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Random-field-like criticality in glass-forming liquids

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We introduce an approach to derive an effective scalar field theory for the glass transition; the fluctuating field is the overlap between equilibrium configurations. We apply it to the case of constrained liquids for which the introduction of a conjugate source to the overlap field was predicted to lead to an equilibrium critical point. We show that the long-distance physics in the vicinity of this critical point is in the same universality class as that of a paradigmatic disordered model: the random-field Ising model. The quenched disorder is provided here by a reference equilibrium liquid configuration. We discuss to what extent this field-theoretical description and the mapping to the random field Ising model hold in the whole supercooled liquid regime, in particular near the glass transition.

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One of the first steps in the analysis of a standard phase transition consists in identifying the correct order parameter. Once this crucial step is done, one can first construct a Landau functional and analyze it to obtain a mean-field description; then, eventually, one can promote the order parameter to a truly fluctuating field and study the associated field theory in order to get a full-fledged theoretical description. In the case of the glass transition establishing what is the correct order parameter is by no means an easy task. The Random First-Order Transition (RFOT) theory[1,2] identifies as order parameter the similarity, also called overlap, between equilibrium configurations. More specifically, take an equilibrium (reference) configuration and then restrict the thermodynamic sampling only to equilibrium configurations constrained to have a high overlap with the reference one on the boundary of the sample. By definition, a RFOT takes place when this boundary condition forces the entire system to have a large value of the overlap instead of the zero (or very low) one characteristic of the liquid phase. This is analogous to forcing a positive boundary magnetization in the case of a ferromagnetic phase transition. The identification of this order parameter was first made in the context of disordered mean-field systems[7] but it was soon realized that the overlap and its fluctuations provide an interesting tool in the study of glassy systems, in particular supercooled liquids, irrespective of the presence or not of quenched disorder[4]. Studying the response to perturbations directly acting on the order parameter, as done in usual phase transitions, has recently allowed an access to the growing static length accompanying the slowing down of the dynamics. In the RFOT context the perturbation is a pinning field that forces configurations to have a high overlap locally; the corresponding correlation length is called "point-to-set".[5] As discussed before, promoting the order parameter to a fully fluctuating field is a way to study fluctuations and correlations beyond mean-field theory: this was recently done to analyze the slow but intermediate β relaxation in the vicinity of the "ideal" dynamical transition that is found both in the mode-coupling approach and in the RFOT theory of liquids.[6,7] In particular, it has been shown that the dynamical transition is in the universality class of the spinodal of the random-field Ising model (RFIM). Both singularities of course can only be present when activated events such as nucleation are not taken into account. Finally, the overlap has also been the focus of an intense numerical research in model supercooled liquids: the distribution of the fluctuations of the uniform overlap between equilibrium configurations has been computed and found to develop a nontrivial, non-Gaussian, shape as one cools the liquid.[8–10]

The aim of our work is to develop an effective field theory of glass-forming systems directly formulated in terms of an overlap field. This is highly desirable for several reasons: first, it allows one to focus directly on what is thought to be the physically relevant field; second, it leads to a scalar field theory in the presence of quenched disorder and should therefore settle the recurrent debate about whether the glass transition is related to random-field, random-bond or spin-glass physics.[2,11,12] Our approach is able to capture nonperturbative effects conjectured to be crucial to describe the glass transition: by perturbatively integrating out irrelevant degrees of freedom only, we derive an effective theory for the relevant field, identified as the the overlap with a reference equilibrium configuration. The model (or field theory) obtained by this procedure can then be nonperturbatively studied

either by renormalization-group analysis or by computer simulations.

In the following we first introduce our method in a general setting for glass-forming liquids. Then, we apply it to the *critical point* that terminates the transition line in an extended phase diagram where one introduces a coupling between liquid configurations and we show that the critical behavior is the same as in the equilibrium RFIM. The motivation for studying this specific region of parameters, besides providing a first progress towards a comprehensive field theory of the glass transition, stems from recent numerical works[8–10] that have directly focused on the behavior of supercooled liquids in the presence of such an attractive coupling and have provided evidence for a first-order transition line and a terminal critical point. In consequence, our predictions are prone to direct tests in the future. (Note that the analysis of Ref. [13] does not apply here since, in the presence of a nonzero coupling, the first-order transition line is *not* a RFOT and the spinodal is *not* equivalent to a dynamical mode-coupling-like transition.[14])

Consider a glass-forming liquid formed by N particles and described by a Hamiltonian $H[\mathbf{r^N}]$ where $\mathbf{r^N}$ denotes a configuration of the N particles. We consider a reference equilibrium configuration $\mathbf{r_0^N}$ and define the overlap at point x between the latter and another configuration $\mathbf{r^N}$ as $\hat{q}_x[\mathbf{r^N},\mathbf{r_0^N}] := \int_{\mathbf{y}} \mathbf{f(y)}[\hat{\rho}(\mathbf{x}+\frac{\mathbf{y}}{\mathbf{2}}|\mathbf{r^N})\hat{\rho}(\mathbf{x}-\frac{\mathbf{y}}{\mathbf{2}}|\mathbf{r_0^N})-\rho^{\mathbf{2}}]$, where $\int_y \equiv \int d^D y$ and $f(y)$ is a smoothing function of short range (typically the cage size); $\rho(x|\mathbf{r^N}) = \sum_{\mathbf{i=1}}^{\mathbf{N}} \delta^{(\mathbf{D})}(\mathbf{x}-\mathbf{r_i})$ is the microscopic density at point x and ρ is the mean liquid density. We can now define an overlap field $p(x)$ and introduce an effective Hamiltonian or action for this field,

$$S[p|\mathbf{r_0^N}] = -\log \int \frac{d\mathbf{r^N}}{N!}\delta[p-\hat{q}[\mathbf{r^N},\mathbf{r_0^N}]]e^{-\beta H[\mathbf{r^N}]} \qquad (1)$$

