qquad e^\times_{ij} = \hat{u}_i \hat{v}_j + \hat{u}_j \hat{v}_i}$$

where $\hat{u}, \hat{v}$ are unit vectors living in the plane perpendicular to $\hat{n}$ and perpendicular among each other.

Choosing: $\hat{n} \parallel \underline{z}$, $\hat{u} \parallel \underline{x}$, $\hat{v} \parallel \underline{y}$ one gets

$$e^+_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab} \qquad e^\times_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ab} \qquad ab = x, y$$ spanning the plane

normalisation: $e^A_{ij}(\hat{n})\, e^{B\, ij}(\hat{n}) = 2\,\delta^{AB}$

Since the polarisation tensors are a basis of the "TT space", in a generic frame we can expand:

$$\frac{f^2}{c^3} A_{ij}(f,\hat{n}) = \sum_{A}^{+,\times} \tilde{h}_A(f,\hat{n})\, e^A_{ij}(\hat{n})$$

going back to (1) and introducing negative frequencies,

$$\boxed{h^{(TT)}_{ab}(t,x) = \sum_{A=+,\times} \int_{-\infty}^{+\infty} df \int d^2\hat{n}\; \tilde{h}_A(f,\hat{n})\, e^A_{ab}(\hat{n})\, e^{-2\pi i f\left(t-\frac{\hat{n}\cdot x}{c}\right)}}$$

and $\tilde{h}_A(-f,\hat{n}) = \tilde{h}^*_A(f,\hat{n})$.

⊙ for Earth based interferometers, the size of the detector is such that it is much smaller than the reduced wavelength of the waves it should detect:

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$L \ll \lambdabar$ with $\lambdabar = \frac{\lambda}{2\pi}$

$\omega = 2\pi f = ck$
$\lambda = \frac{c}{f} = \frac{2\pi c}{\omega} = \frac{2\pi}{k}$
$\lambdabar = 1/k$ TYPICAL LENGTHSCALE
if $f = e^{ikx}$
$\frac{df}{dx} = \frac{1}{\lambdabar}|f|$

therefore, the wave at the detector position $|\underline{x}| \simeq L$ satisfies:

$$\exp\left(2\pi i f\,\frac{\hat{n}\cdot\underline{x}}{c}\right) \simeq \exp\left(i\,\frac{2\pi}{\lambda}L\right) = \exp\left(i\,\frac{L}{\lambdabar}\right) \simeq 1$$

and one can neglect the space dependence:

$$h^{(TT)}_{ab}(t) = \int_{-\infty}^{+\infty} df \left[\tilde{h}_{ab}(f)\, e^{-2\pi i f t} + c.c.\right]$$

Earth-based detectors: $f \simeq 10^2 - 10^3$ Hz $\Rightarrow$ $\lambdabar \simeq 500-50$ km; $L \simeq 4$ km

Therefore, a GW in the TT gauge (the one in which the physical degrees of freedom are apparent) has TWO INDEPENDENT POLARISATIONS, represented by the polarisation tensors e^{+}_{ij} and $e^{\times}_{ij}$.

The polarisation states of a classical radiation field can be related to the SPIN of the massless particle that one expects upon quantisation of the theory. If the polarisation modes are <u>invariant under rotation of an angle Θ</u>, the spin of the particle associated with the radiation field should satisfy $S = \dfrac{2\pi}{\Theta}$.

MISNER, THORNE WHEELER CHAPTER 35.6

EXAMPLE: Electromagnetic radiation has two independent polarisation VECTORS (LINEAR polarisation states). Each of them is invariant under a rotation of 2π. The photon, which is the boson mediating the electromagnetic interaction has spin $S = 1$.

GWs are a radiation field, propagating at the speed of light, and are the waves of the gravitational field: in a quantum theory for the gravitational interaction, one expects the presence of a massless particle mediating the interaction: the GRAVITON.

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What should be its spin?

We have seen that GWs have 2 independent polarisation states represented by the tensors e^{+}_{ij} and $e^{\times}_{ij}$. A GW propagating in the $\hat{z}$ direction

$$h^{TT}_{ij}(z,t) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos\left[\omega\left(t-\frac{z}{c}\right)\right]$$

changes upon rotation around the z-axis of an angle θ as:

$$h^{TT\,'}_{ij} = R_{ik} R_{jl}\, h^{TT}_{kl}$$

with $R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore one obtains

$$\begin{cases} h_+' = h_+ \cos 2\theta - h_\times \sin 2\theta \\ h_\times' = h_+ \sin 2\theta + h_\times \cos 2\theta \end{cases}$$

which is invariant if $\theta = \pi$.

Applying this finding to what stated above, one finds that the GRAVITON has spin

$$S = \frac{2\pi}{\pi} = 2.$$

1) if the particle mediating gravity would have spin 0: one can construct this theory, and finds that photons do not couple to gravity. wrong

2) spin 1: massless vector field for which the coupling must respect G.I. } get like in EM repulsive force among two masses of equal sign

3) spin $j \geq 3$: one cannot couple this to a conserved tensor, there are not conserved tensors with more than 3 indices.

PARTICLES ARE REPRESENTATIONS OF THE POINCARE GROUP

can be MASSIVE REPRESENTATIONS $\rightarrow$ labeled by mass and spin

MASSLESS REPRESENTATIONS $\rightarrow$ labeled by helicity states

dimension of the MASSIVE REPR: $2j+1$

dimension of the MASSLESS REPR: $h = \pm j$ two-dimensional

The entity that describes the field:

PHOTON, SPIN 1, MASSLESS, 2 degrees of freedom:

A_μ has 4, impose G.I. and get 2

GRAVITON, SPIN 2, MASSLESS, 2 degrees of freedom

$h_{\mu\nu}$ symmetric has 10, impose TT gauge and get 2

The general asymmetric tensor $h_{\mu\nu}$, from the point of view of spatial rotations, decomposes into :

- h_{00} and h_{ii}, scalars under rotation: these represent spin 0 particles
- h_{0i}, a spatial vector, representing a spin 1 particle
- traceless h_{ij}, a tensor with space only components, representing a spin 2 field with **5** degrees of freedom.

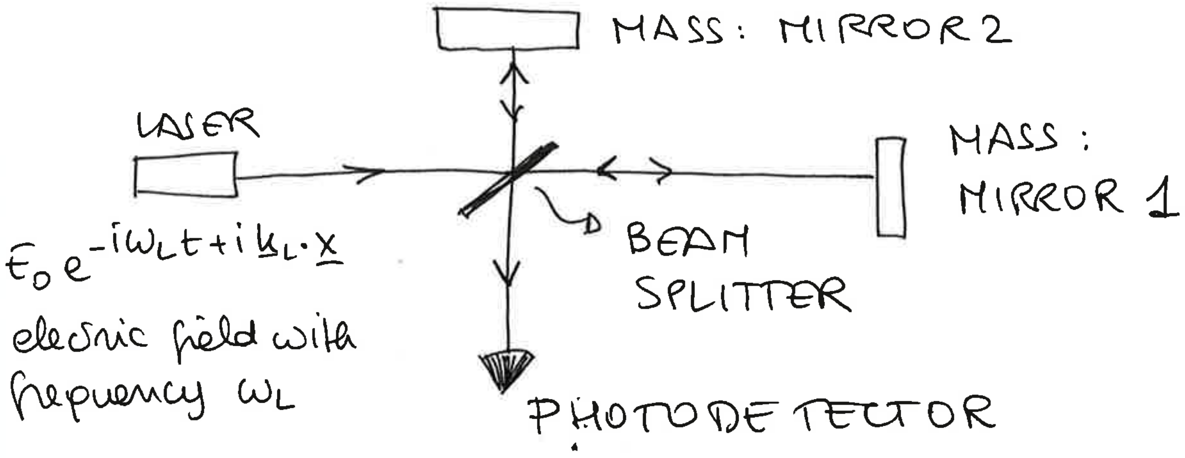
GWs in the TT gauge are described by a traceless h_{ij} which is furthermore transverse (Lorentz gauge) $\partial^i h_{ij}^{TT} = 0$.
From the 5 degrees of freedom initially, one is left with only 2 degrees of freedom: corresponding to a MASSLESS particle with helicity ± 2.

(c.f. Michele Maggiore book, chapter 2.2 and in particular problem 2.1)

Furthermore, if one decomposes a radiation field which associated particle of spin S in SPHERICAL HARMONICS, it turns out that all moments with $\ell < S$ go to zero. The first moment of the gravitational field is the quadrupole $\ell = 2$: sources must have a quadrupole moment to radiate GWs, as we will see.

An "idealised" detector of GWs is given by a Michelson interferometer:

MAGGIORE CHAPTER 9 FOR DETAILS



this instrument measures changes in the travel time of the laser light in the two arms:

At the beam splitter position and at observation time t we have the superposition of a beam that entered the beam splitter at time

$$t_0^{(x)} = t - 2\frac{L_x}{c} \Rightarrow$$ the beam that went through the x-arm

and another beam that entered the beam splitter at time

$$t_0^{(y)} = t - 2\frac{L_y}{c} \Rightarrow$$ the beam that went through the y-arm.

the two electric fields that combine at the beam splitter are

$$\begin{cases} E_1 = -\frac{1}{2} E_0 \, e^{-i\omega_L\left(t - 2\frac{L_x}{c}\right)} \\ \text{reflection and transmission at the mirrors} \\ E_2 = +\frac{1}{2} E_0 \, e^{-i\omega_L\left(t - 2\frac{L_y}{c}\right)} \end{cases}$$

$\boxed{\underline{x} = 0}$ because the phase does not change during the propagation giving an output

$$E_1 + E_2 = -i E_0 \, e^{-i\omega_L t + i \frac{\omega_L}{c}(L_x + L_y)} \sin\left[\frac{\omega_L}{c}(L_y - L_x)\right]$$

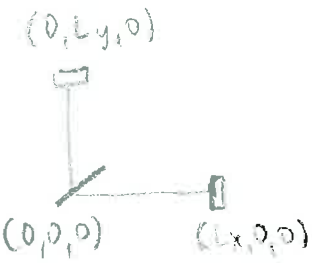
$$|E_1 + E_2|^2 = E_0^2 \sin^2\left[\frac{\omega_L}{c}(L_y - L_x)\right]$$

therefore, the output (the power at the photodetector) varies as the difference in length between the arms varies. THIS PHYSICAL EFFECT must be the same regardless of the reference frame

TT gauge : the mirrors (test masses) are in free-fall (we eliminate the Newtonian slowly varying gravitational effects by choosing the proper frequency range) : their coordinate distance remains constant, and therefore the effect of the GW going through manifests itself affecting

THE PROPAGATION OF LIGHT

between the test masses (fixed points)



coordinates do not change

PROPER DETECTOR FRAME : in this frame, the distance is measured by rigid rulers, while the GW passing through displaces the test masses from their original position.

in the sense that the photon of the laser follows the geodesics of flat spacetime ← Space-time can be taken as flat, and the interaction of the mirrors with the GW can be seen as a Newtonian force.

For example, in the TT gauge, supposing that the wave has only + polarization and comes from the z direction:

$$h_+(t) = h_0 \cos \omega_{GW} t$$

the space time interval becomes:

$$ds^2 = -c^2dt^2 + [1+h_+]dx^2 + [1-h_+]dy^2 + dz^2$$

and the effect is calculated by starting from the fact that photons travel on null geodesics with proper distance $ds^2 = 0$. For the light in the x-arm to first order in h_+ one gets:

$$dx \underset{\substack{\uparrow \\ \text{(Taylor expansion)}}}{=} \pm c dt \left[1 - \frac{1}{2} h_+(t)\right] \Longrightarrow$$

Integrating this equation, accounting for the round trip, and in the limit $\lambda \gg L$ (see Maggiore chapter 9 for details) one obtains that the times $t_0^{(x)}$ and $t_0^{(y)}$ acquire a correction of order h_0 as:

$$t_0^x \simeq t - 2\frac{L_x}{c} - \frac{L_x}{c} h_+\left(t - \frac{L_x}{c}\right)$$

and correspondingly for y

Setting $L_x = L_y = L$, the output is modulated by the presence of the GW as:

$$|E_1+E_2|^2 = E_0^2 \sin^2\left[\omega_L \frac{L}{c} h_0 \cos\left(\omega_{GW}\left(t - \frac{L}{c}\right)\right)\right]$$

this is the basic principle of GW detection with Earth-based interferometers.

$\Longrightarrow$ the full photon path is :

$$\int_0^{L_x} dx = \int_{t_0^x}^{t_1} c\,dt \left[1 - \frac{h_+}{2}\right] \quad + \quad \int_{L_x}^{0} dx = \int_{t_1}^{t} -c\,dt \left[1-\frac{h_+}{2}\right]$$

$$2L_x = \int_{t_0^x}^{t} c\,dt \left[1-\frac{h_+}{2}\right] = c(t - t_0^x) - \frac{c}{2}\int_{t_0^x}^{t} dt\, h_+$$

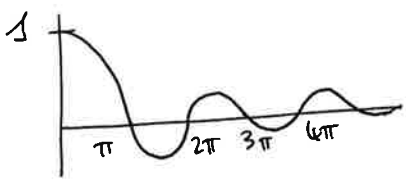
Therefore the total time interval that the photon travels into the arm of length (fixed in TT gauge) L_x is :

$t_0^x + \frac{2L_x}{c}$ $\longrightarrow$ the difference here is $O(h)$

$$t - t_0^x \simeq \frac{2L_x}{c} + \underbrace{\frac{1}{2}\int_{t_0^x} dt'\, h_+(t')}_{\text{correction due to the GW passing}}$$

$$\simeq \frac{2L_x}{c} + \frac{L_x}{c}\underbrace{\frac{\sin\left(\frac{\omega_{GW} L_x}{c}\right)}{\left(\frac{\omega_{GW} L_x}{c}\right)}}\; h_+(\underbrace{t_0^x + \frac{L_x}{c}}_{\text{time when the photon arrives at the mirror}})$$

if $\omega_{GW} \ll c/L_x$ the shift in time is $\simeq h_+(t)\frac{L_x}{c}$



if $\omega_{GW} \gtrsim c/L_x$ the contribution of the oscillations tends to cancel out

Inverting at first order in h one gets :

$$t_0^x \simeq t - \frac{2L_x}{c} - \frac{L_x}{c}\,\mathrm{sinc}\left(\frac{\omega_{GW} L_x}{c}\right) h_+\left(t - \frac{L_x}{c}\right)$$ and analogously

$$t_0^y \simeq t - \frac{2L_y}{c} + \frac{L_y}{c}\,\mathrm{sinc}\left(\frac{\omega_{GW} L_y}{c}\right) h_+\left(t - \frac{L_y}{c}\right)$$

from this one can calculate the total electric field at the beam splitter at time t:

$$E_1 + E_2 = -i E_0 \, e^{-i\omega_L \left(t - \frac{2L}{c}\right)} \sin\left[\phi_0 + \overbrace{\Delta\phi_x(t)}^{\text{effect}}\right]$$

$L_x \simeq L_y \simeq L$ in the part that multiplies h_+

$$\phi_0 = \frac{\omega_L}{c}(L_x - L_y)$$

phase to adjust with small differences in armlength

$$\boxed{\Delta\phi_x(t) = \frac{\omega_L L}{c} \, \mathrm{sinc}\left(\frac{\omega_{GW} L}{c}\right) h_+\left(t - \frac{L}{c}\right)}$$

which corresponds then to a $\frac{\Delta L}{L} \simeq h_+\left(t - \frac{L}{c}\right)$ (c.f. eq on page 34)

If one wants to maximize the phase difference one must choose

$$\mathrm{MAX}\left[\sin(\omega_{GW} L/c) \, \frac{\omega_L}{\omega_{GW}}\right] \Rightarrow L = \frac{\pi}{2} \frac{c}{\omega_{GW}}$$

$$\boxed{L \simeq 750 \text{ km} \left(\frac{100 \text{ Hz}}{f_{GW}}\right)}$$

which is impossible to obtain on Earth: the aLIGO/Virgo have arms of 4 km and 3 km respectively, and use Fabry-Perot cavities that effectively "fold" this length into 4 km. The effective length becomes $\frac{2F}{\pi}$ times longer, where F is the "finesse" of the cavity. Therefore
(storage time per length)

$$L_{eff} = \frac{\pi}{2F} \, 750 \text{ km} = 4 \text{ km} \quad \text{for} \quad F \simeq \mathcal{O}(10^2)$$

The dephasing that one needs to measure is:

$$\Delta\phi \simeq \frac{2\omega_L}{c}\,\frac{2F}{\pi}\,L_{eff}\,h_+ \simeq 6.4\cdot 10^{13}\,h_+ \simeq 10^{-8}\ \text{rad}$$

total dephasing, $\Delta\phi_x = -\Delta\phi_y$

$\lambda_L = 0.1\ \mu m$

$F = 200$

$h_+ \simeq 10^{-21}$ (we will see why)

The sources of noise are seismic and time varying gravitational fields, thermal noise, pressure of the laser, shot noise of the laser...

If we consider the point of view of the PROPER DETECTOR FRAME: eq. of geodesic deviation seen in the Newtonian analogy $\Rightarrow$ the masses move and space-time is flat, as long as $\boxed{\lambda_{GW} \gg L \Rightarrow \frac{\omega_{GW} L}{c} \ll 1}$

(one recovers the results obtained in this limit). As on page 31 one solves

$$\underbrace{\ddot{\xi}_x}_{\text{distance between masses}} = \frac{1}{2}\ddot{h}_+\,\xi_x \Rightarrow \xi_x = L_x + \frac{h_0 L_x}{2}\cos(\omega_{GW} t)$$

The photon reaches the mirror at time t_1 such that

$$c(t_1 - t_0^x) = \xi_x(t_1) \quad \Rightarrow \text{flat spacetime } (dx = c\,dt)$$

$$c(t_1 - t_0^x) = L_x + \frac{h_0 L_x}{2}\cos\left[\omega_{GW}\left(t_0^x + \frac{L_x}{c}\right)\right]$$

accounting for the round trip and inverting:

$$t_0^x = t - \frac{2L_x}{c} - \frac{L_x}{c}\,h_0\cos\left[\omega_{GW}\left(t - \frac{L_x}{c}\right)\right]$$

as on page 34/2 but with $\frac{\omega_{GW} L}{c} \ll 1$.

THE ENERGY OF GWs

What is the energy and momentum carried by GWs?

- It is clear that it is non-zero from the fact that they displace test masses (in the proper detector frame) and we can even describe this in terms of a Newtonian force.
- However, to find an expression for the energy-momentum tensor of GWs is not completely straightforward, because one has to distinguish what is a wave from what is background curvature: this analysis will show us that GWs CAN ONLY BE DEFINED IN A CERTAIN LIMIT, that will become clear soon
- According to GR, every form of energy contributes to the CURVATURE of space-time. The question to answer in order to find the energy-momentum tensor of GWs is therefore:

ARE GWs A SOURCE OF SPACE-TIME CURVATURE?

- First of all to answer this question we must go beyond linearised theory on a flat space-time:

we had $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

1. this is defined as a wave because we have seen that it satisfies a wave equation in the Lorentz gauge.

2. but if we keep as a background $\eta_{\mu\nu}$, we EXCLUDE FROM THE BEGINNING that GWs can generate curvature on the back-ground space-time.

Therefore we generalise

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) \qquad |h_{\mu\nu}| \ll 1$$

this is now a curved dynamical background metric

the same meaning as in lin. theory

•) obviously, within this setting it becomes completely untrivial to decide what is the background and what is the perturbation

For example $\bar{g}_{\mu\nu}(x)$ can have space and time dependent contributions from the time varying Newtonian grav. fields we were considering previously: this should be distinguished from the contribution $h_{\mu\nu}$.

(Analogy: waves in the sea / skin of an orange)

•) The answer can be guessed: the way to distinguish the two contributions is exploiting a possible

SEPARATION OF SCALES / FREQUENCIES

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)$$

1. if this has a typical scale of spatial variations L_B

2. and these are small amplitude perturbations with reduced wavelenght $\bar{\lambda}$ and with $|h_{\mu\nu}| \ll 1$

$$\boxed{\text{IF} \quad \bar{\lambda} = \frac{\lambda}{2\pi} \ll L_B}$$

then $h_{\mu\nu}$ has the meaning of small ripples on a smooth background.

$\left(e^{ik\cdot x} \to k = \frac{2\pi}{\lambda} = \frac{1}{\bar{\lambda}} \text{ to be compared with a typical scale } L_B \right)$

$$R^0{}_{i0j} = \dot{\mathcal{H}}\gamma_{ij} \qquad R^i{}_{00j} = \mathcal{H}\delta^i_j \qquad R^i{}_{jkm} = {}^{(3)}R^i{}_{jkm} + \mathcal{H}^2\left(\delta^i_k\gamma_{jm} - \delta^i_m\gamma_{jk}\right)$$

$$R_{00} = \dot{\mathcal{H}} \qquad R_{ij} = [\dot{\mathcal{H}} + 2(\mathcal{H}^2 + K)]\gamma_{ij} \qquad K(\delta^i_k\gamma_{jm} - \delta^i_m\gamma_{jk})$$

Matter era : $a = \frac{H_0^2}{4}\eta^2$

$\mathcal{H} = \frac{2}{\eta} \quad \dot{\mathcal{H}} = -\frac{2}{\eta^2}$

Components of Ricci today:

$$\dot{\mathcal{H}}_0 = -\frac{H_0^2}{2a_0}, \quad \mathcal{H}^2_0 = \frac{H_0^2}{a_0}, \quad K \simeq 0$$

Radiation era : $a = \frac{\sqrt{\Omega_{rad}}}{\eta_{eq}}\eta$

$\mathcal{H} = \frac{1}{\eta} \quad \dot{\mathcal{H}} = -\frac{1}{\eta^2}$

Components of Ricci :

$$\dot{\mathcal{H}} = -\left(\frac{\sqrt{\Omega_{rad}}}{\eta_{eq}\, a}\right)^2 = -\mathcal{H}^2, \quad K \simeq 0$$

Stoch bck of cosmological origin : $f_* \simeq \mathcal{H}(t_*) \qquad f_B \simeq H_0 \quad (\sqrt{R})$

Riemann in TT gauge

$R^0{}_{i0j} \sim \ddot{h}_{ij} \sim \omega^2 h \sim \frac{h}{\lambda^2} \qquad \lambda_* \simeq \frac{1}{\mathcal{H}(t_*)} \qquad L_B \simeq H_0^{-1} \quad \left(\sqrt{\frac{1}{R}}\right)$

Inflation : bck of GWs generated, stretched outside the horizon, re-enter when $k = \mathcal{H}(t)$ at each time : comparable with the background $\Rightarrow$ one cannot say they are GW once well into the horizon

Earth : Ricci in Newtonian limit $\quad R_{00} \sim -\frac{1}{2}\Delta h_{00} \sim \frac{h_{00}}{L_B^2}$

Riemann in TT gauge: $\quad R^i{}_{0j0} \sim \ddot{h}_{ij} \sim \omega^2 h_{ij} \sim \frac{h}{\lambda^2}$

comparing time : the first is static, $f_B \simeq 0$, the second is $R^i{}_{0j0} \sim f^2 h_{GW}$ distinguishable $f > 0$

comparing length : L_B can have many different scales! Setting $L_B \simeq \lambda$

$$\frac{h_{00}}{L_B^2} \gg \frac{h_{GW}}{\lambda^2} \quad \text{since } 10^{-9} \gg 10^{-21}$$

"curvature" of the Earth and of GW not distinguishable

Alternatively: $g_{\mu\nu} = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)$

1. IF this has frequencies (its F.T.) up to a given frequency f_B

2. AND this is peaked around f, with $|h_{\mu\nu}| \ll 1$

THEN IF $\boxed{f_B \ll f}$

$h_{\mu\nu}$ has the meaning of small, rapidly varying perturbations on a static or slowly varying background.

Since we know that $h_{\mu\nu}$ satisfies a wave equation, we have that $\lambda = \frac{c}{f}$, so in this respect the requirements on $h_{\mu\nu}$ are related. However, in principle L_B and f_B are not related, so it is enough that one of the above conditions is satisfied and the two conditions are independent.

EXAMPLES: •) $f_B \ll f$ is true in a cosmological context if the source of GWs is causal ($f \gtrsim \mathcal{H}(t_*)$) and $f_B \simeq H_0 \ll \mathcal{H}(t_*)$. $L_B \gg \lambda$ is also true for the same reason: $H_0^{-1} \gg \mathcal{H}^{-1}(t_*)$. However, this is not true for INFLATION which is not causal.

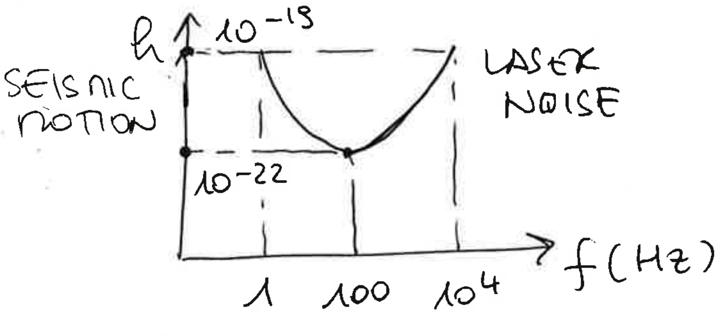
•) $f_B \ll f$ and NOT $L_B \gg \lambda$ is true for Earth-based detectors: the spatial variations of the gravitational field of the Earth do not satisfy $L_B \gg \lambda$; on the other hand, the gravitational field is almost static and does not have temporal variations on the same typical frequency of GWs detected by a Earth based interferometer.

typical frequency for detection: $f \sim 10^2 - 10^3$ Hz $\leftrightarrow$ $\lambda \simeq 500-50$ km

over a length-scale of 500-50 km the gravitational field of the Earth is not smooth (fluctuations such as mountains etc count) and it is also much bigger than the expected GW amplitude: page 27 of first chapter, $h_{00}(\text{EARTH}) \simeq 10^{-9}$, $h_{GW} \sim 10^{-21}$. However, the field is almost static, $\boxed{f_B \ll f}$: one **distinguishes** GWs from **the background** based on this condition.

On top of this, even if we could get rid of the spatial variations of the Earth grav. field and we were effectively on a flat background, a Earth-based interferometer could not monitor spatial variations of the metric due to the GWs because it is much smaller than their typical wavelenght (around few km)

Advanced LIGO SENSITIVITY



SIGNAL: COMPACT BINARIES INSPIRALLING (mainly)

From now on we consider then this situation: there is a reference frame in which we can separate the metric into background and fluctuations because there is a clear distinction in time and/or spatial characteristic scales: we analyse the situation $\lambda\!\!\!^{-} \ll L_B$ and/or $f_B \ll f$. This is referred to as performing a SHORT WAVE EXPANSION.

The aim is to understand

1) how the perturbation $h_{\mu\nu}$ affects the background
2) how the perturbation $h_{\mu\nu}$ propagates

To address these questions, we start with Einstein equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

and we EXPAND them: in this expansion there are two small parameters

1. $|h_{\mu\nu}| \ll 1$ 2. $\frac{\lambda\!\!\!^{-}}{L_B} \ll 1$ and/or $\frac{f_B}{f} \ll 1$

let's start by expanding the Ricci tensor AT SECOND ORDER IN $|h_{\mu\nu}| \ll 1$: ENOUGH TO DEFINE EN. MOM. TENSOR

Note that we need to go to second order if we want to understand how the perturbations affect the background, as it will become clear soon. All this is a manifestation of the non-linearity of GR.

$$\boxed{R_{\mu\nu} = \bar R_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}}$$

term constructed only with the background $\bar g_{\mu\nu}$: it contains only "low frequency - long wavelength" modes: $k \lesssim 4 k_B$.

$\bar g_{\mu\nu}$ varies on a typical scale L_B or larger, corresponding to modes $k < k_B = \frac{2\pi}{L_B}$.
then the $\bar R_{\mu\nu}$ will have modes up to $k \lesssim 4 k_B$, since it is at maximum quartic in the metric $(\bar g_{\mu\nu})^4$

this term is linear in $h_{\mu\nu}$
By definition, then, it contains only "high frequency - short wavelength" modes

$$k \gg 4 k_B$$

$\langle \sin x \rangle = 0$

this term is quadratic in $h_{\mu\nu}$: therefore, it can contain BOTH low and high frequency modes (long and short wavelength).
For example, the high-frequency mode $\underline{k}_1$ can combine in the squared term with the high-frequency mode $\underline{k}_2 \simeq -\underline{k}_1$ and give in total a low-frequency mode.
Note that this is always the case

for CONVOLUTIONS: the F.T. of a product

$$h^2(x) \longrightarrow \int_0^\infty d\bar k \, \tilde h(\bar k)\, \tilde h(k - \bar k) = \tilde H(k)$$

$\langle \sin^2 x \rangle \neq 0$

and $\tilde H(k)$ is such that

$$\tilde H(k \to 0) \simeq \int_0^\infty d\bar k \, \tilde h(\bar k)^2 \neq 0$$

so it has power also on small k, large scales even if $\tilde h(\bar k)$ has not.

