Characterizing flow fluctuations with moments

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We present a complete set of multiparticle correlation observables for ultrarelativistic heavy-ion collisions. These include moments of the distribution of the anisotropic flow in a single harmonic, and also mixed moments, which contain the information on correlations between event planes of different harmonics. We explain how all these moments can be measured using just two symmetric subevents separated by a rapidity gap. This presents a multi-pronged probe of the physics of flow fluctuations. For instance, it allows to test the hypothesis that event-plane correlations are generated by non-linear hydrodynamic response. We illustrate the method with simulations of events in A MultiPhase Transport (AMPT) model.

I. INTRODUCTION

Large anisotropic flow has been observed in ultra-relativistic nucleus-nucleus collisions at the Relativistic Heavy-Ion Collider (RHIC) and the Large Hadron Collider (LHC) [1]. Anisotropic flow is an azimuthal (φ) asymmetry of the single-particle distribution [2]:

$$P(\varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} V_n e^{-in\varphi}, \qquad (1)$$

where V_n is the (complex) anisotropic flow coefficient in the nth harmonic. One usually uses the notation v_n for the magnitude: $v_n \equiv |V_n|$. Anisotropic flow is understood as the hydrodynamic response to spatial deformation of the initial density profile. This profile fluctuates event to event, which implies that the flow also fluctuates [3, 4]. The recognition of the importance of flow fluctuations has led to a wealth of new flow observables, among which triangular flow [5] and higher harmonics, as well as correlations between different Fourier harmonics [6].

Flow fluctuations provide a window [7] into both the early stage dynamics and the transport properties of the quark-gluon plasma. Specifically, the magnitudes of higher-order harmonics (V_3 to V_6) are increasingly sensitive to the shear viscosity to entropy density ratio [8]. The distributions of V_2 and V_3 carry detailed information about the initial density profile [9, 10], while V_4 and higher harmonics are understood as superpositions of linear and nonlinear responses, through which they are correlated with lower-order harmonics [11, 12]. Ideally, one would like to measure the full probability distribution $p(V_1, V_2, \cdots, V_n)$ [13]. So far, only limited information has been obtained, concerning either the distribution of a single V_n [14] or specific angular correlations between different harmonics [6].

We propose to study the distribution $p(V_1, V_2, \cdots, V_n)$ via its moments in various harmonics [15, 16], either single or mixed, and illustrate our point with realistic simulations using the AMPT model [17]. In Sec. II, we recall how moments can be measured simply with a single rapidity gap [18]. This procedure is less demanding in terms of detector acceptance than the one based on several rapidity windows separated pairwise by gaps [6], and can be used to study even four-plane correlators. In Sec. III, we list standard measures of flow fluctuations which have been used in the literature and express them in terms of moments. In Sec. IV, we introduce new observables which shed additional light on the origin of event-plane correlations. For instance, a correlation between $(V_2)^2$ and V_4 has been observed, which increases with impact parameter [6]. This correlation is usually understood [19] as an effect of the non-linear hydrodynamic response which creates a V_4 proportional to $(V_2)^2$ [11, 20, 21]: the increase in the correlation is thus assumed to result from the increase of elliptic flow [22]. We show that this hypothesis can be tested directly by studying how the correlation between $(V_2)^2$ and V_4 is correlated with the magnitude of V_2. We also investigate in a similar way the origin of the three-plane correlation between V_2, V_3 and V_5 [6].

II. MEASURING MOMENTS

The statistical properties of V_n are contained in its moments, which are average values of products of V_n, of the form

$$\mathcal{M} \equiv \left\langle \prod_n (V_n)^{k_n} (V_n^*)^{l_n} \right\rangle, \qquad (2)$$

where k_n and l_n are integers, and angular brackets denote an average value over events. Note that $V_n^* = V_{-n}$ and $V_0 = 1$. Azimuthal symmetry implies that the only nontrivial moments satisfy [23]

$$\sum_n n k_n = \sum_n n l_n. \qquad (3)$$

We now describe a simple procedure for measuring these moments, which applies to harmonics $n \geq 2$, i.e., $k_1 = l_1 = 0$. (We do not study moments involving directed flow V_1 [23].) We define in each collision the flow vector [24] by