where $\delta[\ ]$ is a functional that enforces a delta function at each point x and $\beta = 1/(k_B T)$. The probability to observe a certain profile of the density field is given by $\exp(-S[p|\mathbf{r_0^N}])$. Thermodynamic quantities and correlation functions of the overlap field, e.g. point-to-set ones, are obtained as usual from a "partition function" and the associated functional W,

$$e^{W[\epsilon|\mathbf{r_0^N}]} = \int \mathcal{D}p\, e^{-S[p|\mathbf{r_0^N}]+\int d^D x \epsilon(x)p(x)}, \qquad (2)$$

where we have introduced an auxiliary coupling $\epsilon(x)$ that, in field-theoretical language, plays the role of a "source" for generating the connected correlation functions of the overlap field. A RFOT corresponds to the appearance for $\epsilon = 0$ of long-range order in the overlap field, which acquires a large value in the entire sample. Because of the reference configuration, $\mathbf{r_0^N}$, the action S describes a scalar field theory in the presence of quenched disorder. In order to analyze it and understand in more detail what kind of disorder is generated by $\mathbf{r_0^N}$ one can study the cumulants of S by considering replicas of the original system. As known in the context of the critical behavior of the RFIM,[4,5] (see also Refs. [6,8]) $\exp(-S_{rep}[\{p_a\}]) = \overline{\exp(-\sum_{a=1}^n S[p_a|\mathbf{r_0^N}])}$ generates the cumulants of the action $S[p|\mathbf{r_0^N}]$ through an expansion in increasing number of free replica sums:[4,5]

$$\begin{aligned} S_{rep}[\{p_a\}] = &\sum_{a=1}^n S_1[p_a] - \frac{1}{2}\sum_{a,b=1}^n S_2[p_a,p_b] \\ &+ \frac{1}{3!}\sum_{abc=1}^n S_3[p_a,p_b,p_c] + \cdots \end{aligned} \qquad (3)$$

where $S_l[p_1,\cdots,p_l]$ is the lth cumulant: *e.g.*, $\overline{S[p|\mathbf{r_0^N}]} = S_1[p]$ and $\overline{S[p_1|\mathbf{r_0^N}]S[p_2|\mathbf{r_0^N}]} - \overline{S[p_1|\mathbf{r_0^N}]}\,\overline{S[p_2|\mathbf{r_0^N}]} = S_2[p_1,p_2]$. The Franz-Parisi potential,[7] which is the average free-energy cost to keep two configurations at a fixed global overlap, is the Legendre transform of the first cumulant of $W[\epsilon|\mathbf{r_0^N}]$ (and is not equal to S_1 except in the mean-field limit).

Our goal is to derive the action for the overlap field and its cumulants in glass-forming systems. To proceed, one can formally rewrite

$$\begin{aligned} e^{-S_{rep}[\{p_a\}]} \propto &\int d\mathbf{r_0^N}\prod_{a=1}^n d\mathbf{r_a^N}\delta[p_a-\hat{q}[\mathbf{r_a^N},\mathbf{r_0^N}]] \\ &\times e^{-\beta(H[\mathbf{r_0^N}]+\sum_{a=1}^n H[\mathbf{r_a^N}])}. \end{aligned} \qquad (4)$$

The n replicas plus the reference configuration can now be described by Greek letters $\alpha = 0,1,\cdots,n$ whereas Roman letters are still used for replicas from 1 to n only. We also introduce additional collective fields $q_{\alpha\beta}(x)$ that describe the overlap between two different replicas α and β. Eq. (4) then becomes

$$\begin{aligned} e^{-S_{rep}[\{p_a\}]} \propto &\int \prod_{ab\neq}\mathcal{D}q_{ab}\Big[\int\prod_{\alpha}d\mathbf{r_\alpha^N}\prod_{\alpha\beta\neq}\delta[q_{\alpha\beta}-\hat{q}[\mathbf{r_\alpha^N},\mathbf{r_\beta^N}]] \\ &\times e^{-\beta\sum_\alpha H[\mathbf{r_\alpha^N}]}\Big] \propto \int\prod_{ab\neq}\mathcal{D}q_{ab}\, e^{-\mathcal{S}[\{p_a,q_{ab}\}]} \end{aligned} \qquad (5)$$

where we have used the notation $q_{0a} = q_{a0} \equiv p_a$ and $S[\{p_a,q_{ab}\}]$ is defined as minus the logarithm of the expression between parethensis. Our approach differs from the usual replica one in that it treats the fields $p_a = q_{0a}$ and q_{ab} differently. The underlying working hypothesis, which can at least be checked in the vicinity of the terminal critical point in the presence of a nonzero coupling ϵ, is that the p_a's may develop long-range fluctuations while the q_{ab}'s are harmless for the long-distance physics and can be approximately integrated out. In the following we show how this procedure can be carried out near the terminal critical point.

One first needs a tractable expression for the action $\mathcal{S}[\{p_a,q_{ab}\}]$ in terms of the overlap fields [defined via Eq. (17)]. This action of course depends on the microscopic details of the glass-forming system under study and deriving its expression can be a rather formidable

task. One can formally derive $\mathcal{S}[\{q_{\alpha\beta}\}]$ from the Morita-Hiroike functional of the 1- and 2-particle densities[18] of the replicated $(n+1)$-component liquid mixture along lines similar to those followed by Ref. [7]. A short-cut is however provided by the coarse-grained effective Landau-like functional considered in Ref. [8]:

$$\mathcal{S}[\{q_{\alpha\beta}\}] = \frac{E_0}{k_B T}\int_x \Big\{ \frac{c}{2}\sum_{\alpha\beta\neq}(\partial q_{\alpha\beta}(x))^2 + \sum_{\alpha\beta\neq} V(q_{\alpha\beta}(x)) - \frac{u}{3}\sum_{\alpha\beta\gamma\neq} q_{\alpha\beta}(x)q_{\beta\gamma}(x)q_{\gamma\alpha}(x)\Big\}, \tag{6}$$

where $V(q) = (t/2)q^2 - [(u+w)/3]q^3 + (y/4)q^4$ and the primary dependence on temperature is given by $t \approx k_B(T-T_0)/E_0$ with E_0 the typical energy scale of the liquid and T_0 a constant with dimension of temperature. In numerical applications, we focus on the parameter values found to roughly reproduce properties of glass-forming liquid ortho-terphenyl: $u = 0.385, w = 2.73, y = 1.82$, and $c = 1$ (in appropriate length units).[8]

