...ssification of random matrix models

...et $X = [X_{ij}]$ be an $(N \times N)$ matrix.

In general 'N' eigenvalues $\rightarrow$ complex.

Not all matrices have real spectrum.

Three types of matrices, ~~wom~~ which are guaranteed to have real spectrum.

$$\left.\begin{array}{l}\text{(i) Real symmetric matrix} \\ \text{(ii) Complex Hermitian "} \\ \text{(iii) Quaternion self-dual Hermitian "}\end{array}\right\} \rightarrow \text{Dyson's 3-fold way.}$$

Real Symmetric Matrix

$X_{ij} = X_{ji} \Rightarrow X^t = X$ $\qquad$ $X^t \rightarrow$ transpose of X

All N eigenvalues are real $(\lambda_1, \cdots \lambda_N)$

It can be digonalised via,

$$X = O \Lambda O^t = O \Lambda O^{-1} \quad \text{or} \quad \Lambda = O^t X O$$

$$O^t O = O O^t = I$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} \quad \text{and} \quad O \rightarrow \text{orthogonal matrix } (O^t = O^{-1}) \; (N \times N)$$

$$O = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

columns $\rightarrow$ eigenvectors (normalized to unity) of X

vial example: $X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Two eigenvalues, $(1-\lambda)^2 - 1 = 0. \Rightarrow \lambda = (0, 2)$.

eigenvectors: $\lambda = 0$, $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda = 2$ $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

hence $O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Check: $O^t = O^{-1}$

also $O^t X O = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$.

<u>Complex Hermitian matrix</u>

$X \to$ Complex entries. $(N \times N)$ matrix

$X^{\dagger} = (X^{t})^{*} = X. \to$ Hermitian.

Note that $X_{ii} = X_{ii}^{*}$

$\hookrightarrow$ diagonal elements are real.

All 'N' eigenvalues are real.

X can be diagonalised by

$$X = U \Lambda U^{\dagger} \quad \text{or} \quad \Lambda = U^{\dagger} X U.$$

where $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$ and $U^{\dagger} U = U U^{\dagger} = I.$

$U \to$ unitary matrix $(U^{\dagger} = U^{-1})$ $(N \times N)$

$$U = \begin{bmatrix} \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

columns $\to$ orthonormalized eigenvectors of 'X'

<u>Trivial example:</u> $X = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

Two real eigenvalues: $\lambda = 0, 2.$

$\lambda = 0. \to$ eigenvector $\to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda = 2 \to$ " $\to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$ $\quad U^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$

Check that $U^{\dagger} U = U U^{\dagger} = \mathbb{1}.$

$$U^{\dagger} X U = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \Lambda$$

* For instance one can show that

$$ij = k, \quad ji = -k$$
$$jk = i, \quad kj = -i$$
$$ki = j, \quad ik = -j$$

$$q_1 = a_1 + b_1 i + c_1 j + d_1 k$$
$$q_2 = a_2 + b_2 i + c_2 j + d_2 k$$

$$(q_1 + q_2) = (a_1 + a_2,\ b_1 + b_2,\ c_1 + c_2,\ d_1 + d_2)$$

$$(q_1 q_2) = (a_1, b_1, c_1, d_1)(a_2, b_2, c_2, d_2)$$

$$= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2,\ a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2,\ a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2,\ a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)$$

$$q \equiv \begin{pmatrix} a+bi & c+di \\ \cancel{a-bi}\ \ \cancel{c-di} & \\ -c+di & a-bi \end{pmatrix}$$

Quaternion Self-dual ~~Matrix~~ Hermitian matrix

What are quaternions? Hamilton, 1843

quaternions → generalisations of $\underbrace{\text{complex numbers}}_{(a+bi)}$ → $q = a + bi + cj + dk$ (base of 4)

complex number (base-2): $a+bi$ with $i^2=-1$

quaternion (base-4): $a+bi+cj+dk$, with $i^2=j^2=k^2=ijk=-1$. *

any quaternion can be represented by a 2×2 complex matrix.

$$q \equiv \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

Now consider a $\underbrace{\text{matrix of quaternions}}$: → $\begin{pmatrix} q_{11} & q_{12} \cdots & q_N \\ \vdots \\ q_{N1} & q_{N2} \cdots & q_{NN} \end{pmatrix}$

$(2N\times 2N)$ → block matrix.

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$ → Quaternion matrix.

where A, B, C, D → each $(N\times N)$ complex matrix.

Define the dual matrix: → duality transformation

$$X^R = -Z X^t Z$$ where $\underbrace{Z = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}}$ note $Z^{-1} = Z^T = -Z$.

$$= \begin{bmatrix} D^t & -B^t \\ -C^t & A^t \end{bmatrix}$$

note $(X^R)^R = X$, similarly $(AB)^R = B^R A^R$.

for instance $N=1$, $q^R = \begin{pmatrix} a-bi & -c-di \\ c-di & a+bi \end{pmatrix}$

Quaternion ~~Hermitian~~ self-dual Hermitian matrix X:

$$\hookrightarrow \quad X^R = X^\dagger = X \quad \Rightarrow \quad \begin{bmatrix} D^t & -B^t \\ -C^t & A^t \end{bmatrix} = \begin{bmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\Rightarrow \quad A^\dagger = A = D^t$$

and $\quad C^\dagger = B = -B^t$

Symplectic matrix. $\quad S \rightarrow (2N \times 2N)$ complex matrix such that

$$S^T Z S = Z$$

$$\Rightarrow \quad S^{-1} = -Z S^T Z = S^R$$

S is an unitary symplectic if $\quad S^\dagger = S^{-1} = S^R$

$X \rightarrow$ Quaternion self-dual Hermitian matrix $\quad (X^R = X^\dagger = X)$

can be diagonalised by

$$X = S \Lambda S^R \qquad (\text{where } S^\dagger = S^{-1} = S^R)$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & 0 \\ & \lambda_2 & & & & \\ & & \ddots \lambda_N & & & \\ & & & \lambda_1 & & \\ 0 & & & & \ddots & \\ & & & & & \lambda_N \end{pmatrix} \rightarrow 2N \text{ real eigenvalues ~~and~~ each occurring twice}$$

$$\{\lambda_1, \ldots \lambda_N, \lambda_1, \lambda_2, \ldots \lambda_N\}.$$

Trivial example: $N=1$, $\rightarrow$ a single quaternion

$$q = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

If it has to self-dual Hermitian, we need $\quad q^R = q^\dagger = q$

$$\Rightarrow \begin{pmatrix} a-bi & -c-di \\ c-di & a+bi \end{pmatrix} = \begin{pmatrix} a-bi & -c-id \\ c-di & a+bi \end{pmatrix} = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

$$\Rightarrow \quad b = c = d = 0.$$

$$\Rightarrow \quad q = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \qquad \lambda = \{a, a\}.$$

$$\hookrightarrow S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

2 real eigenvalues.

Real symmetric:

$$dX \equiv \prod_{1\le i\le j\le N} dX_{ij}$$

Ex: $N=2$ $\quad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}$

$$dX \equiv dx_{11}\, dx_{22}\, dx_{12}$$

↳ volume element.

(i) The volume element is invariant under an orthogonal transformation.

$$Y = O^t X O \qquad X = O Y O^t$$

Then $Y \to$ symmetric matrix.

$$\cancel{dY_{11}\, dY_{22}\, dY} \prod_{1\le i\le j\le N} dY_{ij} = \prod_{1\le i\le j\le N} dX_{ij}$$

Proof: $N=2$

$$Y = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\Rightarrow \quad y_{11} = x_{11}\cos^2\theta + x_{22}\sin^2\theta + 2x_{12}\cos\theta\sin\theta$$

$$y_{22} = x_{11}\sin^2\theta + x_{22}\cos^2\theta - 2x_{12}\cos\theta\sin\theta$$

$$y_{12} = x_{12}(\cos^2\theta - \sin^2\theta) + (x_{22}-x_{11})\cos\theta\sin\theta$$

$$dy_{11}\, dy_{22}\, dy_{12} = J\, dx_{11}\, dx_{22}\, dx_{12}$$

$$J = \cancel{\det}\begin{vmatrix} \frac{\partial y_{11}}{\partial x_{11}} & \frac{\partial y_{11}}{\partial x_{22}} & \frac{\partial y_{11}}{\partial x_{12}} \\ \frac{\partial y_{22}}{\partial x_{11}} & \frac{\partial y_{22}}{\partial x_{22}} & \frac{\partial y_{22}}{\partial x_{12}} \\ \frac{\partial y_{12}}{\partial x_{11}} & \frac{\partial y_{12}}{\partial x_{22}} & \frac{\partial y_{12}}{\partial x_{12}} \end{vmatrix} = \det \begin{vmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \sin^2\theta & 2\cos\theta\sin\theta \\ 0 & \cos^2\theta - \sin^2\theta & -4\cos\theta\sin\theta \\ 0 & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{vmatrix} = (\cos^2\theta - \sin^2\theta)^2 + 4\cos^2\theta\sin^2\theta = 1$$

General proof:

Consider ~~the matrix~~ an arbitrary ~~real~~ real matrix X:

One can define a metric in the space of all entries

$$ds^2 = \mathrm{Tr}(dx^t dx) = \sum_{i,j} (dx_{ij})^2 \overset{N=2}{=} (dx_{11})^2 + (dx_{22})^2 + (dx_{12})^2 + (dx_{21})^2$$

$$\rightarrow (dx_{11})^2 + (dx_{22})^2 + 2(dx_{12})^2$$

$$\rightarrow \sum g_{jk}\, dq_j\, dq_k$$

For symmetric matrices $(X^t = X)$

$\frac{N(N+1)}{2} \to$ indep entries

$$\text{metric} \nearrow g = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{bmatrix} \rightarrow g \to \frac{N(N+1)}{2} \times \frac{N(N+1)}{2} \text{ matrix}$$

↳ diagonal.

$$\det g = 2^{1+2+\cdots(N-1)} = 2^{\frac{N(N-1)}{2}}$$

$$\sqrt{\det g} = 2^{N(N-1)/4}$$

$$dX \to \sqrt{\det g} \prod_{i\le j\le N} dx_{ij} = 2^{\frac{N(N-1)}{4}} \prod_{1\le i\le j\le N} dx_{ij}$$

Now since ds^2 is invariant under $Y = O^t X O$

$\Rightarrow dX = \cancel{\text{in}}\ d(O^t X O)$

(5a)

Proof that the flat measure is invariant under orthonormal transformation

$$[y_{ij}] = U[x_{ij}]U^T \quad — ①$$

$y, x \to N\times N$ real symm. matrix

We want to prove that $dy_{11}\cdots dy_{NN}\prod_{i<j} dy_{i,j} = \prod_{i=1}^{N} dx_{ii} \prod_{i<j} dx_{i,j}$ — ②

Proof: $\boxed{\text{Eq. ① is linear}}$ in x

Now, construct the vector $[\tilde{Y}] = \begin{bmatrix} y_{11} \\ y_{22} \\ \vdots \\ y_{NN} \\ y_{12} \\ y_{23} \\ \vdots \\ y_{N,N-1} \end{bmatrix} \to$ with $\frac{N(N+1)}{2}$ entries.

similarly $[\tilde{X}] = \begin{bmatrix} x_{11} \\ \vdots \\ x_{N,N-1} \end{bmatrix} \to$ " "

~~Clearly~~, because, $[\tilde{Y}]$ and $[\tilde{X}]$ are linearly related by ①.

$$[\tilde{Y}] = J[\tilde{X}]$$

$J \to \frac{N(N+1)}{2} \times \frac{N(N+1)}{2}$ matrix. (constant ↳ does not depend on x).

Now, since $[y_{ij}] = U[x_{ij}]U^T$

$\boxed{② \Rightarrow \text{amounts to prove } |\det J| = 1}$

Clearly $Tr[y^T y] = Tr[x^t x]$

$$\Rightarrow \sum y_{ii}^2 + 2\sum_{i<j} y_{ij}^2 = \sum_i x_{i,i}^2 + 2\sum_{i<j} x_{i,j}^2$$

$$\tilde{Y}^t D \tilde{Y} = \tilde{X}^t D \tilde{X}$$

$D = \begin{bmatrix} 1 & & & & 0 \\ & \ddots 1 & & & \\ & & 2 & & \\ 0 & & & \ddots & 2 \end{bmatrix}$

matrix, $\tilde{Y} = J\tilde{X}$.

and $\Rightarrow \tilde{X}^t J^t D J \tilde{X} = \tilde{X}^t D \tilde{X}$ $\qquad J^t J = \mathbb{1}$

$2^{\frac{N(N-1)}{2}}$ true for any vector $[\tilde{X}] \Rightarrow J^t D J = D \Rightarrow$

$\Rightarrow |\det J| = 1$

$\det[J^t D J] = \det D = 2^{\frac{N(N-1)}{2}}$ ✓ $\Rightarrow$ $J \to$ orthogonal matrix $\frac{N(N+1)}{2}$.

$(\det J)^2\, 2^{\frac{N(N-1)}{2}} = 2^{\frac{N(N-1)}{2}}$

(6)

We now want to make the matrices random. i.e. ~~put a~~ introduce a ~~measure~~ probability measure in the space of the entries of the matrix X.

$$\longrightarrow \quad P(\{x_{ij}\})\, dX \quad \rightarrow$$

where

$$dX \equiv \prod_{1\leq i\leq j\leq N} dx_{ij}$$

for real symmetric matrices. $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}$

↳ no. of independent entries

$$= 1+2+\cdots N = \frac{N(N+1)}{2}.$$

$\ast$

$$\equiv \prod_{i=1}^{N} dx_{ii} \prod_{1\leq i<j\leq N} d(\mathrm{Re}\, x_{ij})\, d(\mathrm{Im}\, x_{ij})$$

for complex Hermitian matrices.

$$\hookrightarrow X = \begin{bmatrix} x_{11} & x_{12}+iy_{12} & x_{13}+iy_{13} \\ x_{12}-iy_{12} & x_{22} & x_{23}+iy_{23} \\ x_{13}-iy_{13} & x_{23}-iy_{23} & x_{33} \end{bmatrix}$$

↳ no. of independent real random variables

$$= 2\cdot(1+2+\cdots N-1) + N$$

$$= \cancel{2}\cdot\frac{N(N-1)}{\cancel{2}} + N = N^2.$$

Similarly for symplectic quaternion case.

There are two main categories about the choice of $P[\{x_{ij}\}]$

Ensembles with independent entries. (so called Wigner matrices)

$$\Rightarrow P[\{x_{ij}\}] = \begin{cases} \prod_{i=1}^{N} f_i(x_{ii}) \prod_{1\leq i<j\leq N} f_{ij}(x_{ij}) & \rightarrow \text{real symmetric} \\ \prod_{i=1}^{N} f_i(x_{ii}) \prod_{1\leq i<j\leq N} f_{ij}^{(1)}(x_{ij})\, f_{ij}^{(2)}(y_{ij}) & \rightarrow \text{complex Hermitian} \end{cases}$$

β	TRS Time-reversal	SRS spin-rotation	U	H
1	yes	yes	orthogonal	real
2	no	irrelevant	unitary ~~complex~~	complex.
4	yes	no	symplectic	real quaternion

Pauli spin matrices.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$q = a\,\mathbb{I} + i b \sigma_x + i c \sigma_y + i d \sigma_z$$

$\hookrightarrow$ is real if $(a, b, c, d) \to$ real.

) Ensembles that are invariant under ~~xx~~ rotation

$$P[X] = \begin{cases} P[O X O^{-1}] & \rightarrow \text{invariant under orthogonal transformation} \\ \text{or } P[U X U^{-1}] & \rightarrow \text{" " unitary transformations} \\ P[S X S^{-1}] & \rightarrow \text{" " symplectic transformations} \end{cases}$$

e.g. if $P[X] \propto \exp\left[-a \operatorname{Tr}(X^2)\right]$

~~exp[...]~~

$P[X] = P[O X O^{-1}]$ if $X \rightarrow$ real symmetric matrix
↳ Gaussian orthogonal ensemble (GOE)
↳ models Hamiltonians with time-reversal symmetry & electron spin is conserved

$P[X] = P[U X U^{-1}]$ if $X \rightarrow$ Complex Hermitian matrix. ↳ Gaussian unitary ensemble (GUE)
↳ models Hamiltonians with NO time-reversal symm.

$P[X] = P[S X S^{-1}]$ if $X \rightarrow$ Quaternion self-dual Hermitian ↳ Gaussian symplectic ensemble (GSE).
↳ models Hamiltonians with time-reversal symmetry but NO rotational symmetry (spin-rotation symmetry broken e.g. by strong spin-orbit scattering).

Implication → $P[X]dX$ depends only on eigenvalues and not on eigenvectors.

$$P[X] = P[O X O^{-1}] = P[\Lambda]$$

all the eigenvector dependence is in the flat measure

dX ~~≈~~

↳ ~~all eigenvectors~~

Separation of eigenvalues & eigenvectors.

~~all eigenvectors are equally probable.~~

eigenvectors are uniformly distributed.

$$P[X]dX = P[Y]dY$$

Here $Y = O X O^{-1}$
i.e. similarity transformation
eigenvectors change
not eigenvalues
, $P[X]dX$ can not depend explicitly on eigenvectors, but only on eigenvalues λ_i

only common element betⁿ (i) and (ii) → Gaussian ensembles.

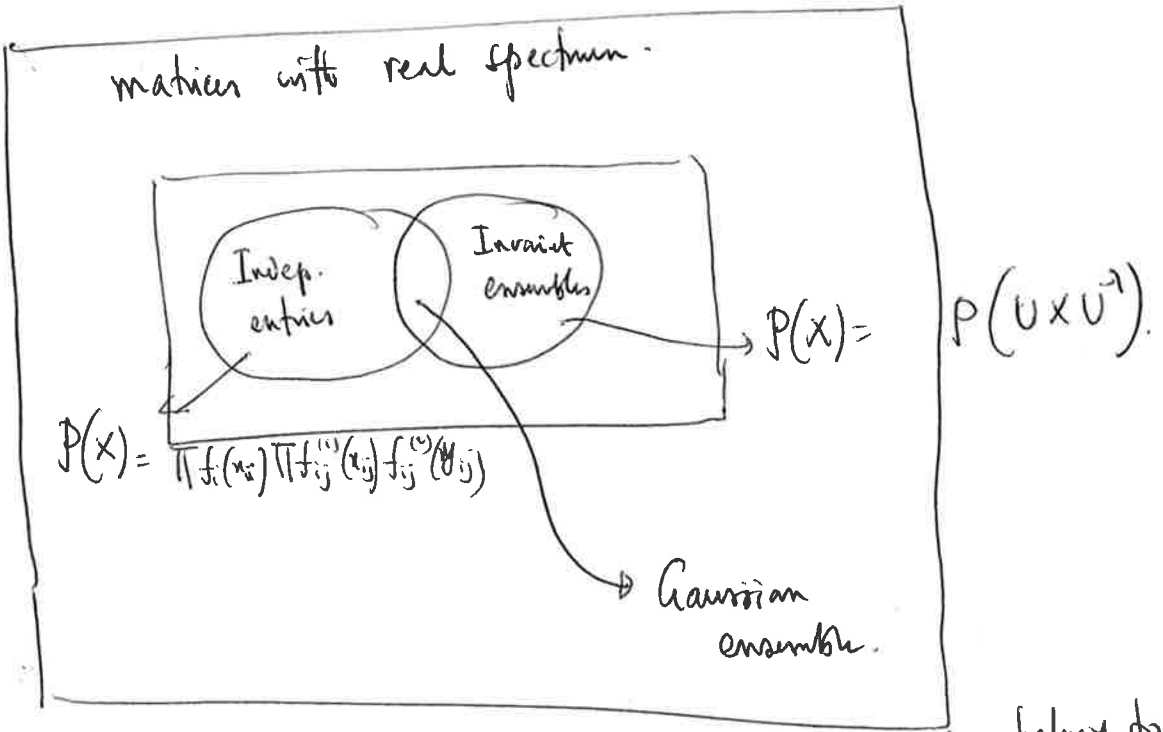
real symmetric matrix:

$$P(X) \propto e^{-\frac{1}{2\sigma^2}\left[\sum_{i,j} x_{ij}^2\right]} \rightarrow \text{i.i.d. indep. entries. diag.} \rightarrow \text{variance } \sigma^2, \text{ off} \rightarrow 2\sigma^2$$

$$= e^{-\frac{1}{2\sigma^2}\mathrm{Tr}\left[X^2\right]}.$$

complex Hermitian.
$$P[X] \propto e^{-\frac{1}{2\sigma^2}\mathrm{Tr}\left[X^2\right]}$$

↳ rigorous proof → Porter-Rosenzweig theorem.



What are the quantities we can typically compute for ensembles belonging to the two classes above.

i) Indep. entries: the jpdf of eigenvalues $P(\lambda_1, \dots \lambda_N)$ → generally difficult
↳ exception → Dumitriu-Edelman generalized β ensemble

ii) Invariant ensembles. Exact jpdf $P(\lambda_1, \dots \lambda_N)$
↓
different spectral properties both for finite N and large N asymptotics.

Calculation of the jpdf $P(\lambda_1, \ldots \lambda_N)$ for Gaussian Ensembles.

~~Let with start with a simple but illustrative example~~

Consider, for instance, Gaussian Orthogonal ensemble. (GOE).

We have a real $(N \times N)$ symmetric matrix

$$X \equiv \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1N} \\ X_{12} & X_{22} & \cdots & X_{2N} \\ \vdots & & \ddots & \\ \vdots & & & X_{NN} \end{pmatrix}$$

$\hookrightarrow$ no. of indep. entries.

$= 1 + 2 + 3 + \cdots N$

$= \frac{N(N+1)}{2}$.

$$\left\{ \begin{matrix} X_{11}, & X_{12}, & \cdots & X_{1N} \\ & X_{22}, & X_{23}, \cdots & X_{2N} \\ & & \ddots & \\ & & & X_{NN} \end{matrix} \right\}$$

Prob. density over the space of indep. entries.

$$\propto e^{-\frac{1}{2}\left(X_{11}^2 + \cdots X_{NN}^2 + 2\left(X_{12}^2 + X_{13}^2 + \cdots X_{1N}^2 + \cdots X_{N-1,N}^2\right)\right)} \, dX_{11} \cdots dX_{NN} \, dX_{12} \, dX_{13}, \cdots dX_{N-1,N}.$$

We can diagonalise X by, $\quad X = O \Lambda O^t$

Where, $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$ $\qquad O \rightarrow \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$

$\longrightarrow$ no. of indep. degrees of freedom $= N$.

$\hookrightarrow$ Columns are eigenvectors of X

O_{ij}

[no. of indep. entries:

$N^2 - \underbrace{(N + N-1 + \cdots 1)}_{\text{constraints of orthonormality}}$

$= N^2 - \frac{N(N+1)}{2} = \frac{N^2 - N}{2}$

We want to make a change of variable.

from. $\{X_{ij}\} \rightarrow \{\Lambda, O_{ij}\}$.

hence $\{\Lambda, O\} \longrightarrow$ total $= N + \frac{N^2 - N}{2} = \frac{N(N+1)}{2}$.

$\{X_{ij}\} \longrightarrow$ total $\longrightarrow \frac{N(N+1)}{2}$.

~~steps~~

oal is to express

- $P[\{X_{ij}\}]\, dX_{11}\cdots dX_{N-1,N} \to P[\lambda_1,\ldots\lambda_N]\, J(\lambda_1,\ldots\lambda_N, O_{ij})\, d\lambda_1\cdots d\lambda_N \prod dO_{ij}$

- Then integrate over the eigenvector degrees of freedom to get the "marginal jpdf of λ_i's" only

$$P[\lambda_1,\ldots\lambda_N]\left[\int\!\!\int J(\lambda_1,\ldots\lambda_N, O_{ij}) \prod dO_{ij}\right] d\lambda_1\cdots d\lambda_N.$$

et us start with a simple but illustrative example of (2×2) matrix:

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}. \quad \longrightarrow \frac{N(N+1)}{2} = 3 \text{ indep. entries.}$$

$$X = O\Lambda O^t,$$

where $O \to$ two by two orthonormal matrix

$$O = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

$$\Rightarrow \quad X = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\cos^2\theta + \lambda_2\sin^2\theta & (\lambda_1-\lambda_2)\cos\theta\sin\theta \\ (\lambda_1-\lambda_2)\cos\theta\sin\theta & \lambda_1\sin^2\theta + \lambda_2\cos^2\theta \end{bmatrix}$$

$\theta \to$ parametizes the eigenvector. $\quad \lambda_1, \lambda_2, \theta \longrightarrow 3$ indep. entries

$$P[x_{11}, x_{12}, x_{22}]\, dx_{11}\, dx_{22}\, dx_{12}$$

$$= e^{-\frac{1}{2}\left[x_{11}^2 + x_{22}^2 + 2x_{12}^2\right]}\, dx_{11}\, dx_{22}\, dx_{12}$$

$$= e^{-\frac{1}{2}\left[\lambda_1^2 + \lambda_2^2\right]}\, J(\lambda_1,\lambda_2,\theta)\, d\lambda_1\, d\lambda_2\, d\theta.$$

calculate the Jacobian $J(\lambda_1, \lambda_2, \theta)$

$$x_{11} = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$$
$$x_{22} = \lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta$$
$$x_{12} = (\lambda_1 - \lambda_2)\cos\theta \sin\theta .$$

$$J = \begin{vmatrix} \frac{\partial x_{11}}{\partial \lambda_1} & \frac{\partial x_{22}}{\partial \lambda_1} & \frac{\partial x_{12}}{\partial \lambda_1} \\ \frac{\partial x_{11}}{\partial \lambda_2} & \frac{\partial x_{22}}{\partial \lambda_2} & \frac{\partial x_{12}}{\partial \lambda_2} \\ \frac{\partial x_{11}}{\partial \theta} & \frac{\partial x_{22}}{\partial \theta} & \frac{\partial x_{12}}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2\theta & \sin^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ -2(\lambda_1-\lambda_2)\cos\theta\sin\theta & 2(\lambda_1-\lambda_2)\cos\theta\sin\theta & (\lambda_1-\lambda_2)(\cos^2\theta - \sin^2\theta) \end{vmatrix}$$

$$= |\lambda_1 - \lambda_2| \begin{vmatrix} 1 & 1 & 0 \\ \sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{vmatrix}$$

$$= |\lambda_1 - \lambda_2| \begin{vmatrix} 1 & 0 & 0 \\ \sin^2\theta & \cos^2\theta - \sin^2\theta & -\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 4\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{vmatrix}$$

$$= |\lambda_1 - \lambda_2| \left[\cos^4\theta + \sin^4\theta - 2\cos^2\theta\sin^2\theta + 4\cos^2\theta\sin^2\theta \right] = |\lambda_1 - \lambda_2|$$

$$P[x_{11}, x_{12}, x_{22}]\, dx_{11}\, dx_{22}\, dx_{12} = e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} |\lambda_1 - \lambda_2| \; d\theta \, . \, d\lambda_1\, d\lambda_2$$

↳ uniformly distributed over θ

It is natural that the Jacobian J does not depend on θ.