$$Q_n \equiv \frac{1}{N} \sum_j e^{in\varphi_j}, \qquad (4)$$

where the sum runs over N particles seen in a reference detector, and φ_j are their azimuthal angles. One typically measures Q_n in two different parts of the detector ("subevents" [25]) A and B, which are symmetric around midrapidity and separated by a gap in pseudorapidity (i.e., polar angle) [26]. The moment (2) is then given by

$$\mathcal{M} \equiv \left\langle \prod_n (V_n)^{k_n} (V_n^*)^{l_n} \right\rangle = \left\langle \prod_n (Q_{nA})^{k_n} (Q_{nB}^*)^{l_n} \right\rangle, \qquad (5)$$

which one can symmetrize over A and B to decrease the statistical error. This configuration, with all factors on Q_n on one side and all factors of Q_n^* on the other side [18], suppresses nonflow correlations and self-correlations as long as only harmonics $n \geq 2$ are involved. An alternative procedure, where self correlations are explicitly subtracted, is described in [15].

In order to illustrate the validity of the method, we perform calculations using the AMPT model [17]. AMPT reproduces quite well LHC data for anisotropic flow (v_2 to v_6) at all centralities [27–29]. The implementation adopted in this paper [30] uses initial conditions from the HIJING 2.0 model [31], which contains nontrivial event-by-event fluctuations. Flow in AMPT is produced by elastic scatterings in the partonic phase. In addition, the model contains resonance decays, and thus nontrivial nonflow effects. In the present work, subevent A consists of all particles in the pseudorapidity range $0.4 < \eta < 4.8$, and subevent B is symmetric around mid-rapidity, so that there is an η gap of 0.8 between A and B [32].

The thumb rule for measuring moments is that smaller values of n are easier to measure because v_n decreases with n for $n \geq 2$. Lower order moments, corresponding to smaller values of k_n and l_n, are also easier because higher-order moments are plagued with large variances, which entail large statistical errors.

III. v_n FLUCTUATIONS, EVENT-PLANE CORRELATIONS, STANDARD CANDLES

We first list observables which have been previously studied in the literature and explain how they can be measured using the method outlined in Sec. II. Fluctuations of v_n have been studied using cumulants [33–36], which are linear combinations of even moments of the distribution of v_n, that is, $\langle (v_n)^{2k} \rangle$. These moments are obtained by keeping only one value of n and setting
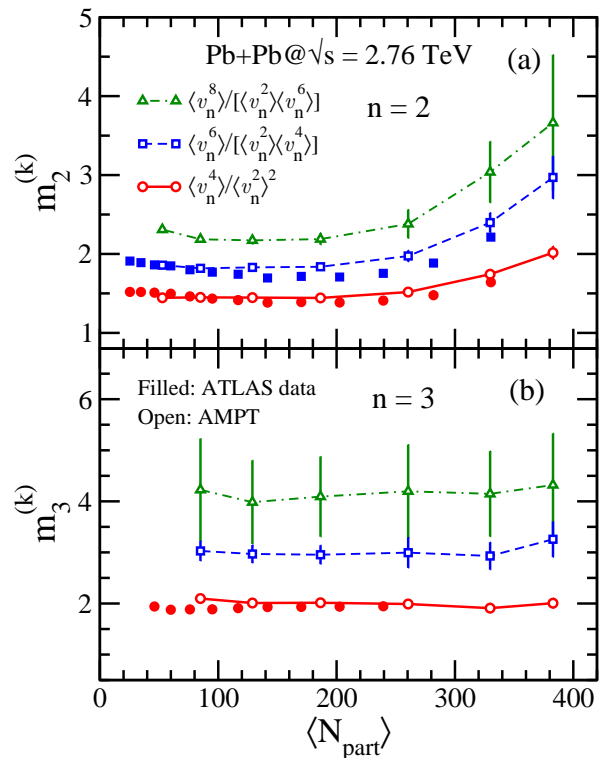

FIG. 1. (Color online) Scaled moments of the distribution of v_n, (see Eq. (6)) for $k = 2, 3, 4$, as a function of centrality, measured with the number of participant nucleons. Results are for (a) elliptic flow, $n = 2$, and (b) triangular flow, $n = 3$, in Pb-Pb collisions at $\sqrt{s} = 2.76$ TeV. Open symbols represent AMPT calculations, and closed symbols are obtained from ATLAS data [10, 14].