To derive an effective field theory for the overlaps $p_a(x)$ with a reference configuration, one needs to perform the functional integration over the $q_{ab}(x)$'s while keeping the fields $p_a(x)$ fixed [see Eq. (17)]. It is clear from Eq. (6) that, as discussed above, nonzero p_a's exert an external source or field on the q_{ab}'s. In Eq. (6) the cubic term generates a contribution $-u\sum_{ab\neq} p_a(x)p_b(x)q_{ab}(x)$. In consequence, the q_{ab}'s do not develop fluctuations on all scales and stay "massive" near the terminal critical point.[19] It is then sufficient to perform the functional integration on these fields through a perturbative treatment. Perturbation is carried out with a saddle-point approximation as zeroth order. The saddle-point equations for the q_{ab}'s with $a \neq b$ read

$$c\,\partial^2 q_{ab*}(x) + V'(q_{ab*}(x)) = up_a(x)p_b(x) + u\sum_{c\neq a,b} q_{ac*}(x)q_{cb^*}(x)\,. \tag{7}$$

The solution of this equation has to be inserted back into the action in order to obtain the final result. Since we are interested in the long wave-length fluctuations of the p_a's it is sufficient to solve the saddle-point equations in an expansion in the gradient term, the zeroth order then corresponding to simply neglecting the gradient.

All quantities can be expanded in increasing number of free replica sums as in Eq. (3), *e.g.*,

$$q_{ab*}(x) = q_x^{[0]}[p_a,p_b] + \sum_c q_x^{[1]}[p_a,p_b|p_c] + \mathcal{O}(\sum_{cd})\,. \tag{8}$$

Such expansions allow algebraic manipulations that lead to well defined and unique expressions of the various orders[4–6]. In the lowest order in the gradient amplitude parameter c, Eqs. (22) and (19) lead for instance to $q_x^{[0]}[p_a,p_b] \equiv q^{[0]}(p_a(x),p_b(x))$ and

$$V'(q^{[0]}(p_a,p_b)) = up_ap_b - uq^{[0]}(p_a,p_b)[q^{[0]}(p_a,p_a)+q^{[0]}(p_b,p_b)]\,. \tag{9}$$

With the above results, one immediately derives from Eqs. (3) and (17) the expressions of the first two cumulants of the action for the p_a's at the level of the saddle-point approximation and including $\mathcal{O}(\partial^2)$ terms only:

$$S_1[p_a] = \frac{E_0}{k_B T}\int_x \Big\{ c(\partial p_a)^2 - \frac{1}{2}c(\partial q^{[0]}(p_a,p_a))^2 + 2V(p_a) - V(q^{[0]}(p_a,p_a)) - \frac{2u}{3}q^{[0]}(p_a,p_a)^3 + up_a^2 q^{[0]}(p_a,p_a)\Big\} \tag{10}$$

and

$$S_2[p_a,p_b] = -\frac{E_0}{k_B T}\int_x \Big\{ \frac{c}{2}(\partial q^{[0]}(p_a,p_b))^2 - 2up_ap_bq^{[0]}(p_a,p_b) + u\left[q^{[0]}(p_a,p_a)+q^{[0]}(p_b,p_b)\right]q^{[0]}(p_a,p_b)^2 + 2V(q^{[0]}(p_a,p_b))\Big\} \tag{11}$$

where the explicit x-dependence has been omitted. This derivation is easily extended to the higher orders but the algebra rapidly becomes tedious and the results are given in the supplementary information (SI).

The above cumulants describe a scalar field theory for a disordered system.[4–6,6] We now more specifically consider the vicinity of the terminal critical point in the (T,ϵ) plane. At the saddle-point level, the critical point is defined by the following conditions:

$$\left.\frac{\partial S_1(p_1)}{\partial p_1}\right|_c = \epsilon_c\,, \left.\frac{\partial^2 S_1(p_1)}{\partial p_1^2}\right|_c = 0\,, \left.\frac{\partial^3 S_1(p_1)}{\partial p_1^3}\right|_c = 0\,. \tag{12}$$

When inserted in Eq. (10), this gives $\epsilon_c = 0.602\,E_0/(k_B T_c)$, $t_c = 1.680$, $p_c = 0.534$ and $q_c^{[0]} = 0.072$ (using the relation[8] between t and T one finds $E_0/(k_B T_c) = 0.952$). One can then expand the cumulants $S_l[p_1,\cdots,p_l]$ around the critical value after defining $\phi_a(x) = p_a(x) - p_c$. Keeping only terms to order ϕ^4 (higher-order terms are expected to be irrelevant at criticality) leads to a Wilson-Ginzburg-Landau action for the replica scalar fields $\phi_a(x)$:

$$S_{rep}[\{\phi_a\}] - \epsilon_c\int_x \phi_a(x) = \int_x \Big\{ \sum_a \Big[c'(\partial\phi_a)^2 + \frac{r_2}{2}\phi_a^2 + \frac{r_3}{3!}\phi_a^3 + \frac{r_4}{4!}\phi_a^4\Big] - \frac{1}{2}\sum_{ab}\phi_a\phi_b\Big[\Delta_{20} + \frac{\Delta_{21}}{2}(\phi_a+\phi_b) + \frac{\Delta_{22}}{4}\phi_a\phi_b + \frac{\Delta_{23}}{2}(\phi_a^2+\phi_b^2)\Big] + \frac{1}{3!}\sum_{abc}\phi_a\phi_b\phi_c\Big[\Delta_{30} + \frac{\Delta_{31}}{3}\times(\phi_a+\phi_b+\phi_c)\Big] - \frac{\Delta_{40}}{4!}\sum_{abcd}\phi_a\phi_b\phi_c\phi_d + \mathcal{O}(\sum_{abcde})\Big\} \tag{13}$$

where r_2 and r_3 are zero at the saddle-point (mean-field) critical point and all other coefficients are evaluated at the latter point: $c' = 0.91$, $r_4 = 23.33$, $\Delta_{20} = 0.22$, $\Delta_{21} = 0.47$, $\Delta_{22} = 1.23$, $\Delta_{23} = 0.04$, $\Delta_{30} = -0.12$, $\Delta_{31} = -0.51$, $\Delta_{40} = 0.33$. We have neglected some square gradient terms of higher orders in the fields because they are either less relevant in the RG sense or are anyhow

generated along the RG flow. Beyond the precise values, the important fact is that r_4, Δ_{20}, Δ_{22}, and Δ_{40} are greater than zero. It is then easily inferred that the above replicated action is obtained from a random one, $S[\phi|h, \delta r_2, \delta r_3]$, which describes the large-scale fluctuations of the overlap field $p(x)$ with a reference equilibrium configuration:

$$S[\phi|h,\delta r_2,\delta r_3] = \int_x \left\{ c'[\partial\phi(x)]^2 + \frac{r_2}{2}\phi(x)^2 + \frac{r_3}{3!}\phi(x)^3 + \frac{r_4}{4!}\phi(x)^4 \right\} + \int_x \left\{ \frac{\delta r_2(x)}{2}\phi(x)^2 + \frac{\delta r_3(x)}{3!}\phi(x)^3 - h(x)\phi(x) \right\}$$

where the random field $h(x)$, the random mass $\delta r_2(x)$, the random cubic coupling $\delta r_3(x)$ are delta-correlated processes with zero mean and higher cumulants given by

$$\overline{h(x)h(x')} = \delta_{xx'}\Delta_{20}\,, \overline{h(x)h(x')h(x'')} = \delta_{xx'x''}\Delta_{30}\,,$$
$$\overline{h(x)h(x')h(x'')h(x''')}|_{cum} = \delta_{xx'x''x'''}\Delta_{40}\,,$$
$$\overline{\delta r_2(x)\delta r_2(x')} = \delta_{xx'}\Delta_{22}\,, \overline{h(x)\delta r_2(x')} = -\delta_{xx'}\Delta_{21}\,,$$
$$\overline{h(x)h(x')\delta r_2(x'')} = \delta_{xx'x''}\Delta_{31}\,, \overline{h(x)\delta r_3(x')} = -\delta_{xx'}\Delta_{23}\,,$$

etc, with $\delta_{xx'}, \delta_{xx'x''}, \cdots$ short-hand notations for delta functions and products of delta functions. The resulting theory is thus a ϕ^4 one in the presence of quenched disorder without statistical inversion (Z_2) symmetry. As is known from the theory of disordered systems,[11] the random field is the most relevant of the above terms at criticality and it leads to a universality class controlled by a nontrivial zero-temperature fixed point. This is so even in the absence of Z_2 symmetry, as can be found from a simple Harris-like criterion and has recently been confirmed by a full nonperturbative RG analysis.[22]

The conclusion of our analysis is that the terminal critical point found in the presence of a conjugate source within the mean-field theory belongs in finite dimension to the universality class of the RFIM. Of course this is valid *if the transition is not destroyed by the disorder*. One therefore needs to compare the "bare" strength of the random field, $\sqrt{\Delta}$, obtained from S_2 to that of the surface tension Y obtained from S_1. We have computed the latter as the free energy cost per unit surface between two regions with high and low overlap[1,9] at coexistence far below the (mean-field) critical point and evaluated Δ at the same temperature, see SI for more details. The output is $\sqrt{\Delta}/Y \simeq 0.47$, which from known numerical results on the RFIM in $d = 3$ (see also SI) is compatible with the existence of a transition.[11] We have repeated the whole analysis for the finite dimensional 3-spin model with weak long-range interactions (see SI) and obtained that the mapping to the RFIM also holds. However, in this case, we have found $\sqrt{\Delta}/Y \simeq 2.24$, a value likely too large for a transition to survive in $d = 3$, in agreement with results and arguments presented in Refs. [25,26].

The Landau-like functional for glass-forming liquids[8,11] used as initial input in the present field-theoretical approach is just a crude approximation. Deriving from first principles a proper starting point with effective parameters that incorporate the microscopic information about glass-formers is then a crucial task. As illustrated by recent numerical work,[9,10] this is now within reach. Indeed, by constraining the overlap to a fixed value in a *small-size system* (less than the point-to-set length), it should be possible to measure the local part of the first cumulants S_1 and S_2 (see above), thus allowing a direct evaluation of the *bare magnitude* of the different sources of quenched disorder in model supercooled liquids.

An important question for the theory of glass-forming systems is to what extent the mapping to the RFIM found near the terminal critical point in the presence of a nonzero coupling ϵ is general? We have verified that it also holds for the transition line in the temperature-$\epsilon > 0$ plane, which is therefore a first-order transition in the presence of a random field. Results of Refs. [13,14] suggest that it also applies for the continuous glass transition taking place at the terminal point in the phase diagram of pinned systems[27]. The more interesting and physical case corresponds to the situation in the absence of coupling ϵ, where an ideal glass transition of RFOT type is predicted at the mean-field level. Although the mapping then holds for completely connected models (one can *e.g.* show that the Random Energy Model[28] maps exactly to a zero-dimensional RFIM), its extension to finite-dimensional systems is not straightforward and it will be the focus of a future publication. Whether the physics of the glass transition is related to the RFIM[2,11], to a spin glass in a field[12] or even to a different kind of universality class still remains an open question, but its answer seems now within reach.