Integrating over θ:

$$\boxed{P[\lambda_1, \lambda_2] \propto e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} |\lambda_1 - \lambda_2|}$$

let us generalize to $(N\times N)$ real symmetric matrix:

have learnt that $J \to$ can not depend on the eigenvectors.
$\hookrightarrow$ property of the volume element.

, compute it. we consider X very close to the diagonal.

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}.$$

then,
$$\boxed{X = O \Lambda O^t}$$

if X is very close to Λ, , $O \to$ very close to $\mathbb{1}$.

$$dX = dO \Lambda O^t + O\, d\Lambda\, O^t + O \Lambda\, dO^t$$

to leading order.
$$= dO\, \Lambda\, \mathbb{1} + d\Lambda + \Lambda\, dO^t$$
$$= dO\, \Lambda + d\Lambda + \Lambda\, dO^t$$
$$= dO\, \Lambda + d\Lambda - \Lambda\, dO.$$
$$= d\Lambda + [dO, \Lambda]$$

$O^t O = \mathbb{1}$.
$dO^t O + O^t dO = 0$
$\Rightarrow dO^t = -dO$

$$(dO\,\Lambda)_{ij} = \sum_k (dO)_{ik} \Lambda_{kj} = \sum_k dO_{ik} \lambda_k \delta_{kj} = \lambda_j\, dO_{ij}$$
$$(\Lambda\, dO)_{ij} = \sum_k \Lambda_{ik}\, dO_{kj} = \sum \lambda_k \delta_{ik}\, dO_{kj} = \lambda_i\, dO_{ij}$$

$$\Rightarrow (dO\,\Lambda)_{ij} - (\Lambda\, dO)_{ij} = (\lambda_j - \lambda_i)\, dO_{ij}$$

$$\Rightarrow \quad dX_{ij} = d\Lambda_{ij} + (\lambda_j - \lambda_i)\, dO_{ij}$$
$$= d\lambda_i\, \delta_{ij} + (\lambda_j - \lambda_i)\, dO_{ij}$$

- example

$$\left| \begin{aligned} dX_{11} &= d\lambda_1 \\ dX_{12} &= (\lambda_2 - \lambda_1)\, dO_{12} \\ dX_{22} &= d\lambda_2 \end{aligned} \right| \quad J = \begin{vmatrix} \frac{\partial X_{11}}{\partial \lambda_1} & \frac{\partial X_{11}}{\partial \lambda_2} & \frac{\partial X_{11}}{\partial O_{12}} \\ \frac{\partial X_{22}}{\partial \lambda_1} & \frac{\partial X_{22}}{\partial \lambda_2} & \frac{\partial X_{22}}{\partial O_{12}} \\ \frac{\partial X_{12}}{\partial \lambda_1} & \frac{\partial X_{12}}{\partial \lambda_2} & \frac{\partial X_{12}}{\partial O_{12}} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_2 - \lambda_1 \end{vmatrix} = |\lambda_2 - \lambda_1|$$

3

$$dX_{11} = d\lambda_1$$
$$dX_{22} = d\lambda_2$$
$$dX_{33} = d\lambda_3$$
$$dX_{12} = (\lambda_2-\lambda_1)\, dO_{12}$$
$$dX_{13} = (\lambda_3-\lambda_1)\, dO_{13}$$
$$dX_{23} = (\lambda_3-\lambda_2)\, dO_{23}$$

$\frac{N(N+1)}{2} = \frac{3.\cancel{X}^2}{\cancel{2}} = 6 \rightarrow$ degrees of freedom.

$J \rightarrow 6\times 6$ determinant

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_2-\lambda_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & (\lambda_3-\lambda_1) & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3-\lambda_2 \end{bmatrix}$$

$$= |\lambda_1-\lambda_2|\,|\lambda_3-\lambda_1|\,|\lambda_3-\lambda_2|.$$

general.

$$P(\lambda_1,\dots\lambda_N) = \frac{1}{Z_N}\, e^{-\frac{1}{2}(\lambda_1^2+\dots\lambda_N^2)} \prod_{j<k} |\lambda_j-\lambda_k|$$

unitary matrices: exactly similar calculation. replace, $O \rightarrow U$.

$$dX_{ij} = d\lambda_i\,\delta_{ij} + (\lambda_j-\lambda_i)\,dU_{ij} \qquad \begin{bmatrix} X_{11} & X_{12}+iY_{12} \\ X_{12}-iY_{12} & X_{22} \end{bmatrix}$$

(2×2) case

$$\left.\begin{aligned} dX_{11} &= d\lambda_1 \\ dX_{22} &= d\lambda_2 \end{aligned}\right\}$$
$$dX_{12}^{R} = (\lambda_2-\lambda_1)\, dU_{1,2}^{R}$$
$$dX_{12}^{IM} = (\lambda_2-\lambda_1)\, dU_{1,2}^{IM}$$

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2-\lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2-\lambda_1 \end{bmatrix} \Rightarrow J = (\lambda_2-\lambda_1)^2.$$

In general

$$P(\lambda_1,\dots\lambda_N) = \frac{1}{Z_N}\, e^{-\frac{1}{2}(\lambda_1^2+\dots\lambda_N^2)} \prod_{j<k} |\lambda_j-\lambda_k|^2.$$

matrix

difficult

Joint dist^n of eigenvalues

given jpdf

analysis of ~~asymptotics for~~ the statistics of physical observables

Statistical mechanics + mathematical physics

Similarly for GSE,

$$P(\lambda_1, \dots \lambda_N) \propto e^{-\frac{1}{2}(\lambda_1^2 + \dots \lambda_N^2)} \prod_{j<k} |\lambda_j - \lambda_k|^4.$$

$$\text{ns} \quad P_\beta(\lambda_1, \dots \lambda_N) \propto e^{-\frac{1}{2}(\lambda_1^2 + \dots \lambda_N^2)} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

with $\beta = 1, 2, 4 \rightarrow$ quantized.

* Coulomb gas + Dyson Brownian Motions

wever, it is possible to generalize to arbitrary real 'β'.

↳ Dumitriu & Edelman →

↳ generalized β- ensemble.

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Coulomb gas interpretation (Dyson, 1962).

$$P(\lambda_1, \dots \lambda_N) = \frac{1}{Z_N} e^{-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

where
$$Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\lambda_1 \cdots d\lambda_N \, e^{-\frac{1}{2}\sum \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

using $\lambda_i \to \sqrt{\beta}\,\lambda_i$

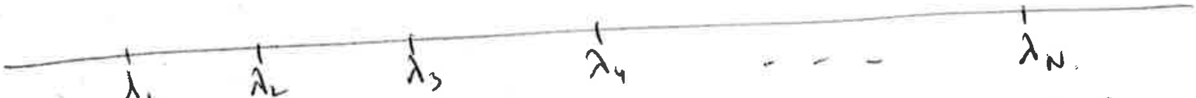
$$P(\lambda_1, \dots \lambda_N) = \frac{1}{Z_N} e^{-\frac{\beta}{2}\sum \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

$$= \frac{1}{Z_N} e^{-\frac{\beta}{2}\sum \lambda_i^2 + \frac{\beta}{2}\sum_{j\neq k} \log|\lambda_j - \lambda_k|}$$

$$= \frac{1}{Z_N} e^{-\beta\left[\frac{1}{2}\sum_i \lambda_i^2 - \frac{1}{2}\sum_{j\neq k}\log|\lambda_j - \lambda_k|\right]} \propto e^{-\beta E[\{\lambda_i\}]}$$

↳ Boltzmann weight

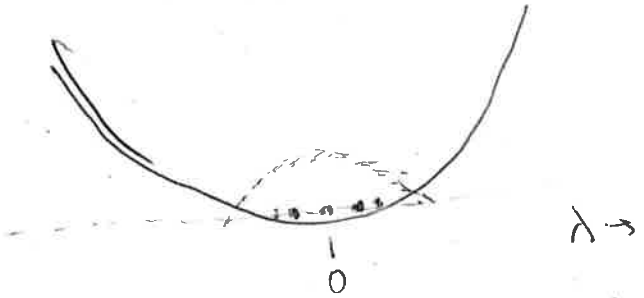
A gas of N interacting charges with locations $\lambda_i \in R$.



These charges are sitting in an external potential : $V_{ext}(\lambda) = \lambda^2$.

Each pair of charges (λ_j, λ_k) repell each other by logarithmic repulsion : $-\log|\lambda_j - \lambda_k|$

These charges are two-dimensional, but are constrained to stay on the real line.



↳ long-range interacting system in presence of an external Harmonic potential.

Note Coulomb gas → particularly suited to derive large 'N' properties
↳ thermodynamic limit

More on Coulomb gas later.

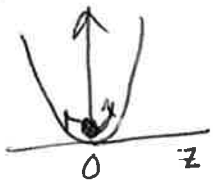
Dyson Brownian Motion

alternative way to arrive at the Joint distn

$$P(\lambda_1, \dots \lambda_N) \propto e^{-\frac{\beta}{2}\sum_i \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

is via the Dyson Brownian Motion.

Consider first the simple problem of a particle moving in an external harmonic potn, $V(z) = \frac{z^2}{2}$, in presence of a thermal noise.



$$\frac{dz}{dt} = -\frac{\partial V}{\partial z} + \eta(t) = -z + \eta(t). \quad \text{— (1)}$$

where $\eta(t)$ is a Gaussian white noise.

$$\begin{cases} \langle \eta(t) \rangle = 0. \\ \langle \eta(t_1)\eta(t_2) \rangle = 2D\,\delta(t_1 - t_2) \end{cases}$$

~~Since eq. (1) is linear in z,~~ $z(t)$.

Integrating (1).
$$z(t) = z_0 e^{-t} + e^{-t}\int_0^t e^{t'}\eta(t')\,dt' \quad \text{— (2)}$$

$\eta \to$ Gaussian process.

since, $z(t) \to$ linear combination of η's and since

$\Rightarrow z(t) \to$ Gaussian process.

$$\langle z(t) \rangle = z_0 e^{-t}$$

$$\langle [z(t) - \langle z \rangle]^2 \rangle = e^{-2t}\int_0^t\int_0^t \langle \eta(t_1)\eta(t_2)\rangle e^{t_1+t_2}\,dt_1\,dt_2$$

$$= 2D\,e^{-2t}\int_0^t\int_0^t \delta(t_1-t_2)\,e^{t_1+t_2}\,dt_1\,dt_2$$

$$= 2D\,e^{-2t}\int_0^t e^{2t_1}\,dt_1 = 2D\,e^{-2t}\frac{(e^{2t}-1)}{2}$$

$$= D(1 - e^{-2t}).$$

Hence,

$$P[z(t) = z, t] = \frac{1}{\sqrt{2\pi D(1-e^{-2t})}}\, e^{-\frac{(z - z_0 e^{-t})^2}{2D(1-e^{-2t})}} \quad \text{— (3)}, \qquad \int_{-\infty}^{\infty} P(z,t)\,dz = 1.$$

In particular, as $t \to \infty$. If we set, $D = k_B T$.

$$P_{st}(z) = P[z, t\to\infty] \longrightarrow \frac{1}{\sqrt{2\pi D}}\, e^{-\frac{z^2}{2D}} \propto e^{-\beta V(z)}.$$

(4) $\quad \hookrightarrow$ equilibrium Boltzmann distribution

ore generally, for arbitrary confining potential $V(z)$.

angevin eq^n: $\frac{dz}{dt} = -\frac{\partial V}{\partial z} + \eta(t) = f(z) + \eta(t)$. — ⑤.

$P(z,t)$ satisfies the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial z^2} - \frac{\partial}{\partial z}\left[f(z)\,P\right] = -\frac{\partial j}{\partial z}. \quad — ⑥$$

$$j = \text{current} = -D\frac{\partial P}{\partial z} + f(z)\,P \quad — ⑦.$$

At equilibrium, $\frac{\partial P}{\partial t} = 0$ and $j = 0$.

$$\hookrightarrow -D\frac{\partial P_{st}}{\partial z} + f(z)\,P_{st}(z) = 0$$

$\Rightarrow$ $\cancel{P_{st}(z)}$

$$\Rightarrow P_{st}(z) \propto e^{\frac{1}{D}\int^{z} f(z')\,dz'}$$

$$\sim e^{-\frac{1}{D}V(z)}.$$

Once again, setting $D = k_BT$, $P_{st}(z) \sim e^{-\beta V(z)}$

$\hookrightarrow$ Boltzmann-Gibbs sol^n.

Single particle $\rightarrow$ multiparticle generalization.

$z_1, z_2, \ldots z_N$, $E[\{z_i\}]$

$$\frac{dz_i}{dt} = -\frac{\partial E}{\partial z_i} + \eta_i(t)$$

$\langle \eta_i(t)\rangle = 0 \;\; \forall\, i$

$\langle \eta_i(t)\,\eta_i(t')\rangle = 2D\,\delta(t-t')$

$\langle \eta_i(t)\,\eta_j(t')\rangle = 0$ for $i \neq j$

Then,

$D = k_BT$.

$$\boxed{P_{st}\left[\{z_i\}\right] \propto e^{-\beta E[\{z_i\}]}}$$

[Cons]ider for instance a real symmetric $(N \times N)$ Gaussian matrix (GOE).

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{12} & \ddots & & \\ \vdots & & \ddots & \\ \vdots & \cdots & & x_{NN} \end{bmatrix}$$

Note,

$$P[\{x_{ij}\}] \propto e^{-\frac{\beta'}{2} Tr(X^2)} = e^{-\frac{\beta'}{2} \sum x_{ij} x_{ji}}$$

$$= e^{-\frac{\beta'}{2} \sum_{i,j} x_{ij}^2}$$

$$= e^{-\frac{\beta'}{2}\left[x_{11}^2 + x_{22}^2 + \cdots x_{NN}^2 + 2\left(x_{12}^2 + x_{13}^2 + \cdots + x_{N-1,N}^2\right)\right]}$$

[W]e can introduce a fictitious time t, and define.

$$X(t) = \begin{bmatrix} x_{11}(t) & x_{12}(t), & \cdots & x_{1N}(t) \\ & \ddots & & \\ & & \ddots & \\ & & & X_{NN}(t) \end{bmatrix}$$

[ ] each entry $x_{ij}(t)$ is an OU process

diagonal elements:

$$\frac{dX_{ii}}{dt} = -X_{ii} + \eta_{i,i}(t),$$

$\langle \eta_{i,i}(t) \rangle = 0$

$\langle \eta_{i,i}(t) \eta_{i,i}(t') \rangle = 2D\delta(t-t')$

set, $D = 1$

[off]-diagonal elements,

$i \neq j$

$$\frac{dX_{i,j}}{dt} = -X_{i,j} + \eta_{i,j}(t),$$

$\langle \eta_{i,j}(t) \rangle = 0$

$\langle \eta_{i,j}(t) \eta_{i,j}(t') \rangle$

$= 2 \cdot D' \delta(t-t')$

$= 2 \cdot \frac{1}{2} \cdot \delta(t-t')$

$= \delta(t-t')$

[ ] we are guaranteed that as $t \to \infty$,

$$P[\{X_{i,j}\}, t \to \infty] \propto e^{-\frac{1}{2} Tr(X^2)}$$

[ ], at finite t:

$$P[\{X_{ij}\}, t] \propto \frac{1}{\sqrt{(1-e^{-2t})^{\frac{N(N+1)}{2}}}} \exp\left[-\frac{1}{2(1-e^{-2t})} Tr\left[\left(X - X_0 e^{-t}\right)^2\right]\right]$$

(4)

t, we want to know how the eigenvalues $\{\lambda_1(t), \lambda_2(t), \dots \lambda_N(t)\}$ evolve with time t ?

$\downarrow$

instantaneous eigenvalues obtained by diagonalising $X(t)$.

te:
$$\frac{dX_{ii}}{dt} = -X_{ii} + \eta_{i,i}(t)$$

$$i \neq j \quad \frac{dX_{i,j}}{dt} = -X_{i,j} + \eta_{i,j}(t).$$

ally.

$$i \leq j, \quad \boxed{\frac{dX_{i,j}}{dt} = -X_{i,j} + \eta_{i,j}(t)}$$

where $\langle \eta_{i,j}(t)\, \eta_{i,j}(t') \rangle = g_{i,j}\, \delta(t-t')$.

where $g_{ii} = 2$

$g_{i,j} = 1 \quad i \neq j$

$$\boxed{\delta(t) \rightarrow \frac{1}{\Delta t}}$$

$$X_{i,j}(t+\Delta t) = X_{i,j}(t) - X_{i,j}(t).\Delta t + \eta_{i,j}(t).\Delta t.$$

$$\boxed{\hat{H} = \hat{H}_0 + \Delta t.\hat{H}_1}$$ where $\hat{H}_1 = -\hat{H}_0 + \hat{\eta}$

assume $\hat{X}_0$ was diagonal with eigenvalues $\lambda_1, \dots \lambda_N$.

$$\hat{X}_0 = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix},$$ let $\hat{X}_0 |u_n\rangle = \lambda_n |u_n\rangle$

We add the perturbation $\Delta t.\hat{H}'$

Then $\hat{H} = \hat{H}_0 + \Delta t\, \hat{H}' \rightarrow$ no longer diagonal.

If we diagonalise $\hat{H}$, ~~new~~ eigenvalues $\rightarrow \{\lambda_1', \lambda_2', \dots \lambda_N'\}$

One can use perturbation theory to compute $\{\lambda_1', \dots \lambda_N'\}$ to leading order.

(5)

...ll Perturbation Theory:

$$\hat{H}|\psi\rangle = \Lambda|\psi\rangle \quad , \quad \hat{H} = \hat{H}_0 + \epsilon \hat{H}_1$$

~~$\hat{H}_0 |u_k\rangle = E_k u_k$~~ $\hat{H}_0 u_k = E_k u_k$

...t

$$\Psi = |\Psi_0\rangle + \epsilon|\Psi_1\rangle + \epsilon^2|\Psi_2\rangle + \cdots$$

$$\Lambda = \Lambda_0 + \epsilon\Lambda_1 + \epsilon^2\Lambda_2 + \cdots$$

$$\left[\hat{H}_0 + \epsilon\hat{H}_1\right]\left[|\Psi_0\rangle + \epsilon|\Psi_1\rangle + \epsilon^2|\Psi_2\rangle + \cdots\right] = \left[\Lambda_0 + \epsilon\Lambda_1 + \epsilon^2\Lambda_2 + \cdots\right]\left[|\Psi_0\rangle + \epsilon|\Psi_1\rangle + \epsilon^2|\Psi_2\rangle + \cdots\right]$$

ϵ^0: $\quad \hat{H}_0|\Psi_0\rangle = \Lambda_0|\Psi_0\rangle$ — (1)

ϵ^1: $\quad (\hat{H}_0 - \Lambda_0)|\Psi_1\rangle = (\Lambda_1 - H_1)|\Psi_0\rangle$ — (2)

ϵ^2: $\quad (\hat{H}_0 - \Lambda_0)|\Psi_2\rangle = (\Lambda_1 - H_1)|\Psi_1\rangle + \Lambda_2|\Psi_0\rangle$ — (3)

from (1), (3), etc. $\quad |\Psi_1\rangle \to |\Psi_1\rangle + a|\Psi_0\rangle \to$ keeps (2) invariant.

So we can choose Ψ_1 to be orthogonal to Ψ_0.

similarly, Ψ_2 " " " " Ψ_0.

in general $\langle\Psi_k|\Psi_0\rangle = 0$.

(2)

$$\underbrace{\langle\Psi_0|\hat{H}_0 - \Lambda_0|\Psi_1\rangle}_{=0} = \langle\Psi_0|\Lambda_1 - H_1|\Psi_0\rangle$$

$$\Rightarrow \boxed{\Lambda_1 = \frac{\langle\Psi_0|H_1|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}} \quad , \quad \boxed{\Lambda_1^{(m)} = \langle u_m|H_1|u_m\rangle}$$

we write $\quad |\Psi_1\rangle = \sum_k a_k^{(1)}|u_k\rangle \quad$, $u_k \to$ eigenvectors of H_0, $a_m^{(1)} = 0$

Λ_3

Λ_2

Λ_1

Λ_0

Consider the perturbation of the m-th eigenvector for $\Lambda_0 = E_m$. from (2)

$$(\hat{H}_0 - E_m)\sum_{k\neq m} a_k^{(1)}|u_k\rangle = (\Lambda_1 - H_1)|u_m\rangle$$

$$\sum_k a_k^{(1)}(E_k - E_m)|u_k\rangle = (\Lambda_1 - H_1)|u_m\rangle$$

multiplying by $\langle u_k|$, $k\neq m$

$$a_k^{(1)}(E_k - E_m) = \langle u_k|\Lambda_1 - H_1|u_m\rangle$$

$$= \cancel{\Lambda_1} - \langle u_k|H_1|u_m\rangle$$

$$\Rightarrow \quad a_k^{(1)} = \frac{\langle u_k|\hat{H}_1|u_m\rangle}{E_m - E_k} \quad k \neq m.$$

$$\longrightarrow \quad \boxed{|\psi_1\rangle_m = \sum_{k\neq m} \frac{\langle u_k|\hat{H}_1|u_m\rangle}{E_m - E_k} |u_k\rangle}$$

...ultiplying ③ by $\langle\psi_0|$

$$0 = \langle\psi_0|\Lambda_1 - H_1|\psi_1\rangle + \Lambda_2\langle\psi_0|\psi_0\rangle$$

$$\Rightarrow \quad \Lambda_2 = \frac{\langle\psi_0|\hat{H}_1 - \Lambda_1|\psi_1\rangle}{\langle\psi_0|\psi_0\rangle} = \frac{\langle\psi_0|\hat{H}_1|\psi_1\rangle - \Lambda_1\overbrace{\langle\psi_0|\psi_1\rangle}^{=0}}{\langle\psi_0|\psi_0\rangle}$$

$$= \frac{\langle\psi_0|\hat{H}_1|\psi_1\rangle}{\langle\psi_0|\psi_0\rangle}$$

$$\boxed{\Lambda_2 = \frac{\langle\psi_0|\hat{H}_1|\psi_1\rangle}{\langle\psi_0|\psi_0\rangle}}$$

... the m-th eigenvalue $\quad |\psi_0\rangle \equiv |u_m\rangle$

...hen

$$\Lambda_2 = \langle u_m|\hat{H}_1|\psi_1\rangle$$

$$= \langle u_m|\hat{H}_1| \sum_{k\neq m} \frac{\langle u_k|\hat{H}_1|u_m\rangle}{E_m - E_k}|u_k\rangle$$

$$\boxed{\Lambda_2 = \sum_{k\neq m} \frac{|\langle u_k|\hat{H}_1|u_m\rangle|^2}{E_m - E_k}}$$

$$\boxed{\Lambda_2^{(m)} = E_m + \epsilon.\langle u_m|H_1|u_m\rangle + \epsilon^2 \sum_{k\neq m} \frac{|\langle u_k|\hat{H}_1|u_m\rangle|^2}{E_m - E_k}} + \cdots$$

Going back to our problem where

$$\hat{H} = \hat{H}_0 + \Delta t \cdot \hat{H}_1 \qquad \text{where} \quad \hat{H}_1 = -\hat{H}_0 + \hat{\eta}.$$

$\epsilon = \Delta t$.

$$\lambda_m' = \lambda_m + \Delta t \left[\langle u_m | -\hat{H}_0 + \hat{\eta} | u_m \rangle \right] + (\Delta t)^2 \sum_{k \neq m} \frac{|\langle u_k | -\hat{H}_0 + \hat{\eta} | u_m \rangle|^2}{\lambda_m - \lambda_k} + O(\Delta t^3)$$

$$\lambda_m' = \lambda_m - \lambda_m \Delta t + \Delta t \cdot \langle u_m | \hat{\eta} | u_m \rangle + (\Delta t)^2 \sum_{k \neq m} \frac{|\langle u_k | \hat{\eta} | u_m \rangle|^2}{\lambda_m - \lambda_k} + O(\Delta t^3)$$

$$\lambda_m' = \lambda_m - \lambda_m \Delta t + \eta_{m,m} \Delta t + (\Delta t)^2 \sum_{k \neq m} \frac{\eta_{k,m}^2}{\lambda_m - \lambda_k} + O(\Delta t^3)$$

$$\eta_{k,m}^2 \to \langle \eta_{k,m}^2 \rangle = \frac{1}{\Delta t}$$

$$\eta_{k,m}^2 = \langle \eta_{k,m}^2 \rangle + O(\Delta t) = \frac{1}{\Delta t} + O(\Delta t)$$

$$\lambda_m' = \lambda_m - \lambda_m \Delta t + \eta_{m,m} \Delta t + \Delta t \sum_{k \neq m} \frac{1}{\lambda_m - \lambda_k}$$

$$\boxed{\frac{d\lambda_m}{dt} = -\lambda_m + \sum_{k \neq m} \frac{1}{\lambda_m - \lambda_k} + \eta_{m,m}} = -\frac{\delta E}{\delta \lambda_m} + \eta_{m,m}(t)$$

$$P[\{\lambda_m\}] \propto e^{-E\{\lambda_i\}} = e^{-\left[\sum_i \frac{\lambda_i^2}{2} - \frac{1}{2} \sum_{i \neq j} \ln|\lambda_i - \lambda_j|\right]}$$

ing back to our problem where

$$\hat{H} = \hat{H}_0 + \Delta t \cdot \hat{H}_1 \qquad \text{where} \qquad \hat{H}_1 = -\hat{H}_0 + \hat{\eta}.$$