$k_n = l_n = k$ in Eq. (5). Figure 1 displays the scaled moments

$$m_n^{(k)} \equiv \frac{\langle v_n^{2k} \rangle}{\langle v_n^{2(k-1)} \rangle \langle v_n^2 \rangle}, \qquad (6)$$

for $k = 2, 3, 4$ as a function of centrality for $n = 2$ and $n = 3$, obtained by using the subevent method of Sec. II. The scaled moment $m_n^{(k)}$ thus defined is invariant if one multiplies v_n by a constant, therefore it reflects the statistics of v_n and should be essentially independent of the detector acceptance. ATLAS data, extracted from the probability distribution of v_n [14],[1] are shown along with AMPT calculations. AMPT calculations are in fair agreement with data, but tend to slightly overpredict $m_n^{(k)}$, i.e., overestimate flow fluctuations.

If flow is solely created by fluctuations and if the statistics of these fluctuations is a 2-dimensional Gaussian [38],

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[1] A recent analysis [37] shows that cumulants obtained from a dedicated analysis are in close agreement with those obtained using the probability distribution of v_n.
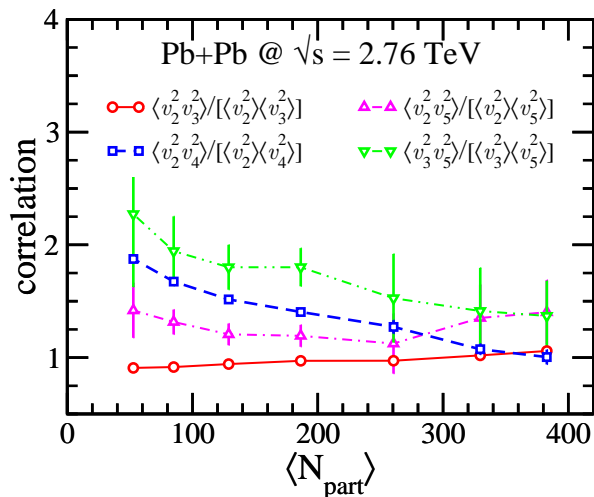
FIG. 2. (Color online) Correlations between v_n^2 and v_m^2 as a function of centrality for $(n,m) = (2,3), (2,4), (2,5), (3,5)$, in Pb-Pb collisions at $\sqrt{s} = 2.76$ TeV calculated in the AMPT model.

then $m_n^{(k)} = k$. As can be seen in Fig. 1(b), $m_3^{(k)} \simeq k$ for all centralities, as expected since v_3 is only from fluctuations in Pb-Pb collisions.[2] Similarly, as seen in Fig. 1(a), $m_2^{(k)}$ is roughly equal to k for central collisions where v_2 is mostly from Gaussian fluctuations, but decreases for mid-central collisions, corresponding to the emergence of a mean elliptic flow in the reaction plane [40].

Event-plane correlations [6] can also be expressed in terms of moments which can be measured using the method outlined in Sec. II, as already discussed in Ref. [18]. Specifically, two-plane correlations are Pearson correlation coefficients between moments.[3] The Pearson correlation coefficient between two complex variables f and g whose average value is 0, $\langle f \rangle = \langle g \rangle = 0$, is defined as

$$r \equiv \frac{\langle fg^* \rangle}{\sqrt{\langle |f|^2 \rangle \langle |g|^2 \rangle}}. \tag{7}$$

$|r| \leq 1$ in general, and $r = 0$ if f and g are uncorrelated. The correlation between the second and fourth harmonic planes, which is denoted by $\langle \cos(4(\Phi_2 - \Phi_4)) \rangle_w$ in Ref. [6], corresponds to $f = V_4$, $g = (V_2)^2$. The largest source of uncertainty in this measurement is the denominator which involves $\langle v_4^2 \rangle$, a measurement quadratic in the small harmonic V_4.