Acknowledgments

After completion of our work we came to know that S. Franz and G. Parisi have also addressed the problem of the critical point of constrained glassy systems (arXiv:1307.4955). Their approach is different but leads to the same conclusion regarding the universality class of the terminal point in the T-ϵ plane.We acknowledge support from the ERC grants NPRGGLASS and (CC) CRIPHERASY (no. 247328).

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Supplementary information

I. CALCULATION OF THE CUMULANTS OF THE EFFECTIVE ACTION FOR THE OVERLAP FIELD

As explained in the main text, the central quantity is the effective Hamiltonian or action $S[p|\mathbf{r_0^N}]$ describing the configurations of a glass-forming system with an overlap $p(x)$ with a reference equilibrium configuration denoted $\mathbf{r_0^N}$. Our aim is to compute the cumulants of this action, which are generated according to

$$\exp(-S_{rep}[\{p_a\}]) = \overline{\exp(-\sum_{a=1}^{n} S[p_a|\mathbf{r_0^N}])} \tag{14}$$

through an expansion in increasing number of free replica sums of the "replicated action" $S_{rep}[\{p_a\}]$:

$$S_{rep}[\{p_a\}] = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l!} \sum_{a_1 \cdots a_l=1}^{n} S_l[p_{a_1}, \cdots, p_{a_l}], \tag{15}$$

where $S_l[p_1, \cdots, p_l] = \overline{S[p_1|\mathbf{r}_0^N] \cdots S[p_l|\mathbf{r}_0^N]}\big|_{cum}$.

For the sake of concreteness, we show the details of the calculation for the p-spin models in the Kac limit[1,2] with $p \geq 3$ [p should not be confused with the overlap field $p(x)$]. All the steps of the derivation can be similarly repeated for the Landau-like functional given in Eq. (6) of the main text. The action $\mathcal{S}[\{p_a, q_{ab}\}]$ that appears in Eq. (5) of the main text and is the starting point of the computations can then be obtained by standard methods and reads in the case of the p-spin Kac model[1,2]

$$\mathcal{S}[\{q_{\alpha\beta}\}] = \int_x \Big\{ \frac{c}{2} \sum_{\alpha\beta\neq} (\partial q_{\alpha\beta}(x))^2 - \frac{\beta^2}{4} \sum_{\alpha\beta\neq} q_{\alpha\beta}(x)^p - \frac{1}{2} \mathrm{Tr} \log[I + U(\{q_{\alpha\beta}(x)\})] \Big\}, \tag{16}$$

where I is the identity and U an $(n+1)\times(n+1)$ matrix with all diagonal elements equal to 0, $U_{0a} = q_{0a} = p_a$, $U_{a0} = q_{a0} = p_a$, and $U_{ab} = q_{ab}$ for $a \neq b$. (we recall that Greek letters are use for the $n+1$ copies of the original system, including the reference configuration $\alpha = 0$, whereas Latin ones are reserved for the n replicas other than the reference one.).[3] The action for the overlaps p_a with the reference configuration is then obtained by integrating out the overlaps q_{ab}:

$$e^{-S_{rep}[\{p_a\}]} \propto \int \prod_{ab\neq} \mathcal{D}q_{ab}\, e^{-\mathcal{S}[\{p_a, q_{ab}\}]}. \tag{17}$$

In the Kac limit, the gradient term in Eq. (16) can be considered as a perturbation and the integral over the q_{ab}'s can be computed through a saddle-point approximation. In the following we therefore only detail the computation of the local part of $S_{rep}[\{p_a\}]$ that is obtained via the saddle-point. The gradient terms can then be trivially added.

When dropping the gradient term and considering spatially uniform overlaps p_a's, the saddle-point equations for the q_{ab}'s with $a \neq b$ read

$$p\frac{\beta^2}{4} q_{ab*}^{p-1} = -P_{ab}(\{p_c, q_{cd*}\}) \tag{18}$$

where, in matrix form, $\mathbf{P} = (\mathbf{I}+\mathbf{U})^{-1}$. All quantities can be expanded in increasing number of free replica sums. The saddle-point solution can be expanded as

$$q_{ab*} = q^{[0]}(p_a, p_b) + \sum_c q^{[1]}(p_a, p_b|p_c) + \mathcal{O}(\sum_{cd}) \tag{19}$$

and, similarly, $P_{ab}(\{p_c, q_{cd*}\}) = \widehat{P}_{a*}(\{p_c\})\delta_{ab} + \widetilde{P}_{ab*}(\{p_c\})$ with

$$\begin{aligned} \widehat{P}_{a*}(\{p_c\}) &= \widehat{P}^{[0]}(p_a) + \sum_c \widehat{P}^{[1]}(p_a, p_b|p_c) + \cdots \\ \widetilde{P}_{ab*}(\{p_c\}) &= \widetilde{P}^{[0]}(p_a, p_b) + \sum_c \widetilde{P}^{[1]}(p_a, p_b|p_c) + \cdots \end{aligned} \tag{20}$$

Such expansions in increasing number of free replica sums allow algebraic manipulations that lead to well defined and unique expressions of the various orders.[4–6]

The algebra however rapidly becomes cumbersome. An efficient way to proceed, which allows the use of a symbolic software like Mathematica, is then as follows. For the calculation of the k-th cumulant of the action, one divides the n replicas into k groups of $n_1, \ldots, n_k$ replicas having the same overlap $p_1, \ldots, p_k$ with the reference configuration. One then computes the replicated action $S_{rep}[p_1, \ldots, p_k]$ corresponding to this ansatz and evaluate the term of order $n_1 \cdots n_k$. In the following we detail how this calculation works up to the third cumulant.