$= \epsilon = \Delta t.$

$$\lambda_m' = \lambda_m + \Delta t \left[\langle u_m | -\hat{H}_0 + \hat{\eta} | u_m \rangle \right] + (\Delta t)^2 \sum_{k \neq m} \frac{|\langle u_k | -\hat{H}_0 + \hat{\eta} | u_m \rangle|^2}{\lambda_m - \lambda_k} + O(\Delta t^3)$$

$$\lambda_m' = \lambda_m - \lambda_m \Delta t + \Delta t \cdot \langle u_m | \hat{\eta} | u_m \rangle + (\Delta t)^2 \sum_{k \neq m} \frac{|\langle u_k | \hat{\eta} | u_m \rangle|^2}{\lambda_m - \lambda_k} + O(\Delta t^3)$$

$$\lambda_m' = \lambda_m - \lambda_m \Delta t + \eta_{m,m} \Delta t + (\Delta t)^2 \sum_{k \neq m} \frac{\eta_{k,m}^2}{\lambda_m - \lambda_k} + O(\Delta t^3)$$

$$\cancel{\eta_{k,m}^2 \to \langle \eta_{k,m}^2 \rangle = \frac{1}{\Delta t}}$$

$$\eta_{k,m}^2 = \langle \eta_{k,m}^2 \rangle + O(\Delta t) = \frac{1}{\Delta t} + O(\Delta t)$$

$$\lambda_m' = \lambda_m - \lambda_m \Delta t + \eta_{m,m} \Delta t + \Delta t \sum_{k \neq m} \frac{1}{\lambda_m - \lambda_k}$$

$$\boxed{\frac{d\lambda_m}{dt} = -\lambda_m + \sum_{k \neq m} \frac{1}{\lambda_m - \lambda_k} + \eta_{m,m}} = -\frac{\delta E}{\delta \lambda_m} + \eta_{m,m}(t)$$

$$P[\{\lambda_m\}] \propto e^{-E\{\lambda_i\}} = e^{-\left[\sum_i \frac{\lambda_i^2}{2} - \frac{1}{2} \sum_{i \neq j} \ln|\lambda_i - \lambda_j|\right]}$$

(8)

$$\lambda_m' = \lambda_m - \lambda_m \Delta t + \eta_{m,m} \Delta t + (\Delta t)^2 \sum_{k \neq m} \frac{\eta_{k,m}^2 - \langle \eta_{k,m}^2 \rangle + \langle \eta_{k,m}^2 \rangle}{\lambda_m - \lambda_k}$$

$$= \lambda_m - \lambda_m \Delta t + (\Delta t)^2 \sum_{k \neq m} \frac{\langle \eta_{k,m}^2 \rangle}{\lambda_m - \lambda_k}$$

$$+ \underbrace{\eta_{m,m} \Delta t + (\Delta t)^2 \sum_{k \neq m} \frac{\eta_{k,m}^2 - \langle \eta_{k,m}^2 \rangle}{\lambda_m - \lambda_k}}_{\xi_m \Delta t}$$

$$\langle \eta_{k,m}^2 \rangle = \frac{1}{\Delta t}$$

$$\lambda_m' = \lambda_m - \left[\lambda_m - \sum_{k \neq m} \frac{1}{\lambda_m - \lambda_k}\right] \Delta t + \xi_m \Delta t$$

$$\xi_m = \eta_{m,m} + \Delta t \sum_{k \neq m} \frac{\eta_{k,m}^2 - \langle \eta_{k,m}^2 \rangle}{\lambda_m - \lambda_k}$$

$$\langle \xi_m \rangle = 0$$

$$\langle \xi_m^2 \rangle = \langle \eta_{m,m}^2 \rangle + \text{smaller order}$$

$$= \frac{1}{\Delta t} + \ldots$$

$$\frac{d\lambda_m}{dt} = -\lambda_m + \sum_{k \neq m} \frac{1}{\lambda_m - \lambda_k} + \xi_m$$

$\hookrightarrow$ Gaussian white noise.

$$P\left[\{\lambda_m\}, t \to \infty\right] \propto e^{-\beta E[\{\lambda_i\}]} \propto e^{-\left[\sum \frac{\lambda_i^2}{2} - \frac{1}{2}\sum_{i \neq j} \ln|\lambda_i - \lambda_j|\right]}$$

$$\propto e^{-\frac{1}{2}\sum_i \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|$$

$$P(\lambda_1, \cdots \lambda_N) = \frac{1}{Z_N} e^{-\frac{\beta}{2}\sum \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

$$Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\lambda_1 \cdots d\lambda_N \; e^{-\frac{\beta}{2}\sum \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$ } Selberg's integral

$$= (2\pi)^{N/2} \beta^{-\frac{N}{2} - \frac{\beta}{4}N(N-1)} \left[\Gamma\left(1+\frac{\beta}{2}\right)\right]^{-N} \prod_{j=1}^{N} \Gamma\left(1+\frac{\beta j}{2}\right)$$

Note in particular that as $N \to \infty$.

check:

$$\ln Z_N \underset{N\to\infty}{\sim} \frac{\beta}{4} \log N . N^2 - \beta N^2 \left[\frac{3}{8} + \frac{1}{4}\log 2\right].$$

Scaling $\lambda_i \sim \sqrt{N}\, \tilde{\lambda}_i$

$$\int d\tilde{\lambda}_1 \cdots d\tilde{\lambda}_N \; e^{-\frac{\beta}{2} N \sum_{i=1}^{N} \tilde{\lambda}_i^2 + \frac{\beta}{2} \sum_{j\neq k} \ln|\tilde{\lambda}_j - \tilde{\lambda}_k|}$$

$$\sim e^{-\beta N^2 \left[\frac{3}{8} + \frac{1}{4}\log 2\right] + O(N \log N)}$$

Generalized β-ensemble [Dumitriu & Edelman, J. Math. Phys. 43, 5830 (2002)]

Consider the tridiagonal matrix:

$$H_\beta = \frac{1}{\sqrt{2}} \begin{bmatrix} N(0,2) & \chi_{(N-1)\beta} & & & 0 \\ \chi_{(N-1)\beta} & N(0,2) & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \chi_\beta \\ 0 & & \ddots & \chi_\beta & N(0,2) \end{bmatrix}$$

- symmetric.
- matrix elements are indep. but non-identically distributed.

$\Omega = \chi_d^2 = X_1^2 + \cdots + X_d^2$, $\quad X_i \to$ standard normal dist^n.

$$\langle e^{-\lambda\Omega}\rangle = \left[\langle e^{-\lambda X^2}\rangle\right]^d = \left[\int_{-\infty}^{\infty} e^{-\lambda x^2 - x^2/2} \frac{dx}{\sqrt{2\pi}}\right]^d = \frac{1}{(2\lambda+1)^{d/2}}$$

$$\int_0^\infty e^{-\lambda\Omega} P(\Omega)\, d\Omega = \frac{1}{(2\lambda+1)^{d/2}}$$

$$\Rightarrow \quad P(\Omega) = \frac{\Omega^{\frac{d}{2}-1} e^{-\Omega/2}}{2^{d/2}\,\Gamma\left(\frac{d}{2}\right)}$$

$\chi_d = \sqrt{\Omega}$, $\quad \Omega = \chi_d^2$

$P(\chi_d)\, d\chi_d = P(\Omega)\, d\Omega$

$$P(\chi_d) \frac{1}{2\sqrt{\Omega}} = \frac{\Omega^{\frac{d}{2}-1} e^{-\Omega/2}}{2^{d/2}\,\Gamma(d/2)}$$

$$\boxed{P(\chi_d) = \frac{\Omega^{\frac{d-1}{2}} e^{-\Omega/2}}{2^{\frac{d}{2}-1}\,\Gamma(d/2)} = \frac{\chi_d^{d-1} e^{-\chi_d^2/2}}{2^{\frac{d}{2}-1}\,\Gamma(d/2)}} \qquad \chi_d > 0.$$

Then

$$\boxed{P(\lambda_1, \ldots \lambda_N) = \frac{1}{Z_N} e^{-\frac{1}{2}\sum \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta}$$

Wishart matrices.

$$W = \begin{cases} X^t X & X \to \text{real } \cancel{\text{symmetric}} \ (M\times N) \text{ rectangular matrix. } M \geq N. \\ \\ X^\dagger X & X \to \text{complex } (M\times N) \to \text{rectangular matrix.} \end{cases}$$

$M < N \to$ Anti-Wishart.
{ M positive eigenvalues, the rest $N-M$ zero eigenvalues }

$W \to N\times N$ matrix.

no. 'N' of non-negative eigenvalues
$\{\lambda_1, \cdots \lambda_N\}$.

[~~XX^t~~ $X \to 2\times 3$
$X^t \to 3\times 2$
$W = X^t X = (3\times 3)$
$\hookrightarrow$ 3 eigen
$\to$ one zero
2 non zero.

Proof:
$X^t X|\lambda\rangle = \lambda|\lambda\rangle$
$X X^t [X|\lambda\rangle] = \lambda [X|\lambda\rangle]$
$\Rightarrow X|\lambda\rangle \to$ eigenvector of XX^t with eigenvalue λ,
unless $\lambda=0$.].

Then it can be shown [James, 1964]
when $X \to$ Gaussian $P[X] \propto e^{-\frac{1}{2}\mathrm{Tr}(X^t X)}$

$$P[\lambda_1, \cdots \lambda_N] = K_N \, e^{-\frac{1}{2}\sum_{i=1}^{N}\lambda_i} \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \times \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

originally for $\beta = 1, 2, 4$.
$\hookrightarrow$ arbitrary β by Dumitriu - Edelman.

$\to \beta$-Laguerre ensemble.

Wishart matrices.

$$W = \begin{cases} X^t X & X \to \text{real } \cancel{\text{symmetric}}\ (M \times N) \text{ rectangular matrix.} \quad M \geq N. \\ \\ X^\dagger X & X \to \text{complex } (M \times N) \to \text{rectangular matrix.} \end{cases}$$

$M < N \to$ Anti-Wishart.

$\left\{ M \text{ positive eigenvalues, the rest } N-M \text{ zero eigenvalues} \right\}$

$W \to N \times N$ matrix.

has 'N' ~~of~~ non-negative eigenvalues $\{\lambda_1, \dots \lambda_N\}$.

[~~$X^t X$~~ $X \to 2\times 3$

$X^t \to 3 \times 2$

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for when $X \to$ Gaussian $P[X] \propto e^{-\frac{1}{2}\mathrm{Tr}(X^t X)}$

$$P[\lambda_1, \dots \lambda_N] = K_N \, e^{-\frac{1}{2}\sum_{i=1}^{N}\lambda_i} \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \times \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

originally for $\beta = 1, 2, 4$.

$\hookrightarrow$ arbitrary β by Dumitriu - Edelman.

$\to$ β-Laguerre ensemble.

$$\cancel{\prod p(a_i - z)}$$

$$D_n(z) = \prod_{i=1}^{N} (z - \lambda_i) =$$

$$D_2(z) = z^2 - (\lambda_1 + \lambda_2) z + \lambda_1 \lambda_2 .$$

$$\mathrm{Tr}(H) = \lambda_1 + \lambda_2$$

$$\mathrm{Tr}(H^2) = \mathrm{Tr} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \mathrm{Tr} \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} = \lambda_1^2 + \lambda_2^2 .$$

$$E_s[\rho_c] = \frac{3}{8} + \frac{s^2}{3} - \frac{s^4}{108} - \left(\frac{s^3}{108} + \frac{5s}{36}\right)\sqrt{6+s^2} - \frac{1}{2}\ln\left[s + \sqrt{6+s^2}\right] + \frac{1}{2}\log 3 + \frac{1}{2}\log 2$$

$$\cancel{E_{\sqrt{2}}[\rho_c]} \qquad \Psi_s = E_s(\rho_c) - E_{\sqrt{2}}(\rho_c)$$

$$\Rightarrow E_{\sqrt{2}}(\rho_c) = E_s(\rho_c) - \Psi_s$$

$$= \frac{3}{8} + \cancel{\frac{s^2}{3}} - \cancel{\frac{s^4}{108}} - \cancel{\left(\frac{s^3}{108} + \frac{5s}{36}\right)\sqrt{6+s^2}} - \cancel{\frac{1}{2}\ln\left(s+\sqrt{6+s^2}\right)}$$

$$+ \cancel{\frac{1}{2}\log 3} + \frac{1}{2}\log 2$$

$$- \cancel{\frac{s^2}{3}} + \cancel{\frac{s^4}{108}} + \cancel{\left(\frac{s^3}{108} + \frac{5s}{36}\right)\sqrt{6+s^2}} + \cancel{\frac{1}{2}\ln\left(s+\sqrt{6+s^2}\right)}$$

$$- \frac{1}{4}\log 2 - \cancel{\frac{\ln 3}{2}}$$

$$= \frac{3}{8} + \frac{1}{4}\log 2$$

$$e^{-\beta N^2\left[\frac{3}{8} + \frac{1}{4}\log 2\right]}$$

$$e^{-\frac{\beta}{4}N^2 \log\beta}$$

$$e^{\sum_{j=1}^{N} \log\Gamma\left(1+\frac{\beta j}{2}\right)}$$

$$\Gamma(z) \sim z^{z-1/2}\, e^{-z} \sqrt{2\pi}$$

$$\ln\Gamma(z) \sim \left(z - \frac{1}{2}\right)\ln z - z + \ln(\sqrt{2\pi})$$

$$\sum_j \frac{\beta j}{2} \ln\left(\frac{\beta j}{2}\right) - \sum_{j=1}^{N} \frac{\beta j}{2}$$

$$- \frac{\beta}{2}\cdot\frac{N^2}{2}$$

$$\frac{\beta}{2}\int^N x \log x \, dx$$

$$\frac{\beta}{2}\left[\log x \cdot \frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}\, dx\right]$$

$$\frac{\beta}{2}\left[\log N \cdot \frac{N^2}{2} - \frac{N^2}{4}\right.$$

$$\frac{\beta}{2}\ln\left(\frac{\beta}{2}\right)\frac{N^2}{2}$$

$$- \frac{\beta N^2}{4}\ln 2$$

$$\frac{1}{4} + \frac{1}{8}$$

$$\frac{2+1}{8}$$

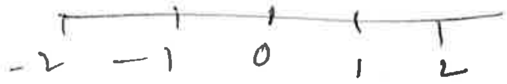
$$\beta N^2\left[-\frac{1}{8} - \frac{1}{4} - \frac{1}{4}\log 2\right]$$

$$-\beta N^2\left[\frac{3}{8} + \frac{1}{4}\log 2\right]$$

$$\frac{\beta}{2\cdot 2}\, N^2 \log N$$

$$\phi(i,t) = \frac{1}{2}\left[\phi(i-1,t-1) + \phi(i+1,t-1)\right].$$

$$\phi(i,0) \to N(0,1)$$

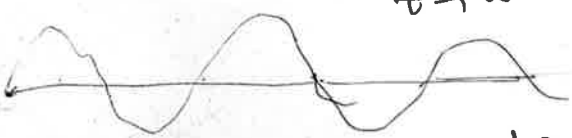


$$\phi(0,t) \quad , \quad \phi(0,0)$$

$$\text{Prob}\left[\begin{array}{l} \phi(0,\tau) \text{ does not change sign} \\ > \phi(0,0) \quad \text{up to time } t \end{array}\right]$$

$$\text{Prob}\left[\phi(0,\tau) > \phi(0,0) \quad \forall\, 0 < \tau \leq t\right]$$

$$\underset{t\to\infty}{\sim} \frac{1}{t^{\theta}} \qquad f(T)$$



$$\theta \simeq .1209\ldots \qquad P\Big[$$

$$\langle \phi(0,t_1)\,\phi(0,t_2)\rangle = \frac{1}{\sqrt{t_1+t_2}}.$$

$$\psi(0,t) = \frac{\phi(0,t)}{\sqrt{\langle \phi^2(0,t)\rangle}}$$

$$\langle \psi(0,t_1)\,\psi(0,t_2)\rangle = \frac{\langle \phi(0,t_1)\,\phi(0,t_2)\rangle}{\sqrt{\langle \phi^2(t_1)\rangle \langle \phi^2(0,t_2)\rangle}} = \left[\frac{\sqrt{t_1 t_2}}{t_1+t_2}\right]^{1/2}$$

$$= \left[\frac{\sqrt{t_2/t_1}}{1+\frac{t_2}{t_1}}\right]^{1/2}$$

$$\cancel{T = \log} \quad t = e^{T} \qquad e^{T_1 -}$$

$$= f\left[(T_1 - T_2)\right]$$

Recap of Lecture-1 (Saclay, 2015)

Random matrices with real spectrum
(real symm. or complex Hermitian).

Gaussian

matrices with indep. entries

$$P[X]dX = \prod_{i=1}^{N} f_i(x_{ii}) \prod_{i>j} f_{ij}(x_{ij})\, dX$$

Wigner matrices

Rotationally invariant matrices

$$P(X)\,dX = P[O^t X O]\, dX.$$

any normalizable positive function

(i) Numerically easy to generate such matrices.

(ii) Very hard to compute the joint distn of eigenvalues $P(\lambda_1, \cdots \lambda_N)$

eigenvalues & eigenvectors $\to$ correlated.

exception: Dimitriu-Edelman β-ensemble.

$$P(\lambda_1, \ldots \lambda_N) \propto e^{-\frac{1}{2}\sum \lambda_i^2} \prod_{i>j} |\lambda_i - \lambda_j|^{\beta}$$

for any real $\beta \geq 0$.

- ex: $P[X] \propto e^{-\frac{1}{2\sigma^2}\mathrm{Tr}(X^2)} \to$ Gaussian ensemble.
- matrix models: $P[X] \propto e^{-\frac{1}{2\sigma^2}\mathrm{Tr}(X^2) - b\,\mathrm{Tr}(X^4) \ldots}$

  in general $P[X] \propto e^{-\mathrm{Tr}[V(X)]}$, V polynomial in X.
- Cauchy ensemble:

$$P[X] \propto \frac{1}{\left[\det(I + X^t X)\right]^{\gamma}}, \quad \gamma > 0$$

Remarks:

(i) Numerically hard to generate such matrices (except Gaussian)

eg. $P[X] \propto e^{-\frac{1}{2\sigma^2}\mathrm{Tr}(X^2) - b\,\mathrm{Tr}(X^4)}$

$$\to e^{-\frac{1}{2\sigma^2}\sum_{i,j} x_{ij}^2 - b \sum_{i,j,k,l} x_{ij}x_{jk}x_{kl}x_{li}}$$

$\hookrightarrow$ entries are horribly correlated.

(ii) however, advantage

$\hookrightarrow$ eigenvalues & eigenvectors $\to$ uncorrelated

$P[X] \to$ depends only on eigenvalues.

(2)

rotationally invar. ensemble: (iii) As a result, the joint pdf of eigenvalues is ~~easy to compute~~ possible to compute.

$$P(\lambda_1, \dots \lambda_N) \propto f\left[\{\lambda_i\}\right] \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

$\beta = 1 \to$ real sym.
$= 2 \to$ complex Hermit.

ex: Matrix β models:

$$P(\lambda_1, \dots \lambda_N) \propto e^{-\sum_{i=1}^N V(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

special case: Gaussian,

$$P(\lambda_1, \dots \lambda_N) \propto e^{-\frac{1}{2\sigma^2}\sum \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

Cauchy:

$$P(\lambda_1, \dots \lambda_N) \propto \frac{1}{\left[\prod_{i=1}^N (1+\lambda_i^2)\right]^\gamma} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

etc.

Note: Porter-Rosenzweig Th:

Gaussian $\to$ special $\to$ indep. entries, rotationally invariant } easy to generate numerically, " " compute the joint dist^n of eigenvalues.

$$\boxed{P(\lambda_1, \dots \lambda_N) = \frac{1}{Z_N} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^N \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta}$$

Remark-1: if the vandermonde is absent:

$$P(\lambda_1, \dots \lambda_N) \propto \prod_{i=1}^N e^{-\frac{1}{2\sigma^2}\lambda_i^2} \to \text{factorize}$$

$\hookrightarrow$ uncorrelated $\to$ trivial.

$\prod_{i<j} |\lambda_i - \lambda_j|^\beta \to$ makes the eigenvalues strongly correlated

Also, Prob. two eigenvalues "close" $\to$ very small

$\hookrightarrow$ "level repulsion"

Remark-2. Rescaling. $\lambda_i \to \sigma\sqrt{\beta}\,\lambda_i$

Then

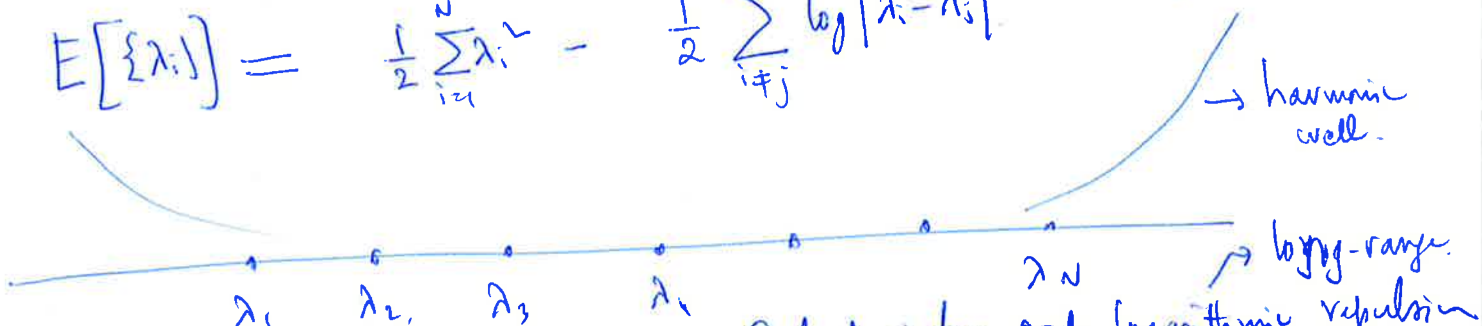
$$P(\lambda_1,\dots\lambda_N) = \frac{1}{Z_N}\, e^{-\frac{\beta}{2}\sum \lambda_i^2} \prod_{i<j} |\lambda_i-\lambda_j|^\beta$$

$$= \frac{1}{Z_N}\, e^{-\frac{\beta}{2}\sum\lambda_i^2 + \frac{\beta}{2}\sum_{i\neq j}\log|\lambda_i-\lambda_j|}$$

$$= \frac{1}{Z_N}\, e^{-\beta E[\{\lambda_i\}]}, \qquad Z_N = \int d\lambda_1\dots d\lambda_N\, e^{-\beta E[\{\lambda_i\}]}$$

$\beta \to$ inverse temp.

$$E[\{\lambda_i\}] = \frac{1}{2}\sum_{i=1}^N \lambda_i^2 - \frac{1}{2}\sum_{i\neq j}\log|\lambda_i-\lambda_j|$$



each pair → repel each other via ~~Gravitational~~ logarithmic repulsion
↳ Dyson gas.

remark: 2.1 Analogy with Ising model.

$$E[\{s_i\}] = -J\sum_{\langle ij\rangle} S_i S_j - h\sum S_i$$

$\sum_{\langle ij\rangle}$ ↓ short range. h ↓ external magnetic field.

Dyson gas → 1-d long range stat. mech. system.

Remark-3. Two terms $\frac{1}{2}\sum\lambda_i^2$ and $-\frac{1}{2}\sum_{i\neq j}\ln|\lambda_i-\lambda_j|$ compete with each other.

Typical size of eigenvalues for large N.

$$\lambda_{typ}^2 N \sim N^2$$

$$\boxed{\lambda_{typ} \sim \sqrt{N}}$$

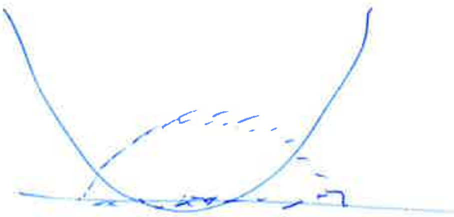
Sometimes. $\lambda_i \to \sqrt{N}\,\lambda_i$

$$P(\lambda_1,\dots\lambda_N) \sim e^{-\frac{\beta N}{2}\sum\lambda_i^2 - \frac{1}{2}\sum\ln|\lambda_i-\lambda_j|}$$

Then $\lambda_i \sim O(1)$.

Consequence $E_{Dyson} \sim O(N^2)$
↑ long range system
For short range, $E \sim O(N)$

mark-4 :



The gas will settle down into an average config.

$$\rho_N(\lambda) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle \to \text{av. density of eigenvalues.}$$

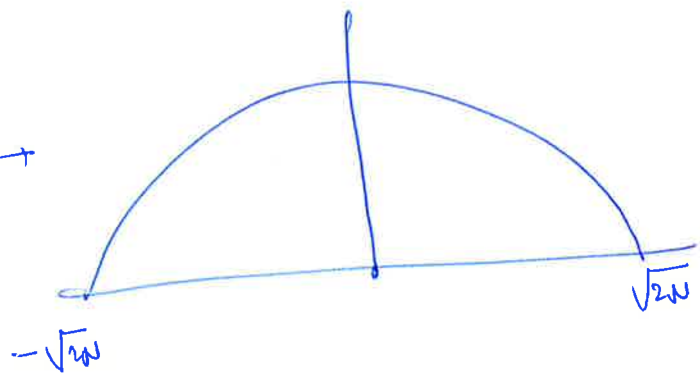
(normalized to unity)

Av. fraction of eigenvalues in $[\lambda, \lambda + d\lambda)$.

We will see later that as $N \to \infty$,

$$\rho_N(\lambda) \longrightarrow \frac{1}{\sqrt{N}} f\left(\frac{\lambda}{\sqrt{N}}\right).$$

where $f(z) = \frac{1}{\pi} \sqrt{2 - z^2}$.



$$\rho_N(\lambda) = \frac{1}{\sqrt{N}} \frac{1}{\pi} \sqrt{2 - \frac{\lambda^2}{N}}$$
$$= \frac{1}{\pi N} \sqrt{2N - \lambda^2}.$$

Wigner semi-circular law.

many methods to compute this.

$\underline{m}$, $P(\lambda_1, \ldots \lambda_N) = \frac{1}{Z_N} e^{-\frac{\beta}{2}\sum \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$;

Question → <u>observables</u> → depends on the applications.

<u>Natural observables:</u>

① ~~n-pointsso~~ Av. density of states

$$\rho_N(\lambda) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle$$

$$= \int P(\lambda, \lambda_2, \ldots \lambda_N) \, d\lambda_2 \ldots d\lambda_N$$

↳ one-point marginal.

More generally, n-point correlation functions:

$$R_n(\lambda_1, \ldots \lambda_n) = \frac{N!}{(N-n)!} \int P(\lambda_1, \ldots \lambda_n, \lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_N) \, d\lambda_{n+1} \ldots d\lambda_N .$$

↳ n-point marginal.