Note that the scaled moments Eq. (6) are of the type $\langle fg \rangle / \langle f \rangle \langle g \rangle$, which is another measure of the correlation between f and g when $\langle f \rangle$ and $\langle g \rangle$ both differ from

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[2] Small deviations from Gaussian statistics are actually seen experimentally and result in a non-zero cumulant $v_3\{4\}$ [39]. Our simulation does not have enough statistics to detect this small non-Gaussianity.
[3] Three- and four-plane [41] correlations are not Pearson coefficients and are not bounded by unity [42].
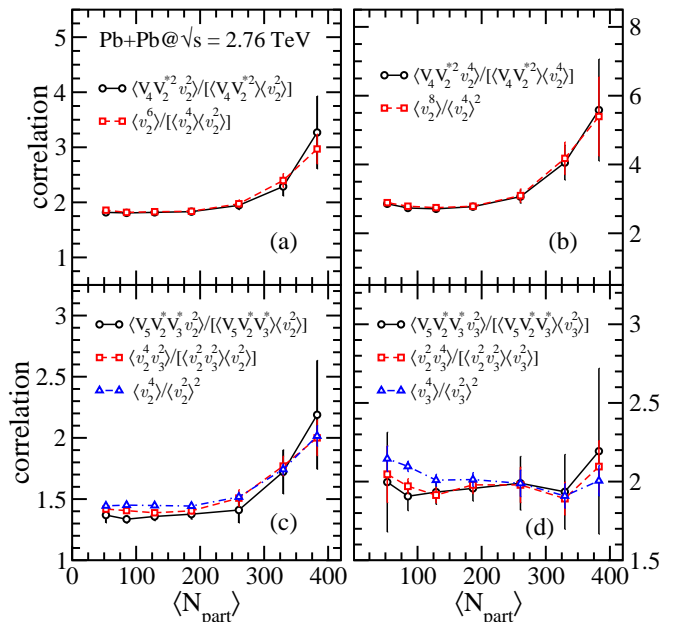

FIG. 3. (Color online) Correlations between event-plane correlations and anisotropic flow calculated in AMPT, and tests of Eqs. (9), (10) and (12). Circles connected by solid lines correspond to the left-hand sides, and squares connected by dashed lines correspond to the right-hand sides of these equations.

0. This correlation measure equals unity if f and g are uncorrelated, and is larger than unity if there is a positive correlation. It is in general easier to measure than the Pearson correlation coefficient, because it does not involve the higher-order moments $\langle |f|^2 \rangle$ and $\langle |g|^2 \rangle$.

The "standard candles" introduced in Ref. [15] correspond to the case $f = v_n^2$, $g = v_m^2$, obtained by keeping only two harmonics, n and m, and setting $k_n = k_m = l_n = l_m = 1$ in Eq. (5). These are correlations between the magnitudes v_n and v_m, which do not involve the angular correlation between event planes. Four of these correlations are displayed in Fig. 2. The correlation between v_2^2 and v_3^2 is small and negative ($\langle v_2^2 v_3^2 \rangle - \langle v_2^2 \rangle \langle v_3^2 \rangle < 0$), while the correlation between the corresponding event planes is small and positive [6]. All other correlations are positive. The correlation between v_4 and v_2, and that between v_5 and v_3, become smaller for more central collisions, which is likely due to the smaller non-linear contributions [21] of v_4 and v_5, respectively.