Let us start with the first cumulant. We thus consider only one group of replicas, namely all n replicas have the same overlap p_1 with the reference configuration. As mentioned above, we focus on the local part of the cumulant and therefore only consider a spatially uniform overlap field p_1. Inserting this ansatz into Eq. (15), we obtain

$$S_{rep}[p_1] = nS_1[p_1] - \frac{n^2}{2} S_2[p_1, p_1] + \frac{n^3}{3!} S_3[p_1, p_1, p_1] + \ldots \tag{21}$$

whereas the saddle-point equation for the overlap q_1 between two replicas having overlap p_1 with the reference configuration [see Eq. (22)] becomes

$$p\frac{\beta^2}{4} q_{1*}^{p-1} = \frac{q_{1*} - p_1^2}{(1-q_{1*})(1-q_{1*}+nq_{1*}-np_1^2)}. \tag{22}$$

As in the conventional replica trick, the first cumulant of the action is simply given by the term of order n of $S_{rep}[p_1]$, which can be obtained as $\partial_n S_{rep}[p_1]|_{n=0}$. This

object, which only in the mean-field limit coincides with the standard mean-field Franz-Parisi potential,[7] can be computed in a straightforward way,

$$L^{-d}S_1(p_1) = -\frac{\beta^2}{4}\left(2p_1^p - q_1^p\right) + \frac{p_1^2 - q_1}{2(1-q_1)} - \frac{1}{2}\log(1-q_1), \tag{23}$$

where L^d is the volume of the system and q_1 is given by the saddle point equation, Eq. (22), when $n \to 0$ (for ease of notation, we drop the star that denotes the saddle-point solution in what follows):

$$\frac{\beta^2 p}{4}[q_1^{(0)}]^{p-1} = \frac{1}{2}\,\frac{q_1^{(0)} - p_1^2}{(1-q_1^{(0)})^2}\,. \tag{24}$$

In order to compute the second cumulant of the action, we now consider two groups of replicas, n_1 replicas having an overlap p_1 with the reference configuration, and n_2 replicas having overlap p_2 with the reference configuration; q_1 denotes the overlap among replicas of the first group, q_2 that among replicas of the second group, and q_{12} the overlap between a replica of the first group and one of the second group. With this ansatz, Eq. (15) becomes

$$\begin{aligned} &S_{rep}[p_1,p_2] = n_1 S_1[p_1] + n_2 S_1[p_2] \\ &- \frac{1}{2}\left(n_1^2 S_2[p_1,p_1] + n_2^2 S_2[p_2,p_2] + 2n_1 n_2 S_2[p_1,p_2]\right) + \dots \end{aligned} \tag{25}$$

As a result, $S_2[p_1,p_2]$ is simply obtained from the term of order $n_1 n_2$ of S_{rep} and is given by $-\partial_{n_1,n_2} S_{rep}[p_1,p_2]|_{n_1=n_2=0}$. The saddle-point solutions can also be expanded in powers of n_1 and n_2, *e.g.*,

$$\begin{aligned} q_1 &= q_1^{(0)} + n_1 \partial_{n_1} q_1|_0 + n_2 \partial_{n_2} q_1|_0 + \dots \\ &= q_1^{(0)} + n_1 q_1^{(10)} + n_2 q_1^{(01)} + \dots\,, \end{aligned} \tag{26}$$

where we have introduced a short-hand notation for the derivative of the overlaps with respect to the numbers of replicas (taken in the limit $n_1 = n_2 = 0$). From the saddle-point equation for q_1, one has for instance

$$\begin{aligned} &\left[p(p-1)\frac{\beta^2}{4}[q_1^{(0)}]^{p-2} - \frac{1-2p_1^2+q_1^{(0)}}{2(1-q_1^{(0)})^3}\right] q_1^{(01)} = \\ &\qquad - \frac{(q_{12}^{(0)} - p_1 p_2)^2}{2(1-q_1^{(0)})^2(1-q_2^{(0)})}\,. \end{aligned} \tag{27}$$

Contributions to $-\partial_{n_1,n_2} S_{rep}[p_1,p_2]|_{n_1=n_2=0}$ therefore come from the term directly proportional to $n_1 n_2$ in Eq. (25) but also from terms that are explicitly linear in n_1 or n_2 but involve q_1 and/or q_2 that themselves carry a dependence on n_1 and n_2 at the saddle-point [see Eq. (26)].

After some manipulations, one then finds

$$L^{-d}S_2[p_1,p_2] = \frac{\beta^2}{2}[q_{12}^{(0)}]^p - \frac{(q_{12}^{(0)} - p_1 p_2)^2}{2(1-q_1^{(0)})(1-q_2^{(0)})}, \tag{28}$$

where $q_{12}^{(0)}$ satisfies the saddle-point equation at this order,

$$\frac{\beta^2 p}{2}[q_{12}^{(0)}]^{p-1} = \frac{q_{12}^{(0)} - p_1 p_2}{(1-q_1^{(0)})(1-q_2^{(0)})}\,, \tag{29}$$

and the contributions involving the first-order derivatives of q_1 and q_2 (see above) exactly cancel due to the saddle-point condition.