② <u>Spacing distribution</u> (nearest neighbour)

$$p_N(s) = \left\langle \frac{1}{N-1} \sum_{i=1}^{N-1} \delta\left[|\lambda_i - \lambda_{i+1}| - s\right] \right\rangle$$

③ <u>Counting statistics</u>: $N_{[L_1, L_2]}$ → no. of eigenvalues in $[L_1, L_2]$

↳ random variables

$N_{[0,\infty]}$ → index

↳ What are its statistics?

④ <u>Largest eigenvalue</u>: $\lambda_{max} = \max(\lambda_1, \ldots, \lambda_N)$

$$P[\lambda_{max}, N] = ?$$

$$\boxed{P_N(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N}\, e^{-\frac{\beta}{2}\sum \lambda_i^2} \prod_{i<j} |\lambda_i-\lambda_j|^\beta}$$

Two approaches.

finite N approach

"harder"

exception: $\boxed{\beta=2}$

GUE

"determinantal process"

↓

many properties can be computed explicitly.

↑

via cold atoms.

↳ physical realization of GUE

large-N, any-β } Coulomb gas approach

↓

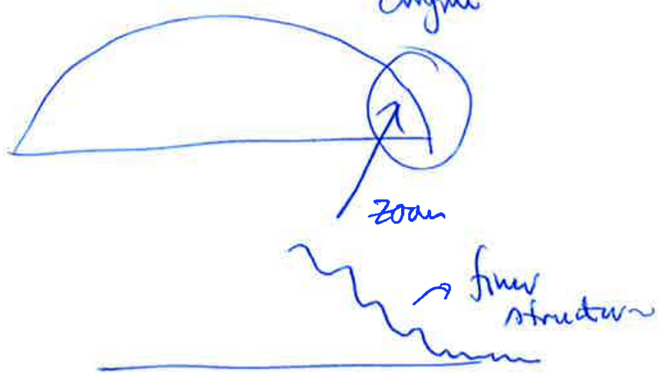
good for global quantities.

↓

ex. semi-circular law.

↓

but local quantities are harder to compute



Spectral properties of eigenvalues

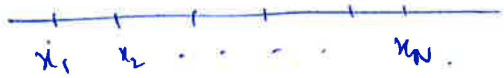
$x_1, x_2, \dots x_N \longrightarrow$ eigenvalues.

$P(x_1, x_2, \dots x_N) \longrightarrow$ Joint distribution of eigenvalues.

Gaussian Random matrices.

$$P(x_1, \dots x_N) = \frac{1}{Z_N} e^{-\frac{\beta}{2}\sum x_i^2} \prod_{i<j} |x_i - x_j|^{\beta}$$

$$\beta = 1 \longrightarrow \text{GOE}$$
$$= 2 \longrightarrow \text{GUE}$$
$$= 4 \longrightarrow \text{GSE}.$$



What knowledge on "physically measurable" variables can ~~do~~ we infer from the joint distn. $P(x_1, \dots x_N)$?

First basic observable:

~~Average~~ Number density:

define: $\hat{n}(x) = \sum_{i=1}^{N} \delta(x - x_i) \longrightarrow$ counts the no. of eigenvalues in $[x, x+dx]$.

$$\Rightarrow \int \hat{n}(x)\, dx = N.$$

Normalized ~~number~~ density: $\hat{\rho}(x) = \frac{1}{N}\hat{n}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i)$.

$\int \hat{\rho}(x)\, dx = 1 \longrightarrow$ counts the fraction of eigenvalues in $[x, x+dx]$.

verage density of eigenvalues:

$$\rho_N(x) = \left\langle \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i) \right\rangle,$$
$$= \langle \hat{\rho}(x) \rangle$$

where $\langle\ \rangle \longrightarrow$ w.r.t. to the joint distn

$$\langle f(\{x_i\}) \rangle = \int f[\{x_i\}]\, P(x_1, \dots x_N)\, dx_1 \dots dx_N$$

Hence,

$$\rho_N(x) = \left\langle \frac{1}{N}\sum_{i=1}^{N} \delta(x-x_i) \right\rangle$$

$$= \int \frac{1}{N}\sum_{i=1}^{N} \delta(x-x_i)\; P(x_1,\dots x_N)\; dx_1\dots dx_N$$

$$= \frac{1}{N}\sum_{i=1}^{N} P(x_1,\dots x_{i-1}, \cancel{x_i}, x_{i+1},\dots x_N)\; dx_1\dots dx_{i-1}\, dx_{i+1}\dots dx_N.$$

↓ using isotropic property.

$$\boxed{\rho_N(x) = \int P(x, x_2, \dots x_N)\; dx_2\dots dx_N}$$

→ one-point marginal.

Average density ⟶ one-point marginal of the joint distribution $P(x_1,\dots x_N)$.

2) n-point correlation function ⇒ n-point marginals

: $$R_n(x_1,\dots x_n) = \frac{N!}{(N-n)!}\int P(x_1,\dots x_n, x_{n+1}, x_{n+2},\dots x_N)\; dx_{n+1}\, dx_{n+2}\dots dx_N$$

Ex: $$R_1(x) = N \int P(x, x_2,\dots x_N)\; dx_2\dots dx_N.$$

$$= N.\, \rho_N(x) = \text{average number density} = \langle \hat{n}(x) \rangle.$$

: $R_N(x_1,\dots x_N) = N!\, P(x_1,\dots x_N)$

Knowledge of $R_n(x_1,\dots x_n)$ for all $n = 1, 2, \dots N$.
provides a full description of the system.

Any physical observable ⇒ can be expressed in terms of R_n's.

Ex: no. of particles in an interval A:
$A = [L_1, L_2]$
→ Later

Calculating n- point correlation function R_n is usually hard for ~~any other~~ ~~β~~ arbitrary 'β'.

However for $\beta = 2$ (GUE) - there is a simplication using methods of orthogonal polynomials $\longrightarrow$ "determinantal structure".

We will illustrate this method using physical example of "N free ~~For~~ fermions in a harmonic trap at $T=0$" in $d=1$."

↳ Quantum system

$\Big\downarrow$ exact correspondence

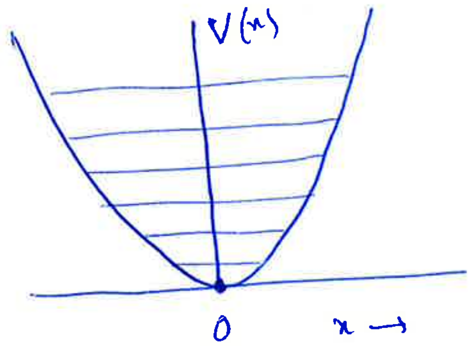
~~GUE~~ GUE

methods of orthogonal polynomials $\longrightarrow$ basic quantum mechanics.

Free Fermion in a harmonic trap in $d=1$ $(T=0)$ $\to$ Quantum system.

$\to$ Exact one to one correspondence to GUE $(\beta=2)$.

Consider first a single ~~particle~~ quantum particle in a 1-d harmonic potential $V(x) = \frac{1}{2} m\omega^2 x^2$.



Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x).$$

Energy eigenfunctions & eigenvalues: (single particle):

Schrödinger eq^n: $\hat{H}\varphi_k = \epsilon_k \varphi$.

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\varphi_k}{dx^2} + \frac{1}{2}m\omega^2 x^2 \varphi_k(x) = \epsilon_k \varphi_k(x).}$$

- $\epsilon_k = (k+\frac{1}{2})\hbar\omega$, $\quad k = 0, 1, 2, \ldots$
- $$\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi}\, 2^k\, k!}\right]^{1/2} e^{-\frac{\alpha^2}{2}x^2} H_k(\alpha x)$$

Where $\alpha = \sqrt{\frac{m\omega}{\hbar}}$ $\to$ dimension of inverse length.

$H_k(x)$ $\to$ Hermite polynomials of degree 'k'.

ex: $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = 4x^2 - 2$, ~~etc~~ $H_3(x) = 8x^3 - 12x$ etc.

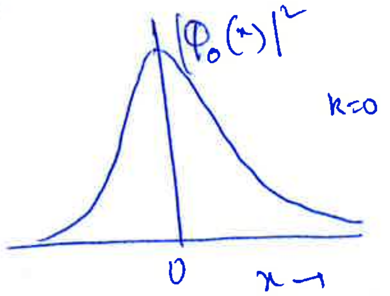
in general $\boxed{H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k}\left(e^{-x^2}\right)}$

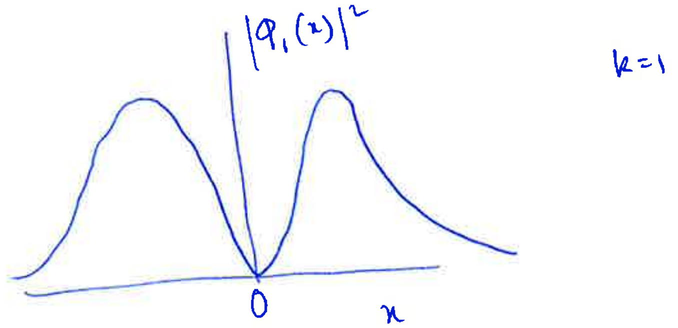
Hermite polynomials ~~too~~ are orthogonal in the sense.

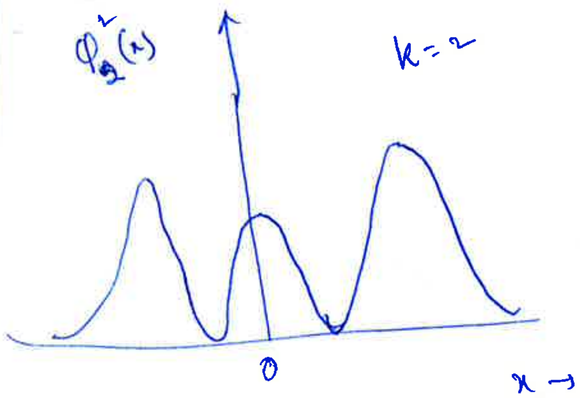
$$\boxed{\int_{-\infty}^{\infty} H_n(x)\, H_m(x)\, e^{-x^2}\, dx = \sqrt{\pi}\; 2^n\, n!\; \delta_{n,m}.}$$

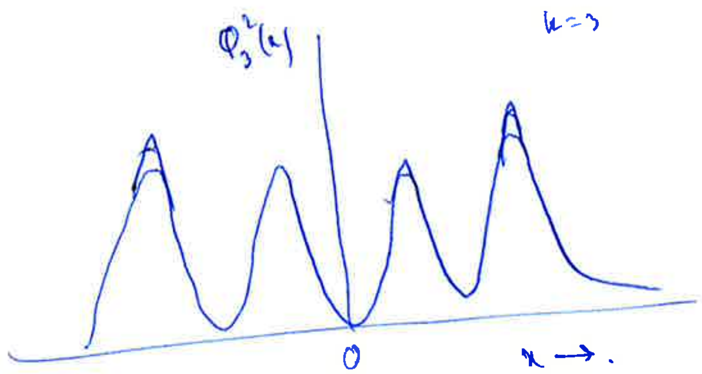
One can then easily show that $\varphi_k(x)$'s are orthonormal.

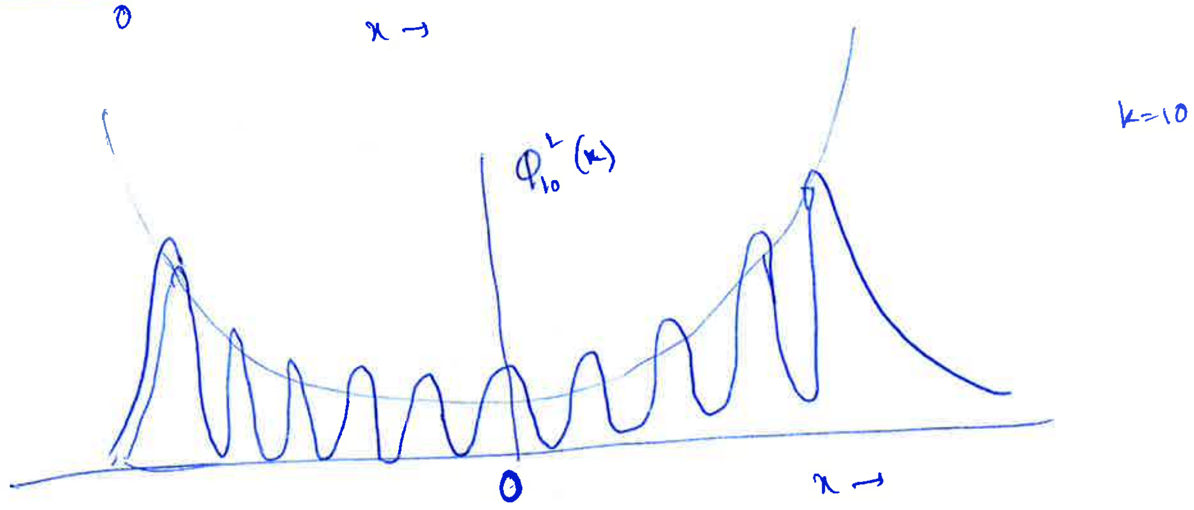
$$\boxed{\int_{-\infty}^{\infty} \varphi_k^*(x)\, \varphi_m(x)\, dx = \delta_{n,m}}$$







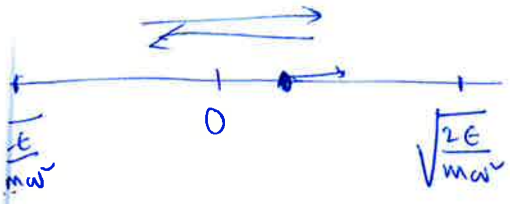




Correspondence Principle: (appendix)

Classical oscillator: $\frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2 = \mathcal{E}$ = energy.

$$v^2 = \dot{x}^2 = \frac{2\mathcal{E}}{m} - \omega^2 x^2$$



$$\frac{dx}{dt} = v = \pm \sqrt{\frac{2\mathcal{E}}{m} - \omega^2 x^2}$$

Time period of oscillation:
↳ 'T'

$$\int_{-\sqrt{\frac{2\mathcal{E}}{m\omega^2}}}^{\sqrt{\frac{2\mathcal{E}}{m\omega^2}}} \frac{dx}{\sqrt{\frac{2\mathcal{E}}{m} - \omega^2 x^2}} = \int_0^{T/2} dt = \frac{T}{2}.$$

$$\int_{-\sqrt{\frac{2\mathcal{E}}{m\omega^2}}}^{\sqrt{\frac{2\mathcal{E}}{m\omega^2}}} \frac{dx}{\sqrt{\frac{2\mathcal{E}}{m\omega^2} - x^2}} = \frac{\omega T}{2}.$$

$\hookrightarrow x = \sqrt{\frac{2\mathcal{E}}{m\omega^2}}\, y.$

$$\int_{-1}^{1} \frac{dy}{\sqrt{1-y^2}} = \frac{\omega T}{2} = \pi$$

$$\Rightarrow \boxed{T = \frac{2\pi}{\omega}}$$

↳ indep. of energy $\mathcal{E}$.

Classical prob. distn:

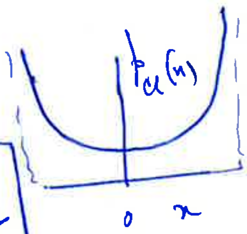
$$P_{cl}(x)\,dx = \frac{dt}{T/2}$$

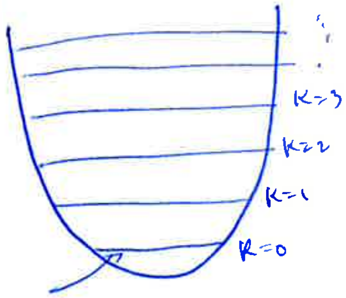
$$\Rightarrow P_{cl}(x) = \frac{2}{T}\frac{dt}{dx} = \frac{\omega}{\pi}\frac{1}{v} = \frac{\omega}{\pi}\frac{1}{\sqrt{\frac{2\mathcal{E}}{m} - \omega^2 x^2}}$$

Note

$$\int P_{cl}(x)\,dx = \frac{\omega}{\pi}\int_{-\sqrt{\frac{2\mathcal{E}}{m\omega^2}}}^{\sqrt{\frac{2\mathcal{E}}{m\omega^2}}} \frac{1}{\sqrt{\frac{2\mathcal{E}}{m} - \omega^2 x^2}}\,dx = \frac{1}{\pi}\int_{-1}^{1}\frac{dy}{\sqrt{1-y^2}} = 1$$

$$\Rightarrow \boxed{P_{cl}(x) = \frac{\omega}{\pi}\frac{1}{\sqrt{\frac{2\mathcal{E}}{m} - \omega^2 x^2}}}$$





$\epsilon_k = (k + \frac{1}{2})\hbar\omega, \quad k = 0, 1, 2, \ldots$

$\phi_k(x) \rightarrow$ single particle energy eigenfunction.

Consider now 'N' particles $\rightarrow$ N spinless free fermions.

$\boxed{\text{Pauli exclusion principle (no two fermions can be in the same state).}}$

N-body Hamiltonian

$$\hat{H}_N \equiv -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \cdots \frac{\partial^2}{\partial x_N^2}\right) + \frac{1}{2}m\omega^2(x_1^2 + \cdots x_N^2)$$

$x_1, x_2, \ldots x_N \rightarrow$ positions of 'N' fermions.

We want to find the many body ~~wavefunction~~ energy eigenfunction

$$\Psi_E(x_1, \ldots x_N).$$

that satisfies $\hat{H}_N \Psi_E = E\,\Psi_E$

$$\boxed{-\frac{\hbar^2}{2m}\sum_i \frac{\partial^2}{\partial x_i^2}\Psi_E + \sum_i V(x_i)\,\Psi_E = E\,\Psi_E.} \quad \text{--- (i)}$$

Since the particles are non-interacting (no cross term in $\hat{H}_N$: $\hat{H}_N = \sum_{i=1}^{N}\hat{H}_i$).

obviously. $\Psi(x_1, \ldots x_N) = \phi_{k_1}(x_1)\,\phi_{k_2}(x_2)\cdots\phi_{k_N}(x_N)$

$\hookrightarrow$ satisfies the Schrödinger eqn

with $E = (\epsilon_{k_1} + \epsilon_{k_2} + \cdots \epsilon_{k_N})$

$= (k_1 + k_2 + \cdots + k_N + \frac{N}{2})\hbar\omega.$

any permutation, e.g.

$\phi_{k_1}(x_2)\,\phi_{k_2}(x_1)\cdots\phi_{k_N}(x_N)$

is also an eigenstate with the same energy $E = (k_1 + \cdots k_N + \frac{N}{2})\hbar\omega.$

$\Rightarrow$ any linear combination of the permutation states $\rightarrow$ eigenstate

...rmion $\Longrightarrow$ $\Psi_E(x_1, x_2, \cdots x_N) = 0$ if any $x_i = x_j$ $i \neq j$

$\hookrightarrow$ Pauli exclusion principle.

...ce for free Fermions, we need to "antisymmetrize" the wavefunction:

$$\Psi_E(x_1, \cdots x_N) = \frac{1}{\sqrt{N!}}\left[\sum_\sigma \epsilon_\sigma \, \phi_{k_1}(x_{\sigma_1}) \phi_{k_2}(x_{\sigma_2}) \cdots \phi_{k_N}(x_{\sigma_N})\right] \quad \text{— (1)}$$

$\downarrow$

eigenstate with
...ergy $E = (k_1 + \cdots k_N + \frac{N}{2})\hbar\omega$

$\sigma_1, \sigma_2, \cdots \sigma_N \to$ permutation of 1, 2, ... N.

$\epsilon_\sigma \to$ sign of the permutation

any pair-wise exchange gives a negative sign.

...: $N = 2$

$$\Psi_E(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\phi_{k_1}(x_1)\phi_{k_2}(x_2) - \phi_{k_1}(x_2)\phi_{k_2}(x_1)\right].$$

...en, $\Psi_E(x_1, x_2 = x_1) = 0$

...q. (1) can be conveniently written as a determinant.

$$\Psi_E(x_1, \cdots x_N) = \frac{1}{\sqrt{N!}} \det\left[\phi_{k_i}(x_j)\right]_{1 \leq i, j \leq N} \longrightarrow \text{Slater determinant}$$

...: $N=2$

$$\Psi_E(x_1, x_2) = \frac{1}{\sqrt{2}} \det\begin{vmatrix} \phi_{k_1}(x_1) & \phi_{k_1}(x_2) \\ \phi_{k_2}(x_1) & \phi_{k_2}(x_2) \end{vmatrix}$$

...general,

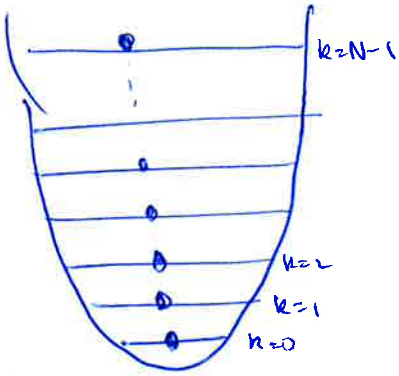
$$\Psi_E(x_1, x_2, \cdots x_N) = \frac{1}{\sqrt{N!}} \det\begin{vmatrix} \phi_{k_1}(x_1) & \phi_{k_1}(x_2) & \cdots & \phi_{k_1}(x_N) \\ \phi_{k_2}(x_1) & \phi_{k_2}(x_2) & \cdots & \phi_{k_2}(x_N) \\ \vdots & & & \vdots \\ \phi_{k_N}(x_1) & \phi_{k_N}(x_2) & \cdots & \phi_{k_N}(x_N) \end{vmatrix}$$

At $T=0$, the N-body system is in the ground state
$\downarrow$
lowest energy many-body state.

since $E = \left(k_1 + k_2 + \cdots k_N + \frac{N}{2}\right)\hbar\omega$, $\quad k_1 = 0, 1, 2, \ldots$ $\quad k_2 = 0, 1, 2, \ldots$ $\quad k_3 = 0, 1, 2, \ldots$

Clearly the ground state corresponds to choosing

$k_1 = 0$
$k_2 = 1$
$k_3 = 2$
$\vdots$
$k_N = N-1$



$\rightarrow$ filling the first N levels of the single particle spectrum:

Hence, E_0 = ground state energy = $\left(0+1+2+\cdots(N-1)+\frac{N}{2}\right)\hbar\omega$

$= \left(\frac{N}{2}(N-1)+\frac{N}{2}\right)\hbar\omega = \frac{N^2}{2}\hbar\omega.$

Fermi level $\rightarrow$ the energy of the ~~last~~ top filled level

$\hookrightarrow$ $\boxed{\mu = \left[(N-1)+\frac{1}{2}\right]\hbar\omega = \left(N-\frac{1}{2}\right)\hbar\omega.}$

Finally, the many body ground state wavefunction with energy E_0:

$$\Psi_0(x_1, \ldots x_N) = \frac{1}{\sqrt{N!}} \det\left[\phi_{k_i}(x_j)\right] \qquad k_i = 0, 1, 2, \ldots N-1, \quad j = 1, 2, \ldots N$$

$$= \frac{1}{\sqrt{N!}} \det \begin{vmatrix} \phi_0(x_1) & \phi_0(x_2) & \cdots & \phi_0(x_N) \\ \phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_N) \\ \vdots & & & \\ \phi_{N-1}(x_1) & \phi_{N-1}(x_2) & \cdots & \phi_{N-1}(x_N) \end{vmatrix}$$

For convenience of notations, let us denote.

$$\phi_0(x) \to \phi_1(x), \to \ldots \quad \phi_{N-1}(x) \to \phi_N(x)$$

$\sigma_i =$

$$\Psi_0(x_1, \ldots x_N) = \frac{1}{\sqrt{N!}} \det\left[\phi_i(x_j)\right]_{1\le i,j \le N}$$

$$= \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_1(x_1) & \cdots & \phi_N(x_N) \\ \vdots & & \\ \phi_N(x_1) & \cdots & \phi_N(x_N) \end{vmatrix}$$

The prefactor $\frac{1}{\sqrt{N!}}$ is needed to normalize the many-body wave function.

$$\int \left|\Psi_0(x_1, \ldots x_N)\right|^2 dx_1 \ldots dx_N = 1 \quad \text{(to be proven later)}.$$

$$\Psi_0(x_1, \ldots x_N) \propto \frac{1}{\sqrt{N!}} \det\left[\phi_{k_i}(x_j)\right] \qquad k_i = 0, 1, 2, \ldots N-1$$

$$\phi_{k_i}(x_j) \propto e^{-\frac{\alpha^2}{2} x_j^2} H_{k_i}(\alpha x_j)$$

Can we evaluate this determinant explicitly?

Ex: $N=2$.

$$\Psi_0(x_1, x_2) \propto \det\begin{bmatrix} e^{-\frac{\alpha^2}{2}x_1^2} H_0(x_1) & e^{-\frac{\alpha^2}{2}x_2^2} H_0(x_2) \\ e^{-\frac{\alpha^2}{2}x_1^2} H_1(x_1) & e^{-\frac{\alpha^2}{2}x_2^2} H_1(x_2) \end{bmatrix}$$

$$\propto e^{-\frac{\alpha^2}{2}(x_1^2 + x_2^2)} \det\begin{vmatrix} H_0(x_1) & H_0(x_2) \\ H_1(x_1) & H_1(x_2) \end{vmatrix}$$

$$\psi_0(x_1,x_2) \propto e^{-\frac{\alpha^2}{2}(x_1^2+x_2^2)} \det\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}$$

$$= e^{-\frac{\alpha^2}{2}(x_1^2+x_2^2)} (x_2-x_1)$$

$$\psi_0(x_1,x_2,x_3) \propto \det\begin{bmatrix} e^{-\frac{\alpha^2}{2}x_1^2} H_0(x_1) & e^{-\frac{\alpha^2}{2}x_2^2} H_0(x_2) & e^{-\frac{\alpha^2}{2}x_3^2} H_0(x_3) \\ e^{-\frac{\alpha^2}{2}x_1^2} H_1(x_1) & e^{-\frac{\alpha^2}{2}x_2^2} H_1(x_2) & e^{-\frac{\alpha^2}{2}x_3^2} H_1(x_3) \\ e^{-\frac{\alpha^2}{2}x_1^2} H_2(x_1) & e^{-\frac{\alpha^2}{2}x_2^2} H_2(x_2) & e^{-\frac{\alpha^2}{2}x_3^2} H_2(x_3) \end{bmatrix}$$

$$= e^{-\frac{\alpha^2}{2}(x_1^2+x_2^2+x_3^2)} \det\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ 4x_1^2-2 & 4x_2^2-2 & 4x_3^2-2 \end{bmatrix}$$

$$\downarrow$$

$$= \det\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ 4x_1^2 & 4x_2^2 & 4x_3^2 \end{bmatrix}$$

$$= 4 \det\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix}$$

$$= 4 (x_2-x_1)(x_3-x_1)(x_3-x_2).$$

$$\psi_0(x_1,x_2,x_3) \propto e^{-\frac{\alpha^2}{2}(x_1^2+x_2^2+x_3^2)} (x_2-x_1)(x_3-x_1)(x_3-x_2)$$

Similarly.

$$\psi_0(x_1,\dots x_N) \propto e^{-\frac{\alpha^2}{2}(x_1^2+x_2^2+\dots x_N^2)} \det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_N \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_N^2 \\ \vdots & & & & \\ x_1^{N-1} & x_2^{N-1} & \cdots & \cdots & x_N^{N-1} \end{bmatrix}$$

$\Delta \to$ Vandermonde

$$\prod_{i>j}(x_i-x_j)$$

$$|\Psi_0(x_1,\cdots x_N)|^2 = \frac{1}{C_N} e^{-\alpha^2(x_1^2+\cdots x_N^2)} \prod_{i<j}(x_i-x_j)^2$$

$\downarrow$

Same as the joint pdf of eigenvalues $(x_1 \ldots x_N)$ of GUE $(\beta=2)$.