IV. TESTING THE NON-LINEAR RESPONSE USING MOMENTS

We now introduce new correlation measures of the type $\langle fg \rangle / \langle f \rangle \langle g \rangle$ in order to study how event-plane correlations are correlated with the magnitude of anisotropic flow. We first consider the case $f = V_4(V_2^*)^2$ and $g = v_2^2$. $\langle fg \rangle$ is obtained by setting $k_2 = k_4 = 1$ and $l_2 = 3$

in Eq. (5). The correlation $\langle fg \rangle / \langle f \rangle \langle g \rangle$ is displayed in Fig. 3(a). There is a significant positive correlation for all centralities, which becomes larger for central collisions. In hydrodynamics, the correlation between V_4 and $(V_2)^2$ originates from a non-linear response [19]. In order to test this hypothesis, we model V_4 as the sum of two terms:

$$V_4 = V_{4l} + \beta V_2^2, \qquad (8)$$

where the non-linear response coefficient β is the same for all events in a centrality class. This corresponds to the separation of V_4 into a linear part, created by fluctuations, and a non-linear part from V_2 [11]. We assume in addition that V_{4l} and V_2 are uncorrelated. Equation (8) then implies the following relation between moments:

$$\frac{\langle V_4 (V_2^*)^2 v_2^2 \rangle}{\langle V_4 (V_2^*)^2 \rangle \langle v_2^2 \rangle} = \frac{\langle v_2^6 \rangle}{\langle v_2^4 \rangle \langle v_2^2 \rangle}. \qquad (9)$$

This equation relates a mixed correlation between V_4 and V_2 to the fluctuations of elliptic flow: the right-hand side is the scaled moment $m_2^{(3)}$ introduced in Eq. (6). The AMPT simulations support Eq. (9) for all centralities, as can be seen in Fig. 3(a). A straightforward generalization of Eq. (9) is obtained using $g = v_2^4$ instead of v_2^2:

$$\frac{\langle V_4 (V_2^*)^2 v_2^4 \rangle}{\langle V_4 (V_2^*)^2 \rangle \langle v_2^4 \rangle} = \frac{\langle v_2^8 \rangle}{\langle v_2^4 \rangle^2}. \qquad (10)$$

This correlation gives a higher weight to events with large elliptic flow. Equation (10) is also supported by AMPT simulations, as shown in Fig. 3(b).

This discussion can be readily extended to the correlation between the 5th harmonic plane and the 2nd and 3rd harmonic planes. We now set $f = V_5 V_2^* V_3^*$ and $g = v_2^2$ or $g = v_3^2$ and write

$$V_5 = V_{5l} + \beta' V_2 V_3, \qquad (11)$$

where V_{5l} is independent of V_2 and V_3 and β' is constant. One thus obtains

$$\frac{\langle V_5 V_2^* V_3^* v_2^2 \rangle}{\langle V_5 V_2^* V_3^* \rangle \langle v_2^2 \rangle} = \frac{\langle v_2^4 v_3^2 \rangle}{\langle v_2^2 v_3^2 \rangle \langle v_2^2 \rangle},$$
$$\frac{\langle V_5 V_2^* V_3^* v_3^2 \rangle}{\langle V_5 V_2^* V_3^* \rangle \langle v_3^2 \rangle} = \frac{\langle v_2^2 v_3^4 \rangle}{\langle v_2^2 v_3^2 \rangle \langle v_3^2 \rangle}. \qquad (12)$$

The numerators in the right-hand side are obtained using Eq. (5) with $k_2 = l_2 = 2$, $k_3 = l_3 = 1$ (first line) and $k_2 = l_2 = 1$, $k_3 = l_3 = 2$ (second line). Figs. 3(c)-(d)

again show that these equalities are very well verified by AMPT simulations. One can also use the fact that v_2 and v_3 are weakly correlated, as seen in Fig. 2, which leads to the following simplified relations:

$$\frac{\langle v_2^4 v_3^2 \rangle}{\langle v_2^2 v_3^2 \rangle \langle v_2^2 \rangle} \simeq \frac{\langle v_2^4 \rangle}{\langle v_2^2 \rangle^2},$$
$$\frac{\langle v_2^2 v_3^4 \rangle}{\langle v_2^2 v_3^2 \rangle \langle v_3^2 \rangle} \simeq \frac{\langle v_3^4 \rangle}{\langle v_3^2 \rangle^2}. \qquad (13)$$

These relations are also satisfied to a good approximation (see Figs. 3(c)-(d)), thus showing that the correlators in Eq. (12) are mostly driven by the fluctuations of v_2 and v_3.