So from this expansion, Einstein eqs can be divided in two separate equations, one for the low-frequency and one for the high-frequency part:

$$\boxed{\begin{aligned}\bar{R}_{\mu\nu} &= \left[-R^{(2)}_{\mu\nu}\right]^{low} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)^{low} \\ R^{(1)}_{\mu\nu} &= \left[-R^{(2)}_{\mu\nu}\right]^{high} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)^{high}\end{aligned}}$$

- we first analyse the first equation, the low-frequency one, and we will find that from this equation we can understand how to compute the GW ENERGY MOMENTUM TENSOR
- we then analyse the second equation, the high-frequency one, which will tell us how GWs PROPAGATE ON A CURVED BACKGROUND

But before this, let's see how to perform mathematically the split $\bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$ on the curved background, and let's find the form of $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$.
We start with the expansion

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu}$$

$\downarrow$

parameter ε that helps us to keep track of the order of the expansion: at the end, $\varepsilon = 1$

[MISNER, THORNE WHEELER CHAPTER 35.7 AND 35.13]

[FLANAGAN & HUGHES gr-qc/0501041]

the inverse matrix become:

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \varepsilon h^{\mu\nu} + \varepsilon^2 h^{\mu\alpha} h_\alpha{}^\nu$$

(indices raised and lowered with $\bar{g}_{\mu\nu}$)

the christoffel symbols become:

$$\Gamma^\mu_{\alpha\beta} = \bar{\Gamma}^\mu_{\alpha\beta} + \frac{1}{2}\varepsilon g^{\mu\gamma}\left(\bar{\nabla}_\beta h_{\alpha\gamma} + \bar{\nabla}_\alpha h_{\gamma\beta} - \bar{\nabla}_\gamma h_{\alpha\beta}\right)$$

christoffel built with the background metric $\bar{g}_{\mu\nu}$

this contains a term in ε ($\mathcal{O}(h)$) and a term in ε^2 ($\mathcal{O}(h^2)$)

covariant derivative built with the bckg. metric $\bar{g}_{\mu\nu}$

$$\boxed{S^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \bar{\Gamma}^\mu_{\alpha\beta}}$$

is a TENSOR containing both FIRST and SECOND order terms in $h_{\mu\nu}$

The Riemann tensor is:

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\nu\alpha}$$

we do it first in a local inertial frame with $\bar{\Gamma}^\mu_{\alpha\beta} = 0$

$$R^\mu{}_{\nu\alpha\beta} = \underbrace{\partial_\alpha \bar{\Gamma}^\mu_{\nu\beta} - \partial_\beta \bar{\Gamma}^\mu_{\nu\alpha}}_{\text{this is } \bar{R}^\mu{}_{\nu\alpha\beta}} + \partial_\alpha S^\mu_{\nu\beta} - \partial_\beta S^\mu_{\nu\alpha} + S^\mu_{\alpha\sigma} S^\sigma_{\nu\beta} - S^\mu_{\beta\sigma} S^\sigma_{\nu\alpha}$$

by general covariance it follows that in the original ref. frame:

$$R^\mu{}_{\nu\alpha\beta} = \bar{R}^\mu{}_{\nu\alpha\beta} + \bar{\nabla}_\alpha S^\mu_{\nu\beta} - \bar{\nabla}_\beta S^\mu_{\nu\alpha} + S^\mu_{\alpha\sigma} S^\sigma_{\nu\beta} - S^\mu_{\beta\sigma} S^\sigma_{\nu\alpha}$$

41/2

note: if we had expanded around $\eta_{\mu\nu}$ and we stayed at linear order (we will see that by definition one cannot go at second order when expanding around $\eta_{\mu\nu}$) the Riemann tensor at linear order would be:

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + O(h^2)$$

Now we can contract to get the Ricci tensor:

$$R^{\mu}{}_{\nu\mu\beta} = R_{\nu\beta} = \bar{R}_{\nu\beta} + \bar{\nabla}_\mu S^{\mu}_{\nu\beta} - \bar{\nabla}_\beta S^{\mu}_{\nu\mu} + S^{\mu}_{\mu\sigma} S^{\sigma}_{\nu\beta} - S^{\mu}_{\beta\sigma} S^{\sigma}_{\nu\mu}$$

And now, using

$$S^{\mu}{}_{\alpha\beta} = \frac{1}{2}\varepsilon\left(\bar{g}^{\mu\gamma} - \varepsilon h^{\mu\gamma}\right)\left(\bar{\nabla}_\beta h_{\alpha\gamma} + \bar{\nabla}_\alpha h_{\gamma\beta} - \bar{\nabla}_\gamma h_{\alpha\beta}\right)$$

we can expand the Ricci tensor as done on page 40

The result is:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left[\bar{\nabla}^\alpha \bar{\nabla}_\mu h_{\nu\alpha} + \bar{\nabla}^\alpha \bar{\nabla}_\nu h_{\mu\alpha} - \bar{\nabla}^\alpha \bar{\nabla}_\alpha h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h\right]$$

$$\begin{aligned} R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[& \frac{1}{2}\bar{\nabla}_\mu h_{\rho\alpha}\bar{\nabla}_\nu h_{\sigma\beta} + \\ & + (\bar{\nabla}_\rho h_{\nu\alpha})(\bar{\nabla}_\sigma h_{\mu\beta} - \bar{\nabla}_\beta h_{\mu\sigma}) + \\ & + h_{\rho\alpha}\left(\bar{\nabla}_\nu \bar{\nabla}_\mu h_{\sigma\beta} + \bar{\nabla}_\beta \bar{\nabla}_\sigma h_{\mu\nu} - \bar{\nabla}_\beta \bar{\nabla}_\nu h_{\mu\sigma} - \bar{\nabla}_\beta \bar{\nabla}_\mu h_{\nu\sigma}\right) \\ & + \left(\frac{1}{2}\bar{\nabla}_\alpha h_{\rho\sigma} - \bar{\nabla}_\rho h_{\alpha\sigma}\right)\left(\bar{\nabla}_\nu h_{\mu\beta} + \bar{\nabla}_\mu h_{\nu\beta} - \bar{\nabla}_\beta h_{\mu\nu}\right)\Big] \end{aligned}$$

Now that we have the form of $R_{\mu\nu}^{(2)}$, let's go back to the analysis of the first equation on page 41:

$$\bar{R}_{\mu\nu} = \left[-R_{\mu\nu}^{(2)}\right]^{low} + \frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right]^{low}$$

here we are mixing different terms of the $|h_{\mu\nu}| \ll 1$ expansion: is this coherent? YES, because there is also the second expansion $\frac{\lambdabar}{L_B} \ll 1$ in the game. In the above eq. the two expansions mix: the relative strength of the two parameters is fixed precisely by Einstein eqs. (DEMONSTRATION THAT ONE CANNOT GO TO SECOND ORDER PERTURBING AROUND $\eta_{\mu\nu}$)

CASE 1, $T_{\mu\nu}^{low} = 0$:
$$\bar{R}_{\mu\nu} = \left[-R_{\mu\nu}^{(2)}\right]^{low}$$

The background curvature is determined UNIQUELY by the back-reaction of the GWs on the back-ground: from the expression on page 41/2 one sees that

$$R_{\mu\nu}^{(2)} \sim (\partial h)^2 + h\,\partial^2 h \sim O\left(\frac{h}{\lambdabar}\right)^2$$

{ the typical scale of variation of h is the reduced wave-length $\lambdabar$

We see therefore that GWs contribute a background curvature of order $O\left(\frac{h}{\lambdabar}\right)^2$. In the case of zero external matter we therefore obtain the following relation among the two expansions: (case that applies e.g. far away from a source)

CASES WITH $T_{\mu\nu}^{low} \neq 0$

1) UNIVERSE : $\bar{R}_{\mu\nu} \approx \Theta(H_0^2) \sim$ MATTER $+ \Theta\left(\frac{h^2}{\lambda^2}\right)$

GW from causal production (today) : $h \overset{?}{\ll} \frac{\lambda}{L_B} = \frac{H_0}{\mathcal{H}_*}$ $\quad \Omega_{GW} \sim \frac{\langle \dot{h}^2 \rangle}{G\rho_c} \sim \frac{\mathcal{H}_*^2 h^2}{G\rho_c}$

$$h \sim \frac{\sqrt{\Omega_{GW}\, G\rho_c}}{\mathcal{H}_*} \sim \sqrt{\Omega_{GW}}\, \frac{H_0}{\mathcal{H}_*} \ll \frac{H_0}{\mathcal{H}_*}$$

$$\frac{3H_0^2}{8\pi G} = \rho_c$$

GW from inflation : (today) $h \overset{?}{\ll} \frac{\lambda}{L_B} = \frac{H_0}{k}$ where $k \simeq H_0$ the mode that enters today (*)

for the mode that enters today, $h \ll 1$, true.

2) EARTH : $\bar{R}_{\mu\nu} \sim \Theta\left(\frac{h_{00}}{L_B^2}\right) \sim$ MATTER $+ \Theta\left(\frac{h^2}{\lambda^2}\right)$

$$h \ll \sqrt{h_{00}}\, \frac{\lambda}{L_B} \simeq 3\cdot 10^{-5}\, \frac{\lambda}{L_B} = 3\cdot 10^{-5}\, \frac{500\,\text{km}}{L_B} \quad \text{true}$$

(*) Note that since $\frac{\lambda}{L_B} \simeq 1$ the modes close to the horizon are not GWs; but since $h \ll 1$ and the background is homogeneous and isotropic they can be described using linearized theory $g_{\mu\nu} = g_{\mu\nu}^{(B)} + h_{\mu\nu}$

I know how to pick this: hom. and isotropic hypersurface

$|h_{\mu\nu}| \ll 1$

$$\bar{R}_{\mu\nu} \sim (\partial \bar{g})^2 + \bar{g}\,\partial^2 \bar{g} \sim \mathcal{O}\left(\frac{1}{L_B}\right)^2 = \left[R^{(2)}_{\mu\nu}\right] \sim \mathcal{O}\left(\frac{h}{\lambda\!\!\!^-}\right)^2$$

and therefore $\boxed{h \sim \dfrac{\lambda\!\!\!^-}{L_B}}$ (bck curvature determined by GWs)

the two small parameters are the same

CASE 2, $T^{low}_{\mu\nu} \neq 0$: In this case, the background curvature is determined by the presence of the matter. By definition then, the curvature due to GWs must be much smaller:

$$\bar{R}_{\mu\nu} \sim \mathcal{O}\left(\frac{1}{L_B}\right)^2 = \mathcal{O}\left(\frac{h}{\lambda\!\!\!^-}\right)^2 + \text{MATTER} \gg \mathcal{O}\left(\frac{h}{\lambda\!\!\!^-}\right)^2$$

and therefore $\boxed{h \ll \dfrac{\lambda\!\!\!^-}{L_B}}$ (bck curvature determined by matter)

two separate expansions

From the results above, one can understand two things:

1) In the linearised theory studied up to now (and in the context of which we will also describe the sources) one cannot go beyond linear order. In fact, setting $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ means that we force $\frac{1}{L_B} \to 0$. Therefore, whatever $h \ll 1$ violates the above condition $h \lesssim \frac{\lambda\!\!\!^-}{L_B}$. So we cannot investigate the low-frequency part of

the second order term $R^{(2)}_{\mu\nu}$; we must stop at $R^{(1)}_{\mu\nu}$. Consequently, we have to go to the expansion on a general bck in order to ask the question "what is the energy momentum tensor of GWs", which is necessarily quadratic in $h_{\mu\nu}$.

2) The notion of GWs is well defined if and only if $h \ll 1$. If h becomes of order one, then the above conditions tell us that also $\frac{\lambda\!\!\!^{-}}{L_B}$ becomes of order one, and one cannot distinguish the wave from the background any longer.

The method to effectively implement this distinction among GWs and the background, based on the difference in their typical length-scales / frequencies is to

$$\boxed{\text{PERFORM AN AVERAGE}}$$

1) Select a scale $\bar{\ell}$ such that $\lambda\!\!\!^{-} \ll \bar{\ell} \ll L_B$, and average quantities over a volume $\bar{\ell}^3$: all the short-wavelength modes $k \gtrsim 1/\bar{\ell} \gg k_B$ will average to zero. OR

2) Select a frequency $\bar{f}$ with $f_B \ll \bar{f} \ll f$ and average quantities over a timescale $\bar{t} = 1/\bar{f}$, i.e. over several periods of the GWs.
All high frequencies will average to zero.

NOTE ADDED

modes such that $k \gtrsim \frac{1}{\bar{\ell}}$ average to zero inside the average: demonstration

let's consider a quantity that has power on all scales, like for example $T_{\mu\nu}$ of the matter, and let's see that by averaging we get only the low-frequency long-wavelength part $\langle T_{\mu\nu}\rangle \equiv \bar{T}_{\mu\nu}$:

$$\langle T_{\mu\nu}\rangle = \frac{1}{V}\int_V d^3x\, T_{\mu\nu}(x) \simeq \frac{1}{\bar{\ell}^3}\int_{\bar{\ell}^3} d^3x\, T_{\mu\nu}(\underline{x}) =$$

Fourier Transform in space

$$= \frac{1}{\bar{\ell}^3}\int_{\bar{\ell}^3} d^3x \int \frac{d^3k}{(2\pi)^3}\,\tilde{T}_{\mu\nu}(\underline{k})\, e^{-i\underline{k}\cdot\underline{x}} =$$

$$= \frac{1}{(2\pi)^3\bar{\ell}^3}\int d^3k\,\tilde{T}_{\mu\nu}(\underline{k})\int_0^{\bar{\ell}} dr\, r^2\int_0^{2\pi} d\varphi\int_{-1}^{1} d\cos\theta\; e^{-ikr\cos\theta}$$

(here we have chosen the $\hat{k}$ direction such that $\underline{k}\cdot\underline{x} = kr\cos\theta$ without loss of generality)

$$= \frac{2}{(2\pi)^2\bar{\ell}^3}\int d^3k\,\tilde{T}_{\mu\nu}(\underline{k})\int_0^{\bar{\ell}} dr\, r^2\,\frac{\sin kr}{kr} =$$

$$= \frac{2}{(2\pi)^2\bar{\ell}^3}\int d\Omega_{\underline{k}}\int_0^{\infty} dk\, k\,\tilde{T}_{\mu\nu}(\underline{k})\left[\frac{\sin(k\bar{\ell}) - k\bar{\ell}\cos(k\bar{\ell})}{k^2}\right]$$

Now for $k \lesssim 1/\bar{\ell}$ the function in $[\dots]$ is smooth; but for $k \gtrsim 1/\bar{\ell}$ it oscillates strongly, and the integral over k averages to zero. We are left with the contribution $k \lesssim 1/\bar{\ell}$

this is a common procedure in physics, denominated renormalization group transformation, which consists in "integrating out" the higher fluctuating part in the fundamental eq. of motions describing a theory in order to obtain an EFFECTIVE THEORY that describes the physics of the low fluctuating part. For example it can be done

- on space : to describe the effective physics at larger scales
- on energy (momentum) : to describe the effective physics at low energies
- on frequency : to describe the effective physics of a slowly evolving phenomenon

•) The effective energy momentum tensor of matter is obtained from the average: (effective: low-frequency part)

$$\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle := \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T}$$

"macroscopic" energy mom. tensor

the trace $\bar{T} = \bar{g}_{\mu\nu} \bar{T}^{\mu\nu}$

•) To get the effective energy momentum tensor of GWs, we define the quantity:

$$\boxed{t_{\mu\nu} := -\frac{c^4}{8\pi G} \left\langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \right\rangle}$$

with $R^{(2)} = \bar{g}_{\mu\nu} R^{(2)\mu\nu}$ (still second order in h)

and the trace is: (note that $\bar{g}_{\mu\nu}$ is purely low-frequency so it can exit and enter the average $\langle\rangle$)

$$t = \bar{g}^{\mu\nu} t_{\mu\nu} = -\frac{c^4}{8\pi G} \langle R^{(2)} - 2 R^{(2)} \rangle = \frac{c^4}{8\pi G} \langle R^{(2)} \rangle$$

So that we get:

$$t_{\mu\nu} = -\frac{c^4}{8\pi G} \left[\langle R^{(2)}_{\mu\nu} \rangle - \frac{1}{2} \bar{g}_{\mu\nu} \frac{8\pi G}{c^4} t \right]$$

$$\Rightarrow \langle R^{(2)}_{\mu\nu} \rangle = -\frac{8\pi G}{c^4} \left[t_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} t \right]$$

Inserting all into the (low-frequency / long-wavelength) part of Einstein eqs we find:

$$\bar{R}_{\mu\nu} = \frac{8\pi G}{c^4} \left[t_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} t \right] + \frac{8\pi G}{c^4} \left[\bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T} \right]$$

and rearranging:

$$\boxed{\underbrace{\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R}}_{1.} = \frac{8\pi G}{c^4} \underbrace{\left[t_{\mu\nu} + \bar{T}_{\mu\nu} \right]}_{2.}}$$

1. the dynamics of the background metric $\bar{g}_{\mu\nu}$ is determined by

2. the low-frequency part of the matter energy momentum tensor $\bar{T}_{\mu\nu}$ plus an energy momentum tensor $t_{\mu\nu}$ which depends only on the presence of the GWs and it is QUADRATIC in $h_{\mu\nu}$

to summarize:

- In principle there is no fundamental distinction between a background metric $\bar{g}_{\mu\nu}$ and the fluctuations over it: the gravitational field is described by all its modes through Einstein eqs.

- In order to define the fluctuations, they must be small in amplitude and with a characteristic wavelength / frequency of variation that must be clearly separated from the one of the background.

- When we are in this situation, to describe the system we perform an average over some length / time scale intermediate among the one peculiar of the background and the one peculiar of the fluctuation. We therefore "integrate out" the short wavelength / high frequency degrees of freedom to obtain a MACROSCOPIC EFFECTIVE description of the system

- The result of this procedure performed over Einstein equations tells us that the dynamics of the background metric $\bar{g}_{\mu\nu}$ is determined by the sum of two terms: a smoothed version of the energy momentum tensor of matter plus the contribution of GWs, which is FORMALLY IDENTICAL to that of matter with energy momentum tensor $t_{\mu\nu}$.

We now proceed in calculating $t_{\mu\nu}$.

ENERGY MOMENTUM TENSOR OF GWS :

We want to get an expression for $t_{\mu\nu}$. From now on we consider that we are far away from the source of GWs and that we can approximate space-time as flat. Therefore, $\bar{D}_\mu \to \partial_\mu$ (for the most general case, see MISNER, THORNE, WHEELER CHAPTER 35.15 AND FLANAGAN HUGHES, gr-qc/0501041).

From the expression of $R^{(2)}_{\mu\nu}$ on page 41/2, making the approximation $\bar{D}_\mu \to \partial_\mu$ we get:

$$\langle R^{(2)}_{\mu\nu} \rangle = \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\alpha\beta} \Big\langle \frac{1}{2} \partial_\mu h_{\rho\alpha} \partial_\nu h_{\sigma\beta} + \partial_\rho h_{\nu\alpha} (\partial_\sigma h_{\mu\beta} - \partial_\beta h_{\mu\sigma}) + h_{\rho\alpha} (\partial_\nu \partial_\mu h_{\sigma\beta} + \underbrace{\partial_\beta \partial_\sigma h_{\mu\nu}}_{\text{parts + lor.}} - \underbrace{\partial_\beta \partial_\nu h_{\mu\sigma}}_{\text{parts + lor}} - \underbrace{\partial_\beta \partial_\mu h_{\nu\sigma}}_{\text{parts + lor.}}) +$$
$$+ (\underbrace{\frac{1}{2} \partial_\alpha h_{\rho\sigma}}_{\text{trace}} - \underbrace{\partial_\rho h_{\alpha\sigma}}_{\text{lorentz}}) (\partial_\nu h_{\mu\beta} + \partial_\mu h_{\nu\beta} - \partial_\beta h_{\mu\nu}) \Big\rangle$$

$\Box h_\nu{}^\beta = 0$

parts + lor.

- This expression is generic and can contain also gauge modes. We pick the Lorentz gauge with $h = 0$, $\partial^\mu h_{\mu\nu} = 0$ (we will see after that this gets rid of all spurious gauge modes)
- Inside the average, we can integrate by parts and neglect the boundary term: by doing so, we make an error of the order $\Theta(\lambda\!\!\!^{-}/\bar{e})$ with $\bar{e}$ the average scale (see note added for a demonstration). This can be done even if the derivative is a time derivative, because $\partial_t = -c\partial_z$ for a function $h(t - \frac{z}{c})$ as is the GW, which

satisfies $\Box h_{\mu\nu} = 0$ outside the source. In summary we get:

$$\langle R^{(2)}_{\mu\nu} \rangle = \frac{1}{2} \left\langle \frac{1}{2} \partial_\mu h^{\sigma\beta} \partial_\nu h_{\sigma\beta} + h_{\rho\alpha} \partial_\mu \partial_\nu h^{\rho\alpha} \right\rangle = (\text{parts})$$

$$= -\frac{1}{4} \left\langle \partial_\mu h^{\sigma\beta} \partial_\nu h_{\sigma\beta} \right\rangle$$

$$\langle R^{(2)} \rangle = \langle \bar{g}_{\mu\nu} R^{(2)\mu\nu} \rangle = 0 \quad (\text{parts} + \text{Lorentz})$$

we have therefore:

$$\boxed{t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\sigma\beta} \partial_\nu h^{\sigma\beta} \rangle}$$

we still have to check that we have removed all gauge contributions from this expression: we do this by demonstrating that it is invariant under slowly varying infinitesimal coord. transformation $x^\mu \to x^\mu + \xi^\mu(x)$

$$\delta t_{\mu\nu} = \frac{c^4}{32\pi G} \left[\langle \partial_\mu h_{\sigma\beta} \partial_\nu (\delta h^{\sigma\beta}) \rangle + \mu \leftrightarrow \nu \right] =$$

$$= \frac{c^4}{32\pi G} \left[\langle \partial_\mu h_{\sigma\beta} \partial_\nu (\partial^\sigma \xi^\beta + \partial^\beta \xi^\sigma) \rangle + \mu \leftrightarrow \nu \right] = 0$$

(by parts)

we have therefore found an expression for the energy-mom. tensor of GWs which is free from gauge modes and therefore is a PHYSICAL QUANTITY.

This can only be defined INSIDE AN AVERAGE, either over space or over time or even over both.

It is therefore not defined in a point but on average.

NOTE ADDED

Integration by parts inside the average:

$$\langle R^{(2)}_{\mu\nu} \rangle = \frac{1}{V} \int_V d^3x \, R^{(2)}_{\mu\nu} \qquad \text{with } V \sim \bar{\ell}^{\,3}$$

take one term of $R^{(2)}_{\mu\nu}$ as an example: $\partial^\sigma h_{\nu\alpha} \partial^\alpha h_{\mu\sigma}$

we would have:

$$\langle \partial^\sigma h_{\nu\alpha} \partial^\alpha h_{\mu\sigma} \rangle = - \langle (\partial^\alpha \partial^\sigma h_{\nu\alpha}) h_{\mu\sigma} \rangle + \langle \partial^\alpha [(\partial^\sigma h_{\nu\alpha}) h_{\mu\sigma}] \rangle$$

If we demonstrate that the last term on the right hand side is negligible, we have demonstrated that we can integrate by parts inside the average $\langle \ldots \rangle$.

We rewrite this term:

$$\frac{1}{V} \int_V d^3x \, \partial^\alpha [(\partial^\sigma h_{\nu\alpha}) h_{\mu\sigma}] = \frac{1}{V} \int_{\text{all space}} d^3x \, f(\underline{x}) \, \partial^\alpha [(\partial^\sigma h_{\nu\alpha}) h_{\mu\sigma}]$$

where $f(\underline{x})$ is a function such that $f(\underline{x}) \to 0$ for $|\underline{x}| > \bar{\ell}$, and which is of order one otherwise: $\int_{\text{all space}} d^3x \, f(\underline{x}) = 1$

then the integral is

$$= \frac{1}{V} \int_{\text{all space}} d^3x \left\{ \partial^\alpha [f (\partial^\sigma h_{\nu\alpha}) h_{\mu\sigma}] - (\partial^\alpha f)(\partial^\sigma h_{\nu\alpha}) h_{\mu\sigma} \right\}$$

The first term can be converted into a surphace integral on a surphace where $f(\underline{x})$ goes to zero: so it goes to zero. We are left with:

$$\langle \partial^\sigma h_{\nu\alpha} \partial^\alpha h_{\mu\sigma} \rangle = - \langle (\partial^\alpha \partial^\sigma h_{\nu\alpha}) h_{\mu\sigma} \rangle +$$

$$- \frac{1}{V} \int_{\text{all space}} d^3x \; (\partial^\alpha f)(\partial^\sigma h_{\nu\alpha}) h_{\mu\sigma}$$

the first two terms are of order $\mathcal{O}\left(\frac{h}{\lambda}\right)^2$. In the last term one has $\partial^\alpha f \sim \mathcal{O}\left(\frac{f}{\ell}\right) \sim \mathcal{O}\left(\frac{1}{\ell}\right)$ and it is therefore of order $\mathcal{O}\left(\frac{h^2}{\ell \lambda}\right)$.

One has therefore from the above expression:

$$\mathcal{O}\left(\frac{h^2}{\lambda^2}\right) \sim \mathcal{O}\left(\frac{h^2}{\lambda^2}\left[1 + \frac{\lambda}{\ell}\right]\right)$$

so if we neglect the last term we make an error of the order $\frac{\lambda}{\ell} \ll 1$. Within this degree of precision, we can integrate by parts into the average $\langle \dots \rangle$.

the GRAVITATIONAL WAVE ENERGY DENSITY is therefore (here we pick the $\underbrace{\text{TT gauge}}_{\text{outside of the source}}$, and we set $\partial_{x_0} = +c\,\partial_t$)

$$\boxed{\rho_{GW} = t^{00} = \frac{c^2}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle}$$

the "CONSERVATION" of the energy momentum tensor can be read off from Einstein eqs:

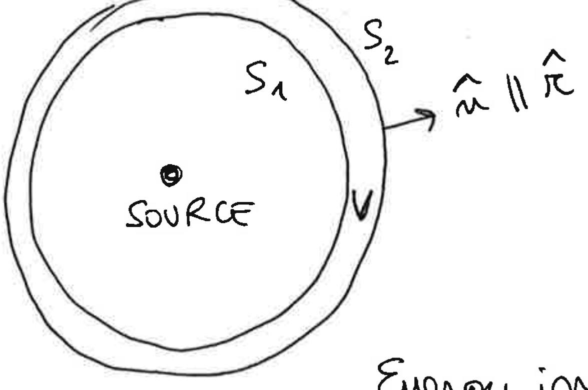
$$\nabla^\mu \left(\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} \right) = \frac{8\pi G}{c^4} \underbrace{\nabla^\mu \left(T_{\mu\nu} + t_{\mu\nu} \right)}_{} = 0$$

the matter and the GWs are not separately "conserved", because in general there is an exchange of energy and momentum among them: the matter sources the GWs, and GWs impact on the matter.

In flat spacetime far away from the sources, we can write

$$\partial^\mu t_{\mu\nu} = 0$$

we can also compute THE ENERGY FLUX and THE MOMENTUM FLUX: the energy (momentum) flowing per unit time and unit surface distant from the source:
(therefore, where the gravitational field is only due to the presence of the GWs, and we are in flat space-time)
This will be useful for calculating the signal from a far-away source.



$$\partial^\mu t_{\mu\nu} = 0$$

$$\int_V d^3x \left(\partial_0 t^{00} + \partial_i t^{i0} \right) = 0 \qquad (*)$$

Energy inside V : $\quad E_V = \int_V d^3x \, t^{00}$

So $(*)$ becomes: $\quad \frac{1}{c}\frac{dE_V}{dt} = -\int_V d^3x \, \partial_i t^{i0} = -\int_{S_2} dA \, n_i t^{i0}$

$$= -\int d\Omega \, r^2 \, t^{0r}$$

$\left(\begin{array}{l}\text{we are outside} \\ \text{the source,} \\ \text{so TT gauge}\end{array}\right)$ $= -\int d\Omega \, r^2 \frac{c^4}{32\pi G} \langle \underbrace{\partial^0 h_{ij}^{TT} \partial_r h_{ij}^{TT}}_{\text{here the position of the indices is due to } \eta_{\mu\nu}} \rangle$

Analogously to EM waves (anticipation of a result which will be demonstrated later on), if the wave is propagating radially and we are sufficiently far away from the source:

$$h_{ij}^{TT}(t,r) = \frac{1}{r} f_{ij}\left(t - \frac{r}{c}\right) \qquad \left[\begin{array}{l} t - \frac{r}{c} = t_{ret} \\ \text{the retarded time} \end{array}\right]$$

$$\partial_r \left(\frac{1}{r} f_{ij}\left(t - \frac{r}{c}\right) \right) = -\frac{1}{r^2} f_{ij}\left(t - \frac{r}{c}\right) + \frac{1}{r} \partial_r f_{ij}\left(t - \frac{r}{c}\right) =$$

$$= -\frac{1}{r^2} f_{ij}\left(t - \frac{r}{c}\right) - \frac{1}{r}\frac{1}{c} \partial_t f_{ij}\left(t - \frac{r}{c}\right)$$

$$\partial_r f\left(t - \frac{r}{c}\right) = \frac{\partial f}{\partial\left(t - \frac{r}{c}\right)} \frac{\partial\left(t - \frac{r}{c}\right)}{\partial r} = -\frac{1}{c} \frac{\partial f}{\partial\left(t - \frac{r}{c}\right)} = -\frac{1}{c} \partial_t f \frac{\partial t}{\partial\left(t - \frac{r}{c}\right)} =$$

$$= -\frac{1}{c} \partial_t f\left(t - \frac{r}{c}\right)$$

$$= +\partial^0 h_{ij}^{TT} + O\left(\frac{1}{r^2}\right)$$

In conclusion, we have found that $t^{00} = t^{0r}$:

$$\frac{dE_V}{dt} = -c \int d\Omega \, r^2 \frac{c^4}{32\pi G} \langle \partial^0 h_{ij}^{TT} \partial^0 h_{ij}^{TT} \rangle$$

the − sign tells us that this corresponds to the energy carried away from the surphace S_2

The energy flux per unit time and unit surface is then:

$$\boxed{\frac{dE}{dt\,dA} = \frac{c^3}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle}$$

$$= \frac{c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

It is also usefull to compute the energy spectrum.