Ground state squared wavefunction of N free Fermions $(d=1)$

$\Longrightarrow$ that characterizes the ground state quantum fluctuations

$$= P(x_1,\ldots x_N) = \frac{1}{C_N} e^{-\alpha^2(x_1^2+\cdots x_N^2)} \prod_{i<j}(x_i-x_j)^2$$

$\hookrightarrow$ Joint pdf of N eigenvalues of a GUE random matrix.

$$\Psi_0(x_1,\ldots x_N) = \frac{1}{\sqrt{N!}} \det\left[\phi_{k_i}(x_j)\right] \qquad k_i = 0,1,\cdots N-1, \quad j=1,2,\cdots N.$$

r convenience of notations we will shift the labels. $\phi_0(x) \to \phi_1(x)$

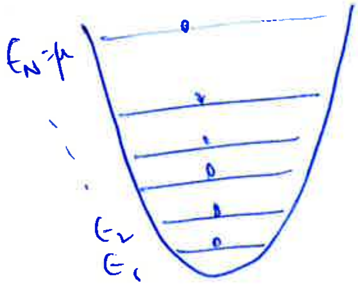
$\vdots$

$\phi_{N-1}(x) \to \phi_N(x)$.

en,
$$\Psi_0(x_1,\ldots x_N) = \frac{1}{\sqrt{N!}} \det\left[\phi_i(x_j)\right]_{1\le i,j\le N}$$

$$= \frac{1}{\sqrt{N!}} \det\begin{bmatrix} \phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_N) \\ \vdots & & & \\ \phi_N(x_1) & \phi_N(x_2) & \cdots & \phi_N(x_N) \end{bmatrix}$$

neralization:



$V(x) \to$ arbitrary confining potential

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_k}{dx^2} + V(x)\,\phi_k(x) = \epsilon_k\,\phi_k(x).$$

$k \to$ label of the single particle state.

nd state many body wavefunction

$$\Psi_0(x_1, \ldots x_N) = \frac{1}{\sqrt{N!}} \det\left[\phi_i(x_j)\right]_{1 \le i,j \le N}$$

$$E_0 = (\epsilon_1 + \cdots \epsilon_N).$$

Fermi energy, $\boxed{\mu = \epsilon_N}$.

nce,

$$\ldots x_N) = |\Psi_0(x_1, \ldots x_N)|^2 = \frac{1}{N!} \det^2\left[\phi_i(x_j)\right]_{1 \le i,j \le N}$$

$\hookrightarrow$ can be interpreted as a prob. distn of N points on a line.

or the special case, $V(x) = \frac{1}{2} m \omega^2 x^2$;

$$P(x_1, x_2, \ldots x_N) \propto e^{-\alpha^2 \sum x_i^2} \prod_{i<j} (x_i - x_j)^2$$

$\hookrightarrow$ GUE random matrix.

but one can also study it for arbitrary potn.

ain lesson: Ground state of N free fermions ($d=1$),

$$\ (x_1, \dots x_N) = \left|\Psi_0(x_1, \dots x_N)\right|^2 = \frac{1}{N!} \det^2\left[\phi_i(x_j)\right]_{1\le i,j \le N} \quad \text{—— (1)}$$

$\uparrow$

rob. distⁿ of N points
n a line
GUE eigenvalues)

$\hookrightarrow$ general $\to$ holds for any potential $V(x)$ $\mu \to$ Fermi level.

We want to now calculate the n-point marginals, starting from (1)

Only using the fact that $\phi_i(x_j)$ are orthonormal

i.e. $\int \phi_i^*(x)\,\phi_j(x)\,dx = \delta_{i,j}$

(without actually using their explicit form), how far can be proceed?

To proceed, it is useful to rewrite (1) in a slightly different way.

We use the property $\det(AB) = \det A \det B$.

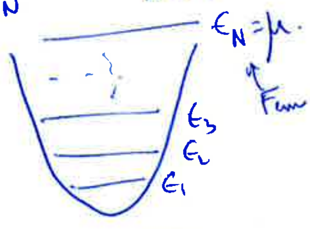
The product matrix $(AB)_{ij} = \sum_k A_{ik} B_{kj}$

putting $A=B$,
$$\det^2 A = \det\left[A^2\right] = \det\left[(A^2)_{ij}\right]$$
$$= \det\left[\sum_k A_{ik} A_{kj}\right]$$
$$= \det\left[\sum_k A^{+}_{ki} A_{kj}\right]$$

where

$$\ (x_1 \dots x_N) = \frac{1}{N!}\det^2\left[\phi_i(x_j)\right] = \frac{1}{N!}\det\left[\sum_{k=1}^{N} \phi_k^*(x_i)\,\phi_k(x_j)\right]_{1\le i,j\le N}$$

$k \to$ label of the single particle state.



Let us introduce the notation

$$K(x,y) = \sum_{k=1}^{N} \phi_k^*(x)\,\phi_k(y)$$

or more generally

$$K_\mu(x,y) = \sum_k \Theta(\mu - \epsilon_k)\,\phi_k^*(x)\,\phi_k(y), \quad \mu \to \text{Fermi level.}$$

$\to$ Kernel $\to$ name

Then,

$$P(x_1, \dots x_N) = |\Psi_0(x_1, \dots x_N)|^2 = \frac{1}{N!} \det\left[K_\mu(x_i, x_j)\right]_{1 \le i,j \le N}$$

where $\boxed{K_\mu(x,y) = \sum_k \Theta(\mu - \epsilon_k)\, \phi_k^*(x)\phi_k(y)}$, $\boxed{\mu = \epsilon_N}$

↓ Fermi level.

Main point → Joint distⁿ can be written as a determinant.

Transitive property of the kernel:

$$\int K_\mu(x,z) K_\mu(z,y)\, dz = \int \sum_k \Theta(\mu - \epsilon_k)\, \phi_k^*(x)\phi_k(z) \sum_{k'} \Theta(\mu - \epsilon_{k'})\, \phi_{k'}^*(z)\phi_{k'}(y)\, dz.$$

$$= \sum_{k,k'} \Theta(\mu - \epsilon_k)\, \Theta(\mu - \epsilon_{k'})\, \phi_k^*(x)\, \phi_{k'}(y) \underbrace{\int \phi_{k'}^*(z)\, \phi_k(z)\, dz}_{\substack{\delta_{k,k'} \\ \text{(orthonormality)}}}.$$

$$= \sum_k \Theta(\mu - \epsilon_k)\, \phi_k^*(x)\, \phi_k(y) = K_\mu(x,y).$$

$$\boxed{\int K_\mu(x,z) K_\mu(z,y)\, dz = K_\mu(x,y)}$$

↳ Kernel is transitive.

Note that $K_\mu(x,x) = \sum_k \Theta(\mu - \epsilon_k) |\phi_k(x)|^2$

$$\int K_\mu(x,x)\, dx = \sum_k \Theta(\mu - \epsilon_k) \cdot \int |\phi_k(x)|^2 dx = \sum_k \Theta(\mu - \epsilon_k) = N$$

ecall n-point correlation function

$$R_n(x_1,\dots x_n) = \frac{N!}{(N-n)!}\int P(x_1,\dots x_n, x_{n+1},\dots, x_N)\, dx_{n+1}\dots dx_N$$

$$= \frac{N!}{(N-n)!}\int \det\left[K_N(x_i,x_j)\right]_{1\le i,j\le N}\, dx_{n+1}\dots dx_N.$$

Can we make progress in explicitly calculating this integral?

Ve make use of the following theorem (Mehta's book):

heorem: Consider any function $f(x,y)$ with two arguments and suppose it satisfies the transitivity property

$$\int f(x,z)\, f(z,y)\, dz = f(x,y).$$

Construct an $(n\times n)$ matrix

$$J_n(x_1,\dots x_n) = \left| f(x_i,x_j)\right|_{1\le i,j\le n}$$

Then

$$\int \det\left[J_n(x_1,\dots x_n)\right] dx_n = \left[q-(n-1)\right]\det\left[J_{n-1}(x_1,\dots x_{n-1})\right]$$

where

$$q = \int f(x,x)\, dx.$$

Proof for $n=2$:

$$J_2(x_1,x_2) = \begin{vmatrix} f(x_1,x_1) & f(x_1,x_2) \\ f(x_2,x_1) & f(x_2,x_2)\end{vmatrix}.$$

$$\det\left[J_2(x_1,x_2)\right] = \det\begin{bmatrix} f(x_1,x_1) & f(x_1,x_2) \\ f(x_1,x_1) & f(x_2,x_2)\end{bmatrix} = f(x_1,x_1)f(x_2,x_2) - f(x_1,x_2)f(x_2,x_1).$$

ence

$$\int \det\left[J_2(x_1,x_2)\right] dx_2 = \int\left[f(x_1,x_1)\, f(x_2,x_2) - \underbrace{f(x_1,x_2)\, f(x_2,x_1)}_{\text{transitivity}}\right] dx_2$$

$$= f(x_1,x_1)\int f(x_2,x_2)\, dx_2 - f(x_1,x_1)$$

$$= (q-1)\, f(x_1,x_1) = (q-1)\det\left[J_1(x_1)\right].$$

roof for general n → see Mehta's book.

use this theorem in our case we choose. $f(x,y) = K_\mu(x,y)$.

$K_\mu(x,y)$ satisfies the transitivity property,

$$\int K_\mu(x,z)\, K_\mu(z,y)\, dz = K_\mu(x,y).$$

Also, $$q = \int K_\mu(x,x)\,dx = N.$$

Hence.

$$J_n(x_1, \dots x_n) = \left| K_\mu(x_i, x_j) \right|_{1 \le i,j \le n}$$

$$\Rightarrow \boxed{\int \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le n} dx_n = (N-n+1) \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le n-1}}$$

choosing $n = N$.

$$\int \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le N} dx_N = \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le N-1}$$

continue recursively,

$$\int \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le N} dx_N\, dx_{N-1} = \int \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le N-1} dx_{N-1}.$$

$$= 2 \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le N-2}$$

$$\boxed{\int \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le N} dx_N\, dx_{N-1} \cdots dx_{n+1} = (N-n)!\, \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le n}}$$

Thus, n-point correlation function.

$$R_n(x_1, \dots x_n) = \frac{N!}{(N-n)!} \int P(x_1, \dots x_n, x_{n+1} \dots x_N)\, dx_{n+1} \dots dx_N.$$

$$= \frac{1}{(N-n)!} \int \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le N} dx_{n+1} \dots dx_N = \det\left[K_\mu(x_i,x_j) \right]_{1 \le i,j \le n}$$

Hence, all n-point correlation functions can be expressed as a determinant.

i) Joint distⁿ can be written as a determinant of a kernel $K_\mu(x,y)$

ii) Kernel $K_\mu(x,y)$ satisfies transitivity property

$\Rightarrow$ all n-point correlation functions can be written as a determinant.

$\Longrightarrow$ "determinantal point process".

Average density $\rightarrow$ 1-point function.

$$\rho_N(x) = \left\langle \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i)\right\rangle = \int P(x, x_2, \dots x_N)\, dx_2 \dots dx_N .$$

$$= R_1(x)/N$$

$$= \frac{1}{N}\det\left[K_\mu(x_i,x_j)\right]_{1\le i,j\le 1} = \frac{1}{N}\det\left[K_\mu(x,x)\right]$$

$$= \frac{1}{N}K_\mu(x,x).$$

$$\Rightarrow \boxed{\rho_N(x) = \frac{1}{N}K_\mu(x,x) = \frac{1}{N}\sum_k \theta(\mu-\epsilon_k)\,|\phi_k(x)|^2}$$

Check the normalization $\int \rho_N(x)\,dx = \frac{1}{N}\int K_\mu(x,x)\,dx = \frac{1}{N}\cdot N. = 1$

$\rightarrow$ Physical interpretation: all single particle states up to the Fermi level contribute equally with prob. $\frac{1}{N}$.

Two-point function.

$$R_2(x_1,x_2) = N(N-1)\int P(x_1,x_2,x_3,\dots x_N)\,dx_3\dots dx_N.$$

$$= \det\left[K_\mu(x_i,x_j)\right]_{1\le i,j\le 2}$$

$$= \det\begin{bmatrix} K_\mu(x_1,x_1) & K_\mu(x_1,x_2) \\ K_\mu(x_2,x_1) & K_\mu(x_2,x_2)\end{bmatrix}$$

$$\boxed{R_2(x_1,x_2) = K_\mu(x_1,x_1)K_\mu(x_2,x_2) - K_\mu^2(x_1,x_2)}$$

$\hookrightarrow$ to be used later.

$$R_N(x_1,\dots x_N) = N!\,P(x_1,\dots x_N) = N!\,\frac{1}{N!}\det\left[K_\mu(x_i,x_j)\right]_{1\le i,j\le N} = \det\left[K_\mu(x_i,x_j)\right]_{1\le i,j\le N}$$

Main lesson: For determinantal processes, the central quantity is the Kernel.

$$\boxed{K_\mu(x,y) = \sum_k \Theta(\mu-\epsilon_k)\,\phi_k^*(x)\,\phi_k(y)}$$

If we can compute the Kernel, in principle, we can compute any n-point correlation function.

$$R_n(x_1,\dots x_n) = \det\left[K_\mu(x_i,x_j)\right]_{1\le i,j\le n}$$

by calculating the $(n\times n)$ determinant whose entries are the Kernels themselves.

Note,

$$P(x_1,\dots x_N) = \frac{1}{N!}\det\left[K_\mu(x_i,x_j)\right]_{1\le i,j\le N}$$

Next step: Computation of the Kernel.

compute the Kernel.

$$K_\mu(x,y) = \sum_k \theta(\mu - \epsilon_k)\, \phi_k^*(x)\phi_k(y)$$

$$\frac{\partial K_\mu}{\partial \mu} = \sum_k \delta(\mu - \epsilon_k)\, \phi_k^*(x)\, \phi_k(y)$$

$$\int \frac{\partial K_\mu}{\partial \mu}\, e^{-\frac{\mu t}{\hbar}}\, d\mu = \sum_k \int \delta(\mu - \epsilon_k)\, e^{-\frac{\mu t}{\hbar}}\, d\mu \;\; \phi_k^*(x)\phi_k(y)$$

$$\underbrace{e^{-\mu t/\hbar} K_\mu \Big|_{\mu=0}^{\infty}}_{= \text{'0'}} + \frac{t}{\hbar}\int K_\mu e^{-\frac{\mu t}{\hbar}} d\mu = \sum_k \phi_k^*(x)\, \phi_k(y)\, e^{-\frac{t\epsilon_k}{\hbar}} = G(x,y;t)$$

$$\Rightarrow \int K_\mu\, e^{-\frac{\mu t}{\hbar}}\, \frac{d\mu}{\hbar} = \frac{1}{t}\, G(x,y,t).$$

Bromwitch inversion formula $\Rightarrow$

$$\boxed{K_\mu(x,y) = \int_\Gamma \frac{dt}{2\pi i}\, \frac{e^{\frac{\mu t}{\hbar}}}{t}\, G(x,y;t)} \quad \text{——} \; (M)$$

where

$$G(x,y;t) = \sum_k \phi_k^*(x)\, \phi_k(y)\, e^{-\frac{\epsilon_k}{\hbar} t} = \langle x | e^{-\hat{H}\frac{t}{\hbar}} | y \rangle$$

↳ single particle propagator in imaginary time.

Quantum propagator $\langle x | e^{i\hat{H}t/\hbar} | y \rangle \xrightarrow[t \to it]{} \langle x | e^{-\frac{\hat{H}t}{\hbar}} | y \rangle = G(x,y;t).$

If we can compute the propagator, we can compute the Kernel $K_\mu(x,y)$ from (M).

$$\langle x | e^{-i\hat{H}t/\hbar} | y \rangle$$

= Quantum propagator

$$= \frac{\alpha}{\sqrt{2\pi i \sin \omega t}} \exp\left[\frac{i\alpha^2}{2 \sin \omega t} \left\{ (x^2+y^2)\cos \omega t - 2xy \right\} \right]$$

~~for arbitrary~~
To compute the propagator:

$$\frac{\partial G}{\partial t} = -\frac{1}{\hbar}\sum_k \epsilon_k \, \phi_k^*(x)\,\phi_k(y)\, e^{-\frac{\epsilon_k}{\hbar}t}.$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi_k^*(x)}{\partial x^2} + V(x)\,\phi_k^*(x) = \epsilon_k\, \phi_k^*(x).$$

$$= -\frac{1}{\hbar}\sum_k \left[-\frac{\hbar^2}{2m}\frac{\partial^2\phi_k^*(x)}{\partial x^2} + V(x)\,\phi_k^*(x)\right]\phi_k(y)\, e^{-\frac{\epsilon_k}{\hbar}t}.$$

$$\boxed{\frac{\partial G}{\partial t} = \frac{\hbar}{2m}\frac{\partial^2 G}{\partial x^2} - \frac{1}{\hbar}V(x)\,G}$$

↳ Feynman-Kac equation

initial condition,
$G(x,y;t=0)$
$= \sum_k \phi_k^*(x)\phi_k(y)$
$= \delta(x-y).$

B.C.
$G(x\to\pm\infty,y,t) = 0$

For arbitrary $V(x)$, 'G' is hard to solve explicitly.

However, for harmonic oscillator, $V(x) = \frac{1}{2}m\omega^2x^2.$
(GUE)

$$\boxed{\frac{\partial G}{\partial t} = \frac{\hbar}{2m}\frac{\partial^2 G}{\partial x^2} - \frac{1}{\hbar}\cdot\frac{1}{2}m\omega^2x^2\,G}$$

; $G(x,y;t=0) = \delta(x-y)$
and b.c.
$G(x\to\pm\infty,y;t) = 0.$

One can find an exact solution, with $\alpha = \sqrt{\frac{m\omega}{\hbar}}$

$$\boxed{G(x,y;t) = \frac{\alpha}{\sqrt{2\pi\sinh(\omega t)}}\exp\left[-\frac{\alpha^2}{2\sinh(\omega t)}\left\{(x^2+y^2)\cosh(\omega t) - 2xy\right\}\right]}$$

Prove it: (i) either by substituting. $G(x,y,t) = e^{-A(y,t)x^2 - B(y,t)x - C(y,t)}.$

(ii) or calculate the quantum propagator $\langle x|e^{i\hat{H}t/\hbar}|y\rangle$ by discretizing ~~Feynman~~ path integral and $t\to it$ } Feynman-Hibbs.

From now on, let us focus on $V(x) = \frac{1}{2} m\omega^2 x^2$. $\Rightarrow$ GUE.

then

$$K_\mu(x,y) = \int_\Gamma \frac{dt}{2\pi i} \frac{e^{\mu t/\hbar}}{t} \cdot \frac{\alpha}{\sqrt{2\pi \sinh(\omega t)}} \exp\left[-\frac{\alpha^2}{2\sinh(\omega t)}\left\{(x^2+y^2)\cosh\omega t - 2xy\right\}\right].$$

$$= \int_\Gamma \frac{dt}{2\pi i} \frac{e^{\mu t/\hbar}}{t} \cdot \frac{\alpha}{\sqrt{2\pi \sinh(\omega t)}} \exp\left[-\frac{\alpha^2}{2\sinh(\omega t)}\left\{(x-y)^2 + (x^2+y^2)(\cosh\omega t - 1)\right\}\right]$$

where recall that $\boxed{\mu = (N - \tfrac{1}{2})\hbar\omega}$ for harmonic oscillator.

<u>Goal:</u> to obtain the scaling behavior of the Kernel $K_\mu(x,y)$ for large N.

Note when $x=y$

- $\rho_N(x) = \frac{1}{N} K_\mu(x,x) \rightarrow$ average density.

We will analyze first, $x=y$, $K_\mu(x,x)$.

and then $x \neq y$. $\rightarrow$ Kernel.

~~Substituting for~~

rage Density: [asymptotics as $N \to \infty$]

$\left[\rho_N(x) = \frac{1}{N} K_N(x,x)\right]$

ostituting $x=y$,

$$K_\mu(x,x) = \int_\Gamma \frac{dt}{2\pi i} \frac{e^{\frac{\mu t}{\hbar}}}{t} \frac{\alpha}{\sqrt{2\pi \sinh \omega t}} \exp\left[-\frac{\alpha^2}{\sinh(\omega t)}\left\{\cosh(\omega t)-1\right\} x^2\right]$$

$$\boxed{K_\mu(x,x) = \int_\Gamma \frac{dt}{2\pi i} \frac{e^{\mu t/\hbar}}{t} \frac{\alpha}{\sqrt{2\pi \sinh(\omega t)}} \exp\left[-\tanh\left(\frac{\omega t}{2}\right) \alpha^2 x^2\right].}$$

where $\alpha = \sqrt{\frac{m\omega}{\hbar}}$, $\mu = (N-\frac{1}{2})\hbar\omega$.

When $N \to \infty$, $\mu \to \infty$, $\Rightarrow$ small 't' behavior of the integrand dominates.

As $t \to 0$,

$$\tanh\left(\frac{\omega t}{2}\right) \to \frac{\omega t}{2} - \frac{1}{3}\left(\frac{\omega t}{2}\right)^3 + O(t^4)$$

$$\frac{1}{\sqrt{\sinh \omega t}} \to \frac{1}{\sqrt{\omega t}}\left[1 - \frac{\omega^2 t^2}{12} + O(t^3)\right].$$

Keeping only leading order terms.

$$K_\mu(x,x) \simeq \frac{\alpha}{\sqrt{2\pi\omega}} \int_\Gamma \frac{dt}{2\pi i} \frac{e^{\frac{\mu t}{\hbar} - \frac{\omega \alpha^2}{2} x^2 t}}{t^{3/2}}$$

$$\simeq \frac{\alpha}{\sqrt{2\pi\omega}} \int_\Gamma \frac{dt}{2\pi i} \frac{e^{\frac{t}{\hbar}\left(\mu - \frac{1}{2} m\omega^2 x^2\right)}}{t^{3/2}}.$$

(24)

Recall: $$\int_0^\infty \sqrt{y}\, e^{-ty}\, dy = \frac{1}{t^{3/2}}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2\, t^{3/2}}\sqrt{\pi}.$$

$$\Rightarrow \frac{2}{\sqrt{\pi}}\sqrt{y} = \int_\Gamma \frac{dt}{2\pi i}\frac{e^{yt}}{t^{3/2}}.$$

$$\Rightarrow \boxed{\int_\Gamma \frac{dt}{2\pi i}\frac{e^{yt}}{t^{3/2}} = \frac{2}{\sqrt{\pi}}\sqrt{y}.}$$

Hence,

$$K_\mu(x,x) \cong \frac{\alpha}{\sqrt{2\pi\omega}}\frac{2}{\sqrt{\pi}}\sqrt{\frac{\mu-\frac{1}{2}m\omega^2x^2}{\hbar}}$$

$$\simeq \frac{1}{\pi}\frac{\sqrt{2}}{\sqrt{\omega}}\frac{1}{\sqrt{\hbar}}\sqrt{\frac{m\omega}{\hbar}}\sqrt{\mu-\frac{1}{2}m\omega^2x^2}.$$

$$= \sqrt{\frac{2m}{\pi^2\hbar^2}}\sqrt{\mu-\frac{1}{2}m\omega^2x^2}$$

Recall, $$\int \rho_N(x)\,dx = 1 \Rightarrow \int K_\mu(x,x)\,dx = \frac{1}{N}.$$

$$\sqrt{\frac{2m}{\pi^2\hbar^2}}\int_{-\sqrt{\frac{2\mu}{m\omega^2}}}^{\sqrt{\frac{2\mu}{m\omega^2}}}\sqrt{\mu-\frac{1}{2}m\omega^2x^2}\,dx = \frac{1}{N}.$$

$$\Rightarrow \sqrt{\frac{2m}{\pi^2\hbar^2}\cdot\frac{1}{2}m\omega^2}\int_{-\sqrt{\frac{2\mu}{m\omega^2}}}^{\sqrt{\frac{2\mu}{m\omega^2}}}\sqrt{\frac{2\mu}{m\omega^2}-x^2}\,dx = \frac{1}{N}.$$

$$\frac{m\omega}{\pi\hbar}\cdot\frac{2\mu}{m\omega^2}\cdot\int_{-1}^{1}\sqrt{1-y^2}\,dy = \frac{1}{N}.$$

$$\frac{2\mu}{\pi\hbar\omega}\cdot\frac{\pi}{2} = \frac{1}{N} \Rightarrow \boxed{\mu \simeq N\hbar\omega} \to \text{Completely Consistent.}$$

i.e.,

$$\frac{1}{N} K_\mu(x,x) \xrightarrow[N\to\infty]{} \frac{1}{N}\sqrt{\frac{2m}{\pi^2\hbar^2}}\sqrt{\hbar\omega N - \frac{1}{2}m\omega^2 x^2}$$

$$= \frac{1}{N}\sqrt{\frac{2m}{\pi^2\hbar^2}\cdot\frac{1}{2}m\omega^2}\sqrt{\frac{2\hbar}{m\omega}N - x^2}$$

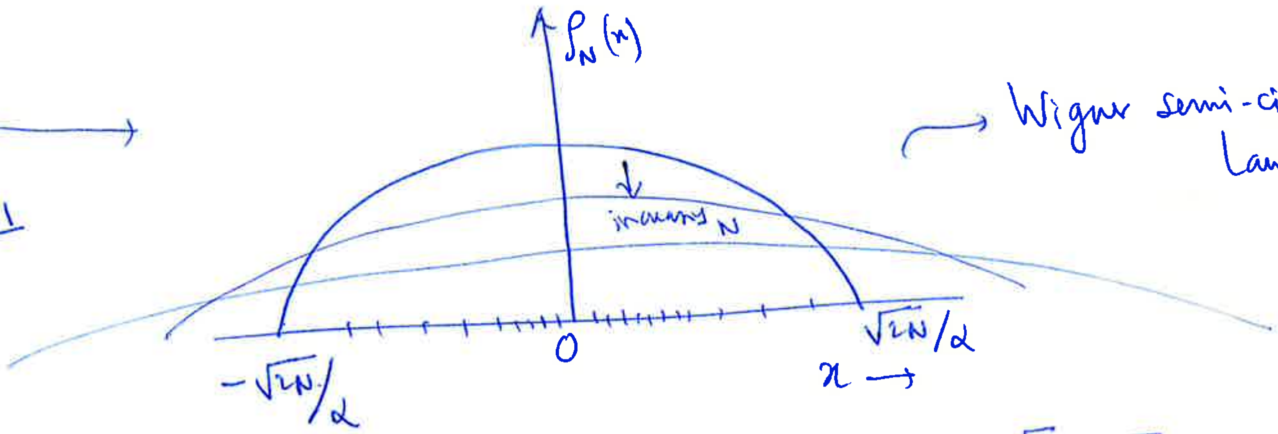
$$= \frac{1}{N\pi}\alpha^2\sqrt{\frac{2N}{\alpha^2} - x^2} \qquad \alpha = \sqrt{\frac{m\omega}{\hbar}}$$

$$= \frac{\alpha}{N\pi}\sqrt{2N-(\alpha x)^2} = \frac{\alpha}{\sqrt{N}} f\left[\frac{\alpha x}{\sqrt{N}}\right]$$

where $f(z) = \frac{1}{\pi}\sqrt{2-z^2}$.