V. CONCLUSION

Moments of the distribution of V_n provide a complete set of multiparticle correlation observables, which can be used to probe the physics of flow fluctuations in unprecedented detail. All these moments can be measured using just two subevents separated by a rapidity gap. Moments yield the full information on the multiparticle correlations without resorting to unfolding procedures [14, 43] or event-shape engineering [13, 44, 45]. They can be measured easily at LHC and even with detectors having smaller acceptance, and can be directly compared with theoretical calculations. In particular, scaled moments [23] as studied in this paper are typically independent of the details of the acceptance, and reflect global fluctuations. They can be used to further probe the physics of initial-state fluctuations and the hydrodynamic response of the quark-gluon plasma.

We have shown that the assumption that the correlation between V_4 and V_2, and that between V_5, V_2 and V_3, are driven by nonlinear response, yields nontrivial relations between moments, Eqs. (9), (10) and (12). Simulations within the AMPT model have shown that these relations are very well satisfied. It is important to test if experimental data confirm these predictions.

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[1] U. Heinz and R. Snellings, Ann. Rev. Nucl. Part. Sci. **63**, 123 (2013) [arXiv:1301.2826 [nucl-th]].

[2] M. Luzum, J. Phys. G **38**, 124026 (2011) [arXiv:1107.0592 [nucl-th]].

[3] M. Miller and R. Snellings, nucl-ex/0312008.

[4] B. Alver *et al.* [PHOBOS Collaboration], Phys. Rev. Lett. **98**, 242302 (2007) [nucl-ex/0610037].

[5] B. Alver and G. Roland, Phys. Rev. C **81**, 054905 (2010)

[Erratum-ibid. C **82**, 039903 (2010)] [arXiv:1003.0194 [nucl-th]].

[6] G. Aad *et al.* [ATLAS Collaboration], Phys. Rev. C **90**, 024905 (2014) [arXiv:1403.0489 [hep-ex]].

[7] M. Luzum and H. Petersen, J. Phys. G **41**, 063102 (2014) [arXiv:1312.5503 [nucl-th]].

[8] M. Luzum and J. Y. Ollitrault, Nucl. Phys. A **904-905**, 377c (2013) [arXiv:1210.6010 [nucl-th]].

[9] T. Renk and H. Niemi, Phys. Rev. C **89**, 064907 (2014) [arXiv:1401.2069 [nucl-th]].

[10] L. Yan, J. Y. Ollitrault and A. M. Poskanzer, arXiv:1408.0921 [nucl-th].

[11] D. Teaney and L. Yan, Phys. Rev. C **86**, 044908 (2012) [arXiv:1206.1905 [nucl-th]].

[12] L. V. Bravina, B. H. Brusheim Johansson, G. K. Eyyubova, V. L. Korotkikh, I. P. Lokhtin, L. V. Malinina, S. V. Petrushanko and A. M. Snigirev *et al.*, Phys. Rev. C **89**, no. 2, 024909 (2014) [arXiv:1311.0747 [hep-ph]].

[13] J. Jia, arXiv:1407.6057 [nucl-ex].

[14] G. Aad *et al.* [ATLAS Collaboration], JHEP **1311**, 183 (2013) [arXiv:1305.2942 [hep-ex]].

[15] A. Bilandzic, C. H. Christensen, K. Gulbrandsen, A. Hansen and Y. Zhou, Phys. Rev. C **89**, 064904 (2014) [arXiv:1312.3572 [nucl-ex]].

[16] A. Bilandzic, arXiv:1409.5636 [nucl-ex].

[17] Z. W. Lin, C. M. Ko, B. A. Li, B. Zhang and S. Pal, Phys. Rev. C **72**, 064901 (2005) [nucl-th/0411110].

[18] R. S. Bhalerao, J. Y. Ollitrault and S. Pal, Phys. Rev. C **88**, 024909 (2013) [arXiv:1307.0980 [nucl-th]].

[19] D. Teaney and L. Yan, Phys. Rev. C **90**, 024902 (2014) [arXiv:1312.3689 [nucl-th]].

[20] N. Borghini and J. Y. Ollitrault, Phys. Lett. B **642**, 227 (2006) [nucl-th/0506045].