For computing the third cumulant, we divide the n replicas in three groups of n_1, n_2, and n_3 replicas, having an overlap with the reference configuration equal to p_1, p_2, and p_3, respectively; q_1, q_2, q_3 are the overlaps inside a given group and q_{12}, q_{13}, q_{23} the overlaps among replicas belonging to distinct groups. As for the first and second cumulants, $S_3[p_1,p_2,p_3]$ is simply given by the term of order $n_1 n_2 n_3$ of S_{rep}, and can be obtained as $\partial_{n_1,n_2,n_3} S_{rep}[p_1,p_2,p_3]|_{n_1=n_2=n_3=0}$. There are several contributions to this term, coming from the formal expression of $\mathcal{S}[\{p_a, q_{ab}\}]$ when three groups of replicas are introduced (considering again uniform overlaps):

$$\begin{aligned} &L^{-d}\mathcal{S}[\{p_1,p_2,p_3,q_1,q_2,q_3,q_{12},q_{13},q_{23}\}] = \\ &\sum_{a=1}^{3} n_a A_a - \frac{1}{2}\sum_{a,b=1}^{3} n_a n_b B_{ab} + \frac{1}{3!}\sum_{a,b,c=1}^{3} n_a n_b n_c C_{abc} + \dots \end{aligned} \tag{30}$$

where $A_a \equiv A(p_a, q_a)$, $B_{ab} \equiv B(p_a, p_b, q_a, q_b, q_{ab})$, and $C_{abc} \equiv C(p_a, p_b, p_c, q_a, q_b, q_c, q_{ab}, q_{bc}, q_{ca})$ (with the convention $q_{aa} = q_a$ when $b = a$).

The simplest term is that having an explicit dependence on $n_1 n_2 n_3$. It is given by

$$\begin{aligned} &C(p_1,p_2,p_3,q_1,q_2,q_3,q_{12},q_{13},q_{23}) \\ &\qquad = -\frac{(q_{12}-p_1p_2)(q_{13}-p_1p_3)(q_{23}-p_3p_3)}{(1-q_1)(1-q_2)(1-q_3)}\,, \end{aligned} \tag{31}$$

where all overlaps can be taken at the lowest order, *i.e.*, $q_1 = q_1^{(0)}$, $q_{12} = q_{12}^{(0)}$, etc.

The second contribution is given by the expansion of the terms of order $n_1 n_2$, $n_1 n_3$, and $n_2 n_3$ of S_{rep} to first order in, respectively, n_3, n_2, and n_1 around the saddle-point values of the overlaps:

$$\begin{aligned} &-\left(\partial_{q_1} B_{12}\right) q_1^{(001)} - \left(\partial_{q_2} B_{12}\right) q_2^{(001)} - \left(\partial_{q_{12}} B_{12}\right) q_{12}^{(001)} \\ &-\left(\partial_{q_1} B_{13}\right) q_1^{(010)} - \left(\partial_{q_3} B_{13}\right) q_2^{(010)} - \left(\partial_{q_{13}} B_{13}\right) q_{13}^{(010)} \\ &-\left(\partial_{q_2} B_{23}\right) q_2^{(100)} - \left(\partial_{q_3} B_{23}\right) q_3^{(100)} - \left(\partial_{q_{23}} B_{23}\right) q_{23}^{(100)}, \end{aligned} \tag{32}$$

where the derivatives are taken for $n_1 = n_2 = n_3 = 0$ and the notation $q_1^{(001)}$, $q_{12}^{(001)}$, etc, is an obvious generalization to three groups of replicas of that introduced above for two groups (and are therefore given by expressions similar to Eq. (27). B_{ab} is expressed as

$$B(p_a,p_b,q_a,q_b,q_{ab}) = \frac{\beta^2}{2} q_{ab}^p - \frac{(q_{ab}-p_a p_a)^2}{2(1-q_a)(1-q_b)}\,. \tag{33}$$

The third and last contribution comes from the expansion of the terms of order n_1, n_2, and n_3 of S_{rep} up to second order in, respectively, $n_2 n_3$, $n_1 n_3$, and $n_1 n_2$ around the saddle-point values of the overlaps:

$$\begin{aligned}&(\partial_{q_1} A_1)\, q_1^{(011)} + (\partial_{q_2} A_2)\, q_2^{(101)} + (\partial_{q_3} A_3)\, q_3^{(110)} \\ &+ \left(\partial^2_{q_1} A_1\right) q_1^{(010)} q_1^{(001)} + \left(\partial^2_{q_2} A_2\right) q_2^{(100)} q_2^{(001)} \\ &+ \left(\partial^2_{q_3} A_3\right) q_3^{(100)} q_3^{(010)}.\end{aligned} \tag{34}$$

After combining the three contributions and using the the saddle-point equations for the overlaps, Eq. (24) and Eq. (29), several terms cancel out and the expression of the third cumulants simplifies to

$$\begin{aligned}S_3[p_1, p_2, p_3] = C^{(0)}_{123} &- (\partial_{q_2} B_{12})\, q_2^{(001)} \\ &- (\partial_{q_1} B_{13})\, q_1^{(010)} - (\partial_{q_3} B_{23})\, q_3^{(100)}\,,\end{aligned} \tag{35}$$

where $C^{(0)}_{123}$ is given by Eq. (31) with all overlaps given by their zeroth-order expression, B_{ab} is given by Eq. (33), and $q_1^{(010)}$ and related terms are given by expressions similar to Eq. (27).

The calculation of the fourth cumulant can be carried out along similar lines by introducing 4 groups of replicas. For simplicity, we do not write down its explicit expression, which can be easily obtained with Mathematica.