Suppose we are in the situation in which $\langle \ \rangle$ denotes a time average. The spectrum is the energy per unit frequency. We therefore write:

$$E = \frac{c^3}{32\pi G} \int_S dA \int_{-\infty}^{+\infty} dt \, \langle \dot{h}_{ij}^{TT} \cdot \dot{h}_{ij}^{TT} \rangle_{time}$$

now the time average is just over a constant an can be ignored

we use now the Fourier expansion:

$$h_{ab}(t) = \int_{-\infty}^{+\infty} df \, \tilde{h}_{ab}(f) \, e^{-2\pi i f t}$$

$$= \int_{-\infty}^{+\infty} df \begin{pmatrix} \tilde{h}_+(f) & \tilde{h}_\times(f) \\ \tilde{h}_\times(f) & -\tilde{h}_+(f) \end{pmatrix}_{ab} e^{-2\pi i f t}$$

which gives (use condition on the delta)

$$E = \frac{\pi c^3}{4G} \int dA \int_{-\infty}^{+\infty} df \, f^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right)$$

writing $dA = r^2 d\Omega$, we get the ENERGY SPECTRUM

$$\boxed{\frac{dE}{df} = \frac{\pi c^3}{2G} f^2 r^2 \int d\Omega \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right)}$$

where we have defined it over the positive frequencies

$$\int_{-\infty}^{+\infty} df = 2 \int_0^{\infty} df$$

the flux of momentum can be computed analogously:

$$P_V^k = \frac{1}{c} \int_V d^3x \, t^{0k}$$

the momentum of the GW inside the spherical shell of volume V

$$\boxed{\frac{dP^k}{dt} = -\frac{c^3}{32\pi G} r^2 \int d\Omega \left\langle \dot{h}_{ij}^{TT} \partial^k h_{ij}^{TT} \right\rangle}$$

momentum flux per unit time.

we focus now on the high-frequency mode eq:

$$\boxed{R^{(1)}_{\mu\nu} = \left[-R^{(2)}_{\mu\nu}\right]^{\text{high}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)^{\text{high}}}$$

see page 41

From this equation we will derive

GW PROPAGATION IN CURVED SPACE TIME

•) First consider the case with no external matter $T_{\mu\nu} \equiv 0$

GWs are the only source of bck curvature in this case, but this is a non-linear phenomenon. At linear order we expect to find the propagation eq. in flat space-time that we are familiar with:

$$R^{(1)}_{\mu\nu} = \left[-R^{(2)}_{\mu\nu}\right]^{\text{high}}$$

$$R^{(1)}_{\mu\nu} \sim \partial^2 h \sim O\left(\frac{h}{\lambda\!\!\!^{-2}}\right) \qquad\qquad R^{(2)}_{\mu\nu} \sim (\partial h)^2 + h\,\partial^2 h \sim O\left(\frac{h}{\lambda\!\!\!^{-}}\right)^2$$

The $R^{(2)}_{\mu\nu}$ term is one order higher in h, and we know that in the absence of matter, the two expansions $h \ll 1$ and $\frac{\lambda\!\!\!^{-}}{L_B} \ll 1$ become equivalent since $h \sim \frac{\lambda\!\!\!^{-}}{L_B}$. So as obvious, we can neglect the second order term in h and we find:

$$\boxed{R^{(1)}_{\mu\nu} \simeq 0} \longleftrightarrow \eta^{\rho\sigma}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\nu\partial_\mu h_{\rho\sigma} - \partial_\rho\partial_\sigma h_{\mu\nu}\right) \simeq 0$$

$$\longleftrightarrow \Box \bar{h}_{\mu\nu} \simeq 0 \quad \text{in the Lorentz gauge.}$$

In the limit in which GWs are the only source of curvature, stopping at $O(h)$, we fall again on the result of

$[R^{(2)}_{\mu\nu}]^{high}$ contains the self-interaction of the GW with itself ⊛

GW generate non-linear corrections to themselves, and to see this more consistently one should have split

$$R^{(2)}_{\mu\nu} = R^{(2)}_{\mu\nu}(j_{\mu\nu}) + R^{(2)}_{\mu\nu}(h_{\mu\nu}) \qquad \text{in page } 40$$

so that one would have got a further piece in Einst. eqs.

$$R^{(2)}(j_{\mu\nu}) + [R^{(2)}_{\mu\nu}(h_{\mu\nu})]^{high} = 0$$

$j_{\mu\nu}$: second order metric perturbation representing the non-linear correction to ~~the~~ the metric due to $(h_{\mu\nu})^2$.

⊛ like wave-wave scattering and scattering off the curvature the waves themselves produce

•) If external matter is present, $T_{\mu\nu} \neq 0$:

In this case, the effect of the LOW FREQUENCY / LONG WAVELENGTH PART $\langle T_{\mu\nu} \rangle = [T_{\mu\nu}]^{low}$ is to generate a background curvature in $\bar{g}_{\mu\nu}$, such that $\bar{R}_{\mu\nu} \sim \mathcal{O}\left(\frac{1}{L_B^2}\right)$

The GWs then propagate on a background which is not Minkowsky : we expect to find a propagation eq. that takes into account the smooth curvature of the bckg.

$$R^{(1)}_{\mu\nu} = [-R^{(2)}_{\mu\nu}]^{high} + \underbrace{\frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right]^{high}}$$

$R^{(1)}_{\mu\nu} \downarrow \mathcal{O}\left(\frac{h}{\lambda}\right)$, $\quad [-R^{(2)}_{\mu\nu}]^{high} \downarrow \mathcal{O}\left(\frac{h^2}{\lambda}\right)$

Between these two terms, we can neglect $[-R^{(2)}_{\mu\nu}]^{high}$ as done before, because it is ONE ORDER HIGHER IN THE h - EXPANSION, and we have already accounted for the bck curvature due to the low-frequency part $[-R^{(2)}_{\mu\nu}]^{low}$ in the previous analysis

Here we are dealing only with the high frequency/ short wavelength part of the en.-mom. tensor of the matter, we already accounted for $\langle T_{\mu\nu} \rangle$ and its effect on the bck : therefore, this part must be AT BEST OF ORDER h, OR HIGHER.

This part can in principle act as a SOURCE of GWs, i.e. GENERATE $R^{(1)}_{\mu\nu}$.

For example, the above equation could describe a GW source operating in the early universe, where the background curvature is due to the smooth distribution of matter, while small tensor-type perturbations (anisotropic stresses) on this smooth distribution can act as a source of GWs. However, this is not the situation of interest here. Here we want to show how the above equation can describe the PROPAGATION OF GWs ON A CURVED BACKGROUND.

We therefore study the limit in which we are outside the matter sources (or in the early universe case, we study the limit after GW generation).

We therefore discard the term $\left[T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} T\right]^{\text{high}}$ in the above equation and rewrite it simply:

$$\boxed{R^{(1)}_{\mu\nu} \simeq 0}$$

on the curved background given by $\bar{g}_{\mu\nu}$

From the expression of $R^{(1)}_{\mu\nu}$ on page 41/2 one gets:

$$\bar{g}^{\rho\sigma}\left(\bar{\nabla}_\rho \bar{\nabla}_\nu h_{\mu\sigma} + \bar{\nabla}_\rho \bar{\nabla}_\mu h_{\nu\sigma} - \bar{\nabla}_\nu \bar{\nabla}_\mu h_{\rho\sigma} - \bar{\nabla}_\rho \bar{\nabla}_\sigma h_{\mu\nu}\right) = 0$$

(note that here we don't substitute $\bar{\nabla}_\mu \to \partial_\mu$ because we want to describe the propagation on $\bar{g}_{\mu\nu}$)

One can introduce the trace-reversed metric

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} h$$

and impose the Lorentz gauge condition

$$\boxed{\bar{\nabla}^\nu \bar{h}_{\mu\nu} = 0}$$

this can be done in exact analogy with the flat-background case: see e.g. MISNER, THORNE, WHEELER CHAPTER 35.14 and M. MAGGIORE, PROBLEM 1.2

Using the commutation rule for the covariant derivative:

$$\bar{\nabla}_\rho \bar{\nabla}_\nu \bar{h}_{\mu\sigma} = \bar{\nabla}_\nu \bar{\nabla}_\rho \bar{h}_{\mu\sigma} - \bar{R}_{\mu\alpha\nu\rho} \bar{h}^{\alpha}{}_{\sigma} - \bar{R}_{\sigma\alpha\nu\rho} \bar{h}_{\mu}{}^{\alpha}$$

one can rewrite the equation as (*)

$$\boxed{\bar{\nabla}^\sigma \bar{\nabla}_\sigma \bar{h}_{\mu\nu} + 2 \bar{R}_{\mu\alpha\nu\rho} \bar{h}^{\alpha\rho} - \bar{R}_{\alpha\mu} \bar{h}^{\alpha}{}_{\nu} - \bar{R}_{\alpha\nu} \bar{h}^{\alpha}{}_{\mu} = 0}$$

The term $\bar{\nabla}^\sigma \bar{\nabla}_\sigma \bar{h}_{\mu\nu}$ is of $O\left(\frac{h}{\lambda^2}\right)$, while the following terms are of order $O\left(\frac{h}{L_B{}^2}\right)$. They can therefore be neglected in the limit $\lambda \ll L_B$.

(NOTE: here a stroke on the metric perturbation $\bar{h}_{\mu\nu}$, meaning the trace-reversed metric, should not be confused with the stroke denoting background quantities, as the one appearing on $\bar{g}_{\mu\nu}$, $\bar{R}_{\mu\nu}$ ...)

(*) Can be used to describe phenomena such as gravitational redshift of GWs and deflection of GWs by a gravitational field (lensing) etc.

We find then, at lowest order in $\frac{\lambda\!\!\!^{-}}{L_B}$:

$$\boxed{\bar{\nabla}^{\sigma} \bar{\nabla}_{\sigma} \bar{h}_{\mu\nu} = 0}$$

this equation determines the propagation of GWs in a curved background in the limit $\lambda\!\!\!^{-} \ll L_B$.

It can be solved in the EIKONAL APPROXIMATION, analogously to what is done in the context of GEOMETRIC OPTICS for electromagnetic waves.

We consider then a solution of the above equation characterised by a rapidly varying phase Θ (which changes on the length-scale $\lambda\!\!\!^{-}$) and a slowly varying amplitude, changing on a scale L_c such that

$$L_B \gtrsim L_c \gg \lambda\!\!\!^{-}$$

Ansatz

$$\boxed{\bar{h}_{\mu\nu} = \left[A_{\mu\nu} + \varepsilon B_{\mu\nu} + \varepsilon^2 C_{\mu\nu} + \dots \right] e^{\frac{i\Theta}{\varepsilon}}}$$

where $\varepsilon \sim \frac{\lambda\!\!\!^{-}}{(L_B, L_c)}$ is a dummy parameter just to keep track of the expansion.

We now substitute the above Ansatz into equations

$$\begin{cases} \nabla^{\nu} \bar{h}_{\mu\nu} = 0 \\ \nabla^{\sigma} \nabla_{\sigma} \bar{h}_{\mu\nu} = 0 \end{cases}$$

and equating the terms order by order in ε one finds the following facts (similar to EM waves):

defining : $k_\alpha = \partial_\alpha \theta$ the wave vector

$$\mathcal{a} = \left(\frac{1}{2} A^*_{\mu\nu} A_{\mu\nu}\right)^{\frac{1}{2}}$$ the amplitude

$$e_{\mu\nu} = \frac{A_{\mu\nu}}{\mathcal{a}}$$ the polarization

1) The rays, i.e. the curves orthogonal to the surfaces of constant phase of which k_α is the tangent vector are NULL GEODESICS :

$$\begin{cases} k_\alpha k^\alpha = 0 \\ k^\beta (\nabla_\beta k_\alpha) = 0 \end{cases}$$

2) The polarization $e_{\mu\nu}$ is orthogonal to the rays and parallel transported along them :

$$\begin{cases} e_{\mu\alpha} k^\alpha = 0 \\ k^\alpha \nabla_\alpha (e_{\mu\nu}) = 0 \end{cases}$$

3) the amplitude decreases as the rays diverge according to :

$$\nabla_\alpha (\mathcal{a}^2 k^\alpha) = 0$$

this is a conservation law (it corresponds to a divergence put to zero, i.e. by Stokes theorem to a quantity that is conserved over a volume) : it represents in quantum language the CONSERVATION OF THE NUMBER OF GRAVITONS

•) Therefore, GW propagation in the Eikonal approximation satisfies the same rules as Electromagnetic waves in geometric optics: GW travel along null geodesics with slowly changing amplitude due to the coupling with the background curvature.

•) This is no longer true when $\lambda \simeq L_B$: in this case there might be strong coupling among the background and the perturbation.

•) One example of the above situation is inflation: tensor metric perturbations (one cannot call them gravitational waves) with $\lambda \simeq L_B$ are amplified during inflation: the above discussion does not apply and the number of graviton is not conserved.
In this case, how can one distinguish the background $\bar{g}_{\mu\nu}$ from the perturbation $h_{\mu\nu}$, given that $\lambda \simeq L_B$? It is possible because the background is HOMOGENEOUS and ISOTROPIC (given by the FRW metric) and so we know how to distinguish a tensor perturbation over it once we have chosen the proper reference frame (the one of comoving observers, who see the background homogeneous and isotropic).

CHIARA CAPRINI

DRAFT LECTURE NOTES ON "GRAVITATIONAL WAVES"

lecture # 2

From the last discussion, it is clear that the TT gauge exhibits the fact that GW have only two polarisation components, but that this fact is true in general.

In particular let's demonstrate that this holds also when $T_{\mu\nu} \neq 0$, i.e. "inside the source" (where the TT gauge cannot be fixed)

(discussion in section 2.2 of Flanagan & Hughes gr-qc/0501041)

[This actually holds also when the background space-time is not Minkowski, as is the case in a Friedmann universe: we will make an analogy at the end. In this case, one is always "inside the source"]

GRAVITATIONAL WAVES IN LINEARIZED THEORY WITH NON-ZERO $T_{\mu\nu}$

$h_{\mu\nu}$ contains in general
1) gauge degrees of freedom
2) physical degrees of freedom which are non-radiating
3) gravitational waves

In order to make these degrees of freedom apparent and recognize them one needs to
1) split the metric into irreducible components under rotations
2) construct gauge invariant variables

by doing so, one obtains that

ONLY THE TT PART OF THE METRIC (which is automatically gauge invariant) OBEYS A WAVE EQUATION IN ALL GAUGES.

the other physical degrees of freedom obey Poisson-like equations. and are therefore non-radiative.

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $h_{\mu\nu}$ decomposed as

$(c = G = 1)$

$$h_{00} = 2\phi$$

$$h_{0i} = \beta_i + \partial_i \gamma$$

$$h_{ij} = h_{ij}^{TT} + \frac{1}{3} H \delta_{ij} + \partial_{(i}\varepsilon_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right)\lambda$$

which satisfy the conditions

$$\begin{cases} \partial_i \beta_i = 0 \\ \partial_i \varepsilon_i = 0 \\ \partial_i h_{ij}^{TT} = 0 \\ \delta_{ij} h_{ij}^{TT} = 0 \end{cases}$$

- ϕ, $H = \delta_{ij} h_{ij}$, γ, λ are scalar components
- β_i, ε_i are vector components
- h_{ij}^{TT} is the tensor component (the TT piece which will give GW)

16 free functions uniquely determined under boundary conditions ($\gamma \to 0$, $\varepsilon_i \to 0$, $\lambda \to 0$, $\Delta\gamma \to 0$)

\- 6 constraints gives

10 independent variables (symmetric 4x4 matrix)

the gauge symmetry of the theory as we have seen are INFINITESIMAL COORDINATE TRANSFORMATIONS (12/6)

(infinitesimal because they have to leave the background metric invariant, These can be seen as the family of diffeomorphisms generated by the vector field ξ, which are infinitesimal such that they deviate from the identity only at first order and are the flow of ξ : $\phi_* = \mathbb{1} + \varepsilon L_\xi + O(\varepsilon^2)$ L_ξ lie derivative in ξ dir.)

(see Durrer chapter 2 and Carrol chapter 7) $\Longrightarrow$ 12/6/bis

---

These can be then parametrised via the vector field ξ with

$$\xi^M = (A, B^i + \partial^i C)$$

with $\partial_i B^i = 0$

$\left(\begin{array}{l} C \text{ scalar} \\ B^i \text{ vector} \end{array}\right)$

the metric transforms as $h_{\mu\nu} \to h_{\mu\nu} - 2\,\partial_{(\mu}\xi_{\nu)}$ which gives:

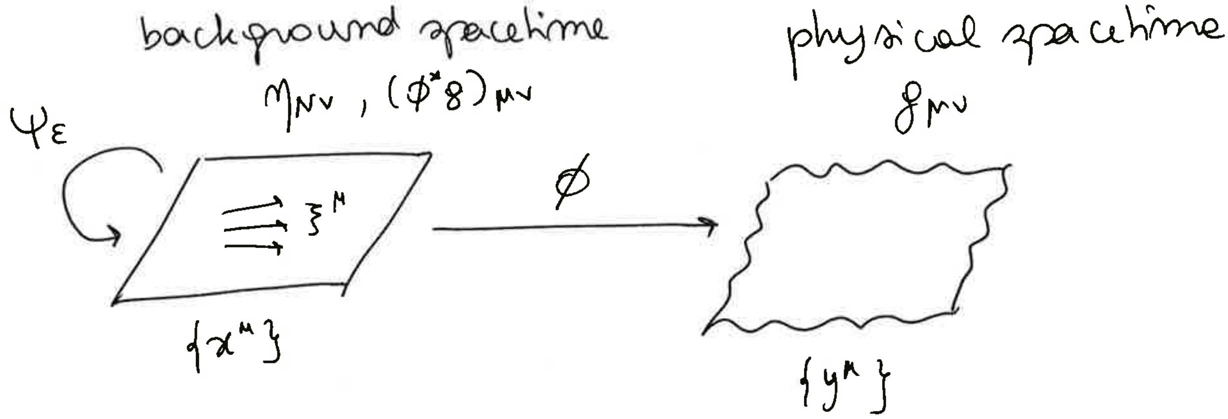
$$\phi \to \phi - \dot{A}$$
$$\beta_i \to \beta_i - \dot{B}_i$$
$$\gamma \to \gamma - A - \dot{C}$$
$$H \to H - 2\Delta C$$
$$\lambda \to \lambda - 2C$$
$$\varepsilon_i \to \varepsilon_i - 2B_i$$
$$h_{ij}^{TT} \to h_{ij}^{TT} \Longrightarrow$$

automatically gauge invariant

since there are not tensor type gauge transformation

Another way to understand why infinitesimal coordinate transformations are a symmetry of the linearised theory (12/6/bis)

(can be generalised easily to the cosmological case: perturb. around curved bckground)



diffeomorphism ϕ allows to move tensors back and forth in the two spacetimes: we need this since we want to write the theory in the background spacetime.

We then define:
$$h_{\mu\nu} = \underbrace{(\phi^* g)_{\mu\nu}}_{\text{the pull-back of the physical metric}} - \eta_{\mu\nu}$$

$$(\phi^* g)(\underbrace{X, Y}_{\text{tg space of bck}}) = g(\underbrace{T_p\phi X, T_p\phi Y}_{\text{tg space of physical}})$$

If the gravitational field in the physical spacetime is weak, then for some diffeomorphism (we choose it) $|h_{\mu\nu}| \ll 1$; but in general there is more than one $\Rightarrow$ this is the gauge freedom: they all lead to a description of the same physical spacetime and are all allowed since they satisfy $|h_{\mu\nu}| \ll 1$

12/6/bis

Suppose that we have a vector field on the background spacetime that generates a 1-parameter family of diffeomorphism ψ_ε ~~~~ One will have that also $|h^{(\varepsilon)}_{\mu\nu}| \ll 1$ with: (ξ vector field)

flow along the integral curves of ξ

$$h^{(\varepsilon)}_{\mu\nu} = [(\phi \circ \psi_\varepsilon)^* g]_{\mu\nu} - \eta_{\mu\nu}$$

$$= [\psi_\varepsilon^* (\phi^* g)]_{\mu\nu} - \eta_{\mu\nu}$$

$$= \psi_\varepsilon^* (h + \eta)_{\mu\nu} - \eta_{\mu\nu}$$

$$= \psi_\varepsilon^* (h_{\mu\nu}) + \psi_\varepsilon^* (\eta_{\mu\nu}) - \eta_{\mu\nu}$$

FIRST ORDER IN ε

$$= h_{\mu\nu} + \varepsilon L_\xi \eta_{\mu\nu} = h_{\mu\nu} + 2\varepsilon \nabla_{(\mu} \xi_{\nu)} =$$

$$\left(\lim_{\varepsilon \to 0} \frac{\psi_\varepsilon^* [T(\psi_\varepsilon(p))] - T(p)}{\varepsilon} \right.$$ change of a tensor along the flow)

$$= h_{\mu\nu} + 2\varepsilon \partial_{(\mu} \xi_{\nu)}$$

This is how the metric $h_{\mu\nu}$ changes under the infinitesimal diffeomorphism caused by the vector field $\varepsilon \xi^\mu$.

It is another way to see why the linearized theory admits infinitesimal coordinate transformations as symmetry $\rightarrow$ the same physical spacetime.

The perturbation of an arbitrary tensor field $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$ obeys: $\delta T_{\mu\nu} \rightarrow \delta T_{\mu\nu} + \varepsilon L_\xi \bar{T}_{\mu\nu}$

and therefore $T_{\mu\nu}$ is gauge invariant iff $\bar{T}_{\mu\nu} \equiv 0$ (STEWART - WALKER lemma
Durrer par. 2.2)

From the above gauge transformations one can construct GAUGE INVARIANT VARIABLES (12/7)

$$\phi \equiv -\varphi + \dot{\gamma} - \frac{1}{2}\ddot{\lambda}$$

$$\theta \equiv \frac{1}{3}(H - \Delta\lambda)$$

$$C_i \equiv \beta_i - \frac{1}{2}\dot{\varepsilon}_i \qquad \text{with } \partial_i C_i = 0$$

there are in total physical degrees of freedom.

$$\underbrace{1+1}_{\text{scalar}} + \underbrace{(3-1)}_{\substack{\text{vector} \\ \text{(div. less)}}} + \underbrace{(6-1-3)}_{\text{tensor TT}} = 6$$

We have therefore eliminated the 4 gauge dof. from the 10 initial ones.

In order to write Einstein equations we first decompose also the ENERGY MOMENTUM TENSOR

$$\begin{cases} T_{00} = \rho & \text{(mass)} \\ T_{0i} = S_i + \partial_i S & \text{(angular and linear momentum)} \\ T_{ij} = P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right)\sigma & \text{(anisotropic stress)} \end{cases}$$

with constraints

$$\begin{cases} \partial_i S_i = 0 \\ \partial_i \sigma_i = 0 \\ \partial_i \sigma_{ij} = 0 \\ \delta^{ij}\sigma_{ij} = 0 \end{cases}$$

note that this quantity ($T_{\mu\nu}$) vanishes in the background and therefore it is gauge invariant automatically (STEWART-WALKER lemma - DURRER chapter 2) however here the constraints are given by energy momentum conservation $\partial_\mu T^{\mu\nu} = 0$:

$$\begin{cases} \Delta S = \dot{\rho} \\ \Delta \sigma = -\frac{3}{2} P + \frac{3}{2} \dot{S} \\ \Delta \sigma_i = 2 \dot{S}_i \end{cases}$$

The Einstein tensor can be computed in terms of the G.I. variables :

$(c = G = 1)$

$$\begin{cases} G_{00} = -\Delta \Theta \\ G_{0i} = -\frac{1}{2} \Delta \Xi_i - \partial_i \dot{\Theta} \\ G_{ij} = -\frac{1}{2} \Box h_{ij}^{TT} - \partial_{(i} \Xi_{j)} - \frac{1}{2} \partial_i \partial_j (2\Phi + \Theta) \\ \qquad + \delta_{ij} \left[\frac{1}{2} \Delta (2\Phi + \Theta) - \ddot{\Theta} \right] \end{cases}$$

Equating the components properly and using the constraint above one gets in the end :

$$\begin{cases} \Delta \Theta = -8\pi \rho \\ \Delta \Phi = 4\pi (\rho + 3P - 3\dot{S}) \\ \Delta \Xi_i = -16\pi S_i \\ \Box h_{ij}^{TT} = -16\pi \sigma_{ij} \end{cases}$$

POISSON-LIKE EQUATIONS (for the first three)

$\Longrightarrow$ WAVE EQUATION

therefore only the TT metric components are radiative : this is exhibited by the TT gauge but it is true in general as demonstrated by this gauge invariant formalism.

IN THE CASE OF COSMOLOGICAL PERTURBATION THEORY:

the observed structure in the Universe (galaxies, clusters...) have grown out of small initial perturbations (seen in the CMB). To study the evolution of the perturbations from their generation to the time when they become of order unity, one has developed cosmological perturbation theory on the FRW background. This is not linearised gravity in the sense that the background is not $\eta_{\mu\nu}$, but it is still a theory of perturbations, that grow under gravitational instability.

DURRER, chapter 2

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \text{"}\varepsilon\text{"} a^2 h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

FRW: $\bar{g}_{00} = -a^2$

$\bar{g}_{ij} = a^2 \gamma_{ij}$

In this case, there is a energy momentum tensor also in the background, which is a perfect fluid

$$T^{\mu}_{\ \nu} = \bar{T}^{\mu}_{\ \nu} + \Theta^{\mu}_{\ \nu} \qquad \frac{|\Theta^{\mu}_{\ \nu}|}{\bar{\rho}} \ll 1$$

$\bar{T}^{0}_{\ 0} = -\bar{\rho}$

$\bar{T}^{i}_{\ j} = \bar{P}\, \delta^{i}_{\ j}$

Since the background is fixed, this theory also has infinitesimal coordinate transformations as symmetries therefore the perturbations are not unique.

So in general one needs to build gauge-inv. variables to understand the physical d.o.f. (12/10)

(BARDEEN 1980)

Since the hypersurfaces of constant times are homogeneous and isotropic, one can decompose tensors irreducibly under rotations as done in the previous case; moreover, in general one also decomposes into eigenfunctions of the Laplacian (harmonic analysis) which are irreducible components of translations.

(each mode $\underline{k}$ evolves independently)

except that one must define G.I. variables also for the matter

Pretty much everything goes through as before: in particular, one finds GAUGE INVARIANT TENSOR PERTURBATIONS that obey the wave equation:

(damped)

$$\ddot{h}_{ij}^{TT}(\underline{k},t) + 2\mathcal{H}\,\dot{h}_{ij}^{TT}(\underline{k},t) + (2K+k^2)\,h_{ij}^{TT}(\underline{k},t) = 8\pi G a^2\,\Pi_{ij}^{TT}(\underline{k},t)$$

expansion

spatial curvature

tensor anisotropic stress σ_{ij}

these are the GWs in this case, once they enter the horizon

In this case the fact that the energy mom. tensor is not zero in the background leads to a difference: one finds also a wave-like eq. for Φ which are SOUND WAVES sourced by the matter-radiation perturbations if $c_s^2 \neq 0$

BARDEEN EQUATION

12/10 bis

In the case of adiabatic perturbations of a perfect fluid (like matter or radiation) so that

$$\begin{cases} \sigma = 0 \\ c_s^2 = \dfrac{\delta P}{\delta \rho} \end{cases} \qquad w = \frac{\bar{P}}{\bar{\rho}}$$

there is only one scalar dynamical degree of freedom satisfying:

$$\ddot{\Phi} + 3\mathcal{H}(1+c_s^2)\dot{\Phi} + \Big[(1+3c_s^2)(\mathcal{H}^2 - K) - (1+3w)(\mathcal{H}^2 + K) + \underbrace{k^2 c_s^2}_{\text{sound-waves}}\Big]\Phi = 0$$

INTERACTION OF GWs WITH TEST MASSES

* how a set of test masses respond to a GW passing through? (Idealized version of a detector)
* the physics of this phenomenon must be independent on the choice of the reference frame (on the choice of coordinate) BUT the way one uses to describe the GW and the detector DO depend on the choice of the reference frame: for example, we have seen how to describe the wave in TT gauge.
* Which reference frame corresponds to the TT gauge? Can we describe the detector in this frame, for which the propagating GW is so simple?