Standard GUE $\Rightarrow \alpha = 1$.

$\alpha = 1$



Wigner semi-circle law.

Density near 0: $\rho_N(0) \simeq \frac{\sqrt{2}}{\pi}\frac{\alpha}{\sqrt{N}}$, $n_N(0) \simeq \frac{\sqrt{2}}{\pi}\alpha\sqrt{N}$.

~~$n_N(x) \sim N\rho_N(x)$~~

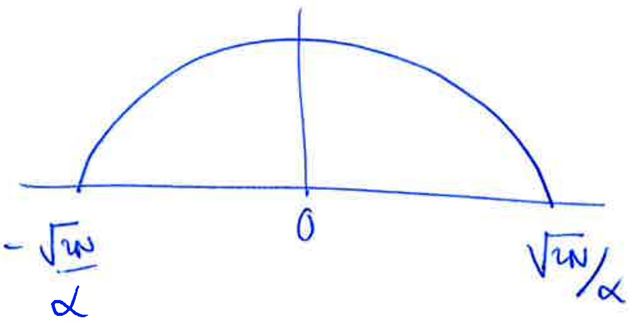
typical spacing in the bulk: $\frac{1}{N}\frac{\sqrt{2N}}{\alpha} \sim O\left(\frac{1}{\sqrt{N}}\right)$ as $N\to\infty$.

$d\, n_N(0) = 1 \Rightarrow d = \frac{1}{N\rho_N(0)} \sim \frac{\pi}{\alpha\sqrt{2N}} \to 0$ as $N\to\infty$.

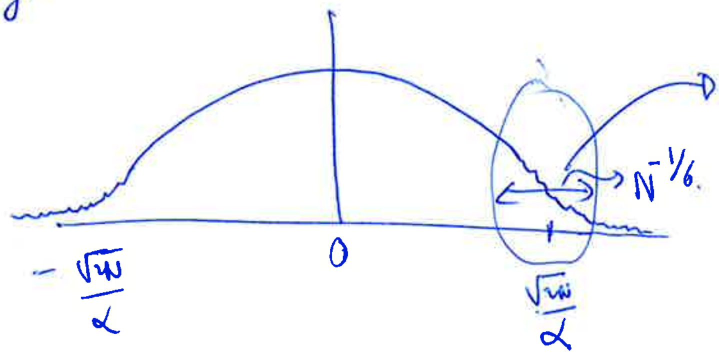
Bulk global density $\longrightarrow$ Wigner semi-circle.

$N \to \infty$,

$$\rho_N(x) = \frac{\alpha}{N\pi}\sqrt{2N - (\alpha x)^2} \quad \to \text{Semi-circle.}$$



finite but large N



Zoom out.
how does the density behave there?

What is the typical separation of bet^n particles near the edge

↳ They are sparse near the edge
dense " the origin.

fschitz argument:

$$\int_W^{\sqrt{2N}/\alpha} \rho_N(x)\,dx \simeq \frac{1}{N}.$$

$$\frac{\alpha}{N\pi}\int_W^{\sqrt{2N}/\alpha}\sqrt{2N-(\alpha x)^2}\,dx \sim \frac{1}{N}.$$

$$\frac{1}{\pi N}\int_{W\alpha}^{\sqrt{2N}}\sqrt{2N-y^2}\,dy \sim \frac{1}{N}.$$

$$\Rightarrow \int_{W\alpha}^{\sqrt{2N}}\sqrt{2N-y^2}\,dy \sim \pi$$

$$\sqrt{2N-y^2} = \left[(\sqrt{2N}-y)(\sqrt{2N}+y)\right]^{1/2}.$$

$$y \to \sqrt{2N} \quad (2\sqrt{2N})^{1/2}\cdot\left[\sqrt{2N}-y\right]^{1/2}.$$

$$\Rightarrow N^{1/4}\left(\sqrt{2N} - W\alpha\right)^{3/2} \sim O(1).$$

$$\sqrt{2N} - W\alpha \sim \left(N^{-1/4}\right)^{2/3} \sim N^{-1/6}$$

$$\Rightarrow W \sim \frac{\sqrt{2N}}{\alpha} + O\left(N^{-1/6}\right).$$

analyse the edge density on this scale.

expect.

We start from,

$$K_\mu(x,x) = \int_\Gamma \frac{dt}{2\pi i}\,\frac{e^{\frac{\mu t}{\hbar}}}{t}\,\underbrace{\frac{\alpha}{\sqrt{2\pi\sinh(\omega t)}}\exp\left[-\tanh\left(\frac{\omega t}{2}\right)\alpha^2x^2\right]}.$$

we set 'x' near the edge,

$\mu = N\hbar\omega.$ $\quad x = \frac{\sqrt{2N}}{\alpha} + \frac{z}{\alpha\sqrt{2}}N^{-\phi}$ $\quad$ (we will see $\phi = \frac{1}{6}$ will emerge).

$z \sim O(1) \;\rightarrow \phi$ to be selected., $\quad N\to\infty$

$$\tanh\left(\frac{\omega t}{2}\right) \longrightarrow \frac{\omega t}{2} - \frac{1}{3}\left(\frac{\omega t}{2}\right)^3 + O(t^4)$$

$$\frac{1}{\sqrt{\sinh\omega t}} \longrightarrow \frac{1}{\sqrt{\omega t}}\left[1-\frac{\omega^2t^2}{12}\right].$$

Expanding the propagator for small 't'.

$$K_\mu(x,x) \simeq \frac{\alpha}{\sqrt{2\pi\omega}}\int_\Gamma\frac{dt}{2\pi i}\,\frac{e^{\frac{\mu t}{\hbar}}}{t^{3/2}}\exp\left[-\overset{\hbar\omega N}{\frac{\mu t}{\hbar}} - \sqrt{\frac{\omega\mu}{m}}\,\alpha\, z\, N^{-\varphi}\,t - \frac{\omega^2t^2}{12} + \frac{\mu\omega^2t^3}{12\hbar}\right]$$

$\mu\sim N$

$$\simeq \frac{\alpha}{\sqrt{2\pi\omega}}\int_\Gamma\frac{dt}{2\pi i}\,\frac{1}{t^{3/2}}\exp\left[-\sqrt{\frac{\mu\alpha^2}{m}}\,z\,N^{-\varphi}\,t - \frac{\omega^2t^2}{12} + \frac{\mu\omega^2t^3}{12\hbar}\right].$$

$\mu\sim N.$

$T_1 \sim N^{\frac{1}{2}-\varphi}\,t \sim \tau \to O(1)$ $\quad\Rightarrow t\sim N^{\varphi-\frac{1}{2}}\tau$

$T_2 \sim t^2 \sim N^{2\varphi-1}\tau^2$

$T_3 \sim N t^3 \sim N^{3\varphi-\frac{1}{2}}\tau^3.$

(i) either $\varphi = \frac{1}{2}$, $T_1\sim\tau$, $T_2\sim\tau^2$

then $T_3 \sim N\tau^3 \longrightarrow$ diverges

$\Longrightarrow$ no sense

(ii) or $\varphi = \frac{1}{6}$.

$T_1\sim\tau$

$T_3\sim\tau^3$

$T_2\sim N^{-2/3}\tau^2 \to 0$ as $N\to\infty$.

Correct choice.

hoosing $\varphi = \frac{1}{6}$, $t \sim N^{\varphi - \frac{1}{2}} \tau \sim N^{-1/3} \tau$.

cale, $$t = \frac{N^{-1/3}}{\omega} \tau,$$

$$Z = \left(x - \frac{\sqrt{2N}}{\alpha}\right) \alpha \sqrt{2}\, N^{1/6}$$
$$= \frac{x - x_{edge}}{W_N}$$
where $W_N = \frac{1}{\alpha\sqrt{2}} N^{-1/6}$.

$$K_{edge}(\cancel{z}, \cancel{z}) \simeq \frac{\alpha N^{1/6}}{\sqrt{2\pi}} \int_\Gamma \frac{d\tau}{2\pi i} \cdot \frac{1}{\tau^{3/2}} \exp\left[- Z \tau + \frac{\tau^3}{12}\right]$$

$$= \frac{\alpha}{\cancel{\alpha}} \frac{1}{W_N} \cdot \frac{1}{\sqrt{4\pi}} \int_\Gamma \frac{d\tau}{2\pi i} \frac{1}{\tau^{3/2}} \exp\left[-\tau Z + \frac{\tau^3}{12}\right].$$

~~rescaling~~

hence
$$\rho_{edge}(x) = \frac{1}{N} K_{edge}(x, x) = \frac{1}{N} \frac{1}{W_N} F_1\left[\frac{x - x_{edge}}{W_N}\right]$$

where $$x_{edge} = \frac{\sqrt{2N}}{\alpha}.$$
$$W_N = \text{width} = \frac{1}{\alpha\sqrt{2}} N^{-1/6}.$$

and the scaling function

$$F_1(z) = \frac{1}{\sqrt{4\pi}} \int_\Gamma \frac{d\tau}{2\pi i} \frac{1}{\tau^{3/2}} \exp\left[-z\tau + \frac{\tau^3}{12}\right]$$

The scaling function $F_1(z)$ can be simplified further:

$$\frac{1}{\tau^{3/2}} = \frac{1}{\Gamma(\frac{3}{2})}\int_0^\infty e^{-\tau x}\, x^{1/2}\, dx.$$

$$\Rightarrow \quad F_1(z) = \frac{1}{\Gamma(\frac{3}{2})\sqrt{4\pi}}\int_0^\infty dx\, x^{1/2}\int_\Gamma \frac{d\tau}{2\pi i}\, e^{-\tau(x+z)+\frac{\tau^3}{12}}.$$

rescale $\tau \to 2^{2/3}\tau$ and use $Ai(z) = \int_\Gamma \frac{d\tau}{2\pi i}\, e^{-\tau x + \frac{\tau^3}{3}}$.

$$F_1(z) = \frac{2^{2/3}}{\Gamma(\frac{3}{2})\sqrt{4\pi}}\int_0^\infty dx\,\sqrt{x}\; Ai\left(2^{2/3}(x+z)\right)$$

$x \to 2^{2/3}u$

$$F_1(z) = \frac{1}{\Gamma(\frac{3}{2})\, 2^{4/3}\sqrt{\pi}}\int_0^\infty du\; u^{1/2}\, Ai\left(u + 2^{2/3}z\right).$$

$$= \frac{1}{\pi\, 2^{1/3}}\int_0^\infty du\,\sqrt{u}\; Ai\left(u+2^{2/3}z\right).$$

Using the identity $\displaystyle\int_0^\infty dt\, Ai(x+t)\sqrt{t} = \pi\, 2^{1/3}\int_{x/2^{2/3}}^\infty du\, Ai^2(u)$

[See Vallee & Soares, book].

$$\Rightarrow \quad F_1(z) = \int_{z}^\infty Ai^2(t)\,dt = Ai^2(t)\cdot t\Big|_z^\infty - \int_z^\infty 2\, Ai(t)\, Ai'(t)\cdot t\; dt.$$

$$= -z\, Ai^2(z) - 2\int_z^\infty Ai'(t)\cdot Ai''(t)\, dt.$$

$$= -z\, Ai^2(z) - \int_z^\infty \frac{d}{dt}\left[Ai'^2(t)\right]dt.$$

$$= Ai'^2(z) - z\, Ai^2(z).$$

[O. Vallée & M. Soares, "Airy functions and applications to physics" (Imperial, London, 2004)]

we, finally,

$$\rho_{edge}(x) = \frac{1}{N}\ \frac{1}{W_N} F_1\left[\left(\frac{x - x_{edge}}{W_N}\right)\right].$$

where
$$x_{edge} = \frac{\sqrt{2N}}{\alpha}, \qquad \left(\alpha = \sqrt{\frac{m\omega}{\hbar}}\right)$$

$$W_N = \frac{1}{\alpha\sqrt{2}} N^{-1/6}.$$

now the scaling function

$$\boxed{F_1(z) = \frac{1}{\pi 2^{1/3}} \int_0^\infty du\, \sqrt{u}\, Ai(u + 2^{2/3} z) = Ai'^2(z) - z\, Ai^2(z)} \quad \text{— (1)}$$

Bowick & Bre'zin, ~~& Braviras~~ 91, Forrester, '93.

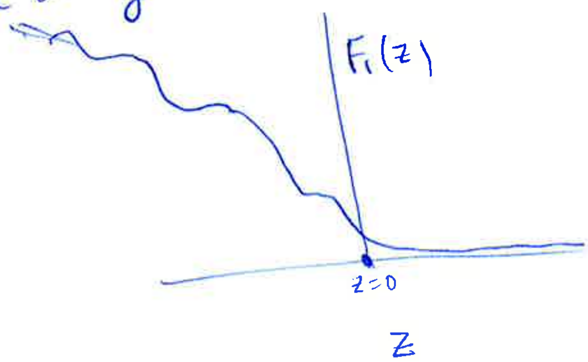
Asymptotics: Recall, $Ai''(z) - z\, Ai(z) = 0$.

$$Ai(z) \begin{cases} \xrightarrow{z\to\infty} \dfrac{1}{2\sqrt{\pi}}\, z^{-1/4}\, e^{-\frac{2}{3} z^{3/2}} \\[2ex] \xrightarrow{z\to-\infty} \dfrac{1}{\sqrt{\pi}}\, \dfrac{1}{|z|^{1/4}}\, \sin\left[\dfrac{\pi}{4} + \dfrac{2}{3}|z|^{3/2}\right] \end{cases}$$

Then one can easily show, from (1)

$$F_1(z) \begin{cases} \xrightarrow{z\to\infty} \dfrac{1}{8\pi z} \exp\left[-\dfrac{4}{3} z^{3/2}\right] \\[2ex] \xrightarrow{z\to-\infty} \dfrac{1}{\pi}\sqrt{|z|} \end{cases}$$

edge density



Matching with the bulk:

when ~~$x \ll x_{edge}$~~ $\frac{x - x_{edge}}{W_N} \ll 0$

$z \to -\infty$.

edge density.

$$\rho_{edge} \to \frac{1}{N} \cdot \frac{1}{W_N} \frac{1}{\pi} \sqrt{\cancel{x \to x_{edge}}\ \frac{x_{edge} - x}{W_N}}$$

$$= \frac{1}{\pi N W_N^{3/2}} \sqrt{x_{edge} - x}. = \cancel{\frac{\alpha\sqrt{2}}{\pi N^{5/6}}\sqrt{x_{edge}-x}}$$

$$= \frac{1}{\pi N}\left(\alpha\sqrt{2} N^{1/6}\right)^{3/2} \sqrt{x_{edge} - x}.$$

$$= \frac{\alpha^{3/2}\, 2^{3/4}}{\pi N^{3/4}} \sqrt{x_{edge} - x}$$

bulk density

$$\rho_N(x) = \frac{\alpha}{N\pi}\sqrt{2N - (\alpha x)^2}.$$

$$= \frac{\alpha^2}{\pi N}\sqrt{\frac{2N}{\alpha} - x^2} = \frac{\alpha^2}{\pi N}\sqrt{x_{edge}^2 - x^2}$$

$$x \to x_{edge} \simeq \frac{\alpha^2}{\pi N} \cdot \sqrt{2 x_{edge}}\sqrt{x_{edge} - x} \simeq$$

$x_{edge} =$

bulk density

$$x_{edge} = \frac{\sqrt{2N}}{\alpha}.$$

$$\rho_N(x) = \frac{\alpha}{N\pi}\sqrt{2N - (\alpha x)^2}$$

$$= \frac{\alpha^2}{N\pi}\sqrt{\frac{2N}{\alpha^2} - x^2}.$$

$$= \frac{\alpha^2}{\pi N}\sqrt{x_{edge}^2 - x^2}$$

$$= \frac{\alpha^2}{\pi N}\sqrt{(x_{edge} - x)(x_{edge} + x)}$$

Then $x \to x_{edge}$

$$\simeq \frac{\alpha^2}{\pi N}\cdot\sqrt{2\,x_{edge}}\cdot\sqrt{(x_{edge} - x)}$$

$$\simeq \frac{\alpha^2}{\pi N}\sqrt{2\cdot\frac{\sqrt{2N}}{\alpha}}\;\sqrt{x_{edge} - x}$$

$$\simeq \frac{\alpha^{3/2}\cdot 2^{3/4}}{\pi\, N^{3/4}}\sqrt{x_{edge} - x}.$$

↳ Perfect matching bet^n bulk & edge density

Kernel $K_\mu(x,y)$ [$x \neq y \to$ different in general] asymptotics as $N\to\infty$

starting point

$$K_\mu(x,y) = \int_\Gamma \frac{dt}{2\pi i}\, \frac{e^{\mu t/\hbar}}{t}\, \frac{\alpha}{\sqrt{2\pi \sinh \omega t}} \exp\left[-\frac{\alpha^2}{2\sinh(\omega t)}\left\{(x-y)^2 + (x^2+y^2)(\cosh\omega t - 1)\right\}\right]$$

here $\mu = (N-\frac{1}{2})\hbar\omega \sim N\hbar\omega$ for large N.

using

$$\frac{1}{\sinh \omega t} \xrightarrow{t\to 0} \frac{1}{\omega t} - \frac{\omega t}{6} + O(t^3).$$

$$\frac{[(\cosh\omega t)-1]}{\sinh \omega t} \xrightarrow{t\to 0} \frac{\omega t}{2} - \frac{(\omega t)^3}{24} + O(t^4).$$

~~keeping terms upto O(t)~~

$$\cancel{K_\mu(x,y) = \int_\Gamma \frac{dt}{2\pi i}\frac{e^{\mu t/\hbar}}{t}}$$

$$\simeq \frac{\alpha}{\sqrt{2\pi\omega}}\int_\Gamma \frac{dt}{2\pi i}\frac{1}{t^{3/2}}\, e^{\left(\mu - \frac{1}{2}m\omega^2 x^2\right)\frac{t}{\hbar} - \frac{\alpha^2 (x-y)^2}{2\omega t}}$$

$-\frac{\sqrt{2N}}{\alpha}$ — x — y — $\frac{\sqrt{2N}}{\alpha}$

Bulk Kernel: We consider two points x & $y \to$ both far from the edge while their relative separation $|x-y| \sim N^{-1/2} \to$ interparticle separation in the bulk.

Then

$$K_\mu(x,y) \simeq \frac{\alpha}{\sqrt{2\pi\omega}}\int_\Gamma \frac{dt}{2\pi i}\frac{1}{t^{3/2}}\, e^{\left[\frac{\mu}{\hbar} - \frac{m\omega^2}{4\hbar}(x^2+y^2)\right]t - \frac{\alpha^2}{2\omega t}(x-y)^2}$$

since $y = x + \underbrace{(x-y)}_{\text{negligible}}$

$$\simeq \frac{\alpha}{\sqrt{2\pi\omega}}\int_\Gamma \frac{dt}{2\pi i}\frac{1}{t^{3/2}}\, e^{\frac{t}{\hbar}\left(\mu - \frac{1}{2}m\omega^2 x^2\right) - \frac{\alpha^2}{2\omega t}(x-y)^2}$$

Note

$$\int_\Gamma \frac{dt}{2\pi i} \frac{1}{t^{3/2}} e^{tz - \frac{a}{t}} = \left(\frac{z}{a}\right)^{1/4} J_{1/2}\left(2\sqrt{az}\right)$$

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z \Rightarrow$$

$$= \left(\frac{z}{a}\right)^{1/4} \sqrt{\frac{\cancel{2}}{\pi \cdot \cancel{2} \sqrt{az}}} \sin\left(2\sqrt{az}\right)$$

$$= \frac{1}{\sqrt{\pi a}} \sin\left(2\sqrt{az}\right)$$

hence

$$K_\mu(x,y) \simeq \frac{\alpha}{\sqrt{2\pi\omega}} \int \frac{dt}{2\pi i} \frac{1}{t^{3/2}} e^{\frac{t}{\hbar}\left[\mu - \frac{1}{2}m\omega^2 x^2\right] - \frac{\alpha^2}{2\omega t}(x-y)^2}$$

$$\sin\left[2 \cdot \frac{\alpha|x-y|}{\sqrt{2\omega}} \cdot \frac{1}{\sqrt{\hbar}} \sqrt{\mu - V(x)}\right].$$

$$\simeq \frac{\cancel{\alpha}}{\sqrt{\cancel{2}\pi\cancel{\omega}}} \cdot \frac{1}{\sqrt{\pi \cdot \frac{\cancel{\alpha^2}(x-y)^2}{\cancel{2}\cancel{\omega}}}}$$

$$\simeq \frac{1}{\pi|x-y|} \sin\left[\frac{2\alpha}{\sqrt{2\hbar\omega}}\sqrt{\mu - V(x)}\ |x-y|\right]$$

$$\rho_N(x) = \frac{1}{N}\sqrt{\frac{2m}{\pi^2\hbar^2}}\sqrt{\mu - V(x)} \Rightarrow N\rho_N(x)\frac{\pi\hbar}{\sqrt{2m}} = \sqrt{\mu - V(x)}.$$

hence

$$\frac{2\alpha}{\sqrt{2\hbar\omega}}\sqrt{\mu - V(x)} = \frac{\cancel{2}\alpha}{\sqrt{\cancel{2}\hbar\omega}} \cdot \frac{\pi\cancel{\hbar}}{\sqrt{\cancel{2}m}} N\rho_N(x).$$

$$= \sqrt{\frac{m\omega\cancel{\hbar}}{\cancel{\hbar}}} \frac{\pi}{\sqrt{m\omega}} N\rho_N(x).$$

$$= \pi N \rho_N(x).$$

$$\boxed{K_{\mu,\text{bulk}}(x,y) = \frac{1}{\pi(x-y)} \sin\left[\pi N \rho_N(x)(x-y)\right]}$$

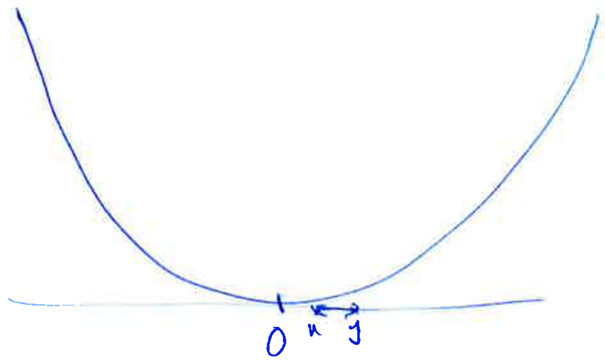
$\rightarrow$ Sine Kernel.

$$= N\rho_N(x)\, K_{\text{Sine}}\left[\pi N \rho_N(x)(x-y)\right]$$

where $K_{\text{Sine}}(z) = \frac{\sin \pi z}{\pi z}$

Note when $x \to y$ $K_\mu(x,x) \simeq N\rho_N(x)$ as expected.

ment



$$\rho_N(x) = \frac{\alpha}{\pi N}\sqrt{2N-\alpha^2 x^2}$$

$$\pi N \rho_N(0) = \alpha\sqrt{2N}$$

If both points $x, y \to 0$ (near the trap center) with $x-y \sim \frac{1}{\sqrt{N}}$,

$$K_0(x,y) \simeq \frac{1}{\pi(x-y)} \sin\left[\pi N \rho_N(0)(x-y)\right]$$

$$\simeq \frac{1}{\pi(x-y)} \sin\left[\alpha\sqrt{2N}(x-y)\right].$$

↳ simple interpretation: the system does not feel the potential.

When both $x \to 0$, $y \to 0$, the system does not feel the potential.