[21] F. G. Gardim, F. Grassi, M. Luzum and J. Y. Ollitrault, Phys. Rev. C **85**, 024908 (2012) [arXiv:1111.6538 [nucl-th]].

[22] K. Aamodt *et al.* [ALICE Collaboration], Phys. Rev. Lett. **105**, 252302 (2010) [arXiv:1011.3914 [nucl-ex]].

[23] R. S. Bhalerao, M. Luzum and J. Y. Ollitrault, Phys. Rev. C **84**, 034910 (2011) [arXiv:1104.4740 [nucl-th]].

[24] A. M. Poskanzer and S. A. Voloshin, Phys. Rev. C **58**, 1671 (1998) [nucl-ex/9805001].

[25] P. Danielewicz and G. Odyniec, Phys. Lett. B **157**, 146 (1985).

[26] S. S. Adler *et al.* [PHENIX Collaboration], Phys. Rev. Lett. **91**, 182301 (2003) [nucl-ex/0305013].

[27] L. X. Han, G. L. Ma, Y. G. Ma, X. Z. Cai, J. H. Chen, S. Zhang and C. Zhong, Phys. Rev. C **84**, 064907 (2011) [arXiv:1105.5415 [nucl-th]].

[28] J. Xu and C. M. Ko, Phys. Rev. C **84**, 044907 (2011) [arXiv:1108.0717 [nucl-th]].

[29] S. Pal and M. Bleicher, J. Phys. Conf. Ser. **420**, 012027 (2013) [arXiv:1209.0335 [nucl-th]].

[30] S. Pal and M. Bleicher, Phys. Lett. B **709**, 82 (2012) [arXiv:1201.2546 [nucl-th]].

[31] W. T. Deng, X. N. Wang and R. Xu, Phys. Rev. C **83**, 014915 (2011) [arXiv:1008.1841 [hep-ph]].

[32] M. Luzum, Phys. Lett. B **696**, 499 (2011) [arXiv:1011.5773 [nucl-th]].

[33] N. Borghini, P. M. Dinh and J. Y. Ollitrault, Phys. Rev. C **64**, 054901 (2001) [nucl-th/0105040].

[34] J. Adams *et al.* [STAR Collaboration], Phys. Rev. C **72**, 014904 (2005) [nucl-ex/0409033].

[35] A. Bilandzic, R. Snellings and S. Voloshin, Phys. Rev. C **83**, 044913 (2011) [arXiv:1010.0233 [nucl-ex]].

[36] A. Bzdak, P. Bozek and L. McLerran, Nucl. Phys. A **927**, 15 (2014) [arXiv:1311.7325 [hep-ph]].

[37] G. Aad *et al.* [ATLAS Collaboration], arXiv:1408.4342 [hep-ex].

[38] S. A. Voloshin, A. M. Poskanzer, A. Tang and G. Wang, Phys. Lett. B **659**, 537 (2008) [arXiv:0708.0800 [nucl-th]].

[39] K. Aamodt *et al.* [ALICE Collaboration], Phys. Rev. Lett. **107**, 032301 (2011) [arXiv:1105.3865 [nucl-ex]].

[40] J. Y. Ollitrault, Phys. Rev. D **46**, 229 (1992).

[41] J. Jia and D. Teaney, Eur. Phys. J. C **73**, 2558 (2013) [arXiv:1205.3585 [nucl-ex]].

[42] M. Luzum and J. Y. Ollitrault, Phys. Rev. C **87**, no. 4, 044907 (2013) [arXiv:1209.2323 [nucl-ex]].

[43] B. Alver *et al.* [PHOBOS Collaboration], Phys. Rev. Lett. **104**, 142301 (2010) [nucl-ex/0702036].

[44] J. Schukraft, A. Timmins and S. A. Voloshin, Phys. Lett. B **719**, 394 (2013) [arXiv:1208.4563 [nucl-ex]].

[45] A. Dobrin [ALICE Collaboration], Nucl. Phys. A **904-905**, 455c (2013) [arXiv:1211.5348 [nucl-ex]].