* In GR, the physical effects are not expressed by what happens to the COORDINATES, because the theory is INVARIANT UNDER COORDINATE CHANGE. So to get the physical effects, for example in this case, one must look e.g. at proper distances, or proper time.
* however $\boxed{\text{to see}}$ what happens to a set of test masses we cannot avoid to pick a reference frame, which needs to be the one of the detector

We have seen in the reminding of GR that by the equivalence principle, it is always possible to define a local inertial system at a given point P of spacetime, such that
$$\begin{cases} g_{\mu\nu}(P) = \eta_{\mu\nu} \\ g_{\mu\nu,\lambda}(P) = 0 \end{cases}$$

- spacetime is locally flat and SR is valid
- the gravitat. field can be eliminated locally.
- curvature effects will only appear at second order

$$g_{\mu\nu} = g_{\mu\nu}(P) + \frac{1}{2} g_{\mu\nu,\alpha\beta}(x^\alpha - x^\alpha_P)(x^\beta - x^\beta_P)$$

Here we will see how to construct such a reference frame, and also that this is actually valid all along a geodesic.

From this, we will then understand better the TT gauge, and define also the PROPER DETECTOR GAUGE to describe GWs.

In the reminding of GR we have seen the concept of geodesic : a curve $\gamma(\lambda)$ is a geodesic if its tangent vector is parallel transported along the curve : $\nabla_{\dot\gamma} \dot\gamma = 0$ and in coordinates:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\sigma} \frac{dx^\alpha}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

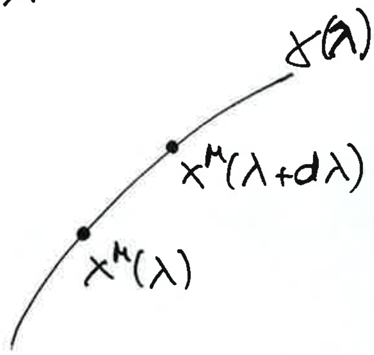
This is the equation of motion of a test mass in the curved background described by the metric $g_{\mu\nu}$ (of which $\Gamma^\mu_{\alpha\sigma}$ are the Christoffel symbols) IN THE ABSENCE OF EXTERNAL NON-GRAVITATIONAL FORCES

This fact can be seen from this derivation of the geodesic equation:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda^2$$

interval between two points on a curve parametrized by λ and separated by $d\lambda$: PROPER DISTANCE

$x^\mu(\lambda)$ are the coordinates along the curve



if the curve is time-like : $ds^2 < 0$

and the $\boxed{\text{proper time}}$ is defined by :

$$c^2 d\tau^2 \equiv -ds^2 = -g_{\mu\nu}\, dx^\mu dx^\nu$$

this is the time measured by a clock carried along $\gamma(\lambda)$ and therefore can be used as the affine parameter.

One defines also the $\boxed{\text{four-velocity}}$

parallel transported along geodesic ← $$u^\mu = \frac{dx^\mu}{d\tau}$$

tangent vector to the curve if parametrized by the proper time.

$$u_\mu u^\mu = -c^2$$

Just as in SR, one can now find the trajectory of a point-like test mass m by extremizing the action — among two fixed boundary conditions x_A^μ, x_B^μ

$$S = -m \int_{\tau_A}^{\tau_B} d\tau$$

the extremization of this action gives the geodesic equation in terms of the proper time τ. (see exercises)

Another derivation of the geodesic equation can be done using the equivalence principle: since with respect to an inertial coordinate system the laws of physics are valid in their special relativistic form, this gives us a rule to find laws in GR:

1) to the quantities already present in SR, we can add in order to formulate a physical law, only the metric and its derivatives. Furthermore, wherever there is $\eta_{\mu\nu} \to g_{\mu\nu}$

2) the laws must be generically COVARIANT and reduce to their SR form in a local inertial system.

let's apply these two principles to find the geodesic eq: start from Newtonian theory of a test mass with no forces acting on it: (made covariant)

$$\frac{d^2 x^\mu}{d\tau^2} = 0$$

Attempt:

$$\left.\frac{d^2 x^\mu}{d\tau^2}\right|_p + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \Rightarrow$$

we can do this, because $\Gamma^\mu_{\alpha\beta}(p) = 0$ so we get back the SR form of the eq.

Now, this is covariant, so it is the good equation!

(For Electromagnetism, see Straumann chap 2) we have also the **EQUATION OF GEODESIC DEVIATION:**

take two nearby geodesics (parametrized by τ), $x^\mu(\tau)$ and $x^\mu(\tau)+\xi^\mu(\tau)$.

$$\begin{cases} \dfrac{d^2(x^\mu+\xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x+\xi)\dfrac{d(x^\nu+\xi^\nu)}{d\tau}\dfrac{d(x^\rho+\xi^\rho)}{d\tau} = 0 \\ \dfrac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x)\dfrac{dx^\nu}{d\tau}\dfrac{dx^\rho}{d\tau} = 0 \end{cases}$$

$\xi^\mu(\tau)$ must be much smaller than the typical scale of variation of the grav. field

the difference of these two eqs to first order in ξ becomes:

$$\left[\Gamma^\mu_{\nu\rho}(x+\xi) = \Gamma^\mu_{\nu\rho}(x) + \partial_\sigma\Gamma^\mu_{\nu\rho}(x)\,\xi^\sigma + O(\xi^2)\right]$$

$$\boxed{\frac{d^2\xi^\mu}{d\tau^2} + 2\,\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0}$$

Using the definition of the Riemann tensor and the expression of the covariant derivative along the geodesic $x^\mu(\tau)$ of $\xi^\mu(\tau)$, one finds in the end:

$$\boxed{\underbrace{\left[\nabla_{\dot\gamma}\left(\nabla_{\dot\gamma}\xi\right)\right]^\beta}_{\text{twice the covariant derivative of } \xi^\mu(\tau) \text{ along } \gamma(\tau)=x^\mu(\tau)} = -\underbrace{R^\beta{}_{\nu\rho\sigma}\,\xi^\rho\,\dot x^\nu\,\dot x^\sigma}_{\text{The Riemann tensor expresses the TIDAL GRAVITATIONAL FORCE experienced by two close-by geodesics.}}}$$

$\dot x^\alpha\left(\nabla_\alpha\left(\dot x^\gamma\nabla_\gamma\xi^\beta\right)\right)$ in components

this means that two nearby freely-falling objects still experience a force, which cannot be transformed away (like the field strength, i.e. the

Christoffel symbols) which is due to the inhomogeneities of the gravitational field

We see now how to construct a LOCAL INERTIAL FRAME and a FREELY FALLING FRAME practically.

LOCAL INERTIAL FRAME: at a given point P of space-time,

$$\Gamma^{\mu}_{\nu\rho}(P) = 0 \quad \Longleftrightarrow \quad \left.\frac{d^2x^\mu}{d\tau^2}\right|_P = 0 \quad \left\{ \begin{array}{l} \text{a test mass moves} \\ \text{freely in } P \end{array} \right.$$

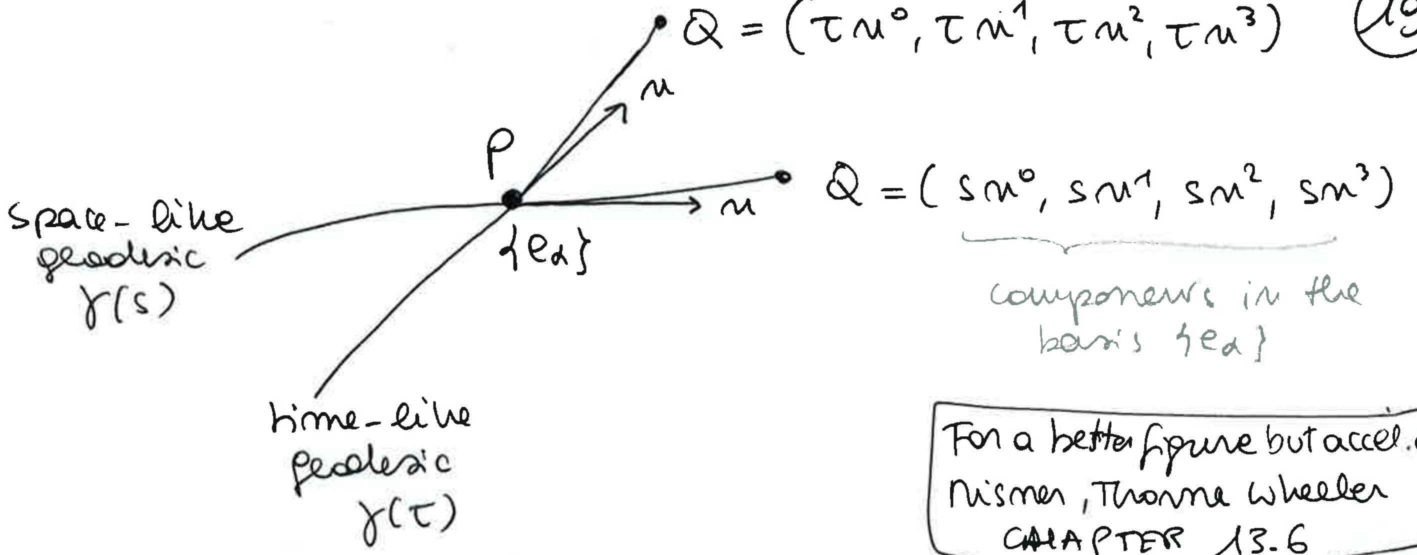
In P, define a basis of 4 orthonormal vectors $\{e^\alpha\}$ with $\eta_{\mu\nu}\, e^\mu_\alpha\, e^\nu_\beta = \eta_{\alpha\beta}$ $\Rightarrow$ we have therefore $g_{\mu\nu}(P) = \eta_{\mu\nu}$ in this coord. system.

From P, send out geodesics, both space-like and time-like, parameterized by proper distance and proper time. Call $\hat{n}$ the four-vector tangent to these geodesics, and call the coordinates of the points on these geodesics as:

proper length

$$\left.\begin{array}{ll} x_Q = (s n^0, s n^1, s n^2, s n^3) & \text{space-like} \\ x_Q = (\tau n^0, \tau n^1, \tau n^2, \tau n^3) & \text{time-like} \end{array}\right\} \begin{array}{l} \text{RIEMANN} \\ \text{NORMAL} \\ \text{COORDINATES} \end{array}$$

Fill all space with geodesics; in the vicinity of P, these do not intersect and therefore we have coordinates for each point of space time in the vicinity of P: this coordinate system gives a realization of a local inertial frame.



For a better figure but accel. obs: Misner, Thorne Wheeler CHAPTER 13.6

the geodesic equation for both these geodesics shows that $\Gamma^\mu_{\alpha\beta}(P) = 0$:

this must be true for each n^α

$$0 = \underbrace{\left.\frac{d^2 x^\mu}{d\tau^2}\right|_P}_{\text{linear in } \tau \text{ by choice}} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 + \Gamma^\mu_{\alpha\beta}\, n^\alpha n^\beta = 0$$

(if we had chosen a space-like geodesic, here τ should be replaced by s)

So we have constructed a local inertial frame at the point P.

Now we want to construct a frame for which a test mass moves freely all along a geodesic : this is a

FREELY FALLING FRAME $\Rightarrow$ masses of eLISA

for which we have $\Gamma^\mu_{\alpha\beta} = 0$ all along the geodesic.

The equation of motion for a $\boxed{\text{gyroscope}}$ along a geodesic (a freely spinning object) is simply

in this case, the intrinsic angular momentum of the turn of the object

$$\boxed{\frac{d\, S^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} S^\nu \frac{dx^\rho}{d\tau} = 0} \iff \nabla_{\dot\gamma} S = 0$$

S^μ is the spin 4-vector. (in the spin rest frame it reduces to $(0, \underline{S})$)

therefore a gyroscope along a geodesic is parallel transported.

This is simply the covariant generalization of the equation of motion $\frac{dS^\mu}{d\tau} = 0$ in flat space-time, that expresses the CONSERVATION OF ANGULAR MOMENTUM: the gyroscope does not precess.

Note that, if we were not along a geodesics, meaning if the reference frame would be accelerated, one would have: (with notation $\dot{\gamma} = u$)

$$\nabla_u S = \langle S, a \rangle u \qquad a = \nabla_u u \left(= u^\alpha \nabla_\alpha u^\beta\right)$$

$$\left(u^\alpha \nabla_\alpha S^\beta = S^\alpha a_\alpha u^\beta\right) \qquad \text{acceleration.}$$

So a gyroscope precesses in an accelerated reference frame.

(this observer is not rotating since it Fermi transport its axis / it is not accelerated since it follows a geodesic)

Now, let's construct the same coordinate system as before, but now use three gyroscopes to mark the directions of the spatial axes. While propagating the reference frame along the time-like geodesic, we always orient the spatial axes in the direction marked by the gyroscopes: therefore, they do not rotate along the timelike geodesic by definition. So their equation is

$$\frac{dS^\mu}{d\tau} = 0 \quad \longleftrightarrow \quad \Gamma^\mu_{\alpha\beta} \equiv 0 \quad \text{all along the geodesic}$$

These axes are called FERMI-TRANSPORTED and the coordinates FERMI NORMAL COORDINATES (see Schaumann Chap. 2.10)

Now we can analyse the TT GAUGE : we call the TT - frame the reference frame corresponding to the TT gauge and see what it means. (21)

Start with a test mass at rest at $\tau = 0$: $\left.\frac{dx^i}{d\tau}\right|_{\tau=0} = 0$

We write the geod. eq. in the frame of interest:

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = \left[-\Gamma^i_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right]_{\tau=0} = \left[-\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2\right]_{\tau=0}$$

Now for a linearised theory $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ the Christoffel symbols are:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}\left(\partial_\nu h_\rho{}^\mu + \partial_\rho h_\nu{}^\mu - \partial^\mu h_{\nu\rho}\right)$$

NOTE: this is only true at lowest order

$$\Gamma^i_{00} = \frac{1}{2}\left(2\partial_0 h^i{}_0 - \partial^i h_{00}\right) \equiv 0 \quad \text{since } h^{TT}_{00} = h^{TT}_{0i} = 0$$

therefore also $\boxed{\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} \equiv 0}$ therefore the mass remains at rest.

(at lowest order in $h_{\mu\nu}$)

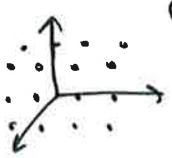
IN THE TT FRAME, MASSES WHICH WERE AT REST BEFORE THE ARRIVAL OF THE WAVE REMAIN AT REST ALSO AFTER ITS ARRIVAL.

Attention: this does not mean that the passing GW has no physical effect: it only means that in the TT gauge, the coordinates are chosen in a way such that they stretch themselves when the GW passes through so that the position of free test masses initially at rest remains the same.

[This is why, if we want to see the effect of a GW on a set of test masses, we need to choose an appropriate frame!]

PHYSICAL IMPLEMENTATION

mark the co-ordinates using



a network of test masses at rest } in (1,0,0) (0,1,0) (0,0,1) (1,1,0) .... also when the wave is passing by.

the axes stretch with them and the above values remain the same also while the wave is passing: freely-falling masses

The same can be seen using the equation for the geodesic deviation: one mass has coordinates x^μ and the other coordinates $x^\mu + \xi^\mu$. Therefore the spatial separation:

$$\left.\frac{d^2\xi^i}{d\tau^2}\right|_{\tau=0} = -\Big[\underbrace{2c\Gamma^i_{0\rho}\frac{d\xi^\rho}{d\tau}}_{\text{the only non-zero term is}} + \underbrace{c^2\xi^\sigma\partial_\sigma\Gamma^i_{00}}_{\text{this is zero}}\Big]_{\tau=0} \qquad \left(\frac{dx^0}{d\tau}=c\right)$$

$$\Gamma^i_{0j} = \frac{1}{2}\partial_0 h_{ij} \qquad\qquad \partial_\sigma\left(\partial_0 h_{0i} - \frac{1}{2}\partial_i h_{00}\right) = 0$$

if $\left.\frac{d\xi^i}{d\tau}\right|_{\tau=0} = 0$

$$\left.\frac{d^2\xi^i}{d\tau^2}\right|_{\tau=0} = -\left[\frac{dh_{ij}}{d\tau}\frac{d\xi^j}{d\tau}\right]_{\tau=0} \overset{\downarrow}{\equiv} 0$$

[if initially the two masses do not move, then their separation stays constant.]

(if we mark the coordinates with freely falling masses, by definition they do not change: no external force is acting on the masses, the GWs are "curvature")

However, we have seen that the ds^2 is not $=0$ in the TT gauge: for a wave propagating along the z-axis, we had found:

$$ds^2 = -c^2dt^2 + dz^2 + \left[1 + h_+ \cos\left(\omega\left(t-\frac{z}{c}\right)\right)\right] dx^2 +$$
$$+ \left[1 - h_+ \cos\left(\omega\left(t-\frac{z}{c}\right)\right)\right] dy^2 + 2 h_\times \cos\left(\omega\left(t-\frac{z}{c}\right)\right) dx\,dy$$

This shows as a neat example that in GR physical effects are not (always) expressed by what happens to the coordinates, because the theory is invariant under coordinate transformation

- ξ^μ of before was the COORDINATE DISTANCE
- ds^2 for two events on a space-like curve is the PROPER DISTANCE

let's see then what happens for the proper distance:

two events, $(t, x_1, 0, 0)$
$(t, x_2, 0, 0)$

NOTE THAT proper time measured by a clock sitting on a test mass initially at rest is the same as coordinate time in the TT gauge

coordinate distance : $\xi = x_2 - x_1$: this remains constant in the TT gauge under the effect of a GW propagating along the z axis.

since nothing happens for him

proper distance : $ds = \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}$

$$s = \sqrt{1 + h_+ \cos\omega t}\;(x_2 - x_1) \simeq \xi \left(1 + \frac{1}{2} h_+ \cos\omega t\right)$$

$\boxed{\text{so the proper distance changes periodically when GW passes}}$

generalizing to arbitrary directions, one finds the geodesic separation equation in terms of proper distance:

two spatial events separated by space-vector L:

$$s^2 = L^2 + h_{ij}(t)\, L_i L_j$$

$$s \simeq L\left(1 + \frac{1}{2} h_{ij} \frac{L_i L_j}{L^2}\right)$$

develop the $\sqrt{\ }$ in h_{ij}

$$\ddot{s} \simeq \frac{1}{2} \ddot{h}_{ij} \frac{L_i}{L} L_j$$

writing $\frac{L_i}{L} = n_i$ and $s_i = s n_i$ one gets

$$\boxed{\ddot{s}_i \simeq \frac{1}{2} \ddot{h}_{ij} L_j \simeq \frac{1}{2} \ddot{h}_{ij} s_j}$$

$$L_j = s_j + \mathcal{O}(h)$$

(*)

With respect to the geodesic deviation equation in terms of coordinate distance, we see that here proper distance changes in time even if $\dot{s}_j(\tau=0) = 0$

Now, the TT frame is simple (the GW has a very simple form in it) but it is not the frame used by an experimentalist to describe a detector. This is the proper detector frame which we next analyse. In this frame, positions are defined by rigid rulers. Note that in the TT gauge, a rigid ruler gets deformed by a GW.

(*) we can therefore monitor the arrival of a GW by measuring the proper distance between two suspended masses: PRINCIPLE OF THE INTERFEROMETER

PROPER DETECTOR FRAME

- In a laboratory, positions are not marked by freely falling masses; rather, one ideally uses a rigid ruler to define positions: in this frame, we expect that a mass initially at rest moves because of the transit of a GW with respect to the rigid ruler.
- The problem is, that a detector on Earth experiences the gravitational acceleration and the apparent forces due to Earth's rotation.
- Let's start with the expression for the metric from an accelerated observer: the coordinate system is chosen like the one to define local inertial coordinates, but now the observer does not move on a geodesic but it is accelerated. The metric in the vicinity of the world-line of this observer takes the form (expanded up to second order in the distance to the world line)

$\underline{a}$: acceleration $a^\nu = u^\mu \nabla_\mu u^\nu$

$\underline{\Omega}$: angular velocity of the observer with respect to **gyroscopes**

$$ds^2 \simeq -c^2 dt^2 \Big[1 + \frac{2}{c^2} \underline{a}\cdot\underline{x} + \frac{1}{c^4} \underbrace{(\underline{a}\cdot\underline{x})^2}_{\text{redshift}} + \frac{1}{c^2} \underbrace{(\underline{\Omega}\times\underline{x})^2}_{\text{time dilatation}} + R_{0i0j} x^i x^j \Big]$$

$$+ 2c\,dt\,dx^i \Big[\frac{1}{c} \varepsilon_{ijk} \Omega^j x^k - \frac{2}{3} R_{0jik} x^j x^k \Big]$$

$$+ dx^i dx^j \Big[\delta_{ij} - \frac{1}{3} R_{ikje} x^k x^e \Big]$$

STRAUMANN PARAGRAPH 1.10.5

1) at zeroth order in x, the metric is flat EVEN IN THE PRESENCE OF GWS AND OF THE EARTH'S GRAV. FIELD.

2) at first order in x, we have the effects due to (Earth's observer) acceleration and rotation, $\underline{a}$, $\underline{\Omega}$

(we are accelerated observers with $\underline{a} = -\underline{g}$ and rotation $\underline{\Omega}$)

3) the effects due to a GW passing through, or to the curvature of the background, only appear at second order in x.

variation of the grav. field

Physically, the expansion in x is an expansion with respect to the typical scale of variation of the metric:

if g varies on scale L_B, the small parameter is $\frac{x}{L_B}$, such that effects due to the CURVATURE (and GWs) only appear at second order in $\frac{x}{L_B}$:

$$R_{\mu\nu\rho\sigma} \sim \partial^2 g \sim \left(\frac{1}{L_B}\right)^2 \implies$$ these are the corrections to the flat metric.

1) for $\frac{x}{L_B} \ll 1$, the observer lives in flat spacetime (this is the equivalence principle)

2) at first order we have Newtonian forces due to the acceleration: from the metric at first order, the geodesic equation becomes (with $\frac{dx^i}{d\tau} = u^i \equiv \underline{v}$)

$$\dot{\underline{v}} = -\underline{a} - 2(\Omega \times \underline{v})$$

$\underline{a} = -\underline{g}$ Earth's gravity

$\Omega \times \underline{v}$ Coriolis force

STRAUMANN 10.1.5

3) at second order we have the effect we are interested in, namely the GWs. Can we isolate this effect?

If we do, it means that we can realize a GW detector on Earth, at least in principle.

NOTE: The reason why at linear order in $\frac{x}{L_B}$ there are no corrections to the flat metric if the observer is in free-fall, is that $g_{\mu\nu,\lambda}(P) = 0$ for free-falling observer.

the answer is YES if two conditions are verified:

1. the gravitational acceleration is compensated by some suspension mechanism, which leaves the test masses free to move in the x-y plane
(not in the z direction)

2. the other terms are eliminated focusing on a frequency range in which these terms are slowly varying, while the GWs have exactly those frequencies:
$\mathcal{O}(\mathrm{Hz} - 10^3\,\mathrm{Hz})$: in this frequency window, it is possible to isolate the detector from external noises due to Earth's rotation, earthquakes etc.
time varying Newtonian gravitational forces, which also appear at second order in x/L_B.

In the plane transverse to the direction of $\underline{g}$, and in the frequency window above mentioned, we can retain in the metric ONLY THE PART PROPORTIONAL TO THE RIEMANN TENSOR - it is as like the observer was in free-fall, in the total grav. field: Earth + GWs.

Consider the eq. of geodesic deviation: (P detector position)

$$\Gamma^{\mu}_{\nu\rho}(P) = 0 \qquad \frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau}$$ since the detector is non relativistic

$$\frac{d^2\xi^i}{d\tau^2} + \xi^\sigma \partial_\sigma \Gamma^i_{00} \left(\frac{dx^0}{d\tau}\right)^2 = 0 \quad \Rightarrow$$

note that this equation has been derived expanding the $\Gamma^{\mu}_{\nu\rho}$ to first order in ξ^ρ. This is only valid if the coordinate separation $|\xi^\sigma|$ is much smaller than the typical scale over which the gravitational field changes substantially; for a GW this is the reduced wave lenght.

Earth-based detectors: $f \simeq 10^2 - 10^3$ Hz $\Rightarrow \lambda \simeq 500 - 50$ km

and the size is $L = 4$ km

Space-based detectors: $f \simeq 10^{-4} - 10^{-2}$ Hz $\Rightarrow$

$\lambda \simeq 5 \cdot 10^6 - 5 \cdot 10^8$ km

and the size of the arm is 10^6 km ; so one is not in the approximation in which the geodesic deviation equation can be used to describe the interaction of the detector with GWs. A full GR description in the TT gauge is necessary.

The advantage of space-based detectors is that one reaches these low frequencies BECAUSE THERE ARE NO NOISES DUE TO THE EARTH GRAVITATIONAL FIELD the "observer" i.e. the detector frame, is really the one of a free falling observer, both in the Earth gravitational field and in the GWs.

No need of suspensions, limits in frequency and so on.

This description is therefore valid only if

$$\boxed{L \ll \lambdabar}$$

typical size of the detector — reduced wave lenght of GWs

we have seen that this is the case on page 12 for Earth based det.

$$\partial_\sigma \Gamma^i{}_{00} = \partial_\sigma \left[\frac{g^{i\rho}}{2} \left(\partial_0 g_{\rho 0} + \partial_0 g_{\rho 0} - \partial_\rho g_{00} \right) \right]$$

since $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^i x^j)$ we need both derivatives to be spatial:

$$\xi^\sigma \partial_\sigma \Gamma^i{}_{00} \equiv \xi^j \partial_j \Gamma^i{}_{00}$$

Alternative derivation
- directly from the deviation eq. on page 17
- see Straumann chap. 2.1

Moreover, the Riemann tensor

$$R^i{}_{0j0} = \partial_j \Gamma^i{}_{00} - \underbrace{\partial_0 \Gamma^i{}_{0j}}_{\substack{\text{this is a}\\ \text{time derivative,}\\ \text{so } = 0}} = \partial_j \Gamma^i{}_{00}$$

To linear order in h, we also have that $dt = d\tau$ since (coordinate time of the proper detector frame)

$$(c d\tau)^2 = -\left(-c dt^2 + dx_i dx^i\right) + \mathcal{O}(h) \Rightarrow dt^2 = d\tau^2 \left[1 + \frac{1}{c^2} \frac{dx^i}{dt} \frac{dx^i}{d\tau}\right]$$

and the last term is $\mathcal{O}(h^2)$ because it is due to the wave and therefore the eq. becomes:

$$\boxed{\frac{d^2 \xi^i}{dt^2} = -c^2 R^i{}_{0j0} \xi^j}$$

Note that with $t \to s$ this is Geodesic deviation eq. in a non rotating ref. frame Straumann (2.11)

Now, consider that the Riemann tensor is <u>INVARIANT</u> in linearized theory under a coordinate transformation. We can therefore write it in any gauge, it has the same form:

(exercise)

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \left(\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho} \right)$$

this does not change under $x^\mu \to x^\mu + \xi^\mu(x)$

we therefore write it in TT gauge, for which it has its simplest form:

$$R^{i}{}_{0j0} \underset{\substack{\uparrow \\ \eta_{\mu\nu} \text{ to lower}}}{=} R_{i0j0} = -\frac{1}{2} \partial_0^2 h_{ij}^{(TT)} = -\frac{1}{2c^2} \ddot{h}_{ij}^{(TT)}$$

In conclusion we have for the geodesic deviation eq. in the PROPER DETECTOR FRAME:

$$\boxed{\ddot{\xi}^{i} = \frac{1}{2} \ddot{h}_{ij}^{TT} \xi^{j}}$$

(applies if $\lambda \gg L$)

where ξ_i is the coordinate distance and a dot is derivative w.r.t. coordinate time:

we can rewrite it in terms of a Force, as a Newtonian equation:

$$F_i = \frac{m}{2} \ddot{h}_{ij}^{TT} \xi^{j}$$

(*) proper detector frame, the one of an experimenter on Earth, has the advantage that Newtonian intuition applies:

- once we account for the suspensions and restrict to a given frequency range, it is as if we were in flat space-time
- the effect of a passing GW can be described on the apparatus in terms of a Newtonian force.

(*) ξ^{i} is the coordinate distance: therefore, this equation is different from the one we have derived in the TT gauge. However, it looks very similar

to the equation for the proper distance in the TT gauge.
this is because of two reasons:

- The Riemann tensor is invariant, so we can compute it in any gauge
- in the proper detector frame, with suspensions and right freq. range, to a first approximation coordinate distances and proper distances are the same thing, since the metric is "flat" up to $\Theta\left(\frac{x}{L_B}\right)^2$.

If we see ξ^i as a proper distance, it means that we can generalise the above equation as describing the evolution of proper distances in any other frame (as long as $\tau = t + \Theta(h)$ and the system is not relativistic).

Therefore, in terms of proper distance, the eq has the same form as the one in TT gauge. It combines Newtonian description and simple GW description.

(*) Now this equation can be used to describe

THE EFFECT OF A PASSING GW ON A CIRCLE OF TEST MASSES

consider a GW propagating in the z-direction.

For both $i = 3$ and $j = 3$ $h_{ij}^{TT} \equiv 0$: a particle that is at rest at $z = 0$ initially remains at $z = 0$: we conclude that

not if it has an initial velocity

GW displace masses transversally with respect to their direction of propagation

which is $\boxed{\text{a consequence of } \partial_i h_{ij} = 0}$ as we have seen.