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi_k}{dx^2} = \epsilon\,\varphi_k(x)$$

$$\Rightarrow \phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{iqx} \quad \text{where} \quad q^2 = \frac{2m\epsilon}{\hbar^2}.$$

↳ plane wave

$$q_F = \sqrt{\frac{2m}{\hbar^2}\mu} = \sqrt{\frac{2m}{\hbar^2} N\hbar\omega} = \alpha\sqrt{2N}.$$

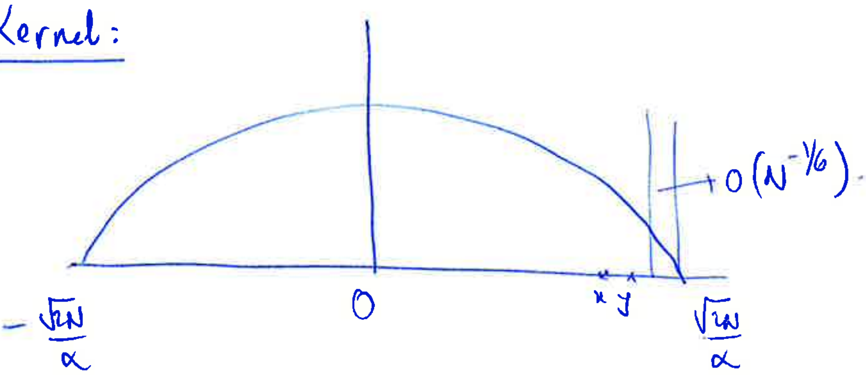
Then

$$K_0(x,y) \simeq \frac{1}{2\pi}\int_{-q_F}^{q_F} e^{iq(x-y)}\,dq$$

$$= \sum_k \theta(\mu-\epsilon_k)\,\phi_k^*(x)\,\phi_k(y)$$

$$\simeq \frac{1}{\pi}\int_0^{q_F} \cos q(x-y)\,dq = \frac{\sin q_F(x-y)}{\pi(x-y)} \simeq \frac{\sin\left[\alpha\sqrt{2N}(x-y)\right]}{\pi(x-y)}.$$

ge Kernel:



o points (x,y) in $K_\mu(x,y)$ $\rightarrow$ close to the edge $\frac{\sqrt{2N}}{\alpha}$. $\Rightarrow$ $|x-y| \sim O(N^{-1/6})$

$$K_\mu(x,y) = \int_\Gamma \frac{dt}{2\pi i}\,\frac{e^{\mu t/\hbar}}{t}\,\frac{\alpha}{\sqrt{2\pi \sinh(\omega t)}}\exp\left[-\frac{\alpha^2}{2\sinh(\omega t)}\left\{(x-y)^2+(x^2+y^2)(\cosh\omega t-1)\right\}\right]$$

$x_{edge} = \frac{\sqrt{2N}}{\alpha}$. $\qquad$ $\alpha = \sqrt{\frac{m\omega}{\hbar}}$ → inverse length

Let. $x = x_{edge} + \frac{1}{\alpha\sqrt{2}}N^{-1/6}\,a = x_{edge} + w_N\, a$

$y = x_{edge} + \frac{1}{\alpha\sqrt{2}}N^{-1/6}\,b = x_{edge} + w_N\, b$. , $w_N = \frac{N^{-1/6}}{\alpha\sqrt{2}}$.

a, b → dimensionless.

Expanding for small 't' and keeping terms up to $O(t^3)$, one gets.

$$K_\mu(x,y) \simeq K_{edge}(a,b) \simeq \frac{1}{2^{4/3}\sqrt{\pi}}\int \frac{d\tau}{2\pi i}\,\frac{1}{\tau^{3/2}}\exp\left[-\frac{(a-b)^2}{2^{8/3}\tau} - \frac{(a+b)\tau}{2^{1/3}} + \frac{\tau^3}{3}\right]$$

Using the integral representation:

$$\frac{e^{-(a-b)^2/4D\tau}}{\sqrt{4\pi D\tau}} = \int_{-\infty}^{\infty}\frac{dq}{2\pi}\,e^{-Dq^2\tau - iq(a-b)}$$

$$\cancel{K^{edge}_{edge}(a,b)\simeq} \quad K_\mu(x,y) \simeq \frac{1}{w_N}K_{edge}\left(\frac{x-x_{edge}}{w_N},\frac{y-x_{edge}}{w_N}\right).$$

where $$K_{edge}(a,b) = \int\frac{dq}{2\pi}\,e^{-iq(a-b)}\int_\Gamma\frac{d\tau}{2\pi i}\,\frac{1}{\tau}\exp\left[-\left(2^{2/3}q^2+2^{-1/3}(a+b)\right)\tau + \frac{\tau^3}{3}\right]$$

Defining $\quad Ai_1(z) = \int_\Gamma \frac{d\tau}{2\pi i} \frac{1}{\tau} e^{-z\tau + \frac{\tau^3}{3}} = \int_z^\infty Ai(u)\, du$

$Ai_1'(z) = - \int \frac{d\tau}{2\pi i} e^{-z\tau + \tau^3/3} = -Ai(z)$

e get $\quad K_{edge}(a,b) = \int \frac{dq}{2\pi} e^{-iq(a-b)} Ai_1\left(2^{2/3} q^2 + \frac{a+b}{2^{1/3}}\right)$

heck that when $a \to b$, $\quad K_{edge}(a,b) \to N \rho_{edge}$

$$Ai_1(z) = \int_\Gamma \frac{d\tau}{2\pi i} \frac{1}{\tau} e^{-z\tau + \tau^3/3}$$

$$Ai_1'(z) = -\int_\Gamma \frac{d\tau}{2\pi i} e^{-z\tau + \tau^3/3} = -Ai(z).$$

$$\Rightarrow \quad Ai_1(z) = \int_z^\infty Ai(u)\,du \;= \int_0^\infty Ai(z+u)\,du$$

Hence

$$K_{edge}(a,b) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{-iq(a-b)} \int_0^\infty Ai\left(u + 2^{2/3} q^2 + (a+b) 2^{-1/3}\right).$$

Nontrivial identity: (Vallee & Soares).

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{iq(v-v')} Ai\left(2^{2/3} q^2 + 2^{-1/3}(v+v')\right) = \pi 2^{-2/3} Ai(v) Ai(v').$$

choosing $v = a + 2^{-2/3} u$, $v' = b + 2^{-2/3} u$ and $u \to u\, 2^{-2/3}$.

we get $\qquad Ai''(z) = z\, Ai(z)$

$$K_{edge}(a,b) = \int_0^\infty du\, Ai(a+u)\, Ai(b+u)$$

$$= \int_0^\infty du\, Ai''(a+u)\, Ai''(b+u) \frac{1}{a+u} \frac{1}{b+u}$$

$$= \int_0^\infty du\, Ai''(u+a)\, Ai''(u+b) \left[\frac{1}{a+u} - \frac{1}{b+u}\right] \frac{1}{(b-a)}.$$

$$= \frac{1}{(b-a)} \left[\int_0^\infty du\, Ai(u+a)\, Ai''(u+b) - (a \to b)\right].$$

& by parts

$$= \frac{1}{(b-a)} \left[Ai(u+a) Ai'(u+b)\Big|_0^\infty - \int_0^\infty du\, Ai'(\cancel{u+a}) Ai'(u+b) \right.$$
$$\left. - Ai(b+u) Ai'(a+u) + \cancel{\int_0^\infty du\, Ai'(u+a) Ai'(u+b)} \right]$$

$$= \frac{1}{(b-a)} \left[Ai(b) Ai'(a) - Ai(a) Ai'(b) \right] = K_{Airy}(a,b).$$

hence

$$K(x,y) \simeq \frac{1}{W_N} K_{edge}\left(\frac{x - x_{edge}}{W_N}, \frac{y - x_{edge}}{W_N}\right)$$

where

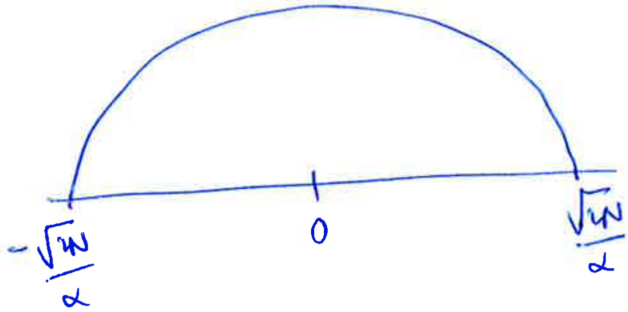
$$K_{edge}(a,b) \equiv K_{Airy}(a,b) = \frac{1}{(a-b)} \left[Ai(a) Ai'(b) - Ai'(a) Ai(b) \right].$$

Summary: 1-d harmonic oscillator at $T=0$

N free Fermions. $\Longrightarrow$ GUE, $\alpha = \sqrt{\frac{m\omega}{\hbar}}$

Global density:

$$\rho_N(x) = \frac{\alpha}{\sqrt{N}} f\left[\frac{\alpha x}{\sqrt{N}}\right], \quad f(z) = \frac{1}{\pi}\sqrt{2-z^2}.$$



$\rightarrow$ Wigner semi-circle.

Edge density:

$$\rho_{edge}(x) \simeq \frac{1}{N} \frac{1}{W_N} F_1\left[\frac{x - x_{edge}}{W_N}\right].$$

$x_{edge} = \sqrt{2N}/\alpha$

W_N = width of the edge $= \frac{1}{\alpha\sqrt{2}} N^{-1/6}$,

$$F_1(z) = Ai'^2(z) - z\,Ai^2(z).$$

- Bulk Kernel:

$$K_{bulk}(x,y) \simeq N \rho_N(x)\, K_{sine}\left[N\rho_N(x)\,(x-y)\right].$$

where $K(z) = \frac{\sin \pi z}{\pi z} \rightarrow$ sine Kernel.

- Edge kernel:

$$K_{edge}(x,y) = \frac{1}{W_N} K_{Airy}\left(\frac{x - x_{edge}}{W_N}, \frac{y - y_{edge}}{W_N}\right)$$

where

$$\cancel{K_{Airy}(u,v) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u-v}} \rightarrow \text{~~Airy Kernel~~}$$

$$K_{Airy}(a,b) = \frac{Ai(a)\,Ai'(b) - Ai'(b)\,Ai(a)}{a-b} \rightarrow \text{Airy Kernel}$$

Generalization to d-dim. harmonic oscillator at $T=0$.

[Dean, Le doussal, S.M, Schehr, 2015].
arXiv: 1505.01543

[illegible] $\Rightarrow$ <u>determinantal process</u>

Kernel. $$K_\mu(\vec{x},\vec{y}) = \sum_{\vec{k}} \theta(\mu-\epsilon_{\vec{k}})\, \phi^*_{\vec{k}}(\vec{x})\, \phi_{\vec{k}}(\vec{y})$$, $\mu \to$ Fermi energy.

$$K_\mu(\vec{x},\vec{y}) = \int_\Gamma \frac{dt}{2\pi i} \frac{e^{\mu t/\hbar}}{t}\, G(\vec{x},\vec{y};t) \quad \text{—— (M)}$$

$\hookrightarrow \langle \vec{x} | e^{-\hat{H}t/\hbar} | \vec{y} \rangle$

$\hookrightarrow$ d-dim. propagator in imaginary time.

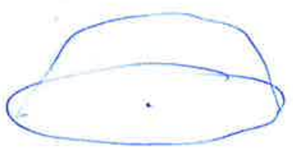
$$G(\vec{x},\vec{y};t) = \left[\frac{\alpha^2}{2\pi \sinh \omega t}\right]^{d/2} \exp\left[-\frac{\alpha^2}{2\sinh(\omega t)}\left\{(\vec{x}-\vec{y})^2 + (\vec{x}^2+\vec{y}^2)(\cosh(\omega t)-1)\right\}\right].$$

Rest of the analysis is similar to 1-d.

<u>Summary</u>:

- Global density: $$\rho_N(\vec{x}) \simeq \frac{1}{N}\left(\frac{m}{2\pi\hbar^2}\right)^{d/2} \frac{1}{\Gamma(\frac{d}{2}+1)}\left[\mu - \frac{1}{2}m\omega^2 r^2\right]^{d/2}$$

$r = |\vec{x}|$



$$\boxed{\mu \simeq \hbar\omega\, (\Gamma(d+1))^{1/d}\, N^{1/d}},$$

$$r_{edge} = \frac{\sqrt{2}}{\alpha}\left[\Gamma(d+1)\right]^{1/2d} \times N^{1/2d}.$$

- Edge density: $$\rho_{edge}(\vec{x}) \simeq \frac{1}{N}\frac{1}{w_N^d} F_d\left(\frac{r-r_{edge}}{w_N}\right)$$

where $$\boxed{w_N = \frac{[\Gamma(1+d)]^{-1/6d}}{\alpha\sqrt{2}}\, N^{-\frac{1}{6d}}},$$ $$F_d(z) = \frac{1}{\Gamma(\frac{d}{2}+1)\, 2^{4d/3}\, \pi^{d/2}} \int_0^\infty du\, u^{d/2}\, Ai(u+2^{2/3}z)$$

Bulk Kernel:

$\vec{x}, \vec{y} \to$ anywhere in the bulk.

$|\vec{x}-\vec{y}| \sim N^{-\frac{1}{2d}} \sim \ell_{typ} \to$ interparticle distance.

$$K_{bulk}(\vec{x},\vec{y}) = N \rho_N(\vec{x}) \gamma_d^d K^{(d)}_{bulk}\left[\gamma_d \left(N \rho_N(\vec{x})\right)^{1/d} |\vec{x}-\vec{y}|\right].$$

where

$$K^{(d)}_{bulk}(z) = \Gamma\left(\tfrac{d}{2}+1\right) \frac{J_{d/2}(2z)}{(\cancel{\pi} z)^{d/2}}, \qquad \gamma_d = \sqrt{\pi}\left[\Gamma\left(\tfrac{d}{2}+1\right)\right]^{1/d}.$$

Edge Kernel:

$$W_N \sim N^{-\frac{1}{6d}}, \qquad r_{edge} = \frac{\sqrt{2}}{\alpha}\left[\Gamma(d+1)\right]^{\frac{1}{2d}} N^{\frac{1}{2d}}.$$

$$K_{edge}(\vec{a},\vec{b}) \simeq \frac{1}{W_N^d} K^{(d)}_{edge}\left(\frac{\vec{x}-\vec{r}_{edge}}{W_N}, \frac{\vec{y}-\vec{r}_{edge}}{W_N}\right).$$

where

$$K^{(d)}_{edge}(\vec{u},\vec{v}) = \int \frac{d^d \vec{q}}{(2\pi)^d} e^{i\vec{q}\cdot(\vec{u}-\vec{v})} \mathrm{Ai}_1\left(q^2 2^{2/3} + 2^{-1/3}(u_n+v_n)\right)$$

$$= \int \frac{d^d q}{(2\pi)^d} e^{i\vec{q}\cdot(\vec{u}-\vec{v})} \int_0^\infty \mathrm{Ai}\left[z + q^2 2^{2/3} + 2^{-1/3}(u_n+v_n)\right] dz.$$

where,

$$u_n = \frac{\vec{u}\cdot\vec{r}_{edge}}{r_{edge}}, \qquad v_n = \frac{\vec{v}\cdot\vec{r}_{edge}}{r_{edge}}.$$

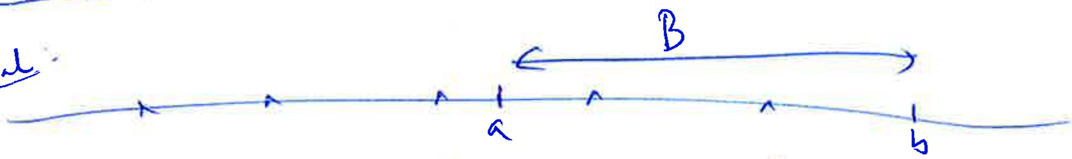
D.S. Dean, P. Le Doussal, S.N.M, G. Schehr.

arXiv: 1505.01543

Counting Statistics

~~Uncorrelated Points on a line~~

General:



$$P(x_1, x_2, \dots x_N) = \cancel{p(x_1)\,p(x_2) \cdots p(x_N)} \to \text{joint pdf.}$$

Consider any interval $B: [a, b]$

Let N_B = no. of points in B ⟶ random variable

What can be a say about the statistics of N_B, knowing the joint pdf $P(x_1, \dots x_N)$?

Simple Case: Uncorrelated points on a line

$$P(x_1, \dots x_N) = p(x_1)\,p(x_2) \cdots p(x_N).$$

Density, $\rho(x) = \left\langle \frac{1}{N} \delta(x - x_i) \right\rangle = p(x).$

Define the indicator function, $I_B(x_i) \begin{cases} \to 1 & \text{if } x_i \in B \\ \to 0 & \text{otherwise.} \end{cases}$

$$\Rightarrow \quad N_B = \sum_{i=1}^{N} I_B(x_i)$$

Moment generating function.

$$X_B(z) = \sum_{N_B} (1-z)^{N_B}\, \mathrm{Prob}[N_B] = \left\langle (1-z)^{N_B} \right\rangle = \left\langle (1-z)^{\sum I_B(x_i)} \right\rangle$$

$$= \left\langle \prod_i (1-z)^{I_B(x_i)} \right\rangle$$

$$= \left\langle \prod_{i=1}^{N} \left(1 - z\, I_B(x_i)\right) \right\rangle$$

↳ indep.

$$= \prod_{i=1}^{N} \left\langle 1 - z\, I_B(x_i) \right\rangle = \left(1 - z \int_B p(x)\,dx\right)^N$$

$$= \left[1 - q_B + s\, q_B\right]$$

$$q_B = \int_B p(x)\,dx$$

$1 - z \to s$

$$\Rightarrow \quad \sum_k s^k\, \mathrm{Prob}[N_B = k] = \sum_k \binom{N}{k} q_B^k (1-q_B)^{N-k} s^k$$

$$\Rightarrow \quad \mathrm{Prob}[N_B = k] = \binom{N}{k} q_B^k (1-q_B)^{N-k} \to \text{Binomial}$$

~~Converges to a Gauss~~

$$\langle N_B \rangle = N q_B$$

$$\mathrm{Var}(N_B) = N q_B (1 - q_B)$$

In particle if the interval 'B' is small,

$$q_B = \int_B p(x)\,dx \to \text{small}.$$

$$N \to \infty$$

Keeping the product $q_B N = s$ fixed

Then

$$\text{Prob}[N_B = k] \to \frac{s^k}{k!} e^{-s} \to \text{Poisson distribution with mean 's'.}$$

$$\langle N_B \rangle = \text{'s'}$$

$$\text{Var}(N_B) \simeq s.$$

What happens when the points are correlated

$$\Rightarrow \quad P(x_1, \ldots x_N) \neq p(x_1) \ldots p(x_N).$$

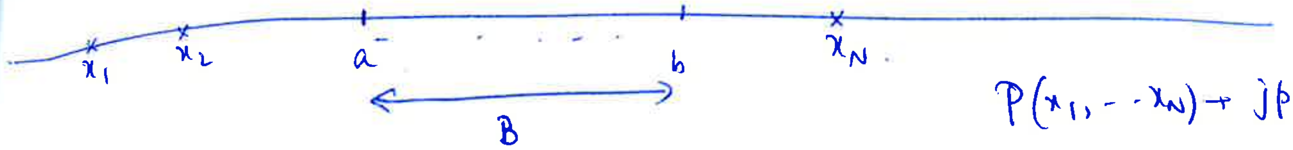
How do we calculate the counting statistics $\text{Prob}[N_B]$?

In particular, consider the case of GUE.

where.

$$P(x_1, \ldots x_N) \propto e^{-\alpha \sum_{i=1}^{N} x_i^2} \underbrace{\prod_{i<j} (x_i - x_j)^2}_{\to \text{ strongly correlated.}}$$

Application: Counting Statistics



$P(x_1, \dots x_N) \to$ jpdf

Consider any interval $B : [a, b]$

Let $N_B \to$ no. of points in $B \longrightarrow$ random variable.

What can ~~I~~ we say about the statistics of N_B, knowing the jpdf $P(x_1 \dots x_N)$?

~~$N_B = n(B)$~~ Recall, $R_n(x_1, \dots x_n) = \frac{N!}{(N-n)!} \int P(x_1, \dots x_n, x_{n+1}, \dots x_N)\, dx_{n+1} \dots dx_N$

$R_1(x) = N \langle \hat{\rho}_N(x) \rangle$

$R_n(x_1, \dots x_n) = \det K_N(x_i, x_j)\big|_{1 \le i,j \le n}$

~~Average: $\langle N_B \rangle = B$~~

Let $\hat{n}(x) = N \hat{\rho}_N(x) = \sum_{i=1}^{N} \delta(x - x_i) \to$ local number density "empirical" (for a given sample).

$R_1(x) = N \langle \hat{\rho}_N(x) \rangle = \langle \sum_{i=1}^{N} \delta(x - x_i) \rangle$

Then N_B = no. of points in $B = \int_B \hat{n}(x)\, dx$.

1) Average:

$$\langle N_B \rangle = \int_B \langle \hat{n}(x) \rangle dx = N \int_B \langle \hat{\rho}_N(x) \rangle dx = \int_B R_1(x)\, dx.$$

$$= \int_B K_N(x, x)\, dx$$

2) Variance:

2nd moment:

$$\langle N_B^2 \rangle = \int_B dx\, dy \, \langle \hat{n}(x) \hat{n}(y) \rangle$$

$$= \int_B dx\, dy \left\langle \sum_{i=1}^{N} \sum_{j=1}^{N} \delta(x - x_i)\, \delta(y - x_j) \right\rangle$$

$$= \int_B dx\, dy \left[\left\langle \sum_{i=j=1}^{N} \delta(x - x_i)\, \delta(y - x_i) + \sum_{i \ne j} \delta(x - x_i)\, \delta(y - x_j) \right\rangle \right]$$

$$= \int_B dx\, dy \left[\left\langle \delta(x - y) \sum_{i=1}^{N} \delta(x - x_i) \right\rangle + \left\langle \sum_{i \ne j} \delta(x - x_i)\, \delta(y - x_j) \right\rangle \right]$$

$$= \int_B dx\, dy\, \delta(x-y)\, R_1(x) + \int dx dy \sum_{i\neq j} \int P(x,y,x_3,\dots x_N)\, dx_3 \cdots dx_N$$

$$= \int_B R_1(x)\, dx + \int_B dx dy \left[N(N-1) \int P(x,y,x_3,\dots x_N)\, dx_3 \cdots dx_N \right.$$

$$\langle N_B^2 \rangle = \int_B R_1(x)\, dx + \int_B dx\, dy\, R_2(x,y).$$

nce

$$\mathrm{Var}(N_B) = \langle N_B^2 \rangle - \langle N_B \rangle^2 = \int_B R_1(x)\, dx + \int_B dx\, dy\, R_2(x,y) - \left[\int_B dx\, R_1(x)\right]\left[\int_B dy\, R_1(y)\right].$$

ow,

$$R_2(x,y) = \det K_N(x_i,x_j)\Big|_{1\leq i,j\leq 2} = \det \begin{vmatrix} K_N(x,x) & K_N(x,y) \\ K_N(y,x) & K_N(y,y) \end{vmatrix}$$

$$= K_N(x,x)K_N(y,y) - K_N(x,y)K_N(y,x).$$

Hence

$$\mathrm{Var}(N_B) = \int_B R_1(x)\, dx + \int_B dx\, dy \left[K_N(x,x)\, K_N(y,y) - K_N(x,y)\, K_N(y,x) \right] - \left[\int_B dx\, R_1(x)\right]^2$$

$$\boxed{R_1(x) = K_N(x,x)}$$

$$= \int_B R_1(x)\, dx + \int_B dx\, \cancel{K_N(x,x) \int dy\, K_N(y,y)} - \int dx\, dy\, K_N^2(x,y) - \cancel{\left[\int dx\, K_N(x,x)\right]^2}$$

$$\boxed{\mathrm{Var}(N_B) = \int_B K_N(x,x)\, dx - \int_B dx\, dy\, K_N^2(x,y)} \rightarrow$$

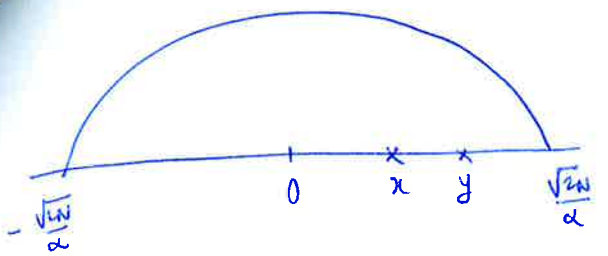
general formula for GUE.

If we know the kernel, we can determine ~~the~~

- $\langle N_B \rangle = \int K_N(x,x)\, dx$
- $\mathrm{Var}(N_B) = \int_B K_N(x,x)\, dx - \int_B dx\, dy\, K_N^2(x,y).$

What about ~~the~~ $\longrightarrow$ higher moments.

(5)



ernel: $K_N(x,y)$

k Kernel:

onsider the interval $[-L, L]$ around the origin, when $L \ll \frac{\sqrt{2N}}{\alpha}$.

$\alpha = \sqrt{\frac{m\omega}{\hbar}}$.

Average: $\langle N_L \rangle = \int_{-L}^{L} K_N(x,x)\, dx$

Variance. $Var(N_L) = \int_{-L}^{L} K_N(x,x)\,dx - \int_{-L}^{L}\int_{-L}^{L} dx\, dy\; K_N^2(x,y)$.

Bulk Kernel: scaling.

$$K_N(x,y) \simeq N \rho_N(x)\, K_{bulk}\left[N \rho_N(x)(x-y)\right]$$

is valid when $|x-y| \sim \frac{1}{N\rho_N(x)}$.

where $K_{bulk}(z) = \frac{\sin \pi z}{\pi z}$.

When $x, y \to$ both close to the origin, $\rho_N(x) \simeq \rho_N(0)$.

$\rho_N(x) = \frac{\alpha}{\pi N}\sqrt{2N - x^2}$

$$K_N(x,y) \simeq N \rho_N(0)\, K_{bulk}\left[N \rho_N(0)(x-y)\right]$$

, $n_N(0) = N\rho_N(0) \simeq \frac{\alpha\sqrt{2N}}{\pi}$

$$\simeq n_N(0)\, K_{bulk}\left[n_N(0)(x-y)\right].$$

$l_{typ} \sim \frac{1}{n_N(0)}$.

Hence.