With the explicit expressions of the cumulants of the action in hand, one can easily compute the values of the coefficients of the different terms at the terminal critical point of the transition line in the T-ϵ diagram. We illustrate this for the case $p = 3$. The first step is to determine the location of the terminal point at the saddle-point level. By imposing that $\partial^2_{p_1} S_1[p_1] = \partial^3_{p_1} S_1[p_1] = 0$ and using Eq (24), we find that $p_{1,c} = 0.285$, $\beta_c = 1.295$, and $q_{1,c} = q_{12,c} = 0.103$. We next expand the overlaps with the reference configuration around the terminal critical point, $p_a = p_{1,c} + \phi_a$, and compute all the coefficients of the expansion of the replicated action $S_{rep}[\{p_a\}]$ up to fourth order in the ϕ_a's. The notations are the same as in the main text and we obtain: $r_4 = 55.05, \Delta_{20} = 0.18, \Delta_{21} = 1.55, \Delta_{22} = 15.45, \Delta_{23} = 2.19, \Delta_{30} = 0.36, \Delta_{31} = 7.18, \Delta_{40} = 5.27$. As for the Landau-like functional considered in the main text, the important point is that r_4, Δ_{20}, Δ_{22}, and Δ_{40} are greater than zero. Finally, we have checked that the eigenvalues of the Hessian at the saddle point are strictly positive: $\lambda_R = 0.052$ and $\lambda_{L,A} = 0.067$. This guarantees the consistency of the whole procedure.

The procedure for the Landau-like functional considered in the main text is analogous (the computation is slightly simpler). All the results are quoted in the main text.

II. SURFACE TENSION AND BARE DISORDER STRENGTH IN THE EFFECTIVE RFIM

The conclusion of the analysis presented in the main text is that the terminal critical point found in the presence of a conjugate source within mean-field theory belongs in finite dimension to the universality class of the RFIM. This is valid *if the transition is not destroyed by the disorder*. In order to assess whether the transition exists when all fluctuations beyond mean-field theory are taken into account we have compared the "bare" strength of the random field, $\sqrt{\Delta}$, obtained from S_2 to that of the surface tension Y obtained from S_1. Clearly, the location of terminal critical point found within mean-field theory is changed by fluctuations. The question is whether it is just shifted down to lower temperature or if the transition is wiped out altogether. In the following we present the computation for the Landau-like functional.[8]

We have computed Y as the free energy cost per unit surface between two regions with high and low overlap[1,9] at coexistence far below the (mean-field) critical point. We have chosen to evaluate Y at the (mean-field) Kauzmann temperature T_K; the idea is that if the disorder is not able to renormalize Y to zero, and hence to destroy the first-order transition close to T_K, then a terminal critical point exists for sure.[10]

The computation of Y is based on a standard instanton computation[1,9]. Let us rewrite S_1 as:

$$S_1[p_a] = \frac{E_0}{k_B T} \int_x \left[c(\partial p_a(x))^2 + \tilde{V}(p_a(x)) \right]$$

where

$$\begin{aligned}\tilde{V}(p_a) \quad &= 2V(p_a) - V(q^{[0]}(p_a, p_a)) \\ &- \frac{2u}{3} q^{[0]}(p_a, p_a)^3 + u p_a^2 q^{[0]}(p_a, p_a)\end{aligned} \tag{36}$$

and, for simplicity, we have neglected the contribution $-\frac{1}{2} c (\partial q^{[0]}(p_a, p_a))^2$ to the kinetic term (at least close to the terminal critical point it leads to a very small correction). The expression for the bare surface tension obtained by an instanton computation at T_K is:

$$Y = \frac{2E_0}{k_B T_K} \sqrt{c} \int_0^{q_{ea}} \sqrt{\tilde{V}(p)}\, dp$$

where 0 and q_{ea} are the values of p corresponding to the two minima of $\tilde{V}(p)$ and $E_0/(k_B T_K) = 3.57$ for the values of the parameters chosen to mimic liquid otho-terphenyl[8] (q_{ea} is equal to 1 in this case). By computing $\tilde{V}(p)$ at T_K (this requires breaking replica symmetry for q_{ab}, see [7]) and taking $c = 1$ we have obtained $Y \simeq 1.02$. When repeating the analysis for the p-spin model with $p = 3$, we have found $Y \simeq 0.095$.

These values must be compared to the strength of the disorder, which has of course to be computed at the same temperature T_K. Because of the quenched disorder the two minima at $p = 0$ and $p = q_{ea}$ have the same height only in average. The height fluctuations can destroy the long-range order found within mean field theory even in the absence of thermal noise, as it is well known from the theory of the random field Ising model (RFIM).[11] In order to evaluate the strength of these fluctuations we

compute for uniform values of the overlap field

$$\Delta = \frac{1}{4} L^{-d} \overline{(S[0|\mathbf{r_0^N}] - S[q_{ea}|\mathbf{r_0^N}])^2} \, .$$

When the overlap field reduces to an Ising variable, *i.e.* it can just take the two values 0 and q_{ea}, which seems a reasonable approximation at T_K, the above expression indeed coincides with the variance of the random field. By using the formalism discussed in the main text we find

$$\Delta = \frac{1}{4} L^{-d} (S_2[0,0] + S_2[q_{ea}, q_{ea}] - 2S_2[0, q_{ea}]) \, .$$

This expression further simplifies further since $S_2[0,0] = S[0, q_{ea}] = 0$. A simple computation then leads to $\Delta = \frac{E_0}{k_B T_K} u q_{ea}^3/6 \simeq 0.23$. (recall that $q_{ea} = 1$ in this case). Repeating the analysis for the 3-spin model then gives $\Delta = q_{ea}^2(2/(3-3q_{ea})-1)/8 \simeq 0.045$ ($q_{ea} = 0.644$ in this case).

From the above values we derive the ratios $\sqrt{\Delta}/Y \simeq 0.47$ for the Landau functional and $\sqrt{\Delta}/Y \simeq 2.24$ for the 3-spin model quoted in the main text. The critical value of the disorder for the RFIM on a cubic lattice is $\sqrt{\Delta}/J|_c \simeq 2.3$, where J is the coupling interaction between spins.[11,12] If the glass-forming system at T_K can be considered as a hard Ising spin model, then Y is twice the coupling J. A transition is therefore expected for the Landau-like functional for ortho-terphenyl but not for the 3-spin model.

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