(but $\partial_i h_{ij} = 0$ is a MORE GENERAL, EXACT condition) we choose the origin of time such that

$$h_{ij}^{TT}(t=0) = 0$$

for the + polarisation we have therefore; at the position $z=0$:

$$h_{ab}^{TT}(z=0,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab} h_+ \sin(\omega t) .$$

with $\xi_a(z=0,t) = (x_0 + \delta x(t), y_0 + \delta y(t))$ and x_0, y_0 the unperturbed positions, $(\delta x(t), \delta y(t))$ the displacements caused by the passing wave:

$$\ddot{\xi}_a(t) = \frac{1}{2} \ddot{h}_{ab}^{TT}(t)\, \xi_b(t)$$

$$\begin{cases} \delta\ddot{x}(t) = -\dfrac{h_+}{2}(x_0 + \delta x(t))\,\omega^2 \sin\omega t \\ \delta\ddot{y}(t) = \dfrac{h_+}{2}(y_0 + \delta y(t))\,\omega^2 \sin\omega t \end{cases}$$

(– sign due to the derivative of the sin)

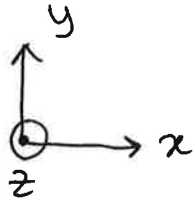
here we neglect the displacements on the right hand side which are of order $\delta x(t) = O(h)$. we have therefore:

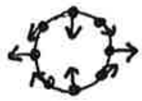
$$\begin{cases} \delta x(t) = \dfrac{h_+}{2} x_0 \sin\omega t \\ \delta y(t) = -\dfrac{h_+}{2} y_0 \sin\omega t \end{cases}$$

For the cross polarisation :

$$\begin{cases} \delta x(t) = \dfrac{h_\times}{2} y_0 \sin\omega t \\ \delta y(t) = \dfrac{h_\times}{2} x_0 \sin\omega t \end{cases}$$

On a ring of test masses we get:



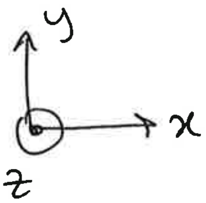
ωt	h_+	$h_\times$
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		

The corresponding newtonian force is such that:

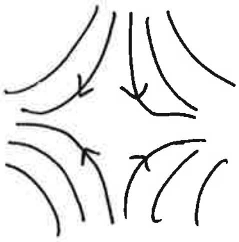
$$F_i = \frac{m}{2} \ddot{h}_{ij}^{TT} \xi^j$$

$$\partial_i F_i = \frac{m}{2} \ddot{h}_{ij}^{TT} \partial_i \xi^j = \frac{m}{2} \ddot{h}_{ii}^{TT} \equiv 0$$

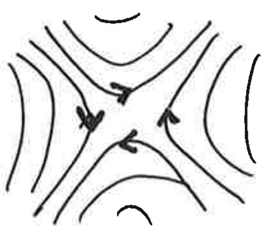
therefore it is $\boxed{\text{divergence-free}}$ as for example the magnetic field: there are no sources or sinks for the FORCE LINES, which look like:



h_+



$h_\times$



$h_+ \to h_\times$ and $h_\times \to h_+$ under a rotation of $\frac{\pi}{4}$, as can be seen also from the transformations derived on page 12/2.

GENERATION OF GWS IN LINEARISED THEORY

Until now we have seen how GWs are defined in linearised theory, what is their energy momentum tensor, how they propagate in vacuum... we now turn to GW generation, always in the context of linearized theory: if we want to keep the expansion around flat spacetime, it means that the gravitational field generated by the source must be sufficiently weak.

For a self-gravitating system, a weak gravitational field means that the typical velocities inside the source are small:

two body system held together by gravitational force:

$$E_{kin} = -\frac{1}{2} U$$

$$\frac{1}{2} \mu v^2 = \frac{1}{2} \frac{G \mu m}{r}$$

μ reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

m: total mass $m_1 + m_2$

$$\boxed{\left(\frac{v}{c}\right)^2 = \frac{R_s}{2r}}$$

$$R_s = 2\frac{Gm}{c^2}$$

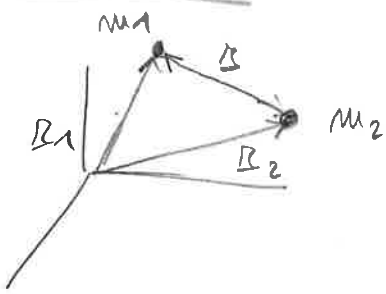
Weak gravitational field means $r \gg R_s$, therefore one has

$$\boxed{v \ll c}$$

$\phi \sim 10^{-1}$ $\phi \sim 1$

This means that in principle, to analyse systems such as NS, BH, compact binaries beyond lowest order, one has to go beyond linear theory and a Newtonian description

SELF GRAVITATING SYSTEM OF TWO MASSES



$$m_1 \ddot{\underline{r}}_1 = -\frac{G m_2 m_1}{|\underline{r}_2 - \underline{r}_1|^3} (\underline{r}_1 - \underline{r}_2)$$

$$m_2 \ddot{\underline{r}}_2 = -\frac{G m_1 m_2}{|\underline{r}_1 - \underline{r}_2|^3} (\underline{r}_2 - \underline{r}_1)$$

$$\ddot{\underline{r}} = -\frac{G}{r^3} (m_1 + m_2) \underline{r}$$

can be seen as a one body problem

$$\mu \ddot{\underline{r}} = -\frac{G}{r^3} m \mu \underline{r} \qquad m = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$E_{kin} = \frac{1}{2} \mu v^2 \quad \text{with} \quad a = \frac{v^2}{r} \quad \text{circular motion}$$

$$E_{kin} = \frac{1}{2} \mu a r = \frac{1}{2} \mu \frac{|F|}{\mu} r = \frac{1}{2} \left(\frac{G m \mu}{r^2} \right) r =$$

$$= \frac{1}{2} \frac{G m \mu}{r} = -\frac{1}{2} E_{POT}$$

$$E_{kin} + E_{POT} = -\frac{G m_1 m_2}{2r}$$

of the sources: this is the post-Newtonian formalism. (2)
we won't treat it in this course: we will stick to the linearized theory.

First we analyze GW production in flat space-time but exact in $\frac{v}{c}$ (this describes for example non-gravitationally bound sources, whose dynamics is determined by other interactions)

(weak-field sources with arbitrary velocity)

then we expand in powers of $\frac{v}{c}$: GW production can be organized in this case in a multipole expansion, of which the first term is the quadrupole of the source.

WEAK FIELD SOURCES WITH ARBITRARY VELOCITY

we want to solve:

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$

($\partial^\mu \bar{h}_{\mu\nu} = 0$, $\partial^\mu T_{\mu\nu} = 0$ satisfied conditions.) the above equation can be solved as any wave equation with the Green function method:

see JACKSON, classical Electrodynamics PARAGRAPH 6.4

for:
$$\Box_x G(x-x') = \delta^4(x-x')$$

the solution is
$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4}\int d^4x' \, G(x-x')\, T_{\mu\nu}(x')$$

and the Green function with the appropriate initial conditions is the retarded one (just as in electrodynamics)

$$G(x-x') = -\frac{1}{4\pi|\underline{x}-\underline{x}'|}\,\delta(x^0_{ret} - x'^0)$$

Reminder of page ⑨ beginning of course :

wave propagating in direction $\hat{n}$

$$P_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j \quad \text{projects orthogonal to } \hat{n}$$

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}$$

$x'^0 = ct'$, $x^0_{ret} = c\,t_{ret}$, $t_{ret} = t - \frac{|\underline{x}-\underline{x}'|}{c}$.

Therefore, the solution becomes:

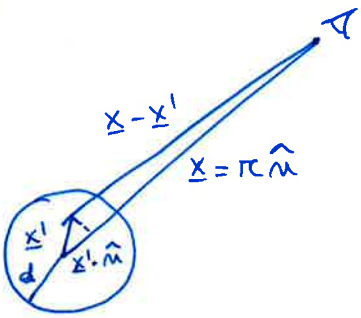
$$\bar{h}_{\mu\nu}(t,\underline{x}) = \frac{4G}{c^4}\int d^3x' \frac{1}{|\underline{x}-\underline{x}'|} T_{\mu\nu}\left(t - \frac{|\underline{x}-\underline{x}'|}{c}, \underline{x}'\right)$$

Now we are interested in this solution OUTSIDE the source, where we can use the TT gauge: $(\hat{x} = \hat{n})$

$$\boxed{h^{TT}_{ij}(t,\underline{x}) = \frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n})\int d^3x' \frac{1}{|\underline{x}-\underline{x}'|} T_{kl}\left(t - \frac{|\underline{x}-\underline{x}'|}{c}, \underline{x}'\right)}$$

Here only the spatial components of the energy momentum tensor enter: but T_{00} and T_{0k} are related to them via energy-momentum conservation.

The above expression can be simplified further since in general the observer will be far away from the source:



$$|\underline{x}-\underline{x}'| = r - \underline{x}'\cdot\hat{n} + O\left(\frac{d^2}{r}\right)$$

and furthermore we make the limit $r \to \infty$ at fixed time t:

$$\boxed{h^{TT}_{ij}(t,\underline{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n})\int d^3x'\, T_{kl}\left(t - \frac{r}{c} + \frac{\underline{x}'\cdot\hat{n}}{c}, \underline{x}'\right)}$$

and we neglect terms of order d/r^2.

It is also useful to rewrite this expression in terms of the Fourier expansion $\tilde{T}_{kl}(\omega, \underline{k})$:

$$\frac{1}{|x - x'|} = \frac{1}{r - x' \cdot \hat{n}} = \frac{1}{r} \left(\frac{1}{1 - \frac{x' \cdot \hat{n}}{r}} \right)$$

$$\frac{x' \cdot \hat{n}}{r} \leq \frac{d}{r} \qquad 1 - \frac{x' \cdot \hat{n}}{r} \geq 1 - \frac{d}{r}$$

$$\frac{1}{1 - \frac{x' \cdot \hat{n}}{r}} \leq \frac{1}{1 - \frac{d}{r}}$$

$$\leq \frac{1}{r} \left(\frac{1}{1 - \frac{d}{r}} \right) \simeq \frac{1}{r} \left(1 + \frac{d}{r} + O\left(\frac{d}{r}\right)^2 \right)$$

4

from $T_{ke}(\underline{x},t) = \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{ke}(\omega,\underline{k})\, e^{-i\omega t + i\underline{k}\cdot\underline{x}}$

here there is still no relation between $\underline{k}$ and ω: the source is general

we get:

$$\int d^3x'\, T_{ke}\left(t-\frac{r}{c}+\frac{\underline{x}'\cdot\hat{n}}{c}, \underline{x}'\right) =$$

$$= \int d^3x' \int \frac{d\omega}{2\pi c}\frac{d^3k}{(2\pi)^3}\tilde{T}_{ke}(\omega,\underline{k})\, e^{-i\omega\left(t-\frac{r}{c}\right)+i\left(\underline{k}-\frac{\omega}{c}\hat{n}\right)\cdot\underline{x}'} =$$

$$= \int \frac{d\omega}{2\pi c}\frac{d^3k}{(2\pi)^3}\tilde{T}_{ke}(\omega,\underline{k})\, e^{-i\omega\left(t-\frac{r}{c}\right)}(2\pi)^3\,\delta^3\left(\underline{k}-\frac{\omega}{c}\hat{n}\right) =$$

$$= \int \frac{d\omega}{2\pi c}\tilde{T}_{ke}\left(\omega,\frac{\omega}{c}\hat{n}\right) e^{-i\omega\left(t-\frac{r}{c}\right)}$$

with this delta function one selects $\underline{k} = \frac{\omega}{c}\hat{n}$, the relevant dispersion relation for GWs: the only components of the source that generate GWs are those satisfying this dispersion relation

giving then:

$$\boxed{h_{ij}^{TT}(t,\underline{x}) = \frac{1}{r}\frac{4G}{c^5}\Lambda_{ij,ke}(\hat{n})\int \frac{d\omega}{2\pi}\tilde{T}_{ke}\left(\omega,\frac{\omega}{c}\hat{n}\right)e^{-i\omega\left(t-\frac{r}{c}\right)}}$$

From this equation we can get the energy radiated per solid angle:

~~recall the formula~~ : $\frac{dE}{d\Omega} = \frac{c^3}{32\pi G} r^2 \int_{-\infty}^{+\infty} dt \langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\rangle$
(page 51 of first part)

~~substituting the expression for h_{ij}^{TT}, recalling that $\Lambda_{ijke}\Lambda_{ijpq} = \Lambda_{kepq}$ and defining $\tilde{T}(-\omega,-\underline{k}) \equiv \tilde{T}^*(\omega,\underline{k})$ one finds, for positive frequencies ω:~~

$$\cancel{\frac{dE}{d\Omega} = \frac{G}{2\pi^2c^7}\Lambda_{kepq}(\hat{n})\int_0^{\infty} d\omega\,\omega^2\,\tilde{T}_{ke}\left(\omega,\omega\frac{\hat{n}}{c}\right)\tilde{T}^*_{pq}\left(\omega,\omega\frac{\hat{n}}{c}\right)}$$

$$\frac{dE}{d\Omega} = \frac{c^3}{32\pi G} r^2 \int_{-\infty}^{+\infty} dt \, \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle =$$

NOTE ADDED

demonstration of the result on the previous page

$$= \frac{c^3}{32\pi G} r^2 \frac{1}{r^2} \frac{16 G^2}{c^{10}} \frac{1}{(2\pi)^2} \Lambda_{ij,kl}(\hat{n}) \Lambda_{ij,pq}(\hat{n})$$

$$\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \langle \tilde{T}_{kl}\left(\omega, \frac{\omega}{c}\hat{n}\right)\left(e^{-i\omega\left(t-\frac{r}{c}\right)}\right)^{\cdot}$$

$$\tilde{T}_{pq}\left(\omega', \frac{\omega'}{c}\hat{n}\right)\left(e^{-i\omega'\left(t-\frac{r}{c}\right)}\right)^{\cdot} \rangle =$$

$$= \frac{G}{8\pi^3 c^7} \Lambda_{kl,pq}(\hat{n}) \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \langle \tilde{T}_{kl}\left(\omega, \frac{\omega}{c}\hat{n}\right) \tilde{T}_{pq}\left(\omega', \frac{\omega'}{c}\hat{n}\right) \rangle$$

$$\int_{-\infty}^{+\infty} dt \, (-\omega\omega') \, e^{-i\left(t-\frac{r}{c}\right)(\omega+\omega')}$$

note that $\langle \ldots \rangle$ is present only if we are averaging over lengthscales $\bar{t}$, otherwise it is redundant with the time integral $\int_{-\infty}^{+\infty} dt$, see page 51

Now we set $\int_{-\infty}^{+\infty} dt \, e^{-it(\omega+\omega')} = (2\pi)\,\delta(\omega+\omega')$

and integrate on $\int_{-\infty}^{+\infty} d\omega'$ with the delta:

$$= \frac{G\,2\pi}{8\pi^3 c^7} \Lambda_{ke,pq}(\hat{n}) \int_{-\infty}^{+\infty} d\omega\, \omega^2 \left\langle \tilde{T}_{ke}\left(\omega, \frac{\omega}{c}\hat{n}\right) \tilde{T}_{pq}\left(-\omega, -\frac{\omega}{c}\hat{m}\right) \right\rangle e^{+i\frac{\pi}{c}\omega - i\frac{\pi}{c}\omega} =$$

setting now $\tilde{T}_{pq}\left(-\omega, -\frac{\omega}{c}\hat{n}\right) = \tilde{T}^*_{pq}\left(\omega, \frac{\omega}{c}\hat{n}\right)$ one gets:

$$\frac{dE}{d\Omega} = \frac{G}{2\pi^2 c^7} \Lambda_{ke,pq}(\hat{n}) \int_0^{\infty} d\omega\, \omega^2 \left\langle \tilde{T}_{ke}\left(\omega, \frac{\omega}{c}\hat{m}\right) \tilde{T}^*_{pq}\left(\omega, \frac{\omega}{c}\hat{m}\right) \right\rangle$$

and from this one gets the energy spectrum : (5)

$$\boxed{\frac{dE}{d\omega} = \frac{G\omega^2}{2\pi^2 c^7} \int d\Omega\, \Lambda_{kepq}(\hat{n})\, \tilde{T}_{ke}\left(\omega, \omega\frac{\hat{n}}{c}\right) \tilde{T}^*_{pq}\left(\omega, \omega\frac{\hat{n}}{c}\right)}$$

here given in terms of the source (compare with page 52).
For an idealized source, exactly monocromatic and therefore that emits forever, we can formally write:

$$\tilde{T}_{ij}(\omega, \underline{k}) = \theta_{ij}(\omega, \underline{k})\, \delta(\omega - \omega_0)\, 2\pi$$

if we insert this expression in the energy radiated per solid angle we get a divergence, since the energy is in principle infinite. Obviously this is only because the above source is unphysical, and we are not accounting for back-reaction of GWs emission on the source. Formally writing $2\pi\,\delta(\omega=0) = T$, the total emission time of the source, and dividing by it, we can get an expression for the power (energy per unit time) radiated istantaneously by a monochromatic source of GW with frequency ω_0 per unit solid angle:

$$\frac{dP}{d\Omega} = \frac{G\omega_0^2}{\pi c^7} \Lambda_{ijke}(\hat{n})\, \theta_{ij}\left(\omega_0, \frac{\omega_0}{c}\hat{n}\right) \theta^*_{ke}\left(\omega_0, \frac{\omega_0}{c}\hat{n}\right)$$

Now in reality, typically the Fourier distribution of a source will be spread around a typical frequency ω_s. This will correspond to the frequency of the motions of matter inside the source.

NOTE ADDED

Idealised monocromatic source: since the energy is formally divergent, we define instead the istantaneous radiated power

$$\frac{dE}{d\omega} = \frac{G\omega^2}{2\pi^2 c^7} \int d\Omega \, \Lambda_{ke,pq}(\hat{n}) \, (4\pi)^2 \, \delta(\omega-\omega_0) \, \delta(\omega-\omega_0) \, \Theta_{ke}(\omega, \underline{k}) \, \Theta^*_{pq}(\omega, \underline{k})$$

$$\frac{dE}{d\Omega} = \frac{G(2\pi)^2}{2\pi^2 c^7} \Lambda_{ke,pq}(\hat{n}) \int d\omega \, \delta(\omega-\omega_0) \, \delta(\omega-\omega_0) \, \omega^2 \, \Theta_{ke}(\omega, \underline{k}) \, \Theta^*_{pq}(\omega, \underline{k})$$

formally divergent, but let's perform formally the integration over $d\omega$ and write

$$2\pi \, \delta(\omega = 0) = T$$

this is the total emission time, which is in principle infinite for a perfectly monocromatic source with frequency ω_0. We then DIVIDE BY T to get the power radiated istantaneously:

$$\frac{dE}{d\Omega} = \frac{G}{\pi c^7} \omega_0^2 \, T \, \Lambda_{ke,pq}(\hat{n}) \, \Theta_{ke}\left(\omega_0, \frac{\omega_0}{c}\hat{n}\right) \Theta^*_{pq}\left(\omega_0, \frac{\omega_0}{c}\hat{n}\right)$$

$$\frac{dP}{d\Omega} = \frac{G\omega_0^2}{\pi c^7} \Lambda_{ke,pq}(\hat{n}) \, \Theta_{ke}\left(\omega_0, \frac{\omega_0}{c}\hat{n}\right) \Theta^*_{pq}\left(\omega_0, \frac{\omega_0}{c}\hat{n}\right)$$

If d is the source size and v the typical velocities inside the source, one has roughly:

$$\boxed{v \sim \omega_s d}$$

the frequency of the emitted GWs also roughly corresponds to ω_s:

$$\omega \sim \omega_s \sim \frac{v}{d}$$

$$\boxed{\lambda = \frac{c}{\omega} \sim \frac{c}{v} d}$$

now if the source is non-relativistic, $v \ll c$ and therefore

$$\boxed{\text{non-rel. source} \Rightarrow \lambda \gg d}$$

(we recover the argument given in the beginning of the course: $f_{GW} \ll \frac{c}{R_s}$ for a non-relativistic source i.e. $R \gg R_s$ where here $R \simeq d$)

Since the emitted wavelength of GWs by a non-relativistic source is typically much larger than the source size, it is clear that

1) we cannot resolve the source, as stated in the first lecture

2) we don't need to know the fine details of the source to evaluate the GW emission: in an expansion in terms of multipoles, only the few lowest will dominate the signal.

(obviously, all this is true for non-relativistic sources $v \ll c$)

LOW VELOCITY EXPANSION OF THE SOURCE

To perform the multipole expansion for gravitational radiation, the key ingredient is to observe that $\tilde{T}_{kl}(\omega, \underline{k})$ for a realistic source is peaked around the typical frequency satisfying (since the source is non-relativistic)

$$\omega_s \sim \frac{v}{d} \ll \frac{c}{d} \quad \Longrightarrow \quad \frac{\omega_s d}{c} \ll 1$$

In order to calculate $h_{ij}^{TT}(t, \underline{x})$ we need: (see equations on page 3)

$$\int d^3x' \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{kl}(\omega, \underline{k}) \, e^{-i\omega\left(t - \frac{r}{c}\right) - i\omega\left(\frac{\underline{x}'\cdot\hat{n}}{c}\right) + i\underline{k}\cdot\underline{x}'}$$

The integral $\int d^3x'$ is restricted to the source, i.e. $|\underline{x}'| \lesssim d$.

Moreover, $\tilde{T}_{kl}(\omega, \underline{k})$ is peaked around ω_s: therefore, the dominant contribution to the integral in $d\omega$ comes from:

$$\frac{\omega}{c}\, \underline{x}'\cdot\hat{n} \lesssim \frac{\omega_s d}{c} \ll 1$$

One can therefore expand the exponential:

$$e^{-i\omega\left(t - \frac{r}{c} + \frac{\underline{x}'\cdot\hat{n}}{c}\right)} = e^{i\omega\left(t - \frac{r}{c}\right)} \cdot$$

$$\cdot \left[1 - i\frac{x'_i n_i}{c}\omega - \frac{1}{2}\frac{(x'_i n_i)(x'_j n_j)}{c^2}\omega^2 + \dots \right]$$

In real space, this is equivalent to the expansion:

$$T_{ke}\left(t-\frac{r}{c}+\frac{x'\cdot\hat{n}}{c},x'\right) \simeq T_{ke}\left(t-\frac{r}{c},x'\right)+$$

$$+\frac{x'_i n_i}{c}\,\partial_0 T_{ke}\Big|_{\left(t-\frac{r}{c},x'\right)}+\frac{1}{2c^2}(x'_i n_i)(x'_j n_j)\,\partial_0^2 T_{ke}\Big|_{\left(t-\frac{r}{c},x'\right)}+\dots$$

The MOMENTA of the stress tensor can be defined:

$$\boxed{\begin{aligned} S^{ij}(t) &= \int d^3x\, T^{ij}(t,x) \\ S^{ij,k}(t) &= \int d^3x\, T^{ij}(t,x)\, x^k \\ S^{ij,ke}(t) &= \int d^3x\, T^{ij}(t,x)\, x^k x^e \end{aligned}}$$

and so on. The usual expression for $h_{ij}^{TT}(t,x)$

$$h_{ij}^{TT}(t,x)=\frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,ke}(\hat{n})\int d^3x'\, T_{ke}\left(t-\frac{r}{c}+\frac{x'\cdot\hat{n}}{c},x'\right)$$

valid far away from the source can therefore be rewritten, using the expansion above (conceptually based on the small parameter $\frac{\omega_s d}{c}\ll 1$) as:

self gravitating system in approximation of weak gravitational field has also $v \ll c$

therefore $\omega_s d \ll c$ $\longrightarrow$ expansion in $\frac{\omega_s d}{c}$

and $\frac{v}{c}$ are equivalent

$$h_{ij}^{TT}(t,\underline{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n}) \cdot$$

$$\cdot \left[S^{kl}\left(t-\frac{r}{c}\right) + \frac{1}{c} n_m \dot{S}^{kl,m}\left(t-\frac{r}{c}\right) + \frac{1}{2c^2} n_m n_p \ddot{S}^{kl,mp}\left(t-\frac{r}{c}\right) + \ldots \right]$$

This is the basic equation for the multipole expansion of the emitted radiation.

Every moment has a factor $x \sim d$ more than the previous; every time derivative brings in a factor ω_s.

The second term in [...] above has therefore a factor $\frac{\omega_s d}{c} \sim \frac{v}{c}$ more than the first; the third, a factor $\left(\frac{v}{c}\right)^2$.

Therefore, we have performed the low-velocity expansion.

MOMENTA OF THE ENERGY DENSITY AND OF THE LINEAR MOMENTUM:

The momenta of the energy density $\rho = T^{00}$ are defined by:

$$M = \frac{1}{c^2}\int d^3x \, T^{00}$$

$$M^i = \frac{1}{c^2}\int d^3x \, T^{00} x^i$$

$$M^{ij} = \frac{1}{c^2}\int d^3x \, T^{00} x^i x^j$$

$(\ldots)$

Note that here we introduce a factor $1/c^2$, so these momenta have the dimensions of a mass density as T^{00}/c^2, but they are more general because they contain other contributions to the total energy and not only the rest-mass contribution the momenta of the linear momentum $\frac{T^{0i}}{c}$ are defined as:

$$p^i = \frac{1}{c}\int d^3x \, T^{0i}$$

$$p^{i,j} = \frac{1}{c}\int d^3x \, T^{0i} x^j$$

$$p^{i,je} = \frac{1}{c}\int d^3x \, T^{0i} x^j x^e$$

$$(\dots)$$

These quantities and the S^{ij}, $S^{ij,k}$ ... defined above are related via energy momentum conservation.

Note that if we set $\partial_\mu T^{\mu\nu} = 0$, it means that the source is NOT LOSING ENERGY as a consequence of GW emission: back-reaction is neglected in the linearized theory.

let's consider a volume V which contains the source.

CONSERVATION OF MASS FOR THE SOURCE:

$$c\dot{M} = \int_V d^3x \, \partial_0 T^{00} \underset{\boxed{\partial_\mu T^{\mu 0} = 0}}{=} -\int_V d^3x \, \partial_i T^{0i} = -\int_{\partial V} dS \, T^{0i} \underset{\substack{T^{0i} \text{ vanishes} \\ \text{on the} \\ \text{boundary}}}{=} 0$$

EXPRESSION FOR THE LINEAR MOMENTUM:

$$c\dot{M}^i = \int_V d^3x \, x^i \partial_0 T^{00} = -\int_V d^3x \, x^i \partial_j T^{0j} \underset{\substack{\text{total derivative} \\ \text{does not contribute}}}{=} \int_V d^3x \, (\partial_j x^i) T^{0j}$$

$$= \int_V d^3x \, \delta_j{}^i \, T^{0j} = c P^i$$

CONSERVATION OF LINEAR MOMENTUM:

$$\dot{P}^i = \int_V d^3x\, \partial_0 T^{0i} = -\int_V d^3x\, \partial_j T^{ij} = \int_{\partial V} dS\, T^{ij} = 0$$

CONSERVATION OF ANGULAR MOMENTUM:

$$\dot{P}^{i,j} = \int_V d^3x\, \partial_0 T^{0i} x^j = -\int_V d^3x\, (\partial_k T^{ki})\, x^j = \int_V d^3x\, T^{ki} \delta_k^j =$$

$$= \int_V d^3x\, T^{ij} = \dot{P}^{j,i} \stackrel{!}{=} S^{ij}$$

$$\dot{P}^{i,j} - \dot{P}^{j,i} = \int_V d^3x \left(\partial_0 T^{0i} x^j - \partial_0 T^{0j} x^i\right) = \partial_t \int_V d^3x\, j^{ji} = \partial_t \int_V d^3x\, \varepsilon^{jik} j^k = 0 \quad \partial_t j^k = 0$$

Proceeding in this way, one can demonstrate all the identities:

$$\dot{M} = 0 \qquad \dot{P}^i = 0$$

$$\dot{M}^i = P^i \qquad \boxed{\dot{P}^{i,j} = S^{ij}}$$

$$\boxed{\dot{M}^{ij} = P^{i,j} + P^{j,i}} \qquad \boxed{\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j}}$$

$$\boxed{\dot{M}^{ijk} = P^{i,jk} + P^{j,ki} + P^{k,ij}}$$

And in particular, one can demonstrate that:

$$\boxed{S^{ij} = \frac{1}{2} \ddot{M}^{ij}}$$

$$\boxed{\dot{S}^{ij,k} = \frac{1}{6} \dddot{M}^{ijk} + \frac{1}{3}\left(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij}\right)}$$

Therefore, the moments of stress tensor involve TIME derivatives of the "mass" and momentum moments:

A STATIC SOURCE DOES NOT RADIATE.