(i) $\langle N_L \rangle = \cancel{n_N(0) L\, K_{bulk}[n_N(0)]}\; \int_{-L}^{L} n_N(0)\, dx = 2\, n_N(0)\, L = s \sim O(1)$

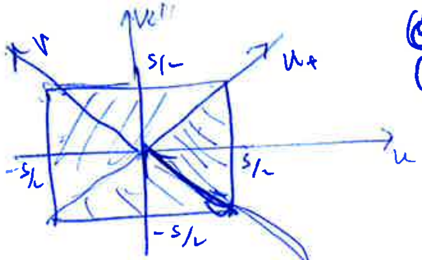
$L \sim \frac{s}{n_N(0)}$

(ii) $Var(N_L) = s - \int_{-L}^{L}\int_{-L}^{L} dx\, dy\; n_N^2(0)\, K_{bulk}^2\left[n_N(0)(x-y)\right]$

$n_N(0)\, x = u$.

$$= s - \int_{-L n_N(0)}^{L n_N(0)} du \int_{-L n_N(0)}^{L n_N(0)} du'\; K_{bulk}^2(u-u')$$

$$= s - \int^{s/2}\int^{s/2} du\, du'\; K_{bulk}^2(u-u')$$



The integral.

$$\int_{-s/2}^{s/2} du \int_{-s/2}^{s/2} du' \; f(u-u')$$

↳ symmetric function

$u_+ = \frac{u+u'}{2}$

$r = u-u'$

for a given 'r'

$u_+ \text{(upper limit)} = \frac{r-s}{2}$

$u_- \text{(lower ")} = \frac{s-r}{2}$

Show that

$$= 2 \int_0^s dr \int_{\frac{r-s}{2}}^{\frac{s-r}{2}} du_+ \, f(r).$$

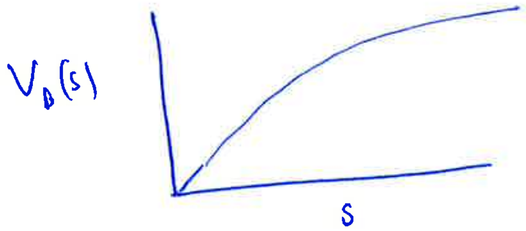
$$= 2 \int_0^s dr \, (s-r) \, f(r).$$

Hence,

$$\mathrm{Var}(N_L) = s - 2 \int_0^s dr \, (s-r) \, K_{\text{bulk}}^2(r).$$

$$= s - 2 \int_0^s dr \, (s-r) \cdot \frac{\sin^2 \pi r}{\pi^2 r^2}$$

$$\mathrm{Var}(N_L) = V_L(s) = s - 2 \int_0^s dr \, (s-r) \left[\frac{\sin \pi r}{\pi r}\right]^2$$



$$V_B(s) \begin{cases} \xrightarrow{s \to 0} & s - \frac{s^2}{2} \\ \xrightarrow[s \to \infty]{} & \frac{1}{\pi^2}\left[\ln(2\pi s) + 1 + \gamma_E\right]. \end{cases},$$

γ_E = Euler constant = 0.577215...

$n_N(0)$

$= \frac{\alpha\sqrt{2N}}{\pi}$

$$G \simeq \frac{1}{\pi^2}\left[\ln\left(2\pi \cdot 2L \frac{\alpha\sqrt{2N}}{\pi}\right) + 1 + \gamma_E\right].$$

$$= \frac{1}{\pi^2}\left[\ln\left(2\alpha L\sqrt{2N}\right) + 1 + \gamma_E + \ln 2\right]$$

$$= \frac{1}{\pi^2} \ln\left[2\alpha L\sqrt{2N}\right] + \underbrace{\frac{1+\gamma_E+\ln 2}{\pi^2}}_{\simeq 0.230036\ldots}$$

$C_{DM} \to$ Dyson-Mehta const.

Note that in the i.i.d. case, (Poisson)

$$P(x_1, \dots x_N) = p(x_1) \dots p(x_N).$$

$N \to \infty$
$L \to \infty$
keeping $NL \to$ fixed

- $\langle N_L \rangle = N \int_{-L}^{L} p(x)\, dx \simeq 2LN\, p(0) = S$

- $\mathrm{Var}(N_B) = N. \int_{-L}^{L} p(x)\, dx \left[1 - \int_{-L}^{L} p(x)\, dx\right].$

$$= N . p(0) . 2L \left[1 - 2L\, p(0)\right].$$

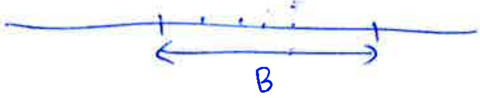
$$= S\left[1 - \frac{S}{N}\right] \xrightarrow[N \to \infty]{} S$$

Thus for the interacting case.

$$V_L(S) \sim \frac{1}{\pi^2}\left[\ln(2\pi S) + 1 + \gamma_E\right] << \underset{\hookrightarrow \text{ i.i.d. case.}}{S}$$

$\Rightarrow$ The eigenvalues are much more ordered in GUE as compared to the IID (Poisson process).

Moment generating function of N_B:



Let $I_B(x_i) \begin{cases} \to 1 & \text{if } x_i \in B \\ \to 0 & \text{otherwise.} \end{cases}$ $\longrightarrow$ indicator function.

Then $N_B = \sum_{i=1}^{N} I_B(x_i)$ $\longrightarrow$ random variable.

Moment generating function:

$$\chi_B(z) = \left\langle (1-z)^{N_B} \right\rangle = \sum_{N_B} (1-z)^{N_B} \, \text{Prob}[N_B]$$

Then,
$$\chi_B(z) = \left\langle (1-z)^{N_B} \right\rangle = \left\langle (1-z)^{\sum_{i=1}^{N} I_B(x_i)} \right\rangle$$
$$= \left\langle \prod_{i=1}^{N} (1-z)^{I_B(x_i)} \right\rangle$$
$$= \left\langle \prod_{i=1}^{N} \left[1 - z\, I_B(x_i) \right] \right\rangle$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \prod_{i=1}^{N} \left[1 - z\, I_B(x_i) \right] \right\} P(x_1, \ldots x_N)\, dx_1 \ldots dx_N.$$

Simple Case:
I.I.D variables: $P(x_1, x_2, \ldots x_N) = p(x_1)\, p(x_2) \ldots p(x_N)$.

Then $\chi_B(z) = \left\langle (1-z)^{N_B} \right\rangle = \prod_{i=1}^{N} \int_{-\infty}^{\infty} \left[1 - z\, I_B(x_i) \right] p(x_i)\, dx_i$

$$= \left[1 - z \int_B p(x)\, dx \right]^N \quad \text{, let } q_B = \int_B p(x)\, dx$$

$$\boxed{\begin{aligned} &1-(1-s)\,q_B \\ &= 1 - q_B + s\, q_B \end{aligned}}$$

$$\sum_{N_B} (1-z)^{N_B} \, \text{Prob}[N_B] = \left[1 - z\, q_B \right]^N$$

let $(1-z) = s$, $\Rightarrow$ $\sum_{N_B} s^{N_B}\, \text{Prob}[N_B] = \left[1 - q_B + s\, q_B \right]^N$

$$= \sum_{k} \binom{N}{k} q_B^k (1-q_B)^{N-k} s^k$$

Comparing powers of s^k:

$$\text{Prob}[N_B = k] = \binom{N}{k} q_B^k (1-q_B)^{N-k} \longrightarrow \text{binomial distribution.}$$

$\hookrightarrow$ Converges to a Gaussian distribution for large k & N.

- $\langle N_B \rangle = N q_B = N \int_B p(x)\, dx$
- $\text{Var}(N_B) = N q_B (1 - q_B)$

$$N_B \xrightarrow[N \to \infty]{} q_B N + \sqrt{N q_B (1-q_B)} \cdot N(0,1)$$

$$\chi_B(z) = \left\langle \prod_{i=1}^{N} \left(1 - z\, I_B(x_i)\right) \right\rangle = \int dx_1 \cdots dx_N \; P(x_1, \dots x_N) \; \prod_{i=1}^{N} \left(1 - z\, I_B(x_i)\right)$$

$N=2$

$$[1 - z\, I_B(x_1)][1 - z\, I_B(x_2)] = 1 - z\left[I_B(x_1) + I_B(x_2)\right] + z^2\, I_B(x_1)\, I_B(x_2)$$

$N=3$

$$[1 - z\, I_B(x_1)][1 - z\, I_B(x_2)][1 - z\, I_B(x_3)] = [1 - z\, I_B(x_1)]\left[1 - z\, (I_B(x_2) + I_B(x_3)) + z^2\, I_B(x_2) I_B(x_3)\right]$$

$$= 1 - z\left(I_B(x_1) + I_B(x_2) + I_B(x_3)\right) + z^2\left[I_B(x_2) I_B(x_3) + I_B(x_1) I_B(x_2) + I_B(x_1) I_B(x_3)\right] - z^3\, I_B(x_1)\, I_B(x_2)\, I_B(x_3)$$

Now

$$\int I_B(x_1) I_B(x_2) \cdots I_B(x_k) \; P(x_1, x_2, \dots x_N)\; dx_1 \cdots dx_N$$

$$= \int_B dx_1 \cdots dx_k \int_{-\infty}^{\infty} P(x_1, x_2, \dots x_N)\; dx_{k+1} \cdots dx_N = \frac{(N-k)!}{N!} \int_B R_k(x_1, \dots x_k)\, dx_1 \cdots dx_k$$

$$\boxed{\text{Recall, } \; R_k(x_1, \dots x_k) = \frac{N!}{(N-k)!} \int_{-\infty}^{\infty} P(x_1, \dots x_N)\; dx_{k+1} \cdots dx_N}$$

Hence, coefficient of z^k in $\prod_{i=1}^{N}\left[1 - z\, I_B(x_i)\right] = (-1)^k \Big[\underbrace{I_B(x_1) \cdots I_B(x_k) + \cdots + \cdots}_{\text{all } \binom{N}{k} \text{ permutations}}\Big]$

Then integrated over $dx_1 \cdots dx_N$, each permutation gives $\frac{(N-k)!}{N!} \int_B R_k(x_1, \dots x_k)\, dx_1 \cdots dx_k$

Hence,

$$\chi_B(z) = \sum_{k=0}^{N} (-1)^k z^k \cdot \frac{(N-k)!}{N!} \binom{N}{k} \cdot \int_B R_k(x_1, \dots x_k)\, dx_1 \cdots dx_k$$

$$= \sum_{k=0}^{N} \frac{(-1)^k z^k}{k!} \int_B R_k(x_1, \dots x_k)\, dx_1 \cdots dx_k$$

$$= \sum_{k=0}^{N} \frac{(-z)^k}{k!} \int_B \det\left[K_N(x_i, x_j)\right]_{1 \le i, j \le k} dx_1 \cdots dx_k = \sum_{N_B=0}^{N} (1-z)^{N_B}\, \mathrm{Prob}[N_B]$$

If we know the kernel, in principle, we can determine $\chi_B(z)$ and hence $\mathrm{Prob}[N_B]$.

$$\left\langle \prod_{i=1}^{N} \left[1 - z I_B(x_i)\right] \right\rangle = \int \prod_{i=1}^{N} \left(1 - z I_B(x_i)\right) P(x_1, \dots x_N)\, dx_1 \dots dx_N .$$

~~If $P(x_1 \dots x_N)$ has a determinantal structure.~~

For GUE, $\quad P(x_1, \dots x_N) = \left|\Psi_0(x_1, \dots x_N)\right|^2 = \frac{1}{N!} \det\left[\varphi_i(x_j)\right] \det\left[\varphi_i(x_j)\right].$

hence

$$\left\langle \prod_{i=1}^{N} \left(1 - z I_B(x_i)\right) \right\rangle = \frac{1}{N!} \int \prod_{i=1}^{N} \left[1 - z I_B(x_i)\right] \det\left[\varphi_i^*(x_j)\right] \det\left[\varphi_i(x_j)\right] dx_1 \dots dx_N .$$

Cauchy-Binet / Andréief identity:

$$\int dx_1 \dots dx_N \prod_{i=1}^{N} h(x_i) \det\left[f_i(x_j)\right]_{1 \le i,j \le N} \det\left[g_i(x_j)\right]_{1 \le i,j \le N} = N! \det\left[\int h(x) f_i(x) g_j(x)\, dx\right]_{1 \le i,j \le N}$$

[for a nice proof for general N, see appendix B of J. Rambeau & G. Schehr, PRE, **83**, 061146 (2011). → arXiv: 1102.1640].

In our problem, $\quad h(x) = 1 - z I_B(x) = 1 - z I_B(x)$

$f_i(x_j) = \varphi_i^*(x_j)$

$g_i(x_j) = \varphi_i(x_j)$

hence,

$$\left\langle \prod_{i=1}^{N} \left(1 - z I_B(x_i)\right) \right\rangle = \det\left[\int_{-\infty}^{\infty} \left[1 - z I_B(x)\right] \phi_i^*(x) \phi_j(x)\, dx\right]$$

$$= \det\left[\int_{-\infty}^{\infty} \varphi_i^*(x) \varphi_j(x)\, dx - z \int_{-\infty}^{\infty} I_B(x) \phi_i^*(x) \varphi_j(x)\, dx\right]$$

↓ orthonormal

$$= \det\left[\delta_{ij} - z \int_B \varphi_i^*(x) \varphi_j(x)\, dx\right]_{N \times N}$$

Def^n: overlap matrix:

$$A \equiv A_{ij} = \int \phi_i(x)\,\phi_j(x)\,dx \rightarrow (N\times N) \text{ matrix}$$

if $B \rightarrow [-\infty,\infty]$ (full space), $A_{ij} = \delta_{ij}$ by orthonormality

hence

$$\left\langle \prod_{i=1}^{N} \left(1 - z\, I_B(x_i)\right)\right\rangle = \det\left[\mathbb{1} - z\,A\right] = \prod_{i=1}^{N}(1 - z\, a_i)$$

$a_i \rightarrow$ eigenvalues of 'A'

hence

$$\chi_B(z) = \langle (1-z)^{N_B}\rangle = \sum_{N_B} (1-z)^{N_B}\, \Pr(N_B)$$

$$= \det\left[\mathbb{1} - z\,A\right]$$

$$= \prod_{i=1}^{N}(1 - z\,a_i)$$

If we can compute the eigenvalues a_i's of the overlap matrix A_{ij}, in principle we can compute the moment generating function. $\chi_B(z)$.

Remark: In particular, taking $z\to 1$ limit

$$\chi_B(z=1) = \lim_{z\to 1} \sum_{N_B} (1-z)^{N_B}\Pr(N_B) = \text{Prob}\left[N_B = 0\right]$$

$\hookrightarrow$ hole probability.

$$= \det\left[\mathbb{1} - A\right] = \prod_{i=1}^{N}(1-a_i)$$

So far we have used the Kernel. overlap matrix A and the Kernel $K_N(x,y)$,

What's the connection bet^n the overlap matrix A and the Kernel $K_N(x,y)$?

Relationship bet^n overlap matrix & Kernel → Fredholm determinant.

Consider the object. $$\chi_B(z) = \det[\mathbb{1} - zA] = \prod_{i=1}^{N}(1 - z a_i)$$

A_{ij} → overlap matrix $(N \times N)$

$$A_{ij} = \int_B dx\, \phi_i^*(x)\, \phi_j(x).$$

Let
$$\sum_j A_{ij}\, C_j^{(a)} = a\, C_i^{(a)}$$

a → eigenvalue of A

C_j^a → $(N \times 1)$ column vector → eigenvector of 'A' associated with the eigenvalue 'a'.

Then $$\chi_B(z) = \det[\mathbb{1} - zA] = \prod_{i=1}^{N}(1 - a_i z)$$

Now, recall the definition of Kernel. [with $\mu = N$; $\hbar = \omega = 1$] (harmonic oscillator GUE).

$\phi_k(x)$ → real.

$$K_N(x, x') = \sum_{k=1}^{N} \phi_k^*(x)\, \phi_k(y)$$

Define a function. $$\psi^{(a)}(x) = \sum_j C_j^{(a)}\, \phi_j(x).$$

Consider the integral
$$\int_B K_N(x, y)\, \psi^{(a)}(y)\, dy = \int_B \left[\sum_{k=1}^{N} \phi_k^*(x)\, \phi_k(y)\right]\left[\sum_{j=1}^{N} C_j^{(a)}\, \phi_j(y)\right] dy$$

$$= \sum_{k,j=1}^{N} \phi_k(x)\, C_j^{(a)} \underbrace{\int_B \phi_k(y)\, \phi_j(y)\, dy}_{A_{kj}}$$

$$= \sum_{k,j} \phi_k(x)\, A_{kj}\, C_j^{(a)}$$

$$= \sum_{k} \phi_k(x)\, a \cdot C_k^{(a)} = a\, \psi^{(a)}(x).$$

$$\boxed{a\,\psi^{(a)}(x) = \int_B K_N(x,y)\,\psi^{(a)}(y)\,dy.}$$

$\psi^{(a)}(x)$ is an eigenvector of ~~of~~ the Fredholm integral operator with Kernel $K_N(x,y)$ associated with its eigenvalue 'a'.

So, one can get the eigenvalues 'a' by solving the Fredholm integral equation.

$$\chi_B(z) = \det\left[\mathbb{1} - zA\right] = \prod_{i=1}^{N}(1 - a_i z) = \det\left[\mathbb{1} - zP_B K_N P_B\right].$$

↑ Fredholm determinant.

The notation $P_B K P_B$ means.

$x \in B$

$$\boxed{a\,\psi^{a}(x) = \int_B K_N(x,y)\,\psi^{(a)}(y)\,dy}$$

Remark: when $B \to [-\infty,\infty]$ full span, $A_{ij} = \delta_{ij} \Rightarrow a = 1$ for all 'N'.

↳ degenerate spectrum

$\psi^{a}(x) = \phi_i(x)$ $i = 1, \dots N$

$\chi_B(z) = \prod_{i=1}^{N}(1 - a_i z)$ can be determined in two ways

a_i's → (i) diagonalizing the overlap matrix
(ii) solving the Fredholm integral equation.

In particular.

$$\mathrm{Prob}[N_B = 0] = \lim_{z\to 1}\chi_B(z) = \prod_{i=1}^{N}(1 - a_i) = \det\left[\mathbb{1} - P_B K_N P_B\right]$$

P_B: projection operator on 'B'.

$P_B K_N P_B$: shorthand K_N

Calculation of moments:

$$\chi_B(z) = \left\langle (1-z)^{N_B} \right\rangle = \prod_{i=1}^{N} (1 - z a_i) = \exp\left[\sum_{i=1}^{N} \ln(1 - z a_i) \right]$$

$$= \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{i=1}^{N} a_i^n \right] = \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}(A^n) \right]$$

$$= \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}(K_N^n) \right]$$

Where

$n=1$: $\mathrm{Tr}[A] = \mathrm{Tr}[K_N] = \int_B K_N(x,x)\,dx = \sum_{i=1}^{N} a_i$

$$A_{ij} = \int_B dx\, \phi_i(x)\phi_j(x)$$

$$\mathrm{Tr}(A) = \sum_{i=1}^{N} A_{ii} = \int_B dx \sum_{i=1}^{N} \phi_i(x)\phi_i(x) = \int_B dx\, K_N(x,x) = \mathrm{Tr}_B[K_N].$$

$$\mathrm{Tr}(A^2) = \sum_{ij} A_{ij} A_{ji} = \sum_{ij} \int_B dx\, \phi_i(x)\phi_j(x) \int_B dy\, \phi_i(y)\phi_j(y)$$

$$= \int_B dx\, dy \sum_i \phi_i(x)\phi_i(y) \sum_j \phi_j(x)\phi_j(y)$$

$$= \int_B dx\, dy\, K_N(x,y) \times K_N(y,x) = \mathrm{Tr}_B[K_N^2]$$

$$\mathrm{Tr}(A^3) = \sum_{ijk} A_{ij} A_{jk} A_{ki} = \int_B dx\, dy\, dz \sum_{ijk} \phi_i(x)\phi_j(x)\phi_j(y)\phi_k(y)\,\phi_k(z)\phi_i(z)$$

$$= \int_B dx\, dy\, dz\, K_N(x,y) K_N(y,z) K_N(z,x)$$

etc.

$$= \mathrm{Tr}_B[K_N^3]$$

...ce,

$$\left\langle (1-z)^{N_B} \right\rangle = \exp\left[- \sum_{n=1}^{\infty} \frac{z^n}{n} \, Tr\left[K_N^n\right] \right].$$

...anding in powers of z

$$L.H.S. = \left\langle (1-z)^{N_B} \right\rangle = 1 - \langle N_B \rangle z + \frac{\langle N_B(N_B-1)\rangle}{2!} z^2 - \frac{\langle N_B(N_B-1)(N_B-2)\rangle}{3!} z^3 + O(z^4)$$

$$R.H.S. = \exp\left[- z\, Tr(K_N) - \frac{z^2}{2} Tr(K_N^2) - \frac{z^3}{3} Tr(K_N^3) - \frac{z^4}{4} Tr(K_N^4) \cdots \right]$$

$$= 1 - Tr(K_N).z + \frac{Tr^2(K_N) - Tr(K_N^2)}{2} z^2 - \left[Tr^3(K_N) - 3\, Tr(K_N) Tr(K_N^2) + 2\, Tr(K_N^3) \right] \frac{z^3}{6} + O(z^4)$$

Comparing powers of z:

$$\langle N_B \rangle = Tr(K_N) = \int_B K_N(x,x)\, dx.$$

$$\langle N_B (N_B - 1) \rangle = Tr^2(K_N) - Tr(K_N^2).$$

$$\Rightarrow \quad Var(N_B) = \langle N_B^2 \rangle - \langle N_B \rangle^2 = \langle N_B \rangle - Tr(K_N^2)$$

$$= \int_B K_N(x,x)\, dx - \int dx\, dy\, K_N^2(x,y).$$

(as derived before)

$$\langle N_B (N_B-1)(N_B-2) \rangle = Tr^3(K_N) - 3\, Tr(K_N) Tr(K_N^2) + 2\, Tr(K_N^3)$$

...that

$$\Rightarrow \quad \langle N_B^3 \rangle = 3\, Tr^2(K_N) - 3\, Tr(K_N^2) + Tr(K_N) + Tr^3(K_N) - 3\, Tr(K) Tr(K^2) + 2\, Tr(K^3).$$

...ntral 3-rd moment,

$$\mu_3 = \left\langle \left[N_B - \langle N_B \rangle \right]^3 \right\rangle = \langle N_B^3 \rangle - 3 \langle N_B \rangle \langle N_B^2 \rangle + 2 \langle N_B \rangle^3$$

show that

$$= Tr\left[K_N\right] - 3\, Tr\left[K_N^2\right] + 2\, Tr\left[K_N^3\right].$$

$$= \int_B K_N(x,x)\, dx - 3 \int_B K_N^2(x,y)\, dx\, dy + 2 \int_B dx\, dy\, dz\; K_N(x,y) K_N(y,z) K_N(z,x)$$

Taking $z \to 1$ limit.

$$\text{Prob}[N_B = 0] = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}[K_N^n]\right]$$

If $B \equiv [M, \infty]$.

$\text{Prob}[N_B = 0] = \text{Prob}[\text{the interval } [M,\infty] \text{ is empty}]$

$= \text{Prob}[\text{all the points or eigenvalues} \leq M]$.

$= \text{Prob}[x_{max} \leq M] = Q_N(M)$

↳ Cumulative distribution of the largest eigenvalue.

Then.

$$Q_N[M] = \text{Prob}[x_{max} \leq M] = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}_B[K_N^n]\right].$$

$B \to [M, \infty]$.

$$\text{Tr}[K_N] = \int_M^\infty K_N(x,x)\,dx$$

$$\text{Tr}[K_N^2] = \int_M^\infty \int_M^\infty K_N^2(x,y)\,dx\,dy$$

$$\text{Tr}[K_N^3] = \int_M^\infty \int_M^\infty \int_M^\infty dx\,dy\,dz\; K_N(x,y)\,K_N(y,z)\,K_N(z,x)$$

etc.

So, if we know the Kernel, in principle we know $Q_N(M)$.

Near the edge. $x_{edge} = \frac{\sqrt{2\alpha}}{N}$, $\quad K_N(x,y) \simeq \frac{1}{w_N} K_{Airy}\left(\frac{x - x_{edge}}{w_N}, \frac{y - x_{edge}}{w_N}\right)$

where $w_N = \frac{1}{\alpha\sqrt{2}} N^{-1/6}$.

$u = \frac{x - x_{edge}}{w_N}$.

$$\text{Tr}[K_N] \simeq \int_{\frac{M - x_{edge}}{w_N}}^{\infty} K_{Airy}(u,u)\,du$$

$$\text{Tr}[K_N^2] \simeq \int_{\frac{M - x_{edge}}{w_N}} \int_{\frac{M - x_{edge}}{w_N}} K_{Airy}^2(u,v)\,du\,dv$$

This implies.

$$Q_N(M) \xrightarrow[\text{close to the edge}]{M\to\frac{\sqrt{2N}}{\alpha}} \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}_z\left[K_{Airy}^n\right]\right].$$

$$z = \frac{M - x_{edge}}{W_N}.$$

Hence

$$Q_N(M) \xrightarrow[\substack{M\to\frac{\sqrt{2N}}{\alpha} \\ N\to\infty}]{} F_2\left[\frac{M - x_{edge}}{W_N}\right] = F_2\left[\left(M - \frac{\sqrt{2N}}{\alpha}\right)\frac{N^{1/6}}{\alpha\sqrt{2}}\right].$$

keeping $z = \dfrac{M - \sqrt{2N}/\alpha}{W_N} = \dfrac{1}{\alpha\sqrt{2}}\left(M - \dfrac{\sqrt{2N}}{\alpha}\right)N^{1/6}$ fixed

Where, the scaling function

$$\boxed{F_2(z) = \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}_z\left[K_{Airy}^n\right]\right]}$$

$$\mathrm{Tr}_z\left[K_{Airy}\right] = \int_z^{\infty} K_{Airy}(x,x)\,dx.$$

$$\mathrm{Tr}_z\left[K_{Airy}^2\right] = \int_z^{\infty}\int_z^{\infty} K_{Airy}(x,y)\,K_{Airy}(y,x)\,dx\,dy$$

$$\mathrm{Tr}_z\left[K_{Airy}^3\right] = \int_z^{\infty}\int_z^{\infty}\int_z^{\infty} K_{Airy}(x,y)\,K_{Airy}(y,z)\,K_{Airy}(z,x)\,dx\,dy\,dz$$

etc.

$$\boxed{K_{Airy}(x,y) = \frac{Ai(x)\,Ai'(y) - Ai(y)\,Ai'(x)}{x-y}}$$

$$1 - F_2(z) \xrightarrow[z\to\infty]{} \frac{e^{-\frac{4}{3}z^{3/2}}}{16\pi z^{3/2}}$$

$$F_2(z) \xrightarrow[z\to-\infty]{} c_2\,\frac{e^{-|z|^3/12}}{|z|^{1/8}}$$

$$c_2 = 2^{1/24}\,e^{\zeta'(-1)}$$

Tracy-Widom (1994) showed that

$$F_2(z) = \exp\left[-\int_z^{\infty}(u-z)\,q^2(u)\,du\right]$$

where

$$q''(u) = 2q^3(u) + u\,q(u) \rightarrow \text{Painlevé-II eq}^n.$$

with the boundary condition, $q(u) \xrightarrow[u\to\infty]{} Ai(u)$.

$\rightarrow$ Hastings-Mcleod sol^n of P-II.