MASS QUADRUPOLE RADIATION

The lowest order of the basis equation for the multipole expansion reads, in terms of the second mass moment:

$$\left[h_{ij}^{TT}(t,\underline{x})\right]_{\text{lowest}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\,\ddot{M}^{kl}\left(t-\frac{r}{c}\right)$$

We can decompose the tensor M^{kl} into a pure trace and a trace-less part, and $\Lambda_{ij,kl}$ will only act on the traceless part:

$$\Lambda_{ij,kl}(\hat{n})\left[\left(M^{kl}-\frac{1}{3}\delta^{kl}M_{pp}\right)+\frac{1}{3}\delta^{kl}M_{pp}\right] = \Lambda_{ij,kl}\left(M^{kl}-\frac{1}{3}\delta^{kl}M_{pp}\right)$$

We can rewrite the traceless part in a more familiar form, i.e. as the quadrupole moment of the mass density.

Since $\rho = \frac{T^{00}}{c^2}$ becomes the mass density for a non-relativistic source, one has:

$$\boxed{Q^{ij} = M^{ij} - \frac{1}{3}\delta^{ij}M_{kk} = \int d^3x\,\rho(\underline{x},t)\left(x^i x^j - \frac{1}{3}r^2\delta^{ij}\right)}$$

The fundamental law for GW emission from a non-relativistic source becomes then:

$$\boxed{\left[h_{ij}^{TT}(t,\underline{x})\right]_{\text{QUAD}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\,\ddot{Q}_{kl}\left(t-\frac{r}{c}\right)}$$

Suppose now that there is a source, and one wants to know the amplitudes of the wave, h_+ and $h_\times$, in a given direction $\hat{n}$, knowing the quadrupole moment of the source (or equivalently, the second mass moment). Can we give a more explicit formula than the one above?

For a generic matrix A, the projector $\Lambda_{ij,ke}(\hat{n})$ is such that:

$$\Lambda_{ij,ke} A_{ke} = \left[P_{ik} P_{je} - \frac{1}{2} P_{ij} P_{ke} \right] A_{ke}$$

$$= (PAP)_{ij} - \frac{1}{2} P_{ij} \, \mathrm{tr}(PA)$$

$$\Lambda_{ij,ke} \ddot{M}_{ke} = (P\ddot{M}P)_{ij} - \frac{1}{2} P_{ij} \, \mathrm{tr}(P\ddot{M})$$

here we can use directly the second mass moment M_{ij} since the trace does not contribute

First consider a wave propagating in the $\hat{n} \parallel \hat{z}$ direction.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{tr}\, P\ddot{M} = \ddot{M}_{11} + \ddot{M}_{22}, \quad P\ddot{M}P = \begin{pmatrix} \ddot{M}_{11} & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & \ddot{M}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_{ij,ke} \ddot{M}_{ke} = \begin{pmatrix} \frac{1}{2}(\ddot{M}_{11} - \ddot{M}_{22}) & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & -\frac{1}{2}(\ddot{M}_{11} - \ddot{M}_{22}) & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

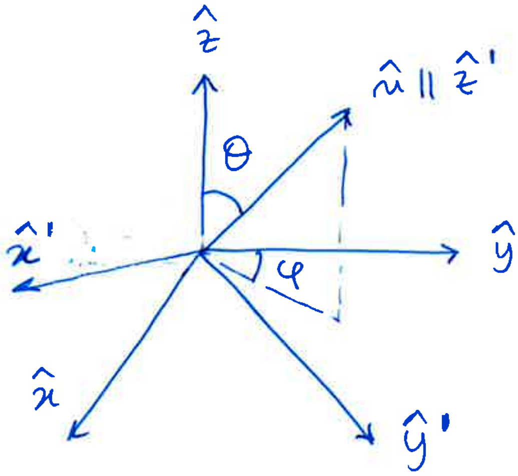
Therefore, the polarisation amplitudes of a wave propagating in the $\hat{z}$ direction become:

$$\boxed{\begin{cases} h_+(t, \hat{z}) = \dfrac{1}{r} \dfrac{G}{c^4} (\ddot{M}_{11} - \ddot{M}_{22})\left(t - \dfrac{r}{c}\right) \\[2ex] h_\times(t, \hat{z}) = \dfrac{2}{r} \dfrac{G}{c^4} \ddot{M}_{21}\left(t - \dfrac{r}{c}\right) \end{cases}}$$

Now let's generalise the above expression for a wave propagating in a generic direction $\hat{n}$:

In a reference frame where $\hat{n} \parallel \hat{z}'$, one has:

$$\begin{cases} h_+(t,\hat{n}) = \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}'_{11} - \ddot{M}'_{22}\right)\left(t-\frac{r}{c}\right) \\ h_\times(t,\hat{n}) = \frac{2}{r}\frac{G}{c^4}\ddot{M}'_{21}\left(t-\frac{r}{c}\right) \end{cases}$$



the rotation matrix that brings the reference frame (x',y',z') into (x,y,z) corresponds to a rotation $-\theta$ around the $\hat{x}$ axis and a rotation $-\varphi$ around the $\hat{z}$ axis:

$$R = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

With this one gets for example: $n_i = R_{ij} z'_j$

and the components of the second mass moment in the (x,y,z) frame become:

$$M_{ij} = R_{ik} R_{jl} M'_{kl}$$

In order to express the above amplitudes $h_+, h_\times$ in the reference frame (x,y,z) we need the inverse relation:

$$M'_{ij} = \left(R^T M R\right)_{ij}$$

(note that M has in principle all components different from zero, since the propagation direction $\hat{n}$ is generic)

Calculating $\ddot{M}_{ij}$ and inserting it into the expressions for h_+ and $h_\times$, we find the amplitudes h_+ and $h_\times$ of a GW propagating in the $\hat{n}$ direction arising from a source with second mass moment M:

$$h_+(t,\theta,\varphi) = \frac{1}{r}\frac{G}{c^4}\Big[\ddot{M}_{11}(\cos^2\varphi - \sin^2\varphi\cos^2\theta) + \ddot{M}_{22}(\sin^2\varphi - \cos^2\varphi\cos^2\theta) + - \ddot{M}_{33}\sin^2\theta + - \ddot{M}_{12}\sin 2\varphi(1+\cos^2\theta) + + \ddot{M}_{13}\sin\varphi\sin 2\theta + + \ddot{M}_{23}\cos\varphi\sin 2\theta\Big]$$

$$h_\times(t,\theta,\varphi) = \frac{1}{r}\frac{G}{c^4}\Big[(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\varphi\cos\theta + + 2\ddot{M}_{12}\cos 2\varphi\cos\theta + - 2\ddot{M}_{13}\cos\varphi\sin\theta + + 2\ddot{M}_{23}\sin\varphi\sin\theta\Big]$$

Once M_{ij} of the source is given, with this equations one computes the angular distribution of the quadrupole radiation.

— o —

It is clear from the above analysis that the leading term of the multipole expansion is the quadrupole.

$\boxed{\text{MONOPOLE AND DIPOLE RADIATION IS ABSENT FOR GWs.}}$

What does this mean?

MONOPOLE : $h_{00} \sim \frac{4G}{rc^4} \int d^3x' \, T^{00}\left(t-\frac{r}{c}, \underline{x}'\right)$

$= \frac{4G}{rc^4} M\left(t-\frac{r}{c}\right)$

since $\dot{M}=0$ h_{00} is a static component, not a radiative one

DIPOLE : $h_{0i} \sim \frac{4G}{rc^4} \int d^3x' \, T^{0i}\left(t-\frac{r}{c}, \underline{x}'\right)$

$= \frac{4G}{rc^4} p^i\left(t-\frac{r}{c}\right)$

idem since $\dot{p}^i=0$ for an isolated system

- a monopole term would depend on M, and a dipole term would depend on P^i (x^i can be set to zero by shifting the origin of the coordinate system). But a static source does not radiate : h_{ij}^{TT} depends on the time derivatives of the momenta. But $\dot{M} = 0$ for mass conservation, and $\dot{P}^i = 0$ for momentum conservation: these contributions must vanish.

- However, $\dot{M} = 0$ and $\dot{P}^i = 0$ are true only in linearized theory, where we assume that back-reaction is zero. In reality, a radiating system loses mass and linear momentum due to the emission of GWs. But even fully accounting for the non-linearities, it arises that the monopole and the dipole DO NOT RADIATE. This characteristic of GWs is more general than linearized theory, it is always valid.

- Actually, this is an expression of the fact that the graviton is a massless spin 2 field. As already stated on page 12/2 of the second chapter, these properties of the graviton are intimately connected with the fact that the only physical degrees of freedom of a GW are the two polarization states of the TT gauge. Property that we have used, together with $\partial_\mu T^{\nu\mu} = 0$, to derive the quadrupole formula. (17)

- As stated in page 12/2, it is also possible to connect the spin of the massless particle mediating the interaction with the lowest non-zero multipole of the radiation: all moments such that $\ell < s$ are always zero

- To summarize, the three properties

  1) the graviton is a massless field with spin 2

  2) the only two physical degrees of freedom of a GW propagating in vacuum are the two polarization states of the TT gauge

  3) the lowest non-zero multipole of the gravitational radiation field is the quadrupole $\ell = 2$, therefore a radiating source must posses at least a quadrupolar distribution to radiate.

are all expression of the same intrinsic physical nature of GWs.

RADIATED ENERGY FROM THE QUADRUPOLE

On page 51 of the second chapter, we calculated the energy flux per unit time and unit surface. Writing $dA = r^2 d\Omega$ this becomes, per unit time and angle:

$$\frac{dE}{dt\,d\Omega} = \frac{c^3 r^2}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle$$

From $h_{ij}^{TT}(t,\underline{x})$ of the QUADRUPOLE (page 12) one gets the power radiated per unit angle:

$$\left[\frac{dP}{d\Omega}\right]_{QUAD} = \frac{G}{8\pi c^5} \Lambda_{ij,kl}(\hat{n}) \langle \dddot{Q}_{ij}(t-\tfrac{r}{c}) \dddot{Q}_{kl}(t-\tfrac{r}{c}) \rangle$$

Integrating over the angle (last formula in 9bis of the second chapter) one gets the TOTAL POWER, the famous QUADRUPOLE FORMULA:

$$\boxed{[P]_{QUAD} = \frac{G}{5c^5} \langle \dddot{Q}_{ij}(t-\tfrac{r}{c}) \dddot{Q}_{ij}(t-\tfrac{r}{c}) \rangle}$$

(derived by Einstein in 1916)

or $$[P]_{QUAD} = \frac{G}{5c^5} \left\langle \dddot{M}_{ij} \dddot{M}_{ij} - \frac{1}{3} (\dddot{M}_{kk})^2 \right\rangle \Big|_{t-\frac{r}{c}}$$

in terms of the second mass moment.

to get the energy radiated per unit angle one F.T. the quadrupole moment of the source:

$$Q_{ij}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \tilde{Q}_{ij}(\omega)\, e^{-i\omega t}$$

$$\left[\frac{dE}{d\Omega}\right]_{QUAD} = \frac{G}{8\pi^2 c^5}\, \Lambda_{ij,kl}(\hat{n}) \int_0^{\infty} d\omega\, \omega^6\, \tilde{Q}_{ij}(\omega)\, \tilde{Q}^*_{kl}(\omega)$$

and then, first integrating over $d\Omega$ and then expressing this per unit frequency, one gets the ENERGY SPECTRUM from the quadrupole:

$$\boxed{\left[\frac{dE}{d\omega}\right]_{QUAD} = \frac{G}{5\pi c^5}\, \omega^6\, \tilde{Q}_{ij}(\omega)\, \tilde{Q}^*_{ij}(\omega)}$$

Note that there is no-loss of linear momentum in the quadrupole approximation:

from the flux of momentum derived on page 52 of the second chapter, inserting $[h_{ij}^{TT}]_{QUAD}$ one obtains:

$$\frac{dp^i}{dt} = -\frac{G}{8\pi c^5} \int d\Omega\, \dddot{Q}^{TT}_{ab}\, \partial^i \dddot{Q}^{TT}_{ab} \equiv 0$$

Q^{TT}_{ab} does not change under reflection $\underline{x} \to -\underline{x}$, while $\partial^i \to -\partial^i$. The above integrand is therefore odd under reflection so the angular integral vanish.

EQUIVALENT FORCE ACTING ON THE SOURCES

The energy carried by the GWs at a large distance from the source r and at time t must come from the source at time $t - \frac{r}{c}$.

(Note that this is only true in linearized gravity where the background spacetime is flat, the wave propagates at c without scattering on the background spacetime and without interacting with itself)

From the quadrupole (dominant term) the instantaneous rate of decrease of energy of the source must be:

$$\left.\frac{dE_{source}}{dt}\right|_{t-\frac{r}{c}} = -\left.\frac{dE_{QUAD}}{dt}\right|_{t} = -\frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle\left(t-\frac{r}{c}\right)$$

Since we are in linearized theory and the source is described by Newtonian theory, we can associate an equivalent force to this energy loss. We look for an expression: $\frac{dE_{source}}{dt} = \langle F_i v_i \rangle$.

Mass quadrupole:

$$Q_{ij} = \int d^3x\, \rho(t,\mathbf{x})\left(x_i x_j - \frac{1}{3}r^2\delta_{ij}\right)$$

[from $\partial_\mu T^{\mu 0} = 0$
$\partial_0(c^2\rho) + \partial_k(c\rho v_k) = 0$]

$$\frac{dQ_{ij}}{dt} = \int d^3x\, \partial_t\rho\left(x_i x_j - \frac{1}{3}r^2\delta_{ij}\right) = -\int d^3x\, \partial_k(\rho v^k)\left(x_i x_j - \frac{1}{3}r^2\delta_{ij}\right)$$

omitting the term proportional to the delta which goes to zero when contracted with the other Q_{ij}:

$$-\int d^3x\, \partial_k(\rho v^k)\, x_i x_j \underset{\substack{\uparrow \\ \text{by parts, the} \\ \text{boundary term} \\ \text{goes to zero}}}{=} +\int d^3x\, \rho\, \dot{x}^k\,(x_j \delta_{ki} + x_i \delta_{kj})$$

$$= \int d^3x\, \rho\,(\dot{x}_i x_j + \dot{x}_j x_i)$$

Now going back to the definition of $\frac{dE_{source}}{dt}$, integrating twice by parts inside the $\langle\ldots\rangle$ we get:

$$\frac{dE_{source}}{dt} = -\frac{G}{5c^5}\langle \dot{Q}_{ij}\, Q_{ij}^{(5)} \rangle = \quad \left(\begin{array}{c}\text{because of} \\ \text{symmetry}\end{array}\right)$$

$$= -\frac{2G}{5c^5}\langle Q_{ij}^{(5)} \int d^3x\, \rho\, \dot{x}_i x_j \rangle$$

$$= \langle \int d^3x \left(-\frac{2G}{5c^5}\rho(t,\underline{x})\, x_j\, Q_{ij}^{(5)}\right)\dot{x}_i$$

$$= \langle \int d^3x\, \frac{dF_i}{dV}\, \dot{x}_i \rangle \qquad \text{force per unit volume.}$$

The total effective force determining the energy loss of the GW source is therefore:

$$F_i = -\frac{2G}{5c^5}\, Q_{ij}^{(5)} \int d^3x'\, \rho(\underline{x}',t)\, x'_j$$

From this force we can also find the <u>angular</u>

momentum lost from the source and carried away by GWs:

$$\frac{dL^i_{source}}{dt} = \langle T^i \rangle$$

where the torque per unit volume is:

$$\frac{dT^i}{dV} = \varepsilon^{ijk} x_j \frac{dF_k}{dV} = -\frac{2G}{5c^5} \varepsilon^{ijk} \rho(t, \underline{x}) x_j x_e Q^{(5)}_{ke}$$

the torque becomes then:

$$T^i = -\frac{2G}{5c^5} \varepsilon^{ijk} Q^{(5)}_{ke} \int d^3x \, \rho(t, \underline{x}) \Big[x_j x_e - \underbrace{\tfrac{1}{3} r^2 \delta_{je}}_{\text{gives zero contracted with } Q^{(5)}_{ke}} \Big]$$

$$= -\frac{2G}{5c^5} \varepsilon^{ijk} Q^{(5)}_{ke} Q_{je}$$

Integrating by parts in the average $\langle T^i \rangle$ one gets

$$\frac{dL^i_{source}}{dt} = -\frac{2G}{5c^5} \varepsilon^{ijk} \langle \ddot{Q}_{ja} \dddot{Q}_{ka} \rangle$$

which is then $-\frac{dL^i}{dt}$ of the GWs: the

ANGULAR MOMENTUM RADIATED FROM THE QUADRUPOLE

$$\boxed{\left[\frac{dL^i}{dt}\right]_{QUAD} = \frac{2G}{5c^5} \varepsilon^{ijk} \langle \ddot{Q}_{ja} \dddot{Q}_{ka} \rangle}$$

THE OCTUPOLE AND CURRENT QUADRUPOLE

Going back to the equation derived in page 9 of this chapter:

$$h^{TT}_{ij}(t,x) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n})\left[S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m} + \dots\right]_{t-\frac{r}{c}}$$

we have analyzed the first term $S^{kl} = \frac{1}{2}\ddot{M}^{kl}$ which gives the quadrupole radiation.
This is the dominant term, and each following term is smaller by a factor $\frac{v}{c}$. ⇒ see back page

The third term gives the MASS OCTUPOLE and the CURRENT QUADRUPOLE. We have stated that: (page 11)

$$\dot{S}^{kl,m} = \underbrace{\frac{1}{6}\dddot{M}^{klm}}\ + \underbrace{\frac{1}{3}\left(\ddot{P}^{k,lm} + \ddot{P}^{l,km} - 2\ddot{P}^{m,kl}\right)}$$

This term gives the mass octupole radiation

$$M^{klm} = \int \frac{d^3x}{c^2}\, T^{00} x^k x^l x^m$$

where the action of $\Lambda_{ij,kl}$ removes all traces from the above expression.

This term gives the current quadrupole, which is the first moment of the angular momentum density symmetrized over k,l.

$$= \varepsilon^{mkp} J^{p,l} + \varepsilon^{mlp} J^{p,k}$$

with $J^{i,j} = \int d^3x\, j^i x^j$ and

j^i the ith component of the angular momentum density vector.

$$h_{QUAD} \sim \frac{\ddot{M}_{ij}}{c^4} \sim \frac{(\omega_s d)^2}{c^4} \sim \frac{1}{c^2}\left(\frac{v}{c}\right)^2$$

$$P_{QUAD} \sim c^3 \dot{h}^2 \sim \frac{\omega_s^2}{c^5}(\omega_s d)^4 \sim \frac{\omega_s^2}{c}\left(\frac{v}{c}\right)^4$$

$$h_{OCT} \sim \frac{\dddot{M}_{ijk}}{c^5} \sim \frac{(\omega_s d)^3}{c^5} \sim \frac{1}{c^2}\left(\frac{v}{c}\right)^3 \sim \frac{v}{c}\, h_{QUAD}$$

$$P_{OCT} \sim \frac{\left(M^{(4)}\right)^2}{c^7} \sim \frac{\omega_s^2}{c^7}(\omega_s d)^6 \sim \frac{\omega_s^2}{c}\left(\frac{v}{c}\right)^6 \sim \left(\frac{v}{c}\right)^2 P_{QUAD}$$

For these terms one can calculate, as for the quadrupole, the emitted power and the angular distribution of the polarisation amplitudes h_+, $h_\times$.
It is also possible to perform a systematic multipole expansion of the radiation field as for the EM case.
But since every higher moment is subdominant in the expansion $\frac{v}{c}$, and since we already have an expression for h_{ij}^{TT} correct at any order in $\frac{v}{c}$ ⊛ (but still in the linearised approximation - derived in page ③), we don't go further in this topic and instead analyse some examples of sources. These examples are idealised cases, but are the basis for analysing real sources.
For more insight, see chapter 3 of MAGGIORE book.

⊛ going beyond linearised theory and one order higher in the low-velocity expansion causes corrections to the quadrupole formula which are of the same order of the octupole and current quadrupole → so one needs to be careful if one wants to be consistent.

GW AND EQUIVALENCE PRINC.

Mass μ orbiting around mass M: an observer on μ sees no GW emission, because in the freely falling frame of μ, the mass is not accelerated.

However, considerations based on the equiv. princ. (the existence of the freely falling frame) are only valid for $r \ll \lambda\!\!\!^{-}, L_B$: this is the near zone. GWs appears in the far zone, so no wonder they are not included in considerations based on the equiv. principle. (RADIATION REACTION)

What about the $\frac{dE_{source}}{dt}$ and $\frac{dL_{source}}{dt}$? these are drowned into a sea of higher order effects: they are of $\left(\frac{v}{c}\right)^5$, while post-Newtonian GR corrections of the grav. potential are of order $\left(\frac{v}{c}\right)^2$ and so on. However, in the far zone all these vanish, because they decrease faster than $\frac{1}{r}$.

In the far zone, there is only the RADIATION FIELD.

RADIATION FROM POINT MASSES

General considerations

The energy momentum tensor of a point particle moving on a trajectory $x_0(t)$ in flat spacetime:
(in linearised theory, the sources are Newtonian, we use flat space-time)

$$T^{\mu\nu}(t,\underline{x}) = \frac{p^\mu p^\nu}{\gamma m}\,\delta^3(\underline{x}-\underline{x}_0(t)) \qquad p^\mu = \gamma m \frac{dx_0^\mu}{dt} = \left(\frac{E}{c}, \underline{p}\right)$$

For a set of point particles this becomes

$$T^{\mu\nu}_{TOT}(t,\underline{x}) = \Sigma_A \gamma_A m_A \frac{dx_A^\mu}{dt}\frac{dx_A^\nu}{dt}\,\delta^3(\underline{x}-\underline{x}_A(t))$$

The second mass moment, which enters the quadrupole formula, would be ($\gamma_A = 1$ in the non-rel. limit)

$$M^{ij}(t) = \Sigma_A m_A \int d^3x\, x^i x^j \underbrace{\delta^3(\underline{x}-\underline{x}_A(t))}_{T^{00}} =$$

$$= \Sigma_A m_A\, x_A^i(t)\, x_A^j(t)$$

In order to be able to use $T^{\mu\nu}_{TOT}$ on the right hand side of the wave equation

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

and to calculate the GWs emitted, the total energy momentum tensor must be conserved: $\partial^\mu T_{\mu\nu}^{TOT} = 0$. This is the case ONLY if the system is closed, i.e. if the trajectories $\underline{x}_A(t)$ are determined by the mutual influence of the particles, i.e. they are geodesics. BUT, if there are external forces acting on the particles which determine the trajectories $\underline{x}_A(t)$, then one cannot simply plug these trajectories into $T_{\mu\nu}^{TOT}$ and calculate the GWs, because $T_{\mu\nu}^{TOT}$ is not conserved: in order for it to be, one should account also for the objects that produce the forces acting on the particles and determining the trajectories $\underline{x}_A(t)$. Equivalently, we cannot trivially use the expression for $M^{ij}(t)$ derived above in the quadrupole formula without first thinking whether the system we are analysing is closed.

Suppose now that we have an ISOLATED two body system, and that $\underline{x}_0$ is the relative coordinate of the two systems in the CENTRE OF MASS FRAME. Then $\underline{x}_0(t)$ describes the relative trajectory of the two bodies as determined by their mutual interaction. In this case, we can use this trajectory into $T_{\mu\nu}$ and compute the GWs emitted.

Suppose the two bodies have masses m_1 and m_2:

$$\underline{x}_{cm} = \frac{m_1 \underline{x}_1 + m_2 \underline{x}_2}{m_1 + m_2} \qquad \underline{x}_0 = \underline{x}_1 - \underline{x}_2$$

$$\mu = \frac{m_1 m_2}{m} \qquad m = m_1 + m_2$$

the second mass moment becomes:

in principle one should demonstrate this also for all multipoles

$$M^{ij}(t) = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = m\, x_{cm}^i x_{cm}^j +$$

$$+ \mu \left(x_{cm}^i x_0^j + x_{cm}^j x_0^i \right) + \mu\, x_0^i x_0^j$$

If we choose $\underline{x}_{cm} \equiv 0$ (the center of mass frame) then this becomes the second mass moment of a particle with mass μ and trajectory $\underline{x}_0(t)$

If we adopt this reference system, we are left with a single particle of mass μ and coordinate $x_0^i(t)$: to calculate the GWs emitted, we can use the $T_{\mu\nu}$ or M^{ij} derived above where we substitute simply μ and $\underline{x}_0(t)$.

Let's see some examples.

QUADRUPOLE RADIATION FROM AN OSCILLATING MASS

We consider here a non-relativistic system with just one degree of freedom that performs harmonic oscillations along the z-axis

$$z_0(t) = a \cos \omega_s t$$

with $a\omega_s \ll c$ and $\omega_s \to 0$. This could be the system of two masses connected by a spring of rest length zero (not realistic but representative): $z_0(t)$ is then the relative coordinate in the center of mass frame, and μ is the reduced mass.

The mass density is (equivalent to a system of mass μ and trajectory $z_0(t)$):

$$\rho = \mu\, \delta(x)\, \delta(y)\, \delta(z - z_0(t))$$

And the second mass moment:

$$M^{ij}(t) = \int d^3x\, \rho\, x^i x^j = \mu\, z_0^2(t)\, \delta^{i3} \delta^{j3}$$

$$= \mu a^2 \cos^2 \omega_s t\, \delta^{i3} \delta^{j3}$$

$$= \mu a^2 \, \frac{1 + \cos 2\omega_s t}{2}\, \delta^{i3} \delta^{j3}$$

We now insert this into the expressions found on page 15 of this chapter

$$\begin{cases} h_+(t,\theta,\varphi) = -\frac{1}{r}\frac{G}{c^4}\ddot{M}_{33}(t-r/c)\sin^2\theta \\ \qquad = \frac{2G\mu a^2\omega_s^2}{r c^4}\sin^2\theta\cos(2\omega_s t) \\ h_\times(t,\theta,\varphi) = 0 \end{cases}$$

We get therefore that A NON-REL. SOURCE PERFORMING HARMONIC OSCILLATIONS WITH FREQUENCY ω_s EMITS MONOCROMATIC QUADRUPOLE RADIATION AT $2\omega_s$.

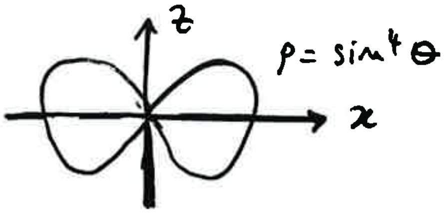
This is a general result; the chosen geometry furthermore tells us that:

- the angular distribution is independent on φ since the source has cylindrical symmetry
- there is only the + polarization in this ref. system.
- radiation vanishes along the z axis: for a wave propagating in direction $\hat{n}$, since $\Lambda_{ij,kl}\hat{n}^k = \Lambda_{ij,kl}\hat{n}^l = 0$ (the Λ tensor projects onto the plane orthogonal to $\hat{n}$), only the components of the motion of the source which are orthogonal to $\hat{n}$ contribute. No motion is orthogonal to $\hat{z}$, so no wave is propagating in the $\hat{z}$ direction. Correspondingly, the maximum of the emission arises for $\theta = \frac{\pi}{2}$.

The radiated power is (from page 51 of the second chapter, or alternatively page 18 of this chapter for the quadrupole)

$$\left[\frac{dP}{d\Omega}\right]_{QUAD} = \frac{r^2c^3}{16\pi G}\langle \dot{h}_+^2\rangle = \frac{G\mu^2 a^4 \omega_s^6}{\pi c^5}\sin^4\theta \,\langle \sin^2(2\omega_s t)\rangle$$

$$= \frac{G\mu^2 a^4 \omega_s^6}{2\pi c^5}\sin^4\theta$$



The total energy radiated during one period of the source motion $T = \frac{2\pi}{\omega_s}$ is therefore:

$$\langle E_{QUAD}\rangle_T = T\int d\Omega\,\frac{dP}{d\Omega} = T\,\frac{16}{15}\,\frac{G\mu^2 a^4\omega_s^6}{c^5} =$$

$$= \boxed{\frac{32\pi}{15}\,\frac{G\mu^2}{a}\left(\frac{v}{c}\right)^5}$$

$v = a\omega_s$ (maximum speed)

gravitational self-energy of an object with mass μ and size a (potential at the surface)

suppression of the energy radiated in the quadrupole approx. with respect to the self-gravitational energy of the emitting body.

If the spring has rest-length L, so that

$$z_0(t) = L + a\cos\omega_s t$$

then $\quad z_0^2(t) = \frac{a^2}{2}\cos(2\omega_s t) + 2La\cos\omega_s t + \text{const}$

and the wave amplitude becomes:

$$h_+(t,\theta,\varphi) = \frac{2G\mu\omega_s^2}{rc^4}\sin^2\theta\left[a^2\cos(2\omega_s t_{ret}) + aL\cos(\omega_s t_{ret})\right]$$

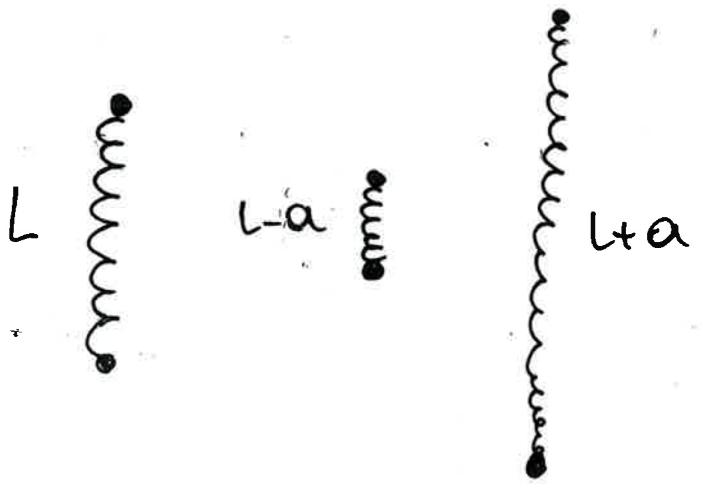
therefore, there is also radiation at $\omega = \omega_s$ and not only at $\omega = 2\omega_s$. The radiation at twice the characteristic frequency of the source is true only for a simple harmonic motion: in general, quadrupole radiation can have also other frequencies.

For example, if the source is a superposition of periodic motions and higher harmonics:

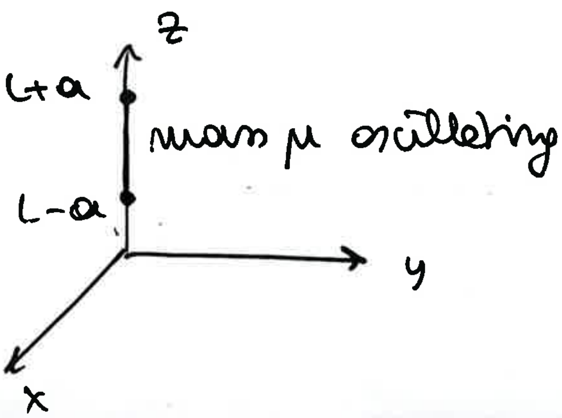
$$z_0(t) = a_1\cos\omega_s t + a_2\cos(2\omega_s t) + \dots$$

one gets quadrupole radiation at all $n\omega_s$.

The system:



$\Rightarrow \quad z_0(t) = L + a\cos(\omega_s t)$



BINARY SYSTEMS

page coherent from here on →

Consider an ISOLATED system composed of two point masses m_1 and m_2 moving on trajectories $\underline{x}_1(t)$ and $\underline{x}_2(t)$ determined only by their mutual interaction. The energy momentum tensor of this system is

$$T^{\mu\nu}_{TOT}(\underline{x},t) = m_1 \frac{dx_1^\mu}{dt}\frac{dx_2^\nu}{dt}\delta^3(\underline{x}-\underline{x}_1(t)) + $$
$$+ m_2 \frac{dx_2^\mu}{dt}\frac{dx_2^\nu}{dt}\delta^3(\underline{x}-\underline{x}_2(t))$$

($\gamma = 1$ for non-relativistic particles)

This energy momentum tensor is $\boxed{\text{conserved}}$, because the system is $\boxed{\text{isolated}}$. It can therefore be used on the right hand side of the wave equation for GW generation.

The corresponding second mass moment is:

$$M^{ij}(t) = \frac{1}{c^2}\int d^3x\, T^{00} x^i x^j = \sum_{A=1,2} m_A \int d^3x\, x^i x^j \delta^3(\underline{x}-\underline{x}_A(t))$$

$$= m_1 x_1^i(t)\, x_1^j(t) + m_2\, x_2^i(t)\, x_2^j(t)$$

We choose now the CENTRE OF MASS REFERENCE FRAME

$\underline{x}_0(t) = \underline{x}_1(t) - \underline{x}_2(t)$ { is the relative trajectory of the two bodies as determined by their mutual interaction.

$\underline{x}_{CM}(t) = \dfrac{m_1 \underline{x}_1 + m_2 \underline{x}_2}{m_1 + m_2} \equiv 0$ { is the position of the centre of mass in its frame

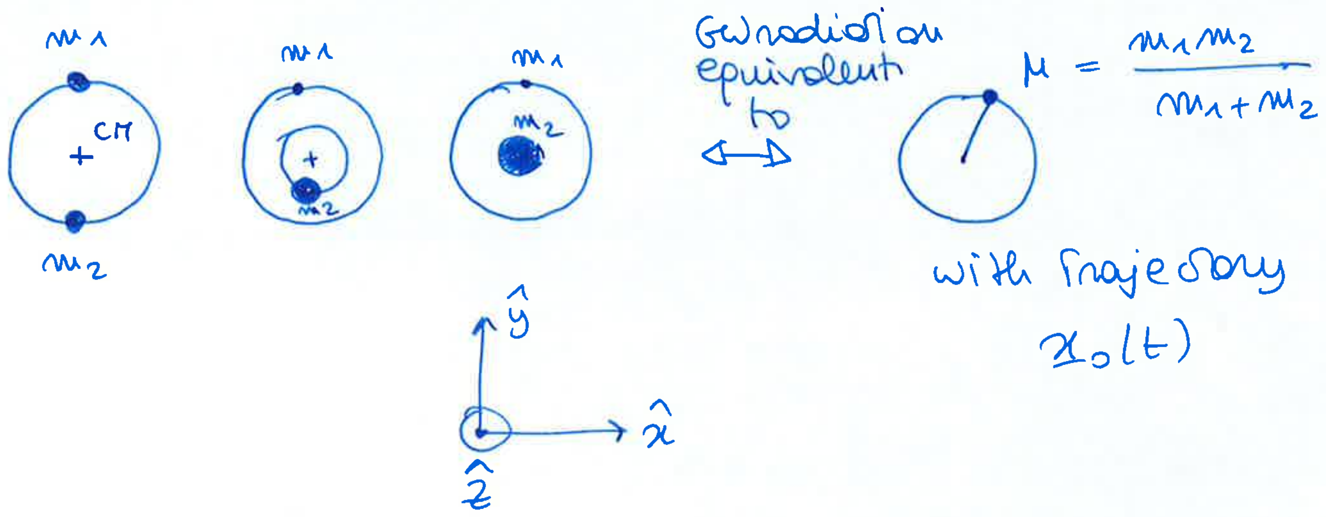
The second mass moment in this case becomes just

$$\boxed{M^{ij} = \mu\, x_0^i(t)\, x_0^j(t)} \qquad (\text{for } x_{cm} \equiv 0)$$

This is the second mass moment of a particle with mass μ (reduced mass) and trajectory $x_0(t)$ (corresponding to the relative trajectory between the two bodies)

To evaluate the quadrupole emission, adopting the CM frame, we can use the M^{ij} above: the GW emission is equivalent to the one of one body of mass μ and trajectory. $x_0(t)$

We use this fact to describe the quadrupole radiation of a binary system in circular orbit: it is equivalent to the quadrupole radiation from a single mass in circular orbit



QUADRUPOLE RADIATION FROM A MASS IN CIRCULAR ORBIT (BINARY SYSTEM IN CM FRAME)

Consider two masses m_1 and m_2, of which the relative coordinate is performing a circular motion.
We assume that the orbital motion is given: therefore, we are neglecting any back-reaction on the motion due to the emission of GWs. The effect of GW back-reaction WITHIN LINEARISED THEORY is accounted for in the part about BINARY INSPIRALS. This can describe the system at lowest order in $\frac{v}{c}$ (flat space-time)

The relative trajectory in the CM frame is

$$\begin{cases} x_0(t) = R\cos\left(\omega_s t + \frac{\pi}{2}\right) \\ y_0(t) = R\sin\left(\omega_s t + \frac{\pi}{2}\right) \\ z_0(t) = 0 \end{cases}$$

the particle of mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is performing circular motion with frequency ω_s, radius R, in the $(\hat{x}, \hat{y})$ plane.
The second mass moment in the CM frame is (page 20):

$$\begin{cases} M_{11} = \mu R^2 \, \frac{1 - \cos(2\omega_s t)}{2} \\ M_{22} = \mu R^2 \, \frac{1 + \cos(2\omega_s t)}{2} \\ M_{12} = \mu R^2 \left(-\frac{\sin(2\omega_s t)}{2}\right) \end{cases}$$

The other components vanish because the trajectory is in the $(\hat{x}, \hat{y})$ plane. To evaluate the GW amplitude one needs the second time derivative:

$$\begin{cases} \ddot{M}_{11} = 2\mu R^2 \omega_s^2 \cos(2\omega_s t) = -\ddot{M}_{22} \\ \ddot{M}_{12} = 2\mu R^2 \omega_s^2 \sin(2\omega_s t) \end{cases}$$

Substituting in the expressions found on page (15) we get the wave signal in direction $\hat{n} = (\theta, \varphi)$:

$$\boxed{\begin{aligned} h_+(t,\theta,\varphi) &= \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \left(\frac{1+\cos^2\theta}{2}\right) \cos(2\omega_s t_{ret} + 2\varphi) \\ h_\times(t,\theta,\varphi) &= \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \cos\theta \sin(2\omega_s t_{ret} + 2\varphi) \end{aligned}}$$

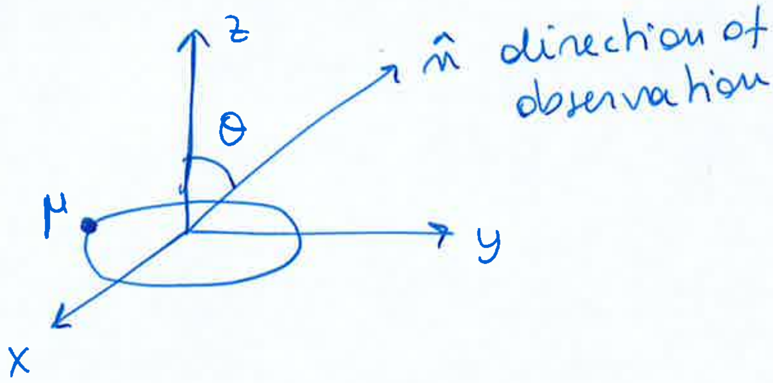
•) we see here one example of the fact that A NON-RELATIVISTIC SOURCE PERFORMING HARMONIC OSCILLATIONS WITH FREQUENCY ω_s EMITS MONOCROMATIC QUADRUPOLE RADIATION AT FREQUENCY $2\omega_s$.

•) the angular dependence on θ is common to all systems with $\ddot{M}_{11} = -\ddot{M}_{22}$ and $M_{13} = M_{23} = M_{33} = 0$.
(it's a good choice of the axis of the binary position)

•) the angular dependence on φ is due to the fact that if the mass μ is at one point on the orbit at time t, a rotation around the $\hat{z}$ axis by an angle $\Delta\varphi$ brings it to the point in the orbit it would reach after a time interval $\omega_s \Delta t = \Delta\varphi$. no proper motion of the source?

Therefore this φ dependence can be re-absorbed into a redefinition of the origin of time, and is not important from the point of view of an observer.

•) On the other hand, θ denotes the angle between the perpendicular to the plane of the orbit and the direction of observation



•) FROM THE DEGREE OF POLARISATION OBSERVED, ONE CAN INFER THE INCLINATION OF THE ORBIT, because the wave polarisations depend on the line of sight:

1) if $\theta = \frac{\pi}{2}$, we see the orbit edge-on: there is only + polarisation.

2) if $\theta = 0$, $h_+ = h_\times$: the radiation is circularly polarized, meaning that it describes a "circle in the plane $(h_+, h_\times)$" in time. The lines of force (see page 32 of the second chapter) rotate in time:

3) for other values of θ, the h_+ and $h_\times$ amplitudes are

different, so the circular polarization becomes elliptic.

RADIATED POWER PER UNIT SOLID ANGLE

From the expressions of h_+ and $h_\times$ we can calculate the radiated power (from page 51 of the second chapter, or page 18 of this chapter for the quadrupole):

$$\left[\frac{dP}{d\Omega}\right]_{QUAD} = \frac{r^2 c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

from the time derivative

$$= \frac{r^2 c^3}{16\pi G} \left(\frac{1}{r} \frac{4 G \mu \omega_s^2 R^2}{c^4} 2\omega_s \right)^2$$

$$\left[\left(\frac{1+\cos^2\theta}{2} \right)^2 \langle \sin^2(2\omega_s t_{ret} + 2\varphi) \rangle + \right.$$

$$\left. + \cos^2\theta \langle \cos^2(2\omega_s t_{ret} + 2\varphi) \rangle \right]$$

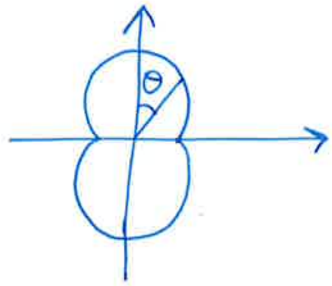
Here the average $\langle \ldots \rangle$ can be considered over several periods of the wave. Therefore, it applies only to the parts of the above expression that depend on time. We use simply:

$$\langle \sin^2(2\omega_s t_{ret} + 2\varphi) \rangle = \langle \cos^2(2\omega_s t_{ret} + 2\varphi) \rangle = \frac{1}{2}$$

and get finally:

$$\boxed{\left[\frac{dP}{d\Omega}\right]_{QUAD} = \frac{2G\mu^2 R^4 \omega_s^6}{\pi c^5}\left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right]}$$

The dependence of the radiated power on the angle of observation θ has the shape:



it is maximal for $\theta = 0$, when one looks at the source from the direction orthogonal to the plane of the orbit. It never vanishes, since from each direction one looks at the system, there is always a component of the motion orthogonal to the line of sight: the components of the motion of the source which are in the direction of observation do not contribute to the radiation, since $\Lambda_{ij,kl}(\hat{n})\,\hat{n}^k = \Lambda_{ij,kl}(\hat{n})\,\hat{n}^l = 0$ (see page 9 of the second chapter).

The TOTAL EMITTED POWER is obtained integrating over the angle $d\Omega$: $\int d\Omega \left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right] = \frac{16\pi}{5}$

$$\boxed{[P]_{QUAD} = \frac{32}{5}\frac{G\mu^2 R^4 \omega_s^6}{c^5}}$$

From this, we can get the ENERGY RADIATED IN ONE PERIOD OF THE SOURCE $\boxed{T = \frac{2\pi}{\omega_s}}$:

$$\langle E_{QUAD} \rangle_T = \frac{64\pi}{5} G\mu^2 R^4 \left(\frac{\omega_s}{c}\right)^5$$

using $v = \omega_s R$

$$= \boxed{\frac{64\pi}{5} \frac{G\mu^2}{R} \left(\frac{v}{c}\right)^5}$$

this corresponds to the gravitational self-energy of the binary system with reduced mass μ and orbital radius R

this factor is the suppression of the energy radiated over a cycle in the quadrupole approximation

We see that the energy emitted in GW in the quadrupole approximation, which is the lowest order of the $\frac{v}{c}$ expansion (see page 9 and following) is in general suppressed with respect to the self energy of the system by a factor $\left(\frac{v}{c}\right)^5$.

Correspondingly, we could calculate the energy emitted in GW by the same system accounting for the following term in the expansion on page 9 this term is composed by the mass octupole and current quadrupole :

$$\left[h_{ij}^{TT}(t, x)\right]_{\substack{\text{MASS}\\ \text{OCTUPOLE}\\ \text{AND CURRENT}\\ \text{QUADRUPOLE}}} = \frac{1}{r}\frac{4G}{c^5}\Lambda_{ij,kl}(\hat{n})\, n_m \dot{S}^{kl,m}$$

with $$\dot{S}^{kl,m} = \underbrace{\frac{1}{6}\dddot{M}^{klm}}_{\substack{\text{mass}\\ \text{octupole}}} + \underbrace{\frac{1}{3}\left(\ddot{P}^{k,lm} + \ddot{P}^{l,km} - 2\ddot{P}^{m,kl}\right)}_{\substack{\text{current quadrupole}\\ \text{(it is related to the first moment of}\\ \text{the angular momentum density)}}}$$

(see the expression derived on page 11 of this chapter).

It is simpler to calculate h_{ij}^{TT} from this contribution directly from the definition $S^{kl,m} = \int d^3x\, T^{kl} x^m$ (page 8).

From the energy momentum tensor given on page 19, one carries on the calculation on the same lines as done for the quadrupole. The most important results are

1) The frequencies of the radiation are now ω_s and $3\omega_s$: the spectrum of the GW radiation gets enriched with respect to the pure quadrupole emission at $2\omega_s$.

2) The amplitude of the emitted power is however SMALLER than the one of the quadrupole by a factor $\left(\frac{v}{c}\right)^2$: this is coherent with the meaning of the multipole expansion

$$\boxed{[P]_{\substack{OCT+\\C.Q.}}(\omega_s) \simeq [P]_{\substack{OCT+\\C.Q.}}(3\omega_s) \simeq \left(\frac{v}{c}\right)^2 [P]_{QUAD}(2\omega_s)}$$

NOTE : the leading term of the quadrupole gets corrections of the order $\left(\frac{v}{c}\right)^2$ also if one goes beyond linear theory, and performs the POST NEWTONIAN expansion of the source. To be consistent at order $\left(\frac{v}{c}\right)^2$ beyond the quadrupole (therefore, order $\left(\frac{v}{c}\right)^7$ in the emitted power) these post-newtonian corrections should be accounted for, in addition to the mass octupole and current quadrupole corrections.

PN expansion is equivalent to going beyond linearized theory which is equivalent to going beyond linear order in the v/c expansion.

INSPIRAL OF COMPACT BINARIES

We now study in a bit more detail the system of two point-like bodies orbiting around each other. With respect to what analyzed previously, here we go a bit beyond the Newtonian trajectory, evaluating the back-reaction of the GW on the emitting system, but always in the context of linearized theory. We will see that the GW emission affects the motion of the source inducing the inspiral and the coalescence; and we will see how the inspiral affects in turns the GW emission.

CIRCULAR ORBIT : THE CHIRP MASS

The physical system is the same as analysed previously. We rewrite the GW amplitudes derived on page 22 in terms of observable quantities defining:

1. $$\boxed{f_{GW} = \frac{\omega_{GW}}{2\pi} = \frac{\omega_s}{\pi}}$$

2. THE CHIRP MASS
$$\mathcal{M}_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

In the CM frame the system is equivalent to a one-body problem for a particle of mass μ with acceleration $\ddot{\underline{x}}_0 = -\frac{Gm}{|\underline{x}_0|^2}\hat{x}_0$

3. From the Newtonian dynamics in CM frame:

$$v = \omega_s R \qquad a = \frac{v^2}{R} = \omega_s v \qquad m = m_1 + m_2$$

$$\frac{v^2}{R} = \frac{Gm}{R^2} \Rightarrow \boxed{\omega_s^2 = \frac{Gm}{R^3}}$$

(Note that this relation is the one given on page 3 of the beginning of this course, with $2\omega_s = \omega_{GW}$)

..

$$\boxed{\begin{aligned} h_+(t) &= \frac{4}{\pi}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{GW}}{c}\right)^{\frac{2}{3}}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\pi f_{GW} t_{ret} + 2\varphi\right) \\ h_\times(t) &= \frac{4}{\pi}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{GW}}{c}\right)^{\frac{2}{3}}\cos\theta \;\sin\left(2\pi f_{GW} t_{ret} + 2\varphi\right)\end{aligned}}$$

so that the GW amplitudes depend on the masses of the binary ONLY through the combination given by the CHIRP MASS defined above. Consequently, this is true also for the emitted power: from the expression given in page 25 we find:

$$\begin{cases} \left[\dfrac{dP}{d\Omega}\right]_{QUAD} = \dfrac{2}{\pi}\dfrac{c^5}{G}\left(\dfrac{GM_c\,\omega_{GW}}{2c^3}\right)^{\frac{10}{3}}\left[\left(\dfrac{1+\cos^2\theta}{2}\right)^2+\cos^2\theta\right] \\ \left[P\right]_{QUAD} = \dfrac{32}{5}\dfrac{c^5}{G}\left(\dfrac{GM_c\,\omega_{GW}}{2c^3}\right)^{\frac{10}{3}} \end{cases}$$

Up to now we have assumed that the motion of the sources is on a given, fixed, circular keplerian orbit. However, the emission of GWs carries away energy from the source, also in simple linearized theory: the way to account for this back-reaction of GWs on the source in linearized theory, is to postulate that the energy lost from the source in unit time is equal and opposite to the power carried by the GWs in the (*) wave-zone far away from the observer (as we have defined it). 31

We now study the effect of the GW emission on the orbit of the source.

CIRCULAR ORBIT : TIME TO COALESCENCE

The source for the GW emission is the total energy of the binary:

$$v^2 = \frac{Gm}{R} \quad \text{(true for circular orbit)}$$

$$E_{kin} + E_{pot} = \frac{1}{2}\mu v^2 - \frac{Gm_1m_2}{R} = -\frac{Gm_1m_2}{2R} = E_{orbit}$$

$$= \frac{1}{2}E_{POT}$$

the total energy must diminish, so R must diminish as well. From:
(compensate)

1. $\omega_s^2 = \frac{Gm}{R^3} \Rightarrow$ R decreases, ω_s increases

2. $[P]_{QUAD} \propto \omega_{GW}^{10/3} = (2\omega_s)^{\frac{10}{3}} \Rightarrow$ ω_s increases, the emitted power increases.

3. Emitted power increases $\Rightarrow$ R decreases further

THE RESULT OF THIS RUNAWAY PROCESS IS THE COALESCENCE OF THE BINARY SYSTEM.

let's analyse the situation in the approximation that the orbit is always circular with a slowly varying radius. From the equation $\omega_s^2 = Gm/R^3$

(*) Note that this is not so obvious outside linear theory: in that case one has to take into account non-linear effects in the propagation of the GW like scattering (represented by $[R^{(2)}_{\mu\nu}]^{\text{high}}$ on page (54))

on the background and with itself: a result of these effects at higher order in the PN expansion is that part of the GW is delayed and besides the wavefront traveling at c the GW develops a tail arriving later. Therefore it is not obvious that one can equate the power in GW at large distances at time t and the rate of energy loss per unit time at the source position and at corresponding tret.

one gets:

$$v = \omega_s R$$

$$R \propto \omega_s^{-2/3} \Longrightarrow \dot{R} = -\frac{2}{3} R \frac{\dot{\omega}_s}{\omega_s} = -\frac{2}{3} v \frac{\dot{\omega}_s}{\omega_s^2}$$

therefore $\boxed{|\dot{R}| \ll v}$ with v the tangential velocity iff $\dot{\omega}_s \ll \omega_s^2$

therefore, the orbit stays circular and the variations of the radius $|\dot{R}|$ are much smaller than the tangential velocity if

$$\boxed{\dot{\omega}_s \ll \omega_s^2}$$

From $R = \frac{(Gm)^{\frac{1}{3}}}{\omega_s^{2/3}}$ one can rewrite the total energy

$$E_{orbit} = -\left(\frac{G^2 M_c^5 \omega_s^2}{32}\right)^{\frac{1}{3}}$$, and using

BACK-REACTION IN LINEARISED THEORY:

$$\boxed{-\frac{dE_{orbit}}{dt} = [P]_{QUAD}}$$

THE ENERGY LOST FROM THE ORBIT IS EQUAL TO THE ENERGY EMITTED IN GW

and rewriting everything in terms of $f_{GW} = \frac{\omega_{GW}}{2\pi} = \frac{\omega_s}{\pi}$

one gets the differential equation which tells us how the GW frequency increases in time due to the emission of the GWs and consequent shrinking of the orbit radius: (both at retarded time! $[P]_{QUAD} = \dots \langle \dddot{Q} \dddot{Q} \rangle (t_{ret})$)

$$\boxed{\dot{f}_{GW} = \frac{96}{5} \pi^{\frac{8}{3}} \left(\frac{G M_c}{c^3}\right)^{\frac{5}{3}} f_{GW}^{\frac{11}{3}}}$$

Integrating this differential equation one sees that there

there is a divergence at a given value of time: in reality, the divergence is cut off by the fact that, when the separation among the two objects becomes smaller than a given critical size, the two objects merge. In our setting we cannot describe this merging; we therefore write the solution in terms of $\boxed{\tau = t_{coal} - t}$ where t_{coal} denotes the coalescence time:

$$\boxed{f_{GW}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{\frac{3}{8}}\left(\frac{G M_c}{c^3}\right)^{-\frac{5}{8}}\left(\frac{1}{\tau}\right)^{\frac{3}{8}}}$$

time to coalescence

Inserting numbers one gets:

$$\boxed{\begin{aligned} f_{GW}(\tau) &\simeq 134\,\text{Hz}\left(\frac{1.21\,M_\odot}{M_c}\right)^{\frac{5}{8}}\left(\frac{1\,\text{sec}}{\tau}\right)^{\frac{3}{8}} \\ \tau &\simeq 2.18\,\text{sec}\left(\frac{1.21\,M_\odot}{M_c}\right)^{\frac{5}{3}}\left(\frac{100\,\text{Hz}}{f_{GW}}\right)^{\frac{8}{3}} \end{aligned}}$$

The normalization at $1.21\,M_\odot$ corresponds to the chirp mass of two neutron stars at $1.4\,M_\odot$.

Typical frequency ranges of detection for earth-based detectors are $10\,\text{Hz} < f < 10^3\,\text{Hz}$.

This corresponds, for a NS binary, to

$$\boxed{5\,\text{msec} < \tau < 17\,\text{min}}$$

⊛ Example at eLISA: $M_c = 10^5 M_\odot$

$$10^{-4}\,Hz < f_{GW} < 10^{-2}\,Hz$$

$$\tau = 2.18\,sec \cdot 10^{-\frac{25}{3}} \begin{cases} 10^{6\cdot\frac{8}{3}} \\ 10^{4\cdot\frac{8}{3}} \end{cases} = 2.18\,sec \cdot \begin{cases} 10^{\frac{48-25}{3}} = \frac{23}{3} \\ 10^{\frac{32-25}{3}} = \frac{7}{3} \end{cases}$$

$$= \begin{cases} 10^8\,sec \\ 463\,sec \end{cases} \simeq \begin{cases} 3\text{ years} \\ 8\text{ min} \end{cases}$$

$\tau \propto \left(\frac{M_\odot}{M_c}\right)^{5/3} \left(\frac{Hz}{f_{GW}}\right)^{8/3}$: at the same frequency, if M increases then the observable time to coalescence decreases.

LIGO has seen 1) $M_c = 28$ $\begin{pmatrix} M_1 = 36 \\ M_2 = 29 \end{pmatrix}$ 8 cycles 0.2 sec

2) $M_c = 8$ $\begin{pmatrix} M_1 = 14 \\ M_2 = 7.5 \end{pmatrix}$ 55 cycles 1 sec

we can therefore observe the last stages of the inspiral. ⊛ Example at eLISA.

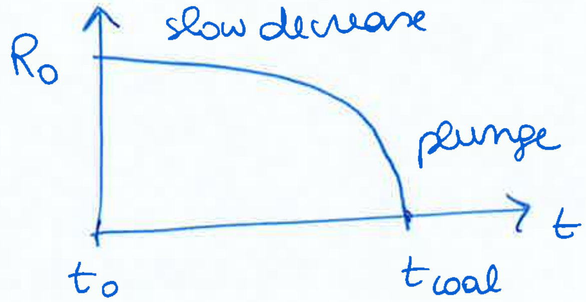
We can also find how the radius of the orbit evolves in time! From page (32) we have:

$$\frac{\dot{R}}{R} = -\frac{2}{3}\frac{\dot{\omega}_s}{\omega_s} = -\frac{2}{3}\frac{\dot{f}_{GW}}{f_{GW}} = -\frac{1}{4}\frac{1}{\tau}$$

from eq. on page (32) for $\dot{f}_{GW}$

this equation can be integrated, calling R_0 the radius at initial time t_0 one gets:

$$R(t) = R_0 \left(\frac{t_{coal} - t}{t_{coal} - t_0} \right)^{\frac{1}{4}}$$



there is a long phase during which R decreases smoothly (quasi-circular orbit, the limit in which we are considering the problem) followed by a PLUNGE PHASE: the approximation of Newtonian (point like objects) source, flat spacetime, quasi-circular orbit in linearized theory cannot describe this phase properly.

Inserting the expression for $f_{GW}(t)$ page (33) taken at initial time t_0 into $\omega_s^2 = \frac{Gm}{R^3}$, one finds the following relationship between the total time to coalescence

$$\tau_0 = t_0 - t_{coal}$$

and the initial radius R_0 :

$$\boxed{\tau_0 = \frac{5}{256} \frac{c^5 R_0^4}{G^3 m^2 \mu}}$$

If one defines also the initial orbital period $T_0 = \frac{2\pi}{\omega_s(\tau_0)}$, then $R_0^3 = Gm\left(\frac{T_0}{2\pi}\right)^2$. Inserting numerical values, the final result for τ_0 can be written as

$$\boxed{\tau_0 \simeq 10^7 \text{ years} \left(\frac{T_0}{1\,\text{hour}}\right)^{\frac{8}{3}} \left(\frac{M_\odot}{m}\right)^{2/3} \left(\frac{M_\odot}{\mu}\right)}$$

The approximative age of galaxies similar to the Milky way is of 10^{10} years. Considering a binary system of masses of the order of the $M_\odot$, it must have an initial period smaller than about one day to have coalesced within the present age of galaxies (and being therefore observable, at least in principle, through GWs: remember that an earth-based interferometers observes the last stages of the inspiral, page 33).

In Schwarzschild geometry, there is a minimum value of the radial distance beyond which stable circular orbits are no longer allowed called the

Innermost Stable Circular Orbit. In the test-mass limit (test mass around a BH) the radius is:

$$R_{isco} = \frac{6Gm}{c^2}$$

In our case we can set $m = m_1 + m_2$ the total mass of the binary, and assign a source frequency to it: (avoid the divergence of the formula on page 33)

$$\boxed{f_{isco} = \frac{1}{6\sqrt{6}(2\pi)} \frac{c^3}{Gm} \simeq 2.2 \text{ kHz} \left(\frac{M_\odot}{m}\right)}$$

NS binary with $m = 2.8 M_\odot$: $f_{isco} \simeq 800$ Hz
BH binary with $m = 10 M_\odot$: $f_{isco} \simeq 200$ Hz
SMBH binary with $m = 10^6 M_\odot$: $f_{isco} \simeq 2$ mHz

The coalescence frequency falls into the sensitivity range of Earth-based interferometers for the first two binary types, while the last binary type falls into the sensitivity range of a space-based, eLISA type interferometer.

Up to now we have considered the changes in the GW frequency due to the shrinking of the orbit. However, the GW amplitudes h_+ and $h_\times$ also change in time with respect to the prediction made on page 30 - 22.

Let's see how to model this.

CIRCULAR ORBIT: THE CHIRP AMPLITUDE

In order to calculate the GW emission from a particle on a circular orbit, we have evaluated the second time derivative of the second mass moment (quadrupole), see page 22. However, if the radius and the angular velocity are now a function of time, the orbit (always in the (x,y) plane) becomes:

$$x(t) = R(t)\cos\left(\frac{\phi(t)}{2}\right), \quad y(t) = R(t)\sin\left(\frac{\phi(t)}{2}\right)$$

$$\phi(t) = 2\int_{t_0}^{t} dt'\,\omega_s(t') = \int_{t_0}^{t} dt'\,\omega_{GW}(t')$$

(t_0: some initial time)

Therefore, when we take the second derivative of the mass moment with respect to time, we should also include in the derivation the dependence on time of $R(t)$ and $\omega_s(t)$. However, if we are in the situation for which $\dot{\omega}_s \ll \omega_s^2$, implying $|\dot{R}| \ll \omega_s R$ this means $\frac{G m_c \omega_{GW}}{c^3} \ll 0.5$ (from the relation $\dot{f}_{GW}$) giving then $f_{GW} \ll 13\,\mathrm{kHz}\left(\frac{1.2 M_\odot}{m_c}\right)$. It turns out that the plunge phase (to be compared with f_{ISCO} on page 36) occurs already at lower frequency: effectively all terms proportional to $\dot{\omega}_s$ and $\dot{R}$ can be dropped, at least in a first approximation.

$\dot{\omega}_s \ll \omega_s^2$

$$\frac{96}{5} \pi^{8/3} \left(\frac{G M_c}{c^3}\right)^{5/3} f_{GW}^{11/3} \ll \frac{1}{2} (2\pi) f_{GW}^2$$

$$f_{GW} \ll \left(\frac{5}{96 \pi^{5/3}} \left(\frac{c^3}{G M_c}\right)^{5/3}\right)^{3/5} = 13 \text{ kHz} \left(\frac{1.2 M_\odot}{M_c}\right)$$

$f_{isco} = 2.2 \text{ kHz} \frac{M_\odot}{m}$

in general the break-down of the stable circular orbit occurs before $\dot{\omega}_s \simeq \omega_s^2$

As a consequence, the GW amplitudes become as those on page 30, with f_{GW} which now depends on time and $\phi(t)$ in the argument of the sin, cos:

$$h_+(t) = \frac{4}{\pi}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{GW}(t_{ret})}{c}\right)^{\frac{2}{3}}\frac{1+\cos^2\theta}{2}\cos(\phi(t_{ret}))$$

$$h_\times(t) = \frac{4}{\pi}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{GW}(t_{ret})}{c}\right)^{\frac{2}{3}}\cos\theta\,\sin(\phi(t_{ret}))$$

where the 2φ dependence has been reabsorbed into a shift in the origin of time, or a reorientation of the axes

Now the quantity $\phi(t)$ can be evaluated from the expression of $f_{GW}(\tau)$ given in page 33:

$$\boxed{\phi(\tau) = -2\left(\frac{5GM_c}{c^3}\right)^{-\frac{5}{8}}\tau^{\frac{5}{8}} + \phi_0}$$

where we have fixed the integration boundary such that ϕ_0 is the frequency at coalescence. Therefore, we can reexpress the GW amplitudes directly in terms of $\tau = t_{coal} - t$:

$$h_+(t) = \frac{1}{\pi}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{4}}\left(\frac{5}{c\tau}\right)^{\frac{1}{4}}\left(\frac{1+\cos^2\theta}{2}\right)\cos\phi(\tau)$$

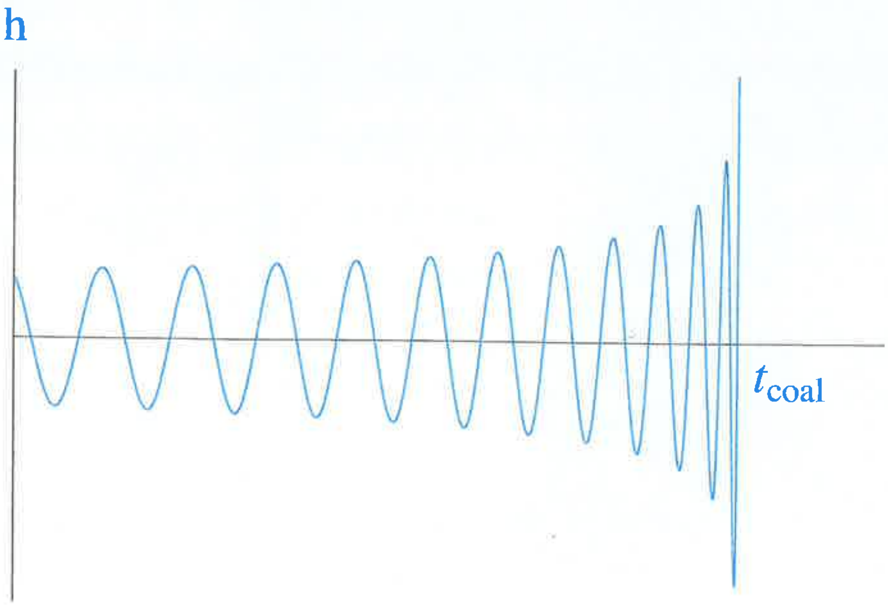
$$h_\times(t) = \frac{1}{\pi}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{4}}\left(\frac{5}{c\tau}\right)^{\frac{1}{4}}\cos\theta\,\sin\phi(\tau)$$

here we can forget about retarded time, since it gives a constant shift between t and t_{coal}, so $\tau = (t_{coal})_{ret} - t_{ret} = t_{coal} - t$.

From the above expressions, it appears that as $t \to t_{coal}$ so that τ decreases, we have

1. the amplitude increases
2. the frequency increases
   (in the sense that $\phi(t)$ approaches ϕ_0)

The waveforms of $h_+(t)$ and $h_\times(t)$ look qualitatively like this:



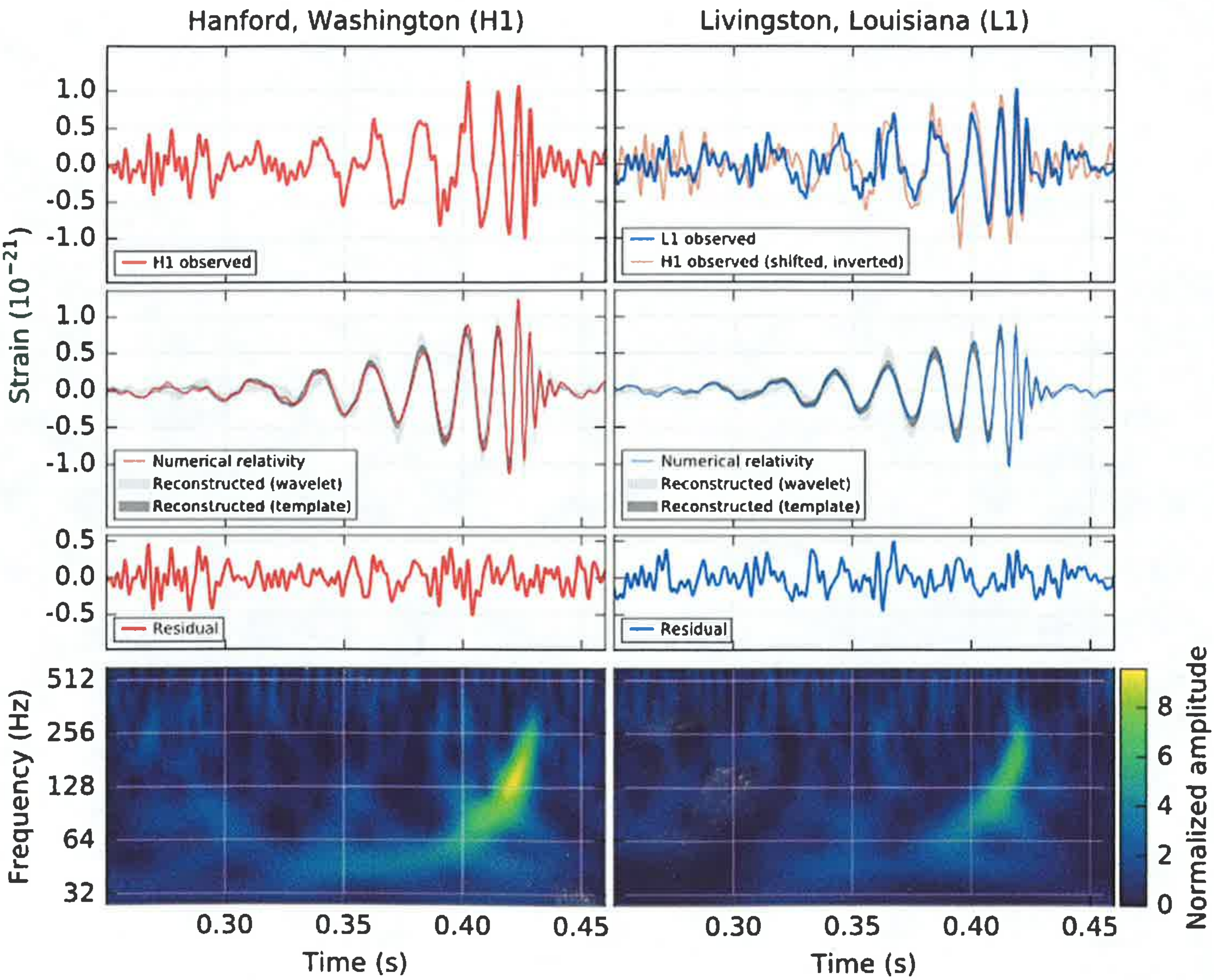
This behaviour is known as "chirping", as the chirp of a bird. The F.T. of this signal is not easy to calculate because of the finite time interval of definition, but the frequency spectrum is $dE/df \propto f^{-1/3}$ (c.f. MAGGIORE)

One recognizes the gross features of the aLIGO detections in 2015: the chirping waveform (though here it is from numerical relativity) and the behaviour with time to coalescence as $f_{GW}(t) \propto \tau^{-3/8}$ (cp. page 33)

$\boxed{\text{GW150914}}$

SNR = 24

$>$ 5.1σ detection



$M_1 = 36\, M_\odot\ (+5, -4)$

$M_2 = 29\, M_\odot\ (\pm 4)$

$M_{BH} = 62\, M_\odot\ (\pm 4)$

$d_L = 410\, M_\odot\ (+160, -180)$

$z = 0.09\ (+0.03, -0.04)$

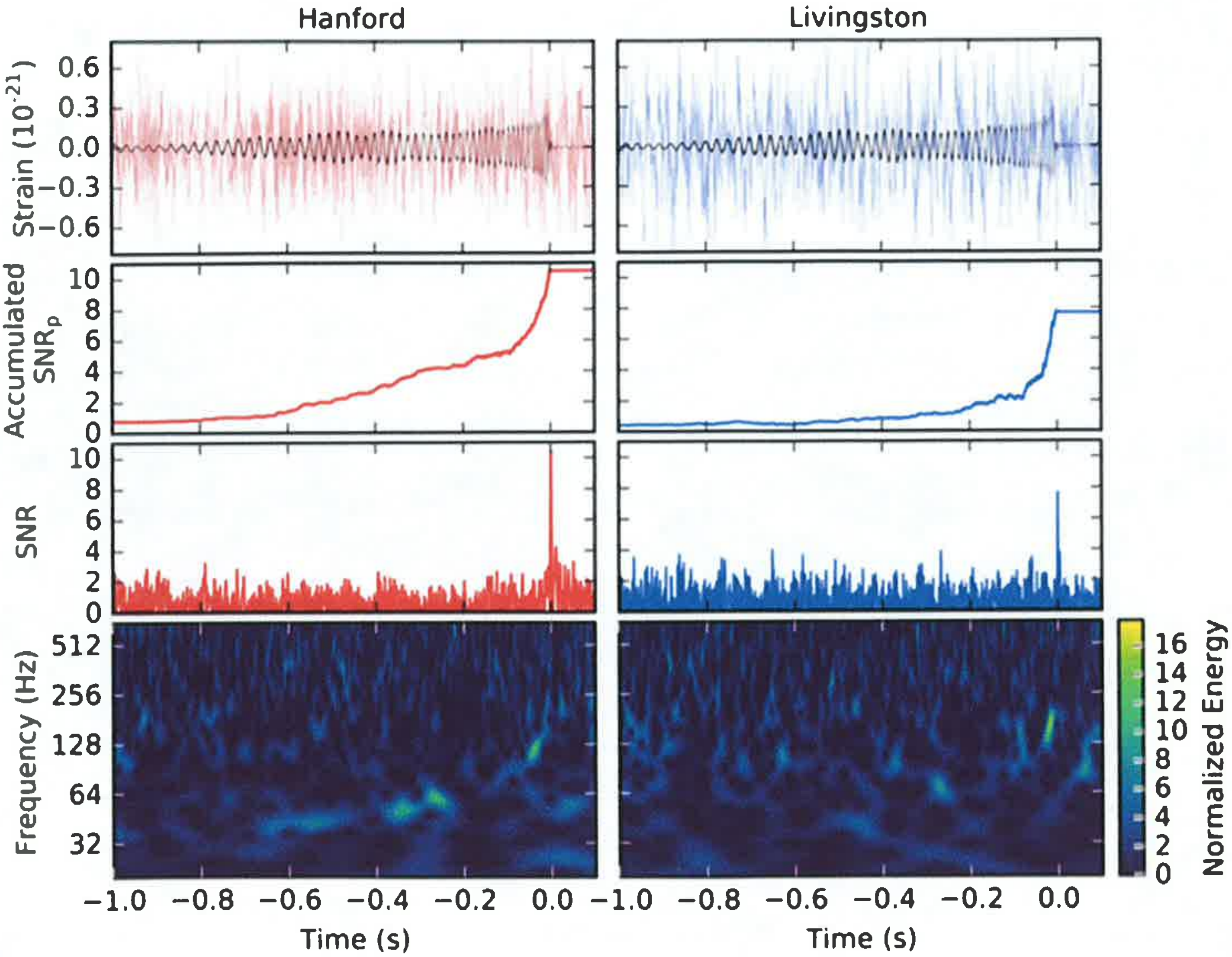
8 cycles

0.2 sec duration

from 35 Hz to 250 Hz

$\boxed{\text{GW 15/12/26}}$

SNR = 13
5σ event



$M_1 \simeq 14{,}2\, M_\odot$ $(+8 -4)$

$M_2 \simeq 7.5\, M_\odot$ (± 2)

$M_{BH} \simeq 20{,}8\, M_\odot$ $(^{+6}_{-2})$

spin $\gtrsim 0.2$

$d_L \simeq 440$ Mpc (± 180)

$z \simeq 0.09$ (± 0.04)

55 cycles

1 sec duration

from 35 to 450 Hz

we now turn to the analysis of Elliptic orbits.

The following results apply for elliptic orbits (we will only derive the first two):

- the radiated power in GWs is ENHANCED with respect to the one which would be emitted by a system in circular orbit with radius corresponding to the semi-major axis of the ellipse.

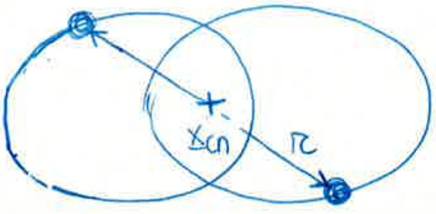
- the elliptic orbit has period $T = \frac{2\pi}{\omega_0}$ with $\omega_0^2 = \frac{Gm}{a^3}$ and m the total mass of the system, a the semi-major axis. As done for the circular orbit case, one can derive the variation of the orbital period due to the emission of the GWs: this can be applied to the Hulse-Taylor binary with great precision as we will see.

- The frequency spectrum is difficult to calculate (see MAGGIORE) but it is obviously much richer than just $2\omega_0$, and depends also on the eccentricity: the higher the eccentricity, the higher the main frequency of emission with respect to $2\omega_0$.

- the evolution of the orbit as the system loses energy and angular momentum via GW emission is such that the orbit becomes more and more circular: long before the coalescence phase, the eccentricity has gone to zero. Even though most commonly the systems are in elliptic orbits, the analysis of circular orbits done previously is useful.

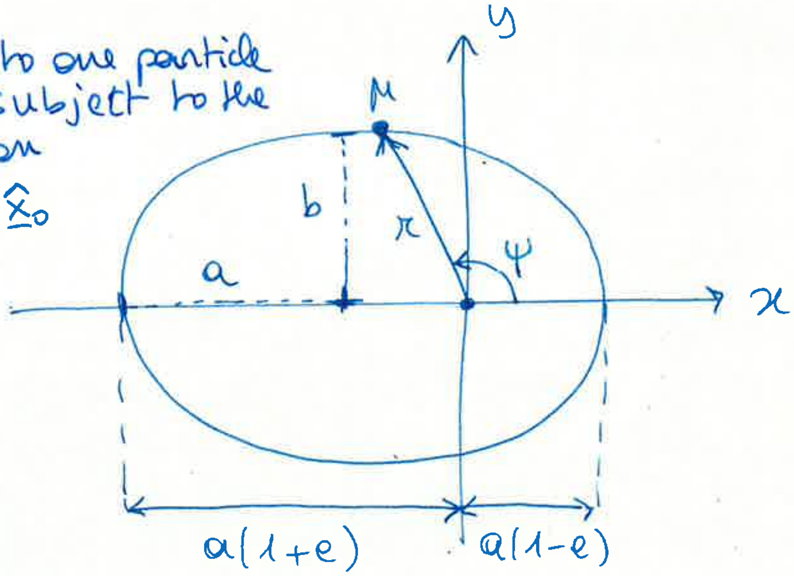
ELLIPTIC ORBIT : RADIATED POWER

We analyse the system in the center of mass frame with $\underline{x}_{cm}=0$. There are two conserved quantities, the angular momentum and the Energy. The motion is equivalent to a particle of reduced mass μ in motion on a elliptic orbit; the origin of the center of mass frame is in the focus of the ellipse:



equivalent to one particle of mass μ subject to the acceleration

$$\ddot{\underline{x}}_0 = -\frac{Gm}{x_0^2}\,\hat{\underline{x}}_0$$



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad \text{(cartesian coordinates)}$$

$$\begin{cases} \text{Angular momentum}: & L = \mu r^2 \dot{\varphi} \\ \text{Energy}: & E = E_{kin} + E_{pot} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{G\mu m}{r} \end{cases}$$

One can solve the system in terms of $r(t), \varphi(t)$, and find that the trajectory is an ellipse with:

$$\boxed{\frac{1}{r} = \frac{Gm\mu^2}{L^2}\left(1+e\cos\varphi\right)}, \quad \begin{cases} a = \frac{Gm\mu}{2|E|} \\ b = a\sqrt{1-e^2} \end{cases}, \quad \boxed{e^2 = 1 + \frac{2EL^2}{G^2m^2\mu^3}}$$

the eccentricity e is such that: $\begin{cases} e=0 & : \text{circular orbit} \\ e=1 & : \text{parabola} \end{cases}$

(note that for a bound orbit $E<0$)

The motion is periodic with $T = \frac{2\pi}{\omega_0}$, $\boxed{\omega_0^2 = \frac{Gm}{a^3}}$

we want to calculate the radiated power, so we need to know the second mass moment: $M_{ij} = \mu x^i x^j$ which has components only in the plane orthogonal to $\hat{z}$ since we have chosen that the orbit is in the (x,y) plane:

$$M_{ab} = \mu r^2 \begin{pmatrix} \cos^2\varphi & \cos\varphi \sin\varphi \\ \cos\varphi \sin\varphi & \sin^2\varphi \end{pmatrix}_{ab}$$

For the radiated power, one needs the third time derivative (see expression on page 18 of this chapter).

$$M_{11} = \mu r^2 \cos^2\varphi = \frac{L^4}{G^2 m^2 \mu^3} \frac{\cos^2\varphi}{(1+e\cos\varphi)^2}$$ and then one uses

$$\dot{\varphi} = \frac{L}{\mu r^2} = \frac{G^2 m^2 \mu^3}{L^3}(1+e\cos\varphi)$$

(using the trajectory $r = \frac{L^2}{Gm\mu^2}\frac{1}{(1+e\cos\varphi)}$)

for calculating $\dddot{M}_{11}(\varphi)$

Doing the same for M_{22} and M_{12}, one finds:

$$\begin{cases} \dddot{M}_{11}(\varphi) = \dfrac{G^4 m^4 \mu^6}{L^5}(1+e\cos\varphi)^2 (4+3e\cos\varphi)\sin 2\varphi \\ \dddot{M}_{12}(\varphi) = -\dfrac{G^4 m^4 \mu^6}{2L^5}(1+e\cos\varphi)^2 (5e\cos\varphi + 8\cos 2\varphi + 3e\cos 3\varphi) \\ \dddot{M}_{22}(\varphi) = -\dfrac{G^4 m^4 \mu^6}{L^5}(1+e\cos\varphi)^2 (8\cos\varphi + 5e + 3e\cos 2\varphi)\sin\varphi \end{cases}$$

The power radiated in the quadrupole approximation as a function of the position φ along the orbit is obtained plugging the above expressions into the formula on page 18

$$\left[P(\varphi)\right]_{QUAD} = \frac{G}{5c^5}\left\langle \dddot{M}_{11}^2 + \dddot{M}_{22}^2 + \dddot{M}_{12}^2 - \frac{1}{3}(\dddot{M}_{11} + \dddot{M}_{22})^2 \right\rangle$$

Here the average is on several periods of the wave.
It is possible to see that the GW frequencies are multiples of the source frequency $\omega_0 = \sqrt{\frac{Gm}{a^3}}$ defined on page 41. An average over several periods of the wave is therefore equivalent to an average over 1 period of the source $T = \frac{2\pi}{\omega_0}$:

$$[P]_{QUAD} = \frac{1}{T}\int_0^T [P]_{QUAD}(\psi)\,dt = \frac{\omega_0}{2\pi}\int_0^{2\pi}\frac{d\psi}{\dot\psi}[P]_{QUAD}(\psi)$$

$$= \frac{32\,G^4\mu^2 m^3}{5c^5a^5}\underbrace{\frac{1}{(1-e^2)^{\frac{7}{2}}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)}_{f(e)}$$

(using the expression $\dot\psi(\psi)$ on the previous page and $1-e^2 = \frac{2|E|L^2}{G^2m^2\mu^3}$ for a bound orbit)

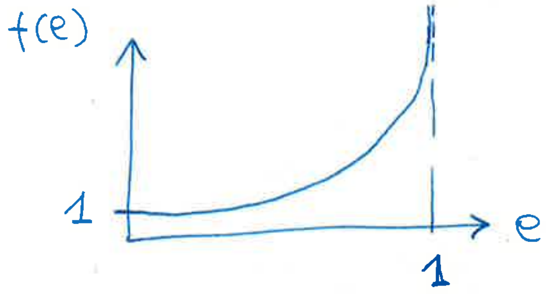
Using $m = \frac{a^3\omega_0^2}{G}$ and defining the function $f(e)$ this becomes:

$$\boxed{[P]_{QUAD} = \frac{32}{5}\frac{G\mu^2}{c^5}a^4\omega_0^6 f(e)}$$

•) when $e=0$ one gets a circular orbit of radius a with $\omega_0 = \omega_s$, and $f(e)=1$: this expression becomes the same as the one found on page 25:

$$[P]_{QUAD,\,CIRCULAR} = \frac{32}{5}\frac{G\mu^2}{c^5}R^4\omega_s^6$$

•) The shape of the function $f(e)$, which depends only on the eccentricity e of the orbit, is :



$$f(e) \geq 1$$

The power radiated by an elliptic orbit is therefore larger than the one radiated by a circular orbit with radius a. Since $a = \frac{GmM}{2|E|}$, a circular orbit with radius a has the same energy as the elliptic one : $|E|_{circular} = \frac{GmM}{2a}$. There is therefore a net enhancement : for example the Hulse-Taylor binary pulsar has $e \simeq 0.617 \Rightarrow f(e) \simeq 11.8$: the radiated power is one order of magnitude larger than it would be for a corresponding system with the same characteristics but on a circular orbit, and the time to coalescence τ_0 is smaller.

•) The limit $e \to 1^-$ in this analysis corresponds to a divergence : since $a = \frac{L^2}{G\mu^2 m} \frac{1}{1-e^2}$, doing this limit keeping a fixed means $\frac{1}{r} = \frac{Gm\mu^2}{L^2}(1+e\cos\psi) \to \infty$ therefore $r \to 0$. The acceleration $\frac{Gm}{r^2}$ diverges and so do the emitted GWs. On the other hand, one can do the calculation for $e \to 1$ keeping $\frac{L^2}{G\mu^2 m}$

fixed: this would correspond to $a \to \infty$ and the trajectory becomes a parabola: $r = \frac{L^2}{G\mu^2 m} \frac{1}{1+\cos\varphi}$.

[See MAGGIORE paragraph 4.1.2 for details]

We now turn to the ORBITAL PERIOD VARIATION:

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{Gm}} a^{3/2} = \frac{\pi}{\sqrt{2}} \frac{Gm\,\mu^{3/2}}{|E|^{3/2}}$$

from this relation one can write

$$\frac{\dot{T}}{T} = -\frac{3}{2}\frac{\dot{E}}{E} = \frac{3}{2}\frac{[P]_{QUAD}}{|E|} = -\frac{3a[P]_{QUAD}}{Gm\mu}$$

from the definition of the emitted power: we implicitly assume an averaging over one period of the source

$a = \frac{Gm\mu}{2|E|}$

and therefore, from $[P]_{QUAD}$ calculated on page (43)

$$\frac{\dot{T}}{T} = -\frac{96}{5}\frac{G^3\mu m^2}{c^5 a^4} f(e) = -\frac{96}{5}\frac{G^{5/3}\mu m^{2/3}}{c^5}\left(\frac{2\pi}{T}\right)^{8/3} f(e)$$

$$\boxed{\dot{T} = -\frac{192\pi}{5}\frac{G^{5/3}\mu m^{2/3}}{c^5}\left(\frac{2\pi}{T}\right)^{5/3} f(e)}$$

this equation expresses the variation of the orbital period for an elliptic orbit. Here it has been derived within the Newtonian approximation for the source, and in linearised theory. It can be derived however also in the context of the post-Newtonian approach, in the same form, without going through the argument of the energetic balance that $\dot{E}_{source} = - P_{GW}$.

It corresponds therefore really to the GR prediction for the decrease of the orbital period due to GW emission.

We can therefore apply it to realistic systems, as the

HULSE TAYLOR BINARY

Pulsar PSR B1913+16 orbiting around a companion (most probably another NS, due to the mass*): the relative coordinate describes an ellipsis with eccentricity e, with orbital velocity $v \simeq 10^{-3} c$, inclined at $\sin\theta \simeq 0.72$ from the line of sight. The values of the parameters which interest us are: (from astro-ph/0407149)

$$e = 0.6171338 \qquad m_p = 1.4414\ M_\odot \qquad m_c = 1.3867\ M_\odot$$

* the companion is also a NS most probabely

$$T_{obs} = 0.322997448930 \text{ days}$$

$$\dot{T}_{obs}^{corr} = -2.4055 \cdot 10^{-12}$$

(corrected by the relative acceleration between the solar system and the binary)

If one inserts the above values into the formula for $\dot{T}$ obtained on page 45, one gets the theoretical value of $\dot{T}$, to be compared with $\dot{T}_{obs}^{corr}$ as a test of

the prediction from GENERAL RELATIVITY:

Note however that the theoretical value of $\dot{T}$ depends on the value of G, which has an intrinsic uncertainty due to the measurement. Inserting the following values (From the particle data group in 2014):

$$G = 6.70838 \cdot 10^{-39} \ \mathrm{GeV}^{-2}$$

$$M_\odot = 1.11547 \cdot 10^{57} \ \mathrm{GeV}$$

one gets the prediction $\dot{T} = -2.402 \cdot 10^{-12}$

this has to be compared with the value $\dot{T}^{\mathrm{corr}}_{\mathrm{obs}}$ to get:

$$\boxed{\frac{\dot{T}^{\mathrm{corr}}_{\mathrm{obs}}}{\dot{T}} = 1.001}$$

there is therefore a very good agreement, as already shown by the plot given at the beginning of the course, just after page 36 of the first part. (also taken from astro-ph/0407149).

This has been the first experimental evidence of gravitational radiation and was the Nobel Prize in 1993.